# Game Theoretic Models of Connectivity Among Internet Access Providers

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(ABSTRACT)

The Internet has a loosely hierarchical structure. At the top of the hierarchy are the backbones, also called Internet Access Providers (hereafter IAPs). The second layer of the hierarchy is comprised of Internet Service Providers (hereafter ISPs). At the bottom of the hierarchy are the end users, consumers, who browse the web, and websites. To provide access to the whole Internet, the providers must interconnect with each other and share their network infrastructure. Two main forms of interconnection have emerged — peering under which the providers carry each other's traffic without any payments and transit under which the downstream provider pays the upstream provider a certain settlement payment for carrying its traffic.

This dissertation develops three game theoretical models to describe the interconnection agreements among the providers, and analysis of those models from two alternative modeling perspectives: a purely non-cooperative game and a network perspective. There are two original contributions of the dissertation. First, we model the formation of peering/transit contracts explicitly as a decision variable in a non-cooperative game, while the current literature does not employ such modelling techniques. Second, we apply network analysis to examine interconnection decisions of the providers, which yields much realistic results.

Chapter 1 provides a brief description of the Internet history, architecture and infrastructure as well as the economic literature. In Chapter 2 we develop a model, in which IAPs decide on private peering agreements, comparing the benefits of private peering relative to being connected only through National Access Points (hereafter NAPs). The model is formulated as a multistage game. Private peering agreements reduce congestion in the Internet, and so improve the quality of IAPs. The results show that even though the profits are lower with private peerings, due to large investments, the network where all the providers privately peer is the stable network.

Chapter 3 discusses the interconnection arrangements among ISPs. Intra-backbone peering refers to peering between ISPs connected to the same backbone, whereas inter-backbone peering refers to peering between ISPs connected to different backbones. We formulate the model as a two-stage game. Peering affects profits through two channels - reduction of backbone congestion and ability to send traffic circumventing congested backbones. The relative magnitude of these factors helps or hinders peering. In Chapter 4 we develop a game theoretic model to examine how providers decide who they want to peer with and who has to pay transit. There is no regulation with regard to interconnection policies of providers, though there is a general convention that the providers peer if they perceive equal benefits from peering, and have transit arrangements otherwise. The model discusses a set of conditions, which determine the formation of peering and transit agreements. We argue that market forces determine the terms of interconnection, and there is no need for regulation to encourage peering. Moreover, Pareto optimum is achieved under the transit arrangements.

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# Chapter 1

# Introduction

The Internet is a worldwide system of interconnected computer networks, in which users at any computer can (if they have permission) communicate with any other computer in the Internet (Hall, 2000, http://whatis.techtarget.com). The Internet started largely as a public sector endeavor, but subsequently became commercialized. Today, the Internet is constructed of many distinct networks, which are operated by different companies. Each company has its own network, where the connected users can communicate with each other. But the users are not interested in being able to access only the other users of the same network. Rather, they want to have access to all possible users, regardless of the network they are attached to. To provide such global connectivity to their users, the companies must interconnect with each other and share their network infrastructure.

The companies interconnect with each other and exchange traffic under two types of agreements: peering and transit. Under the *peering agreement* companies carry each other's traffic without paying each other. Under the *transit agreement* one company pays the other one a certain settlement payment for carrying its traffic.

The structure of the Internet is loosely hierarchical. At the top of the hierarchy are the backbones, also called Internet Access Providers (IAPs), that own or lease national and/or international high-speed fiber optic networks. These companies exchange traffic with each other under peering agreements either at the National Access Points (NAPs) or through direct private peerings. National Access Points are major interconnection points that serve to tie the providers together. An NAP is basically an Internet exchange point where the providers meet to exchange traffic with the other attached providers. *Public Peering* is peering at the NAPs. *Private peering* is peering between parties that are bypassing NAPs. The second layer of the hierarchy consists of Internet Service Providers (ISPs) that have mostly transit agreements with the IAPs. These providers can also form private peering agreements with each other. At the bottom of the hierarchy are the end users: consumers, who browse the web, and content providers or websites.

This dissertation uses game theoretic tools to investigate interconnection arrangements among providers at different layers of the Internet hierarchy. We develop game theoretical models to describe the interconnection agreements among providers and analyze those models from two alternative modeling perspectives: a purely non-cooperative game and a network perspective. The dissertation consists of three main parts.

In Chapter 2 we discuss the interconnection agreements among Internet Access Providers at the top of the Internet hierarchy. IAPs generally exchange traffic with each other under peering arrangements at the NAPs as well as directly under private peering agreements. We develop a model, in which IAPs decide on private peering agreements, comparing the benefits of private peering relative to being connected only through National Access Points. The model is formulated as a multistage game. IAPs compete by setting capacities for their networks, capacities on the private peering links, if they chose to peer privately, and access prices. We examine the model from two alternative modeling perspectives — a purely non-cooperative game, where we solve for Subgame Perfect Nash Equilibria through backward induction, and a network theoretic perspective, where we examine pairwise stable and efficient networks. Private peering agreements reduce congestion in the Internet, and so improve the quality of the providers. While there are a large number of Subgame Perfect Nash Equilibria, both the pairwise stable and the efficient networks are unique and the stable network is not efficient and vice versa. The stable network is the complete network, where all the backbone providers choose to peer with each other, while the efficient network is the one, where the backbones are connected to each other only through the National Access Points. Thus, the results show that even though the profits are lower with private peerings, due to large investments, the network where all the providers privately peer is the stable network. The results are consistent with the observed behavior of the IAPs.

Chapter 3 examines the interconnection arrangements among Internet Service Providers, that are on the second layer of the Internet hierarchy. ISPs are connected to Internet Access Providers mostly through transit agreements. On the other hand, Internet Service Providers can choose to peer among themselves both within and across IAPs. We refer to private peering between ISPs at the same backbone as Intra-backbone peering, whereas Inter-backbone peering refers to peering between ISPs connected to different backbones. We formulate the model as a two-stage game with two Internet Access Providers and a finite number of Internet Service Providers connected to the IAPs. In the first stage, ISPs decide on Inter-backbone and Intra-backbone peering agreements. In the second stage they compete in prices. Inter- and Intra-backbone peerings have two distinct though not unrelated effects on the quality of service and ultimately profits of the ISPs. Peering affects profits through two channels — reduction of backbone congestion, which we term symmetric effect, and ability to send traffic circumventing congested backbones, which we call asymmetric effect. The asymmetric effect generally increases gross profits, while the symmetric effect has a negative or ambiguous impact on gross profits. The effects often work against each other making the net effect ambiguous. The relative magnitude of these factors helps or hinders peering.

Chapter 4 presents a simple game theoretic model of two providers (IAPs or ISPs) who choose between peering and transit agreements. There is no regulation with regard to interconnection policies of providers, though there is a general convention that the providers peer if they perceive equal benefits from peering, and have transit arrangements otherwise. In the literature there is a debate whether the large providers gain and exploit market power through the terms of interconnection that they offer to smaller providers. In this chapter we develop a game theoretic model to examine how providers decide who they want to peer with and who has to pay transit. The model discusses a set of conditions, which determine the formation of peering and transit agreements. The model takes into account the costs of carrying traffic by peering partners. Those costs should be roughly equal for the providers to have incentives to peer, otherwise the larger provider believes that the smaller provider might free ride on its infrastructure investments. The analysis suggests that the providers do not necessarily exploit market power when refusing to peer. Moreover, Pareto optimum is achieved under transit arrangements only. The chapter argues that the market forces determine the decisions of peering and transit, and there is no need for regulation to encourage peering. Furthermore, to increase efficiency, the regulators might actually need to promote transit arrangements.

Most of the recent literature on the economic aspects of the interconnection arrangements are focused on duopoly models which do not capture the dynamics involved in interconnection decisions (Little and Wright, 2000; De Palma et al, 1989; Laffont et al, 2003; Cremer et al, 2000). There are two main original contributions in this dissertation. First, we model the formation of peering/transit contracts explicitly as a decision variable in the game, while other papers do not use game theoretic formulations of this decision variable. Second, we apply network analysis to examine peering/transit decisions of the providers, which yields much more realistic results. Peering decisions are made if both parties agree to peer. So the non-cooperative game theory may not be the best tool to examine peering decisions, while the network analysis combines the elements of both cooperative and non-cooperative game theory.

The rest of this chapter is organized as follows. Section 1.1 presents a brief overview of the history of the Internet and the Internet infrastructure. Section 1.2 discusses the hierarchical structure of the Internet and the interconnection arrangements among providers. Section 1.3 presents the economics literature on Internet provision.

## 1.1 A Brief History of the Internet

In this section we briefly outline the evolution of the Internet and the features that make it unique relative to other markets. In early 1960s, as the computers became crucial to the national defense, the U.S. Department of Defense began to search ways to share computing resources of major research centers and institutions. The purpose was to create a worldwide network that would not require a centralized control, so that the network would operate, even if some parts of it fail. On the other hand, it was important to exchange resources despite having different systems, different languages, hardware and network devices.

In 1969 the Advanced Research Projects Agency (ARPA) of the Department of Defense developed ARPANET, the first wide area packet switching network, which allowed individual units of data to be transmitted from one computer to another as independent entities. Messages could be routed and rerouted in more than one directions, so the network could operate even if some parts of it fail.

At first four computer centers were connected through this network, University of California at Los Angeles, SRI International, the University of California at Santa Barbara, and the University of Utah. Then over the following years many researchers and academic institutions sought connection to ARPANET. The researchers at other universities developed their own networks. The networking software became more widely used by academic and research institutions, as the use of personal computers increased in 1980s. In late 1980s the independent networks merged into one. The Department of Defense and most of the academic networks comprising the Internet had been receiving funds from National Science Foundation, which restricted commercial traffic on its networks. In response a few commercial backbones established the Commercial Internet Exchange (CIX), that enabled them to exchange traffic with each other. In 1991 restrictions on Internet commercial traffic were lessened and by 1995 NSF completed the privatization of the Internet. In order to facilitate the operations of commercial backbones, NSF designed a system of geographically dispersed National Access Points (NAPs) (Kende, 2000). NAPs are major interconnection points where backbones meet and exchange traffic. After the privatization the management of the Internet was left entirely to commercial backbones, and four companies, Pacific Bell, Sprint, Ameritech and MFS Corporation, became owners of four Network Access Points, located in San Francisco, New York, Chicago and Washington D.C..

The networks that comprise the Internet are self deterministic and autonomous, and communicate with each other without being controlled by a central authority. The role of each network cannot be easily predicted in advance, as the Internet is based on connectionless transmission technology. No dedicated connection is required and no dedicated route has to be set up between the sender and the receiver, because the Internet uses packet switching technology[1] to transfer data across the network. The outgoing data is converted to a format, usable by the local network medium, then data files are broken down into so-called packets or datagrams, labeled with codes, which have information on their origin and destination. Each packet is transmitted over the Internet and reassembled at the destination. A datagram formatting and

[1] An alternative of packet switched networks is a circuit-switched networks. In circuit switched networks (like the telephone network) each connection between the sender and the reciever requires a dedicated path for the duration of the connection.

addressing mechanism is independent of any specific characteristics of the individual networks comprising the Internet (Hall, 2000).

The operation of the Internet is supported mainly by two basic protocols: Transmission Control Protocol (TCP) and the Internet Protocol (IP) (Schneider and Perry, 2001), a software based set of networking protocols that allow any system to connect to any other system using any network topology (Hall, 2000). The IP protocol is responsible for routing individual packets from their origin to their destination. Each computer has at least one globally unique address, called its IP address, that identifies it from all other computers in the Internet. The IP address has information on both the network, that the computer belongs to, as well as its location in that network. The currently used version, IPv4, uses a 32 bit number for an IP address. The next generation Internet, IPv6, will use a 128 bit number for an IP address. Each packet transmitted over the Internet contains both the sender's IP address and the receiver's IP address.

The datagrams are transmitted from one host to another, one network at a time (Hall, 2000). Each packet of a data file might take a different path, but it will end up at the destination ready to be reassembled. The best route for transmitting a packet from the origin to its destination is determined at each router-computer that the packet passes on its trip. The router's decision about where to send the packet depends on its current understanding of the state of the networks it is connected to. This includes information on available routes, their conditions, distance and cost. The packets, having the same origin and destination, travel across any network path that the routers or the sending system consider most suitable for that packet at each point of time. If at some point in time some parts of the network do not function, the sending system or a router between the origin and destination will detect the failure and would forward the packet via a different route (Telegeography, 2000). TCP controls the assembly of data into packets before the transmission, keeps track of the individual packets of the data and controls reassembly of the packets at the destination.

The networks in the Internet interconnect and exchange data based on several agreements, which are referred as settlements (Telegeography, 2000):

- Sender Keeps All (SKA), neither network counts or charges for traffic exchange;
- Unilateral settlement or transit, the downstream customer pays the upstream provider to

carry its traffic;

- Bilateral settlement, two providers agree on price, taking into account the imbalance in exchanged traffic;
- Multilateral settlement, several providers construct shared facilities and share the costs.

The type of settlement chosen depends on the providers' size, domestic and international capacity, network quality, content and customer profile, and routing and interconnection topology. (Telegeography, 2000). At the early stages of the Internet development the providers were closer in size and had comparable traffic flows. So they used to exchange traffic as "peers", i.e. not paying each other for the exchange of traffic (SKA settlement). As the Internet became more commercial, the size of the networks changed. Subsequently, the larger providers started to change peering agreements. Now smaller providers pay larger providers for connectivity (transit), but larger providers still exchange traffic under peering.

Cukier (1998a) proposed a functional classification of providers based on four classes, which shows the asymmetry in traffic interchange that occurs between providers and, it determines pretty much the bases for the types of settlements among providers:

- backbone ISPs (or IAPs as they are referred in this dissertation),
- downstream ISPs,
- online service providers,
- ISPs specializing in web hosting.

*Backbone ISPs* provide connectivity and manage network infrastructure. The four largest backbone ISPs are UUNET/MCI, AT&T, SPRINT and GENUITY/LEVEL3 (Haynal, 2003). Since the late 90s, the large backbone ISPs began changing their interconnection terms. These providers or otherwise called "Tier-1" ISPs have several connections dedicated to inter-connecting their backbones without going through the NAPs. They have increased the amount of "private peering" (SKA settlement) they do between themselves and a few of the other ISPs. On the other hand, they have "transit" (exchange traffic for fee) services with smaller ISPs (Haynal, 2003). The Backbone ISPs do not form an exclusive category. The backbone ISPs may also provide web hosting services or online services (like AT&T and FrenchTelecom). These are referred to as integrated ISPs. In the dissertation we refer to backbone ISPs as backbones or Internet Access Providers.

*Downstream ISPs* serve individuals, businesses and even smaller providers. They pay upstream backbone ISPs for connectivity, the price of which depends on the location and amount of data (Telegeography 2000). Downstream ISPs pay for leasing certain amount of circuits per month as well as a connection fee (unilateral settlement or transit), which lets the downstream ISPs' customers to reach other destinations in the Internet. Most downstream ISPs do not pay based on their actual usage. The payment is instead based on a usage profile, determined by the overall traffic pattern.

*Online service providers,* like AOL, earn revenues by providing Internet access, focusing on the content and ease of use. Online service providers lease connectivity from backbones or other upstream ISPs and manage the network points of presence (POPs) that connect dial-up customers to the Internet. The online service providers are either paid a flat monthly fee by customers for unlimited service or charge additional fees after a certain limit of usage is exceeded. However, much of their revenue comes from selling content and advertising space.

*Web hosting* companies, like Exodus, host websites that are accessed by the Internet consuming public. It is important to note that the web hosting ISPs create unidirectional traffic, as websites originate a lot of traffic, while not requesting much. As a result, backbone ISPs demand that web hosting providers, which typically do not maintain a national network, purchase connectivity from a backbone or downstream ISPs (Cukier, 1998a).

## 1.2 Internet Hierarchy and Interconnection

In this section we discuss the interconnection arrangements of the providers. The Internet has a loosely hierarchical structure (Ross and Kurose, 2000). It is comprised of many distinct networks, which are operated by different firms. At the top of the hierarchy are the backbones, also called Internet Access Providers (IAPs). The second layer of the hierarchy consists of so called retail Internet Service Providers (ISPs). At the bottom of the hierarchy are the end users. Each provider has its own network, where the connected users can communicate with each other.

The end-users of the Internet are content providers (or websites) and consumers, who browse the web, purchase products and services from content providers, or communicate with each other using the Internet. End-users generally want to have access to all other possible end-users, regardless of the network they are attached to. To provide such universal connectivity to their users, the providers must interconnect with each other and share their network infrastructure. Two main forms of interconnection emerged — peering under which the providers carry each other's traffic without any payments and transit under which the downstream company pays the upstream company a certain settlement payment for carrying its traffic.

At the early stages of the Internet the providers exchanged traffic mostly at the National Access Points (NAPs). NAPs are major interconnection points, where providers meet to exchange traffic with the other attached providers. Each provider has to provide connection only to these NAPs, instead of having individual connections to every other provider. They exchange traffic under peering arrangements. Such traffic exchange arrangements at the NAPs are called public peerings (Kende, 2000). In Figure 1-1, for instance, providers 1 and 3 are connected to NAPs in both Washington D.C. and San Francisco, where they can exchange traffic with each other as well as with providers 2 and 4. On the other hand, 2 and 4 are connected to each other and the rest of the providers only at the NAP in Washington D.C. It was cost-efficient to provide connections only to NAPs instead of having direct connections with each other, due to the cost of large investments in the fiber optic capacity.

As the number of users increased rapidly over the last few years, the NAPs became congested, so users experienced a lot of delay. The growing congestion at the NAPs have increasingly necessitated private peerings between the providers, which refer direct connections between providers bypassing NAPs. So many providers started to interconnect with each other directly through private peering arrangements. For example, providers 2 and 4 have entered into private peering agreements with each other and can exchange traffic for each other through their private connection in Figure 1-2, while still using NAP in D.C. to exchange traffic with 1 and 3. Peering agreements have some distinctive characteristics. The peering providers exchange only the traffic that originates with the customer of one provider and terminates with the customer of the other. Thus, under the existing peering agreement, providers 2 and 4 cannot route traffic from their other peering partners through the direct connection they have between

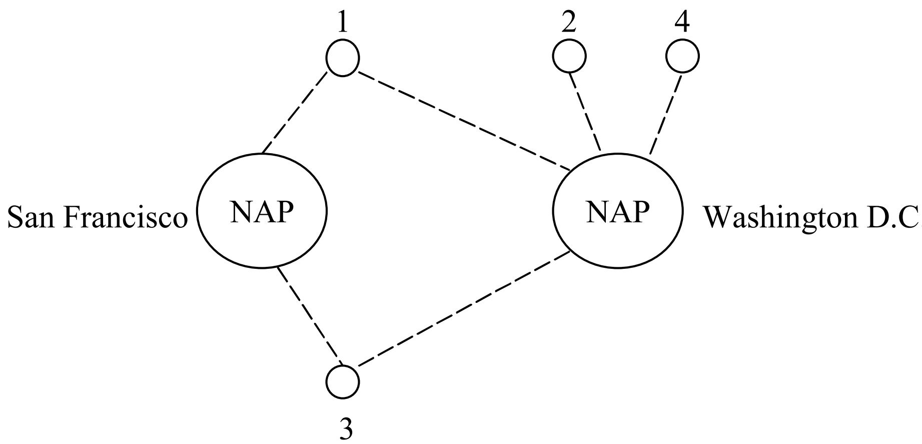


Figure 1-1: Public Peering

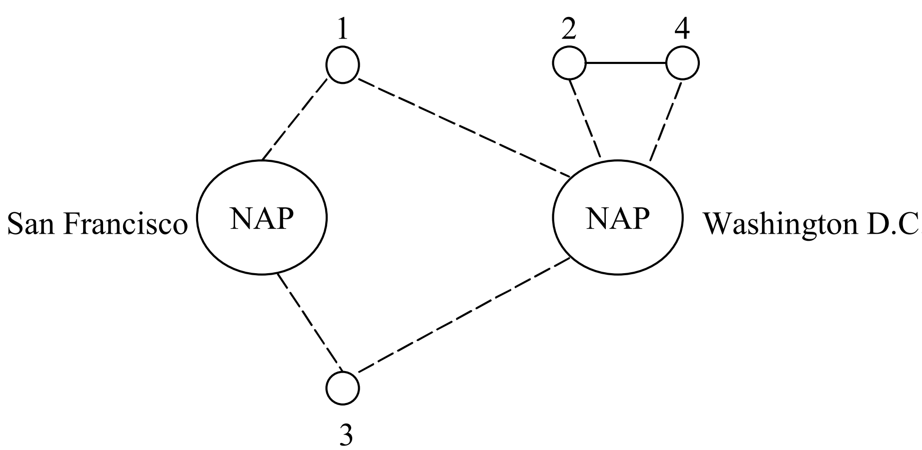


Figure 1-2: Public and Private Peerings

themselves, i.e., 2 will not accept traffic from 4 which is destined to 3 on their private link. However 2 and 4 can exchange traffic of their transit customers on their private link. Under the peering agreement the providers exchange traffic on a settlement free basis. Thus, each provider bills its own customers for the traffic and does not pay the other provider for transporting its traffic. On the other hand, the providers have developed a system called "hot-potato routing", under which the providers pass off traffic to another provider at the earliest point of exchange. For example, providers 1 and 3 are connected to each other in both Washington D.C. and San Francisco. When a customer of the provider 1 in San Francisco requests a webpage from a website connected to the provider 3 in Washington D.C., then the provider 3 passes the traffic to the provider 1 in D.C. and the provider 1 is the one that carries that request from D.C. to San Francisco.

With regard to connectivity agreements, large IAPs, that own national and/or international high speed networks, exchange traffic with each other under peering agreements either at the National Access Points or through direct private peerings. The four largest IAPs in the U.S. are UUNET/MCI (27.9%), AT&T (10%), SPRINT (6.5%) and GENUITY/LEVEL3 (6.3%) (Haynal, 2003). ISPs, on the other hand, generally have transit agreements with the IAPs, but peer also privately with other ISPs, which are referred to as retail peerings as illustrated in Figure 1-3.

The Internet backbone market today remains free of telecommunications regulation (Kende, 2000), which allows providers to make peering decisions freely. Hence the criteria for the peering decisions are not very specific and are made subjectively case by case. Several important criteria for peering decisions include geographic spread, capacity, traffic volume and customer profile. The providers can choose not to peer or even discriminate between other providers in making their peering decisions. This contrasts with other telecommunication industries, where such discrimination is prohibited by FCC regulations.

## 1.3 Economic Literature on Internet Provision

The economic literature on the Internet provision is very recent. Most of the economic research on Internet provision is focused on pricing and sharing the infrastructure. Mackie-Mason and

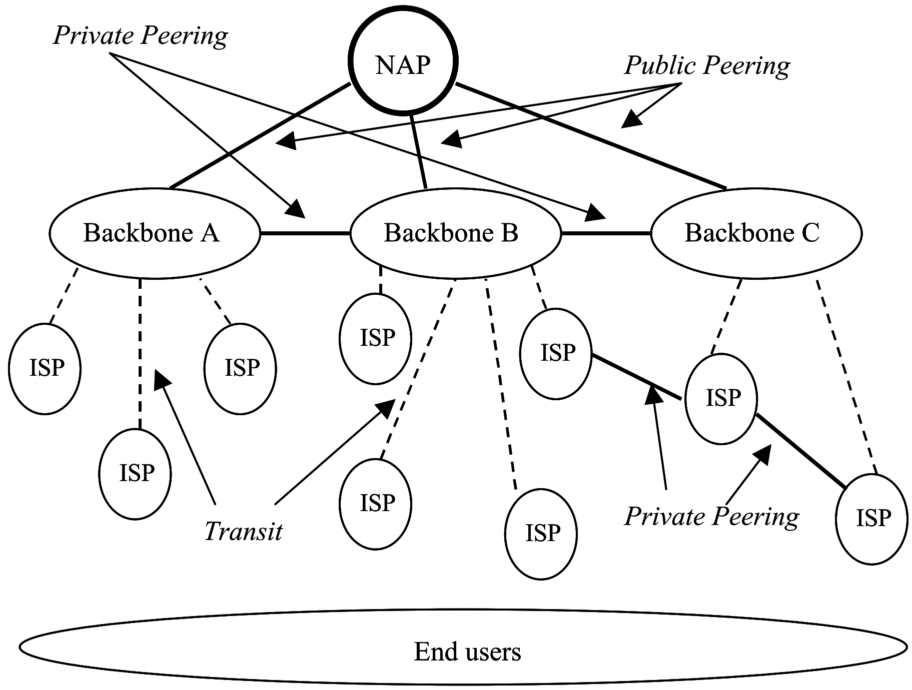


Figure 1-3: Hierarchical Structure of the Internet

Varian (1995) have pointed out that the current practice of flat-rate pricing (charging a fixed fee to consumers unrelated to usage) encourages overusage and hence congestion, a simple application of the tragedy of commons. Users are not confronted by marginal usage costs. Their solution to the problem is setting up a smart market to price consumers according to usage. Of course, this involves a technological leap but may become feasible in the future.

Currently, Internet services offered by providers have little horizontal differentiation offering the same basic services of web-browsing, emails, real time conversation and some Internet telephony, but there is some vertical differentiation. Some authors have examined price and product differentiation in services. Odlyzko (1997) suggests multiservice mechanism, where users can choose between the first and second class services and pay accordingly, even though the quality is not necessarily different. On the other hand Gibbens et al (2000) discusses the competition between two Internet service providers, when either or both of them choose to offer multiple service classes. Assuming a uniform distribution of user preferences towards congestion, a linear function of congestion, and finite number of networks, they prove that, even when Internet service providers are free to set capacities as well as prices, multiproduct competition is not sustainable in a profit maximizing equilibrium. Mason (2001) develops a model where firms are vertically and horizontally differentiated and consumers have different preferences for the firm's size and location. He considers a two stage game, where two firms first decide whether to make their goods compatible or not and then choose prices, given their rival's price. The author concludes that the firms make their goods compatible, the competition increases due to a decrease in vertical differentiation, but at the same time the importance of market share increases, so the competition decreases. The dominance of each effect depends on the relative importance of horizontal and vertical aspects in consumers' utilities.

De Palma and Leruth (1989) discuss a duopoly model, where firms compete on capacities and prices. They consider cases with homogeneous and heterogeneous consumers. They show that competition among firms decreases in the presence of congestion and firms gain monopoly power by selling congested goods. As a result congested goods might be offered at a high price and at a lower quality to consumers.

Cremer et al (2000) describe a model, which analyzes the competition among backbones. The backbones have some installed base of customers, and they compete for new customers. The model incorporates positive external effects on increasing number of customers. The more customers are attached to the backbone the better is the quality of service. On the other hand, the quality of service improves with better interconnection quality of backbones. The demands are based on prices and qualities of service. In the paper, the authors conclude that in the case with backbones of different sizes, the larger backbone prefers a lower quality of interconnection than the smaller backbone. Moreover, they show that if the quality of interconnection is costly, perfect connectivity is not efficient socially or privately. In the absence of the cost for the interconnection, the dominant backbone's best strategy is to refuse interconnection with the smaller one. The authors also discuss the case with equal-size backbones. In this case the backbones prefer high quality of interconnection and obtain identical profits at the equilibrium. The results of the case with equal-size backbones are somewhat similar to what we get in the model presented in the second chapter, particularly the backbones do prefer higher quality of interconnection, when they serve homogeneous customers, and there are no exogenous assumptions about their sizes.

DangNguyen and Penard (1999) consider a model of vertical differentiation with two asymmetric backbones and identical retail ISPs connected to those backbones. The retail ISPs peer with other ISPs connected to the same backbone (intra-backbone peering) and also with ISPs connected to the other backbone (inter-backbone peering). Intra-backbone peering reduces congestion in that backbone only and raises the quality of all ISPs connected to that backbone. Inter-backbone peering reduces congestion in both backbones and raises quality of both backbones. The authors show that ISPs connected to the high quality backbone will always peer with each other. But they may or may not peer with ISPs connected to the low quality backbone. Also, in the latter case, the ISPs of the low quality backbone will peer with each other. The results are illustrated with some evidence from the French Internet market.

Little and Wright (1999) make a strong case against regulator enforced peering, where regulation forbids payments between access providers as well as their right to refuse to peer. There are two providers whose demands are determined by a model of horizontal product differentiation with a built in asymmetry. Costs include a cost for providing capacity, a fixed cost per customer and a marginal cost of usage. Usage exceeding the capacity is assumed to render zero utility. The authors compare the solution under regulator enforced peering in which firms choose their investments in capacity in the first stage and prices in the second stage, with the welfare maximizing solution in which welfare is measured by the sum of producer and consumer surplus. The former leads to congestion owing to under investment in capacity. If, however, firms peer with settlement payments, namely, net users pay net providers at a rate equal to the marginal cost of providing capacity, the solution obtained is precisely the welfare maximizing one. The same is the case when firms refuse to peer with anyone who is a net user of infrastructure.

Laffont et al (2003) develop a model to analyze the impact of an access charge on Internet backbone competition. They distinguish between two types of end users: websites and consumers, and concentrate their analysis on the traffic from websites to consumers. In the absence of direct payments between websites and consumers, the access charge, determined either by regulation or through a bilateral agreement, allocates communication costs between websites and consumers and affects the volume of traffic. The model results in a unique equilibrium, where the prices charged to consumers as well as the prices charged to websites are the same across the backbones. The model does not, however, take into account the externalities associated with the asymmetry of traffic flows between the providers.

Gorman and Malecki (2000) examined the network structure and the performance of ten backbone provider networks in the USA, based on the basic graph theoretic measures together with the median downloading time of those backbone providers. They concluded that even though the basic graph theoretic measures are useful tools, when analyzing the efficiency of the networks, however, they might not be good tools in comparing different networks in one infrastructure. Their analysis show that for an Internet provider network having high graph theoretic measures does not necessarily mean high technical performance. Thus even the complete network, having the highest and most efficient graph-theoretic measures, is not necessarily a very efficient Internet network in terms of median downloading time, which is the measure of performance in their paper. To connect its customers to the whole Internet the Internet backbone provider depends on other backbones' networks through public or private peering. The characteristics of a certain network and its performance in relation to other networks in the Internet determine the demand for the services of that network.

# Chapter 2

# Private Peering Among Internet Access Providers[1]

## 2.1 Introduction

In this chapter we examine the interconnection arrangements of Internet Access Providers (hereafter IAPs). The Internet Access Providers own or lease national and international fiber optic networks, through which they offer their customers access to the whole Internet. The large IAPs interconnect with each other under public and/or private peering agreements. Interconnecting directly under a private peering agreement is associated with costly investments in fiber optic capacity. Nevertheless, nowadays more and more IAPs engage in private peering agreements. In this chapter we explore the incentives behind the IAPs' decisions to engage in private peering. First a four stage game is considered. In the first stage, IAPs decide how they want to be connected to the other IAPs, i.e., the IAPs decide whether to connect to other IAPs through private connections and National Access Points (hereafter NAPs) or only through NAPs. National Access Points are major interconnection points, where IAPs meet to exchange traffic with each other. The NAP provides major switching facilities, which channel incoming traffic from any of multiple IAPs to the specific IAP that will take the traffic to its intended destination. Once an IAP has made a decision on the types of connection with other IAPs, it

[1] **This chapter is based on "Private Peering Among Internet Backbone Providers", Badasyan, N. and Chakrabarti, S. (2003), mimeo.**

chooses a capacity for its network, determining how many customers it can handle at a certain point in time. The difference between the IAP's network capacity and its demand determines connection failure rates. In the third stage, if an IAP engages in private peering, then the IAP chooses capacities for the links to connect to other IAPs, which determines usage congestion on the private link. This, along with the congestion at the NAPs, determines the overall congestion. In the last stage the IAPs compete in prices. If the IAPs choose not to engage in private peering in the first stage, then they compete by determining their network capacities and prices only. We consider a decentralized decision making model, where the choices made by each IAP affect the outcomes and choices made by the other IAPs, and equilibrium is reached without any central control. Subsequently, we look at the model from a network perspective by redefining the first stage as a link formation stage. We examine the properties of pairwise stable and efficient networks.

We concentrate our analysis on the networks with three IAPs only. In such networks, the possibility of peering between two providers makes a third non-peering provider vulnerable to the loss of demand and, hence, profitability. Such a possibility is nonexistent in two-provider networks, making peering unlikely. In other words, our results capture the difference of the dynamics in a three provider network relative to a two-provider network, which most papers do not address (Little and Wright, 2000; De Palma et al, 1989; Cremer et al, 2000). Our main results are that while there is a multitude of subgame perfect Nash equilibria resulting in a multitude of network configurations, network analysis leads to a network configuration, where all IAPs peer with one another.

There are two original contributions of this chapter. First, we model the formation of a peering contract explicitly as a decision variable in a game, while the current literature does not address such a formulation. Second, we apply network analysis to examine peering decisions of IAPs. Peering decisions are made, if both parties agree to peer. So non cooperative game theory may not be the best tool to examine peering decisions, while network analysis combines elements of both cooperative and non—cooperative game theory. Besides, given that a lot of IAPs do privately peer with each other,[2] network analysis leads to more realistic conclusions. Thus, the latter is better suited to analyze such issues.

[2] See *http://navigators.com/sessphys.html* for details.

The rest of this chapter is organized as follows. Section 2.2 introduces preliminaries. Subsequently we develop three parts in our analysis. First, in Section 2.3, we discuss a model with no discrimination in the form of a non-cooperative game and find the subgame perfect Nash equilibria through backward induction. Then we introduce the possibility of discrimination in the model in Section 2.4. Finally, in Section 2.5 we do a complete network analysis and determine the pairwise stable and efficient networks. The conclusion follows in Section 2.6.

## 2.2 Preliminaries

Let $N = \{1, 2, ..., n\}$ be a finite set of Internet Access Providers with $n \geq 3$. The connections among IAPs are represented by undirected links forming a network. The nodes (vertices) of the networks represent the location of IAPs. All IAPs are connected to National Access Points through which they are connected to other IAPs in the Internet. For simplicity we assume there is only one NAP. We suppose that each IAP is connected to the NAP with a given uniform capacity $k$, which is the maximum amount of data that can be handled over that link between the IAP and NAP at a certain point in time. Thus $k$ is the link capacity of the publicly provided network. IAPs may also decide to enter into private peering agreements with one another.

Formally, a link $ij = \{i, j\}$ is a subset of $N$ that contains $i$ and $j$. For any two providers, $i \in N$ and $j \in N$, $ij$ refers to the private peering agreement between $i$ and $j$. The collection of all links on $N$, $g^N = \{ij \mid i, j \in N,\ i \neq j\}$, is called the complete network on $N$, where $|g^N| = \frac{n(n-1)}{2}$. In the complete network each IAP has formed private peering agreements with all the other IAPs. Any arbitrary collection of links $g \subset g^N$ is called a network on $N$. The set of all possible networks on $N$ is denoted by $G = \{g \mid g \subset g^N\}$. $g^0 = \emptyset$ is the empty network, i.e., IAPs connect to each other only through the NAP. The network $g + ij$, where $i, j \notin g$, denotes the new network formed by addition of the link $ij$ to the network $g$. The network $g - ij$, where $i, j \in g$, denotes the new network formed by removal of the link $ij$ from the network $g$. Figure 2-1(a) and 2-1(b) illustrate the empty network and the complete network, respectively, with three IAPs.

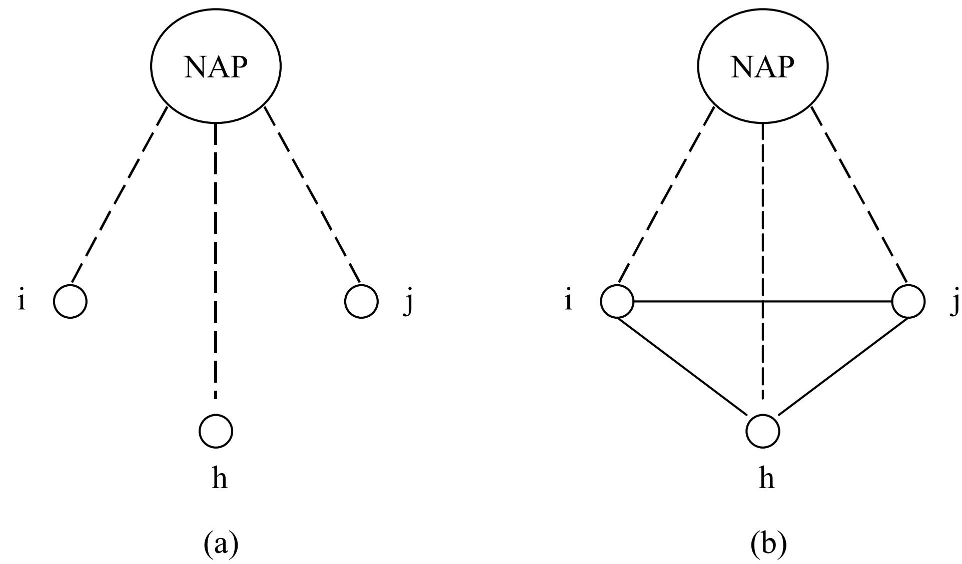


Figure 2-1: Empty and Complete Networks

## 2.3 The Model without Discrimination

We consider a four stage non-cooperative game where in the first stage the IAPs decide whether to have private peering agreements or not. If they decide to peer in the first stage, then, subsequently, they choose investments in capacities or links connecting them to the other IAPs. In the last stage they compete in prices. We explicitly assume that IAPs are not allowed to discriminate against other IAPs, i.e., any IAP who wants to enter into private peering agreements must do so with all other IAPs who are willing to peer as well.

**Stage 1**

In this stage each IAP decides to signal its willingness to engage in private peering. Let $\gamma_i = 1$, if $i$ signals to the other IAPs that it is willing to peer with any of the other IAPs, and gives permission to the other IAPs to build private links to itself if they want to. $\gamma_i = 0$, if it does not want to have any private peerings with other IAPs. Peering agreements materialize only between any pair of IAPs, who are willing to peer and give others permission to connect to themselves. The peering agreements result in creation of a (peering) network $g \in G$.

**Stage 2**

In this stage all IAPs decide how much to invest in the capacities of their own "internal" networks. Each IAP $i$ chooses a capacity level for its network, denoted by $s_i$. $s_i$ determines how many customers IAP $i$ can serve at a certain point in time. (As we will see later we assume that every customer demands one unit of service).

**Stage 3**

Next, every IAP chooses investments in the links that connect its network to the other IAPs' networks with whom peering agreements have resulted in Stage 1 of the game. Let $s_i^j$ denote the investment of IAP $i$ in the link capacity of $ij \in g$, and $s_j^i$ denote the investment of IAP $j$ in the link capacity of $ij$. Then the capacity of the link $ij$, $k_{ij}$, is determined as a sum of $s_i^j$ and $s_j^i$.

$$k_{ij} = s_i^j + s_j^i$$

We assume that, to connect its customers to IAP $j$, IAP $i$ uses either the private direct link it has with the server $j$ and/or the NAP. Based on peering arrangements of backbone providers, no backbone provider can transfer traffic from one of its peering partner to another peering partner, i.e., if $i$ has a peering agreement with $j$ and $h$, then $i$ cannot transfer traffic intended for $h$ which is coming from $j$ (Kende, 2000). Note that even if an IAP is willing to peer and gives permission to accept any connection to itself in the first stage, it still has an option of not making any investments in the private link $ij$ by selecting $s_j^i = 0$.

**Stage 4**

In this last stage of the game IAPs compete in prices. An IAP $i$ sets up a per unit price, $p_i$, for its services. The consumers choose an IAP, given the prices quoted and the quality of service offered by the IAPs. We assume that consumers are homogeneous and each consumer buys one unit of service. Thus, the utility a consumer derives from an IAP is strictly positive. Consumer preferences are presented in more detail in Section 2.3.3.

Before we continue with the model we need to define a few concepts, which we call connection failure and congestion.

### 2.3.1 Connection Failure

In our paper we define the connection failure of the IAP $i$ as follows:

$$F_i = \max\{0, d_i - s_i\}$$

where $d_i$ is the demand for IAP $i$'s services. Hence the connection failure of the IAP $i$ depends on its own network's capacity and the demand for its services, namely

$$F_i = \begin{cases} 0, & \text{if } s_i > d_i \\ d_i - s_i, & \text{if } s_i \leqslant d_i \end{cases}$$

The rationale is straightforward. If $s_i > d_i$, all the customers intending to connect to IAP $i$ are able to connect. If, however, $s_i \leqslant d_i$, then $s_i$ provides an upper limit to the amount of traffic that can be handled by IAP $i$ and some customers face connection failures.

Thus, there is no connection failure, if the demand does not exceed IAP $i$'s network capacity, i.e., all the customers of IAP $i$ will be connected to IAP $i$. On the other hand, if IAP $i$'s demand exceeds its network capacity, then some consumers will not be able to connect to IAP $i$ at all. However, we can rule out the possibility that $s_i > d_i$. We are assuming that it is costly for IAP $i$ to invest in $s_i$. So the profit maximizing IAP will not invest in $s_i$, which exceeds its demand. Hence $F_i$ can be reduced to

$$F_i = d_i - s_i$$

Obviously, lower connection failure indicates higher chance of connection and so better service.

### 2.3.2 Congestion

Even if the customers do not have difficulties connecting to IAPs, they might experience service problems if they want to connect to customers outside IAP $i$'s network. In this paper we introduce a measure of congestion on the link $ij$.

Assuming the number of IAP $i'$s customers, who want to connect to any given IAP including $i$ itself is the same across IAPs, the total traffic intended for $j$ through $i$ would be $\frac{s_i}{n}$.

On the other hand assume that each IAP uses $\frac{1}{n-1}$ of the publicly provided capacity, $k$, for connecting to another IAP.[3] Define the congestion of the link as $l_{ij}$ by:

$$l_{ij} = \begin{cases} \frac{s_i}{n} + \frac{s_j}{n} - \min\{\gamma_i, \gamma_j\}(s_i^j + s_j^i) - \frac{k}{n-1}, & \text{if } \frac{s_i}{n} + \frac{s_j}{n} \geq \min\{\gamma_i, \gamma_j\}(s_i^j + s_j^i) + \frac{k}{n-1} \\ 0, & \text{otherwise} \end{cases}$$

where

$$\min\{\gamma_i, \gamma_j\} = \begin{cases} 1, & \text{if } i \text{ and } j \text{ enter into private peering agreements} \\ 0, & \text{otherwise} \end{cases}$$

It is not unrealistic to assume that the number of $i$'s customers who want to connect to any IAP is the same across IAPs. We assume that customers' interest in other customers is unrelated to the latter's choice of an IAP.

Note that if

$$\frac{s_i}{n} + \frac{s_j}{n} < \min\{\gamma_i, \gamma_j\}(s_i^j + s_j^i) + \frac{k}{n-1},$$

then IAPs $i$ and $j$ invested more than they require to handle the traffic. Consequently, we rule out this possibility. Hence

$$l_{ij} = \frac{s_i}{n} + \frac{s_j}{n} - \min\{\gamma_i, \gamma_j\}(s_i^j + s_j^i) - \frac{k}{n-1}$$

Thus, there is no congestion on the link $ij$ until the capacity of $ij$ is reached. So the overall congestion factor for IAP $i$ will be

$$L_i = \sum_{j \in N \setminus \{i\}} l_{ij}$$

Customers experience congestion, if the total capacity available through public and private peerings is less than the traffic. The lower the congestion is, the better connected is the IAP.

### 2.3.3 Consumer Preferences

In our analysis we assume that consumers are homogeneous. Consumers select an IAP through which they want to connect to others on the Internet. When making decisions, consumers consider the prices, connection failure, and overall quality of connectivity of IAPs. Each consumer

[3] As we assume that customers are homogeneous, then at certain point in time each customer is equally desirable to be connected by all the rest of the customers in the Internet.

consumes either 0 or 1 unit of service. Let $\mu$ and $\lambda$ represent the weights consumers put on the connection failure and congestion, when connecting to the Internet through IAP $i$. Denote $U_i$ to be the utility of a consumer, who connects to the Internet through IAP $i$ then:

$$U_i = V - \mu F_i - \lambda L_i - p_i;\ i = 1, 2, \ldots, n$$

where $V$,[4] which is assumed constant for all consumers, represents the reservation value of the consumer for connecting to the Internet and $p_i$ is the per unit price charged by IAP $i$ for its services. We assume $0 < \lambda < \mu \leq 1$, i.e., customers would prefer to connect even to a congested network than not to connect at all.

Denote the demand for IAP $i$ by $d_i$. Then in equilibrium, a consumer is indifferent between the three IAPs if

$$\mu F_1 + \lambda L_1 + p_1 = \mu F_2 + \lambda L_2 + p_2 = \mu F_3 + \lambda L_3 + p_3$$

This gives us equilibrium values of $d_i$ in terms of $L_i$, $p_i$ and $s_i$, say, $d_i^*$

### 2.3.4 Parameter Constraints

In our model we assume throughout

$$0 < s_i \leqslant d_i \text{ and } \frac{s_i}{n} + \frac{s_j}{n} \geqslant \min\{\gamma_i, \gamma_j\}(s_i^j + s_j^i) - \frac{k}{n-1}$$

which holds for values of $\lambda, \mu$ and $k$ given by the shaded area in Figure 2-2.[5] We call the set of values of $\lambda$, $\mu$ and $k$ represented by the shaded area in Figure 2-2 the feasible parameter range. Figure 2-2 shows that, given $\lambda$ and $\mu$, the value of $k$ must not exceed a certain maximum amount. Except Section 2.3.7 we will always assume that $k$ lies in the feasible parameter range. We refer to Appendix $A$.1 for detailed explanations of these parameter constraints.

[4] We assume V is large enough compared to $\mu F_i + \lambda L_i + p_i$, so that each consumer buys one unit. In this case $U_i$'s are strictly positive.

[5] Figure 2.2 is derived by solving for the equlibrium and imposing the aforesaid conditions on the equilibrium values.

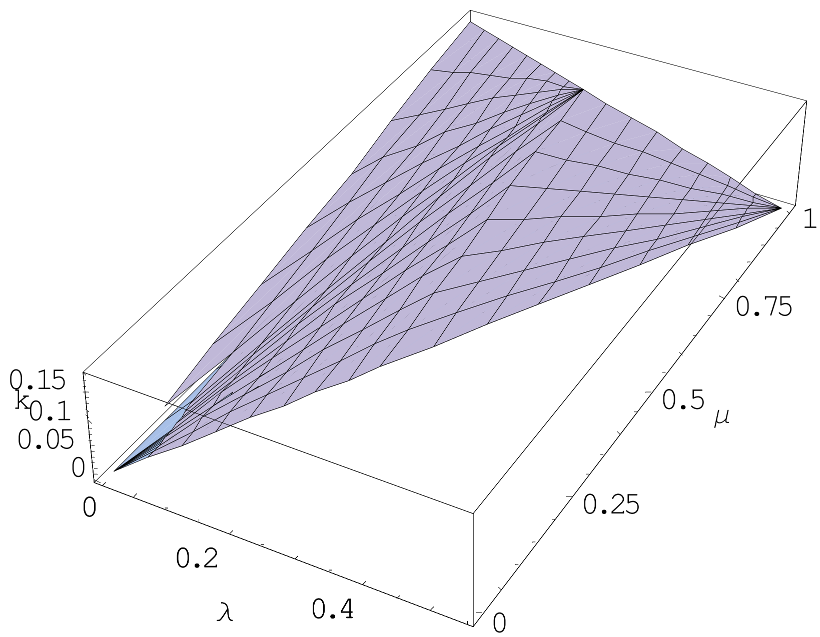


Figure 2-2: Parameter Constraints

### 2.3.5 Equilibrium Analysis

Equilibrium analysis is based on the method of backward induction. Throughout the analysis we assume that the potential market size is normalized to one, i.e., $\sum_{i \in N} d_i = 1$.

We have three cases to consider.

- CASE 1: All IAPs decide to enter into private peering agreements with the rest of IAPs, $\gamma_i = 1$, for all $i \in N$, so we have the complete network, $g^N$.
- CASE 2: None of the IAPs enter into private peering agreements with the rest of IAPs $\gamma_i = 0$, so we have the empty network, $g^0$.
- CASE 3: Some of the IAPs enter into private peering agreements, while others do not.

Investments in both server and link capacities are costly for IAPs. We assume quadratic costs. We denote costs incurred by IAP $i$ by

$$c_i = s_i^2 + \sum_{j \in N \setminus \{i\}} \min\{\gamma_i, \gamma_j\}(s_i^j)^2$$

Then the profits of IAP $i$ are determined by

$$\pi_i = p_i d_i - c_i$$

We consider a case with three IAPs, as this is the minimum number of IAPs with which we can capture essence of all three cases, thus, $N = \{1, 2, 3\}$.

