Appendix D. Integrals

The Psin function are used as trial functions to approximate the displacement field in different models. Let us use *u* and *w* as two functions of displacement.

$$u(x)=\tfrac{1}{2}\left[cos\left(\alpha_p\frac{2x}{L_b}+\beta_p\right)-cos\left(\gamma_p\frac{2x}{L_b}+\delta_p\right)\right]=P\,sin_p\left(\frac{2x}{L_b}\right) \tag{D.1}$$

$$w(x)=\tfrac{1}{2}\left[cos\left(\alpha_q\frac{2x}{L_b}+\beta_q\right)-cos\left(\gamma_q\frac{2x}{L_b}+\delta_q\right)\right]=P\,sin_q\left(\frac{2x}{L_b}\right) \tag{D.2}$$

This appendix present some integrals required when using these functions in a variational model. The first integrals concern one displacement function only.

$$\int_{x_a-\frac{L_a}{2}}^{x_a+\frac{L_a}{2}} w(x)dx=\frac{L_b}{2}G1_q(x_a,L_a) \tag{D.3}$$

$$\int_{x_a-\frac{L_a}{2}}^{x_a+\frac{L_a}{2}} \frac{\partial w(x)}{\partial x}dx=\frac{1}{4}G2_q(x_a,L_a) \tag{D.4}$$

$$\int_{x_a-\frac{L_a}{2}}^{x_a+\frac{L_a}{2}} \frac{\partial^2 w(x)}{\partial x^2}dx=\frac{1}{L_b^3}G3_q(x_a,L_a) \tag{D.5}$$

These integrals are in term of dimensionless functions *G1*, *G2* and *G3*, very similar in their form.

$$G1_q(x_a,L_a)=\frac{1}{\alpha_q}sin\left(\alpha_q\frac{L_a}{L_b}\right)cos\left(\alpha_q\frac{2x_a}{L_b}+\beta_q\right) -\frac{1}{\gamma_q}sin\left(\gamma_q\frac{L_a}{L_b}\right)cos\left(\gamma_q\frac{2x_a}{L_b}+\delta_q\right)$$

$$\begin{aligned} &if \quad \alpha_q = 0, \ the \ first \ the \ term \ is \ \frac{L_a}{L_b} cos(\beta_q), \\ &if \ \gamma_q = 0, \ the \ second \ term \ is \ \frac{L_a}{L_b} cos(\delta_q) \end{aligned} \tag{D.6}$$

$$\begin{aligned} G2_q(x_a, L_a) = &sin\left(\alpha_q \frac{L_a}{L_b}\right) sin\left(\alpha_q \frac{2x_a}{L_b} + \beta_q\right) \\ &+ sin\left(\gamma_q \frac{L_a}{L_b}\right) sin\left(\gamma_q \frac{2x_a}{L_b} + \delta_q\right) \end{aligned} \tag{D.7}$$

$$\begin{aligned} G3_q(x_a, L_a) = &-\alpha_q \ sin\left(\alpha_q \frac{L_a}{L_b}\right) sin\left(\alpha_q \frac{2x_a}{L_b} + \beta_q\right) \\ &+ \gamma_q \ sin\left(\gamma_q \frac{L_a}{L_b}\right) sin\left(\gamma_q \frac{2x_a}{L_b} + \delta_q\right) \end{aligned} \tag{D.8}$$

The next list of integral permits to determine the coeficient of the mass and stiffness matices:

$$\int_{x_a - \frac{L_a}{2}}^{x_a + \frac{L_a}{2}} u(x) w(x) dx = \frac{L_b}{8} F4_{pq}(x_a, L_a) \tag{D.9}$$

$$\int_{x_a - \frac{L_a}{2}}^{x_a + \frac{L_a}{2}} \frac{\partial u(x)}{\partial x} \frac{\partial w(x)}{\partial x} dx = \frac{1}{2L_b} F1_{pq}(x_a, L_a) \tag{D.10}$$

$$\int_{x_a - \frac{L_a}{2}}^{x_a + \frac{L_a}{2}} \frac{\partial^2 u(x)}{\partial x^2} \frac{\partial^2 w(x)}{\partial x^2} dx = \frac{2}{L_b^3} F2_{pq}(x_a, L_a) \tag{D.11}$$

$$\int_{x_a - \frac{L_a}{2}}^{x_a + \frac{L_a}{2}} \frac{\partial^2 u(x)}{\partial x^2} w(x) dx = \frac{1}{2L_b} F5_{pq}(x_a, L_a) \tag{D.12}$$