Let $\widehat{\boldsymbol{\pi}}_i^l$ be the reduced profit of IAP $i$, where $i = 1, 2, 3$ in Case $l$, $l = 1, 2, 3$ in the first stage. Refer to Appendix $A.2$

#### Case 1

In this case all three IAPs decide to enter into private peering agreements with the rest of IAPs, so we have a four stage game. Solving the model by backward induction leads us to a symmetric solution. In equilibrium all IAPs share the market equally, i.e.,

$$\widehat{d}_i^1 = \frac{1}{3}$$

and charge the same price, which depends only on how much consumers value the capacity of the IAP $i$'s own network. The prices do not depend on the value consumers put on the link capacities, as the link capacities are common for IAPs:

$$\widehat{p}_i^1 = \frac{\mu}{2}.$$

IAPs' optimal investments in link capacities are same for all IAPs and depend on the value the customers put on the interconnection or link capacities:

$$\widehat{s_i^j}^1 = \frac{\lambda}{15}.$$

The network capacities of IAPs are given by

$$\widehat{s}_i^1 = \frac{2(3\mu - \lambda)}{45} \frac{(75\mu - 4\lambda^2)}{(75\mu - 6\lambda^2)} = \widehat{s}^1$$

and they make profits equal to

$$\begin{aligned} \widehat{\pi}_i^1 &= \frac{-1424\lambda^6 + 768\lambda^5\mu - 12\lambda^4\mu(96\mu - 5125) - 28800\lambda^3\mu^2}{36450(2\lambda^2 - 25\mu)^2} \\ &\quad + \frac{1800\lambda^2\mu^2(24\mu - 475) + 270000\lambda\mu^3 - 50625\mu^3(8\mu - 75)}{36450(2\lambda^2 - 25\mu)^2} \\ &= \widehat{\pi}^1 \end{aligned}$$

**Case 2**

In this case when all three IAPs decide not to enter into private peering agreements with the rest of IAPs, we have a three stage game. Then there is no third stage of the game, as there are no private peering links, and link capacity choices are irrelevant. Solving the model by backward induction we again get a symmetric solution. At the equilibrium all IAPs charge the same price, which is the same as in Case 1, namely

$$\widehat{p}_i^2 = \frac{\mu}{2}$$

and again they share the market equally, i.e.

$$\widehat{d}_i^2 = \frac{1}{3}.$$

Their optimal investments in network capacities are

$$\widehat{s}_i^2 = \frac{2(3\mu - \lambda)}{45} = \widehat{s}^2.$$

The profits are given by

$$\widehat{\pi}_i^2 = \frac{-8\lambda^2 + 48\lambda\mu - 72\mu^2 + 675\mu}{4050} = \widehat{\pi}^2 \text{ (say)}.$$

As we can see the prices and demands are the same in both Cases 1 and 2, i.e., $\widehat{p}_i^2 = \widehat{p}_i^1$ and $\widehat{d}_i^2 = \widehat{d}_i^1$. When we compare the investments in network capacities, we can see that the investments are higher in the case with no private peering than that in the case with private peering, namely

$$\widehat{s}^1 = \frac{2(3\mu - \lambda)}{45} \frac{(75\mu - 4\lambda^2)}{(75\mu - 6\lambda^2)} > \frac{2(3\mu - \lambda)}{45} = \widehat{s}^2$$

as $0 < \lambda < \mu \leq 1$.

On the other hand in the case with no private peering the IAPs do not have to invest in link capacities at all. So the profits in the case with no private peerings is unambiguously higher than that in the case with private peerings, i.e., for all values of $\lambda$ and $\mu$ satisfying $0 < \lambda < \mu \leq 1$,

$$\widehat{\boldsymbol{\pi}}^2 > \widehat{\boldsymbol{\pi}}^1$$

Consequently we get the following proposition.

**Proposition 2.1** *If $0 < \lambda < \mu \leq 1$, profits are higher in the empty network $g^0$, where none of the IAPs enter into private peering agreements (Case 2), than in the complete network $g^N$, where all of them do (Case 1).*

The intuition is straightforward. Given the symmetry of the model, no additional demand can be obtained with the same prices by competing for customers through engaging in private peering. Hence, private peering basically results in additional costs through additional investments in both network and link capacities.

**Case 3**

In Case 3 two of the IAPs enter into private peering agreement, while the third one does not. Then the prices, market shares and both network and link investments of those IAPs that peer are the same. Their profits are also identical and given by $\widehat{\boldsymbol{\pi}}^{3'}$. Let the profits of the non-peering IAP be given by $\widehat{\boldsymbol{\pi}}^{3}$. All the equilibrium values for this case are given in Appendix $A$.3.

It is the case that for all values of $\lambda$ and $\mu$ satisfying $0 < \lambda < \mu \leq 1$, and $k$ lying in the feasible parameter range:

$$\widehat{\boldsymbol{\pi}}^{3'} > \widehat{\boldsymbol{\pi}}^{2} > \widehat{\boldsymbol{\pi}}^{1} > \widehat{\boldsymbol{\pi}}^{3} \tag{2.1}$$

### 2.3.6 Subgame Perfect Nash Equilibria

We can represent the first stage in the following normal form.

2 peers ($\gamma_2 = 1$)

| 1\3 | peer ($\gamma_3 = 1$) | not peer ($\gamma_3 = 0$) |
|---|---|---|
| peer ($\gamma_1 = 1$) | $\widehat{\boldsymbol{\pi}}^{1}, \widehat{\boldsymbol{\pi}}^{1}, \widehat{\boldsymbol{\pi}}^{1}$ * | $\widehat{\boldsymbol{\pi}}^{3'}, \widehat{\boldsymbol{\pi}}^{3'}, \widehat{\boldsymbol{\pi}}^{3}$ |
| not peer ($\gamma_1 = 0$) | $\widehat{\boldsymbol{\pi}}^{3}, \widehat{\boldsymbol{\pi}}^{3'}, \widehat{\boldsymbol{\pi}}^{3'}$ | $\widehat{\boldsymbol{\pi}}^{2}, \widehat{\boldsymbol{\pi}}^{2}, \widehat{\boldsymbol{\pi}}^{2}$ |

2 does not peer ($\gamma_2 = 0$)

| 1\3 | peer ($\gamma_3 = 1$) | not peer ($\gamma_3 = 0$) |
|---|---|---|
| peer ($\gamma_1 = 1$) | $\widehat{\boldsymbol{\pi}}^{3'}, \widehat{\boldsymbol{\pi}}^{3}, \widehat{\boldsymbol{\pi}}^{3'}$ | $\widehat{\boldsymbol{\pi}}^{2}_1, \widehat{\boldsymbol{\pi}}^{2}_2, \widehat{\boldsymbol{\pi}}^{2}_3$ |
| not peer ($\gamma_1 = 0$) | $\widehat{\boldsymbol{\pi}}^{2}, \widehat{\boldsymbol{\pi}}^{2}, \widehat{\boldsymbol{\pi}}^{2}$ | $\widehat{\boldsymbol{\pi}}^{2}, \widehat{\boldsymbol{\pi}}^{2}, \widehat{\boldsymbol{\pi}}^{2*}$ |

In Case 3 the IAPs that peer earn higher profits compared to the Case 2, the empty network. On the other hand, the IAP that does not engage in peering in Case 3 earns lower profit compared to the profits it can get in both the empty network and the complete network. The IAPs engaging in private peering offer lower connection failures and less congestion in Case 3, as they invest in link capacities compared to the empty network, and capture a larger share of the market at the expense of the IAP, which is not peering.[6] Consequently the complete network is consistent with a Nash equilibrium. Needless to say, the empty network is also a Nash equilibrium. The subgame perfect Nash equilibria are indicated by an asterisk This is summarized in the following proposition.

**Proposition 2.2** *If $0 < \lambda < \mu \leq 1$ and $k$ lies in the feasible parameter range, then there are two subgame perfect Nash equilibria: one resulting in the complete network, $g^N$, which entails in private peering agreements among all the IAPs, and one resulting in the empty network $g^0$. The empty network is Pareto superior to the complete network.*

In the case with only two providers, one can show that the subgame perfect Nash equilibrium would not involve any peering.[7] However, in the three provider case, we find an equilibrium with peering. The reason is that in a three provider case the possibility of peering between two providers makes the third non-peering provider vulnerable to the loss of demand and, thus, profitability. This is nonexistent in a two-provider networks, making peering unlikely.

### 2.3.7 The Role of Public Infrastructure

In this section we examine the role of public infrastructure, denoted by $k$ in the model. Without private peering, congestion on the line connecting $i$ and $j$ is $\frac{s_i}{3} + \frac{s_j}{3} - \frac{k}{2}$. Given that at the equilibrium $s_i + s_j \leq \frac{1}{3}$, if $k > \frac{4}{9}$, there is no congestion on the National Access grid, even without peering. The obvious impact of this is that all incentives for private peering are eliminated and IAPs compete only in network capacities and prices. Consequently, the model is reduced to a two stage game in which IAPs first choose their networks capacities and then choose prices. We call this Case N.

We have solved for the subgame perfect Nash equilibria using backward induction and obtained the following values:

$$\widehat{s}_i^5 = \frac{2\mu}{15}$$

$$\widehat{p}_i^5 = \frac{\mu}{2}$$

[6] Computations are available at the following link: www.filebox.vt.edu/users/schakrab/computations.htm

[7] In fact, if forced to peer, they will not invest anything in link capacities because of the classic "tragedy of the commons" argument. Little and Wright (2000) reach similar conclusions using a model of horizontal product differentiation.

$$\widehat{d}_i^5 = \frac{1}{3}$$

$$\widehat{\pi}_i^5 = \frac{\mu(75-8\mu)}{450}$$

It is important to note that for all values of $0 < \lambda < \mu \leq 1$, profitability is unambiguously lower. Hence we get the following proposition.

**Proposition 2.3** *If the investment in public infrastructure is sufficiently large, i.e., $k > \frac{4}{9}$, then in the unique SPNE, IAPs do not peer and profits are unambiguously lower compared to those in the complete and empty networks with congestion at the NAP.*

The reason is that even though firms do not invest in link capacities, investments in network capacities are much higher. In other words: IAPs compete mainly with low connection failure rates. The proposition shows why increases in publicly provided infrastructure may not be in the best interests of firms and IAPs may lobby against such increases.

### 2.3.8 Welfare Analysis

In this section we conduct some welfare analysis on the model discussed above. First, we compare joint profits and, subsequently, consumer surplus across different cases. Define $\Pi^l$ as the joint profits under the Case $l \in \{1,2,3,4\}$. Then

$$\begin{aligned}
\Pi^1 &= 3\,\widehat{\boldsymbol{\pi}}^1 \\
\Pi^2 &= 3\,\widehat{\boldsymbol{\pi}}^2 \\
\Pi^3 &= 2\widehat{\boldsymbol{\pi}}^{3'} + \widehat{\boldsymbol{\pi}}^3 \\
\Pi^4 &= \widehat{\boldsymbol{\pi}}^{4'} + 2\,\widehat{\boldsymbol{\pi}}^4
\end{aligned}$$

**Conjecture 2.1** *If $0 < \lambda < \mu \leq 1$ and $k$ lies in the feasible parameter range, then the following holds*

$$\Pi^2 > \Pi^3 > \Pi^4 > \Pi^1$$

The computations are available in Appendix $A.6$. Thus, the joint profits are maximized in the empty network, while the complete network results in the lowest joint profits.

Next we compare consumer utilities across the different cases. Define $U_i^l$ be the utility a consumer derives when connected to $i$ under the Case $l \in \{1, 2, 3, 4\}$ and $U^l = \sum_{i \in N} d_i^l U_i^l$ be the weighted consumer surplus under the Case $l$, where $d_i^l$ is the demand for $i$'s services under the Case $l$.

**Conjecture 2.2** *If* $0 < \lambda < \mu \leq 1$ *and* $k$ *lies in the feasible parameter range, then the following holds*

$$U^1 > U^4 > U^3 > U^2$$

Computations are again available in Appendix $A.6$.

The consumer surplus is maximized under Case 1. It is higher in Case 1 compared to Case 2, because while demands and prices are the same, investments in the network capacities are higher in Case 1 compared to Case 2. Also, investments in the link capacities are positive in Case 1 and zero in Case 2. Thus the connection failure and congestion are lower in Case 1 than in Case 2. So we can say that the subgame perfect Nash equilibrium results in an efficient outcome from the consumers' point of view.

Thus, we conclude that the consumer surplus is maximized under Case 1, when all the IAPs have private peering agreements with each other, while the joint profits are maximized under the Case 2, when none of the IAPs peer. Depending on the values of $\lambda$, $\mu$ and $k$, the total welfare is maximized under different networks.

**Example 2.1:** For instance, if $\lambda = 0.4$, $\mu = 0.9$ and $k = 0.02$, then

$$\begin{aligned} \Pi^1 &= 0.414 \\ \Pi^2 &= 0.420 \\ \Pi^3 &= 0.418 \\ \Pi^4 &= 0.416 \end{aligned}$$

and

$$\begin{aligned} U^1 &= V - 0.24767 \\ U^2 &= V - 0.28452 \\ U^3 &= V - 0.27409 \\ U^4 &= V - 0.25981 \end{aligned}$$

Let $\Omega^l$ be the total surplus under Case $l$, i.e., $\Omega^l = \Pi^l + U^l$, where $l \in \{1, 2, 3, 4\}$. Then we have

$$\begin{aligned} \Omega^1 &= V + 0.16633 \\ \Omega^2 &= V + 0.13548 \\ \Omega^3 &= V + 0.14391 \\ \Omega^4 &= V + 0.15619 \end{aligned}$$

Hence, in this example the total surplus is at maximal in Case 1, that is, if all the IAPs peer.

## 2.4 The Model with Discrimination

Currently, unlike some other areas of telecommunications, there are no regulations prohibiting discrimination between IAPs with regard to peering. In the basic model, any IAP who wants to enter into peering agreements must do so with all other IAPs who are willing to peer as well. In this section we modify the model, where we take into account the possibility that an IAP may choose to peer with one but not the other IAP. Stages 2, 3 and 4 remain completely unchanged. However we redefine the strategies in the first stage as follows.

Let $\gamma_{ij}$ be 1, if $i$ signals its willingness to peer with $j$ and 0, if it does not. Peering agreements materialize between any pair of IAPs, who are willing to give permission to connect to each other. The peering agreements result in a creation of a network $g \in G$. Then four possible network configurations are possible. These are represented in Figure 2-3. We have already analyzed cases 1, 2 and 3. Hence we have to analyze one additional case namely Case 4.

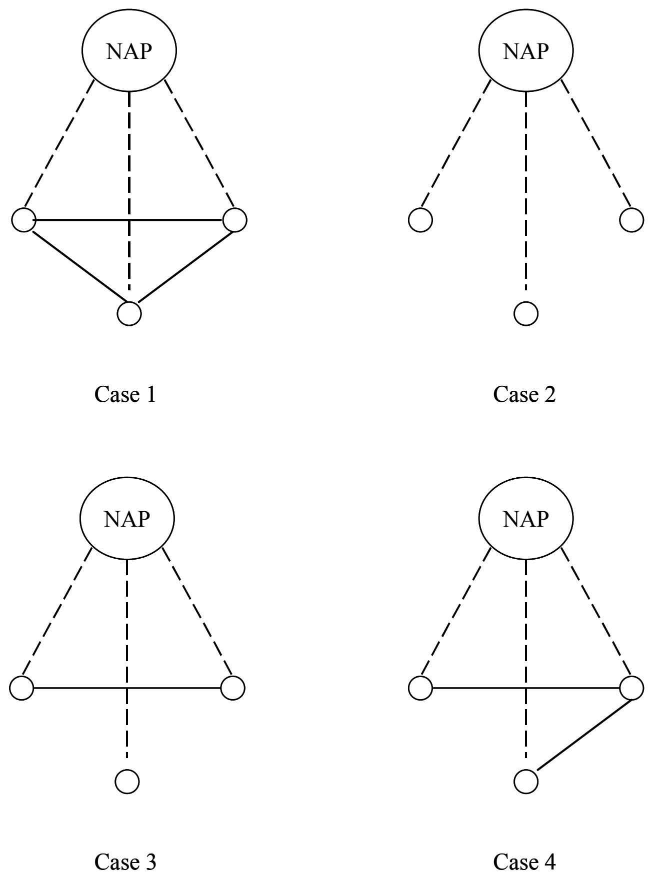


Figure 2-3: Four Possible Network Configurations

**Case 4**

In Case 4 two of the IAPs do not enter into mutual private peering agreement, but they both peer with the third IAP. Then the prices, market shares and both network and link investments and profits of those IAPs that do not peer with each other are the same. If $i$ and $j$ don't peer with each other, but both peer with $h$, then let the reduced profits of $i$ and $j$ be denoted by $\widehat{\boldsymbol{\pi}}^4$, while the profit of $h$ is $\widehat{\boldsymbol{\pi}}^{4'}$.The equilibrium values for this case are given in Appendix $A.4$.

### 2.4.1 Subgame Perfect Nash Equilibria

It is the case that for all values of $\lambda$ and $\mu$ satisfying $0 < \lambda < \mu \leq 1$, and $k$ lying in the feasible parameter range. Refer to Appendix $A.6$ for details.

$$\widehat{\boldsymbol{\pi}}^{4'} > \widehat{\boldsymbol{\pi}}^{3'} > \widehat{\boldsymbol{\pi}}^{2} > \widehat{\boldsymbol{\pi}}^{1} > \widehat{\boldsymbol{\pi}}^{4} > \widehat{\boldsymbol{\pi}}^{3} \tag{2.2}$$

Let $\vartheta_i(g)$ denote the profit of $i$ in the network $g \in G$. For example $\vartheta_1(g) = \widehat{\boldsymbol{\pi}}^1$ is IAP 1's profit in the complete network. The following proposition follows:

**Proposition 2.4** *Let $\vartheta_i(g)$ denote the reduced profits of $i$ in Stage 1 for any given network $g \in G$. Then*

$$\begin{aligned} &\text{(a) for all } ij \in g,\ \vartheta_i(g) > \vartheta_i(g - ij) \text{ and } \vartheta_j(g) > \vartheta_j(g - ij) \\ &\text{(b) for all } ij \notin g,\ \vartheta_i(g) < \vartheta_i(g + ij), \text{ and } \vartheta_j(g) < \vartheta_j(g + ij) \end{aligned}$$

The proof is in Appendix $A.5$. The intuition behind the proposition is as follows. Private peering agreement confers a competitive advantage to the pair engaging in that peering agreement, while breaking an agreement results in a disadvantage to the IAP which is breaking it. Hence irrespective of the network there is always an incentive to form a private peering agreement and disincentive to break such an agreement.

We can represent the first stage in the following normal form.

2

$\gamma_{21} = \gamma_{23} = 1$

| 1/3 | $\gamma_{31} = \gamma_{32} = 1$ | $\gamma_{31} = 1, \gamma_{32} = 0$ | $\gamma_{31} = 0, \gamma_{32} = 1$ | $\gamma_{31} = \gamma_{32} = 0$ |
|---|---|---|---|---|
| $\gamma_{12} = \gamma_{13} = 1$ | $\widehat{\pi}^1, \widehat{\pi}^1, \widehat{\pi}^1$ * | $\widehat{\pi}^{4'}, \widehat{\pi}^4, \widehat{\pi}^4$ | $\widehat{\pi}^4, \widehat{\pi}^{4'}, \widehat{\pi}^4$ | $\widehat{\pi}^{3'}, \widehat{\pi}^{3'}, \widehat{\pi}^3$ |
| $\gamma_{12} = 1, \gamma_{13} = 0$ | $\widehat{\pi}^4, \widehat{\pi}^{4'}, \widehat{\pi}^4$ | $\widehat{\pi}^{3'}, \widehat{\pi}^{3'}, \widehat{\pi}^3$ | $\widehat{\pi}^4, \widehat{\pi}^{4'}, \widehat{\pi}^4$* | $\widehat{\pi}^{3'}, \widehat{\pi}^{3'}, \widehat{\pi}^3$ |
| $\gamma_{12} = 0, \gamma_{13} = 1$ | $\widehat{\pi}^4, \widehat{\pi}^4, \widehat{\pi}^{4'}$ | $\widehat{\pi}^{3'}, \widehat{\pi}^3, \widehat{\pi}^{3'}$ | $\widehat{\pi}^3, \widehat{\pi}^{3'}, \widehat{\pi}^{3'}$ | $\widehat{\pi}^2, \widehat{\pi}^2, \widehat{\pi}^2$ |
| $\gamma_{12} = \gamma_{13} = 0$ | $\widehat{\pi}^3, \widehat{\pi}^{3'}, \widehat{\pi}^{3'}$ | $\widehat{\pi}^2, \widehat{\pi}^2, \widehat{\pi}^2$ | $\widehat{\pi}^3, \widehat{\pi}^{3'}, \widehat{\pi}^{3'}$ | $\widehat{\pi}^2, \widehat{\pi}^2, \widehat{\pi}^2$ |

2

$\gamma_{21} = 1, \gamma_{23} = 0$

| 1/3 | $\gamma_{31} = \gamma_{32} = 1$ | $\gamma_{31} = 1, \gamma_{32} = 0$ | $\gamma_{31} = 0, \gamma_{32} = 1$ | $\gamma_{31} = \gamma_{32} = 0$ |
|---|---|---|---|---|
| $\gamma_{12} = \gamma_{13} = 1$ | $\widehat{\pi}^{4'}, \widehat{\pi}^4, \widehat{\pi}^4$ | $\widehat{\pi}^{4'}, \widehat{\pi}^4, \widehat{\pi}^4$* | $\widehat{\pi}^{3'}, \widehat{\pi}^{3'}, \widehat{\pi}^3$ | $\widehat{\pi}^{3'}, \widehat{\pi}^{3'}, \widehat{\pi}^3$ |
| $\gamma_{12} = 1, \gamma_{13} = 0$ | $\widehat{\pi}^{3'}, \widehat{\pi}^{3'}, \widehat{\pi}^3$ | $\widehat{\pi}^{3'}, \widehat{\pi}^{3'}, \widehat{\pi}^3$ | $\widehat{\pi}^{3'}, \widehat{\pi}^{3'}, \widehat{\pi}^3$ | $\widehat{\pi}^{3'}, \widehat{\pi}^{3'}, \widehat{\pi}^3$* |
| $\gamma_{12} = 0, \gamma_{13} = 1$ | $\widehat{\pi}^{3'}, \widehat{\pi}^3, \widehat{\pi}^{3'}$ | $\widehat{\pi}^{3'}, \widehat{\pi}^3, \widehat{\pi}^{3'}$ | $\widehat{\pi}^2, \widehat{\pi}^2, \widehat{\pi}^2$ | $\widehat{\pi}^2, \widehat{\pi}^2, \widehat{\pi}^2$ |
| $\gamma_{12} = \gamma_{13} = 0$ | $\widehat{\pi}^2, \widehat{\pi}^2, \widehat{\pi}^2$ | $\widehat{\pi}^2, \widehat{\pi}^2, \widehat{\pi}^2$ | $\widehat{\pi}^2, \widehat{\pi}^2, \widehat{\pi}^2$ | $\widehat{\pi}^2, \widehat{\pi}^2, \widehat{\pi}^2$ |

2

$\gamma_{21} = 0, \gamma_{23} = 1$

| 1/3 | $\gamma_{31} = \gamma_{32} = 1$ | $\gamma_{31} = 1, \gamma_{32} = 0$ | $\gamma_{31} = 0, \gamma_{32} = 1$ | $\gamma_{31} = \gamma_{32} = 0$ |
|---|---|---|---|---|
| $\gamma_{12} = \gamma_{13} = 1$ | $\widehat{\pi}^4, \widehat{\pi}^4, \widehat{\pi}^{4'}$ | $\widehat{\pi}^{3'}, \widehat{\pi}^3, \widehat{\pi}^{3'}$ | $\widehat{\pi}^3, \widehat{\pi}^{3'}, \widehat{\pi}^{3'}$ | $\widehat{\pi}^2, \widehat{\pi}^2, \widehat{\pi}^2$ |
| $\gamma_{12} = 1, \gamma_{13} = 0$ | $\widehat{\pi}^3, \widehat{\pi}^{3'}, \widehat{\pi}^{3'}$ | $\widehat{\pi}^2, \widehat{\pi}^2, \widehat{\pi}^2$ | $\widehat{\pi}^3, \widehat{\pi}^{3'}, \widehat{\pi}^{3'}$ | $\widehat{\pi}^2, \widehat{\pi}^2, \widehat{\pi}^2$ |
| $\gamma_{12} = 0, \gamma_{13} = 1$ | $\widehat{\pi}^4, \widehat{\pi}^4, \widehat{\pi}^{4'}$* | $\widehat{\pi}^{3'}, \widehat{\pi}^3, \widehat{\pi}^{3'}$ | $\widehat{\pi}^3, \widehat{\pi}^{3'}, \widehat{\pi}^{3'}$ | $\widehat{\pi}^2, \widehat{\pi}^2, \widehat{\pi}^2$ |
| $\gamma_{12} = \gamma_{13} = 0$ | $\widehat{\pi}^3, \widehat{\pi}^{3'}, \widehat{\pi}^{3'}$ | $\widehat{\pi}^2, \widehat{\pi}^2, \widehat{\pi}^2$ | $\widehat{\pi}^3, \widehat{\pi}^{3'}, \widehat{\pi}^{3'}$* | $\widehat{\pi}^2, \widehat{\pi}^2, \widehat{\pi}^2$ |

2

$\gamma_{21} = \gamma_{23} = 0$

| 1/3 | $\gamma_{31} = \gamma_{32} = 1$ | $\gamma_{31} = 1, \gamma_{32} = 0$ | $\gamma_{31} = 0, \gamma_{32} = 1$ | $\gamma_{31} = \gamma_{32} = 0$ |
|---|---|---|---|---|
| $\gamma_{12} = \gamma_{13} = 1$ | $\widehat{\pi}^{3'}, \widehat{\pi}^{3}, \widehat{\pi}^{3'}$ | $\widehat{\pi}^{3'}, \widehat{\pi}^{3}, \widehat{\pi}^{3'}$ | $\widehat{\pi}^{2}, \widehat{\pi}^{2}, \widehat{\pi}^{2}$ | $\widehat{\pi}^{2}, \widehat{\pi}^{2}, \widehat{\pi}^{2}$ |
| $\gamma_{12} = 1, \gamma_{13} = 0$ | $\widehat{\pi}^{2}, \widehat{\pi}^{2}, \widehat{\pi}^{2}$ | $\widehat{\pi}^{2}, \widehat{\pi}^{2}, \widehat{\pi}^{2}$ | $\widehat{\pi}^{2}, \widehat{\pi}^{2}, \widehat{\pi}^{2}$ | $\widehat{\pi}^{2}, \widehat{\pi}^{2}, \widehat{\pi}^{2}$ |
| $\gamma_{12} = 0, \gamma_{13} = 1$ | $\widehat{\pi}^{3'}, \widehat{\pi}^{3}, \widehat{\pi}^{3'}$ | $\widehat{\pi}^{3'}, \widehat{\pi}^{3}, \widehat{\pi}^{3'*}$ | $\widehat{\pi}^{2}, \widehat{\pi}^{2}, \widehat{\pi}^{2}$ | $\widehat{\pi}^{2}, \widehat{\pi}^{2}, \widehat{\pi}^{2}$ |
| $\gamma_{12} = \gamma_{13} = 0$ | $\widehat{\pi}^{2}, \widehat{\pi}^{2}, \widehat{\pi}^{2}$ | $\widehat{\pi}^{2}, \widehat{\pi}^{2}, \widehat{\pi}^{2}$ | $\widehat{\pi}^{2}, \widehat{\pi}^{2}, \widehat{\pi}^{2}$ | $\widehat{\pi}^{2}, \widehat{\pi}^{2}, \widehat{\pi}^{2*}$ |

Then one can verify that there are eight subgame perfect Nash equilibria. In fact, all cases, Case 1, Case 2, Case 3 and Case 4 are consistent with a subgame perfect Nash equilibrium. We have indicated all the equilibria with an asterisk. So the following proposition follows.

**Proposition 2.5** *Under conditions $0 < \lambda < \mu \leq 1$ and $k$ lying in the feasible parameter range, if we allow the possibility of discrimination with regard to peering, besides the complete network and the empty network, any incomplete network is also consistent with a subgame perfect Nash equilibrium.*

Unlike certain other telecommunication industries, large backbones are not regulated by any sort of regulatory framework. Hence, discrimination is a realistic assumption. The surfeit of subgame perfect Nash equilibria make it imperative to use other equilibrium concepts to figure out plausible equilibrium configurations.

## 2.5 Network Analysis

In this section, we perform a formal network analysis. Peering is a collaborative effort. Hence, noncooperative games cannot fully analyze such a setting in its full complexity. Network analysis is a useful tool to analyze such an effort, because it combines elements of both cooperative and non-cooperative game theory.

The first stage is redefined as a link formation stage. A link $ij \in g$ indicates collaboration in form of a peering agreement. The emphasis is on stability. A stable network is one in which IAPs want to maintain existing links, but do not want to form new ones. This is captured by the concept of pairwise stability (introduced by Jackson and Wolinsky (1996) in the context of normal form games). We modify this definition in the set up of an extensive form game.

**Definition 2.1:** *Let $\vartheta_i(g)$ denote the reduced profits in Stage 1 for a network $g \in G$. Then the network $g$ is* ***pairwise stable*** *if*

$$\begin{aligned} &\text{(a) for all } ij \in g,\ \vartheta_i(g) \geqslant \vartheta_i(g-ij) \text{ and } \vartheta_j(g) \geqslant \vartheta_j(g-ij) \\ &\text{(b) for all } ij \notin g,\ \vartheta_i(g) < \vartheta_i(g+ij), \text{ implies } \vartheta_j(g) > \vartheta_j(g+ij) \end{aligned}$$

The pairwise stable network is the one in which all the IAPs want to maintain the existing links, while at the same time there isn't any pair of IAPs that would want to form a new private peering link with each other.

The next definition is one of efficiency. In our model, an efficient network is the one, which maximizes joint profits. Hence, formally,

**Definition 2.2:** *Let $\vartheta_i(g)$ denote the reduced profits in Stage 1 for a network $g \in G$. Let $W(g) = \sum_{i \in N} \vartheta_i(g)$. A network $g$ is* ***efficient*** *if $W(g) \geqslant W(g')$ for all $g' \in G$.*

Stages 2, 3 and 4 essentially remain unchanged. Next we come to the main result of our paper.

**Theorem 2.6** *(a) The unique pairwise stable network is the complete network $g^N$. (b) The unique efficient network is the empty network $g^0$.*

**Proof.**

**(a)** Consider any network other than the complete network. From Proposition 2-4, for all $ij \notin g$, $\vartheta_i(g) < \vartheta_i(g+ij)$.and $\vartheta_j(g) < \vartheta_j(g+ij)$, which violates part (b) of Definition 2.1. Hence, no network other than the complete network is a candidate for pairwise stability. Now consider the complete network. There are no links to form. Again from Proposition 2-4, for all $ij \in g$, $\vartheta_i(g) > \vartheta_i(g-ij)$ and $\vartheta_j(g) > \vartheta_j(g-ij)$. Hence part (a) of Definition 2.1 is satisfied. and that completes the proof.

**(b)** First we can show that for all values of $\lambda$ and $\mu$ satisfying $0 < \lambda < \mu \leq 1$,

$$3\ \widehat{\boldsymbol{\pi}}^2 > 2\widehat{\boldsymbol{\pi}}^{3'} + \widehat{\boldsymbol{\pi}}^3 > \widehat{\boldsymbol{\pi}}^{4'} + 2\ \widehat{\boldsymbol{\pi}}^4 > 3\ \widehat{\boldsymbol{\pi}}^1 \tag{2.3}$$

Thus, joint profits are strictly decreasing in the number of links. Consequently the result follows. The computations are available in Appendix $A.6$.

■

Hence, we arrive at the result that stable networks are not efficient and vice versa. The intuition is straight forward. Each link benefits the IAPs forming the link at the expense of the third IAP. But the gain through the link formation is more than offset by the loss to the third party.

We find that in our model, all networks are Nash equilibria outcomes of the signaling game with discrimination, but only one of them is pairwise stable. The large number of equilibria in a noncooperative game setting is drastically reduced in a network setting. This is because Nash equilibria focus on individual deviations. On the contrary, pairwise stability focusses on both individual and pairwise deviations. To engage in a private peering agreement both parties must agree, so creating a link is a pairwise deviation, while any IAP can break a private peering agreement unilaterally, so breaking an agreement is an individual deviation.

## 2.6 Conclusion

This chapter examined the interconnection arrangements among the Internet Access Providers that are on the top of the Internet hierarchy. We found that in this relatively simple model, where demand is fixed, and customers do not drop out with the declining quality of service, there is still a case for peering. When we incorporate network externalities, and the customer participation constraint, then the case for peering is even stronger for congested NAPs. We further show that a congested NAP is not necessarily a bad thing as far as IAPs are concerned, because it increases their profits by opening up the possibility of peering.

With regard to welfare analysis, large publicly provided infrastructure benefits consumers and hurts providers. Hence whether such infrastructure would be provided depends on the relative lobbying power of the IAPs vs. consumer groups. On the other hand if the capacities provided by NAPs is low, consumer surplus is maximized when all IAPs peer. Given that the only pairwise stable network is the complete network, there is no need for regulations to promote peering in this case.

The chapter enables comparison of pure non-cooperative game theoretic set-up with a networks set-up, which combines elements of both cooperative and non-cooperative game theory. Given that, in fact, there is extensive private peering among large backbones, it follows that the network approach results in both stronger and more realistic conclusions. Hence this chapter illustrates the advantages of using a mixed approach over a purely non-cooperative approach.

# Chapter 3

# Inter and Intra Backbone Peering[1]

## 3.1 Introduction

In this chapter we examine incentives for peering among the retail Internet Service Providers (ISPs) that are the second tier of the Internet hierarchy. ISPs are connected to Internet Access Providers (IAPs or backbones as they are often referred in this chapter) mostly through transit agreements. On the other hand, they can choose to peer among themselves both within and across backbones. We refer to peering between two ISPs connected to the same backbone as intra-backbone peering and to that between two ISPs connected to different backbones as inter-backbone peering.

A two stage game is considered in this chapter. In the first stage, ISPs decide on peering and in the second stage, given the contractual configuration, they compete in prices. Inter- and intra-backbone peerings among Internet service providers have two distinct (though not completely unrelated effects) on quality of service and ultimately profits of the ISPs. First, intra-backbone peering reduces the traffic within the backbone and raises the quality of the service provided by all ISPs connected to that backbone. Inter-backbone peering reduces the traffic between backbones and raises the quality of the service provided by all ISPs connected to both backbones. We call this the symmetric effect or the traffic diverting effect. Due to the symmetric effect, peering in general has strong positive externalities.

[1] **This chapter is based on "Inter and Intra Backbone Peering Decisions Among Internet Service Providers", Badasyan, N. and Chakrabarti, S. (2004), mimeo.**

The chief reason why ISPs may be interested in peering is not the symmetric effect but the asymmetric effect or the circumventing effect. Users (both web sites and consumers) who connect through privately peered lines avoid the congestion/delays associated with going through backbones and National Access Points. This raises the quality of the ISPs who peer and increases the demand for those ISPs' services. We call this the asymmetric or circumventing effect. We find that symmetric and asymmetric effects of peering often have opposite effects on ISPs' profits making the net effect ambiguous. Hence, the decision to peer or not largely depends on underlying parameters such as network capacity, number of ISPs connected to each backbone and number of consumers and websites connected to each ISP. In the example with six ISPs and two backbones, we find that a variety of configurations emerge in equilibrium depending on the values of the aforesaid parameters.

The symmetric effect has been captured by DangNyugen and Penard (1999) using a model of club behavior and vertical differentiation. They consider a model with two asymmetric backbones and a finite number of identical retail ISPs connected to those backbones. The retail ISPs engage in both intra-backbone peering and inter-backbone peering. They make the assumption that ISPs connected to the same backbone behave like clubs and collude with regard to pricing and connectivity behavior. The authors extend the standard model of vertical product differentiation to show that ISPs connected to the high quality backbone will always peer with each other. But they may or may not peer with the ISPs connected to the low quality backbone. Also, in the latter case, the ISPs of the low quality backbone will peer with each other. The results are illustrated with some evidence from the French Internet market. The asymmetric effect has been captured, among others, by the model presented in the second chapter of this dissertation, which shows that threat of traffic diversion creates strong incentives for peering among backbones or IAPs.

The rest of the chapter is organized as follows. In section 3.2, we discuss the assumptions underlying the model. In section 3.3, we analyze the second stage and we determine equilibrium prices. In section 3.4, we analyze the first stage and determine the equilibrium network. Finally, we conclude with Section 3.5.

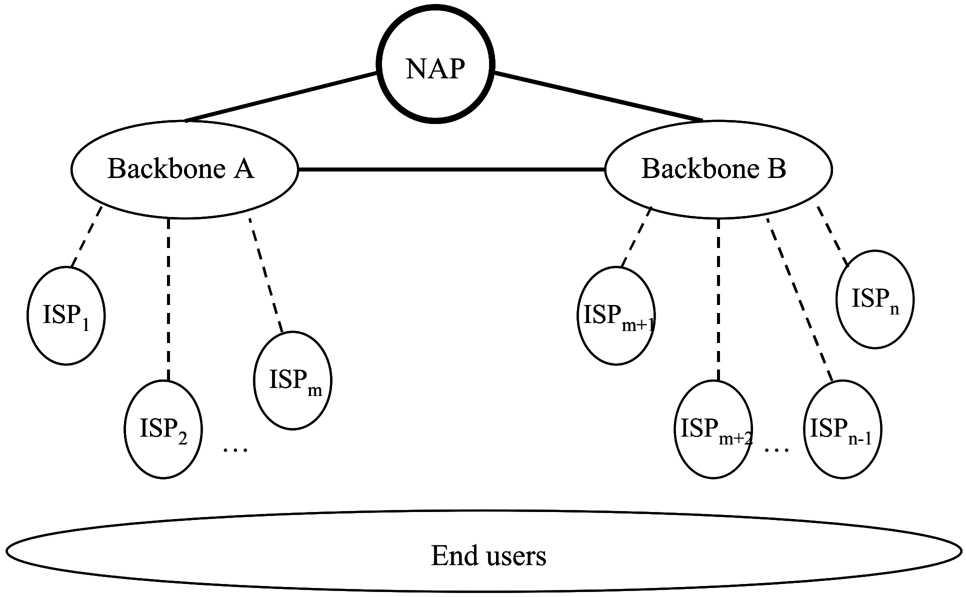


Figure 3-1: Wholesale and Retail Internet Service Providers

## 3.2 The Model

We consider two backbones $A$ and $B$ and a finite set of ISPs denoted by $N = \{1, 2, 3, \ldots, n\}$, where $n \in \mathbb{N}$. We assume that each ISP is connected to one of the backbones, but not both. Every ISP has a transit agreement with the backbone it is connected to. According to that agreement the ISP pays the backbone a certain settlement for interconnection, which is given exogenously. Without loss of generality, we assume that ISPs $1, \ldots, m$ are connected to backbone $A$ and ISPs $m+1, \ldots, n$ are connected to backbone $B$. Let $N_A = \{1, \ldots, m\}$ and $N_B = \{m+1, \ldots, n\}$. Generally we will index backbones by $l$, $l \in \{A, B\}$, and ISPs by $i, j \in N$. We illustrate this in Figure 3-1. We assume that each ISP pays an access fee $c_l$ (transit), if it is connected to backbone $l$. We distinguish between two types of end-users — consumers and websites. There is a continuum of consumers of mass 1 and a continuum of websites of mass 1 as well. We assume away any micro-payments from websites to consumers and vice versa. Let $\alpha_i$ denote the proportion of consumers connected to ISP $i$ and $d_i$ be the proportion of websites connected to ISP $i$. Obviously,

$$\sum_{i=1}^{n} \alpha_i = 1 = \sum_{i=1}^{n} d_i$$

Also denote

$$\sum_{i=1}^{m} \alpha_i = \alpha$$

$\alpha_i$'s are given exogenously, while $d_i$'s will be determined endogenously by explicitly modelling website preferences. Thus, $\alpha$ is the proportion of consumers, who have chosen the ISPs connected to backbone $A$. Similarly, $1 - \alpha$ is the proportion of consumers who have chosen the ISPs connected to backbone $B$.

In general it can be asserted that there is little traffic between websites. Traffic between consumers, while not negligible, is insignificant compared to traffic from websites to consumers. Thus, most of the traffic between the websites and the consumers is unidirectional, i.e., from websites to consumers. To capture this traffic pattern in its simplest form, we ignore all traffic between consumers, and from consumers to websites, and focus exclusively on the traffic from websites to consumers. Following Laffont et al (2003) we assume that consumers are interested in all websites independently of their network choices. A consumer is as likely to request a page from a given website belonging to her network and another website belonging to a rival network. This is referred in the aforesaid paper as a "balanced calling pattern". Hence, we assume each consumer requests one unit of traffic from each website. This gives precise measure to the proportion of traffic originating in ISP $i$ and terminating in ISP $j$, namely,

$$t_{ij} = d_i \cdot \alpha_j$$

To be consistent with the standard terminology, we refer to traffic between ISPs belonging to the same backbone as on-net traffic and traffic going from one backbone to another as off-net traffic. With regard to traffic movement between backbones, backbone providers follow what is called "hot potato routing" — pass off net traffic as soon as possible. Given this pattern of traffic, it is not unrealistic to assume that all off-net traffic is borne by the receiving backbone. Hence, traffic requested by consumers in $A$ from websites in $B$ will be borne by backbone $A$. Similarly, traffic requested by consumers in $B$ from websites in $A$ will be borne by $B$.

Let $U_i$ be the utility derived by a website connected to ISP $i$:

$$U_i = V - \delta_i - p_i$$

where $p_i$ is the price charged by ISP $i$ and $V$ is the value from the connectivity to the Internet. We assume $V$ to be sufficiently large so that no consumer drops out of the market. $\delta_i$ is the delay associated with ISP $i$, which depends on the congestion of the respective backbone that ISP $i$ is connected to as well as $i$'s private peerings.

Let $t_l$ be the amount of traffic coming into backbone $l \in \{A, B\}$. Given the balanced calling pattern and the hot potato routing principle,

$$\begin{aligned} t_A &= \sum_{i=1}^{m}\sum_{j=1}^{m}\alpha_i \cdot d_j + \sum_{i=1}^{m}\sum_{j=m+1}^{n}\alpha_i \cdot d_j = \alpha \\ t_B &= \sum_{i=m+1}^{n}\sum_{j=1}^{m}\alpha_i \cdot d_j + \sum_{i=m+1}^{n}\sum_{j=m+1}^{n}\alpha_i \cdot d_j = 1-\alpha \end{aligned}$$

ISPs can enter into private peering agreements with each other. If two ISPs build a private peering link, then that peering is referred to as intra-backbone peering, if the ISPs are connected to the same backbone, and inter-backbone peering, if the ISPs are connected to different backbones. For example, in Figure 3-2, ISP 2 has an intra-backbone peering agreement with ISP 1 and an inter-backbone peering agreement with ISP $m+2$. For any two distinct ISPs $i$ and $j$, define a binary variable $\gamma_{ij} \in \{0, 1\}$ where

$$\gamma_{ij} = \begin{cases} 1 \text{ if a private peering arrangement exists between } i \text{ and } j \\ 0 \text{ if no such arrangement exists} \end{cases}$$

Obviously, $\gamma_{ij} = \gamma_{ji}$. If $\gamma_{ij} = 1$, we say a link $ij$ exists or a link is formed between $i$ and $j$. The network $g = \{ij|\ \gamma_{ij} = 1,\ i,j \in N,\ i \neq j\}$ is then a collection of links. Let $g - ij$ denote the network obtained by severing an existing link between $i$ and $j$ from the network $g$, while $g + ij$ is the network obtained by adding a new link between $i$ and $j$ in the network $g$. The network $g$ for which $\gamma_{ij} = 1$ for all $i, j \in N$, $i \neq j$ is called *the complete network*. The network $g$ for which $\gamma_{ij} = 0$ for all $i, j \in N$, $i \neq j$ is called *the empty network*.

We assume each private peering link entails a capacity amounting to $\sigma$ between $i$ and $j$. Thus the capacity of each private peering link is exogenously given and set equal to $\sigma$. Usage is shared equally and costs are borne equally. Assuming quadratic costs, each such link costs $\sigma^2$ and the cost borne by each of the two ISPs forming a link is $\sigma^2/2$. We assume that the private

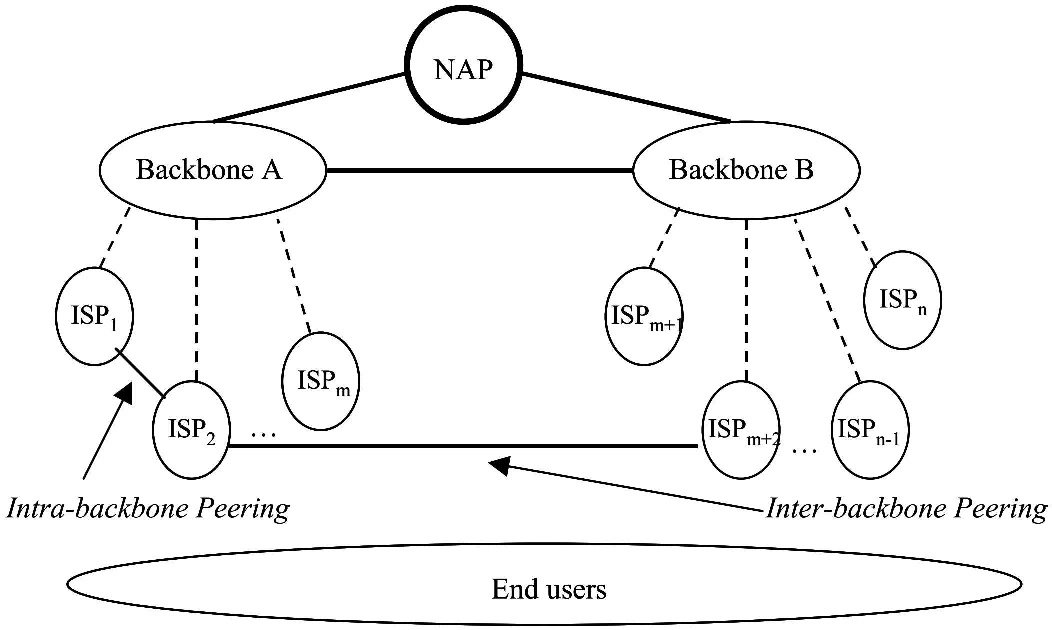


Figure 3-2: Inter and Intra Backbone Peerings

lines are completely uncongested and hence traffic traversing such links experience zero delays. Given that private links carry traffic that otherwise would have been carried by backbones, private peering links also reduce backbone congestion by reducing backbone traffic.

Now we can compute the average delay experienced by ISP $i$, namely, $\delta_i$. Let $n_{AA}$ denote the number of intra-backbone links for backbone $A$, $n_{BB}$ be the number of intra-backbone links for backbone $B$, and $n_{AB}$ be the number of inter-backbone links. Obviously,

$$\begin{aligned} n_{ll} &= \sum_{j \in N_l} \sum_{i \in N_l,\, i<j} \gamma_{ij} \\ n_{AB} &= \sum_{j \in N_B} \sum_{i \in N_A} \gamma_{ij} \end{aligned}$$

where $l \in \{A, B\}$.

Further, let $n_{iA}$ be the number of links ISP $i$ has with members of $N_A$, and $n_{iB}$ be the number of links ISP $i$ has with members of $N_B$. Let $n_i$ be the total number of private links of

ISP $i$. Hence,

$$\begin{aligned} n_{il} &= \sum_{j \in N_l,\, j \neq i} \gamma_{ij} \\ n_i &= n_{iA} + n_{iB} \end{aligned}$$

where $l \in \{A, B\}$.

Let $\omega_l$ be the reduction or leakage in the traffic of backbone $l$ as a result of private peerings. Then,

$$\begin{aligned} \omega_A &= (n_{AA}) \cdot \sigma + (n_{AB}) \cdot (\sigma/2) \\ \omega_B &= (n_{BB}) \cdot \sigma + (n_{AB}) \cdot (\sigma/2) \end{aligned}$$

Delays occur due to excessive traffic in the backbone. Define $s_l$ to be the network capacity of backbone $l \in \{A, B\}$. The network capacity is the maximum amount of traffic that the backbone can handle without experiencing delay. We assume that the network capacities are exogenously given. The traffic of backbone $l$ is reduced by the amount of on-net traffic diverted through intra-backbone private peering links and by the amount of the outgoing off-net traffic diverted through inter-backbone peering links. Each intra-backbone link reduces on-net traffic by $\sigma$ and each inter-backbone link reduces off-net outgoing traffic by $\sigma/2$ given that usage of peered links is shared equally.

Hence, if $\theta_l$ is the amount of congestion in backbone $l$, then

$$\theta_l = t_l - s_l - \omega_l$$

$\theta_l$ is similar to $F_l$ (connection failure) in chapter 2 in a sense that it measures congestion at certain point in time given the limited capacity $s_l$ and the total traffic in the backbone $l$ . However, $F_l$ accounts for the traffic originated in backbone $l$ only, while $\theta_l$ takes into account also the incoming traffic of the other backbone. In chapter 2 the later is captured separately in the congestion of the link between the backbones. In this chapter we separate the congestion caused by the traffic travelling across the backbones, depending on the direction of the traffic.

Traffic going through private peering links experience zero delays, because we explicitly assume that ISPs keep their private links uncongested. All extra traffic is routed through the backbones. The traffic going through backbone $A$ experiences congestion of $\theta_A$ and the traffic going through backbone $B$ experiences a congestion of $\theta_B$. If $\theta_A > \theta_B$, we refer to $A$ as the low quality backbone and $B$ as the high quality backbone and vice versa.

Consider an ISP connected to backbone $A$. Given that this ISP has $n_{iA}$ intra-backbone links and each such link can carry $\sigma/2$ of traffic, on-net traffic circulating through privately peered lines is equal to $n_{iA} \cdot (\sigma/2)$ and this traffic experiences zero delay. Given that total volume of on-net traffic going out from this ISP is $\alpha \cdot d_i$, the traffic that traverses backbone $A$ and experiences a delay of $\theta_A$ has a volume of $\alpha \cdot d_i - n_{iA} \cdot (\sigma/2)$. Next, the off-net traffic going out from this ISP is $(1-\alpha) \cdot d_i$ of which again $n_{iB} \cdot (\sigma/2)$ amount of traffic that travels through privately peered lines and experiences zero delay. Hence, a volume of $(1-\alpha) \cdot d_i - n_{iB} \cdot (\sigma/2)$ traffic traverses both backbones and is largely borne by the receiver backbone $B$ and hence experiences delay $\theta_B$. Therefore, if $i \in N_A$, then[2]

$$\delta_i = \theta_A \cdot (\alpha \cdot d_i - n_{iA} \cdot (\sigma/2)) + \theta_B \cdot ((1-\alpha) \cdot d_i - n_{iB} \cdot (\sigma/2)).$$

Similarly, for $j \in N_B$,

$$\delta_j = \theta_B \cdot ((1-\alpha) \cdot d_j - n_{jB} \cdot (\sigma/2)) + \theta_A \cdot (\alpha \cdot d_j - n_{jA} \cdot (\sigma/2)).$$

Equilibrium demands are determined by equating utilities of all websites given that in equilibrium all websites must derive an identical utility from each ISP. As far as the ISPs are concerned, we will analyze a two-stage game. In the first stage, ISPs form links that determine the network structure $g$. In the second stage they compete in prices. Consistent with the logic of backward induction, we start with the second stage.

[2] Traffic going from websites to consumers connected to the same ISP do not have to traverse any backbone or private peering link. So strictly speaking, we have to deduct this leakage from traffic flows through backbones. We assume that this leakage is relatively small and can be ignored.

## 3.3 Analysis of the Second Stage

When ISPs decide on private peering arrangements in the first stage, a network, $g$, is formed and $n_{AA}$, $n_{BB}$ and $n_{AB}$ are determined. Given a certain network $g$ and the fact that ISPs compete in prices, we can solve for equilibrium prices in the second stage. We refer for the details to Appendix $B$.1. Here we just present the results in the form of a proposition.

**Proposition 3.1** *Suppose the following parameter constraints*

$$0 < s_A < \alpha < 1 \tag{3.1}$$

$$0 < s_B < 1 - \alpha \tag{3.2}$$

$$0 < \sigma < \min\{\frac{2(\alpha - s_A)}{m(n-1)}, \frac{2(1-\alpha-s_B)}{(n-m)(2\cdot n - 2\cdot m - 1)}\} \tag{3.3}$$

*Denote*

$$\begin{aligned}
\theta_A &= \alpha - s_A - \sigma\left(n_{AA} + \frac{n_{AB}}{2}\right) \\
\theta_B &= (1-\alpha) - s_B - \sigma\left(n_{BB} + \frac{n_{AB}}{2}\right) \\
\phi_i &= \frac{\sigma\left(n_{iA}\cdot\theta_A + n_{iB}\cdot\theta_B\right)}{2} \text{ for all } i \in N \\
\widetilde{\theta} &= \alpha\cdot\theta_A + (1-\alpha)\cdot\theta_B
\end{aligned}$$

*Then there is a unique interior solution represented by an equilibrium price, demand and profit of $i \in N_l$ in stage 2 given, respectively, by*

$$p_i^* = \left(\frac{\widetilde{\theta}}{n-1}\right) + \frac{1}{2n-1}\left(n\cdot\phi_i - \sum_{j=1}^{n}\phi_j\right) \tag{3.4}$$

$$d_i^* = \left(\frac{n-1}{n\cdot\widetilde{\theta}}\right)\left[\left(\frac{\widetilde{\theta}}{n-1}\right) + \frac{1}{2n-1}\left(n\cdot\phi_i - \sum_{j=1}^{n}\phi_j\right)\right] \tag{3.5}$$

$$\pi_i^* = \left(\frac{n-1}{n\cdot\widetilde{\theta}}\right)\left[\left(\frac{\widetilde{\theta}}{n-1}\right) + \frac{1}{2n-1}\left(n\cdot\phi_i - \sum_{j=1}^{n}\phi_j\right)\right]^2 - \left(\frac{\sigma^2}{2}\right)n_i - c_l \tag{3.6}$$

The above proposition represents the unique interior solution. For the interior solution to exist and be applicable, we need certain parameter constraints. Specifically, $\theta_A > 0$ and $\theta_B > 0$

requires

$$
\begin{aligned}
\sigma &< \frac{2(\alpha - s_A)}{(2 \cdot n_{AA} + n_{AB})} \\
\sigma &< \frac{2(1 - \alpha - s_B)}{(2 \cdot n_{BB} + n_{AB})} \\
s_A &< \alpha \\
s_B &< 1 - \alpha
\end{aligned}
$$

Given that

$$
\begin{aligned}
n_{AA} &\leqslant \frac{m(m-1)}{2} \\
n_{BB} &\leqslant \frac{(n-m)(n-m-1)}{2} \\
n_{AB} &\leqslant m(n-m)
\end{aligned}
$$

we have

$$
\begin{aligned}
\frac{2 \cdot n_{AA} + n_{AB}}{2} &\leqslant \frac{m(n-1)}{2} \\
\frac{2 \cdot n_{BB} + n_{AB}}{2} &\leqslant \frac{(n-m)(2 \cdot n - 2 \cdot m - 1)}{2}
\end{aligned}
$$

The above constraints are satisfied for all possible networks, under (3.1)—(3.3). To allow the possibility of link formation within each backbone, we also assume that $m > 1$, $n > 3$.

We have stated before that $\theta_A$ is the measure of congestion in backbone $A$ and $\theta_B$ is the measure of congestion in backbone $B$. The backbone with the lower level of congestion will be referred to as the high quality backbone, and that with the higher level of congestion will be referred to as the low quality backbone.

Hence, $\widetilde{\theta} = \alpha \cdot \theta_A + (1 - \alpha) \cdot \theta_B$ is the level of congestion in each backbone weighted by the traffic flowing through each backbone. It is an average measure of congestion in the whole network. $\widetilde{\theta}$ captures the symmetric effect of peering because changes in backbone congestion primarily affect this parameter.

As far as the asymmetric effect is concerned, $\phi_i$ will be referred to as the "link density factor" of ISP $i$. It is proportional to the number of links ISP $i$ has in each backbone weighted by the level of congestion. The weights reflect the fact that forming links in congested backbones to divert traffic is more valuable than forming links in uncongested backbones. Next, define

$$\kappa_i = n\left(\phi_i - \frac{1}{n}\sum_{j=1}^{n}\phi_j\right)$$

which is proportional to the difference between the link density factor of ISP $i$ and the average link density factor. $\kappa_i$, captures the asymmetric effect of peering because it is proportional to $\phi_i$, which reflects the impact of circumventing congested backbones on delay.

To see this clearly, we can express the delay or congestion for ISP $i$, in terms of $\widetilde{\theta}$ and $\phi_i$ :

$$\delta_i = d_i \cdot \widetilde{\theta} - \phi_i$$

The higher the overall backbone congestion is, the greater is the delay, but delay can be reduced by increasing the link density factor, because traffic can now circulate through uncongested lines. Increasing demand also increases delay, because a larger amount of the traffic passes through congested backbones.

We can express prices, demands and profits in terms of $\widetilde{\theta}$ and $\kappa_i$.

$$\begin{aligned}
p_i^* &= \left(\frac{\widetilde{\theta}}{n-1}\right) + \left(\frac{\kappa_i}{2n-1}\right) \\
d_i^* &= \left(\frac{n-1}{n\cdot\widetilde{\theta}}\right)\left[\left(\frac{\widetilde{\theta}}{n-1}\right) + \left(\frac{\kappa_i}{2n-1}\right)\right] = \left(\frac{n-1}{n\cdot\widetilde{\theta}}\right)p_i^* \\
\pi_i^* &= \left(\frac{n-1}{n\cdot\widetilde{\theta}}\right)\left[\left(\frac{\widetilde{\theta}}{n-1}\right) + \left(\frac{\kappa_i}{2n-1}\right)\right]^2 - \left(\frac{\sigma^2}{2}\right)n_i - c_l = p_i^* d_i^* - -\left(\frac{\sigma^2}{2}\right)n_i - c_l
\end{aligned}$$

By gross profits we mean profits not taking into account link formation costs. Hence, gross profit is simply the product of price and demand minus the transit fee paid to the backbones. We can examine the impact that symmetric and asymmetric effects of peering have on gross profits.

Next, let us examine how peering affects gross profits. First note that

$$\begin{aligned}\frac{\partial p_i}{\partial \widetilde{\theta}} &= \frac{1}{n-1} > 0 \\ \frac{\partial d_i}{\partial \widetilde{\theta}} &= \frac{-(n-1)\cdot \kappa_i}{(2\cdot n-1)\cdot n\cdot \left(\widetilde{\theta}\right)^2} < 0 \text{ if } \kappa_i > 0 \text{ and } > 0 \text{ if } \kappa_i < 0.\end{aligned}$$

Hence, increasing the level of overall congestion increases prices.[3] For ISPs with above average link density, increasing the level of congestion reduces demand, while reducing the link density increases demand. For ISPs with below average link density, the opposite is the case. Increasing the level of overall congestion increases demand and vice versa.

Now an intra-backbone link reduces $\widetilde{\theta}$ by $\alpha\sigma$ through its effect on the backbone the ISPs belong to. An inter-backbone link reduces $\widetilde{\theta}$ by $\sigma/2$ through its impact on both backbones. While those boost consumer utilities, they have either a negative or an ambiguous effect on ISPs' profits. For ISPs with below average link density ($\kappa_i < 0$) the impact is definitely negative, because given the signs of the derivatives above, both prices and demands fall and, hence, gross profits fall. For ISPs with above average link density ($\kappa_i < 0$), the impact is ambiguous. While prices fall, demands rise and hence the net effect depends on the relative magnitudes of the two effects.

It is somewhat counter-intuitive that peering should have a negative or ambiguous effect on gross profit. This is because the effect of reducing overall congestion on price is always negative. This is because,

$$\delta_i - \delta_j = (d_i - d_j)\cdot \widetilde{\theta} - (\phi_i - \phi_j)$$

Hence, any increase in $\widetilde{\theta}$ accentuates the difference in quality or congestion or the level of vertical differentiation between ISPs $i$ and $j$. The increase in vertical differentiation softens price competition and enables both firms to charge higher prices. In fact, both prices rise by an exactly equal amount in equilibrium and, hence, final differences in prices as well as differences in congestion remain unchanged. Hence, the initial increase in the difference in congestion is compensated by appropriate changes in demands.

[3] This is a standard result for congested goods—see De Palma and Leruth (1989).

Next, consider the asymmetric effects of peering. We have

$$\begin{aligned}\frac{\partial p_i}{\partial \kappa_i} &= \frac{1}{2\cdot n-1} > 0\\ \frac{\partial d_i}{\partial \kappa_i} &= \frac{(n-1)}{(2\cdot n-1)\cdot n\cdot \widetilde{\theta}} > 0\end{aligned}$$

Consider the impact of intra and inter-backbone links on $\kappa_i$. There is a direct effect owing to the fact that traffic can now travel through uncongested lines and there is an indirect effect due to the impact of link formation on congestion in the backbones $\theta_A$ and $\theta_B$. We ignore the indirect effect because it involves terms $\sigma^2$ which is negligible if $\sigma$ is small. Thus, an intra-backbone link for $i \in N_l$ boosts $\kappa_i$ by $\frac{n\cdot\sigma\cdot\theta_l}{2}$ and, hence, has a positive impact on gross profits. An inter-backbone link increases $\kappa_i$ by $\frac{\sigma[(n-1)\theta_{-l}-\theta_l]}{2}$, where $-l$ refers to the backbone other than $l$. If $i$ belongs to the high quality backbone, namely, $\theta_{-l} > \theta_l$, inter-backbone links increase gross profits. If $i$ belongs to the low quality backbone, $\theta_{-l} < \theta_l$, inter-backbone links increase gross profits, provided that the quality difference between the two backbones is not very high, i.e. $(n-1)\theta_{-l} > \theta_l$. However, if the quality difference between the two backbones is very high, $(n-1)\theta_{-l} < \theta_l$, then inter-backbone peering has indeed a negative effect on gross profits.

Hence, we get Proposition 3.2.

**Proposition 3.2** *(a) For ISPs with link density, below the average link density, the symmetric effects of intra-backbone or inter-backbone peering on gross profits are always negative. For those with link density above the average, the effect is ambiguous.*
*(b) For all ISPs, the asymmetric effect of intra-backbone peering on gross profits is always positive. For ISPs connected to high quality backbones, the effect of inter-backbone peering is also positive. However, for ISPs connected to low quality backbones, the effect is positive, if the difference in quality between the two backbones is sufficiently small, and negative otherwise.*

We can summarize these facts in the form of the following tables. Let HH denote the fact that the ISP in question belongs to the high quality backbone and has a higher than average link density. Let HL denote the fact that the ISP in question belongs to the high quality backbone and has a lower than average link density. Let LH denote the fact that the ISP in question belongs to the low quality backbone and has a higher than average link density. Let LL denote

| | Intra-backbone Peering | | Inter-backbone Peering | |
|---|---|---|---|---|
| | Symmetric Effect | Asymmetric Effect | Symmetric Effect | Asymmetric Effect |
| HH | Ambiguous | Positive | Ambiguous | Positive |
| HL | Negative | Positive | Negative | Positive |
| LH | Ambiguous | Positive | Ambiguous | Positive |
| LL | Negative | Positive | Negative | Positive |

Table 3.1: Effect of Peering on Gross Profits under Condition 3.1

| | Intra-backbone Peering | | Inter-backbone Peering | |
|---|---|---|---|---|
| | Symmetric Effect | Asymmetric Effect | Symmetric Effect | Asymmetric Effect |
| HH | Ambiguous | Positive | Ambiguous | Positive |
| HL | Negative | Positive | Negative | Positive |
| LH | Ambiguous | Positive | Ambiguous | Negative |
| LL | Negative | Positive | Negative | Negative |

Table 3.2: Effect of Peering on Gross Profits under Condition 3.2

the fact that the ISP in question belongs to the low quality backbone and has a lower than average link density. We analyze the effect of peering on gross profit under the two following conditions: Condition 3.1 requiring that $(n-1)\theta_{-l} > \theta_l$ and Condition 3.2 requiring that $(n-1)\theta_{-l} < \theta_l$.

Hence, one finds that quite often the symmetric and asymmetric effects actually work against each other. This is because, while peering improves quality and hence prices, demands and profits, its effect on backbone congestion has the unintended effect of reducing vertical differentiation, stiffening price competition and reducing prices and profits. The net effect is almost always ambiguous and stating anything further would require comparison of relative magnitudes, which in turn would depend on relative values of parameters $\alpha, s_l$ and $m$. Hence, one would expect to find a multitude of equilibrium networks depending on the values of these parameters, a fact that we illustrate in the next section with an example.

However, we can state one thing with certainty. If the quality difference between the two backbones is substantial, namely $(n-1)\theta_A < \theta_B$ or vice versa, then ISPs connected to the quality backbone have no interest in inter-peering, if they have also lower than average link densities. This is because gross profits reduce with inter-backbone peering given that both the symmetric and asymmetric effects are negative, and the added cost of link formation further reduces net profits. This actually forms the basis of Proposition 3.3 formulated in the next section.

## 3.4 Analysis of the First Stage

We will analyze the first stage using the notion of pairwise stability, which was introduced by Jackson and Wolinsky (1996). A network is pairwise stable if given the network, there is no incentive to either form links or destroy links. Since links are formed bilaterally, but can be broken unilaterally, we can formally define it as follows:

**Definition 3.1**: *Let $\pi_i(g)$ denoted the reduced profits of stage 1 for a network $g$. The network $g$ is pairwise stable if for all $i, j \in N$ :*

$$\begin{aligned}
\text{(a) If } \gamma_{ij} &= 1, \text{then } \pi_i(g) \geqslant \pi_i(g-ij) \text{ and } \pi_j(g) \geqslant \pi_j(g-ij) \\
\text{(b) If } \gamma_{ij} &= 0 \text{ and } \pi_i(g+ij) > \pi_i(g), \text{ then } \pi_j(g+ij) < \pi_j(g)
\end{aligned}$$

The intuition here is quite simple. Links are formed bilaterally, but can be broken unilaterally. Hence, in a pairwise stable network, neither player should gain by breaking a link, while at least one player must lose or remain indifferent through forming a new link.

Before we continue with an example, let us formalize our analysis in the previous section in the form of Proposition 3.3.

**Proposition 3.3** *Let $(n-1)\theta_{-l} < \theta_l$, where $l$ represents the low quality backbone, namely the quality differential between the two backbones is sufficiently large. Then, in any pairwise stable* *network, there is no inter-backbone peering between two ISPs with different link densities, if the ISP belonging to the low quality backbone has a lower than average link density.*

In all other cases, given that net effects of peering are ambiguous, we cannot state anything for certain. Thus, we will consider an example. We only study six-provider networks. The choice of six is not entirely arbitrary. Since the number of ISPs attached to each backbone has to be greater than or equal to two for any meaningful analysis, six is the smallest number that allows us to consider both symmetry as well asymmetry in the number of firms attached to individual backbones. We only consider the following eight possible cases or eight possible networks.[4]

**1.** All ISPs peer with each other. The resulting network, called a complete network, is denoted by $g_{AB1}$.

**2.** There is no peering whatsoever. The resulting network, called an empty network, is denoted by $g_{000}$.

**3.** All ISPs belonging to both backbones engage in intra-backbone peering, but there is no inter-backbone peering. We refer to the resulting network as $g_{AB0}$.

**4.** All ISPs belonging to backbone A engage in intra-backbone peering. However, there is neither any inter-backbone peering nor any intra-backbone peering in backbone B. We refer to this network as $g_{A00}$.

**5.** All ISPs belonging to backbone B engage in intra-backbone peering. However, there is neither any inter-backbone peering nor any intra-backbone peering in backbone A. We refer to this network as $g_{0B0}$.

**6.** All ISPs belonging to backbone A engage in intra-backbone peering. There is no intra-backbone peering in backbone B, but there is inter-backbone peering. We refer to this network as $g_{A01}$.

**7.** All ISPs belonging to backbone B engage in intra-backbone peering. There is no intra-backbone peering in backbone A, but there is inter-backbone peering. We refer to this network as $g_{0B1}$.

**8.** No intra-backbone peering whatsoever, but there is inter-backbone peering. We refer to this network as $g_{001}$.

[4] Checking more complicated networks for pairwise stability requires some programming which is reserved as a future endeavour.

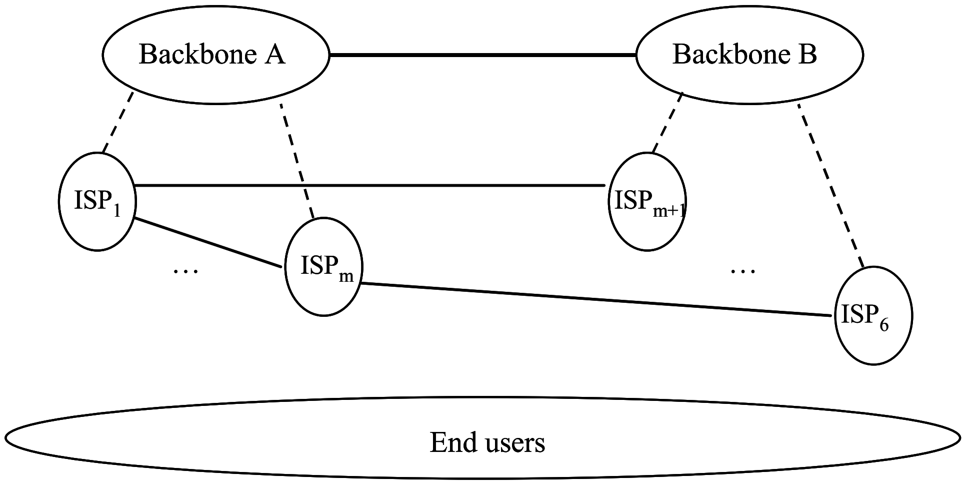


Figure 3-3: Network $g_{A01}$

Figure 3.3, for example, illustrates network $g_{A01}$.

**Case 3.1**: We start with perfect symmetry, namely, $\alpha = 0.5$, $m = 3$, $s_A = s_B = s$ (say). Then, the area, where all parameter constraints (3.1)-(3.3) are satisfied, is represented by Figure $B-1.1$, where we plot $\sigma$ and $s$ along the two axes. This figure and all subsequent figures are in Appendix $B-1$. We will refer to the parameter range for which constraints (3.1)-(3.3) are satisfied as the feasible parameter range or the feasible range. The figures have by developed using Mathematica. Refer to Appendix $B-1$ for details.

We find that with complete symmetry, two networks, namely, the complete network and the empty network, are pairwise stable in the feasible parameter range. Figure $B-1.2$ represents the area where the complete network is pairwise stable. Figure $B-1.3$ represents the area where the empty network is pairwise stable. The two areas represent mutually exclusive parameter ranges, hence, for a given set of parameter values, there is an unique pairwise stable network which is either the complete or the empty network. This is clear from Figure $B-1.4$, where we represent the two areas in the same figure. Under complete symmetry, all firms face exactly the same benefits and costs. Hence, it is expected that we get symmetric outcomes, namely either all firms will peer or nobody will peer. Depending on the magnitudes of $\sigma$ and $s$ either outcome is feasible.

**Case 3.2:** Next we will introduce asymmetries. We find that introduction of asymmetries results in other pairwise stable networks. We begin with an asymmetry in the network capacity. Specifically, assume that $s_A > s_B = s$. We plot $\sigma, s_A$ and $s_B = s$ along the three axes. We find that there are two additional pairwise stable network configurations within the feasible parameter range besides the complete and the empty network, namely, the network $g_{AB0}$ and the network $g_{0B0}$. We summarize this in Figures $B-1.5$, $B-1.6$, $B-1.7$, $B-1.8$ and $B-1.9$. The figures are three dimensional to account for the fact that $s_A$ and $s_B$ are represented in two different axes. Figure $B-1.5$ represents the feasible parameter range. Figure $B-1.6$ represents the area where the complete network is pairwise stable. Figure $B-1.7$ represents the area, where the empty network is pairwise stable. Figure $B-1.8$ represents the area, where the network $g_{AB0}$ is pairwise stable. Figure $B-1.9$ represents the area where the network $g_{0B0}$ is pairwise stable.

**Case 3.3:** Next, starting from perfect symmetry, let us introduce an asymmetry in the number of ISPs connected to each backbone. Here our options are quite limited. We can consider only the case where $m = 4$ and $n = 6$. We plot $\sigma$ and $s(= s_A = s_B)$ on the two axes. We find that again there are two possible pairwise stable networks, besides the complete and the empty network in the feasible parameter range, namely, the network $g_{AB0}$ and the network $g_{0B0}$. We represent this situation in Figures $B-1.10$, $B-1.11$, $B-1.12$, $B-1.13$, $B-1.14$ and $B-1.15$. Figure $B-1.10$ represents the feasible parameter range. Figure $B-1.11$ represents the area, where the complete network is pairwise stable. Figure $B-1.12$ represents the area, where the empty network is pairwise stable. Figure $B-1.13$ represents the area, where the network $g_{AB0}$ is pairwise stable. Figure $B-1.14$ represents the area, where the network $g_{0B0}$ is pairwise stable. The areas are mutually exclusive, hence, for a certain parameter value we get an unique stable network. This can be seen from Figure $B-1.15$, where we bring all the different areas in one figure.

**Case 3.4**: Finally, let us explore asymmetries in the consumer base. Assume, for instance, starting from a perfectly symmetric setup with $n = 6$, that $\alpha > 0.5$. We represent this situation in Figures $B-1.16$, $B-1.17$, $B-1.18$, $B-1.19$ and $B-1.20$. The figures are three dimensional and we plot $\alpha, \sigma$ and $s(= s_A = s_B)$ along the three axes. This time we find that besides the

| **Nature of asymmetry** | **Pairwise Stable Networks** |
|---|---|
| None | $g_{AB1}, g_{000}$ |
| Asymmetry in network capacity $s_A > s_B$ | $g_{AB1}, g_{000}, g_{AB0}, g_{0B0}$ |
| Asymmetry in the the number of firms $m > n/2$ | $g_{AB1}, g_{000}, g_{AB0}, g_{0B0}$ |
| Asymmetry in the website base $\alpha > 1/2$ | $g_{AB1}, g_{000}, g_{AB0}, g_{A00}$ |

Table 3.3: Results of the Example

complete and the empty network, two more networks namely $g_{AB0}$ and $g_{A00}$ are pairwise stable within the feasible parameter range. Figure $B - 1.16$ represents the feasible parameter range. Figure $B - 1.17$ represents the area, where the complete network is pairwise stable. Figure $B - 1.18$ represents the area, where the empty network is pairwise stable. Figure $B - 1.19$ represents the area, where the network $g_{AB0}$ is pairwise stable. Figure $B - 1.20$ represents the area, where the network $g_{A00}$ is pairwise stable.

We summarize these results of our example in the form of Observation 3.1.

**Observation 3.1** *Let $n = 6$ and parameter values satisfy constraints (3.1)-(3.3). Then, (1) for complete symmetry, namely, $\alpha = 0.5, m = 3$ and $s_A = s_B$, there are two pairwise stable networks, namely the complete network and the empty network; (2) for an asymmetry in the network capacity, namely, $\alpha = 0.5, m = 3$ and $s_A > s_B$, there are four pairwise stable networks, namely the complete network, the empty network, $g_{AB0}$ and $g_{0B0}$. (3) for an asymmetry in the number of firms connected to each backbone, namely, $\alpha = 0.5, m = 4$ and $s_A = s_B$, there are four pairwise stable networks, namely the complete network, the empty network, $g_{AB0}$ and $g_{0B0}$ (4) for an asymmetry in the consumer base, namely, $\alpha > 0.5, m = 3$ and $s_A = s_B$, there are four pairwise stable networks, namely the complete network, the empty network, $g_{AB0}$ and $g_{A00}$.*
We summarize this in Table 3.3.
We find there are two major trends in our example:

**(a)** First, when ISPs connected to backbones, whose congestion without taking into account leakage due to peering $(\theta_l + \omega_l)$ is higher, tend to intrapeer, the ISPs belonging to the other backbone do not intrapeer.

**(b)** There is no interpeering in presence of asymmetries in networks other than the complete network.

Now, (b) could be partly due to the proposition 3.2, especially given the fact that $n$ is small. But there are likely other effects that we are unable to capture in a formal manner.

We will briefly compare our results with that of DangNyugen and Penard (1999). In their model, ISPs connected to each backbone collude with each other and behave like a club, while ours is a purely non-cooperative game. Further, in their model, the only asymmetry is with regard to an exogenously given quality, while in ours there are many sources of asymmetry and quality is determined endogenously by a complex interaction of several factors. Their model only takes the symmetric effect into account, while ours takes both effects into account. If one extends their results in the context of pairwise stability, one would be likely to observe the network configuration $g_{A00}$, $g_{AB0}$ and $g_{0B0}$ depending on whether $A$ or $B$ is the high quality backbone. We also find that in our example, these three networks recur with some regularity. However, depending on parameter values, two other networks namely, the complete and empty network are also pairwise stable.

## 3.5 Conclusion

We find that there are two main avenues by which peering affects gross profits of ISPs. The impact on the quality of service offered by the ISP given that traffic can circumvent congested backbones, which we term the asymmetric effect, and the impact on backbone congestion, which we term the symmetric effect. The asymmetric effect generally increases gross profits, while the symmetric effect has a negative or ambiguous impact on gross profits. The two effects often work against each other making the net effect ambiguous as well.

One may object to this paper on the grounds that it does not have a clear point as a result of our equilibrium analysis. Yet, it is precisely that absence of a punch line that we strive to show. Retail peering has quite complicated effects on ISPs' profits, and it is by no means certain that improving the quality of service by forming peering agreements would automatically increase gross profits, even if we disregard the costs of peering. Peering among retail ISPs has both positive and negative effects with regard to gross profits. On the positive side, peering improves quality, increases demand and enables ISPs to charge higher prices. On the negative side, it reduces differentiation and promotes stiffer price competition. Also, it may lead to overusage of one's network without adequate reciprocity. The relative magnitude of these factors helps or hinders peering. In complex settings, such magnitudes are also quite complicated to analyze and give rise to a multitude of configurations depending on the various exogenously determined factors.

# Chapter 4

# A Simple Game Theoretic Analysis of Peering and Transit Contracting among Internet Access Providers

## 4.1 Introduction

Interconnection of Internet Access Providers (IAPs) has been a concern of regulators since the late 1990s. The providers exchange traffic with each other based on either peering agreements, under which the providers do not charge each other for the exchange of traffic, or transit agreements, under which the downstream provider pays the upstream provider a certain settlement payment for carrying its traffic. There is no regulation with regard to interconnection policies of providers, but there is a general convention that the providers peer if they perceive equal benefits from peering, and have transit arrangements otherwise. There has been a growing concern with regard to the increasing number of transit agreements replacing previously peering agreements. There is a debate whether the large providers are unwilling to peer with small providers, or that transit arrangements are related to the actual differences in the relative costs incurred by the providers.

This chapter presents a simple game theoretic model of two providers who choose between peering and transit agreements. The model discusses a set of conditions, which determine the formation of peering and transit agreements. It is important to recognize that there are costs of carrying traffic by peering partners and those costs should be roughly equal for the providers to have incentives to peer, otherwise the larger provider believes that the smaller provider might free ride on its infrastructure investments.

Norton (2002) generalizes the peering vs. transit tradeoff based on purely average cost analysis. He defines a certain "Peering Breaking Point" based on peering and transit costs, and argues that if the downstream provider exchanges more traffic than indicated by that point, then it should prefer peering with the upstream provider as it saves in transit costs. On the other hand, if the provider exchanges little traffic, then there is a possibility of not realizing sufficient peering traffic volume to offset the cost of peering, and transit is a more cost effective solution for the provider. This partial analysis of interconnection terms is based only on the costs of peering and transit. The author does not take into account the overall effects of peering and transit arrangements on providers' profits.

On the other hand, Laffont et al (2003) develop a model to analyze the impact of an access charge on Internet backbone competition. They distinguish between two types of end users: websites and consumers, and concentrate their analysis on the traffic from websites to consumers. Given the access charge, determined either by regulation or through a bilateral agreement, the backbones set prices to compete for websites and consumers. Their model results in a unique equilibrium, where the prices charged to consumers as well as the prices charged to websites are the same across the backbones. The model does not take into account the additional benefits or costs associated with the asymmetry of traffic flows between the backbones.

In this chapter we consider a one stage game, where providers make their transit and peering decisions based on the costs associated with transit and peering as well as the additional benefits and costs associated with the traffic imbalances between the providers. Our analysis suggests that the providers do not necessarily exploit market power when refusing to peer. Our results show that the market forces determine the terms of interconnection. Hence, there is no need for regulation to encourage peering. Moreover, we show that the unique Pareto optimum is achieved under transit arrangements. Thus to increase efficiency, the regulators might actually need to promote transit arrangements.

Section 4.2 presents preliminaries. We develop the formal model in section 4.3. Section 4.4 discusses the results. Section 4.5 describes future research which includes the extended version of the model from a network's perspective. Section 4.6 concludes.

## 4.2 Preliminaries

Consider two providers, $\{A, B\}$, with established networks and market shares. The providers are connected to each other, and, given their respective networks' as well as customers' characteristics, they have to decide whether to peer with each other, or one has to pay transit to the other. If the providers peer, then they follow so called "hot potato routing" —each provider passes off net traffic as soon as possible, and the receiving provider bears the cost of transporting the traffic. On the other hand, if one provider pays transit to the other one, then the latter charges for the traffic travelling in both directions. We follow the assumptions of Laffont et al (2003) to capture the traffic imbalances between the providers. They distinguish between two types of customers: websites and consumers. The traffic between websites, between consumers (emails) and from consumers to websites (the requests for webpage/file downloads) is neglected in our model, as it is much smaller than the traffic from websites to consumers (the actual downloads of webpages/files). We will concentrate our analysis on the traffic from websites to consumers. Consumers receive traffic, while websites send traffic. We assume that each consumer receives one unit of traffic, and each website sends one unit of traffic.

Denote provider $i$'s market share for consumers by $\alpha_i$, and its market share for websites by $\widetilde{\alpha}_i$, where $i \in \{A, B\}$. Further assume that the providers charge their customers the same prices for the same services, which are given exogenously. Let $p$ be the per unit price charged to consumers, while $\widetilde{p}$ be the per unit price charged to websites. Under the peering agreement, when traffic is handed over from one provider to the other, the latter incurs per unit transport cost, $c^p$(cost per Mega-bits-per second (Mbps)). On the other hand, if one provider charges transit to the other one, then the latter pays per unit price of traffic, $c^t$ (transit price per Mbps), for the traffic in both directions[1]. The connection between a larger and a smaller

[1]To simplify our analysis we assume that the per unit cost of transit doesn't vary with the total traffic exchanged. Typically the cost of transit is based on a 95th percentile traffic sampling technique. Prices depend on the total traffic exchanged and are charged on per Megabit-per-second. For further reference refer to W.

provider is much more valuable to a smaller provider, than it is to the larger provider. We introduce $0 < \theta < 1$ to capture additional benefits providers receive from the connection with each other. It can be interpreted as an additional benefit received by the transit customer, and an additional negative benefit (such as congestion) imposed on the upstream provider under the transit agreement. $c^p$, $c^t$ and $\theta$ are exogenous in our model.

## 4.3 The Game

We consider a simple $2 \times 2$ normal form game, where providers decide on peering and transit. The providers are connected to each other and they have the information about market shares, prices charged to both consumers and websites, as well as the costs of peering and transit. In the game they simultaneously announce a transfer arrangement for the connection they already have. The transfer choice of a provider shows whether the provider is willing to pay for the traffic exchanged or demand the other party to pay for it. Let $t_{ij}^i$ represent $i$'s transfer decision on the connection between $i$ and $j$, where $i, j \in \{A, B\}$,.

$$t_{ij}^i = \begin{cases} -1, & \text{if } i \text{ demands } j \text{ to pay for the traffic exchanged} \\ 1, & \text{if } i \text{ offers } j \text{ to pay for the traffic exchanged} \end{cases}$$

If both providers simultaneously seek the same type of transfer arrangement, they end up with a peering agreement. If they choose different transfers, they end up with a transit agreement. Let $\gamma_{ij}$, where $i, j \in \{A, B\}$, and $i \neq j$, determine the peering/transit agreement resulting from the transfer arrangement choices of the providers , i.e.,

$$\gamma_{ij} = \begin{cases} 0, & \text{for transit} \\ 1, & \text{for peering} \end{cases}$$

The outcomes of possible combinations of the decisions are given in the matrix below.

---

Norton (2002).

| | | $B$ | |
|---|---|---|---|
| | | demand ($t^B_{AB} = -1$) | offer ($t^B_{AB} = 1$) |
| $A$ | demand ($t^A_{AB} = -1$) | peer | transit ($B$ pays transit to $A$) |
| | offer ($t^A_{AB} = 1$) | transit ($A$ pays transit to $B$) | peer |

We concentrate our analysis only on the traffic from websites to consumers. Consumers do not know in advance which website they might want to visit and receive traffic. They assume universal connectivity, regardless of which network they are attached to. We assume following Laffont et al (2003) that consumers are interested in all websites independent of the websites' network choices, which is referred to as a "balanced calling pattern" in the aforesaid paper. The total traffic that the consumers of $i$ receive from $j$'s websites is $\alpha_i\widetilde{\alpha}_j$, while the total traffic $i$'s websites send to $j$'s consumers is $\widetilde{\alpha}_i\alpha_j$. We assume that under the peering agreement $i$ bears the per unit transport cost, $0 < c^p \leq 1$, on the traffic coming from $j$, while $j$ bears the per unit transport cost, $c^p$, on the traffic coming from $i$. This is a consequence of so called "hot-potato routing". The providers pass on off-net traffic as soon as possible, and as a result, most of the transporting cost is borne by the receiving provider. On the other hand, when $i$ pays transit to $j$, then $i$ pays per unit price $0 < c^t \leq 1$, for the traffic both to and from $j$. At the same time, under the transit relationship, if $i$ pays transit to $j$, then $i$ receives positive benefit for the whole traffic exchanged, while $j$ incurs negative benefit from the whole traffic exchanged.

The profit of $i$ from its connection with $j$, where, $i \neq j$ and $i, j \in \{A, B\}$, is now given by

$$\pi_i = \alpha_i\widetilde{\alpha}_j(p - \gamma_{ij}c^p - t^i_{ij}(1-\gamma_{ij})(c^t - \theta)) + \widetilde{\alpha}_i\alpha_j(\widetilde{p} - t^i_{ij}(1-\gamma_{ij})(c^t - \theta)) \tag{4.1}$$

## 4.4 Equilibrium Analysis

There are three possible outcomes of the game. When both providers demand payments for the traffic exchanged, i.e., $t^A_{AB} = -1$ and $t^B_{AB} = -1$, then $\gamma_{AB} = \gamma_{BA} = 1$, and the providers end up with peering agreement. Similarly, the providers will end up peering, when both of them offer to pay for the traffic exchanged, i.e., $t^A_{AB} = 1$ and $t^B_{AB} = 1$. The profits of the providers is

| $A\backslash B$ | $-1$ | $1$ |
|---|---|---|
| $-1$ | $\pi_A = \alpha_A\widetilde{\alpha}_B(p-c^p)+\widetilde{\alpha}_A\alpha_B\widetilde{p}$<br>$\pi_B = \alpha_B\widetilde{\alpha}_A(p-c^p)+\widetilde{\alpha}_B\alpha_A\widetilde{p}$ | $\pi_A = \alpha_A\widetilde{\alpha}_B(p+c^t-\theta)+\widetilde{\alpha}_A\alpha_B(\widetilde{p}+c^t-\theta)$<br>$\pi_B = \alpha_B\widetilde{\alpha}_A(p-c^t+\theta)+\widetilde{\alpha}_B\alpha_A(\widetilde{p}-c^t+\theta)$ |
| $1$ | $\pi_A = \alpha_A\widetilde{\alpha}_B(p-c^t+\theta)+\widetilde{\alpha}_A\alpha_B(\widetilde{p}-c^t+\theta)$<br>$\pi_B = \alpha_B\widetilde{\alpha}_A(p+c^t-\theta)+\widetilde{\alpha}_B\alpha_A(\widetilde{p}+c^t-\theta)$ | $\pi_A = \alpha_A\widetilde{\alpha}_B(p-c^p)+\widetilde{\alpha}_A\alpha_B\widetilde{p}$<br>$\pi_B = \alpha_B\widetilde{\alpha}_A(p-c^p)+\widetilde{\alpha}_B\alpha_A\widetilde{p}$ |

Table 4.1: The Payoff Matrix of the Game

now reduced to

$$\begin{aligned}\pi_A &= \alpha_A\widetilde{\alpha}_B(p-c^p)+\widetilde{\alpha}_A\alpha_B\widetilde{p}\\ \pi_B &= \alpha_B\widetilde{\alpha}_A(p-c^p)+\widetilde{\alpha}_B\alpha_A\widetilde{p}\end{aligned} \tag{4.2}$$

If the provider $A$ offers to pay, $t^A_{AB}=1$, and the provider $B$ demands payment, $t^B_{AB}=-1$, for the traffic exchanged, then $\gamma_{AB}=\gamma_{BA}=0$, a transit agreement will be written and the respective profits are

$$\begin{aligned}\pi_A &= \alpha_A\widetilde{\alpha}_B(p-c^t+\theta)+\widetilde{\alpha}_A\alpha_B(\widetilde{p}-c^t+\theta)\\ \pi_B &= \alpha_B\widetilde{\alpha}_A(p+c^t-\theta)+\widetilde{\alpha}_B\alpha_A(\widetilde{p}+c^t-\theta)\end{aligned} \tag{4.3}$$

On the other hand, if the provider $B$ offers to pay, $t^B_{AB}=1$, and the provider $A$ demands payment, $t^A_{AB}=-1$, for the traffic exchanged, then $\gamma_{AB}=\gamma_{BA}=0$, a transit agreement will be written and the respective profits are

$$\begin{aligned}\pi_A &= \alpha_A\widetilde{\alpha}_B(p+c^t-\theta)+\widetilde{\alpha}_A\alpha_B(\widetilde{p}+c^t-\theta)\\ \pi_B &= \alpha_B\widetilde{\alpha}_A(p-c^t+\theta)+\widetilde{\alpha}_B\alpha_A(\widetilde{p}-c^t+\theta)\end{aligned} \tag{4.4}$$

We can represent the game in the payoff matrix given in Table 4.1

All three outcomes can be equilibrium under certain conditions. We consider two cases:

*Case 4.1:* $\theta > c^t$, i.e., the benefit received by the downstream provider generated by the connection between the two providers under the transit agreement is greater than the cost of the transit.

**Proposition 4.1** *Suppose* $\theta > c^t$

*(a)* $t^A_{AB} = 1$ *and* $t^B_{AB} = 1$ *is a Nash equilibrium, i.e.,*

*the providers will end up with peering agreement,*

$$\text{if } \frac{\alpha_B \widetilde{\alpha}_A}{\alpha_B \widetilde{\alpha}_A + \widetilde{\alpha}_B \alpha_A} < \frac{\theta - c^t}{c^p} \text{ and } \frac{\alpha_A \widetilde{\alpha}_B}{\alpha_A \widetilde{\alpha}_B + \widetilde{\alpha}_A \alpha_B} < \frac{\theta - c^t}{c^p} \tag{4.5}$$

*(b)* $t^A_{AB} = 1$ *and* $t^B_{AB} = -1$ *is a Nash equilibrium, i.e.,*

*the provider* $A$ *will pay transit to the provider* $B$*,*

$$\text{if } \frac{\alpha_B \widetilde{\alpha}_A}{\alpha_B \widetilde{\alpha}_A + \widetilde{\alpha}_B \alpha_A} > \frac{\theta - c^t}{c^p} \text{ and } \frac{\alpha_A \widetilde{\alpha}_B}{\alpha_A \widetilde{\alpha}_B + \widetilde{\alpha}_A \alpha_B} < \frac{\theta - c^t}{c^p} \tag{4.6}$$

*(c)* $t^A_{AB} = -1$ *and* $t^B_{AB} = 1$ *is a Nash equilibrium, i.e.,*

*the provider* $B$ *will pay transit to the provider* $A$*,*

$$\text{if } \frac{\alpha_B \widetilde{\alpha}_A}{\alpha_B \widetilde{\alpha}_A + \widetilde{\alpha}_B \alpha_A} < \frac{\theta - c^t}{c^p} \text{ and } \frac{\alpha_A \widetilde{\alpha}_B}{\alpha_A \widetilde{\alpha}_B + \widetilde{\alpha}_A \alpha_B} > \frac{\theta - c^t}{c^p} \tag{4.7}$$

**Proof.** Let's start with assuming (4.5) condition. Then $A$'s best response to the given strategy of $B$, $t^B_{AB} = 1$, is $t^A_{AB} = 1$, because $\alpha_A \widetilde{\alpha}_B (p + c^t - \theta) + \widetilde{\alpha}_A \alpha_B (\widetilde{p} + c^t - \theta) < \alpha_A \widetilde{\alpha}_B (p - c^p) + \widetilde{\alpha}_A \alpha_B \widetilde{p}$. On the other hand, $A$'s best response to the given strategy of $B$, $t^B_{AB} = -1$, is $t^A_{AB} = 1$, because $\alpha_A \widetilde{\alpha}_B (p - c^p) + \widetilde{\alpha}_A \alpha_B \widetilde{p} < \alpha_A \widetilde{\alpha}_B (p - c^t + \theta) + \widetilde{\alpha}_A \alpha_B (\widetilde{p} - c^t + \theta)$. Similarly, $B$'s best response to the given strategy of $A$, $t^A_{AB} = 1$, is $t^B_{AB} = 1$, because $\alpha_B \widetilde{\alpha}_A (p + c^t - \theta) + \widetilde{\alpha}_B \alpha_A (\widetilde{p} + c^t - \theta) < \alpha_B \widetilde{\alpha}_A (p - c^p) + \widetilde{\alpha}_B \alpha_A \widetilde{p}$, and $B$'s best response to the given strategy of $A$, $t^A_{AB} = -1$, is $t^B_{AB} = 1$, because $\alpha_B \widetilde{\alpha}_A (p - c^p) + \widetilde{\alpha}_B \alpha_A \widetilde{p} < \alpha_B \widetilde{\alpha}_A (p - c^t + \theta) + \widetilde{\alpha}_B \alpha_A (\widetilde{p} - c^t + \theta)$. The other parts of the proposition follow the same way. ■

Thus, Proposition 4.1 states that, given that the benefit is greater than the cost of transit, then the providers choose to peer with each other, if the proportion of the traffic travelling in each direction with respect to the total traffic is less than $\frac{\theta - c^t}{c^p}$. On the other hand, the providers will end up with a transit agreement, and $i$ will pay transit to $j$, if $j$ receives more traffic from $i$, than it sends, and if the proportion of the traffic received by $j$ with respect to the total traffic is greater than $\frac{\theta - c^t}{c^p}$, while the proportion of the traffic send to $i$ with respect to the total traffic is less than $\frac{\theta - c^t}{c^p}$, where $i, j \in \{A, B\}$, and $i \neq j$.

**Corollary 4.1** *If* $\theta > c^t + \frac{c^p}{2}$*, then only peering occurs in equilibrium.*

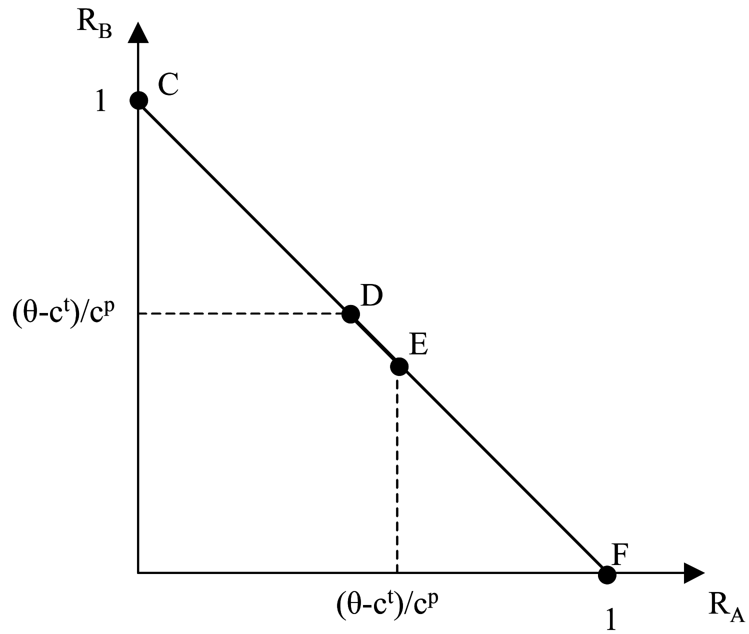


Figure 4-1: The constraints under Proposition 4.1

Denote $R_A$ to be the proportion of the total traffic coming to $A$ from $B$, and $R_B$ to be the proportion of the total traffic coming to $B$ from $A$, i.e.,

$$\begin{aligned} R_A &= \frac{\alpha_A \widetilde{\alpha}_B}{\alpha_A \widetilde{\alpha}_B + \widetilde{\alpha}_A \alpha_B} \\ R_B &= \frac{\alpha_B \widetilde{\alpha}_A}{\alpha_B \widetilde{\alpha}_A + \widetilde{\alpha}_B \alpha_A} \end{aligned} \tag{4.8}$$

Note that $R_A + R_B = 1$. Then, if $\frac{\theta - c^t}{c^p} < 1$, we can summarize Proposition 4.1 in Figure 4.1. $DE$ shows all the combinations of $R_A$ and $R_B$, where condition 4.5 holds. Thus, for all combination of $R_A$ and $R_B$ on $DE$ the providers will end up with a peering agreement. Similarly, $CD$ and $EF$ show, the combinations of $R_A$ and $R_B$, where conditions 4.6 and 4.7 hold, respectively, and the providers end up with a transit agreement.*Case 4.2:* $\theta < c^t$, the benefit received by the downstream provider generated by the connection between the two providers under a transit agreement is less than the cost of the transit.

**Proposition 4.2** *Suppose $\theta < c^t$*

*(a) $t^A_{AB} = -1$ and $t^B_{AB} = -1$ is a Nash equilibrium, i.e., the providers will end up with peering agreement,*

$$\text{if } \frac{\alpha_B \widetilde{\alpha}_A}{\alpha_B \widetilde{\alpha}_A + \widetilde{\alpha}_B \alpha_A} < \frac{c^t - \theta}{c^p} \text{ and } \frac{\alpha_A \widetilde{\alpha}_B}{\alpha_A \widetilde{\alpha}_B + \widetilde{\alpha}_A \alpha_B} < \frac{c^t - \theta}{c^p} \tag{4.9}$$

*(b) $t^A_{AB} = 1$ and $t^B_{AB} = -1$ is a Nash equilibrium, i.e., the provider $A$ will pay transit to the provider $B$,*

$$\text{if } \frac{\alpha_B \widetilde{\alpha}_A}{\alpha_B \widetilde{\alpha}_A + \widetilde{\alpha}_B \alpha_A} < \frac{c^t - \theta}{c^p} \text{ and } \frac{\alpha_A \widetilde{\alpha}_B}{\alpha_A \widetilde{\alpha}_B + \widetilde{\alpha}_A \alpha_B} > \frac{c^t - \theta}{c^p} \tag{4.10}$$

*(c) $t^A_{AB} = -1$ and $t^B_{AB} = 1$ is a Nash equilibrium, i.e., the provider $B$ will pay transit to the provider $A$,*

$$\text{if } \frac{\alpha_B \widetilde{\alpha}_A}{\alpha_B \widetilde{\alpha}_A + \widetilde{\alpha}_B \alpha_A} > \frac{c^t - \theta}{c^p} \text{ and } \frac{\alpha_A \widetilde{\alpha}_B}{\alpha_A \widetilde{\alpha}_B + \widetilde{\alpha}_A \alpha_B} < \frac{c^t - \theta}{c^p} \tag{4.11}$$

**Proof.** The proof is obvious, and follows from the payoff matrix of the game. ■

Proposition 4.2. states that, given that the benefit is less than the cost of transit, the providers choose to peer with each other, if the proportion of the traffic travelling in each direction with respect to the total traffic is less than $\frac{c^t-\theta}{c^p}$. On the other hand, the providers end up with a transit agreement, and $i$ will pay transit to $j$, if $j$ receives less traffic from $i$, than it sends, and if the proportion of traffic received by $j$ with respect to the total traffic is less than $\frac{c^t-\theta}{c^p}$, while the proportion of traffic send to $i$ with respect to the total traffic is greater than $\frac{\theta-c^t}{c^p}$, where $i, j \in \{A, B\}$, and $i \neq j$.