These integrals are in term of four dimentionless functions *F4*, *F1*, *F2*, and *F5* very similar in their form:

$$F4_{pq}(x_a, L_a) = \left[SC11_{pq}(x_a, L_a) + SC12_{pq}(x_a, L_a)\right]$$

$$
\begin{aligned}
&+\left[SC21_{pq}(x_a,L_a)+SC22_{pq}(x_a,L_a)\right]\\
&-\left[SC31_{pq}(x_a,L_a)+SC32_{pq}(x_a,L_a)\right]\\
&-\left[SC41_{pq}(x_a,L_a)+SC42_{pq}(x_a,L_a)\right]
\end{aligned} \tag{D.13}
$$

$$
\begin{aligned}
F1_{pq}(x_a,L_a)=&-\alpha_p\alpha_q\left[SC11_{pq}(x_a,L_a)+SC12_{pq}(x_a,L_a)\right]\\
&-\gamma_p\gamma_q\left[SC21_{pq}(x_a,L_a)+SC22_{pq}(x_a,L_a)\right]\\
&+\alpha_p\gamma_q\left[SC31_{pq}(x_a,L_a)+SC32_{pq}(x_a,L_a)\right]\\
&+\gamma_p\alpha_q\left[SC41_{pq}(x_a,L_a)+SC42_{pq}(x_a,L_a)\right]
\end{aligned} \tag{D.14}
$$

$$
\begin{aligned}
F2_{pq}(x_a,L_a)=&\,\alpha_p^2\alpha_q^2\left[SC11_{pq}(x_a,L_a)+SC12_{pq}(x_a,L_a)\right]\\
&+\gamma_p^2\gamma_q^2\left[SC21_{pq}(x_a,L_a)+SC22_{pq}(x_a,L_a)\right]\\
&-\alpha_p^2\gamma_q^2\left[SC31_{pq}(x_a,L_a)+SC32_{pq}(x_a,L_a)\right]\\
&-\gamma_p^2\alpha_q^2\left[SC41_{pq}(x_a,L_a)+SC42_{pq}(x_a,L_a)\right]
\end{aligned} \tag{D.15}
$$

$$
\begin{aligned}
F5_{pq}(x_a,L_a)=&-\alpha_p^2\left[SC11_{pq}(x_a,L_a)+SC12_{pq}(x_a,L_a)\right]\\
&+\gamma_p^2\left[SC21_{pq}(x_a,L_a)+SC22_{pq}(x_a,L_a)\right]\\
&-\alpha_p^2\left[SC31_{pq}(x_a,L_a)+SC32_{pq}(x_a,L_a)\right]\\
&-\gamma_p^2\left[SC41_{pq}(x_a,L_a)+SC42_{pq}(x_a,L_a)\right]
\end{aligned} \tag{D.16}
$$

These functions are based on 8 elementary functions *SC11*, *SC12*, *SC21*, *SC22*, *SC31*, SC32, *SC41*, and *SC42*. They also are very similar and use the Psin coefficients α_n, β_n, γ_n, and δ_n.