**Corollary 4.2** *If $\theta < c^t$ and $c^t - \frac{c^p}{2} > \theta$, then only peering occurs in equilibrium.*

If $\frac{c^t-\theta}{c^p} < 1$, then we can summarize the constraints under Proposition 4.2 in Figure 4.2. $DE$ shows all the combinations of $R_A$ and $R_B$, where condition 4.9 holds. Thus, for all combination of $R_A$ and $R_B$ on $DE$ the providers will end up with a peering agreement. Similarly, $CD$ and $EF$ show, the combinations of $R_A$ and $R_B$, where conditions 4.10 and 4.11 hold, respectively, and the providers end up with a transit agreement.Thus, given the providers network characteristics

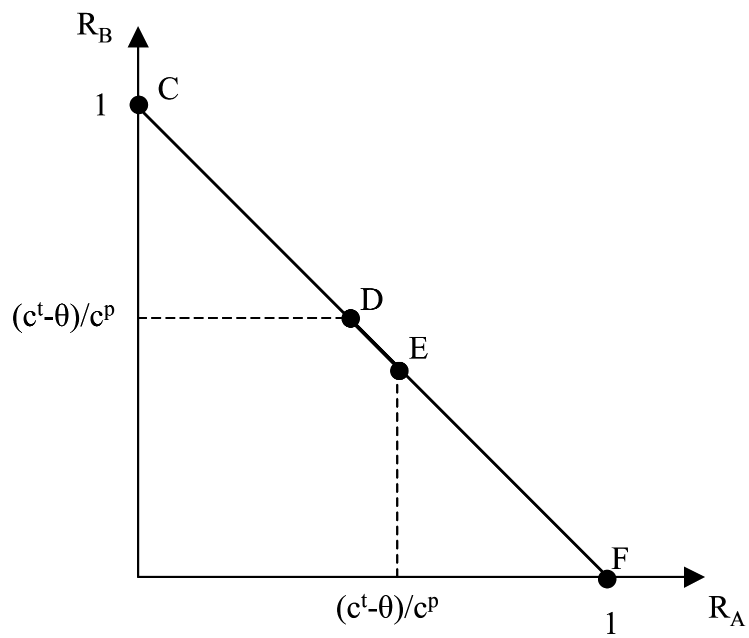


Figure 4-2: The constraints under Proposition 4.2

(which determines $\theta$), customers' characteristics, as well as costs and the prices charged to customers, the model provides conditions that determine the peering/transit decisions of the providers.

**Proposition 4.3** *The joint profits are maximized under a transit agreement.*

**Proof.** Under the peering arrangements, whether $t_{AB}^A = -1$ and $t_{AB}^B = -1$, or $t_{AB}^A = 1$ and $t_{AB}^B = 1$, the the joint profits are $(p + \widetilde{p} - c^p)(\alpha_A \widetilde{\alpha}_B + \widetilde{\alpha}_A \alpha_B)$. On the other hand, the joint profits under transit agreements, i.e., when $t_{AB}^A = -1$ and $t_{AB}^B = 1$, or $t_{AB}^A = 1$ and $t_{AB}^B = -1$, are $(p + \widetilde{p}^p)(\alpha_A \widetilde{\alpha}_B + \widetilde{\alpha}_A \alpha_B)$, which is greater than the joint profits under the peering agreements. ■

**Corollary 4.3** *The Pareto optimum coincides with the Nash equilibrium if any of these con-*

*ditions hold*

$$
\begin{aligned}
(a)\ \theta &> c^t,\ R_A < \frac{\theta - c^t}{c^p}\ \textit{and}\ R_B > \frac{\theta - c^t}{c^p}\ \textit{or}\\
(b)\ \theta &> c^t,\ R_A > \frac{\theta - c^t}{c^p}\ \textit{and}\ R_B < \frac{\theta - c^t}{c^p}\ \textit{or}\\
(c)\ \theta &< c^t,\ R_A > \frac{c^t - \theta}{c^p}\ \textit{and}\ R_B < \frac{c^t - \theta}{c^p}\ \textit{or}\\
(d)\ \theta &< c^t,\ R_A < \frac{c^t - \theta}{c^p}\ \textit{and}\ R_B > \frac{c^t - \theta}{c^p}.
\end{aligned}
$$

Thus, if there is a large traffic imbalance between two providers, a transit arrangement is a Nash equilibrium, and Pareto optimum is achieved. In this case there is no need for a regulation to increase efficiency.

## 4.5 Future Research

The model can be extended to many players. Consider a finite set of $N$ providers denoted by $N = \{1, 2, 3, ...n\}$, where $n \in \mathbb{N}$. We assume a complete network, where every pair of providers is connected to each other. In the game providers simultaneously announce a transfer for each of the connection, or henceforth link, they already have with other providers. The transfer chosen by a provider $i$ on each link it has with another provider shows whether $i$ is willing to pay for the traffic exchanged on that link, or $i$ demands the other party to pay for it. Let $t^i_{ij}$ represent $i$'s transfer decision on the connection between $i$ and $j$, where $i, j \in \{1, 2, 3, ...N\}$ and $i \neq j$,

$$
t^i_{ij} = \begin{cases} -1, & \text{if } i \text{ demands } j \text{ to pay for the traffic exchanged} \\ 1, & \text{if } i \text{ offers } j \text{ to pay for the traffic exchanged} \end{cases}
$$

If every pair of providers simultaneously seeks the same transfers on the link connecting each other, then they end up with a peering agreement. If they choose different transfers, they end up with a transit agreement. In the network $t^i_{ij} = -1$ reflects $i$'s demand for a link, directed to itself, i.e., $i \leftarrow j$, while $t^i_{ij} = 1$ reflects $i$'s offer for a directed link to $j$, i.e., $i \rightarrow j$. Thus, if for some $i$ and $j$, where $i \neq j$, $t^i_{ij} = -1$ and $t^j_{ij} = -1$, then both providers demand the link to be directed to themselves, and they will end up with a two directional link, $i \longleftrightarrow j$, which represents the peering agreement in the network. Similarly, if for some $i$ and $j$, where $i \neq j$, $t^i_{ij} = 1$ and $t^j_{ij} = 1$, then both providers offer that the link be directed towards the other provider, and they will end up with the peering agreement.

Let $\gamma_{ij}$, where $i, j \in \{1, 2, 3, ...N\}$ and $i \neq j$, determine the peering/transit agreement resulting from the transfer choices of the providers, i.e.,

$$\gamma_{ij} = \begin{cases} 0, & \text{for transit} \\ 1, & \text{for peering} \end{cases}$$

Transfer decisions on a link, determine the explicit direction of the link. The network $g$ resulting from the game is a collection of all $t^i_{ij}$s, i.e., $g = \{t^i_{ij} | i, j \in N\}$. The profit of $i$ from the network $g$ is defined as

$$\pi_i(g) = \alpha_i \sum_{j=1}^{n} \widetilde{\alpha}_j (p - \gamma_{ij} c^p - t^i_{ij}(1 - \gamma_{ij})(c^t - \theta)) + \widetilde{\alpha}_i \sum_{j=1}^{n} \alpha_j (\widetilde{p} - t^i_{ij}(1 - \gamma_{ij})(c^t - \theta)) \quad (4.12)$$

To be more realistic we should assume $\theta_{ij}$, the benefit from transit agreement to be different for each link. Here, besides conducting Nash equilibrium analysis, we would be interested in defining a stability concept, which determines the formation of networks with unidirectional and bidirectional links, depending on the transfer decisions of the providers.

## 4.6 Conclusion

As there is no regulation with regard to interconnection polices, policymakers are concerned about possible anti-competitive behavior of larger providers. It can be questioned whether larger providers gain and exploit market power through terms of interconnection that they offer to smaller providers. The purpose of this chapter was to develop a theoretical framework for modelling the peering/transit decisions of the providers, which examines the basis for the interconnection policies. The model discusses a set of conditions, which determine the formation of peering and transit agreements, which might have some policy implications. The providers weigh the benefits and costs of entering into a particular interconnection agreement. The decisions are based on how one variable weighs against the other. The larger provider may demand a transit arrangement, if it believes that the smaller provider might free ride on its infrastructure investments, because the larger provider might be the one transporting most of the traffic between them. The results suggest that the providers do not necessarily exploit market power when refusing to peer. Moreover, the joint profits are maximized under the transit arrangement. Peering partners do not get any compensations for carrying each other's traffic, while incurring costs of transporting each other's traffic. Under the transit agreement, on the other hand, the downstream provider pays for the traffic carried by the upstream provider, while getting benefits of dumping most of the traffic on the upstream provider. These two effects cancel out when maximizing the joint profits of the transit partners. The chapter argues that the market forces determine the decisions of peering and transit, and, given the current system of peering and transit arrangements, there is no need for a regulation on interconnection policies.

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# Appendix A

# Appendix to Chapter 2

## A.1 Parameter constraints

We require three restrictions on the equilibrium values.

(a) All the values of the variables at the equilibrium must be positive.

(b) There must be some connection failure for each IAP.

(c) There must be some congestion on each link as well as the NAP.

We have already imposed the constraints, $0 < \lambda < \mu \leq 1$ and $k \geq 0$. However for (a), (b) and (c) to hold, the aforesaid constraints are not sufficient. In fact, a sufficient condition is that the values of $\lambda, \mu$, $k$ must lie in the area defined by Figure 2-3.

## A.2 Solution Methodology

Given a network $g$, we first illustrate how we have solved stages 2, 3 and 4 using backward induction. Assume that values of the primitives of the model $\lambda, \mu$ and $k$ lie in the area shown by Figure 2-2.

**Solving for prices**

Profits for IAP $i$ are given by

$$\pi_i = p_i \cdot d_i^* - c_i$$

Hence, our first order conditions are given by

$$\frac{\partial\left(p_i \cdot d_i^* - c_i\right)}{\partial p_i} = 0,\ i = 1, 2, 3$$

which gives us a linear system of three simultaneous linear equations with three unknowns, which can be solved to obtain the equilibrium prices.

Next we plug in the equilibrium prices, $p_i^*$, and the values of $L_i$s to obtain the reduced form of profits in terms of network capacities and link capacities. Note that the value of $L_i$ will vary from one network to another. Define a variable $q_{ij}$, $j \neq i$ to be equal to 1, if there is a peering arrangement between $i$ and $j$ in the network $g$ and zero otherwise. Then,

$$L_i = \frac{s_i}{3} + \frac{s_j}{3} - q_{ij} \cdot (s_i^j + s_j^i) - \frac{k}{2} + \frac{s_i}{3} + \frac{s_h}{3} - q_{ih} \cdot (s_i^h + s_h^i) - \frac{k}{2}$$

where $h \neq i, j \neq i$.

**Solving for link capacities**

This stage is relevant if and only if there is some peering i.e. $q_{ij} = 1$ for some $i, j \in N, i \neq j$. The first order conditions are given by

$$\frac{\partial \pi_i}{\partial s_{ij}} = 0$$

This gives us $\sum\limits_{i \in N} \sum\limits_{j \neq i} q_{ij}$ simultaneous linear equations in an equal number of unknowns which can be solved to obtain optimal link capacities. We plug in the optimal link capacities in the reduced profits for the third stage to obtain reduced profits for the second stage in terms of the network capacity only.

**Solving for network capacities**

Finally, we solve for network capacities. The first order conditions are given by

$$\frac{\partial \pi_i}{\partial s_i} = 0$$

This gives us three linear equations in three unknowns which can be solved to obtain the equilibrium network capacities. We plug in the optimal network capacity in the reduced profits for the second stage to obtain reduced profits for the first stage.

## A.3 Equilibrium Values for Case 3

### Case 3

Assume $i$ and $j$ peer with each other while $h$ does not peer with anyone. Then the equilibrium prices in the subgame perfect Nash equilibrium are given by

$$
\begin{aligned}
\widehat{p}_i^3 &= \frac{5625\mu^3(16\lambda^2-24\lambda\mu+9\mu(4\mu-25))}{2(8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-4500\lambda^2(\mu-35)\mu^2-135000\lambda\mu^3+50265\mu^3(4\mu-25)+4\lambda^4\mu(18\mu-1025))}\\
\widehat{p}_j^3 &= \frac{5625\mu^3(16\lambda^2-24\lambda\mu+9\mu(4\mu-25))}{2(8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-4500\lambda^2(\mu-35)\mu^2-135000\lambda\mu^3+50265\mu^3(4\mu-25)+4\lambda^4\mu(18\mu-1025))}\\
\widehat{p}_h^3 &= \frac{3\mu(8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-45000\lambda\mu^3-1500\lambda^2\mu^2(3\mu-65)+16875\mu^3(4\mu-25)+4\lambda^4\mu(18\mu-1025))}{2(8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-4500\lambda^2(\mu-35)\mu^2-135000\lambda\mu^3+50265\mu^3(4\mu-25)+4\lambda^4\mu(18\mu-1025))}
\end{aligned}
$$

The demands in equilibrium are given by

$$
\begin{aligned}
\widehat{d}_i^3 &= \frac{1875\mu^3(16\lambda^2-24\lambda\mu+9\mu(4\mu-25))}{2(8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-4500\lambda^2(\mu-35)\mu^2-135000\lambda\mu^3+50265\mu^3(4\mu-25)+4\lambda^4\mu(18\mu-1025))}\\
\widehat{d}_j^3 &= \frac{1875\mu^3(16\lambda^2-24\lambda\mu+9\mu(4\mu-25))}{2(8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-4500\lambda^2(\mu-35)\mu^2-135000\lambda\mu^3+50265\mu^3(4\mu-25)+4\lambda^4\mu(18\mu-1025))}\\
\widehat{d}_h^3 &= \frac{8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-45000\lambda\mu^3-1500\lambda^2\mu^2(3\mu-65)+16875\mu^3(4\mu-25)+4\lambda^4\mu(18\mu-1025)}{2(8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-4500\lambda^2(\mu-35)\mu^2-135000\lambda\mu^3+50265\mu^3(4\mu-25)+4\lambda^4\mu(18\mu-1025))}
\end{aligned}
$$

Their optimal investments in network capacities are

$$
\begin{aligned}
\widehat{s}_i^3 &= \frac{10(\lambda-3\mu)\mu(32\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-45000\lambda\mu^3+16875\mu^3(4\mu-25)-375\lambda^2\mu^2(12\mu-155)+2\lambda^4\mu(36\mu-1225))}{(4\lambda^2-75\mu)(8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-4500\lambda^2(\mu-35)\mu^2-135000\lambda\mu^3+50265\mu^3(4\mu-25)+4\lambda^4\mu(18\mu-1025))}\\
\widehat{s}_j^3 &= \frac{10(\lambda-3\mu)\mu(32\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-45000\lambda\mu^3+16875\mu^3(4\mu-25)-375\lambda^2\mu^2(12\mu-155)+2\lambda^4\mu(36\mu-1225))}{(4\lambda^2-75\mu)(8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-4500\lambda^2(\mu-35)\mu^2-135000\lambda\mu^3+50265\mu^3(4\mu-25)+4\lambda^4\mu(18\mu-1025))}\\
\widehat{s}_h^3 &= \frac{10(\lambda-3\mu)\mu(8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-45000\lambda\mu^3-1500\lambda^2\mu^2+16875\mu^3(4\mu-25)+4\lambda^4\mu(18\mu-1025))}{(4\lambda^2-75\mu)(8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-4500\lambda^2(\mu-35)\mu^2-135000\lambda\mu^3+50265\mu^3(4\mu-25)+4\lambda^4\mu(18\mu-1025))}
\end{aligned}
$$

The optimal investments in link capacities are

$$
\begin{aligned}
\widehat{s_i^j}^3 &= \frac{375\lambda\mu^2(16\lambda^2-24\lambda\mu+9\mu(4\mu-25))}{(8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-4500\lambda^2(\mu-35)\mu^2-135000\lambda\mu^3+50265\mu^3(4\mu-25)+4\lambda^4\mu(18\mu-1025))}\\
\widehat{s_j^i}^3 &= \frac{375\lambda\mu^2(16\lambda^2-24\lambda\mu+9\mu(4\mu-25))}{(8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-4500\lambda^2(\mu-35)\mu^2-135000\lambda\mu^3+50265\mu^3(4\mu-25)+4\lambda^4\mu(18\mu-1025))}
\end{aligned}
$$

The profits are given by

$$\widehat{\pi}_i^3 = \left(\frac{-(25\mu^2(16\lambda^2-24\lambda\mu+9\mu(4\mu-25))^2)}{(8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-4500\lambda^2(\mu-35)\mu^2-135000\lambda\mu^3+50265\mu^3(4\mu-25)+4\lambda^4\mu(18\mu-1025))^2}\right)$$
$$\left(\frac{(32\lambda^{10}-192\lambda^9\mu+24000\lambda^7\mu^2-1110000\lambda^5\mu^3+22500000\lambda^3\mu^4-168750000\lambda\mu^5+(31640625\mu^5(8\mu-75)}{2(4\lambda^2-75\mu)^2}+\right.$$
$$\left.\frac{32\lambda^8\mu(9\mu-125)-1406250\lambda^2\mu^4(24\mu-245)-1000\lambda^6\mu^2(36\mu-365))+15000\lambda^4\mu^3(111\mu-1150))}{2(4\lambda^2-75\mu)^2}\right)$$

$$\widehat{\pi}_j^3 = \left(\frac{-(25\mu^2(16\lambda^2-24\lambda\mu+9\mu(4\mu-25))^2)}{(8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-4500\lambda^2(\mu-35)\mu^2-135000\lambda\mu^3+50265\mu^3(4\mu-25)+4\lambda^4\mu(18\mu-1025))^2}\right)$$
$$\left(\frac{(32\lambda^{10}-192\lambda^9\mu+24000\lambda^7\mu^2-1110000\lambda^5\mu^3+22500000\lambda^3\mu^4-168750000\lambda\mu^5+(31640625\mu^5(8\mu-75)}{2(4\lambda^2-75\mu)^2}+\right.$$
$$\left.\frac{32\lambda^8\mu(9\mu-125)-1406250\lambda^2\mu^4(24\mu-245)-1000\lambda^6\mu^2(36\mu-365))+15000\lambda^4\mu^3(111\mu-1150))}{2(4\lambda^2-75\mu)^2}\right)$$

$$\widehat{\pi}_h^3 = \left(\frac{8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-45000\lambda\mu^3-1500\lambda^2\mu^2(3\mu-65)+16875\mu^3(4\mu-25)+4\lambda^4\mu(18\mu-1025)^2}{(8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-4500\lambda^2(\mu-35)\mu^2-135000\lambda\mu^3+50265\mu^3(4\mu-25)+4\lambda^4\mu(18\mu-1025))^2}\right)$$
$$\left(\frac{\mu(48\lambda^4-2000\lambda^2\mu+1200\lambda\mu^2-225\mu^2(-75+8\mu))}{2(4\lambda^2-75\mu)^2}\right)$$

## A.4 Equilibrium Values for Case 4

**Case 4**

Assume $i$ and $j$ peer with each other, so do $i$ and $h$. But $j$ and $h$ do not peer. Then the equilibrium prices in the subgame perfect Nash equilibrium are given by

$$\begin{aligned}
\widehat{p}_i^4 &= \frac{-3\mu(64\lambda^6+48\lambda^5\mu-3000\lambda^3\mu^2+45000\lambda\mu^3-16875\mu^3(4\mu-25)+750\lambda^2\mu^2(6\mu-25)-8\lambda^4\mu(9\mu+275))}{2(144\lambda^6-432\lambda^5\mu+15000\lambda^3\mu^2-135000\lambda\mu^3-11250\lambda^2\mu^2(2\mu-17)+50625\mu^3(4\mu-25)+8\lambda^4\mu(81\mu-1100))}\\
\widehat{p}_j^4 &= \frac{3\mu(104\lambda^6-195\lambda^5\mu+6000\lambda^3\mu^2-45000\lambda\mu^3+16875\mu^3(4\mu-25)-750\lambda^2\mu^2(12\mu-115)+4\lambda^4\mu(72\mu-1375)}{2(144\lambda^6-432\lambda^5\mu+15000\lambda^3\mu^2-135000\lambda\mu^3-11250\lambda^2\mu^2(2\mu-17)+50625\mu^3(4\mu-25)+8\lambda^4\mu(81\mu-1100))}\\
\widehat{p}_h^4 &= \frac{3\mu(104\lambda^6-195\lambda^5\mu+6000\lambda^3\mu^2-45000\lambda\mu^3+16875\mu^3(4\mu-25)-750\lambda^2\mu^2(12\mu-115)+4\lambda^4\mu(72\mu-1375)}{2(144\lambda^6-432\lambda^5\mu+15000\lambda^3\mu^2-135000\lambda\mu^3-11250\lambda^2\mu^2(2\mu-17)+50625\mu^3(4\mu-25)+8\lambda^4\mu(81\mu-1100))}
\end{aligned}$$

The demands in equilibrium are given by

$$\begin{aligned}
\widehat{d}_i^4 &= \frac{-(64\lambda^6+48\lambda^5\mu-3000\lambda^3\mu^2+45000\lambda\mu^3-16875\mu^3(4\mu-25)+750\lambda^2\mu^2(6\mu-25)-8\lambda^4\mu(9\mu+275))}{(144\lambda^6-432\lambda^5\mu+15000\lambda^3\mu^2-135000\lambda\mu^3-11250\lambda^2\mu^2(2\mu-17)+50625\mu^3(4\mu-25)+8\lambda^4\mu(81\mu-1100))}\\
\widehat{d}_j^4 &= \frac{(104\lambda^6-195\lambda^5\mu+6000\lambda^3\mu^2-45000\lambda\mu^3+16875\mu^3(4\mu-25)-750\lambda^2\mu^2(12\mu-115)+4\lambda^4\mu(72\mu-1375)}{(144\lambda^6-432\lambda^5\mu+15000\lambda^3\mu^2-135000\lambda\mu^3-11250\lambda^2\mu^2(2\mu-17)+50625\mu^3(4\mu-25)+8\lambda^4\mu(81\mu-1100))}\\
\widehat{d}_h^4 &= \frac{(104\lambda^6-195\lambda^5\mu+6000\lambda^3\mu^2-45000\lambda\mu^3+16875\mu^3(4\mu-25)-750\lambda^2\mu^2(12\mu-115)+4\lambda^4\mu(72\mu-1375)}{(144\lambda^6-432\lambda^5\mu+15000\lambda^3\mu^2-135000\lambda\mu^3-11250\lambda^2\mu^2(2\mu-17)+50625\mu^3(4\mu-25)+8\lambda^4\mu(81\mu-1100))}
\end{aligned}$$

Their optimal investments in network capacities are

$$
\begin{aligned}
\widehat{s}_i^4 &= \frac{256\lambda^7-576\lambda^6\mu+101250\mu^4(4\mu-25)+144\lambda^4\mu^2(25+6\mu)-5400\lambda^2\mu^3(7\mu-50)}{15(144\lambda^6-432\lambda^5\mu+15000\lambda^3\mu^2-135000\lambda\mu^3-11250\lambda^2\mu^2(2\mu-17)+50625\mu^3(4\mu-25)+8\lambda^4\mu(81\mu-1100))} \\
&\quad + \frac{-33750\lambda\mu^3(12\mu-25)-32\lambda^5\mu(125+27\mu)+600\lambda^3\mu^2(63\mu-100)}{15(144\lambda^6-432\lambda^5\mu+15000\lambda^3\mu^2-135000\lambda\mu^3-11250\lambda^2\mu^2(2\mu-17)+50625\mu^3(4\mu-25)+8\lambda^4\mu(81\mu-1100))} \\
\widehat{s}_j^4 &= \frac{-2(\lambda-3\mu)(52\lambda^6-96\lambda^5\mu+4200\lambda^3\mu^2-45000\lambda\mu^3+16875\mu^3(4\mu-25)+8\lambda^4\mu(18\mu-425)-75\lambda^2\mu^2(84\mu-925))}{15(144\lambda^6-432\lambda^5\mu+15000\lambda^3\mu^2-135000\lambda\mu^3-11250\lambda^2\mu^2(2\mu-17)+50625\mu^3(4\mu-25)+8\lambda^4\mu(81\mu-1100))} \\
\widehat{s}_h^4 &= \frac{-2(\lambda-3\mu)(52\lambda^6-96\lambda^5\mu+4200\lambda^3\mu^2-45000\lambda\mu^3+16875\mu^3(4\mu-25)+8\lambda^4\mu(18\mu-425)-75\lambda^2\mu^2(84\mu-925))}{15(144\lambda^6-432\lambda^5\mu+15000\lambda^3\mu^2-135000\lambda\mu^3-11250\lambda^2\mu^2(2\mu-17)+50625\mu^3(4\mu-25)+8\lambda^4\mu(81\mu-1100))}
\end{aligned}
$$

Their optimal investments in link capacities are

$$
\begin{aligned}
\widehat{s_i^j}^4 &= \frac{-\lambda(64\lambda^6+48\lambda^5\mu-3000\lambda^3\mu^2+45000\lambda\mu^3-16875\mu^3(4\mu-25)+750\lambda^2\mu^2(6\mu-25)-8\lambda^4\mu(9\mu+275))}{5(144\lambda^6-432\lambda^5\mu+15000\lambda^3\mu^2-135000\lambda\mu^3-11250\lambda^2\mu^2(2\mu-17)+50625\mu^3(4\mu-25)+8\lambda^4\mu(81\mu-1100))} \\
\widehat{s_i^h}^4 &= \frac{-\lambda(64\lambda^6+48\lambda^5\mu-3000\lambda^3\mu^2+45000\lambda\mu^3-16875\mu^3(4\mu-25)+750\lambda^2\mu^2(6\mu-25)-8\lambda^4\mu(9\mu+275))}{5(144\lambda^6-432\lambda^5\mu+15000\lambda^3\mu^2-135000\lambda\mu^3-11250\lambda^2\mu^2(2\mu-17)+50625\mu^3(4\mu-25)+8\lambda^4\mu(81\mu-1100))} \\
\widehat{s_j^i}^4 &= \frac{\lambda(104\lambda^6-195\lambda^5\mu+6000\lambda^3\mu^2-45000\lambda\mu^3+16875\mu^3(4\mu-25)-750\lambda^2\mu^2(12\mu-115)+4\lambda^4\mu(72\mu-1375)}{5(144\lambda^6-432\lambda^5\mu+15000\lambda^3\mu^2-135000\lambda\mu^3-11250\lambda^2\mu^2(2\mu-17)+50625\mu^3(4\mu-25)+8\lambda^4\mu(81\mu-1100))} \\
\widehat{s_h^i}^4 &= \frac{\lambda(104\lambda^6-195\lambda^5\mu+6000\lambda^3\mu^2-45000\lambda\mu^3+16875\mu^3(4\mu-25)-750\lambda^2\mu^2(12\mu-115)+4\lambda^4\mu(72\mu-1375)}{5(144\lambda^6-432\lambda^5\mu+15000\lambda^3\mu^2-135000\lambda\mu^3-11250\lambda^2\mu^2(2\mu-17)+50625\mu^3(4\mu-25)+8\lambda^4\mu(81\mu-1100))}
\end{aligned}
$$

The reduced profits are given by

$$
\begin{aligned}
\widehat{\pi}_i^4 &= \left(\frac{-(32\lambda^4+23\lambda^3\mu-600\lambda\mu^2+225\mu^2(4\mu-25)+4\lambda^2\mu(9\mu-25))^2}{(144\lambda^6-432\lambda^5\mu+15000\lambda^3\mu^2-135000\lambda\mu^3-11250\lambda^2\mu^2(2\mu-17)+50625\mu^3(4\mu-25)+8\lambda^4\mu(81\mu-1100))^2}\right) \\
&\quad \left(\frac{272\lambda^6-768\lambda^5\mu+28800\lambda^3\mu^2-270000\lambda\mu^3+50625\mu^3(8\mu-75)-3600\lambda^2\mu^2(12\mu-125)+12\lambda^4\mu(96\mu-1525)}{450}\right)
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\pi}_j^4 &= \left(\frac{-(52\lambda^4-96\lambda^3\mu+1800\lambda\mu^2-675\mu^2(4\mu-25)+12\lambda^2\mu(12\mu-175)^2}{(144\lambda^6-432\lambda^5\mu+15000\lambda^3\mu^2-135000\lambda\mu^3-11250\lambda^2\mu^2(2\mu-17)+50625\mu^3(4\mu-25)+8\lambda^4\mu(81\mu-1100))^2}\right) \\
&\quad \left(\frac{80\lambda^6-48\lambda^5\mu+2400\lambda^3\mu^2-30000\lambda\mu^3+5625\mu^3(8\mu-75)+4\lambda^4\mu(18\mu-1225)-50\lambda^2\mu^2(72\mu-1265)}{450}\right)
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\pi}_h^4 &= \left(\frac{-(52\lambda^4-96\lambda^3\mu+1800\lambda\mu^2-675\mu^2(4\mu-25)+12\lambda^2\mu(12\mu-175)^2}{(144\lambda^6-432\lambda^5\mu+15000\lambda^3\mu^2-135000\lambda\mu^3-11250\lambda^2\mu^2(2\mu-17)+50625\mu^3(4\mu-25)+8\lambda^4\mu(81\mu-1100))^2}\right) \\
&\quad \left(\frac{80\lambda^6-48\lambda^5\mu+2400\lambda^3\mu^2-30000\lambda\mu^3+5625\mu^3(8\mu-75)+4\lambda^4\mu(18\mu-1225)-50\lambda^2\mu^2(72\mu-1265)}{450}\right)
\end{aligned}
$$

## A.5 Proof of Proposition 2-4.

We have

$$
\widehat{\boldsymbol{\pi}}^{4'} > \widehat{\boldsymbol{\pi}}^{3'} > \widehat{\boldsymbol{\pi}}^{2} > \widehat{\boldsymbol{\pi}}^{1} > \widehat{\boldsymbol{\pi}}^{4} > \widehat{\boldsymbol{\pi}}^{3} \tag{A.1}
$$

*Case 1*: Consider the complete network $g^N$. Each IAP $i$ earns a profit of $\vartheta_i(g^N) = \widehat{\boldsymbol{\pi}}^1$. Each IAP has private peering agreements with all the rest of IAPs. So there are no links to form. On the other hand if $i$ and $j$ delete their mutual link, they earn $\vartheta_i(g^N - ij) = \vartheta_j(g^N - ij) = \widehat{\boldsymbol{\pi}}^4$. Since, $\widehat{\boldsymbol{\pi}}^1 > \widehat{\boldsymbol{\pi}}^4$, $\vartheta_i(g^N - ij) < \vartheta_i(g^N)$ and $\vartheta_j(g^N - ij) < \vartheta_j(g^N)$.

*Case 2*: Consider an empty network $g^0$. Each IAP earns a profit given by $\vartheta_i(g^0) = \widehat{\boldsymbol{\pi}}^2$. There are no links to break. On the other hand if $i$ and $j$ form a link, they earn $\vartheta_i(g^0 + ij) = \vartheta_j(g^0 + ij) = \widehat{\boldsymbol{\pi}}^{3'}$. Since $\widehat{\boldsymbol{\pi}}^{3'} > \widehat{\boldsymbol{\pi}}^2$, $\vartheta_i(g^0 + ij) > \vartheta_i(g^0)$, and $\vartheta_j(g^0 + ij) > \vartheta_j(g^0)$.

*Case 3*: Consider the network represented by Case 3, say, $g = \{12\}$. Then the payoffs of the IAPs are $\vartheta_1(g) = \vartheta_2(g) = \widehat{\boldsymbol{\pi}}^{3'}$, $\vartheta_3(g) = \widehat{\boldsymbol{\pi}}^3$. If 1 and 3 form a link, then 1 earns $\vartheta_1(g+13) = \widehat{\boldsymbol{\pi}}^{4'}$ and 3 earns $\vartheta_3(g + 13) = \widehat{\boldsymbol{\pi}}^4$. Now $\widehat{\boldsymbol{\pi}}^{4'} > \widehat{\boldsymbol{\pi}}^{3'}$ and $\widehat{\boldsymbol{\pi}}^4 > \widehat{\boldsymbol{\pi}}^3$. Hence, $\vartheta_1(g + 13) > \vartheta_1(g)$ and $\vartheta_3(g + 13) > \vartheta_3(g)$. If 1 and 2 break a link, then 1 earns $\vartheta_1(g - 12) = \widehat{\boldsymbol{\pi}}^2$ and 2 earns $\vartheta_2(g - 12) = \widehat{\boldsymbol{\pi}}^2$. Since $\widehat{\boldsymbol{\pi}}^{3'} > \widehat{\boldsymbol{\pi}}^2$, $\vartheta_1(g - 12) < \vartheta_1(g)$ and $\vartheta_2(g - 12) < \vartheta_2(g)$

*Case 4*: Consider the network represented by Case 4, say, $g = \{13, 12\}$. Then the payoffs of the IAPs are $\vartheta_3(g) = \vartheta_2(g) = \widehat{\boldsymbol{\pi}}^4$, $\vartheta_1(g) = \widehat{\boldsymbol{\pi}}^{4'}$. If 2 and 3 form a link, then 2 earns $\vartheta_2(g + 23) = \widehat{\boldsymbol{\pi}}^1$ and 3 earns $\vartheta_3(g + 23) = \widehat{\boldsymbol{\pi}}^1$. As $\widehat{\boldsymbol{\pi}}^1 > \widehat{\boldsymbol{\pi}}^4$ , $\vartheta_2(g + 23) > \vartheta_2(g)$ and $\vartheta_3(g+23) > \vartheta_3(g)$. On the other hand, if 1 and 2 break a link then 1 earns $\vartheta_1(g-12) = \widehat{\boldsymbol{\pi}}^{3'}$ and 2 earns $\vartheta_2(g-12) = \widehat{\boldsymbol{\pi}}^3$. Since $\widehat{\boldsymbol{\pi}}^{4'} > \widehat{\boldsymbol{\pi}}^{3'}$ and $\widehat{\boldsymbol{\pi}}^4 > \widehat{\boldsymbol{\pi}}^3$, $\vartheta_1(g-12) < \vartheta_1(g)$ and $\vartheta_2(g-12) < \vartheta_2(g)$.

That completes the proof.

## A.6 Computations

### COMPUTATIONS FOR CASE

$$U[i] == V - \mu * (F[i]) - \lambda * L[i] - p[i]$$

$$F[i] == d[i] - s[i]$$

$$L[1] == \left(\frac{s[1]}{3} + \frac{s[2]}{3} - (s[1, 2] + s[2, 1]) - \frac{k}{2}\right) + \left(\frac{s[1]}{3} + \frac{s[3]}{3} - (s[1, 3] + s[3, 1]) - \frac{k}{2}\right)$$

$$L[2] == \left(\frac{s[1]}{3} + \frac{s[2]}{3} - (s[1, 2] + s[2, 1]) - \frac{k}{2}\right) + \left(\frac{s[2]}{3} + \frac{s[3]}{3} - (s[2, 3] + s[3, 2]) - \frac{k}{2}\right)$$

$$L[3] == \left(\frac{s[3]}{3} + \frac{s[2]}{3} - (s[3, 2] + s[2, 3]) - \frac{k}{2}\right) + \left(\frac{s[1]}{3} + \frac{s[3]}{3} - (s[1, 3] + s[3, 1]) - \frac{k}{2}\right)$$

$$c[1] == (s[1])^2 + (s[1, 2])^2 + (s[1, 3])^2$$

$$c[2] == (s[2])^2 + (s[2, 1])^2 + (s[2, 3])^2$$

$$c[3] == (s[3])^2 + (s[3, 1])^2 + (s[3, 2])^2$$

### FIND THE DEMANDS

$$\text{Solve}[\{-\mu * (d[1] - s[1]) - \lambda * L[1] - p[1] == -\mu * (d[2] - s[2]) - \lambda * L[2] - p[2],\ -\mu * (d[1] - s[1]) - \lambda * L[1] - p[1] == -\mu * (d[3] - s[3]) - \lambda * L[3] - p[3],\ d[1] + d[2] + d[3] == 1\}, \{d[1], d[2], d[3]\}]$$

$$\Big\{\Big\{d[2] \to -\frac{1}{3\mu}(-\mu - \lambda L[1] + 2\lambda L[2] - \lambda L[3] - p[1] + 2p[2] - p[3] + \mu s[1] - 2\mu s[2] + \mu s[3]),$$
$$d[1] \to -\frac{1}{3\mu}(-\mu + 2\lambda L[1] - \lambda L[2] - \lambda L[3] + 2p[1] - p[2] - p[3] - 2\mu s[1] + \mu s[2] + \mu s[3]),$$
$$d[3] \to -\frac{1}{3\mu}(-\mu - \lambda L[1] - \lambda L[2] + 2\lambda L[3] - p[1] - p[2] + 2p[3] + \mu s[1] + \mu s[2] - 2\mu s[3])\Big\}\Big\}$$

$$\pi[1] == p[1] * \left(-\frac{1}{3\mu}(-\mu + 2\lambda L[1] - \lambda L[2] - \lambda L[3] + 2p[1] - p[2] - p[3] - 2\mu s[1] + \mu s[2] + \mu s[3])\right) - c[1]$$

$$\pi[2] == p[2] * \left(-\frac{1}{3\mu}(-\mu - \lambda L[1] + 2\lambda L[2] - \lambda L[3] - p[1] + 2p[2] - p[3] + \mu s[1] - 2\mu s[2] + \mu s[3])\right) - c[2]$$

$$\pi[3] == p[3] * \left(-\frac{1}{3\mu}(-\mu - \lambda L[1] - \lambda L[2] + 2\lambda L[3] - p[1] - p[2] + 2p[3] + \mu s[1] + \mu s[2] - 2\mu s[3])\right) - c[3]$$

## SOLVE FOR THE LAST STAGE, FIND PRICES

$$\mathrm{D}\left[\mathrm{p}[1]*\left(-\frac{1}{3\mu}\,(-\mu+2\lambda\mathrm{L}[1]-\lambda\mathrm{L}[2]-\lambda\mathrm{L}[3]+2\,\mathrm{p}[1]-\mathrm{p}[2]-\mathrm{p}[3]-2\mu\,\mathrm{s}[1]+\mu\,\mathrm{s}[2]+\mu\,\mathrm{s}[3])\right)-\mathrm{c}[1],\ \mathrm{p}[1]\right]$$

$$-\frac{2\,\mathrm{p}[1]}{3\mu}-\frac{1}{3\mu}\,(-\mu+2\lambda\mathrm{L}[1]-\lambda\mathrm{L}[2]-\lambda\mathrm{L}[3]+2\,\mathrm{p}[1]-\mathrm{p}[2]-\mathrm{p}[3]-2\mu\,\mathrm{s}[1]+\mu\,\mathrm{s}[2]+\mu\,\mathrm{s}[3])$$

$$\mathrm{D}\left[\mathrm{p}[2]*\left(-\frac{1}{3\mu}\,(-\mu-\lambda\mathrm{L}[1]+2\lambda\mathrm{L}[2]-\lambda\mathrm{L}[3]-\mathrm{p}[1]+2\,\mathrm{p}[2]-\mathrm{p}[3]+\mu\,\mathrm{s}[1]-2\mu\,\mathrm{s}[2]+\mu\,\mathrm{s}[3])\right)-\mathrm{c}[2],\ \mathrm{p}[2]\right]$$

$$-\frac{2\,\mathrm{p}[2]}{3\mu}-\frac{1}{3\mu}\,(-\mu-\lambda\mathrm{L}[1]+2\lambda\mathrm{L}[2]-\lambda\mathrm{L}[3]-\mathrm{p}[1]+2\,\mathrm{p}[2]-\mathrm{p}[3]+\mu\,\mathrm{s}[1]-2\mu\,\mathrm{s}[2]+\mu\,\mathrm{s}[3])$$

$$\mathrm{D}\left[\mathrm{p}[3]*\left(-\frac{1}{3\mu}\,(-\mu-\lambda\mathrm{L}[1]-\lambda\mathrm{L}[2]+2\lambda\mathrm{L}[3]-\mathrm{p}[1]-\mathrm{p}[2]+2\,\mathrm{p}[3]+\mu\,\mathrm{s}[1]+\mu\,\mathrm{s}[2]-2\mu\,\mathrm{s}[3])\right)-\mathrm{c}[3],\ \mathrm{p}[3]\right]$$

$$-\frac{2\,\mathrm{p}[3]}{3\mu}-\frac{1}{3\mu}\,(-\mu-\lambda\mathrm{L}[1]-\lambda\mathrm{L}[2]+2\lambda\mathrm{L}[3]-\mathrm{p}[1]-\mathrm{p}[2]+2\,\mathrm{p}[3]+\mu\,\mathrm{s}[1]+\mu\,\mathrm{s}[2]-2\mu\,\mathrm{s}[3])$$

$$\begin{aligned}\mathrm{Solve}\Big[&\Big\{-\frac{2\,\mathrm{p}[3]}{3\mu}-\frac{1}{3\mu}\,(-\mu-\lambda\mathrm{L}[1]-\lambda\mathrm{L}[2]+2\lambda\mathrm{L}[3]-\mathrm{p}[1]-\mathrm{p}[2]+2\,\mathrm{p}[3]+\mu\,\mathrm{s}[1]+\mu\,\mathrm{s}[2]-2\mu\,\mathrm{s}[3])==0,\\ &-\frac{2\,\mathrm{p}[2]}{3\mu}-\frac{1}{3\mu}\,(-\mu-\lambda\mathrm{L}[1]+2\lambda\mathrm{L}[2]-\lambda\mathrm{L}[3]-\mathrm{p}[1]+2\,\mathrm{p}[2]-\mathrm{p}[3]+\mu\,\mathrm{s}[1]-2\mu\,\mathrm{s}[2]+\mu\,\mathrm{s}[3])==0,\\ &-\frac{2\,\mathrm{p}[1]}{3\mu}-\frac{1}{3\mu}\,(-\mu+2\lambda\mathrm{L}[1]-\lambda\mathrm{L}[2]-\lambda\mathrm{L}[3]+2\,\mathrm{p}[1]-\mathrm{p}[2]-\mathrm{p}[3]-2\mu\,\mathrm{s}[1]+\mu\,\mathrm{s}[2]+\mu\,\mathrm{s}[3])==0\Big\},\ \{\mathrm{p}[1],\mathrm{p}[2],\mathrm{p}[3]\}\Big]\end{aligned}$$

$$\begin{aligned}\Big\{\Big\{&\mathrm{p}[1]\to\frac{1}{10}\,(5\mu-4\lambda\mathrm{L}[1]+2\lambda\mathrm{L}[2]+2\lambda\mathrm{L}[3]+4\mu\,\mathrm{s}[1]-2\mu\,\mathrm{s}[2]-2\mu\,\mathrm{s}[3]),\\ &\mathrm{p}[2]\to\frac{1}{10}\,(5\mu+2\lambda\mathrm{L}[1]-4\lambda\mathrm{L}[2]+2\lambda\mathrm{L}[3]-2\mu\,\mathrm{s}[1]+4\mu\,\mathrm{s}[2]-2\mu\,\mathrm{s}[3]),\\ &\mathrm{p}[3]\to\frac{1}{10}\,(5\mu+2\lambda\mathrm{L}[1]+2\lambda\mathrm{L}[2]-4\lambda\mathrm{L}[3]-2\mu\,\mathrm{s}[1]-2\mu\,\mathrm{s}[2]+4\mu\,\mathrm{s}[3])\Big\}\Big\}\end{aligned}$$

```
Simplify[
 Solve[{π[1] == p[1] * (- 1/(3 μ) (-μ + 2 λ L[1] - λ L[2] - λ L[3] + 2 p[1] - p[2] - p[3] - 2 μ s[1] +
         μ s[2] + μ s[3])) - c[1], π[2] == p[2] *
     (- 1/(3 μ) (-μ - λ L[1] + 2 λ L[2] - λ L[3] - p[1] + 2 p[2] - p[3] + μ s[1] - 2 μ s[2] + μ s[3])) -
    c[2], π[3] == p[3] * (- 1/(3 μ) (-μ - λ L[1] - λ L[2] + 2 λ L[3] - p[1] -
         p[2] + 2 p[3] + μ s[1] + μ s[2] - 2 μ s[3])) - c[3],
   L[1] == (s[1]/3 + s[2]/3 - (s[1, 2] + s[2, 1]) - k/2) + (s[1]/3 + s[3]/3 - (s[1, 3] + s[3, 1]) - k/2),
   L[2] == (s[1]/3 + s[2]/3 - (s[1, 2] + s[2, 1]) - k/2) + (s[2]/3 + s[3]/3 - (s[2, 3] + s[3, 2]) - k/2),
   L[3] == (s[3]/3 + s[2]/3 - (s[3, 2] + s[2, 3]) - k/2) + (s[1]/3 + s[3]/3 - (s[1, 3] + s[3, 1]) - k/2),
   c[1] == (s[1])^2 + (s[1, 2])^2 + (s[1, 3])^2,
   c[2] == (s[2])^2 + (s[2, 1])^2 + (s[2, 3])^2,
   c[3] == (s[3])^2 + (s[3, 1])^2 + (s[3, 2])^2,
   p[1] == 1/10 (5 μ - 4 λ L[1] + 2 λ L[2] + 2 λ L[3] + 4 μ s[1] - 2 μ s[2] - 2 μ s[3]),
   p[2] == 1/10 (5 μ + 2 λ L[1] - 4 λ L[2] + 2 λ L[3] - 2 μ s[1] + 4 μ s[2] - 2 μ s[3]),
   p[3] == 1/10 (5 μ + 2 λ L[1] + 2 λ L[2] - 4 λ L[3] - 2 μ s[1] - 2 μ s[2] + 4 μ s[3])},
  {π[1], π[2], π[3]}, {p[1], p[2], p[3], L[1], L[2], L[3], c[1], c[2], c[3]}]]

{{π[1] →
   1/(1350 μ) (9 μ^2 (5 + 4 s[1] - 2 s[2] - 2 s[3])^2 + 4 λ^2 (-2 s[1] + s[2] + s[3] + 3 s[1, 2] + 3 s[1, 3] +
         3 s[2, 1] - 6 s[2, 3] + 3 s[3, 1] - 6 s[3, 2])^2 -
      6 μ (225 (s[1]^2 + s[1, 2]^2 + s[1, 3]^2) + 2 λ (5 + 4 s[1] - 2 s[2] - 2 s[3]) (2 s[1] -
           s[2] - s[3] - 3 s[1, 2] - 3 s[1, 3] - 3 s[2, 1] + 6 s[2, 3] - 3 s[3, 1] + 6 s[3, 2]))),
  π[2] → 1/(1350 μ) (9 μ^2 (5 - 2 s[1] + 4 s[2] - 2 s[3])^2 + 4 λ^2 (s[1] - 2 s[2] + s[3] +
         3 s[1, 2] - 6 s[1, 3] + 3 s[2, 1] + 3 s[2, 3] - 6 s[3, 1] + 3 s[3, 2])^2 -
      6 μ (225 (s[2]^2 + s[2, 1]^2 + s[2, 3]^2) + 2 λ (-5 + 2 s[1] - 4 s[2] + 2 s[3]) (s[1] -
           2 s[2] + s[3] + 3 s[1, 2] - 6 s[1, 3] + 3 s[2, 1] + 3 s[2, 3] - 6 s[3, 1] + 3 s[3, 2]))),
  π[3] → 1/(1350 μ) (9 μ^2 (5 - 2 s[1] - 2 s[2] + 4 s[3])^2 + 4 λ^2 (s[1] + s[2] - 2 s[3] -
         6 s[1, 2] + 3 s[1, 3] - 6 s[2, 1] + 3 s[2, 3] + 3 s[3, 1] + 3 s[3, 2])^2 -
      6 μ (2 λ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) (s[1] + s[2] - 2 s[3] - 6 s[1, 2] + 3 s[1, 3] -
           6 s[2, 1] + 3 s[2, 3] + 3 s[3, 1] + 3 s[3, 2]) + 225 (s[3]^2 + s[3, 1]^2 + s[3, 2]^2)))}}
```