$$
SC11_{pq}(x_a,L_a)=\frac{sin\left[(\alpha_p+\alpha_q)\frac{L_a}{L_b}\right]cos\left[(\alpha_p+\alpha_q)\frac{2x_a}{L_b}+(\beta_p+\beta_q)\right]}{\alpha_p+\alpha_q}
$$

$$
for \quad \alpha_p+\alpha_q\neq 0 \quad else \qquad SC11_{pq}(x_a,L_a)=\frac{L_a}{L_b}cos(\beta_p+\beta_q) \tag{D.17}
$$

$$SC12_{pq}(x_a,L_a)=\frac{sin\left[(\alpha_p-\alpha_q)\frac{L_a}{L_b}\right]cos\left[(\alpha_p-\alpha_q)\frac{2x_a}{L_b}+(\beta_p-\beta_q)\right]}{\alpha_p-\alpha_q}$$

$$for\quad \alpha_p-\alpha_q\neq 0\quad else\qquad SC12_{pq}(x_a,L_a)=\frac{L_a}{L_b}cos(\beta_p-\beta_q)\tag{D.18}$$

$$SC21_{pq}(x_a,L_a)=\frac{sin\left[(\gamma_p+\gamma_q)\frac{L_a}{L_b}\right]cos\left[(\gamma_p+\gamma_q)\frac{2x_a}{L_b}+(\delta_p+\delta_q)\right]}{\gamma_p+\gamma_q}$$

$$for\quad \gamma_p+\gamma_q\neq 0\quad else\qquad SC21_{pq}(x_a,L_a)=\frac{L_a}{L_b}cos(\delta_p+\delta_q)\tag{D.19}$$

$$SC22_{pq}(x_a,L_a)=\frac{sin\left[(\gamma_p-\gamma_q)\frac{L_a}{L_b}\right]cos\left[(\gamma_p-\gamma_q)\frac{2x_a}{L_b}+(\delta_p-\delta_q)\right]}{\gamma_p-\gamma_q}$$

$$for\quad \gamma_p-\gamma_q\neq 0\quad else\qquad SC22_{pq}(x_a,L_a)=\frac{L_a}{L_b}cos(\delta_p-\delta_q)\tag{D.20}$$

$$SC31_{pq}(x_a,L_a)=\frac{sin\left[(\alpha_p+\gamma_q)\frac{L_a}{L_b}\right]cos\left[(\alpha_p+\gamma_q)\frac{2x_a}{L_b}+(\beta_p+\delta_q)\right]}{\alpha_p+\gamma_q}$$

$$for\quad \alpha_p+\gamma_q\neq 0\quad else\qquad SC31_{pq}(x_a,L_a)=\frac{L_a}{L_b}cos(\beta_p+\delta_q)\tag{D.21}$$

$$SC32_{pq}(x_a,L_a)=\frac{sin\left[(\alpha_p-\gamma_q)\frac{L_a}{l_b}\right]cos\left[(\alpha_p-\gamma_q)\frac{2x_a}{l_b}+(\beta_p-\delta_q)\right]}{\alpha_p-\gamma_q}$$

$$for\quad \alpha_p-\gamma_q\neq 0\quad else\qquad SC32_{pq}(x_a,L_a)=\frac{L_a}{L_b}cos(\beta_p-\delta_q)\tag{D.22}$$

$$SC41_{pq}(x_a,L_a)=\frac{sin\left[(\gamma_p+\alpha_q)\frac{L_a}{L_b}\right]cos\left[(\gamma_p+\alpha_q)\frac{2x_a}{L_b}+(\delta_p+\beta_q)\right]}{\gamma_p+\alpha_q}$$

$$for \quad \gamma_p+\alpha_q\neq 0 \quad else \qquad SC41_{pq}(x_a,L_a)=\frac{L_a}{L_b}cos(\delta_p+\beta_q) \tag{D.23}$$

$$SC42_{pq}(x_a,L_a)=\frac{sin\left[(\gamma_p-\alpha_q)\frac{L_a}{l_b}\right]cos\left[(\gamma_p-\alpha_q)\frac{2x_a}{l_b}+(\delta_p-\beta_q)\right]}{\gamma_p-\alpha_q}$$

$$for \quad \gamma_p-\alpha_q\neq 0 \quad else \qquad SC42_{pq}(x_a,L_a)=\frac{L_a}{L_b}cos(\delta_p-\beta_q) \tag{D.24}$$