### SOLVE THE THIRD STAGE, FIND LINK INVESTMENTS

```
D[1/(1350 μ) (9 μ^2 (5 + 4 s[1] - 2 s[2] - 2 s[3])^2 + 4 λ^2
      (-2 s[1] + s[2] + s[3] + 3 s[1, 2] + 3 s[1, 3] + 3 s[2, 1] - 6 s[2, 3] + 3 s[3, 1] - 6 s[3, 2])^2 -
    6 μ (225 (s[1]^2 + s[1, 2]^2 + s[1, 3]^2) + 2 λ (5 + 4 s[1] - 2 s[2] - 2 s[3]) (2 s[1] - s[2] -
          s[3] - 3 s[1, 2] - 3 s[1, 3] - 3 s[2, 1] + 6 s[2, 3] - 3 s[3, 1] + 6 s[3, 2]))), s[1, 2]]
```

```
1/(1350 μ) (-6 μ (-6 λ (5 + 4 s[1] - 2 s[2] - 2 s[3]) + 450 s[1, 2]) + 24 λ^2
    (-2 s[1] + s[2] + s[3] + 3 s[1, 2] + 3 s[1, 3] + 3 s[2, 1] - 6 s[2, 3] + 3 s[3, 1] - 6 s[3, 2]))
```

```
D[1/(1350 μ) (9 μ^2 (5 + 4 s[1] - 2 s[2] - 2 s[3])^2 + 4 λ^2
      (-2 s[1] + s[2] + s[3] + 3 s[1, 2] + 3 s[1, 3] + 3 s[2, 1] - 6 s[2, 3] + 3 s[3, 1] - 6 s[3, 2])^2 -
    6 μ (225 (s[1]^2 + s[1, 2]^2 + s[1, 3]^2) + 2 λ (5 + 4 s[1] - 2 s[2] - 2 s[3]) (2 s[1] - s[2] -
          s[3] - 3 s[1, 2] - 3 s[1, 3] - 3 s[2, 1] + 6 s[2, 3] - 3 s[3, 1] + 6 s[3, 2]))), s[1, 3]]
```

```
1/(1350 μ) (-6 μ (-6 λ (5 + 4 s[1] - 2 s[2] - 2 s[3]) + 450 s[1, 3]) + 24 λ^2
    (-2 s[1] + s[2] + s[3] + 3 s[1, 2] + 3 s[1, 3] + 3 s[2, 1] - 6 s[2, 3] + 3 s[3, 1] - 6 s[3, 2]))
```

```
D[1/(1350 μ) (9 μ^2 (5 - 2 s[1] + 4 s[2] - 2 s[3])^2 + 4 λ^2
      (s[1] - 2 s[2] + s[3] + 3 s[1, 2] - 6 s[1, 3] + 3 s[2, 1] + 3 s[2, 3] - 6 s[3, 1] + 3 s[3, 2])^2 -
    6 μ (225 (s[2]^2 + s[2, 1]^2 + s[2, 3]^2) + 2 λ (-5 + 2 s[1] - 4 s[2] + 2 s[3]) (s[1] - 2 s[2] +
          s[3] + 3 s[1, 2] - 6 s[1, 3] + 3 s[2, 1] + 3 s[2, 3] - 6 s[3, 1] + 3 s[3, 2]))), s[2, 1]]
```

```
1/(1350 μ) (-6 μ (6 λ (-5 + 2 s[1] - 4 s[2] + 2 s[3]) + 450 s[2, 1]) +
   24 λ^2 (s[1] - 2 s[2] + s[3] + 3 s[1, 2] - 6 s[1, 3] + 3 s[2, 1] + 3 s[2, 3] - 6 s[3, 1] + 3 s[3, 2]))
```

```
D[1/(1350 μ) (9 μ^2 (5 - 2 s[1] + 4 s[2] - 2 s[3])^2 + 4 λ^2
      (s[1] - 2 s[2] + s[3] + 3 s[1, 2] - 6 s[1, 3] + 3 s[2, 1] + 3 s[2, 3] - 6 s[3, 1] + 3 s[3, 2])^2 -
    6 μ (225 (s[2]^2 + s[2, 1]^2 + s[2, 3]^2) + 2 λ (-5 + 2 s[1] - 4 s[2] + 2 s[3]) (s[1] - 2 s[2] +
          s[3] + 3 s[1, 2] - 6 s[1, 3] + 3 s[2, 1] + 3 s[2, 3] - 6 s[3, 1] + 3 s[3, 2]))), s[2, 3]]
```

```
1/(1350 μ) (-6 μ (6 λ (-5 + 2 s[1] - 4 s[2] + 2 s[3]) + 450 s[2, 3]) +
   24 λ^2 (s[1] - 2 s[2] + s[3] + 3 s[1, 2] - 6 s[1, 3] + 3 s[2, 1] + 3 s[2, 3] - 6 s[3, 1] + 3 s[3, 2]))
```

```
D[1/(1350 μ) (9 μ^2 (5 - 2 s[1] - 2 s[2] + 4 s[3])^2 + 4 λ^2
      (s[1] + s[2] - 2 s[3] - 6 s[1, 2] + 3 s[1, 3] - 6 s[2, 1] + 3 s[2, 3] + 3 s[3, 1] + 3 s[3, 2])^2 -
    6 μ (2 λ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) (s[1] + s[2] - 2 s[3] - 6 s[1, 2] + 3 s[1, 3] - 6 s[2, 1] +
          3 s[2, 3] + 3 s[3, 1] + 3 s[3, 2]) + 225 (s[3]^2 + s[3, 1]^2 + s[3, 2]^2))), s[3, 1]]
```

```
1/(1350 μ) (-6 μ (6 λ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) + 450 s[3, 1]) +
   24 λ^2 (s[1] + s[2] - 2 s[3] - 6 s[1, 2] + 3 s[1, 3] - 6 s[2, 1] + 3 s[2, 3] + 3 s[3, 1] + 3 s[3, 2]))
```

$$\text{D}\Big[\frac{1}{1350\,\mu}\,(9\,\mu^2\,(5-2\,s[1]-2\,s[2]+4\,s[3])^2+4\,\lambda^2\,(s[1]+s[2]-2\,s[3]-6\,s[1,2]+3\,s[1,3]-6\,s[2,1]+3\,s[2,3]+3\,s[3,1]+3\,s[3,2])^2-6\,\mu\,(2\,\lambda\,(-5+2\,s[1]+2\,s[2]-4\,s[3])\,(s[1]+s[2]-2\,s[3]-6\,s[1,2]+3\,s[1,3]-6\,s[2,1]+3\,s[2,3]+3\,s[3,1]+3\,s[3,2])+225\,(s[3]^2+s[3,1]^2+s[3,2]^2))),\ s[3,2]\Big]$$

$$\frac{1}{1350\,\mu}\,(24\,\lambda^2\,(s[1]+s[2]-2\,s[3]-6\,s[1,2]+3\,s[1,3]-6\,s[2,1]+3\,s[2,3]+3\,s[3,1]+3\,s[3,2])-6\,\mu\,(6\,\lambda\,(-5+2\,s[1]+2\,s[2]-4\,s[3])+450\,s[3,2]))$$

$$\begin{aligned}
&\text{Simplify}\Big[\\
&\ \text{Solve}\Big[\Big\{\frac{1}{1350\,\mu}\,(24\,\lambda^2\,(s[1]+s[2]-2\,s[3]-6\,s[1,2]+3\,s[1,3]-6\,s[2,1]+3\,s[2,3]+3\,s[3,1]+3\,s[3,2])-6\,\mu\,(6\,\lambda\,(-5+2\,s[1]+2\,s[2]-4\,s[3])+450\,s[3,2]))=0,\\
&\quad \frac{1}{1350\,\mu}\,(-6\,\mu\,(6\,\lambda\,(-5+2\,s[1]+2\,s[2]-4\,s[3])+450\,s[3,1])+24\,\lambda^2\,(s[1]+s[2]-2\,s[3]-6\,s[1,2]+3\,s[1,3]-6\,s[2,1]+3\,s[2,3]+3\,s[3,1]+3\,s[3,2]))=0,\\
&\quad \frac{1}{1350\,\mu}\,(-6\,\mu\,(6\,\lambda\,(-5+2\,s[1]-4\,s[2]+2\,s[3])+450\,s[2,3])+24\,\lambda^2\,(s[1]-2\,s[2]+s[3]+3\,s[1,2]-6\,s[1,3]+3\,s[2,1]+3\,s[2,3]-6\,s[3,1]+3\,s[3,2]))=0,\\
&\quad \frac{1}{1350\,\mu}\,(-6\,\mu\,(6\,\lambda\,(-5+2\,s[1]-4\,s[2]+2\,s[3])+450\,s[2,1])+24\,\lambda^2\,(s[1]-2\,s[2]+s[3]+3\,s[1,2]-6\,s[1,3]+3\,s[2,1]+3\,s[2,3]-6\,s[3,1]+3\,s[3,2]))=0,\\
&\quad \frac{1}{1350\,\mu}\,(-6\,\mu\,(-6\,\lambda\,(5+4\,s[1]-2\,s[2]-2\,s[3])+450\,s[1,3])+24\,\lambda^2\,(-2\,s[1]+s[2]+s[3]+3\,s[1,2]+3\,s[1,3]+3\,s[2,1]-6\,s[2,3]+3\,s[3,1]-6\,s[3,2]))=0,\\
&\quad \frac{1}{1350\,\mu}\,(-6\,\mu\,(-6\,\lambda\,(5+4\,s[1]-2\,s[2]-2\,s[3])+450\,s[1,2])+24\,\lambda^2\,(-2\,s[1]+s[2]+s[3]+3\,s[1,2]+3\,s[1,3]+3\,s[2,1]-6\,s[2,3]+3\,s[3,1]-6\,s[3,2]))=0\Big\},\\
&\quad \{s[1,2],s[1,3],s[2,1],s[2,3],s[3,1],s[3,2]\}\Big]\Big]
\end{aligned}$$

$$\begin{aligned}
\Big\{\Big\{&s[1,2]\to\frac{\lambda\,(6\,\lambda^2+10\,\lambda\,(2\,s[1]-s[2]-s[3])+15\,\mu\,(-5-4\,s[1]+2\,s[2]+2\,s[3]))}{45\,(2\,\lambda^2-25\,\mu)},\\
&s[1,3]\to\frac{\lambda\,(6\,\lambda^2+10\,\lambda\,(2\,s[1]-s[2]-s[3])+15\,\mu\,(-5-4\,s[1]+2\,s[2]+2\,s[3]))}{45\,(2\,\lambda^2-25\,\mu)},\\
&s[2,1]\to\frac{\lambda\,(6\,\lambda^2-10\,\lambda\,(s[1]-2\,s[2]+s[3])+15\,\mu\,(-5+2\,s[1]-4\,s[2]+2\,s[3]))}{45\,(2\,\lambda^2-25\,\mu)},\\
&s[2,3]\to\frac{\lambda\,(6\,\lambda^2-10\,\lambda\,(s[1]-2\,s[2]+s[3])+15\,\mu\,(-5+2\,s[1]-4\,s[2]+2\,s[3]))}{45\,(2\,\lambda^2-25\,\mu)},\\
&s[3,1]\to\frac{\lambda\,(6\,\lambda^2+15\,\mu\,(-5+2\,s[1]+2\,s[2]-4\,s[3])-10\,\lambda\,(s[1]+s[2]-2\,s[3]))}{45\,(2\,\lambda^2-25\,\mu)},\\
&s[3,2]\to\frac{\lambda\,(6\,\lambda^2+15\,\mu\,(-5+2\,s[1]+2\,s[2]-4\,s[3])-10\,\lambda\,(s[1]+s[2]-2\,s[3]))}{45\,(2\,\lambda^2-25\,\mu)}\Big\}\Big\}
\end{aligned}$$

```
Simplify[Solve[
  {π[1] == 1/(1350 μ) (9 μ^2 (5 + 4 s[1] - 2 s[2] - 2 s[3])^2 + 4 λ^2 (-2 s[1] + s[2] + s[3] + 3 s[1, 2] +
          3 s[1, 3] + 3 s[2, 1] - 6 s[2, 3] + 3 s[3, 1] - 6 s[3, 2])^2 -
       6 μ (225 (s[1]^2 + s[1, 2]^2 + s[1, 3]^2) + 2 λ (5 + 4 s[1] - 2 s[2] - 2 s[3]) (2 s[1] -
            s[2] - s[3] - 3 s[1, 2] - 3 s[1, 3] - 3 s[2, 1] + 6 s[2, 3] - 3 s[3, 1] + 6 s[3, 2]))),
   π[2] == 1/(1350 μ) (9 μ^2 (5 - 2 s[1] + 4 s[2] - 2 s[3])^2 + 4 λ^2 (s[1] - 2 s[2] + s[3] +
          3 s[1, 2] - 6 s[1, 3] + 3 s[2, 1] + 3 s[2, 3] - 6 s[3, 1] + 3 s[3, 2])^2 -
       6 μ (225 (s[2]^2 + s[2, 1]^2 + s[2, 3]^2) + 2 λ (-5 + 2 s[1] - 4 s[2] + 2 s[3]) (s[1] - 2 s[2] +
            s[3] + 3 s[1, 2] - 6 s[1, 3] + 3 s[2, 1] + 3 s[2, 3] - 6 s[3, 1] + 3 s[3, 2]))),
   π[3] == 1/(1350 μ) (9 μ^2 (5 - 2 s[1] - 2 s[2] + 4 s[3])^2 + 4 λ^2 (s[1] + s[2] - 2 s[3] -
          6 s[1, 2] + 3 s[1, 3] - 6 s[2, 1] + 3 s[2, 3] + 3 s[3, 1] + 3 s[3, 2])^2 -
       6 μ (2 λ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) (s[1] + s[2] - 2 s[3] - 6 s[1, 2] + 3 s[1, 3] -
            6 s[2, 1] + 3 s[2, 3] + 3 s[3, 1] + 3 s[3, 2]) + 225 (s[3]^2 + s[3, 1]^2 + s[3, 2]^2))),
   s[1, 2] == 1/(45 (2 λ^2 - 25 μ)) (λ (6 λ^2 + 10 λ (2 s[1] - s[2] - s[3]) +
        15 μ (-5 - 4 s[1] + 2 s[2] + 2 s[3]))), s[1, 3] ==
    1/(45 (2 λ^2 - 25 μ)) (λ (6 λ^2 + 10 λ (2 s[1] - s[2] - s[3]) + 15 μ (-5 - 4 s[1] + 2 s[2] + 2 s[3]))),
   s[2, 1] == 1/(45 (2 λ^2 - 25 μ))
     (λ (6 λ^2 - 10 λ (s[1] - 2 s[2] + s[3]) + 15 μ (-5 + 2 s[1] - 4 s[2] + 2 s[3]))), s[2, 3] ==
    1/(45 (2 λ^2 - 25 μ)) (λ (6 λ^2 - 10 λ (s[1] - 2 s[2] + s[3]) + 15 μ (-5 + 2 s[1] - 4 s[2] + 2 s[3]))),
   s[3, 1] == 1/(45 (2 λ^2 - 25 μ))
     (λ (6 λ^2 + 15 μ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) - 10 λ (s[1] + s[2] - 2 s[3]))), s[3, 2] ==
    1/(45 (2 λ^2 - 25 μ)) (λ (6 λ^2 + 15 μ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) - 10 λ (s[1] + s[2] - 2 s[3])))},
  {π[1], π[2], π[3]},
  {s[1, 2],
   s[1, 3],
   s[2, 1],
   s[2, 3],
   s[3, 1],
   s[3, 2]}]]
```

$$
\begin{aligned}
\{\{\pi[1] \to\ & \frac{1}{4050\,(2\,\lambda^2 - 25\,\mu)^2}\,(-144\,\lambda^6 + 16875\,\mu^2\,(-150\,s[1]^2 + \mu\,(5 + 4\,s[1] - 2\,s[2] - 2\,s[3])^2) - \\
& 480\,\lambda^5\,(2\,s[1] - s[2] - s[3]) - 22500\,\lambda\,\mu^2 \\
& (8\,s[1]^2 + 2\,s[2]^2 + s[3]\,(-5 + 2\,s[3]) + s[2]\,(-5 + 4\,s[3]) - 2\,s[1]\,(-5 + 4\,s[2] + 4\,s[3])) + \\
& 600\,\lambda^3\,\mu\,(16\,s[1]^2 + 4\,s[2]^2 + s[3]\,(-25 + 4\,s[3]) + s[2]\,(-25 + 8\,s[3]) - \\
& 2\,s[1]\,(-25 + 8\,s[2] + 8\,s[3])) + 20\,\lambda^4\,(9\,\mu\,(35 + 16\,s[1] - 8\,s[2] - 8\,s[3]) - \\
& 10\,(89\,s[1]^2 - 8\,s[1]\,(s[2] + s[3]) + 2\,(s[2] + s[3])^2)) - \\
& 300\,\lambda^2\,\mu\,(-25\,(58\,s[1]^2 - 4\,s[1]\,(s[2] + s[3]) + (s[2] + s[3])^2) + 6\,\mu\,(50 + 8\,s[1]^2 + \\
& 2\,s[2]^2 + s[1]\,(50 - 8\,s[2] - 8\,s[3]) - 25\,s[3] + 2\,s[3]^2 + s[2]\,(-25 + 4\,s[3]))))),
\end{aligned}
$$

$$
\begin{aligned}
\pi[2] \to\ & \frac{1}{4050\,(2\,\lambda^2 - 25\,\mu)^2}\,(-144\,\lambda^6 + 16875\,\mu^2\,(-150\,s[2]^2 + \mu\,(5 - 2\,s[1] + 4\,s[2] - 2\,s[3])^2) + \\
& 480\,\lambda^5\,(s[1] - 2\,s[2] + s[3]) - 22500\,\lambda\,\mu^2 \\
& (2\,s[1]^2 + 8\,s[2]^2 + s[2]\,(10 - 8\,s[3]) + s[3]\,(-5 + 2\,s[3]) + s[1]\,(-5 - 8\,s[2] + 4\,s[3])) + \\
& 600\,\lambda^3\,\mu\,(4\,s[1]^2 + 16\,s[2]^2 + s[2]\,(50 - 16\,s[3]) + \\
& s[3]\,(-25 + 4\,s[3]) + s[1]\,(-25 - 16\,s[2] + 8\,s[3])) - \\
& 300\,\lambda^2\,\mu\,(-25\,(s[1]^2 + 58\,s[2]^2 - 4\,s[2]\,s[3] + s[3]^2 + s[1]\,(-4\,s[2] + 2\,s[3])) + \\
& 6\,\mu\,(50 + 2\,s[1]^2 + 8\,s[2]^2 + s[2]\,(50 - 8\,s[3]) - 25\,s[3] + 2\,s[3]^2 + \\
& s[1]\,(-25 - 8\,s[2] + 4\,s[3]))) - 20\,\lambda^4\,(9\,\mu\,(-35 + 8\,s[1] - 16\,s[2] + 8\,s[3]) + \\
& 10\,(2\,s[1]^2 + 89\,s[2]^2 - 8\,s[2]\,s[3] + 2\,s[3]^2 + s[1]\,(-8\,s[2] + 4\,s[3])))),
\end{aligned}
$$

$$
\begin{aligned}
\pi[3] \to\ & \frac{1}{4050\,(2\,\lambda^2 - 25\,\mu)^2}\,(-144\,\lambda^6 + 480\,\lambda^5\,(s[1] + s[2] - 2\,s[3]) + \\
& 16875\,\mu^2\,(-150\,s[3]^2 + \mu\,(5 - 2\,s[1] - 2\,s[2] + 4\,s[3])^2) - 22500\,\lambda\,\mu^2 \\
& (2\,s[1]^2 + 2\,s[2]^2 + s[1]\,(-5 + 4\,s[2] - 8\,s[3]) + 2\,s[3]\,(5 + 4\,s[3]) - s[2]\,(5 + 8\,s[3])) + \\
& 600\,\lambda^3\,\mu\,(4\,s[1]^2 + 4\,s[2]^2 + s[1]\,(-25 + 8\,s[2] - 16\,s[3]) + 2\,s[3]\,(25 + 8\,s[3]) - \\
& s[2]\,(25 + 16\,s[3])) - 20\,\lambda^4\,(9\,\mu\,(-35 + 8\,s[1] + 8\,s[2] - 16\,s[3]) + \\
& 10\,(2\,s[1]^2 + 2\,s[2]^2 + 4\,s[1]\,(s[2] - 2\,s[3]) - 8\,s[2]\,s[3] + 89\,s[3]^2)) - 300\,\lambda^2\,\mu \\
& (-25\,(s[1]^2 + s[2]^2 + 2\,s[1]\,(s[2] - 2\,s[3]) - 4\,s[2]\,s[3] + 58\,s[3]^2) + 6\,\mu\,(50 + 2\,s[1]^2 + \\
& 2\,s[2]^2 + s[1]\,(-25 + 4\,s[2] - 8\,s[3]) + 50\,s[3] + 8\,s[3]^2 - s[2]\,(25 + 8\,s[3]))))\}\}
\end{aligned}
$$

## SOLVE THE SECOND STAGE, FIND NETWORK CAPACITIES

$$
\begin{aligned}
\mathbf{D}\Big[ & (-144\,\lambda^6 + 16875\,\mu^2\,(-150\,s[1]^2 + \mu\,(5 + 4\,s[1] - 2\,s[2] - 2\,s[3])^2) - \\
& 480\,\lambda^5\,(2\,s[1] - s[2] - s[3]) - 22500\,\lambda\,\mu^2 \\
& (8\,s[1]^2 + 2\,s[2]^2 + s[3]\,(-5 + 2\,s[3]) + s[2]\,(-5 + 4\,s[3]) - 2\,s[1]\,(-5 + 4\,s[2] + 4\,s[3])) + \\
& 600\,\lambda^3\,\mu\,(16\,s[1]^2 + 4\,s[2]^2 + s[3]\,(-25 + 4\,s[3]) + s[2]\,(-25 + 8\,s[3]) - \\
& 2\,s[1]\,(-25 + 8\,s[2] + 8\,s[3])) + 20\,\lambda^4 \\
& (9\,\mu\,(35 + 16\,s[1] - 8\,s[2] - 8\,s[3]) - 10\,(89\,s[1]^2 - 8\,s[1]\,(s[2] + s[3]) + 2\,(s[2] + s[3])^2)) - \\
& 300\,\lambda^2\,\mu\,(-25\,(58\,s[1]^2 - 4\,s[1]\,(s[2] + s[3]) + (s[2] + s[3])^2) + \\
& 6\,\mu\,(50 + 8\,s[1]^2 + 2\,s[2]^2 + s[1]\,(50 - 8\,s[2] - 8\,s[3]) - 25\,s[3] + \\
& 2\,s[3]^2 + s[2]\,(-25 + 4\,s[3]))))) \Big/ \left(4050\,(2\,\lambda^2 - 25\,\mu)^2\right),\ s[1]\Big]
\end{aligned}
$$

$$
\begin{aligned}
& (-960\,\lambda^5 + 16875\,\mu^2\,(-300\,s[1] + 8\,\mu\,(5 + 4\,s[1] - 2\,s[2] - 2\,s[3])) - \\
& 22500\,\lambda\,\mu^2\,(16\,s[1] - 2\,(-5 + 4\,s[2] + 4\,s[3])) + 600\,\lambda^3\,\mu\,(32\,s[1] - 2\,(-25 + 8\,s[2] + 8\,s[3])) + \\
& 20\,\lambda^4\,(144\,\mu - 10\,(178\,s[1] - 8\,(s[2] + s[3]))) - 300\,\lambda^2\,\mu \\
& (6\,\mu\,(50 + 16\,s[1] - 8\,s[2] - 8\,s[3]) - 25\,(116\,s[1] - 4\,(s[2] + s[3])))) \Big/ \left(4050\,(2\,\lambda^2 - 25\,\mu)^2\right)
\end{aligned}
$$

```
D[(-144 λ^6 + 16875 μ^2 (-150 s[2]^2 + μ (5 - 2 s[1] + 4 s[2] - 2 s[3])^2) +
    480 λ^5 (s[1] - 2 s[2] + s[3]) - 22500 λ μ^2
     (2 s[1]^2 + 8 s[2]^2 + s[2] (10 - 8 s[3]) + s[3] (-5 + 2 s[3]) + s[1] (-5 - 8 s[2] + 4 s[3])) +
    600 λ^3 μ (4 s[1]^2 + 16 s[2]^2 + s[2] (50 - 16 s[3]) + s[3] (-25 + 4 s[3]) +
       s[1] (-25 - 16 s[2] + 8 s[3])) - 300 λ^2 μ
     (-25 (s[1]^2 + 58 s[2]^2 - 4 s[2] s[3] + s[3]^2 + s[1] (-4 s[2] + 2 s[3])) + 6 μ (50 + 2 s[1]^2 +
          8 s[2]^2 + s[2] (50 - 8 s[3]) - 25 s[3] + 2 s[3]^2 + s[1] (-25 - 8 s[2] + 4 s[3]))) -
    20 λ^4 (9 μ (-35 + 8 s[1] - 16 s[2] + 8 s[3]) + 10 (2 s[1]^2 + 89 s[2]^2 - 8 s[2] s[3] +
          2 s[3]^2 + s[1] (-8 s[2] + 4 s[3])))) / (4050 (2 λ^2 - 25 μ)^2), s[2]]
```

```
(-960 λ^5 - 20 λ^4 (-144 μ + 10 (-8 s[1] + 178 s[2] - 8 s[3])) -
   300 λ^2 μ (6 μ (50 - 8 s[1] + 16 s[2] - 8 s[3]) - 25 (-4 s[1] + 116 s[2] - 4 s[3])) +
   16875 μ^2 (-300 s[2] + 8 μ (5 - 2 s[1] + 4 s[2] - 2 s[3])) +
   600 λ^3 μ (50 - 16 s[1] + 32 s[2] - 16 s[3]) -
   22500 λ μ^2 (10 - 8 s[1] + 16 s[2] - 8 s[3])) / (4050 (2 λ^2 - 25 μ)^2)
```

```
D[(-144 λ^6 + 480 λ^5 (s[1] + s[2] - 2 s[3]) +
    16875 μ^2 (-150 s[3]^2 + μ (5 - 2 s[1] - 2 s[2] + 4 s[3])^2) - 22500 λ μ^2
     (2 s[1]^2 + 2 s[2]^2 + s[1] (-5 + 4 s[2] - 8 s[3]) + 2 s[3] (5 + 4 s[3]) - s[2] (5 + 8 s[3])) +
    600 λ^3 μ (4 s[1]^2 + 4 s[2]^2 + s[1] (-25 + 8 s[2] - 16 s[3]) + 2 s[3] (25 + 8 s[3]) -
       s[2] (25 + 16 s[3])) - 20 λ^4 (9 μ (-35 + 8 s[1] + 8 s[2] - 16 s[3]) +
       10 (2 s[1]^2 + 2 s[2]^2 + 4 s[1] (s[2] - 2 s[3]) - 8 s[2] s[3] + 89 s[3]^2)) -
    300 λ^2 μ (-25 (s[1]^2 + s[2]^2 + 2 s[1] (s[2] - 2 s[3]) - 4 s[2] s[3] + 58 s[3]^2) +
       6 μ (50 + 2 s[1]^2 + 2 s[2]^2 + s[1] (-25 + 4 s[2] - 8 s[3]) + 50 s[3] +
          8 s[3]^2 - s[2] (25 + 8 s[3])))) / (4050 (2 λ^2 - 25 μ)^2), s[3]]
```

```
(-960 λ^5 - 22500 λ μ^2 (-8 s[1] - 8 s[2] + 8 s[3] + 2 (5 + 4 s[3])) +
   16875 μ^2 (-300 s[3] + 8 μ (5 - 2 s[1] - 2 s[2] + 4 s[3])) +
   600 λ^3 μ (-16 s[1] - 16 s[2] + 16 s[3] + 2 (25 + 8 s[3])) -
   300 λ^2 μ (6 μ (50 - 8 s[1] - 8 s[2] + 16 s[3]) - 25 (-4 s[1] - 4 s[2] + 116 s[3])) -
   20 λ^4 (-144 μ + 10 (-8 s[1] - 8 s[2] + 178 s[3]))) / (4050 (2 λ^2 - 25 μ)^2)
```

```
Simplify[Solve[{(-960 λ^5 - 22500 λ μ^2 (-8 s[1] - 8 s[2] + 8 s[3] + 2 (5 + 4 s[3])) +
      16875 μ^2 (-300 s[3] + 8 μ (5 - 2 s[1] - 2 s[2] + 4 s[3])) +
      600 λ^3 μ (-16 s[1] - 16 s[2] + 16 s[3] + 2 (25 + 8 s[3])) -
      300 λ^2 μ (6 μ (50 - 8 s[1] - 8 s[2] + 16 s[3]) - 25 (-4 s[1] - 4 s[2] + 116 s[3])) -
      20 λ^4 (-144 μ + 10 (-8 s[1] - 8 s[2] + 178 s[3]))) / (4050 (2 λ^2 - 25 μ)^2) == 0,
   (-960 λ^5 - 20 λ^4 (-144 μ + 10 (-8 s[1] + 178 s[2] - 8 s[3])) -
      300 λ^2 μ (6 μ (50 - 8 s[1] + 16 s[2] - 8 s[3]) - 25 (-4 s[1] + 116 s[2] - 4 s[3])) +
      16875 μ^2 (-300 s[2] + 8 μ (5 - 2 s[1] + 4 s[2] - 2 s[3])) +
      600 λ^3 μ (50 - 16 s[1] + 32 s[2] - 16 s[3]) - 22500 λ μ^2 (10 - 8 s[1] + 16 s[2] - 8 s[3])) /
     (4050 (2 λ^2 - 25 μ)^2) == 0, (-960 λ^5 + 16875 μ^2 (-300 s[1] + 8 μ (5 + 4 s[1] - 2 s[2] - 2 s[3])) -
      22500 λ μ^2 (16 s[1] - 2 (-5 + 4 s[2] + 4 s[3])) + 600 λ^3 μ
       (32 s[1] - 2 (-25 + 8 s[2] + 8 s[3])) + 20 λ^4 (144 μ - 10 (178 s[1] - 8 (s[2] + s[3]))) -
      300 λ^2 μ (6 μ (50 + 16 s[1] - 8 s[2] - 8 s[3]) - 25 (116 s[1] - 4 (s[2] + s[3])))) /
     (4050 (2 λ^2 - 25 μ)^2) == 0}, {s[1], s[2], s[3]}]]
```

$$\left\{\left\{s[1] \to \frac{8\lambda^3 - 150\lambda\mu - 24\lambda^2\mu + 450\mu^2}{-270\lambda^2 + 3375\mu},\; s[2] \to -\frac{2\,(4\lambda^2 - 75\mu)\,(\lambda - 3\mu)}{135\,(2\lambda^2 - 25\mu)},\; s[3] \to -\frac{2\,(4\lambda^2 - 75\mu)\,(\lambda - 3\mu)}{135\,(2\lambda^2 - 25\mu)}\right\}\right\}$$

```
Solve[{s[1, 2] == 1/(45 (2 λ^2 - 25 μ))
     (λ (6 λ^2 + 10 λ (2 s[1] - s[2] - s[3]) + 15 μ (-5 - 4 s[1] + 2 s[2] + 2 s[3]))), s[1, 3] ==
    1/(45 (2 λ^2 - 25 μ)) (λ (6 λ^2 + 10 λ (2 s[1] - s[2] - s[3]) + 15 μ (-5 - 4 s[1] + 2 s[2] + 2 s[3]))),
   s[2, 1] == 1/(45 (2 λ^2 - 25 μ))
     (λ (6 λ^2 - 10 λ (s[1] - 2 s[2] + s[3]) + 15 μ (-5 + 2 s[1] - 4 s[2] + 2 s[3]))), s[2, 3] ==
    1/(45 (2 λ^2 - 25 μ)) (λ (6 λ^2 - 10 λ (s[1] - 2 s[2] + s[3]) + 15 μ (-5 + 2 s[1] - 4 s[2] + 2 s[3]))),
   s[3, 1] == 1/(45 (2 λ^2 - 25 μ))
     (λ (6 λ^2 + 15 μ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) - 10 λ (s[1] + s[2] - 2 s[3]))), s[3, 2] ==
    1/(45 (2 λ^2 - 25 μ)) (λ (6 λ^2 + 15 μ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) - 10 λ (s[1] + s[2] - 2 s[3]))),
   s[1] == (8 λ^3 - 150 λ μ - 24 λ^2 μ + 450 μ^2)/(-270 λ^2 + 3375 μ), s[2] == -(2 (4 λ^2 - 75 μ) (λ - 3 μ))/(135 (2 λ^2 - 25 μ)),
   s[3] == -(2 (4 λ^2 - 75 μ) (λ - 3 μ))/(135 (2 λ^2 - 25 μ))},
  {s[1, 2], s[1, 3], s[2, 1], s[2, 3], s[3, 1], s[3, 2]}]
```

$$\left\{\left\{s[1,2] \to \frac{\lambda}{15},\; s[1,3] \to \frac{\lambda}{15},\; s[2,1] \to \frac{\lambda}{15},\; s[2,3] \to \frac{\lambda}{15},\; s[3,1] \to \frac{\lambda}{15},\; s[3,2] \to \frac{\lambda}{15}\right\}\right\}$$

```
Solve[{p[1] == 1/10 (5 μ - 4 λ L[1] + 2 λ L[2] + 2 λ L[3] + 4 μ s[1] - 2 μ s[2] - 2 μ s[3]),
  p[2] == 1/10 (5 μ + 2 λ L[1] - 4 λ L[2] + 2 λ L[3] - 2 μ s[1] + 4 μ s[2] - 2 μ s[3]),
  p[3] == 1/10 (5 μ + 2 λ L[1] + 2 λ L[2] - 4 λ L[3] - 2 μ s[1] - 2 μ s[2] + 4 μ s[3]),
  L[1] == (s[1]/3 + s[2]/3 - (s[1, 2] + s[2, 1]) - k/2) + (s[1]/3 + s[3]/3 - (s[1, 3] + s[3, 1]) - k/2),
  L[2] == (s[1]/3 + s[2]/3 - (s[1, 2] + s[2, 1]) - k/2) + (s[2]/3 + s[3]/3 - (s[2, 3] + s[3, 2]) - k/2),
  L[3] == (s[3]/3 + s[2]/3 - (s[3, 2] + s[2, 3]) - k/2) + (s[1]/3 + s[3]/3 - (s[1, 3] + s[3, 1]) - k/2),
  s[1] == (8 λ^3 - 150 λ μ - 24 λ^2 μ + 450 μ^2)/(-270 λ^2 + 3375 μ), s[2] == -(2 (4 λ^2 - 75 μ) (λ - 3 μ))/(135 (2 λ^2 - 25 μ)),
  s[3] == -(2 (4 λ^2 - 75 μ) (λ - 3 μ))/(135 (2 λ^2 - 25 μ)), s[1, 2] == λ/15,
  s[1, 3] == λ/15,
  s[2, 1] == λ/15,
  s[2, 3] == λ/15,
  s[3, 1] == λ/15, s[3, 2] == λ/15}, {p[1], p[2], p[3]},
 {L[1], L[2], L[3], s[1], s[2], s[3], s[1, 2], s[1, 3], s[2, 1], s[2, 3], s[3, 1], s[3, 2]}]
```

$$\left\{\left\{p[1] \to \frac{\mu}{2},\ p[2] \to \frac{\mu}{2},\ p[3] \to \frac{\mu}{2}\right\}\right\}$$

$$\texttt{Solve}\Big[\Big\{p[1] == \frac{\mu}{2},\ p[2] == \frac{\mu}{2},\ p[3] == \frac{\mu}{2},$$

$$L[1] == \left(\frac{s[1]}{3} + \frac{s[2]}{3} - (s[1,2] + s[2,1]) - \frac{k}{2}\right) + \left(\frac{s[1]}{3} + \frac{s[3]}{3} - (s[1,3] + s[3,1]) - \frac{k}{2}\right),$$

$$L[2] == \left(\frac{s[1]}{3} + \frac{s[2]}{3} - (s[1,2] + s[2,1]) - \frac{k}{2}\right) + \left(\frac{s[2]}{3} + \frac{s[3]}{3} - (s[2,3] + s[3,2]) - \frac{k}{2}\right),$$

$$L[3] == \left(\frac{s[3]}{3} + \frac{s[2]}{3} - (s[3,2] + s[2,3]) - \frac{k}{2}\right) + \left(\frac{s[1]}{3} + \frac{s[3]}{3} - (s[1,3] + s[3,1]) - \frac{k}{2}\right),$$

$$s[1] == \frac{8\lambda^3 - 150\lambda\mu - 24\lambda^2\mu + 450\mu^2}{-270\lambda^2 + 3375\mu},\ s[2] == -\frac{2\,(4\lambda^2 - 75\mu)\,(\lambda - 3\mu)}{135\,(2\lambda^2 - 25\mu)},$$

$$s[3] == -\frac{2\,(4\lambda^2 - 75\mu)\,(\lambda - 3\mu)}{135\,(2\lambda^2 - 25\mu)},\ s[1,2] == \frac{\lambda}{15},$$

$$s[1,3] == \frac{\lambda}{15},$$

$$s[2,1] == \frac{\lambda}{15},$$

$$s[2,3] == \frac{\lambda}{15},$$

$$s[3,1] == \frac{\lambda}{15},\ s[3,2] == \frac{\lambda}{15},$$

$$d[2] == -\frac{1}{3\mu}\,(-\mu - \lambda L[1] + 2\lambda L[2] - \lambda L[3] - p[1] + 2p[2] - p[3] + \mu s[1] - 2\mu s[2] + \mu s[3]),$$

$$d[1] == -\frac{1}{3\mu}\,(-\mu + 2\lambda L[1] - \lambda L[2] - \lambda L[3] + 2p[1] - p[2] - p[3] - 2\mu s[1] + \mu s[2] + \mu s[3]),$$

$$d[3] == -\frac{1}{3\mu}\,(-\mu - \lambda L[1] - \lambda L[2] + 2\lambda L[3] - p[1] - p[2] + 2p[3] + \mu s[1] + \mu s[2] - 2\mu s[3])\Big\},$$

$$\{d[1], d[2], d[3]\}, \{L[1], L[2], L[3], s[1], s[2], s[3], s[1,2],$$
$$s[1,3], s[2,1], s[2,3], s[3,1], s[3,2], p[1], p[2], p[3]\}\Big]$$

$$\left\{\left\{d[1] \to \frac{1}{3},\ d[2] \to \frac{1}{3},\ d[3] \to \frac{1}{3}\right\}\right\}$$

```
Simplify[Solve[{p[1] == μ/2, p[2] == μ/2, p[3] == μ/2, s[1] == (8 λ^3 - 150 λ μ - 24 λ^2 μ + 450 μ^2)/(-270 λ^2 + 3375 μ),
    s[2] == -(2 (4 λ^2 - 75 μ) (λ - 3 μ))/(135 (2 λ^2 - 25 μ)), s[3] == -(2 (4 λ^2 - 75 μ) (λ - 3 μ))/(135 (2 λ^2 - 25 μ)), s[1, 2] == λ/15,
    s[1, 3] == λ/15,
    s[2, 1] == λ/15,
    s[2, 3] == λ/15,
    s[3, 1] == λ/15, s[3, 2] == λ/15, d[1] == 1/3,
    d[2] == 1/3, d[3] == 1/3, c[3] == (s[3])^2 + (s[3, 1])^2 + (s[3, 2])^2,
    c[2] == (s[2])^2 + (s[2, 1])^2 + (s[2, 3])^2, c[1] == (s[1])^2 + (s[1, 2])^2 + (s[1, 3])^2,
    π[1] == p[1] * d[1] - c[1], π[2] == p[2] * d[2] - c[2], π[3] == p[3] * d[3] - c[3]},
   {π[1], π[2], π[3], c[1], c[2], c[3]}, {L[1], L[2], L[3], s[1], s[2], s[3], s[1, 2],
    s[1, 3], s[2, 1], s[2, 3], s[3, 1], s[3, 2], p[1], p[2], p[3], d[1], d[2], d[3]}]]
```

$$\Big\{\Big\{\pi[1] \to \frac{1}{36450\,(2\lambda^2-25\mu)^2}\,(-1424\lambda^6+768\lambda^5\mu-28800\lambda^3\mu^2+270000\lambda\mu^3-50625\mu^3\,(-75+8\mu)+1800\lambda^2\mu^2\,(-475+24\mu)-12\lambda^4\mu\,(-5125+96\mu)),$$

$$\pi[2] \to \frac{1}{36450\,(2\lambda^2-25\mu)^2}\,(-1424\lambda^6+768\lambda^5\mu-28800\lambda^3\mu^2+270000\lambda\mu^3-50625\mu^3\,(-75+8\mu)+1800\lambda^2\mu^2\,(-475+24\mu)-12\lambda^4\mu\,(-5125+96\mu)),$$

$$\pi[3] \to \frac{1}{36450\,(2\lambda^2-25\mu)^2}\,(-1424\lambda^6+768\lambda^5\mu-28800\lambda^3\mu^2+270000\lambda\mu^3-50625\mu^3\,(-75+8\mu)+1800\lambda^2\mu^2\,(-475+24\mu)-12\lambda^4\mu\,(-5125+96\mu)),$$

$$c[1] \to \frac{1}{18225\,(2\lambda^2-25\mu)^2}\,(2\,(356\lambda^6-192\lambda^5\mu+7200\lambda^3\mu^2-67500\lambda\mu^3+101250\mu^4+12\lambda^4\mu\,(-775+24\mu)-225\lambda^2\mu^2\,(-275+48\mu))),$$

$$c[2] \to \frac{1}{18225\,(2\lambda^2-25\mu)^2}\,(2\,(356\lambda^6-192\lambda^5\mu+7200\lambda^3\mu^2-67500\lambda\mu^3+101250\mu^4+12\lambda^4\mu\,(-775+24\mu)-225\lambda^2\mu^2\,(-275+48\mu))),$$

$$c[3] \to \frac{1}{18225\,(2\lambda^2-25\mu)^2}\,(2\,(356\lambda^6-192\lambda^5\mu+7200\lambda^3\mu^2-67500\lambda\mu^3+101250\mu^4+12\lambda^4\mu\,(-775+24\mu)-225\lambda^2\mu^2\,(-275+48\mu)))\Big\}\Big\}$$

## SUMMARY OF FINAL RESULTS

$$s[1] \to \frac{8\lambda^3-150\lambda\mu-24\lambda^2\mu+450\mu^2}{-270\lambda^2+3375\mu}$$

$$s[2] \to -\frac{2\,(4\lambda^2-75\mu)\,(\lambda-3\mu)}{135\,(2\lambda^2-25\mu)}$$

$$s[3] \to -\frac{2\,(4\lambda^2-75\mu)\,(\lambda-3\mu)}{135\,(2\lambda^2-25\mu)}$$

$s[1,2] \to \frac{\lambda}{15}$,

$s[1,3] \to \frac{\lambda}{15}$,

$s[2,1] \to \frac{\lambda}{15}$,

$s[2,3] \to \frac{\lambda}{15}$,

$s[3,1] \to \frac{\lambda}{15}$, $s[3,2] \to \frac{\lambda}{15}$

$p[1] \to \frac{\mu}{2}$,

$p[2] \to \frac{\mu}{2}$,

$p[3] \to \frac{\mu}{2}$

$d[1] \to \frac{1}{3}$,

$d[2] \to \frac{1}{3}$,

$d[3] \to \frac{1}{3}$

$$\pi[1] \to (-1424\,\lambda^6 + 768\,\lambda^5\,\mu - 28800\,\lambda^3\,\mu^2 + 270000\,\lambda\,\mu^3 - 50625\,\mu^3\,(-75 + 8\,\mu) + 1800\,\lambda^2\,\mu^2\,(-475 + 24\,\mu) - 12\,\lambda^4\,\mu\,(-5125 + 96\,\mu))\Big/\left(36450\,(2\,\lambda^2 - 25\,\mu)^2\right)$$

$$\pi[2] \to (-1424\,\lambda^6 + 768\,\lambda^5\,\mu - 28800\,\lambda^3\,\mu^2 + 270000\,\lambda\,\mu^3 - 50625\,\mu^3\,(-75 + 8\,\mu) + 1800\,\lambda^2\,\mu^2\,(-475 + 24\,\mu) - 12\,\lambda^4\,\mu\,(-5125 + 96\,\mu))\Big/\left(36450\,(2\,\lambda^2 - 25\,\mu)^2\right)$$

$$\pi[3] \to (-1424\,\lambda^6 + 768\,\lambda^5\,\mu - 28800\,\lambda^3\,\mu^2 + 270000\,\lambda\,\mu^3 - 50625\,\mu^3\,(-75 + 8\,\mu) + 1800\,\lambda^2\,\mu^2\,(-475 + 24\,\mu) - 12\,\lambda^4\,\mu\,(-5125 + 96\,\mu))\Big/\left(36450\,(2\,\lambda^2 - 25\,\mu)^2\right)$$

$$c[1] \to (2\,(356\,\lambda^6 - 192\,\lambda^5\,\mu + 7200\,\lambda^3\,\mu^2 - 67500\,\lambda\,\mu^3 + 101250\,\mu^4 + 12\,\lambda^4\,\mu\,(-775 + 24\,\mu) - 225\,\lambda^2\,\mu^2\,(-275 + 48\,\mu)))\Big/\left(18225\,(2\,\lambda^2 - 25\,\mu)^2\right)$$

$$c[2] \to (2\,(356\,\lambda^6 - 192\,\lambda^5\,\mu + 7200\,\lambda^3\,\mu^2 - 67500\,\lambda\,\mu^3 + 101250\,\mu^4 + 12\,\lambda^4\,\mu\,(-775 + 24\,\mu) - 225\,\lambda^2\,\mu^2\,(-275 + 48\,\mu)))\Big/\left(18225\,(2\,\lambda^2 - 25\,\mu)^2\right)$$

$$c[3] \to (2\,(356\,\lambda^6 - 192\,\lambda^5\,\mu + 7200\,\lambda^3\,\mu^2 - 67500\,\lambda\,\mu^3 + 101250\,\mu^4 + 12\,\lambda^4\,\mu\,(-775 + 24\,\mu) - 225\,\lambda^2\,\mu^2\,(-275 + 48\,\mu)))\Big/\left(18225\,(2\,\lambda^2 - 25\,\mu)^2\right)$$

## COMPUTATONS FOR CASE 2

$U[i] == V - \mu * (F[i]) - \lambda * L[i] - p[i]$

$F[i] == d[i] - s[i]$

$$L[1] == \left(\frac{s[1]}{3} + \frac{s[2]}{3} - \frac{k}{2}\right) + \left(\frac{s[1]}{3} + \frac{s[3]}{3} - \frac{k}{2}\right)$$

$$L[2] == \left(\frac{s[1]}{3} + \frac{s[2]}{3} - \frac{k}{2}\right) + \left(\frac{s[2]}{3} + \frac{s[3]}{3} - \frac{k}{2}\right)$$

$$L[3] == \left(\frac{s[3]}{3} + \frac{s[2]}{3} - \frac{k}{2}\right) + \left(\frac{s[1]}{3} + \frac{s[3]}{3} - \frac{k}{2}\right)$$

$$c[1] == (s[1])^2$$

$$c[2] == (s[2])^2$$

$$c[3] == (s[3])^2$$

## FIND THE DEMANDS

$$\text{Solve}[\{-\mu * (d[1] - s[1]) - \lambda * L[1] - p[1] == -\mu * (d[2] - s[2]) - \lambda * L[2] - p[2], \\ -\mu * (d[1] - s[1]) - \lambda * L[1] - p[1] == -\mu * (d[3] - s[3]) - \lambda * L[3] - p[3], \\ d[1] + d[2] + d[3] == 1\}, \{d[1], d[2], d[3]\}]$$

$$\Big\{\Big\{d[2] \to -\frac{1}{3\mu}(-\mu - \lambda L[1] + 2\lambda L[2] - \lambda L[3] - p[1] + 2p[2] - p[3] + \mu s[1] - 2\mu s[2] + \mu s[3]), \\ d[1] \to -\frac{1}{3\mu}(-\mu + 2\lambda L[1] - \lambda L[2] - \lambda L[3] + 2p[1] - p[2] - p[3] - 2\mu s[1] + \mu s[2] + \mu s[3]), \\ d[3] \to -\frac{1}{3\mu}(-\mu - \lambda L[1] - \lambda L[2] + 2\lambda L[3] - p[1] - p[2] + 2p[3] + \mu s[1] + \mu s[2] - 2\mu s[3])\Big\}\Big\}$$

$$\pi[1] == p[1] * \left(-\frac{1}{3\mu}(-\mu + 2\lambda L[1] - \lambda L[2] - \lambda L[3] + 2p[1] - p[2] - p[3] - 2\mu s[1] + \mu s[2] + \mu s[3])\right) - c[1]$$

$$\pi[2] == p[2] * \left(-\frac{1}{3\mu}(-\mu - \lambda L[1] + 2\lambda L[2] - \lambda L[3] - p[1] + 2p[2] - p[3] + \mu s[1] - 2\mu s[2] + \mu s[3])\right) - c[2]$$

$$\pi[3] == p[3] * \left(-\frac{1}{3\mu}(-\mu - \lambda L[1] - \lambda L[2] + 2\lambda L[3] - p[1] - p[2] + 2p[3] + \mu s[1] + \mu s[2] - 2\mu s[3])\right) - c[3]$$

## SOLVE FOR THE LAST STAGE, FIND PRICES

$$D\left[p[1] * \left(-\frac{1}{3\mu}(-\mu + 2\lambda L[1] - \lambda L[2] - \lambda L[3] + 2p[1] - p[2] - p[3] - 2\mu s[1] + \mu s[2] + \mu s[3])\right) - \\ c[1], p[1]\right]$$

$$-\frac{2p[1]}{3\mu} - \frac{1}{3\mu}(-\mu + 2\lambda L[1] - \lambda L[2] - \lambda L[3] + 2p[1] - p[2] - p[3] - 2\mu s[1] + \mu s[2] + \mu s[3])$$

$$\mathbf{D}\left[\mathbf{p[2]} * \left(-\frac{1}{3\,\mu}\,(-\mu - \lambda\,\mathbf{L[1]} + 2\,\lambda\,\mathbf{L[2]} - \lambda\,\mathbf{L[3]} - \mathbf{p[1]} + 2\,\mathbf{p[2]} - \mathbf{p[3]} + \mu\,\mathbf{s[1]} - 2\,\mu\,\mathbf{s[2]} + \mu\,\mathbf{s[3]})\right) - \mathbf{c[2]},\ \mathbf{p[2]}\right]$$

$$-\frac{2\,p[2]}{3\,\mu} - \frac{1}{3\,\mu}\,(-\mu - \lambda\,L[1] + 2\,\lambda\,L[2] - \lambda\,L[3] - p[1] + 2\,p[2] - p[3] + \mu\,s[1] - 2\,\mu\,s[2] + \mu\,s[3])$$

$$\mathbf{D}\left[\mathbf{p[3]} * \left(-\frac{1}{3\,\mu}\,(-\mu - \lambda\,\mathbf{L[1]} - \lambda\,\mathbf{L[2]} + 2\,\lambda\,\mathbf{L[3]} - \mathbf{p[1]} - \mathbf{p[2]} + 2\,\mathbf{p[3]} + \mu\,\mathbf{s[1]} + \mu\,\mathbf{s[2]} - 2\,\mu\,\mathbf{s[3]})\right) - \mathbf{c[3]},\ \mathbf{p[3]}\right]$$

$$-\frac{2\,p[3]}{3\,\mu} - \frac{1}{3\,\mu}\,(-\mu - \lambda\,L[1] - \lambda\,L[2] + 2\,\lambda\,L[3] - p[1] - p[2] + 2\,p[3] + \mu\,s[1] + \mu\,s[2] - 2\,\mu\,s[3])$$

$$\begin{aligned}&\mathbf{Solve}\Big[\\&\Big\{-\frac{2\,\mathbf{p[3]}}{3\,\mu} - \frac{1}{3\,\mu}\,(-\mu - \lambda\,\mathbf{L[1]} - \lambda\,\mathbf{L[2]} + 2\,\lambda\,\mathbf{L[3]} - \mathbf{p[1]} - \mathbf{p[2]} + 2\,\mathbf{p[3]} + \mu\,\mathbf{s[1]} + \mu\,\mathbf{s[2]} - 2\,\mu\,\mathbf{s[3]}) == 0,\\&-\frac{2\,\mathbf{p[2]}}{3\,\mu} - \frac{1}{3\,\mu}\,(-\mu - \lambda\,\mathbf{L[1]} + 2\,\lambda\,\mathbf{L[2]} - \lambda\,\mathbf{L[3]} - \mathbf{p[1]} + 2\,\mathbf{p[2]} - \mathbf{p[3]} + \mu\,\mathbf{s[1]} - 2\,\mu\,\mathbf{s[2]} + \mu\,\mathbf{s[3]}) == 0,\\&-\frac{2\,\mathbf{p[1]}}{3\,\mu} - \frac{1}{3\,\mu}\,(-\mu + 2\,\lambda\,\mathbf{L[1]} - \lambda\,\mathbf{L[2]} - \lambda\,\mathbf{L[3]} + 2\,\mathbf{p[1]} - \mathbf{p[2]} - \mathbf{p[3]} - 2\,\mu\,\mathbf{s[1]} + \mu\,\mathbf{s[2]} + \mu\,\mathbf{s[3]}) == 0\Big\},\ \{\mathbf{p[1]},\ \mathbf{p[2]},\ \mathbf{p[3]}\}\Big]\end{aligned}$$

$$\begin{aligned}\Big\{\Big\{&p[1] \to \frac{1}{10}\,(5\,\mu - 4\,\lambda\,L[1] + 2\,\lambda\,L[2] + 2\,\lambda\,L[3] + 4\,\mu\,s[1] - 2\,\mu\,s[2] - 2\,\mu\,s[3]),\\&p[2] \to \frac{1}{10}\,(5\,\mu + 2\,\lambda\,L[1] - 4\,\lambda\,L[2] + 2\,\lambda\,L[3] - 2\,\mu\,s[1] + 4\,\mu\,s[2] - 2\,\mu\,s[3]),\\&p[3] \to \frac{1}{10}\,(5\,\mu + 2\,\lambda\,L[1] + 2\,\lambda\,L[2] - 4\,\lambda\,L[3] - 2\,\mu\,s[1] - 2\,\mu\,s[2] + 4\,\mu\,s[3])\Big\}\Big\}\end{aligned}$$

$$
\begin{aligned}
&\mathbf{Simplify}\Big[\mathbf{Solve}\Big[\Big\{\pi[1] == p[1] * \\
&\quad\left(-\frac{1}{3\,\mu}\,(-\mu + 2\,\lambda\,L[1] - \lambda\,L[2] - \lambda\,L[3] + 2\,p[1] - p[2] - p[3] - 2\,\mu\,s[1] + \mu\,s[2] + \mu\,s[3])\right) - \\
&\quad c[1],\ \pi[2] == p[2] * \left(-\frac{1}{3\,\mu}\,(-\mu - \lambda\,L[1] + 2\,\lambda\,L[2] - \lambda\,L[3] - p[1] + \right. \\
&\qquad \left. 2\,p[2] - p[3] + \mu\,s[1] - 2\,\mu\,s[2] + \mu\,s[3])\right) - c[2],\ \pi[3] == p[3] * \\
&\quad\left(-\frac{1}{3\,\mu}\,(-\mu - \lambda\,L[1] - \lambda\,L[2] + 2\,\lambda\,L[3] - p[1] - p[2] + 2\,p[3] + \mu\,s[1] + \mu\,s[2] - 2\,\mu\,s[3])\right) - \\
&\quad c[3],\ L[1] == \left(\frac{s[1]}{3} + \frac{s[2]}{3} - \frac{k}{2}\right) + \left(\frac{s[1]}{3} + \frac{s[3]}{3} - \frac{k}{2}\right), \\
&\quad L[2] == \left(\frac{s[1]}{3} + \frac{s[2]}{3} - \frac{k}{2}\right) + \left(\frac{s[2]}{3} + \frac{s[3]}{3} - \frac{k}{2}\right), \\
&\quad L[3] == \left(\frac{s[3]}{3} + \frac{s[2]}{3} - \frac{k}{2}\right) + \left(\frac{s[1]}{3} + \frac{s[3]}{3} - \frac{k}{2}\right), \\
&\quad c[1] == (s[1])^2,\ c[2] == (s[2])^2,\ c[3] == (s[3])^2, \\
&\quad p[1] == \frac{1}{10}\,(5\,\mu - 4\,\lambda\,L[1] + 2\,\lambda\,L[2] + 2\,\lambda\,L[3] + 4\,\mu\,s[1] - 2\,\mu\,s[2] - 2\,\mu\,s[3]), \\
&\quad p[2] == \frac{1}{10}\,(5\,\mu + 2\,\lambda\,L[1] - 4\,\lambda\,L[2] + 2\,\lambda\,L[3] - 2\,\mu\,s[1] + 4\,\mu\,s[2] - 2\,\mu\,s[3]), \\
&\quad p[3] == \frac{1}{10}\,(5\,\mu + 2\,\lambda\,L[1] + 2\,\lambda\,L[2] - 4\,\lambda\,L[3] - 2\,\mu\,s[1] - 2\,\mu\,s[2] + 4\,\mu\,s[3])\Big\}, \\
&\quad \{\pi[1], \pi[2], \pi[3]\},\ \{p[1], p[2], p[3], L[1], L[2], L[3], c[1], c[2], c[3]\}\Big]\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\Big\{\Big\{\pi[1] \to \frac{1}{1350\,\mu}\,(9\,\mu^2\,(5 + 4\,s[1] - 2\,s[2] - 2\,s[3])^2 + \\
&\qquad 4\,\lambda^2\,(-2\,s[1] + s[2] + s[3])^2 - 6\,\mu\,(225\,s[1]^2 + 2\,\lambda\,(8\,s[1]^2 + 2\,s[2]^2 + \\
&\qquad\quad s[3]\,(-5 + 2\,s[3]) + s[2]\,(-5 + 4\,s[3]) - 2\,s[1]\,(-5 + 4\,s[2] + 4\,s[3])))), \\
&\quad \pi[2] \to \frac{1}{1350\,\mu}\,(9\,\mu^2\,(5 - 2\,s[1] + 4\,s[2] - 2\,s[3])^2 + 4\,\lambda^2\,(s[1] - 2\,s[2] + s[3])^2 - \\
&\qquad 6\,\mu\,(225\,s[2]^2 + 2\,\lambda\,(2\,s[1]^2 + 8\,s[2]^2 + s[2]\,(10 - 8\,s[3]) + \\
&\qquad\quad s[3]\,(-5 + 2\,s[3]) + s[1]\,(-5 - 8\,s[2] + 4\,s[3])))), \\
&\quad \pi[3] \to \frac{1}{1350\,\mu}\,(4\,\lambda^2\,(s[1] + s[2] - 2\,s[3])^2 + 9\,\mu^2\,(5 - 2\,s[1] - 2\,s[2] + 4\,s[3])^2 - \\
&\qquad 6\,\mu\,(225\,s[3]^2 + 2\,\lambda\,(2\,s[1]^2 + 2\,s[2]^2 + s[1]\,(-5 + 4\,s[2] - 8\,s[3]) + \\
&\qquad\quad 2\,s[3]\,(5 + 4\,s[3]) - s[2]\,(5 + 8\,s[3]))))\Big\}\Big\}
\end{aligned}
$$

## SOLVE FOR NETWORK CAPACITIES

$$
\begin{aligned}
&\mathbf{D}\Big[\frac{1}{1350\,\mu}\,(9\,\mu^2\,(5 + 4\,s[1] - 2\,s[2] - 2\,s[3])^2 + \\
&\quad 4\,\lambda^2\,(-2\,s[1] + s[2] + s[3])^2 - 6\,\mu\,(225\,s[1]^2 + 2\,\lambda\,(8\,s[1]^2 + 2\,s[2]^2 + \\
&\qquad s[3]\,(-5 + 2\,s[3]) + s[2]\,(-5 + 4\,s[3]) - 2\,s[1]\,(-5 + 4\,s[2] + 4\,s[3])))),\ s[1]\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{1350\,\mu}\,(72\,\mu^2\,(5 + 4\,s[1] - 2\,s[2] - 2\,s[3]) - \\
&\quad 16\,\lambda^2\,(-2\,s[1] + s[2] + s[3]) - 6\,\mu\,(450\,s[1] + 2\,\lambda\,(16\,s[1] - 2\,(-5 + 4\,s[2] + 4\,s[3]))))
\end{aligned}
$$

$$\text{D}\Big[\frac{1}{1350\,\mu}\left(9\,\mu^2\,(5-2\,s[1]+4\,s[2]-2\,s[3])^2+4\,\lambda^2\,(s[1]-2\,s[2]+s[3])^2-6\,\mu\,(225\,s[2]^2+2\,\lambda\,(2\,s[1]^2+8\,s[2]^2+s[2]\,(10-8\,s[3])+s[3]\,(-5+2\,s[3])+s[1]\,(-5-8\,s[2]+4\,s[3])))\right),\ s[2]\Big]$$

$$\frac{1}{1350\,\mu}\,(-6\,\mu\,(450\,s[2]+2\,\lambda\,(10-8\,s[1]+16\,s[2]-8\,s[3]))+72\,\mu^2\,(5-2\,s[1]+4\,s[2]-2\,s[3])-16\,\lambda^2\,(s[1]-2\,s[2]+s[3]))$$

$$\text{D}\Big[\frac{1}{1350\,\mu}\left(4\,\lambda^2\,(s[1]+s[2]-2\,s[3])^2+9\,\mu^2\,(5-2\,s[1]-2\,s[2]+4\,s[3])^2-6\,\mu\,(225\,s[3]^2+2\,\lambda\,(2\,s[1]^2+2\,s[2]^2+s[1]\,(-5+4\,s[2]-8\,s[3])+2\,s[3]\,(5+4\,s[3])-s[2]\,(5+8\,s[3])))\right),\ s[3]\Big]$$

$$\frac{1}{1350\,\mu}\,(-16\,\lambda^2\,(s[1]+s[2]-2\,s[3])+72\,\mu^2\,(5-2\,s[1]-2\,s[2]+4\,s[3])-6\,\mu\,(450\,s[3]+2\,\lambda\,(-8\,s[1]-8\,s[2]+8\,s[3]+2\,(5+4\,s[3]))))$$

$$\begin{aligned}\text{Solve}\Big[\Big\{&\frac{1}{1350\,\mu}\,(-16\,\lambda^2\,(s[1]+s[2]-2\,s[3])+72\,\mu^2\,(5-2\,s[1]-2\,s[2]+4\,s[3])-6\,\mu\,(450\,s[3]+2\,\lambda\,(-8\,s[1]-8\,s[2]+8\,s[3]+2\,(5+4\,s[3]))))==0,\\ &\frac{1}{1350\,\mu}\,(-6\,\mu\,(450\,s[2]+2\,\lambda\,(10-8\,s[1]+16\,s[2]-8\,s[3]))+72\,\mu^2\,(5-2\,s[1]+4\,s[2]-2\,s[3])-16\,\lambda^2\,(s[1]-2\,s[2]+s[3]))==0,\\ &\frac{1}{1350\,\mu}\,(72\,\mu^2\,(5+4\,s[1]-2\,s[2]-2\,s[3])-16\,\lambda^2\,(-2\,s[1]+s[2]+s[3])-6\,\mu\,(450\,s[1]+2\,\lambda\,(16\,s[1]-2\,(-5+4\,s[2]+4\,s[3]))))==0\Big\},\ \{s[1],s[2],s[3]\}\Big]\end{aligned}$$

$$\left\{\left\{s[1]\to-\frac{2}{45}\,(\lambda-3\,\mu),\ s[2]\to-\frac{2}{45}\,(\lambda-3\,\mu),\ s[3]\to-\frac{2}{45}\,(\lambda-3\,\mu)\right\}\right\}$$

$$\begin{aligned}\text{Solve}\Big[\Big\{&L[1]==\left(\frac{s[1]}{3}+\frac{s[2]}{3}-\frac{k}{2}\right)+\left(\frac{s[1]}{3}+\frac{s[3]}{3}-\frac{k}{2}\right),\\ &L[2]==\left(\frac{s[1]}{3}+\frac{s[2]}{3}-\frac{k}{2}\right)+\left(\frac{s[2]}{3}+\frac{s[3]}{3}-\frac{k}{2}\right),\\ &L[3]==\left(\frac{s[3]}{3}+\frac{s[2]}{3}-\frac{k}{2}\right)+\left(\frac{s[1]}{3}+\frac{s[3]}{3}-\frac{k}{2}\right),\\ &p[1]==\frac{1}{10}\,(5\,\mu-4\,\lambda\,L[1]+2\,\lambda\,L[2]+2\,\lambda\,L[3]+4\,\mu\,s[1]-2\,\mu\,s[2]-2\,\mu\,s[3]),\\ &p[2]==\frac{1}{10}\,(5\,\mu+2\,\lambda\,L[1]-4\,\lambda\,L[2]+2\,\lambda\,L[3]-2\,\mu\,s[1]+4\,\mu\,s[2]-2\,\mu\,s[3]),\\ &p[3]==\frac{1}{10}\,(5\,\mu+2\,\lambda\,L[1]+2\,\lambda\,L[2]-4\,\lambda\,L[3]-2\,\mu\,s[1]-2\,\mu\,s[2]+4\,\mu\,s[3]),\\ &s[1]==-\frac{2}{45}\,(\lambda-3\,\mu),\ s[2]==-\frac{2}{45}\,(\lambda-3\,\mu),\ s[3]==-\frac{2}{45}\,(\lambda-3\,\mu)\Big\},\\ &\{p[1],p[2],p[3]\},\ \{L[1],L[2],L[3],s[1],s[2],s[3]\}\Big]\end{aligned}$$

$$\left\{\left\{p[1]\to\frac{\mu}{2},\ p[2]\to\frac{\mu}{2},\ p[3]\to\frac{\mu}{2}\right\}\right\}$$

$$
\begin{aligned}
&\textbf{Solve}\Big[\\
&\Big\{d[2] == -\frac{1}{3\,\mu}\,(-\mu - \lambda\,L[1] + 2\,\lambda\,L[2] - \lambda\,L[3] - p[1] + 2\,p[2] - p[3] + \mu\,s[1] - 2\,\mu\,s[2] + \mu\,s[3]),\\
&\ d[1] == -\frac{1}{3\,\mu}\,(-\mu + 2\,\lambda\,L[1] - \lambda\,L[2] - \lambda\,L[3] + 2\,p[1] - p[2] - p[3] - 2\,\mu\,s[1] + \mu\,s[2] + \mu\,s[3]),\\
&\ d[3] == -\frac{1}{3\,\mu}\,(-\mu - \lambda\,L[1] - \lambda\,L[2] + 2\,\lambda\,L[3] - p[1] - p[2] + 2\,p[3] + \mu\,s[1] + \mu\,s[2] - 2\,\mu\,s[3]),\\
&\ L[1] == \left(\frac{s[1]}{3} + \frac{s[2]}{3} - \frac{k}{2}\right) + \left(\frac{s[1]}{3} + \frac{s[3]}{3} - \frac{k}{2}\right),\\
&\ L[2] == \left(\frac{s[1]}{3} + \frac{s[2]}{3} - \frac{k}{2}\right) + \left(\frac{s[2]}{3} + \frac{s[3]}{3} - \frac{k}{2}\right),\\
&\ L[3] == \left(\frac{s[3]}{3} + \frac{s[2]}{3} - \frac{k}{2}\right) + \left(\frac{s[1]}{3} + \frac{s[3]}{3} - \frac{k}{2}\right),\ s[1] == -\frac{2}{45}\,(\lambda - 3\,\mu),\\
&\ s[2] == -\frac{2}{45}\,(\lambda - 3\,\mu),\ s[3] == -\frac{2}{45}\,(\lambda - 3\,\mu),\ p[1] == \frac{\mu}{2},\ p[2] == \frac{\mu}{2},\ p[3] == \frac{\mu}{2}\Big\},\\
&\{d[1], d[2], d[3]\}, \{L[1], L[2], L[3], s[1], s[2], s[3], p[1], p[2], p[3]\}\Big]
\end{aligned}
$$

$$
\left\{\left\{d[1] \to \frac{1}{3},\ d[2] \to \frac{1}{3},\ d[3] \to \frac{1}{3}\right\}\right\}
$$

$$\text{Solve}\Big[\Big\{d[1] == \frac{1}{3},\ d[2] == \frac{1}{3},\ d[3] == \frac{1}{3},\ s[1] == -\frac{2}{45}\ (\lambda - 3\,\mu),$$
$$s[2] == -\frac{2}{45}\ (\lambda - 3\,\mu),\ s[3] == -\frac{2}{45}\ (\lambda - 3\,\mu),\ p[1] == \frac{\mu}{2},\ p[2] == \frac{\mu}{2},$$
$$p[3] == \frac{\mu}{2},\ c[1] == (s[1])^2,\ c[2] == (s[2])^2,\ c[3] == (s[3])^2,$$
$$\pi[1] == p[1] * d[1] - c[1],\ \pi[2] == p[2] * d[2] - c[2],\ \pi[3] == p[3] * d[3] - c[3]\Big\},$$
$$\{\pi[1],\ \pi[2],\ \pi[3],\ c[1],\ c[2],\ c[3]\},\ \{s[1],\ s[2],\ s[3],\ p[1],\ p[2],\ p[3]\}\Big]$$

$$\Big\{\Big\{\pi[1] \to \frac{-8\,\lambda^2 + 675\,\mu + 48\,\lambda\,\mu - 72\,\mu^2}{4050},$$
$$\pi[2] \to \frac{-8\,\lambda^2 + 675\,\mu + 48\,\lambda\,\mu - 72\,\mu^2}{4050},\ \pi[3] \to \frac{-8\,\lambda^2 + 675\,\mu + 48\,\lambda\,\mu - 72\,\mu^2}{4050},$$
$$c[1] \to \frac{4\ (\lambda^2 - 6\,\lambda\,\mu + 9\,\mu^2)}{2025},\ c[2] \to \frac{4\ (\lambda^2 - 6\,\lambda\,\mu + 9\,\mu^2)}{2025},\ c[3] \to \frac{4\ (\lambda^2 - 6\,\lambda\,\mu + 9\,\mu^2)}{2025}\Big\}\Big\}$$

## AT THE EQUILIBRIUM

$$s[1] \to -\frac{2}{45}\ (\lambda - 3\,\mu),$$
$$s[2] \to -\frac{2}{45}\ (\lambda - 3\,\mu),\ s[3] \to -\frac{2}{45}\ (\lambda - 3\,\mu)$$

$$p[1] \to \frac{\mu}{2},$$
$$p[2] \to \frac{\mu}{2},$$
$$p[3] \to \frac{\mu}{2}$$

$$d[1] \to \frac{1}{3},$$
$$d[2] \to \frac{1}{3},$$
$$d[3] \to \frac{1}{3}$$

$$\pi[1] \to \frac{-8\,\lambda^2 + 675\,\mu + 48\,\lambda\,\mu - 72\,\mu^2}{4050},\ \pi[2] \to \frac{-8\,\lambda^2 + 675\,\mu + 48\,\lambda\,\mu - 72\,\mu^2}{4050},$$
$$\pi[3] \to \frac{-8\,\lambda^2 + 675\,\mu + 48\,\lambda\,\mu - 72\,\mu^2}{4050},$$
$$c[1] \to \frac{4\ (\lambda^2 - 6\,\lambda\,\mu + 9\,\mu^2)}{2025},$$
$$c[2] \to \frac{4\ (\lambda^2 - 6\,\lambda\,\mu + 9\,\mu^2)}{2025},\ c[3] \to \frac{4\ (\lambda^2 - 6\,\lambda\,\mu + 9\,\mu^2)}{2025}$$

## COMPUTATIONS FOR CASE 3

$$U[i] == V - \mu * (F[i]) - \lambda * L[i] - p[i]$$

$$F[i] == d[i] - s[i]$$

$$L[1] == \left(\frac{s[1]}{3} + \frac{s[2]}{3} - (s[1, 2] + s[2, 1]) - \frac{k}{2}\right) + \left(\frac{s[1]}{3} + \frac{s[3]}{3} - \frac{k}{2}\right)$$

$$L[2] == \left(\frac{s[1]}{3} + \frac{s[2]}{3} - (s[1,2] + s[2,1]) - \frac{k}{2}\right) + \left(\frac{s[2]}{3} + \frac{s[3]}{3} - \frac{k}{2}\right)$$

$$L[3] == \left(\frac{s[3]}{3} + \frac{s[2]}{3} - \frac{k}{2}\right) + \left(\frac{s[1]}{3} + \frac{s[3]}{3} - \frac{k}{2}\right)$$

$$c[1] == (s[1])^2 + (s[1,2])^2$$

$$c[2] == (s[2])^2 + (s[2,1])^2$$

$$c[3] == (s[3])^2$$

## FIND THE DEMANDS

$$\text{Solve}[\{-\mu * (d[1] - s[1]) - \lambda * L[1] - p[1] == -\mu * (d[2] - s[2]) - \lambda * L[2] - p[2],$$
$$-\mu * (d[1] - s[1]) - \lambda * L[1] - p[1] == -\mu * (d[3] - s[3]) - \lambda * L[3] - p[3],$$
$$d[1] + d[2] + d[3] == 1\}, \{d[1], d[2], d[3]\}]$$

$$\Big\{\Big\{d[2] \to -\frac{1}{3\mu} (-\mu - \lambda L[1] + 2\lambda L[2] - \lambda L[3] - p[1] + 2p[2] - p[3] + \mu s[1] - 2\mu s[2] + \mu s[3]),$$
$$d[1] \to -\frac{1}{3\mu} (-\mu + 2\lambda L[1] - \lambda L[2] - \lambda L[3] + 2p[1] - p[2] - p[3] - 2\mu s[1] + \mu s[2] + \mu s[3]),$$
$$d[3] \to -\frac{1}{3\mu} (-\mu - \lambda L[1] - \lambda L[2] + 2\lambda L[3] - p[1] - p[2] + 2p[3] + \mu s[1] + \mu s[2] - 2\mu s[3])\Big\}\Big\}$$

$$\pi[1] == p[1] * \left(-\frac{1}{3\mu} (-\mu + 2\lambda L[1] - \lambda L[2] - \lambda L[3] + 2p[1] - p[2] - p[3] - 2\mu s[1] + \mu s[2] + \mu s[3])\right) - c[1]$$

$$\pi[2] == p[2] * \left(-\frac{1}{3\mu} (-\mu - \lambda L[1] + 2\lambda L[2] - \lambda L[3] - p[1] + 2p[2] - p[3] + \mu s[1] - 2\mu s[2] + \mu s[3])\right) - c[2]$$

$$\pi[3] == p[3] * \left(-\frac{1}{3\mu} (-\mu - \lambda L[1] - \lambda L[2] + 2\lambda L[3] - p[1] - p[2] + 2p[3] + \mu s[1] + \mu s[2] - 2\mu s[3])\right) - c[3]$$

## SOLVE FOR THE LAST STAGE, FIND PRICES

$$D\Big[p[1] * \left(-\frac{1}{3\mu} (-\mu + 2\lambda L[1] - \lambda L[2] - \lambda L[3] + 2p[1] - p[2] - p[3] - 2\mu s[1] + \mu s[2] + \mu s[3])\right) - c[1], p[1]\Big]$$

$$-\frac{2p[1]}{3\mu} - \frac{1}{3\mu} (-\mu + 2\lambda L[1] - \lambda L[2] - \lambda L[3] + 2p[1] - p[2] - p[3] - 2\mu s[1] + \mu s[2] + \mu s[3])$$

$$D\Big[p[2] * \left(-\frac{1}{3\mu} (-\mu - \lambda L[1] + 2\lambda L[2] - \lambda L[3] - p[1] + 2p[2] - p[3] + \mu s[1] - 2\mu s[2] + \mu s[3])\right) - c[2], p[2]\Big]$$

$$-\frac{2p[2]}{3\mu} - \frac{1}{3\mu} (-\mu - \lambda L[1] + 2\lambda L[2] - \lambda L[3] - p[1] + 2p[2] - p[3] + \mu s[1] - 2\mu s[2] + \mu s[3])$$

$$\mathbf{D}\Big[\mathbf{p[3]*}\left(-\frac{1}{3\,\mu}\,(-\mu-\lambda\,\mathbf{L[1]}-\lambda\,\mathbf{L[2]}+2\,\lambda\,\mathbf{L[3]}-\mathbf{p[1]}-\mathbf{p[2]}+2\,\mathbf{p[3]}+\mu\,\mathbf{s[1]}+\mu\,\mathbf{s[2]}-2\,\mu\,\mathbf{s[3]})\right)-\mathbf{c[3]},\,\mathbf{p[3]}\Big]$$

$$-\frac{2\,p[3]}{3\,\mu}-\frac{1}{3\,\mu}\,(-\mu-\lambda\,L[1]-\lambda\,L[2]+2\,\lambda\,L[3]-p[1]-p[2]+2\,p[3]+\mu\,s[1]+\mu\,s[2]-2\,\mu\,s[3])$$

$$\mathbf{Solve}\Big[\Big\{-\frac{2\,\mathbf{p[3]}}{3\,\mu}-\frac{1}{3\,\mu}\,(-\mu-\lambda\,\mathbf{L[1]}-\lambda\,\mathbf{L[2]}+2\,\lambda\,\mathbf{L[3]}-\mathbf{p[1]}-\mathbf{p[2]}+2\,\mathbf{p[3]}+\mu\,\mathbf{s[1]}+\mu\,\mathbf{s[2]}-2\,\mu\,\mathbf{s[3]})==0,\,-\frac{2\,\mathbf{p[2]}}{3\,\mu}-\frac{1}{3\,\mu}\,(-\mu-\lambda\,\mathbf{L[1]}+2\,\lambda\,\mathbf{L[2]}-\lambda\,\mathbf{L[3]}-\mathbf{p[1]}+2\,\mathbf{p[2]}-\mathbf{p[3]}+\mu\,\mathbf{s[1]}-2\,\mu\,\mathbf{s[2]}+\mu\,\mathbf{s[3]})==0,\,-\frac{2\,\mathbf{p[1]}}{3\,\mu}-\frac{1}{3\,\mu}\,(-\mu+2\,\lambda\,\mathbf{L[1]}-\lambda\,\mathbf{L[2]}-\lambda\,\mathbf{L[3]}+2\,\mathbf{p[1]}-\mathbf{p[2]}-\mathbf{p[3]}-2\,\mu\,\mathbf{s[1]}+\mu\,\mathbf{s[2]}+\mu\,\mathbf{s[3]})==0\Big\},\,\{\mathbf{p[1]},\,\mathbf{p[2]},\,\mathbf{p[3]}\}\Big]$$

$$\Big\{\Big\{p[1]\to\frac{1}{10}\,(5\,\mu-4\,\lambda\,L[1]+2\,\lambda\,L[2]+2\,\lambda\,L[3]+4\,\mu\,s[1]-2\,\mu\,s[2]-2\,\mu\,s[3]),$$
$$p[2]\to\frac{1}{10}\,(5\,\mu+2\,\lambda\,L[1]-4\,\lambda\,L[2]+2\,\lambda\,L[3]-2\,\mu\,s[1]+4\,\mu\,s[2]-2\,\mu\,s[3]),$$
$$p[3]\to\frac{1}{10}\,(5\,\mu+2\,\lambda\,L[1]+2\,\lambda\,L[2]-4\,\lambda\,L[3]-2\,\mu\,s[1]-2\,\mu\,s[2]+4\,\mu\,s[3])\Big\}\Big\}$$

```
Simplify[
 Solve[{π[1] == p[1] * (- 1/(3 μ) (-μ + 2 λ L[1] - λ L[2] - λ L[3] + 2 p[1] - p[2] - p[3] - 2 μ s[1] +
          μ s[2] + μ s[3])) - c[1], π[2] == p[2] *
       (- 1/(3 μ) (-μ - λ L[1] + 2 λ L[2] - λ L[3] - p[1] + 2 p[2] - p[3] + μ s[1] - 2 μ s[2] + μ s[3])) -
     c[2], π[3] == p[3] * (- 1/(3 μ) (-μ - λ L[1] - λ L[2] + 2 λ L[3] - p[1] -
          p[2] + 2 p[3] + μ s[1] + μ s[2] - 2 μ s[3])) - c[3],
    L[1] == (s[1]/3 + s[2]/3 - (s[1, 2] + s[2, 1]) - k/2) + (s[1]/3 + s[3]/3 - k/2),
    L[2] == (s[1]/3 + s[2]/3 - (s[1, 2] + s[2, 1]) - k/2) + (s[2]/3 + s[3]/3 - k/2),
    L[3] == (s[3]/3 + s[2]/3 - k/2) + (s[1]/3 + s[3]/3 - k/2),
    c[1] == (s[1])^2 + (s[1, 2])^2,
    c[2] == (s[2])^2 + (s[2, 1])^2, c[3] == (s[3])^2,
    p[1] == 1/10 (5 μ - 4 λ L[1] + 2 λ L[2] + 2 λ L[3] + 4 μ s[1] - 2 μ s[2] - 2 μ s[3]),
    p[2] == 1/10 (5 μ + 2 λ L[1] - 4 λ L[2] + 2 λ L[3] - 2 μ s[1] + 4 μ s[2] - 2 μ s[3]),
    p[3] == 1/10 (5 μ + 2 λ L[1] + 2 λ L[2] - 4 λ L[3] - 2 μ s[1] - 2 μ s[2] + 4 μ s[3])},
  {π[1], π[2], π[3]}, {p[1], p[2], p[3], L[1], L[2], L[3], c[1], c[2], c[3]}]]
```

```
{{π[1] → 1/(1350 μ) (9 μ^2 (5 + 4 s[1] - 2 s[2] - 2 s[3])^2 - 6 μ (225 (s[1]^2 + s[1, 2]^2) +
          2 λ (5 + 4 s[1] - 2 s[2] - 2 s[3]) (2 s[1] - s[2] - s[3] - 3 s[1, 2] - 3 s[2, 1])) +
       4 λ^2 (-2 s[1] + s[2] + s[3] + 3 s[1, 2] + 3 s[2, 1])^2), π[2] →
    1/(1350 μ) (9 μ^2 (5 - 2 s[1] + 4 s[2] - 2 s[3])^2 + 4 λ^2 (s[1] - 2 s[2] + s[3] + 3 s[1, 2] + 3 s[2, 1])^2 -
       6 μ (2 λ (-5 + 2 s[1] - 4 s[2] + 2 s[3]) (s[1] - 2 s[2] + s[3] + 3 s[1, 2] + 3 s[2, 1]) +
          225 (s[2]^2 + s[2, 1]^2))), π[3] → 1/(1350 μ) (9 μ^2 (5 - 2 s[1] - 2 s[2] + 4 s[3])^2 +
       4 λ^2 (s[1] + s[2] - 2 (s[3] + 3 (s[1, 2] + s[2, 1])))^2 - 6 μ (225 s[3]^2 +
          2 λ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) (s[1] + s[2] - 2 (s[3] + 3 (s[1, 2] + s[2, 1])))))}}
```

## SOLVE FOR LINK CAPACITIES

```
D[1/(1350 μ) (9 μ^2 (5 + 4 s[1] - 2 s[2] - 2 s[3])^2 - 6 μ (225 (s[1]^2 + s[1, 2]^2) +
        2 λ (5 + 4 s[1] - 2 s[2] - 2 s[3]) (2 s[1] - s[2] - s[3] - 3 s[1, 2] - 3 s[2, 1])) +
     4 λ^2 (-2 s[1] + s[2] + s[3] + 3 s[1, 2] + 3 s[2, 1])^2), s[1, 2]]
```

```
1/(1350 μ) (-6 μ (-6 λ (5 + 4 s[1] - 2 s[2] - 2 s[3]) + 450 s[1, 2]) +
   24 λ^2 (-2 s[1] + s[2] + s[3] + 3 s[1, 2] + 3 s[2, 1]))
```

```
D[1/(1350 μ) (9 μ^2 (5 - 2 s[1] + 4 s[2] - 2 s[3])^2 + 4 λ^2 (s[1] - 2 s[2] + s[3] + 3 s[1, 2] + 3 s[2, 1])^2 -
    6 μ (2 λ (-5 + 2 s[1] - 4 s[2] + 2 s[3]) (s[1] - 2 s[2] + s[3] + 3 s[1, 2] + 3 s[2, 1]) +
      225 (s[2]^2 + s[2, 1]^2))), s[2, 1]]
```