Some other integrals are needed especially to compute non-diagonal terms:

$$\int_{x_a-\frac{L_a}{2}}^{x_a+\frac{L_a}{2}} u(x)\frac{\partial w(x)}{\partial x}dx=\frac{1}{4}F6_{pq}(x_a,L_a) \tag{D.25}$$

$$\int_{x_a-\frac{L_a}{2}}^{x_a+\frac{L_a}{2}} \frac{\partial u(x)}{\partial x}\frac{\partial^2 w(x)}{\partial x^2}dx=\frac{1}{L_b^2}F3_{pq}(x_a,L_a) \tag{D.26}$$

These integrals are based on a different type of functions *F6* and *F3*:

$$\begin{aligned}F6_{pq}(x_a,L_a)=&-\alpha_q\left[SS11_{pq}(x_a,L_a)+SS12_{pq}(x_a,L_a)\right]\\&-\gamma_q\left[SS21_{pq}(x_a,L_a)+SS22_{pq}(x_a,L_a)\right]\\&+\gamma_q\left[SS31_{pq}(x_a,L_a)+SS32_{pq}(x_a,L_a)\right]\\&+\alpha_q\left[SS41_{pq}(x_a,L_a)+SS42_{pq}(x_a,L_a)\right]\end{aligned} \tag{D.27}$$

$$\begin{aligned}F3_{pq}(x_a,L_a)=&\,\alpha_p\alpha_q^2\left[SS11_{pq}(x_a,L_a)+SS12_{pq}(x_a,L_a)\right]\\&+\gamma_p\gamma_q^2\left[SS21_{pq}(x_a,L_a)+SS22_{pq}(x_a,L_a)\right]\\&-\alpha_p\gamma_q^2\left[SS31_{pq}(x_a,L_a)+SS32_{pq}(x_a,L_a)\right]\\&-\gamma_p\alpha_q^2\left[SS41_{pq}(x_a,L_a)+SS42_{pq}(x_a,L_a)\right]\end{aligned} \tag{D.28}$$

These functions are in term of 8 other elementary functions *SS11*, *SS12*, *SS21*, *SS22*, *SS31*, *SS32*, *SS41*, and *SS42* also based on the Psin coefficients α_n, β_n, γ_n, and δ_n.

$$SS11_{pq}(x_a,L_a)=\frac{sin\left[(\alpha_p+\alpha_q)\frac{L_a}{L_b}\right]sin\left[(\alpha_p+\alpha_q)\frac{2x_a}{L_b}+(\beta_p+\beta_q)\right]}{\alpha_p+\alpha_q}$$
$$for\quad \alpha_p+\alpha_q\neq 0\quad else\qquad SS11_{pq}(x_a,L_a)=\frac{L_a}{L_b}sin(\beta_p+\beta_q)\tag{D.29}$$

$$SS12_{pq}(x_a,L_a)=\frac{sin\left[(\alpha_p-\alpha_q)\frac{L_a}{L_b}\right]sin\left[(\alpha_p-\alpha_q)\frac{2x_a}{L_b}+(\beta_p-\beta_q)\right]}{\alpha_p-\alpha_q}$$
$$for\quad \alpha_p-\alpha_q\neq 0\quad else\qquad SS12_{pq}(x_a,L_a)=\frac{L_a}{L_b}sin(\beta_p-\beta_q)\tag{D.30}$$

$$SS21_{pq}(x_a,L_a)=\frac{sin\left[(\gamma_p+\gamma_q)\frac{L_a}{L_b}\right]sin\left[(\gamma_p+\gamma_q)\frac{2x_a}{L_b}+(\delta_p+\delta_q)\right]}{\gamma_p+\gamma_q}$$
$$for\quad \gamma_p+\gamma_q\neq 0\quad else\qquad SS21_{pq}(x_a,L_a)=\frac{L_a}{L_b}sin(\delta_p+\delta_q)\tag{D.31}$$