$$\frac{1}{1350\,\mu}\left(24\,\lambda^2\,(s[1]-2\,s[2]+s[3]+3\,s[1,2]+3\,s[2,1])-6\,\mu\,(6\,\lambda\,(-5+2\,s[1]-4\,s[2]+2\,s[3])+450\,s[2,1])\right)$$

```
Solve[{1/(1350 μ) (24 λ^2 (s[1] - 2 s[2] + s[3] + 3 s[1, 2] + 3 s[2, 1]) -
      6 μ (6 λ (-5 + 2 s[1] - 4 s[2] + 2 s[3]) + 450 s[2, 1])) == 0,
   1/(1350 μ) (-6 μ (-6 λ (5 + 4 s[1] - 2 s[2] - 2 s[3]) + 450 s[1, 2]) +
      24 λ^2 (-2 s[1] + s[2] + s[3] + 3 s[1, 2] + 3 s[2, 1])) == 0}, {s[1, 2], s[2, 1]}]
```

```
{{s[1, 2] →
   -(λ (375 μ^2 + 4 λ^3 s[1] - 100 λ μ s[1] - 12 λ^2 μ s[1] + 300 μ^2 s[1] - 4 λ^3 s[2] + 50 λ μ s[2] +
        12 λ^2 μ s[2] - 150 μ^2 s[2] + 50 λ μ s[3] - 150 μ^2 s[3])) / (75 (4 λ^2 - 75 μ) μ), s[2, 1] →
   -(375 λ μ^2 - 4 λ^4 s[1] + 50 λ^2 μ s[1] + 12 λ^3 μ s[1] - 150 λ μ^2 s[1] + 4 λ^4 s[2] - 100 λ^2 μ s[2] -
      12 λ^3 μ s[2] + 300 λ μ^2 s[2] + 50 λ^2 μ s[3] - 150 λ μ^2 s[3]) / (75 (4 λ^2 - 75 μ) μ)}}
```

```
Simplify[
 Solve[{π[1] == 1/(1350 μ) (9 μ^2 (5 + 4 s[1] - 2 s[2] - 2 s[3])^2 - 6 μ (225 (s[1]^2 + s[1, 2]^2) + 2 λ
        (5 + 4 s[1] - 2 s[2] - 2 s[3]) (2 s[1] - s[2] - s[3] - 3 s[1, 2] - 3 s[2, 1])) +
      4 λ^2 (-2 s[1] + s[2] + s[3] + 3 s[1, 2] + 3 s[2, 1])^2), π[2] == 1/(1350 μ)
     (9 μ^2 (5 - 2 s[1] + 4 s[2] - 2 s[3])^2 + 4 λ^2 (s[1] - 2 s[2] + s[3] + 3 s[1, 2] + 3 s[2, 1])^2 -
      6 μ (2 λ (-5 + 2 s[1] - 4 s[2] + 2 s[3]) (s[1] - 2 s[2] + s[3] + 3 s[1, 2] + 3 s[2, 1]) +
        225 (s[2]^2 + s[2, 1]^2))), π[3] == 1/(1350 μ) (9 μ^2 (5 - 2 s[1] - 2 s[2] + 4 s[3])^2 +
      4 λ^2 (s[1] + s[2] - 2 (s[3] + 3 (s[1, 2] + s[2, 1])))^2 - 6 μ (225 s[3]^2 +
        2 λ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) (s[1] + s[2] - 2 (s[3] + 3 (s[1, 2] + s[2, 1]))))),
   s[1, 2] == -(λ (375 μ^2 + 4 λ^3 s[1] - 100 λ μ s[1] - 12 λ^2 μ s[1] + 300 μ^2 s[1] - 4 λ^3 s[2] + 50
          λ μ s[2] + 12 λ^2 μ s[2] - 150 μ^2 s[2] + 50 λ μ s[3] - 150 μ^2 s[3])) / (75 (4 λ^2 - 75 μ) μ),
   s[2, 1] == -(375 λ μ^2 - 4 λ^4 s[1] + 50 λ^2 μ s[1] + 12 λ^3 μ s[1] - 150 λ μ^2 s[1] + 4 λ^4 s[2] -
       100 λ^2 μ s[2] - 12 λ^3 μ s[2] + 300 λ μ^2 s[2] + 50 λ^2 μ s[3] - 150 λ μ^2 s[3]) /
     (75 (4 λ^2 - 75 μ) μ)}, {π[1], π[2], π[3]}, {s[1, 2], s[2, 1]}]]
```

```
{{π[1] → 1/(11250 (4 λ^2 - 75 μ)^2 μ^2) (-32 λ^8 (s[1] - s[2])^2 +
      192 λ^7 μ (s[1] - s[2])^2 + 421875 μ^4 (-150 s[1]^2 + μ (5 + 4 s[1] - 2 s[2] - 2 s[3])^2) -
      1200 λ^5 μ^2 (s[1] - s[2]) (5 + 14 s[1] - 10 s[2] - 4 s[3]) -
      16 λ^6 μ (s[1] - s[2]) ((-175 + 18 μ) s[1] + (125 - 18 μ) s[2] + 50 s[3]) +
      15000 λ^3 μ^3 (32 s[1]^2 + 14 s[2]^2 + s[1] (25 - 44 s[2] - 20 s[3]) +
         s[3] (-5 + 2 s[3]) + 4 s[2] (-5 + 4 s[3])) - 562500 λ μ^4
       (8 s[1]^2 + 2 s[2]^2 + s[3] (-5 + 2 s[3]) + s[2] (-5 + 4 s[3]) - 2 s[1] (-5 + 4 s[2] + 4 s[3])) +
      200 λ^4 μ^2 (18 μ (s[1] - s[2]) (5 + 7 s[1] - 5 s[2] - 2 s[3]) -
         25 (52 s[1]^2 + 7 s[2]^2 + 8 s[2] s[3] + s[3]^2 - 2 s[1] (11 s[2] + 5 s[3]))) -
      3750 λ^2 μ^3 (-50 (40 s[1]^2 - 4 s[1] (s[2] + s[3]) + (s[2] + s[3])^2) + 3 μ (64 s[1]^2 +
            28 s[2]^2 + (5 - 2 s[3])^2 + 16 s[2] (-5 + 2 s[3]) - 4 s[1] (-25 + 22 s[2] + 10 s[3])))),
  π[2] → 1/(11250 (4 λ^2 - 75 μ)^2 μ^2) (-32 λ^8 (s[1] - s[2])^2 + 192 λ^7 μ (s[1] - s[2])^2 +
      421875 μ^4 (-150 s[2]^2 + μ (5 - 2 s[1] + 4 s[2] - 2 s[3])^2) -
      16 λ^6 μ (s[1] - s[2]) ((-125 + 18 μ) s[1] + (175 - 18 μ) s[2] - 50 s[3]) -
      1200 λ^5 μ^2 (s[1] - s[2]) (-5 + 10 s[1] - 14 s[2] + 4 s[3]) + 15000 λ^3 μ^3 (14 s[1]^2 + 32 s[2]^2 -
         4 s[1] (5 + 11 s[2] - 4 s[3]) + s[3] (-5 + 2 s[3]) - 5 s[2] (-5 + 4 s[3])) - 562500 λ μ^4
       (2 s[1]^2 + 8 s[2]^2 + s[2] (10 - 8 s[3]) + s[3] (-5 + 2 s[3]) + s[1] (-5 - 8 s[2] + 4 s[3])) -
      3750 λ^2 μ^3 (3 μ (28 s[1]^2 + 64 s[2]^2 - 8 s[1] (10 + 11 s[2] - 4 s[3]) +
            (5 - 2 s[3])^2 - 20 s[2] (-5 + 2 s[3])) -
         50 (s[1]^2 + 40 s[2]^2 - 4 s[2] s[3] + s[3]^2 + s[1] (-4 s[2] + 2 s[3]))) +
      200 λ^4 μ^2 (18 μ (s[1] - s[2]) (-5 + 5 s[1] - 7 s[2] + 2 s[3]) -
         25 (7 s[1]^2 + 52 s[2]^2 - 10 s[2] s[3] + s[3]^2 + s[1] (-22 s[2] + 8 s[3])))),
  π[3] → 1/(6 (4 λ^2 - 75 μ)^2) (-240 λ^3 μ (s[1] + s[2] - 2 s[3]) + 48 λ^4 (3 μ - 2 s[3]^2) +
      225 μ^2 (-150 s[3]^2 + μ (5 - 2 s[1] - 2 s[2] + 4 s[3])^2) - 300 λ μ^2
       (2 s[1]^2 + 2 s[2]^2 + s[1] (-5 + 4 s[2] - 8 s[3]) + 2 s[3] (5 + 4 s[3]) - s[2] (5 + 8 s[3])) +
      20 λ^2 μ (18 μ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) +
         5 (s[1]^2 + s[2]^2 + 2 s[1] (s[2] - 2 s[3]) - 4 s[2] s[3] + 40 s[3]^2)))}}
```

```
D[(-32 λ^8 (s[1] - s[2])^2 + 192 λ^7 μ (s[1] - s[2])^2 +
    421875 μ^4 (-150 s[1]^2 + μ (5 + 4 s[1] - 2 s[2] - 2 s[3])^2) -
    1200 λ^5 μ^2 (s[1] - s[2]) (5 + 14 s[1] - 10 s[2] - 4 s[3]) -
    16 λ^6 μ (s[1] - s[2]) ((-175 + 18 μ) s[1] + (125 - 18 μ) s[2] + 50 s[3]) +
    15000 λ^3 μ^3 (32 s[1]^2 + 14 s[2]^2 + s[1] (25 - 44 s[2] - 20 s[3]) +
       s[3] (-5 + 2 s[3]) + 4 s[2] (-5 + 4 s[3])) - 562500 λ μ^4
     (8 s[1]^2 + 2 s[2]^2 + s[3] (-5 + 2 s[3]) + s[2] (-5 + 4 s[3]) - 2 s[1] (-5 + 4 s[2] + 4 s[3])) +
    200 λ^4 μ^2 (18 μ (s[1] - s[2]) (5 + 7 s[1] - 5 s[2] - 2 s[3]) -
       25 (52 s[1]^2 + 7 s[2]^2 + 8 s[2] s[3] + s[3]^2 - 2 s[1] (11 s[2] + 5 s[3]))) -
    3750 λ^2 μ^3 (-50 (40 s[1]^2 - 4 s[1] (s[2] + s[3]) + (s[2] + s[3])^2) +
       3 μ (64 s[1]^2 + 28 s[2]^2 + (5 - 2 s[3])^2 + 16 s[2] (-5 + 2 s[3]) -
          4 s[1] (-25 + 22 s[2] + 10 s[3])))) / (11250 (4 λ^2 - 75 μ)^2 μ^2), s[1]]
```

```
(-64 λ^8 (s[1] - s[2]) + 384 λ^7 μ (s[1] - s[2]) - 16800 λ^5 μ^2 (s[1] - s[2]) -
   16 λ^6 μ (-175 + 18 μ) (s[1] - s[2]) + 421875 μ^4 (-300 s[1] + 8 μ (5 + 4 s[1] - 2 s[2] - 2 s[3])) +
   15000 λ^3 μ^3 (25 + 64 s[1] - 44 s[2] - 20 s[3]) - 1200 λ^5 μ^2 (5 + 14 s[1] - 10 s[2] - 4 s[3]) -
   16 λ^6 μ ((-175 + 18 μ) s[1] + (125 - 18 μ) s[2] + 50 s[3]) -
   562500 λ μ^4 (16 s[1] - 2 (-5 + 4 s[2] + 4 s[3])) + 200 λ^4 μ^2 (126 μ (s[1] - s[2]) +
      18 μ (5 + 7 s[1] - 5 s[2] - 2 s[3]) - 25 (104 s[1] - 2 (11 s[2] + 5 s[3]))) -
   3750 λ^2 μ^3 (-50 (80 s[1] - 4 (s[2] + s[3])) + 3 μ (128 s[1] - 4 (-25 + 22 s[2] + 10 s[3])))) /
 (11250 (4 λ^2 - 75 μ)^2 μ^2)
```

```
D[(-32 λ^8 (s[1] - s[2])^2 + 192 λ^7 μ (s[1] - s[2])^2 +
    421875 μ^4 (-150 s[2]^2 + μ (5 - 2 s[1] + 4 s[2] - 2 s[3])^2) -
    16 λ^6 μ (s[1] - s[2]) ((-125 + 18 μ) s[1] + (175 - 18 μ) s[2] - 50 s[3]) -
    1200 λ^5 μ^2 (s[1] - s[2]) (-5 + 10 s[1] - 14 s[2] + 4 s[3]) + 15000 λ^3 μ^3 (14 s[1]^2 + 32 s[2]^2 -
       4 s[1] (5 + 11 s[2] - 4 s[3]) + s[3] (-5 + 2 s[3]) - 5 s[2] (-5 + 4 s[3])) - 562500 λ μ^4
     (2 s[1]^2 + 8 s[2]^2 + s[2] (10 - 8 s[3]) + s[3] (-5 + 2 s[3]) + s[1] (-5 - 8 s[2] + 4 s[3])) -
    3750 λ^2 μ^3 (3 μ (28 s[1]^2 + 64 s[2]^2 - 8 s[1] (10 + 11 s[2] - 4 s[3]) + (5 - 2 s[3])^2 - 20 s[2]
          (-5 + 2 s[3])) - 50 (s[1]^2 + 40 s[2]^2 - 4 s[2] s[3] + s[3]^2 + s[1] (-4 s[2] + 2 s[3]))) +
    200 λ^4 μ^2 (18 μ (s[1] - s[2]) (-5 + 5 s[1] - 7 s[2] + 2 s[3]) - 25 (7 s[1]^2 + 52 s[2]^2 -
          10 s[2] s[3] + s[3]^2 + s[1] (-22 s[2] + 8 s[3])))) / (11250 (4 λ^2 - 75 μ)^2 μ^2), s[2]]
```

```
(64 λ^8 (s[1] - s[2]) - 384 λ^7 μ (s[1] - s[2]) - 16 λ^6 (175 - 18 μ) μ (s[1] - s[2]) +
   16800 λ^5 μ^2 (s[1] - s[2]) + 421875 μ^4 (-300 s[2] + 8 μ (5 - 2 s[1] + 4 s[2] - 2 s[3])) +
   16 λ^6 μ ((-125 + 18 μ) s[1] + (175 - 18 μ) s[2] - 50 s[3]) -
   562500 λ μ^4 (10 - 8 s[1] + 16 s[2] - 8 s[3]) + 1200 λ^5 μ^2 (-5 + 10 s[1] - 14 s[2] + 4 s[3]) +
   200 λ^4 μ^2 (-126 μ (s[1] - s[2]) - 25 (-22 s[1] + 104 s[2] - 10 s[3]) -
      18 μ (-5 + 5 s[1] - 7 s[2] + 2 s[3])) + 15000 λ^3 μ^3 (-44 s[1] + 64 s[2] - 5 (-5 + 4 s[3])) -
   3750 λ^2 μ^3 (-50 (-4 s[1] + 80 s[2] - 4 s[3]) + 3 μ (-88 s[1] + 128 s[2] - 20 (-5 + 2 s[3])))) /
 (11250 (4 λ^2 - 75 μ)^2 μ^2)
```

```
D[1/(6 (4 λ^2 - 75 μ)^2) (-240 λ^3 μ (s[1] + s[2] - 2 s[3]) +
    48 λ^4 (3 μ - 2 s[3]^2) + 225 μ^2 (-150 s[3]^2 + μ (5 - 2 s[1] - 2 s[2] + 4 s[3])^2) - 300 λ μ^2
     (2 s[1]^2 + 2 s[2]^2 + s[1] (-5 + 4 s[2] - 8 s[3]) + 2 s[3] (5 + 4 s[3]) - s[2] (5 + 8 s[3])) +
    20 λ^2 μ (18 μ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) +
       5 (s[1]^2 + s[2]^2 + 2 s[1] (s[2] - 2 s[3]) - 4 s[2] s[3] + 40 s[3]^2))), s[3]]
```

```
1/(6 (4 λ^2 - 75 μ)^2) (480 λ^3 μ - 192 λ^4 s[3] - 300 λ μ^2 (-8 s[1] - 8 s[2] + 8 s[3] + 2 (5 + 4 s[3])) +
   225 μ^2 (-300 s[3] + 8 μ (5 - 2 s[1] - 2 s[2] + 4 s[3])) +
   20 λ^2 μ (-72 μ + 5 (-4 s[1] - 4 s[2] + 80 s[3])))
```

```
Simplify[Solve[
  {1/(6 (4 λ^2 - 75 μ)^2) (480 λ^3 μ - 192 λ^4 s[3] - 300 λ μ^2 (-8 s[1] - 8 s[2] + 8 s[3] + 2 (5 + 4 s[3])) +
       225 μ^2 (-300 s[3] + 8 μ (5 - 2 s[1] - 2 s[2] + 4 s[3])) +
       20 λ^2 μ (-72 μ + 5 (-4 s[1] - 4 s[2] + 80 s[3]))) == 0,
   (64 λ^8 (s[1] - s[2]) - 384 λ^7 μ (s[1] - s[2]) - 16 λ^6 (175 - 18 μ) μ (s[1] - s[2]) +
       16800 λ^5 μ^2 (s[1] - s[2]) + 421875 μ^4 (-300 s[2] + 8 μ (5 - 2 s[1] + 4 s[2] - 2 s[3])) +
       16 λ^6 μ ((-125 + 18 μ) s[1] + (175 - 18 μ) s[2] - 50 s[3]) -
       562500 λ μ^4 (10 - 8 s[1] + 16 s[2] - 8 s[3]) + 1200 λ^5 μ^2 (-5 + 10 s[1] - 14 s[2] + 4 s[3]) +
       200 λ^4 μ^2 (-126 μ (s[1] - s[2]) - 25 (-22 s[1] + 104 s[2] - 10 s[3]) -
          18 μ (-5 + 5 s[1] - 7 s[2] + 2 s[3])) + 15000 λ^3 μ^3 (-44 s[1] + 64 s[2] - 5 (-5 + 4 s[3])) -
       3750 λ^2 μ^3 (-50 (-4 s[1] + 80 s[2] - 4 s[3]) + 3 μ (-88 s[1] + 128 s[2] - 20 (-5 + 2 s[3])))) /
     (11250 (4 λ^2 - 75 μ)^2 μ^2) == 0, (-64 λ^8 (s[1] - s[2]) + 384 λ^7 μ (s[1] - s[2]) -
       16800 λ^5 μ^2 (s[1] - s[2]) - 16 λ^6 μ (-175 + 18 μ) (s[1] - s[2]) +
       421875 μ^4 (-300 s[1] + 8 μ (5 + 4 s[1] - 2 s[2] - 2 s[3])) +
       15000 λ^3 μ^3 (25 + 64 s[1] - 44 s[2] - 20 s[3]) - 1200 λ^5 μ^2 (5 + 14 s[1] - 10 s[2] - 4 s[3]) -
       16 λ^6 μ ((-175 + 18 μ) s[1] + (125 - 18 μ) s[2] + 50 s[3]) -
       562500 λ μ^4 (16 s[1] - 2 (-5 + 4 s[2] + 4 s[3])) + 200 λ^4 μ^2 (126 μ (s[1] - s[2]) +
          18 μ (5 + 7 s[1] - 5 s[2] - 2 s[3]) - 25 (104 s[1] - 2 (11 s[2] + 5 s[3]))) -
       3750 λ^2 μ^3 (-50 (80 s[1] - 4 (s[2] + s[3])) + 3 μ (128 s[1] - 4 (-25 + 22 s[2] + 10 s[3])))) /
     (11250 (4 λ^2 - 75 μ)^2 μ^2) == 0}, {s[1], s[2], s[3]}]]
```

```
{{s[1] → (10 (λ - 3 μ) μ (32 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 +
        16875 μ^3 (-25 + 4 μ) - 375 λ^2 μ^2 (-155 + 12 μ) + 2 λ^4 μ (-1225 + 36 μ))) /
     ((4 λ^2 - 75 μ) (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 +
        50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))),
   s[2] → (10 (λ - 3 μ) μ (32 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) -
        375 λ^2 μ^2 (-155 + 12 μ) + 2 λ^4 μ (-1225 + 36 μ))) /
     ((4 λ^2 - 75 μ) (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 +
        50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))),
   s[3] → (10 (λ - 3 μ) μ (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 - 1500 λ^2 μ^2 (-65 + 3 μ) +
        16875 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))) /
     ((4 λ^2 - 75 μ) (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 +
        50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ)))}}
```

```
Simplify[
 Solve[{s[1, 2] == -(λ (375 μ^2 + 4 λ^3 s[1] - 100 λ μ s[1] - 12 λ^2 μ s[1] + 300 μ^2 s[1] - 4 λ^3 s[2] + 50
          λ μ s[2] + 12 λ^2 μ s[2] - 150 μ^2 s[2] + 50 λ μ s[3] - 150 μ^2 s[3])) / (75 (4 λ^2 - 75 μ) μ),
   s[2, 1] == -(375 λ μ^2 - 4 λ^4 s[1] + 50 λ^2 μ s[1] + 12 λ^3 μ s[1] - 150 λ μ^2 s[1] + 4 λ^4 s[2] - 100
        λ^2 μ s[2] - 12 λ^3 μ s[2] + 300 λ μ^2 s[2] + 50 λ^2 μ s[3] - 150 λ μ^2 s[3]) / (75 (4 λ^2 - 75 μ) μ),
   s[3] == (10 (λ - 3 μ) μ (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 - 1500 λ^2 μ^2 (-65 + 3 μ) +
        16875 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))) /
     ((4 λ^2 - 75 μ) (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 +
        50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))),
   s[2] == (10 (λ - 3 μ) μ (32 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) -
        375 λ^2 μ^2 (-155 + 12 μ) + 2 λ^4 μ (-1225 + 36 μ))) /
     ((4 λ^2 - 75 μ) (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 +
        50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))),
   s[1] == (10 (λ - 3 μ) μ (32 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) -
        375 λ^2 μ^2 (-155 + 12 μ) + 2 λ^4 μ (-1225 + 36 μ))) /
     ((4 λ^2 - 75 μ) (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625
         μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ)))}, {s[1, 2], s[2, 1]}, {s[1], s[2], s[3]}]]
```

```
{{s[1, 2] → (375 λ μ^2 (16 λ^2 - 24 λ μ + 9 μ (-25 + 4 μ))) / (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
      4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ)),
  s[2, 1] → (375 λ μ^2 (16 λ^2 - 24 λ μ + 9 μ (-25 + 4 μ))) / (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
      4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))}}
```

```
Simplify[Solve[{p[1] == 1/10 (5 μ - 4 λ L[1] + 2 λ L[2] + 2 λ L[3] + 4 μ s[1] - 2 μ s[2] - 2 μ s[3]),
   p[2] == 1/10 (5 μ + 2 λ L[1] - 4 λ L[2] + 2 λ L[3] - 2 μ s[1] + 4 μ s[2] - 2 μ s[3]),
   p[3] == 1/10 (5 μ + 2 λ L[1] + 2 λ L[2] - 4 λ L[3] - 2 μ s[1] - 2 μ s[2] + 4 μ s[3]),
   L[1] == (s[1]/3 + s[2]/3 - (s[1, 2] + s[2, 1]) - k/2) + (s[1]/3 + s[3]/3 - k/2),
   L[2] == (s[1]/3 + s[2]/3 - (s[1, 2] + s[2, 1]) - k/2) + (s[2]/3 + s[3]/3 - k/2),
   L[3] == (s[3]/3 + s[2]/3 - k/2) + (s[1]/3 + s[3]/3 - k/2),
   s[1] == (10 (λ - 3 μ) μ (32 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) -
         375 λ^2 μ^2 (-155 + 12 μ) + 2 λ^4 μ (-1225 + 36 μ))) / ((4 λ^2 - 75 μ) (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
         4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))),
   s[2] == (10 (λ - 3 μ) μ (32 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) -
         375 λ^2 μ^2 (-155 + 12 μ) + 2 λ^4 μ (-1225 + 36 μ))) / ((4 λ^2 - 75 μ) (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
         4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))),
   s[3] == (10 (λ - 3 μ) μ (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 - 1500 λ^2 μ^2 (-65 + 3 μ) +
         16875 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))) / ((4 λ^2 - 75 μ) (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
         4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))),
   s[1, 2] == (375 λ μ^2 (16 λ^2 - 24 λ μ + 9 μ (-25 + 4 μ))) / (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
       4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ)),
   s[2, 1] == (375 λ μ^2 (16 λ^2 - 24 λ μ + 9 μ (-25 + 4 μ))) / (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
       4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))},
  {p[1], p[2], p[3]}, {s[1], s[2], s[3], s[1, 2], s[2, 1], L[1], L[2], L[3]}]]
```

```
{{p[1] →
    (5625 μ^3 (16 λ^2 - 24 λ μ + 9 μ (-25 + 4 μ))) / (2 (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 4500 λ^2 (-35 + μ) μ^2 -
         135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))),
  p[2] → (5625 μ^3 (16 λ^2 - 24 λ μ + 9 μ (-25 + 4 μ))) / (2 (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
         4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))),
  p[3] → (3 μ (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 - 1500 λ^2 μ^2 (-65 + 3 μ) +
         16875 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))) / (2 (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
         4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ)))}}
```

```
Simplify[Solve[
  {d[2] == -1/(3 μ) (-μ - λ L[1] + 2 λ L[2] - λ L[3] - p[1] + 2 p[2] - p[3] + μ s[1] - 2 μ s[2] + μ s[3]),
   d[1] == -1/(3 μ) (-μ + 2 λ L[1] - λ L[2] - λ L[3] + 2 p[1] - p[2] - p[3] - 2 μ s[1] + μ s[2] + μ s[3]),
   d[3] == -1/(3 μ) (-μ - λ L[1] - λ L[2] + 2 λ L[3] - p[1] - p[2] + 2 p[3] + μ s[1] + μ s[2] - 2 μ s[3]),
   p[1] == 1/10 (5 μ - 4 λ L[1] + 2 λ L[2] + 2 λ L[3] + 4 μ s[1] - 2 μ s[2] - 2 μ s[3]),
   p[2] == 1/10 (5 μ + 2 λ L[1] - 4 λ L[2] + 2 λ L[3] - 2 μ s[1] + 4 μ s[2] - 2 μ s[3]),
   p[3] == 1/10 (5 μ + 2 λ L[1] + 2 λ L[2] - 4 λ L[3] - 2 μ s[1] - 2 μ s[2] + 4 μ s[3]),
   L[1] == (s[1]/3 + s[2]/3 - (s[1, 2] + s[2, 1]) - k/2) + (s[1]/3 + s[3]/3 - k/2),
   L[2] == (s[1]/3 + s[2]/3 - (s[1, 2] + s[2, 1]) - k/2) + (s[2]/3 + s[3]/3 - k/2),
   L[3] == (s[3]/3 + s[2]/3 - k/2) + (s[1]/3 + s[3]/3 - k/2),
   s[1] == (10 (λ - 3 μ) μ (32 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) -
        375 λ^2 μ^2 (-155 + 12 μ) + 2 λ^4 μ (-1225 + 36 μ))) / ((4 λ^2 - 75 μ) (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
        4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))),
   s[2] == (10 (λ - 3 μ) μ (32 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) -
        375 λ^2 μ^2 (-155 + 12 μ) + 2 λ^4 μ (-1225 + 36 μ))) / ((4 λ^2 - 75 μ) (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
        4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))),
   s[3] == (10 (λ - 3 μ) μ (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 - 1500 λ^2 μ^2 (-65 + 3 μ) +
        16875 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))) / ((4 λ^2 - 75 μ) (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
        4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))),
   s[1, 2] == (375 λ μ^2 (16 λ^2 - 24 λ μ + 9 μ (-25 + 4 μ))) / (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
      4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ)),
   s[2, 1] == (375 λ μ^2 (16 λ^2 - 24 λ μ + 9 μ (-25 + 4 μ))) / (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
      4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))},
  {d[1], d[2], d[3]}, {s[1], s[2], s[3], s[1, 2], s[2, 1], L[1], L[2],
   L[3], p[1], p[2], p[3]}]]

{{d[1] → (1875 μ^2 (16 λ^2 - 24 λ μ + 9 μ (-25 + 4 μ))) / (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
      4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ)),
  d[2] → (1875 μ^2 (16 λ^2 - 24 λ μ + 9 μ (-25 + 4 μ))) / (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
      4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ)),
  d[3] → (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 - 1500 λ^2 μ^2 (-65 + 3 μ) +
      16875 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ)) / (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
      4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))}}
```

```
Simplify[
 Solve[{d[1] == (1875 μ^2 (16 λ^2 - 24 λ μ + 9 μ (-25 + 4 μ))) / (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 4500
       λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ)),
   d[2] == (1875 μ^2 (16 λ^2 - 24 λ μ + 9 μ (-25 + 4 μ))) / (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
       4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ)),
   d[3] == (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 - 1500 λ^2 μ^2 (-65 + 3 μ) +
       16875 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ)) / (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
       4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ)),
   p[1] == (5625 μ^3 (16 λ^2 - 24 λ μ + 9 μ (-25 + 4 μ))) / (2 (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
         4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))),
   p[2] == (5625 μ^3 (16 λ^2 - 24 λ μ + 9 μ (-25 + 4 μ))) / (2 (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
         4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))),
   p[3] == (3 μ (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 - 1500 λ^2 μ^2 (-65 + 3 μ) +
         16875 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))) / (2 (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
         4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))),
   s[1] == (10 (λ - 3 μ) μ (32 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) -
         375 λ^2 μ^2 (-155 + 12 μ) + 2 λ^4 μ (-1225 + 36 μ))) / ((4 λ^2 - 75 μ) (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
         4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))),
   s[2] == (10 (λ - 3 μ) μ (32 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) -
         375 λ^2 μ^2 (-155 + 12 μ) + 2 λ^4 μ (-1225 + 36 μ))) / ((4 λ^2 - 75 μ) (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
         4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))),
   s[3] == (10 (λ - 3 μ) μ (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 - 1500 λ^2 μ^2 (-65 + 3 μ) +
         16875 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))) / ((4 λ^2 - 75 μ) (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
         4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))),
   s[1, 2] == (375 λ μ^2 (16 λ^2 - 24 λ μ + 9 μ (-25 + 4 μ))) / (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
       4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ)),
   s[2, 1] == (375 λ μ^2 (16 λ^2 - 24 λ μ + 9 μ (-25 + 4 μ))) / (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
       4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 + 50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ)),
   π[1] == p[1] * d[1] - c[1], π[2] == p[2] * d[2] - c[2], π[3] == p[3] * d[3] - c[3],
   c[1] == (s[1])^2 + (s[1, 2])^2,
   c[2] == (s[2])^2 + (s[2, 1])^2,
   c[3] == (s[3])^2},
  {π[1], π[2], π[3], c[1], c[2], c[3]},
  {s[1], s[2], s[3], s[1, 2], s[2, 1],
   p[1], p[2], p[3], d[1], d[2], d[3]}]]
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$$\{\{\pi[1] \to -\left(25\,\mu^2\,(16\,\lambda^2 - 24\,\lambda\,\mu + 9\,\mu\,(-25 + 4\,\mu))^2\,(32\,\lambda^{10} - 192\,\lambda^9\,\mu + 24000\,\lambda^7\,\mu^2 - 1110000\,\lambda^5\,\mu^3 + 22500000\,\lambda^3\,\mu^4 - 168750000\,\lambda\,\mu^5 + 31640625\,\mu^5\,(-75 + 8\,\mu) + 32\,\lambda^8\,\mu\,(-125 + 9\,\mu) - 1406250\,\lambda^2\,\mu^4\,(-245 + 24\,\mu) - 1000\,\lambda^6\,\mu^2\,(-365 + 36\,\mu) + 15000\,\lambda^4\,\mu^3\,(-1150 + 111\,\mu))\right) / \left(2\,(4\,\lambda^2 - 75\,\mu)^2\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35 + \mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25 + 4\,\mu) + 4\,\lambda^4\,\mu\,(-1025 + 18\,\mu))^2\right),$$

$$\pi[2] \to -\left(25\,\mu^2\,(16\,\lambda^2 - 24\,\lambda\,\mu + 9\,\mu\,(-25 + 4\,\mu))^2\,(32\,\lambda^{10} - 192\,\lambda^9\,\mu + 24000\,\lambda^7\,\mu^2 - 1110000\,\lambda^5\,\mu^3 + 22500000\,\lambda^3\,\mu^4 - 168750000\,\lambda\,\mu^5 + 31640625\,\mu^5\,(-75 + 8\,\mu) + 32\,\lambda^8\,\mu\,(-125 + 9\,\mu) - 1406250\,\lambda^2\,\mu^4\,(-245 + 24\,\mu) - 1000\,\lambda^6\,\mu^2\,(-365 + 36\,\mu) + 15000\,\lambda^4\,\mu^3\,(-1150 + 111\,\mu))\right) / \left(2\,(4\,\lambda^2 - 75\,\mu)^2\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35 + \mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25 + 4\,\mu) + 4\,\lambda^4\,\mu\,(-1025 + 18\,\mu))^2\right),$$

$$\pi[3] \to \left(\mu\,(48\,\lambda^4 - 2000\,\lambda^2\,\mu + 1200\,\lambda\,\mu^2 - 225\,\mu^2\,(-75 + 8\,\mu))\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 45000\,\lambda\,\mu^3 - 1500\,\lambda^2\,\mu^2\,(-65 + 3\,\mu) + 16875\,\mu^3\,(-25 + 4\,\mu) + 4\,\lambda^4\,\mu\,(-1025 + 18\,\mu))^2\right) / \left(2\,(4\,\lambda^2 - 75\,\mu)^2\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35 + \mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25 + 4\,\mu) + 4\,\lambda^4\,\mu\,(-1025 + 18\,\mu))^2\right),$$

$$c[1] \to \left(25\,\mu^2\,(16\,\lambda^2 - 24\,\lambda\,\mu + 9\,\mu\,(-25 + 4\,\mu))^2\,(16\,\lambda^{10} - 96\,\lambda^9\,\mu + 12000\,\lambda^7\,\mu^2 - 555000\,\lambda^5\,\mu^3 + 11250000\,\lambda^3\,\mu^4 - 84375000\,\lambda\,\mu^5 + 126562500\,\mu^6 + 16\,\lambda^8\,\mu\,(-125 + 9\,\mu) - 703125\,\lambda^2\,\mu^4\,(-65 + 24\,\mu) - 500\,\lambda^6\,\mu^2\,(-365 + 36\,\mu) + 7500\,\lambda^4\,\mu^3\,(-700 + 111\,\mu))\right) / \left((4\,\lambda^2 - 75\,\mu)^2\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35 + \mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25 + 4\,\mu) + 4\,\lambda^4\,\mu\,(-1025 + 18\,\mu))^2\right),$$

$$c[2] \to \left(25\,\mu^2\,(16\,\lambda^2 - 24\,\lambda\,\mu + 9\,\mu\,(-25 + 4\,\mu))^2\,(16\,\lambda^{10} - 96\,\lambda^9\,\mu + 12000\,\lambda^7\,\mu^2 - 555000\,\lambda^5\,\mu^3 + 11250000\,\lambda^3\,\mu^4 - 84375000\,\lambda\,\mu^5 + 126562500\,\mu^6 + 16\,\lambda^8\,\mu\,(-125 + 9\,\mu) - 703125\,\lambda^2\,\mu^4\,(-65 + 24\,\mu) - 500\,\lambda^6\,\mu^2\,(-365 + 36\,\mu) + 7500\,\lambda^4\,\mu^3\,(-700 + 111\,\mu))\right) / \left((4\,\lambda^2 - 75\,\mu)^2\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35 + \mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25 + 4\,\mu) + 4\,\lambda^4\,\mu\,(-1025 + 18\,\mu))^2\right),$$

$$c[3] \to \left(100\,\mu^2\,(-8\,\lambda^7 + 72\,\lambda^6\,\mu + 4\,\lambda^5\,(1025 - 54\,\mu)\,\mu - 13500\,\lambda^2\,(-25 + \mu)\,\mu^3 + 50625\,\mu^4\,(-25 + 4\,\mu) + 36\,\lambda^4\,\mu^2\,(-425 + 6\,\mu) + 1500\,\lambda^3\,\mu^2\,(-65 + 9\,\mu) - 16875\,\lambda\,\mu^3\,(-25 + 12\,\mu))^2\right) / \left((4\,\lambda^2 - 75\,\mu)^2\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35 + \mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25 + 4\,\mu) + 4\,\lambda^4\,\mu\,(-1025 + 18\,\mu))^2\right)\}\}$$

## AT THE EQUILIBRIUM

$$\mathbf{s[1]} \to (10\,(\lambda - 3\,\mu)\,\mu\,(32\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 45000\,\lambda\,\mu^3 + 16875\,\mu^3\,(-25 + 4\,\mu) - 375\,\lambda^2\,\mu^2\,(-155 + 12\,\mu) + 2\,\lambda^4\,\mu\,(-1225 + 36\,\mu))) / ((4\,\lambda^2 - 75\,\mu)\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35 + \mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25 + 4\,\mu) + 4\,\lambda^4\,\mu\,(-1025 + 18\,\mu)))$$

$$\mathbf{s[2]} \to (10\,(\lambda - 3\,\mu)\,\mu\,(32\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 45000\,\lambda\,\mu^3 + 16875\,\mu^3\,(-25 + 4\,\mu) - 375\,\lambda^2\,\mu^2\,(-155 + 12\,\mu) + 2\,\lambda^4\,\mu\,(-1225 + 36\,\mu))) / ((4\,\lambda^2 - 75\,\mu)\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35 + \mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25 + 4\,\mu) + 4\,\lambda^4\,\mu\,(-1025 + 18\,\mu)))$$

$$\mathbf{s[3]} \to (10\,(\lambda - 3\,\mu)\,\mu\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 45000\,\lambda\,\mu^3 - 1500\,\lambda^2\,\mu^2\,(-65 + 3\,\mu) + 16875\,\mu^3\,(-25 + 4\,\mu) + 4\,\lambda^4\,\mu\,(-1025 + 18\,\mu))) / ((4\,\lambda^2 - 75\,\mu)\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35 + \mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25 + 4\,\mu) + 4\,\lambda^4\,\mu\,(-1025 + 18\,\mu)))$$

$$s[1,2] \to (375\,\lambda\,\mu^2\,(16\,\lambda^2 - 24\,\lambda\,\mu + 9\,\mu\,(-25+4\,\mu)))\,/\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35+\mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25+4\,\mu) + 4\,\lambda^4\,\mu\,(-1025+18\,\mu))$$

$$s[2,1] \to (375\,\lambda\,\mu^2\,(16\,\lambda^2 - 24\,\lambda\,\mu + 9\,\mu\,(-25+4\,\mu)))\,/\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35+\mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25+4\,\mu) + 4\,\lambda^4\,\mu\,(-1025+18\,\mu))$$

$$p[1] \to (5625\,\mu^3\,(16\,\lambda^2 - 24\,\lambda\,\mu + 9\,\mu\,(-25+4\,\mu)))\,/\,(2\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35+\mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25+4\,\mu) + 4\,\lambda^4\,\mu\,(-1025+18\,\mu)))$$

$$p[2] \to (5625\,\mu^3\,(16\,\lambda^2 - 24\,\lambda\,\mu + 9\,\mu\,(-25+4\,\mu)))\,/\,(2\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35+\mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25+4\,\mu) + 4\,\lambda^4\,\mu\,(-1025+18\,\mu)))$$

$$p[3] \to (3\,\mu\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 45000\,\lambda\,\mu^3 - 1500\,\lambda^2\,\mu^2\,(-65+3\,\mu) + 16875\,\mu^3\,(-25+4\,\mu) + 4\,\lambda^4\,\mu\,(-1025+18\,\mu)))\,/\,(2\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35+\mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25+4\,\mu) + 4\,\lambda^4\,\mu\,(-1025+18\,\mu)))$$

$$d[1] \to (1875\,\mu^2\,(16\,\lambda^2 - 24\,\lambda\,\mu + 9\,\mu\,(-25+4\,\mu)))\,/\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35+\mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25+4\,\mu) + 4\,\lambda^4\,\mu\,(-1025+18\,\mu))$$

$$d[2] \to (1875\,\mu^2\,(16\,\lambda^2 - 24\,\lambda\,\mu + 9\,\mu\,(-25+4\,\mu)))\,/\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35+\mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25+4\,\mu) + 4\,\lambda^4\,\mu\,(-1025+18\,\mu))$$

$$d[3] \to (8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 45000\,\lambda\,\mu^3 - 1500\,\lambda^2\,\mu^2\,(-65+3\,\mu) + 16875\,\mu^3\,(-25+4\,\mu) + 4\,\lambda^4\,\mu\,(-1025+18\,\mu))\,/\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35+\mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25+4\,\mu) + 4\,\lambda^4\,\mu\,(-1025+18\,\mu))$$

$$\pi[1] \to -\Big(25\,\mu^2\,(16\,\lambda^2 - 24\,\lambda\,\mu + 9\,\mu\,(-25+4\,\mu))^2\,(32\,\lambda^{10} - 192\,\lambda^9\,\mu + 24000\,\lambda^7\,\mu^2 - 1110000\,\lambda^5\,\mu^3 + 22500000\,\lambda^3\,\mu^4 - 168750000\,\lambda\,\mu^5 + 31640625\,\mu^5\,(-75+8\,\mu) + 32\,\lambda^8\,\mu\,(-125+9\,\mu) - 1406250\,\lambda^2\,\mu^4\,(-245+24\,\mu) - 1000\,\lambda^6\,\mu^2\,(-365+36\,\mu) + 15000\,\lambda^4\,\mu^3\,(-1150+111\,\mu))\Big)\Big/\Big(2\,(4\,\lambda^2 - 75\,\mu)^2\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35+\mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25+4\,\mu) + 4\,\lambda^4\,\mu\,(-1025+18\,\mu))^2\Big)$$

$$\pi[2] \to -\Big(25\,\mu^2\,(16\,\lambda^2 - 24\,\lambda\,\mu + 9\,\mu\,(-25+4\,\mu))^2\,(32\,\lambda^{10} - 192\,\lambda^9\,\mu + 24000\,\lambda^7\,\mu^2 - 1110000\,\lambda^5\,\mu^3 + 22500000\,\lambda^3\,\mu^4 - 168750000\,\lambda\,\mu^5 + 31640625\,\mu^5\,(-75+8\,\mu) + 32\,\lambda^8\,\mu\,(-125+9\,\mu) - 1406250\,\lambda^2\,\mu^4\,(-245+24\,\mu) - 1000\,\lambda^6\,\mu^2\,(-365+36\,\mu) + 15000\,\lambda^4\,\mu^3\,(-1150+111\,\mu))\Big)\Big/\Big(2\,(4\,\lambda^2 - 75\,\mu)^2\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35+\mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25+4\,\mu) + 4\,\lambda^4\,\mu\,(-1025+18\,\mu))^2\Big)$$

$$\pi[3] \to \Big(\mu\,(48\,\lambda^4 - 2000\,\lambda^2\,\mu + 1200\,\lambda\,\mu^2 - 225\,\mu^2\,(-75+8\,\mu))\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 45000\,\lambda\,\mu^3 - 1500\,\lambda^2\,\mu^2\,(-65+3\,\mu) + 16875\,\mu^3\,(-25+4\,\mu) + 4\,\lambda^4\,\mu\,(-1025+18\,\mu))^2\Big)\Big/\Big(2\,(4\,\lambda^2 - 75\,\mu)^2\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35+\mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25+4\,\mu) + 4\,\lambda^4\,\mu\,(-1025+18\,\mu))^2\Big)$$

$$c[1] \to \Big(25\,\mu^2\,(16\,\lambda^2 - 24\,\lambda\,\mu + 9\,\mu\,(-25 + 4\,\mu))^2\,(16\,\lambda^{10} - 96\,\lambda^9\,\mu + 12000\,\lambda^7\,\mu^2 - 555000\,\lambda^5\,\mu^3 + 11250000\,\lambda^3\,\mu^4 - 84375000\,\lambda\,\mu^5 + 126562500\,\mu^6 + 16\,\lambda^8\,\mu\,(-125 + 9\,\mu) - 703125\,\lambda^2\,\mu^4\,(-65 + 24\,\mu) - 500\,\lambda^6\,\mu^2\,(-365 + 36\,\mu) + 7500\,\lambda^4\,\mu^3\,(-700 + 111\,\mu))\Big) \Big/ \Big((4\,\lambda^2 - 75\,\mu)^2\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35 + \mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25 + 4\,\mu) + 4\,\lambda^4\,\mu\,(-1025 + 18\,\mu))^2\Big)$$

$$c[2] \to \Big(25\,\mu^2\,(16\,\lambda^2 - 24\,\lambda\,\mu + 9\,\mu\,(-25 + 4\,\mu))^2\,(16\,\lambda^{10} - 96\,\lambda^9\,\mu + 12000\,\lambda^7\,\mu^2 - 555000\,\lambda^5\,\mu^3 + 11250000\,\lambda^3\,\mu^4 - 84375000\,\lambda\,\mu^5 + 126562500\,\mu^6 + 16\,\lambda^8\,\mu\,(-125 + 9\,\mu) - 703125\,\lambda^2\,\mu^4\,(-65 + 24\,\mu) - 500\,\lambda^6\,\mu^2\,(-365 + 36\,\mu) + 7500\,\lambda^4\,\mu^3\,(-700 + 111\,\mu))\Big) \Big/ \Big((4\,\lambda^2 - 75\,\mu)^2\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35 + \mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25 + 4\,\mu) + 4\,\lambda^4\,\mu\,(-1025 + 18\,\mu))^2\Big)$$

$$c[3] \to \Big(100\,\mu^2\,(-8\,\lambda^7 + 72\,\lambda^6\,\mu + 4\,\lambda^5\,(1025 - 54\,\mu)\,\mu - 13500\,\lambda^2\,(-25 + \mu)\,\mu^3 + 50625\,\mu^4\,(-25 + 4\,\mu) + 36\,\lambda^4\,\mu^2\,(-425 + 6\,\mu) + 1500\,\lambda^3\,\mu^2\,(-65 + 9\,\mu) - 16875\,\lambda\,\mu^3\,(-25 + 12\,\mu))^2\Big) \Big/ \Big((4\,\lambda^2 - 75\,\mu)^2\,(8\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 4500\,\lambda^2\,(-35 + \mu)\,\mu^2 - 135000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-25 + 4\,\mu) + 4\,\lambda^4\,\mu\,(-1025 + 18\,\mu))^2\Big)$$

## COMPUTATIONS FOR CASE 4

$$U[i] == V - \mu * (F[i]) - \lambda * L[i] - p[i]$$

$$F[i] == d[i] - s[i]$$

$$L[1] == \left(\frac{s[1]}{3} + \frac{s[2]}{3} - (s[1,2] + s[2,1]) - \frac{k}{2}\right) + \left(\frac{s[1]}{3} + \frac{s[3]}{3} - (s[1,3] + s[3,1]) - \frac{k}{2}\right)$$

$$L[2] == \left(\frac{s[1]}{3} + \frac{s[2]}{3} - (s[1,2] + s[2,1]) - \frac{k}{2}\right) + \left(\frac{s[2]}{3} + \frac{s[3]}{3} - \frac{k}{2}\right)$$

$$L[3] == \left(\frac{s[3]}{3} + \frac{s[2]}{3} - \frac{k}{2}\right) + \left(\frac{s[1]}{3} + \frac{s[3]}{3} - (s[1,3] + s[3,1]) - \frac{k}{2}\right)$$

$$c[1] == (s[1])^2 + (s[1,2])^2 + (s[1,3])^2$$

$$c[2] == (s[2])^2 + (s[2,1])^2$$

$$c[3] == (s[3])^2 + (s[3,1])^2$$

## FIND THE DEMANDS

```
Solve[{-μ * (d[1] - s[1]) - λ * L[1] - p[1] == -μ * (d[2] - s[2]) - λ * L[2] - p[2],
  -μ * (d[1] - s[1]) - λ * L[1] - p[1] == -μ * (d[3] - s[3]) - λ * L[3] - p[3],
  d[1] + d[2] + d[3] == 1}, {d[1], d[2], d[3]}]
```

$$\left\{\left\{d[2]\to-\frac{1}{3\,\mu}\,(-\mu-\lambda\,L[1]+2\,\lambda\,L[2]-\lambda\,L[3]-p[1]+2\,p[2]-p[3]+\mu\,s[1]-2\,\mu\,s[2]+\mu\,s[3]),\right.\right.$$
$$d[1]\to-\frac{1}{3\,\mu}\,(-\mu+2\,\lambda\,L[1]-\lambda\,L[2]-\lambda\,L[3]+2\,p[1]-p[2]-p[3]-2\,\mu\,s[1]+\mu\,s[2]+\mu\,s[3]),$$
$$\left.\left.d[3]\to-\frac{1}{3\,\mu}\,(-\mu-\lambda\,L[1]-\lambda\,L[2]+2\,\lambda\,L[3]-p[1]-p[2]+2\,p[3]+\mu\,s[1]+\mu\,s[2]-2\,\mu\,s[3])\right\}\right\}$$

$$\pi[1]==p[1]*\left(-\frac{1}{3\,\mu}\,(-\mu+2\,\lambda\,L[1]-\lambda\,L[2]-\lambda\,L[3]+2\,p[1]-p[2]-p[3]-2\,\mu\,s[1]+\mu\,s[2]+\mu\,s[3])\right)-c[1]$$

$$\pi[2]==p[2]*\left(-\frac{1}{3\,\mu}\,(-\mu-\lambda\,L[1]+2\,\lambda\,L[2]-\lambda\,L[3]-p[1]+2\,p[2]-p[3]+\mu\,s[1]-2\,\mu\,s[2]+\mu\,s[3])\right)-c[2]$$

$$\pi[3]==p[3]*\left(-\frac{1}{3\,\mu}\,(-\mu-\lambda\,L[1]-\lambda\,L[2]+2\,\lambda\,L[3]-p[1]-p[2]+2\,p[3]+\mu\,s[1]+\mu\,s[2]-2\,\mu\,s[3])\right)-c[3]$$

## SOLVE FOR THE LAST STAGE, FIND PRICES

$$D\left[p[1]*\left(-\frac{1}{3\,\mu}\,(-\mu+2\,\lambda\,L[1]-\lambda\,L[2]-\lambda\,L[3]+2\,p[1]-p[2]-p[3]-2\,\mu\,s[1]+\mu\,s[2]+\mu\,s[3])\right)-c[1],\,p[1]\right]$$

$$-\frac{2\,p[1]}{3\,\mu}-\frac{1}{3\,\mu}\,(-\mu+2\,\lambda\,L[1]-\lambda\,L[2]-\lambda\,L[3]+2\,p[1]-p[2]-p[3]-2\,\mu\,s[1]+\mu\,s[2]+\mu\,s[3])$$

$$D\left[p[2]*\left(-\frac{1}{3\,\mu}\,(-\mu-\lambda\,L[1]+2\,\lambda\,L[2]-\lambda\,L[3]-p[1]+2\,p[2]-p[3]+\mu\,s[1]-2\,\mu\,s[2]+\mu\,s[3])\right)-c[2],\,p[2]\right]$$

$$-\frac{2\,p[2]}{3\,\mu}-\frac{1}{3\,\mu}\,(-\mu-\lambda\,L[1]+2\,\lambda\,L[2]-\lambda\,L[3]-p[1]+2\,p[2]-p[3]+\mu\,s[1]-2\,\mu\,s[2]+\mu\,s[3])$$

$$D\left[p[3]*\left(-\frac{1}{3\,\mu}\,(-\mu-\lambda\,L[1]-\lambda\,L[2]+2\,\lambda\,L[3]-p[1]-p[2]+2\,p[3]+\mu\,s[1]+\mu\,s[2]-2\,\mu\,s[3])\right)-c[3],\,p[3]\right]$$

$$-\frac{2\,p[3]}{3\,\mu}-\frac{1}{3\,\mu}\,(-\mu-\lambda\,L[1]-\lambda\,L[2]+2\,\lambda\,L[3]-p[1]-p[2]+2\,p[3]+\mu\,s[1]+\mu\,s[2]-2\,\mu\,s[3])$$

## SOLVE FOR THE LAST STAGE, FIND PRICES

$$D\left[p[1]*\left(-\frac{1}{3\,\mu}\,(-\mu+2\,\lambda\,L[1]-\lambda\,L[2]-\lambda\,L[3]+2\,p[1]-p[2]-p[3]-2\,\mu\,s[1]+\mu\,s[2]+\mu\,s[3])\right)-c[1],\,p[1]\right]$$

$$-\frac{2\,p[1]}{3\,\mu}-\frac{1}{3\,\mu}\,(-\mu+2\,\lambda\,L[1]-\lambda\,L[2]-\lambda\,L[3]+2\,p[1]-p[2]-p[3]-2\,\mu\,s[1]+\mu\,s[2]+\mu\,s[3])$$

```
D[p[2] * (-1/(3 μ) (-μ - λ L[1] + 2 λ L[2] - λ L[3] - p[1] + 2 p[2] - p[3] + μ s[1] - 2 μ s[2] + μ s[3])) -
   c[2], p[2]]
```

$$-\frac{2\,p[2]}{3\,\mu}-\frac{1}{3\,\mu}\,(-\mu-\lambda\,L[1]+2\,\lambda\,L[2]-\lambda\,L[3]-p[1]+2\,p[2]-p[3]+\mu\,s[1]-2\,\mu\,s[2]+\mu\,s[3])$$

```
D[p[3] * (-1/(3 μ) (-μ - λ L[1] - λ L[2] + 2 λ L[3] - p[1] - p[2] + 2 p[3] + μ s[1] + μ s[2] - 2 μ s[3])) -
   c[3], p[3]]
```

$$-\frac{2\,p[3]}{3\,\mu}-\frac{1}{3\,\mu}\,(-\mu-\lambda\,L[1]-\lambda\,L[2]+2\,\lambda\,L[3]-p[1]-p[2]+2\,p[3]+\mu\,s[1]+\mu\,s[2]-2\,\mu\,s[3])$$

```
Solve[
 {-2 p[3]/(3 μ) - 1/(3 μ) (-μ - λ L[1] - λ L[2] + 2 λ L[3] - p[1] - p[2] + 2 p[3] + μ s[1] + μ s[2] - 2 μ s[3]) ==
   0, -2 p[2]/(3 μ) -
    1/(3 μ) (-μ - λ L[1] + 2 λ L[2] - λ L[3] - p[1] + 2 p[2] - p[3] + μ s[1] - 2 μ s[2] + μ s[3]) == 0,
  -2 p[1]/(3 μ) - 1/(3 μ) (-μ + 2 λ L[1] - λ L[2] - λ L[3] + 2 p[1] - p[2] - p[3] - 2 μ s[1] + μ s[2] + μ s[3]) ==
   0}, {p[1], p[2], p[3]}]
```

$$\Big\{\Big\{p[1]\to\frac{1}{10}\,(5\,\mu-4\,\lambda\,L[1]+2\,\lambda\,L[2]+2\,\lambda\,L[3]+4\,\mu\,s[1]-2\,\mu\,s[2]-2\,\mu\,s[3]),$$
$$p[2]\to\frac{1}{10}\,(5\,\mu+2\,\lambda\,L[1]-4\,\lambda\,L[2]+2\,\lambda\,L[3]-2\,\mu\,s[1]+4\,\mu\,s[2]-2\,\mu\,s[3]),$$
$$p[3]\to\frac{1}{10}\,(5\,\mu+2\,\lambda\,L[1]+2\,\lambda\,L[2]-4\,\lambda\,L[3]-2\,\mu\,s[1]-2\,\mu\,s[2]+4\,\mu\,s[3])\Big\}\Big\}$$

```
Simplify[
 Solve[{π[1] == p[1] * (-1/(3 μ) (-μ + 2 λ L[1] - λ L[2] - λ L[3] + 2 p[1] - p[2] - p[3] - 2 μ s[1] +
         μ s[2] + μ s[3])) - c[1], π[2] == p[2] *
     (-1/(3 μ) (-μ - λ L[1] + 2 λ L[2] - λ L[3] - p[1] + 2 p[2] - p[3] + μ s[1] - 2 μ s[2] + μ s[3])) -
    c[2], π[3] == p[3] * (-1/(3 μ) (-μ - λ L[1] - λ L[2] + 2 λ L[3] - p[1] -
         p[2] + 2 p[3] + μ s[1] + μ s[2] - 2 μ s[3])) - c[3],
   L[1] == (s[1]/3 + s[2]/3 - (s[1, 2] + s[2, 1]) - k/2) + (s[1]/3 + s[3]/3 - (s[1, 3] + s[3, 1]) - k/2),
   L[2] == (s[1]/3 + s[2]/3 - (s[1, 2] + s[2, 1]) - k/2) + (s[2]/3 + s[3]/3 - k/2),
   L[3] == (s[3]/3 + s[2]/3 - k/2) + (s[1]/3 + s[3]/3 - (s[1, 3] + s[3, 1]) - k/2),
   c[1] == (s[1])^2 + (s[1, 2])^2 + (s[1, 3])^2,
   c[2] == (s[2])^2 + (s[2, 1])^2, c[3] == (s[3])^2 + (s[3, 1])^2,
   p[1] == 1/10 (5 μ - 4 λ L[1] + 2 λ L[2] + 2 λ L[3] + 4 μ s[1] - 2 μ s[2] - 2 μ s[3]),
   p[2] == 1/10 (5 μ + 2 λ L[1] - 4 λ L[2] + 2 λ L[3] - 2 μ s[1] + 4 μ s[2] - 2 μ s[3]),
   p[3] == 1/10 (5 μ + 2 λ L[1] + 2 λ L[2] - 4 λ L[3] - 2 μ s[1] - 2 μ s[2] + 4 μ s[3])},
  {π[1], π[2], π[3]}, {p[1], p[2], p[3], L[1], L[2], L[3], c[1], c[2], c[3]}]]
```

```
{{π[1] → 1/(1350 μ) (9 μ^2 (5 + 4 s[1] - 2 s[2] - 2 s[3])^2 -
      6 μ (225 (s[1]^2 + s[1, 2]^2 + s[1, 3]^2) + 2 λ (5 + 4 s[1] - 2 s[2] - 2 s[3])
           (2 s[1] - s[2] - s[3] - 3 s[1, 2] - 3 s[1, 3] - 3 s[2, 1] - 3 s[3, 1])) +
      4 λ^2 (-2 s[1] + s[2] + s[3] + 3 s[1, 2] + 3 s[1, 3] + 3 s[2, 1] + 3 s[3, 1])^2),
  π[2] → 1/(1350 μ) (9 μ^2 (5 - 2 s[1] + 4 s[2] - 2 s[3])^2 -
      6 μ (225 (s[2]^2 + s[2, 1]^2) + 2 λ (-5 + 2 s[1] - 4 s[2] + 2 s[3])
           (s[1] - 2 s[2] + s[3] + 3 s[1, 2] - 6 s[1, 3] + 3 s[2, 1] - 6 s[3, 1])) +
      4 λ^2 (s[1] - 2 s[2] + s[3] + 3 s[1, 2] - 6 s[1, 3] + 3 s[2, 1] - 6 s[3, 1])^2),
  π[3] → 1/(1350 μ) (9 μ^2 (5 - 2 s[1] - 2 s[2] + 4 s[3])^2 +
      4 λ^2 (s[1] + s[2] - 2 s[3] - 6 s[1, 2] + 3 s[1, 3] - 6 s[2, 1] + 3 s[3, 1])^2 -
      6 μ (2 λ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) (s[1] + s[2] - 2 s[3] - 6 s[1, 2] +
             3 s[1, 3] - 6 s[2, 1] + 3 s[3, 1]) + 225 (s[3]^2 + s[3, 1]^2)))}}
```

## SOLVE THE THIRD STAGE, FIND LINK INVESTMENTS

```
D[1/(1350 μ) (9 μ^2 (5 + 4 s[1] - 2 s[2] - 2 s[3])^2 -
    6 μ (225 (s[1]^2 + s[1, 2]^2 + s[1, 3]^2) + 2 λ (5 + 4 s[1] - 2 s[2] - 2 s[3])
        (2 s[1] - s[2] - s[3] - 3 s[1, 2] - 3 s[1, 3] - 3 s[2, 1] - 3 s[3, 1])) +
    4 λ^2 (-2 s[1] + s[2] + s[3] + 3 s[1, 2] + 3 s[1, 3] + 3 s[2, 1] + 3 s[3, 1])^2), s[1, 2]]
```

$$\frac{1}{1350\,\mu}\,(-6\,\mu\,(-6\,\lambda\,(5+4\,s[1]-2\,s[2]-2\,s[3])+450\,s[1,2])+24\,\lambda^2\,(-2\,s[1]+s[2]+s[3]+3\,s[1,2]+3\,s[1,3]+3\,s[2,1]+3\,s[3,1]))$$

```
D[1/(1350 μ) (9 μ^2 (5 + 4 s[1] - 2 s[2] - 2 s[3])^2 -
    6 μ (225 (s[1]^2 + s[1, 2]^2 + s[1, 3]^2) + 2 λ (5 + 4 s[1] - 2 s[2] - 2 s[3])
        (2 s[1] - s[2] - s[3] - 3 s[1, 2] - 3 s[1, 3] - 3 s[2, 1] - 3 s[3, 1])) +
    4 λ^2 (-2 s[1] + s[2] + s[3] + 3 s[1, 2] + 3 s[1, 3] + 3 s[2, 1] + 3 s[3, 1])^2), s[1, 3]]
```

$$\frac{1}{1350\,\mu}\,(-6\,\mu\,(-6\,\lambda\,(5+4\,s[1]-2\,s[2]-2\,s[3])+450\,s[1,3])+24\,\lambda^2\,(-2\,s[1]+s[2]+s[3]+3\,s[1,2]+3\,s[1,3]+3\,s[2,1]+3\,s[3,1]))$$

```
D[1/(1350 μ) (9 μ^2 (5 - 2 s[1] + 4 s[2] - 2 s[3])^2 -
    6 μ (225 (s[2]^2 + s[2, 1]^2) + 2 λ (-5 + 2 s[1] - 4 s[2] + 2 s[3])
        (s[1] - 2 s[2] + s[3] + 3 s[1, 2] - 6 s[1, 3] + 3 s[2, 1] - 6 s[3, 1])) +
    4 λ^2 (s[1] - 2 s[2] + s[3] + 3 s[1, 2] - 6 s[1, 3] + 3 s[2, 1] - 6 s[3, 1])^2), s[2, 1]]
```

$$\frac{1}{1350\,\mu}\,(-6\,\mu\,(6\,\lambda\,(-5+2\,s[1]-4\,s[2]+2\,s[3])+450\,s[2,1])+24\,\lambda^2\,(s[1]-2\,s[2]+s[3]+3\,s[1,2]-6\,s[1,3]+3\,s[2,1]-6\,s[3,1]))$$

```
D[1/(1350 μ) (9 μ^2 (5 - 2 s[1] - 2 s[2] + 4 s[3])^2 +
    4 λ^2 (s[1] + s[2] - 2 s[3] - 6 s[1, 2] + 3 s[1, 3] - 6 s[2, 1] + 3 s[3, 1])^2 -
    6 μ (2 λ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) (s[1] + s[2] - 2 s[3] - 6 s[1, 2] +
          3 s[1, 3] - 6 s[2, 1] + 3 s[3, 1]) + 225 (s[3]^2 + s[3, 1]^2))), s[3, 1]]
```

$$\frac{1}{1350\,\mu}\,(24\,\lambda^2\,(s[1]+s[2]-2\,s[3]-6\,s[1,2]+3\,s[1,3]-6\,s[2,1]+3\,s[3,1])-6\,\mu\,(6\,\lambda\,(-5+2\,s[1]+2\,s[2]-4\,s[3])+450\,s[3,1]))$$

```
Simplify[
 Solve[{1/(1350 μ) (24 λ^2 (s[1] + s[2] - 2 s[3] - 6 s[1, 2] + 3 s[1, 3] - 6 s[2, 1] + 3 s[3, 1]) -
        6 μ (6 λ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) + 450 s[3, 1])) == 0,
    1/(1350 μ) (-6 μ (6 λ (-5 + 2 s[1] - 4 s[2] + 2 s[3]) + 450 s[2, 1]) +
        24 λ^2 (s[1] - 2 s[2] + s[3] + 3 s[1, 2] - 6 s[1, 3] + 3 s[2, 1] - 6 s[3, 1])) == 0,
    1/(1350 μ) (-6 μ (-6 λ (5 + 4 s[1] - 2 s[2] - 2 s[3]) + 450 s[1, 3]) +
        24 λ^2 (-2 s[1] + s[2] + s[3] + 3 s[1, 2] + 3 s[1, 3] + 3 s[2, 1] + 3 s[3, 1])) == 0,
    1/(1350 μ) (-6 μ (-6 λ (5 + 4 s[1] - 2 s[2] - 2 s[3]) + 450 s[1, 2]) +
        24 λ^2 (-2 s[1] + s[2] + s[3] + 3 s[1, 2] + 3 s[1, 3] + 3 s[2, 1] + 3 s[3, 1])) ==
     0}, {s[1, 2], s[1, 3], s[2, 1], s[3, 1]}]]
```

```
{{s[1, 2] → -(λ (6 λ^2 + 15 μ (5 + 4 s[1] - 2 s[2] - 2 s[3]) + 10 λ (-2 s[1] + s[2] + s[3])))/(15 (2 λ^2 - 75 μ)),
  s[1, 3] → -(λ (6 λ^2 + 15 μ (5 + 4 s[1] - 2 s[2] - 2 s[3]) + 10 λ (-2 s[1] + s[2] + s[3])))/(15 (2 λ^2 - 75 μ)), s[2, 1] →
   (λ (12 λ^4 + 60 λ^2 μ (-5 + s[1] - s[2]) - 20 λ^3 (s[1] - s[2]) + 250 λ μ (s[1] - 2 s[2] + s[3]) -
       375 μ^2 (-5 + 2 s[1] - 4 s[2] + 2 s[3]))) / (15 (4 λ^4 - 200 λ^2 μ + 1875 μ^2)),
  s[3, 1] → (λ (12 λ^4 - 375 μ^2 (-5 + 2 s[1] + 2 s[2] - 4 s[3]) + 250 λ μ (s[1] + s[2] - 2 s[3]) +
       60 λ^2 μ (-5 + s[1] - s[3]) - 20 λ^3 (s[1] - s[3]))) / (15 (4 λ^4 - 200 λ^2 μ + 1875 μ^2))}}
```