$$SS22_{pq}(x_a,L_a)=\frac{sin\left[(\gamma_p-\gamma_q)\frac{L_a}{L_b}\right]sin\left[(\gamma_p-\gamma_q)\frac{2x_a}{L_b}+(\delta_p-\delta_q)\right]}{\gamma_p-\gamma_q}$$
$$for\quad \gamma_p-\gamma_q\neq 0\quad else\qquad SS22_{pq}(x_a,L_a)=\frac{L_a}{L_b}sin(\delta_p-\delta_q)\tag{D.32}$$

$$SS31_{pq}(x_a,L_a)=\frac{sin\left[(\alpha_p+\gamma_q)\frac{L_a}{L_b}\right]sin\left[(\alpha_p+\gamma_q)\frac{2x_a}{L_b}+(\beta_p+\delta_q)\right]}{\alpha_p+\gamma_q}$$

$$for \quad \alpha_p + \gamma_q \neq 0 \quad else \qquad SS31_{pq}(x_a, L_a) = \frac{L_a}{L_b} sin(\beta_p + \delta_q) \tag{D.33}$$

$$SS32_{pq}(x_a, L_a) = \frac{sin\left[(\alpha_p - \gamma_q)\frac{L_a}{l_b}\right] sin\left[(\alpha_p - \gamma_q)\frac{2x_a}{l_b} + (\beta_p - \delta_q)\right]}{\alpha_p - \gamma_q}$$

$$for \quad \alpha_p - \gamma_q \neq 0 \quad else \qquad SS32_{pq}(x_a, L_a) = \frac{L_a}{L_b} sin(\beta_p - \delta_q) \tag{D.34}$$

$$SS41_{pq}(x_a, L_a) = \frac{sin\left[(\gamma_p + \alpha_q)\frac{L_a}{L_b}\right] sin\left[(\gamma_p + \alpha_q)\frac{2x_a}{L_b} + (\delta_p + \beta_q)\right]}{\gamma_p + \alpha_q}$$

$$for \quad \gamma_p + \alpha_q \neq 0 \quad else \qquad SS41_{pq}(x_a, L_a) = \frac{L_a}{L_b} sin(\delta_p + \beta_q) \tag{D.35}$$

$$SS42_{pq}(x_a, L_a) = \frac{sin\left[(\gamma_p - \alpha_q)\frac{L_a}{l_b}\right] sin\left[(\gamma_p - \alpha_q)\frac{2x_a}{l_b} + (\delta_p - \beta_q)\right]}{\gamma_p - \alpha_q}$$

$$for \quad \gamma_p - \alpha_q \neq 0 \quad else \qquad SS42_{pq}(x_a, L_a) = \frac{L_a}{L_b} sin(\delta_p - \beta_q) \tag{D.36}$$

Some other integrals of third order are needed for modeling the distributed absorber:

$$\int_{x_a - \frac{L_a}{2}}^{x_a + \frac{L_a}{2}} u(x) v(x) w(x) dx = \frac{L_b}{32} E4_{pq}(x_a, L_a) \tag{D.37}$$

$$\int_{x_a - \frac{L_a}{2}}^{x_a + \frac{L_a}{2}} u(x) v(x) \frac{\partial w(x)}{\partial x} dx = \frac{1}{16} E6_{pq}(x_a, L_a) \tag{D.38}$$

$$\int_{x_a - \frac{L_a}{2}}^{x_a + \frac{L_a}{2}} u(x) \frac{\partial v(x)}{\partial x} \frac{\partial w(x)}{\partial x} dx = \frac{1}{8L_b} E1_{pq}(x_a, L_a) \tag{D.39}$$

The detail of these integrals would be too long to present in this appendix since the the size of the expressions is multipled by 4. However once the previous integrals are programed, (D,37), (D,38) and (D,39) are not to difficult to programed.