```
Simplify[Solve[{π[1] == 1/(1350 μ) (9 μ^2 (5 + 4 s[1] - 2 s[2] - 2 s[3])^2 -
        6 μ (225 (s[1]^2 + s[1, 2]^2 + s[1, 3]^2) + 2 λ (5 + 4 s[1] - 2 s[2] - 2 s[3])
            (2 s[1] - s[2] - s[3] - 3 s[1, 2] - 3 s[1, 3] - 3 s[2, 1] - 3 s[3, 1])) +
        4 λ^2 (-2 s[1] + s[2] + s[3] + 3 s[1, 2] + 3 s[1, 3] + 3 s[2, 1] + 3 s[3, 1])^2),
    π[2] == 1/(1350 μ) (9 μ^2 (5 - 2 s[1] + 4 s[2] - 2 s[3])^2 -
        6 μ (225 (s[2]^2 + s[2, 1]^2) + 2 λ (-5 + 2 s[1] - 4 s[2] + 2 s[3])
            (s[1] - 2 s[2] + s[3] + 3 s[1, 2] - 6 s[1, 3] + 3 s[2, 1] - 6 s[3, 1])) +
        4 λ^2 (s[1] - 2 s[2] + s[3] + 3 s[1, 2] - 6 s[1, 3] + 3 s[2, 1] - 6 s[3, 1])^2),
    π[3] == 1/(1350 μ) (9 μ^2 (5 - 2 s[1] - 2 s[2] + 4 s[3])^2 +
        4 λ^2 (s[1] + s[2] - 2 s[3] - 6 s[1, 2] + 3 s[1, 3] - 6 s[2, 1] + 3 s[3, 1])^2 -
        6 μ (2 λ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) (s[1] + s[2] - 2 s[3] - 6 s[1, 2] +
              3 s[1, 3] - 6 s[2, 1] + 3 s[3, 1]) + 225 (s[3]^2 + s[3, 1]^2))), s[1, 2] ==
     -1/(15 (2 λ^2 - 75 μ)) (λ (6 λ^2 + 15 μ (5 + 4 s[1] - 2 s[2] - 2 s[3]) + 10 λ (-2 s[1] + s[2] + s[3]))),
    s[1, 3] == -1/(15 (2 λ^2 - 75 μ))
      (λ (6 λ^2 + 15 μ (5 + 4 s[1] - 2 s[2] - 2 s[3]) + 10 λ (-2 s[1] + s[2] + s[3]))), s[2, 1] ==
     (λ (12 λ^4 + 60 λ^2 μ (-5 + s[1] - s[2]) - 20 λ^3 (s[1] - s[2]) + 250 λ μ (s[1] - 2 s[2] + s[3]) -
         375 μ^2 (-5 + 2 s[1] - 4 s[2] + 2 s[3]))) / (15 (4 λ^4 - 200 λ^2 μ + 1875 μ^2)),
    s[3, 1] == (λ (12 λ^4 - 375 μ^2 (-5 + 2 s[1] + 2 s[2] - 4 s[3]) + 250 λ μ (s[1] + s[2] - 2 s[3]) +
         60 λ^2 μ (-5 + s[1] - s[3]) - 20 λ^3 (s[1] - s[3]))) / (15 (4 λ^4 - 200 λ^2 μ + 1875 μ^2))},
  {π[1], π[2], π[3]}, {s[1, 2], s[1, 3], s[2, 1], s[3, 1]}]]
```

```
{{π[1] → 1/(450 (2 λ^2 - 75 μ)^2) (-144 λ^6 + 16875 μ^2 (-150 s[1]^2 + μ (5 + 4 s[1] - 2 s[2] - 2 s[3])^2) +
      480 λ^5 (2 s[1] - s[2] - s[3]) - 22500 λ μ^2
       (8 s[1]^2 + 2 s[2]^2 + s[3] (-5 + 2 s[3]) + s[2] (-5 + 4 s[3]) - 2 s[1] (-5 + 4 s[2] + 4 s[3])) +
      600 λ^3 μ (16 s[1]^2 + 4 s[2]^2 + s[3] (5 + 4 s[3]) + s[2] (5 + 8 s[3]) -
         2 s[1] (5 + 8 s[2] + 8 s[3])) - 20 λ^4 (9 μ (5 + 16 s[1] - 8 s[2] - 8 s[3]) +
         10 (17 s[1]^2 - 8 s[1] (s[2] + s[3]) + 2 (s[2] + s[3])^2)) -
      300 λ^2 μ (-25 (22 s[1]^2 - 4 s[1] (s[2] + s[3]) + (s[2] + s[3])^2) + 6 μ (-25 + 8 s[1]^2 +
            2 s[2]^2 + 5 s[3] + 2 s[3]^2 + s[2] (5 + 4 s[3]) - 2 s[1] (5 + 4 s[2] + 4 s[3])))),
  π[2] → (-144 λ^8 + 480 λ^7 (s[1] - s[2]) - 80 λ^6 (18 μ (-5 + s[1] - s[2]) +
         5 (s[1]^2 - 2 s[1] s[2] + 10 s[2]^2)) -
      140625 μ^3 (-150 s[2]^2 + μ (5 - 2 s[1] + 4 s[2] - 2 s[3])^2) +
      1200 λ^5 μ (2 s[1]^2 + 20 s[2] + 2 s[2]^2 - s[1] (15 + 4 s[2]) - 5 s[3]) - 15000 λ^3 μ^2
       (4 s[1]^2 + 8 s[2]^2 + s[2] (25 - 4 s[3]) - 10 s[3] + s[1] (-15 - 12 s[2] + 4 s[3])) + 187500 λ
       μ^3 (2 s[1]^2 + 8 s[2]^2 + s[2] (10 - 8 s[3]) + s[3] (-5 + 2 s[3]) + s[1] (-5 - 8 s[2] + 4 s[3])) -
      200 λ^4 μ (9 μ (75 + 2 s[1]^2 + 40 s[2] + 2 s[2]^2 - 2 s[1] (15 + 2 s[2]) - 10 s[3]) -
         25 (2 s[1]^2 + s[2] (49 s[2] - 2 s[3]) + s[1] (-6 s[2] + 2 s[3]))) +
      2500 λ^2 μ^2 (18 μ (-5 + s[1] - s[2]) (-5 + 2 s[1] - 4 s[2] + 2 s[3]) -
         25 (s[1]^2 + 67 s[2]^2 - 4 s[2] s[3] + s[3]^2 + s[1] (-4 s[2] + 2 s[3])))) /
    (450 (2 λ^2 - 75 μ) (2 λ^2 - 25 μ)^2), π[3] → (-144 λ^8 + 480 λ^7 (s[1] - s[3]) +
      1200 λ^5 μ (2 s[1]^2 - 5 s[2] + 2 s[3] (10 + s[3]) - s[1] (15 + 4 s[3])) -
      140625 μ^3 (-150 s[3]^2 + μ (5 - 2 s[1] - 2 s[2] + 4 s[3])^2) + 187500 λ μ^3
       (2 s[1]^2 + 2 s[2]^2 + s[1] (-5 + 4 s[2] - 8 s[3]) + 2 s[3] (5 + 4 s[3]) - s[2] (5 + 8 s[3])) -
      80 λ^6 (18 μ (-5 + s[1] - s[3]) + 5 (s[1]^2 - 2 s[1] s[3] + 10 s[3]^2)) +
      2500 λ^2 μ^2 (18 μ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) (-5 + s[1] - s[3]) -
         25 (s[1]^2 + s[2]^2 + 2 s[1] (s[2] - 2 s[3]) - 4 s[2] s[3] + 67 s[3]^2)) - 15000 λ^3 μ^2
       (4 s[1]^2 - 2 s[2] (5 + 2 s[3]) + s[3] (25 + 8 s[3]) + s[1] (4 s[2] - 3 (5 + 4 s[3]))) -
      200 λ^4 μ (9 μ (75 + 2 s[1]^2 - 10 s[2] + 40 s[3] + 2 s[3]^2 - 2 s[1] (15 + 2 s[3])) -
         25 (2 s[1]^2 + 2 s[1] (s[2] - 3 s[3]) + s[3] (-2 s[2] + 49 s[3])))) /
    (450 (2 λ^2 - 75 μ) (2 λ^2 - 25 μ)^2)}}
```

## SOLVING FOR THE SECOND STAGE: INVESTMENTS IN NETWORK CAPACITY

```
D[(-144 λ^6 + 16875 μ^2 (-150 s[1]^2 + μ (5 + 4 s[1] - 2 s[2] - 2 s[3])^2) +
     480 λ^5 (2 s[1] - s[2] - s[3]) - 22500 λ μ^2 (8 s[1]^2 + 2 s[2]^2 + s[3] (-5 + 2 s[3]) +
        s[2] (-5 + 4 s[3]) - 2 s[1] (-5 + 4 s[2] + 4 s[3])) + 600 λ^3 μ
      (16 s[1]^2 + 4 s[2]^2 + s[3] (5 + 4 s[3]) + s[2] (5 + 8 s[3]) - 2 s[1] (5 + 8 s[2] + 8 s[3])) - 20 λ^4
      (9 μ (5 + 16 s[1] - 8 s[2] - 8 s[3]) + 10 (17 s[1]^2 - 8 s[1] (s[2] + s[3]) + 2 (s[2] + s[3])^2)) -
     300 λ^2 μ (-25 (22 s[1]^2 - 4 s[1] (s[2] + s[3]) + (s[2] + s[3])^2) +
        6 μ (-25 + 8 s[1]^2 + 2 s[2]^2 + 5 s[3] + 2 s[3]^2 + s[2] (5 + 4 s[3]) -
           2 s[1] (5 + 4 s[2] + 4 s[3])))) / (450 (2 λ^2 - 75 μ)^2), s[1]]
```

```
(960 λ^5 + 16875 μ^2 (-300 s[1] + 8 μ (5 + 4 s[1] - 2 s[2] - 2 s[3])) -
   22500 λ μ^2 (16 s[1] - 2 (-5 + 4 s[2] + 4 s[3])) + 600 λ^3 μ (32 s[1] - 2 (5 + 8 s[2] + 8 s[3])) -
   20 λ^4 (144 μ + 10 (34 s[1] - 8 (s[2] + s[3]))) - 300 λ^2 μ
    (-25 (44 s[1] - 4 (s[2] + s[3])) + 6 μ (16 s[1] - 2 (5 + 4 s[2] + 4 s[3])))) / (450 (2 λ^2 - 75 μ)^2)
```

```
D[(-144 λ^8 + 480 λ^7 (s[1] - s[2]) -
    80 λ^6 (18 μ (-5 + s[1] - s[2]) + 5 (s[1]^2 - 2 s[1] s[2] + 10 s[2]^2)) -
    140625 μ^3 (-150 s[2]^2 + μ (5 - 2 s[1] + 4 s[2] - 2 s[3])^2) +
    1200 λ^5 μ (2 s[1]^2 + 20 s[2] + 2 s[2]^2 - s[1] (15 + 4 s[2]) - 5 s[3]) - 15000 λ^3 μ^2
     (4 s[1]^2 + 8 s[2]^2 + s[2] (25 - 4 s[3]) - 10 s[3] + s[1] (-15 - 12 s[2] + 4 s[3])) + 187500 λ
     μ^3 (2 s[1]^2 + 8 s[2]^2 + s[2] (10 - 8 s[3]) + s[3] (-5 + 2 s[3]) + s[1] (-5 - 8 s[2] + 4 s[3])) -
    200 λ^4 μ (9 μ (75 + 2 s[1]^2 + 40 s[2] + 2 s[2]^2 - 2 s[1] (15 + 2 s[2]) - 10 s[3]) -
       25 (2 s[1]^2 + s[2] (49 s[2] - 2 s[3]) + s[1] (-6 s[2] + 2 s[3]))) +
    2500 λ^2 μ^2 (18 μ (-5 + s[1] - s[2]) (-5 + 2 s[1] - 4 s[2] + 2 s[3]) -
       25 (s[1]^2 + 67 s[2]^2 - 4 s[2] s[3] + s[3]^2 + s[1] (-4 s[2] + 2 s[3])))) /
  (450 (2 λ^2 - 75 μ) (2 λ^2 - 25 μ)^2), s[2]]

(-480 λ^7 + 1200 λ^5 μ (20 - 4 s[1] + 4 s[2]) - 80 λ^6 (-18 μ + 5 (-2 s[1] + 20 s[2])) -
   140625 μ^3 (-300 s[2] + 8 μ (5 - 2 s[1] + 4 s[2] - 2 s[3])) -
   200 λ^4 μ (9 μ (40 - 4 s[1] + 4 s[2]) - 25 (-6 s[1] + 98 s[2] - 2 s[3])) +
   187500 λ μ^3 (10 - 8 s[1] + 16 s[2] - 8 s[3]) - 15000 λ^3 μ^2 (25 - 12 s[1] + 16 s[2] - 4 s[3]) +
   2500 λ^2 μ^2 (-72 μ (-5 + s[1] - s[2]) - 25 (-4 s[1] + 134 s[2] - 4 s[3]) -
      18 μ (-5 + 2 s[1] - 4 s[2] + 2 s[3]))) / (450 (2 λ^2 - 75 μ) (2 λ^2 - 25 μ)^2)

D[(-144 λ^8 + 480 λ^7 (s[1] - s[3]) +
    1200 λ^5 μ (2 s[1]^2 - 5 s[2] + 2 s[3] (10 + s[3]) - s[1] (15 + 4 s[3])) -
    140625 μ^3 (-150 s[3]^2 + μ (5 - 2 s[1] - 2 s[2] + 4 s[3])^2) + 187500 λ μ^3
     (2 s[1]^2 + 2 s[2]^2 + s[1] (-5 + 4 s[2] - 8 s[3]) + 2 s[3] (5 + 4 s[3]) - s[2] (5 + 8 s[3])) -
    80 λ^6 (18 μ (-5 + s[1] - s[3]) + 5 (s[1]^2 - 2 s[1] s[3] + 10 s[3]^2)) +
    2500 λ^2 μ^2 (18 μ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) (-5 + s[1] - s[3]) -
       25 (s[1]^2 + s[2]^2 + 2 s[1] (s[2] - 2 s[3]) - 4 s[2] s[3] + 67 s[3]^2)) -
    15000 λ^3 μ^2 (4 s[1]^2 - 2 s[2] (5 + 2 s[3]) + s[3] (25 + 8 s[3]) + s[1] (4 s[2] - 3 (5 + 4 s[3]))) -
    200 λ^4 μ (9 μ (75 + 2 s[1]^2 - 10 s[2] + 40 s[3] + 2 s[3]^2 - 2 s[1] (15 + 2 s[3])) -
       25 (2 s[1]^2 + 2 s[1] (s[2] - 3 s[3]) + s[3] (-2 s[2] + 49 s[3])))) /
  (450 (2 λ^2 - 75 μ) (2 λ^2 - 25 μ)^2), s[3]]

(-480 λ^7 - 15000 λ^3 μ^2 (25 - 12 s[1] - 4 s[2] + 16 s[3]) + 1200 λ^5 μ (-4 s[1] + 2 s[3] + 2 (10 + s[3])) +
   187500 λ μ^3 (-8 s[1] - 8 s[2] + 8 s[3] + 2 (5 + 4 s[3])) -
   140625 μ^3 (-300 s[3] + 8 μ (5 - 2 s[1] - 2 s[2] + 4 s[3])) - 80 λ^6 (-18 μ + 5 (-2 s[1] + 20 s[3])) -
   200 λ^4 μ (9 μ (40 - 4 s[1] + 4 s[3]) - 25 (-6 s[1] - 2 s[2] + 98 s[3])) +
   2500 λ^2 μ^2 (-18 μ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) - 72 μ (-5 + s[1] - s[3]) -
      25 (-4 s[1] - 4 s[2] + 134 s[3]))) / (450 (2 λ^2 - 75 μ) (2 λ^2 - 25 μ)^2)
```

```
Simplify[
 Solve[{(-480 λ^7 - 15000 λ^3 μ^2 (25 - 12 s[1] - 4 s[2] + 16 s[3]) + 1200 λ^5 μ (-4 s[1] + 2 s[3] +
          2 (10 + s[3])) + 187500 λ μ^3 (-8 s[1] - 8 s[2] + 8 s[3] + 2 (5 + 4 s[3])) - 140625 μ^3
        (-300 s[3] + 8 μ (5 - 2 s[1] - 2 s[2] + 4 s[3])) - 80 λ^6 (-18 μ + 5 (-2 s[1] + 20 s[3])) -
       200 λ^4 μ (9 μ (40 - 4 s[1] + 4 s[3]) - 25 (-6 s[1] - 2 s[2] + 98 s[3])) +
       2500 λ^2 μ^2 (-18 μ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) - 72 μ (-5 + s[1] - s[3]) -
          25 (-4 s[1] - 4 s[2] + 134 s[3]))) / (450 (2 λ^2 - 75 μ) (2 λ^2 - 25 μ)^2) == 0,
   (-480 λ^7 + 1200 λ^5 μ (20 - 4 s[1] + 4 s[2]) - 80 λ^6 (-18 μ + 5 (-2 s[1] + 20 s[2])) -
       140625 μ^3 (-300 s[2] + 8 μ (5 - 2 s[1] + 4 s[2] - 2 s[3])) -
       200 λ^4 μ (9 μ (40 - 4 s[1] + 4 s[2]) - 25 (-6 s[1] + 98 s[2] - 2 s[3])) +
       187500 λ μ^3 (10 - 8 s[1] + 16 s[2] - 8 s[3]) - 15000 λ^3 μ^2 (25 - 12 s[1] + 16 s[2] - 4 s[3]) +
       2500 λ^2 μ^2 (-72 μ (-5 + s[1] - s[2]) - 25 (-4 s[1] + 134 s[2] - 4 s[3]) -
          18 μ (-5 + 2 s[1] - 4 s[2] + 2 s[3]))) / (450 (2 λ^2 - 75 μ) (2 λ^2 - 25 μ)^2) == 0,
   (960 λ^5 + 16875 μ^2 (-300 s[1] + 8 μ (5 + 4 s[1] - 2 s[2] - 2 s[3])) -
       22500 λ μ^2 (16 s[1] - 2 (-5 + 4 s[2] + 4 s[3])) +
       600 λ^3 μ (32 s[1] - 2 (5 + 8 s[2] + 8 s[3])) - 20 λ^4 (144 μ + 10 (34 s[1] - 8 (s[2] + s[3]))) -
       300 λ^2 μ (-25 (44 s[1] - 4 (s[2] + s[3])) + 6 μ (16 s[1] - 2 (5 + 4 s[2] + 4 s[3])))) /
     (450 (2 λ^2 - 75 μ)^2) == 0}, {s[1], s[2], s[3]}]]

{{s[1] → (256 λ^7 - 576 λ^6 μ + 101250 μ^4 (-25 + 4 μ) + 144 λ^4 μ^2 (25 + 6 μ) - 5400 λ^2 μ^3 (-50 + 7 μ) -
       33750 λ μ^3 (-25 + 12 μ) - 32 λ^5 μ (125 + 27 μ) + 600 λ^3 μ^2 (-100 + 63 μ)) /
     (15 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
          50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))),
   s[2] → -(2 (λ - 3 μ) (52 λ^6 - 96 λ^5 μ + 4200 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) +
           8 λ^4 μ (-425 + 18 μ) - 75 λ^2 μ^2 (-925 + 84 μ))) / (15 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 -
           135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) + 50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))),
   s[3] → (2 (-52 λ^7 + 252 λ^6 μ + 8 λ^5 (425 - 54 μ) μ + 144 λ^4 μ^2 (-100 + 3 μ) + 50625 μ^4 (-25 + 4 μ) -
          16875 λ μ^3 (-25 + 12 μ) - 675 λ^2 μ^3 (-375 + 28 μ) + 75 λ^3 μ^2 (-925 + 252 μ))) /
     (15 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
          50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ)))}}
```

```
Simplify[Solve[{s[1, 2] ==
    - 1/(15 (2 λ^2 - 75 μ)) (λ (6 λ^2 + 15 μ (5 + 4 s[1] - 2 s[2] - 2 s[3]) + 10 λ (-2 s[1] + s[2] + s[3]))),
   s[1, 3] == - 1/(15 (2 λ^2 - 75 μ))
     (λ (6 λ^2 + 15 μ (5 + 4 s[1] - 2 s[2] - 2 s[3]) + 10 λ (-2 s[1] + s[2] + s[3]))), s[2, 1] ==
    (λ (12 λ^4 + 60 λ^2 μ (-5 + s[1] - s[2]) - 20 λ^3 (s[1] - s[2]) + 250 λ μ (s[1] - 2 s[2] + s[3]) -
        375 μ^2 (-5 + 2 s[1] - 4 s[2] + 2 s[3]))) / (15 (4 λ^4 - 200 λ^2 μ + 1875 μ^2)),
   s[3, 1] == (λ (12 λ^4 - 375 μ^2 (-5 + 2 s[1] + 2 s[2] - 4 s[3]) + 250 λ μ (s[1] + s[2] - 2 s[3]) +
        60 λ^2 μ (-5 + s[1] - s[3]) - 20 λ^3 (s[1] - s[3]))) / (15 (4 λ^4 - 200 λ^2 μ + 1875 μ^2)),
   s[1] == (256 λ^7 - 576 λ^6 μ + 101250 μ^4 (-25 + 4 μ) + 144 λ^4 μ^2 (25 + 6 μ) - 5400 λ^2 μ^3 (-50 + 7 μ) -
      33750 λ μ^3 (-25 + 12 μ) - 32 λ^5 μ (125 + 27 μ) + 600 λ^3 μ^2 (-100 + 63 μ)) /
     (15 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
        50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))),
   s[2] == - (2 (λ - 3 μ) (52 λ^6 - 96 λ^5 μ + 4200 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) +
         8 λ^4 μ (-425 + 18 μ) - 75 λ^2 μ^2 (-925 + 84 μ))) / (15 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 -
        135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) + 50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))),
   s[3] == (2 (-52 λ^7 + 252 λ^6 μ + 8 λ^5 (425 - 54 μ) μ + 144 λ^4 μ^2 (-100 + 3 μ) + 50625 μ^4 (-25 + 4 μ) -
        16875 λ μ^3 (-25 + 12 μ) - 675 λ^2 μ^3 (-375 + 28 μ) + 75 λ^3 μ^2 (-925 + 252 μ))) /
     (15 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
        50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ)))}, {s[1, 2], s[1, 3], s[2, 1], s[3, 1]}]]

{{s[1, 2] →
   - (λ (64 λ^6 + 48 λ^5 μ - 3000 λ^3 μ^2 + 45000 λ μ^3 - 16875 μ^3 (-25 + 4 μ) + 750 λ^2 μ^2 (-25 + 6 μ) -
         8 λ^4 μ (275 + 9 μ))) / (5 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 -
         11250 λ^2 μ^2 (-17 + 2 μ) + 50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))),
  s[1, 3] → - (λ (64 λ^6 + 48 λ^5 μ - 3000 λ^3 μ^2 + 45000 λ μ^3 - 16875 μ^3 (-25 + 4 μ) +
         750 λ^2 μ^2 (-25 + 6 μ) - 8 λ^4 μ (275 + 9 μ))) / (5 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 -
         135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) + 50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))),
  s[2, 1] → (λ (104 λ^6 - 192 λ^5 μ + 6000 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) -
        750 λ^2 μ^2 (-115 + 12 μ) + 4 λ^4 μ (-1375 + 72 μ))) /
     (5 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
        50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))),
  s[3, 1] → (λ (104 λ^6 - 192 λ^5 μ + 6000 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) -
        750 λ^2 μ^2 (-115 + 12 μ) + 4 λ^4 μ (-1375 + 72 μ))) /
     (5 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
        50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ)))}}
```

```
Simplify[
 Solve[{π[1] == (-144 λ^6 + 16875 μ^2 (-150 s[1]^2 + μ (5 + 4 s[1] - 2 s[2] - 2 s[3])^2) + 480 λ^5
        (2 s[1] - s[2] - s[3]) - 22500 λ μ^2 (8 s[1]^2 + 2 s[2]^2 + s[3] (-5 + 2 s[3]) +
          s[2] (-5 + 4 s[3]) - 2 s[1] (-5 + 4 s[2] + 4 s[3])) + 600 λ^3 μ
        (16 s[1]^2 + 4 s[2]^2 + s[3] (5 + 4 s[3]) + s[2] (5 + 8 s[3]) - 2 s[1] (5 + 8 s[2] + 8 s[3])) -
       20 λ^4 (9 μ (5 + 16 s[1] - 8 s[2] - 8 s[3]) +
          10 (17 s[1]^2 - 8 s[1] (s[2] + s[3]) + 2 (s[2] + s[3])^2)) -
       300 λ^2 μ (-25 (22 s[1]^2 - 4 s[1] (s[2] + s[3]) + (s[2] + s[3])^2) + 6 μ (-25 + 8 s[1]^2 +
             2 s[2]^2 + 5 s[3] + 2 s[3]^2 + s[2] (5 + 4 s[3]) - 2 s[1] (5 + 4 s[2] + 4 s[3]))))) /
     (450 (2 λ^2 - 75 μ)^2), π[2] == (-144 λ^8 + 480 λ^7 (s[1] - s[2]) -
       80 λ^6 (18 μ (-5 + s[1] - s[2]) + 5 (s[1]^2 - 2 s[1] s[2] + 10 s[2]^2)) -
       140625 μ^3 (-150 s[2]^2 + μ (5 - 2 s[1] + 4 s[2] - 2 s[3])^2) +
       1200 λ^5 μ (2 s[1]^2 + 20 s[2] + 2 s[2]^2 - s[1] (15 + 4 s[2]) - 5 s[3]) - 15000 λ^3 μ^2 (4 s[1]^2 +
          8 s[2]^2 + s[2] (25 - 4 s[3]) - 10 s[3] + s[1] (-15 - 12 s[2] + 4 s[3])) + 187500 λ μ^3
        (2 s[1]^2 + 8 s[2]^2 + s[2] (10 - 8 s[3]) + s[3] (-5 + 2 s[3]) + s[1] (-5 - 8 s[2] + 4 s[3])) -
       200 λ^4 μ (9 μ (75 + 2 s[1]^2 + 40 s[2] + 2 s[2]^2 - 2 s[1] (15 + 2 s[2]) - 10 s[3]) -
          25 (2 s[1]^2 + s[2] (49 s[2] - 2 s[3]) + s[1] (-6 s[2] + 2 s[3]))) +
       2500 λ^2 μ^2 (18 μ (-5 + s[1] - s[2]) (-5 + 2 s[1] - 4 s[2] + 2 s[3]) -
          25 (s[1]^2 + 67 s[2]^2 - 4 s[2] s[3] + s[3]^2 + s[1] (-4 s[2] + 2 s[3])))) /
     (450 (2 λ^2 - 75 μ) (2 λ^2 - 25 μ)^2), π[3] == (-144 λ^8 + 480 λ^7 (s[1] - s[3]) +
       1200 λ^5 μ (2 s[1]^2 - 5 s[2] + 2 s[3] (10 + s[3]) - s[1] (15 + 4 s[3])) -
       140625 μ^3 (-150 s[3]^2 + μ (5 - 2 s[1] - 2 s[2] + 4 s[3])^2) + 187500 λ μ^3
        (2 s[1]^2 + 2 s[2]^2 + s[1] (-5 + 4 s[2] - 8 s[3]) + 2 s[3] (5 + 4 s[3]) - s[2] (5 + 8 s[3])) -
       80 λ^6 (18 μ (-5 + s[1] - s[3]) + 5 (s[1]^2 - 2 s[1] s[3] + 10 s[3]^2)) +
       2500 λ^2 μ^2 (18 μ (-5 + 2 s[1] + 2 s[2] - 4 s[3]) (-5 + s[1] - s[3]) -
          25 (s[1]^2 + s[2]^2 + 2 s[1] (s[2] - 2 s[3]) - 4 s[2] s[3] + 67 s[3]^2)) - 15000 λ^3 μ^2
        (4 s[1]^2 - 2 s[2] (5 + 2 s[3]) + s[3] (25 + 8 s[3]) + s[1] (4 s[2] - 3 (5 + 4 s[3]))) -
       200 λ^4 μ (9 μ (75 + 2 s[1]^2 - 10 s[2] + 40 s[3] + 2 s[3]^2 - 2 s[1] (15 + 2 s[3])) -
          25 (2 s[1]^2 + 2 s[1] (s[2] - 3 s[3]) + s[3] (-2 s[2] + 49 s[3])))) /
     (450 (2 λ^2 - 75 μ) (2 λ^2 - 25 μ)^2), s[1] == (256 λ^7 - 576 λ^6 μ + 101250 μ^4 (-25 + 4 μ) +
       144 λ^4 μ^2 (25 + 6 μ) - 5400 λ^2 μ^3 (-50 + 7 μ) - 33750 λ μ^3 (-25 + 12 μ) -
       32 λ^5 μ (125 + 27 μ) + 600 λ^3 μ^2 (-100 + 63 μ)) /
     (15 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
          50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))),
   s[2] == -(2 (λ - 3 μ) (52 λ^6 - 96 λ^5 μ + 4200 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) +
          8 λ^4 μ (-425 + 18 μ) - 75 λ^2 μ^2 (-925 + 84 μ))) /
     (15 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
          50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))),
   s[3] == (2 (-52 λ^7 + 252 λ^6 μ + 8 λ^5 (425 - 54 μ) μ + 144 λ^4 μ^2 (-100 + 3 μ) + 50625 μ^4 (-25 + 4 μ) -
         16875 λ μ^3 (-25 + 12 μ) - 675 λ^2 μ^3 (-375 + 28 μ) + 75 λ^3 μ^2 (-925 + 252 μ))) /
     (15 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) + 50625 μ^3
          (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ)))}, {π[1], π[2], π[3]}, {s[1], s[2], s[3]}]]
```

$$
\begin{aligned}
\{\{\pi[1] \to -\Big(&(32\lambda^4 + 24\lambda^3\mu - 600\lambda\mu^2 + 225\mu^2(-25+4\mu) - 4\lambda^2\mu(-25+9\mu))^2 \\
&(272\lambda^6 - 768\lambda^5\mu + 28800\lambda^3\mu^2 - 270000\lambda\mu^3 + 50625\mu^3(-75+8\mu) - \\
&\quad 3600\lambda^2\mu^2(-125+12\mu) + 12\lambda^4\mu(-1525+96\mu))\Big) \Big/ \\
&\Big(450\,(144\lambda^6 - 432\lambda^5\mu + 15000\lambda^3\mu^2 - 135000\lambda\mu^3 - 11250\lambda^2\mu^2(-17+2\mu) + \\
&\quad 50625\mu^3(-25+4\mu) + 8\lambda^4\mu(-1100+81\mu))^2\Big), \\
\pi[2] \to -\Big(&(52\lambda^4 - 96\lambda^3\mu + 1800\lambda\mu^2 - 675\mu^2(-25+4\mu) + 12\lambda^2\mu(-175+12\mu))^2 \\
&(80\lambda^6 - 48\lambda^5\mu + 2400\lambda^3\mu^2 - 30000\lambda\mu^3 + 5625\mu^3(-75+8\mu) + \\
&\quad 4\lambda^4\mu(-1225+18\mu) - 50\lambda^2\mu^2(-1675+72\mu))\Big) \Big/ \\
&\Big(450\,(144\lambda^6 - 432\lambda^5\mu + 15000\lambda^3\mu^2 - 135000\lambda\mu^3 - 11250\lambda^2\mu^2(-17+2\mu) + \\
&\quad 50625\mu^3(-25+4\mu) + 8\lambda^4\mu(-1100+81\mu))^2\Big), \\
\pi[3] \to -\Big(&(52\lambda^4 - 96\lambda^3\mu + 1800\lambda\mu^2 - 675\mu^2(-25+4\mu) + 12\lambda^2\mu(-175+12\mu))^2 \\
&(80\lambda^6 - 48\lambda^5\mu + 2400\lambda^3\mu^2 - 30000\lambda\mu^3 + 5625\mu^3(-75+8\mu) + \\
&\quad 4\lambda^4\mu(-1225+18\mu) - 50\lambda^2\mu^2(-1675+72\mu))\Big) \Big/ \\
&\Big(450\,(144\lambda^6 - 432\lambda^5\mu + 15000\lambda^3\mu^2 - 135000\lambda\mu^3 - 11250\lambda^2\mu^2(-17+2\mu) + \\
&\quad 50625\mu^3(-25+4\mu) + 8\lambda^4\mu(-1100+81\mu))^2\Big)\}\}
\end{aligned}
$$

```
Simplify[Solve[{p[1] == 1/10 (5 μ - 4 λ L[1] + 2 λ L[2] + 2 λ L[3] + 4 μ s[1] - 2 μ s[2] - 2 μ s[3]),
   p[2] == 1/10 (5 μ + 2 λ L[1] - 4 λ L[2] + 2 λ L[3] - 2 μ s[1] + 4 μ s[2] - 2 μ s[3]),
   p[3] == 1/10 (5 μ + 2 λ L[1] + 2 λ L[2] - 4 λ L[3] - 2 μ s[1] - 2 μ s[2] + 4 μ s[3]),
   s[1, 2] == -(λ (64 λ^6 + 48 λ^5 μ - 3000 λ^3 μ^2 + 45000 λ μ^3 - 16875 μ^3 (-25 + 4 μ) +
            750 λ^2 μ^2 (-25 + 6 μ) - 8 λ^4 μ (275 + 9 μ))) / (5 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 -
          135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) + 50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))),
   s[1, 3] == -(λ (64 λ^6 + 48 λ^5 μ - 3000 λ^3 μ^2 + 45000 λ μ^3 - 16875 μ^3 (-25 + 4 μ) +
            750 λ^2 μ^2 (-25 + 6 μ) - 8 λ^4 μ (275 + 9 μ))) / (5 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 -
          135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) + 50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))),
   s[2, 1] == (λ (104 λ^6 - 192 λ^5 μ + 6000 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) -
          750 λ^2 μ^2 (-115 + 12 μ) + 4 λ^4 μ (-1375 + 72 μ))) / (5 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 -
          135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) + 50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))),
   s[3, 1] == (λ (104 λ^6 - 192 λ^5 μ + 6000 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) -
          750 λ^2 μ^2 (-115 + 12 μ) + 4 λ^4 μ (-1375 + 72 μ))) / (5 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 -
          135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) + 50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))),
   s[1] == (256 λ^7 - 576 λ^6 μ + 101250 μ^4 (-25 + 4 μ) + 144 λ^4 μ^2 (25 + 6 μ) - 5400 λ^2 μ^3 (-50 + 7 μ) -
        33750 λ μ^3 (-25 + 12 μ) - 32 λ^5 μ (125 + 27 μ) + 600 λ^3 μ^2 (-100 + 63 μ)) /
      (15 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
          50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))),
   s[2] == -(2 (λ - 3 μ) (52 λ^6 - 96 λ^5 μ + 4200 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) +
            8 λ^4 μ (-425 + 18 μ) - 75 λ^2 μ^2 (-925 + 84 μ))) / (15 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 -
          135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) + 50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))),
   s[3] == (2 (-52 λ^7 + 252 λ^6 μ + 8 λ^5 (425 - 54 μ) μ + 144 λ^4 μ^2 (-100 + 3 μ) + 50625 μ^4 (-25 + 4 μ) -
          16875 λ μ^3 (-25 + 12 μ) - 675 λ^2 μ^3 (-375 + 28 μ) + 75 λ^3 μ^2 (-925 + 252 μ))) /
      (15 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
          50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))),
   L[1] == (s[1]/3 + s[2]/3 - (s[1, 2] + s[2, 1]) - k/2) + (s[1]/3 + s[3]/3 - (s[1, 3] + s[3, 1]) - k/2),
   L[2] == (s[1]/3 + s[2]/3 - (s[1, 2] + s[2, 1]) - k/2) + (s[2]/3 + s[3]/3 - k/2),
   L[3] == (s[3]/3 + s[2]/3 - k/2) + (s[1]/3 + s[3]/3 - (s[1, 3] + s[3, 1]) - k/2)},
  {p[1], p[2], p[3]}, {s[1], s[2], s[3], L[1],
   L[2], L[3], s[1, 2],
   s[1, 3], s[2, 1], s[3, 1]}]]
```

```
{{p[1] →
   -(3 μ (64 λ^6 + 48 λ^5 μ - 3000 λ^3 μ^2 + 45000 λ μ^3 - 16875 μ^3 (-25 + 4 μ) + 750 λ^2 μ^2 (-25 + 6 μ) - 8
            λ^4 μ (275 + 9 μ))) / (2 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 -
          11250 λ^2 μ^2 (-17 + 2 μ) + 50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))),
  p[2] → (3 μ (104 λ^6 - 192 λ^5 μ + 6000 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) -
        750 λ^2 μ^2 (-115 + 12 μ) + 4 λ^4 μ (-1375 + 72 μ))) /
     (2 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
        50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))),
  p[3] → (3 μ (104 λ^6 - 192 λ^5 μ + 6000 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) -
        750 λ^2 μ^2 (-115 + 12 μ) + 4 λ^4 μ (-1375 + 72 μ))) /
     (2 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
        50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ)))}}
```

$$\mathbf{Simplify}\Big[\mathbf{Solve}\Big[$$

$$\Big\{d[2] == -\frac{1}{3\mu}\,(-\mu - \lambda L[1] + 2\lambda L[2] - \lambda L[3] - p[1] + 2\,p[2] - p[3] + \mu\, s[1] - 2\mu\, s[2] + \mu\, s[3]),$$

$$d[1] == -\frac{1}{3\mu}\,(-\mu + 2\lambda L[1] - \lambda L[2] - \lambda L[3] + 2\,p[1] - p[2] - p[3] - 2\mu\, s[1] + \mu\, s[2] + \mu\, s[3]),$$

$$d[3] == -\frac{1}{3\mu}\,(-\mu - \lambda L[1] - \lambda L[2] + 2\lambda L[3] - p[1] - p[2] + 2\,p[3] + \mu\, s[1] + \mu\, s[2] - 2\mu\, s[3]),$$

$$p[1] == \frac{1}{10}\,(5\mu - 4\lambda L[1] + 2\lambda L[2] + 2\lambda L[3] + 4\mu\, s[1] - 2\mu\, s[2] - 2\mu\, s[3]),$$

$$p[2] == \frac{1}{10}\,(5\mu + 2\lambda L[1] - 4\lambda L[2] + 2\lambda L[3] - 2\mu\, s[1] + 4\mu\, s[2] - 2\mu\, s[3]),$$

$$p[3] == \frac{1}{10}\,(5\mu + 2\lambda L[1] + 2\lambda L[2] - 4\lambda L[3] - 2\mu\, s[1] - 2\mu\, s[2] + 4\mu\, s[3]),$$

$$\begin{aligned}s[1,2] == -(&\lambda\,(64\lambda^6 + 48\lambda^5\mu - 3000\lambda^3\mu^2 + 45000\lambda\mu^3 - 16875\mu^3\,(-25 + 4\mu) +\\ &750\lambda^2\mu^2\,(-25 + 6\mu) - 8\lambda^4\mu\,(275 + 9\mu)))\,/\,(5\,(144\lambda^6 - 432\lambda^5\mu + 15000\lambda^3\mu^2 -\\ &135000\lambda\mu^3 - 11250\lambda^2\mu^2\,(-17 + 2\mu) + 50625\mu^3\,(-25 + 4\mu) + 8\lambda^4\mu\,(-1100 + 81\mu))),\end{aligned}$$

$$\begin{aligned}s[1,3] == -(&\lambda\,(64\lambda^6 + 48\lambda^5\mu - 3000\lambda^3\mu^2 + 45000\lambda\mu^3 - 16875\mu^3\,(-25 + 4\mu) +\\ &750\lambda^2\mu^2\,(-25 + 6\mu) - 8\lambda^4\mu\,(275 + 9\mu)))\,/\,(5\,(144\lambda^6 - 432\lambda^5\mu + 15000\lambda^3\mu^2 -\\ &135000\lambda\mu^3 - 11250\lambda^2\mu^2\,(-17 + 2\mu) + 50625\mu^3\,(-25 + 4\mu) + 8\lambda^4\mu\,(-1100 + 81\mu))),\end{aligned}$$

$$\begin{aligned}s[2,1] == (&\lambda\,(104\lambda^6 - 192\lambda^5\mu + 6000\lambda^3\mu^2 - 45000\lambda\mu^3 + 16875\mu^3\,(-25 + 4\mu) -\\ &750\lambda^2\mu^2\,(-115 + 12\mu) + 4\lambda^4\mu\,(-1375 + 72\mu)))\,/\,(5\,(144\lambda^6 - 432\lambda^5\mu + 15000\lambda^3\mu^2 -\\ &135000\lambda\mu^3 - 11250\lambda^2\mu^2\,(-17 + 2\mu) + 50625\mu^3\,(-25 + 4\mu) + 8\lambda^4\mu\,(-1100 + 81\mu))),\end{aligned}$$

$$\begin{aligned}s[3,1] == (&\lambda\,(104\lambda^6 - 192\lambda^5\mu + 6000\lambda^3\mu^2 - 45000\lambda\mu^3 + 16875\mu^3\,(-25 + 4\mu) -\\ &750\lambda^2\mu^2\,(-115 + 12\mu) + 4\lambda^4\mu\,(-1375 + 72\mu)))\,/\,(5\,(144\lambda^6 - 432\lambda^5\mu + 15000\lambda^3\mu^2 -\\ &135000\lambda\mu^3 - 11250\lambda^2\mu^2\,(-17 + 2\mu) + 50625\mu^3\,(-25 + 4\mu) + 8\lambda^4\mu\,(-1100 + 81\mu))),\end{aligned}$$

$$\begin{aligned}s[1] == (&256\lambda^7 - 576\lambda^6\mu + 101250\mu^4\,(-25 + 4\mu) + 144\lambda^4\mu^2\,(25 + 6\mu) - 5400\lambda^2\mu^3\,(-50 + 7\mu) -\\ &33750\lambda\mu^3\,(-25 + 12\mu) - 32\lambda^5\mu\,(125 + 27\mu) + 600\lambda^3\mu^2\,(-100 + 63\mu))\,/\\ &(15\,(144\lambda^6 - 432\lambda^5\mu + 15000\lambda^3\mu^2 - 135000\lambda\mu^3 - 11250\lambda^2\mu^2\,(-17 + 2\mu) +\\ &50625\mu^3\,(-25 + 4\mu) + 8\lambda^4\mu\,(-1100 + 81\mu))),\end{aligned}$$

$$\begin{aligned}s[2] == -(&2\,(\lambda - 3\mu)\,(52\lambda^6 - 96\lambda^5\mu + 4200\lambda^3\mu^2 - 45000\lambda\mu^3 + 16875\mu^3\,(-25 + 4\mu) +\\ &8\lambda^4\mu\,(-425 + 18\mu) - 75\lambda^2\mu^2\,(-925 + 84\mu)))\,/\,(15\,(144\lambda^6 - 432\lambda^5\mu + 15000\lambda^3\mu^2 -\\ &135000\lambda\mu^3 - 11250\lambda^2\mu^2\,(-17 + 2\mu) + 50625\mu^3\,(-25 + 4\mu) + 8\lambda^4\mu\,(-1100 + 81\mu))),\end{aligned}$$

$$\begin{aligned}s[3] == (&2\,(-52\lambda^7 + 252\lambda^6\mu + 8\lambda^5\,(425 - 54\mu)\,\mu + 144\lambda^4\mu^2\,(-100 + 3\mu) + 50625\mu^4\,(-25 + 4\mu) -\\ &16875\lambda\mu^3\,(-25 + 12\mu) - 675\lambda^2\mu^3\,(-375 + 28\mu) + 75\lambda^3\mu^2\,(-925 + 252\mu)))\,/\\ &(15\,(144\lambda^6 - 432\lambda^5\mu + 15000\lambda^3\mu^2 - 135000\lambda\mu^3 - 11250\lambda^2\mu^2\,(-17 + 2\mu) +\\ &50625\mu^3\,(-25 + 4\mu) + 8\lambda^4\mu\,(-1100 + 81\mu))),\end{aligned}$$

$$L[1] == \left(\frac{s[1]}{3} + \frac{s[2]}{3} - (s[1,2] + s[2,1]) - \frac{k}{2}\right) + \left(\frac{s[1]}{3} + \frac{s[3]}{3} - (s[1,3] + s[3,1]) - \frac{k}{2}\right),$$

$$L[2] == \left(\frac{s[1]}{3} + \frac{s[2]}{3} - (s[1,2] + s[2,1]) - \frac{k}{2}\right) + \left(\frac{s[2]}{3} + \frac{s[3]}{3} - \frac{k}{2}\right),$$

$$L[3] == \left(\frac{s[3]}{3} + \frac{s[2]}{3} - \frac{k}{2}\right) + \left(\frac{s[1]}{3} + \frac{s[3]}{3} - (s[1,3] + s[3,1]) - \frac{k}{2}\right)\Big\},$$

$$\{d[1], d[2], d[3]\}, \{s[1], s[2], s[3], L[1], L[2], L[3], s[1,2], s[1,3], s[2,1], s[3,1], p[1], p[2], p[3]\}\Big]\Big]$$

$$\{\{d[1] \to (-64\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 45000\,\lambda\,\mu^3 + 16875\,\mu^3\,(-25 + 4\,\mu) - 750\,\lambda^2\,\mu^2\,(-25 + 6\,\mu) + 8\,\lambda^4\,\mu\,(275 + 9\,\mu))\,/\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 - 135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17 + 2\,\mu) + 50625\,\mu^3\,(-25 + 4\,\mu) + 8\,\lambda^4\,\mu\,(-1100 + 81\,\mu)),$$

$$d[2] \to (104\,\lambda^6 - 192\,\lambda^5\,\mu + 6000\,\lambda^3\,\mu^2 - 45000\,\lambda\,\mu^3 + 16875\,\mu^3\,(-25 + 4\,\mu) - 750\,\lambda^2\,\mu^2\,(-115 + 12\,\mu) + 4\,\lambda^4\,\mu\,(-1375 + 72\,\mu))\,/\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 - 135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17 + 2\,\mu) + 50625\,\mu^3\,(-25 + 4\,\mu) + 8\,\lambda^4\,\mu\,(-1100 + 81\,\mu)),$$

$$d[3] \to (104\,\lambda^6 - 192\,\lambda^5\,\mu + 6000\,\lambda^3\,\mu^2 - 45000\,\lambda\,\mu^3 + 16875\,\mu^3\,(-25 + 4\,\mu) - 750\,\lambda^2\,\mu^2\,(-115 + 12\,\mu) + 4\,\lambda^4\,\mu\,(-1375 + 72\,\mu))\,/\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 - 135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17 + 2\,\mu) + 50625\,\mu^3\,(-25 + 4\,\mu) + 8\,\lambda^4\,\mu\,(-1100 + 81\,\mu))\}\}$$

## AT THE EQUILIBRIUM

$$p[1] \to -(3\,\mu\,(64\,\lambda^6 + 48\,\lambda^5\,\mu - 3000\,\lambda^3\,\mu^2 + 45000\,\lambda\,\mu^3 - 16875\,\mu^3\,(-25 + 4\,\mu) + 750\,\lambda^2\,\mu^2\,(-25 + 6\,\mu) - 8\,\lambda^4\,\mu\,(275 + 9\,\mu)))\,/\,(2\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 - 135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17 + 2\,\mu) + 50625\,\mu^3\,(-25 + 4\,\mu) + 8\,\lambda^4\,\mu\,(-1100 + 81\,\mu)))$$

$$p[2] \to (3\,\mu\,(104\,\lambda^6 - 192\,\lambda^5\,\mu + 6000\,\lambda^3\,\mu^2 - 45000\,\lambda\,\mu^3 + 16875\,\mu^3\,(-25 + 4\,\mu) - 750\,\lambda^2\,\mu^2\,(-115 + 12\,\mu) + 4\,\lambda^4\,\mu\,(-1375 + 72\,\mu)))\,/\,(2\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 - 135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17 + 2\,\mu) + 50625\,\mu^3\,(-25 + 4\,\mu) + 8\,\lambda^4\,\mu\,(-1100 + 81\,\mu)))$$

$$p[3] \to (3\,\mu\,(104\,\lambda^6 - 192\,\lambda^5\,\mu + 6000\,\lambda^3\,\mu^2 - 45000\,\lambda\,\mu^3 + 16875\,\mu^3\,(-25 + 4\,\mu) - 750\,\lambda^2\,\mu^2\,(-115 + 12\,\mu) + 4\,\lambda^4\,\mu\,(-1375 + 72\,\mu)))\,/\,(2\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 - 135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17 + 2\,\mu) + 50625\,\mu^3\,(-25 + 4\,\mu) + 8\,\lambda^4\,\mu\,(-1100 + 81\,\mu)))$$

$$d[1] \to (-64\,\lambda^6 - 48\,\lambda^5\,\mu + 3000\,\lambda^3\,\mu^2 - 45000\,\lambda\,\mu^3 + 16875\,\mu^3\,(-25 + 4\,\mu) - 750\,\lambda^2\,\mu^2\,(-25 + 6\,\mu) + 8\,\lambda^4\,\mu\,(275 + 9\,\mu))\,/\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 - 135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17 + 2\,\mu) + 50625\,\mu^3\,(-25 + 4\,\mu) + 8\,\lambda^4\,\mu\,(-1100 + 81\,\mu))$$

$$d[2] \to (104\,\lambda^6 - 192\,\lambda^5\,\mu + 6000\,\lambda^3\,\mu^2 - 45000\,\lambda\,\mu^3 + 16875\,\mu^3\,(-25 + 4\,\mu) - 750\,\lambda^2\,\mu^2\,(-115 + 12\,\mu) + 4\,\lambda^4\,\mu\,(-1375 + 72\,\mu))\,/\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 - 135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17 + 2\,\mu) + 50625\,\mu^3\,(-25 + 4\,\mu) + 8\,\lambda^4\,\mu\,(-1100 + 81\,\mu))$$

$$d[3] \to (104\,\lambda^6 - 192\,\lambda^5\,\mu + 6000\,\lambda^3\,\mu^2 - 45000\,\lambda\,\mu^3 + 16875\,\mu^3\,(-25 + 4\,\mu) - 750\,\lambda^2\,\mu^2\,(-115 + 12\,\mu) + 4\,\lambda^4\,\mu\,(-1375 + 72\,\mu))\,/\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 - 135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17 + 2\,\mu) + 50625\,\mu^3\,(-25 + 4\,\mu) + 8\,\lambda^4\,\mu\,(-1100 + 81\,\mu))$$

$$\pi[1] \to -\left((32\,\lambda^4 + 24\,\lambda^3\,\mu - 600\,\lambda\,\mu^2 + 225\,\mu^2\,(-25 + 4\,\mu) - 4\,\lambda^2\,\mu\,(-25 + 9\,\mu))^2\,(272\,\lambda^6 - 768\,\lambda^5\,\mu + 28800\,\lambda^3\,\mu^2 - 270000\,\lambda\,\mu^3 + 50625\,\mu^3\,(-75 + 8\,\mu) - 3600\,\lambda^2\,\mu^2\,(-125 + 12\,\mu) + 12\,\lambda^4\,\mu\,(-1525 + 96\,\mu))\right)\Big/\left(450\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 - 135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17 + 2\,\mu) + 50625\,\mu^3\,(-25 + 4\,\mu) + 8\,\lambda^4\,\mu\,(-1100 + 81\,\mu))^2\right)$$

$$\pi[2] \to -\left((52\,\lambda^4 - 96\,\lambda^3\,\mu + 1800\,\lambda\,\mu^2 - 675\,\mu^2\,(-25 + 4\,\mu) + 12\,\lambda^2\,\mu\,(-175 + 12\,\mu))^2\,(80\,\lambda^6 - 48\,\lambda^5\,\mu + 2400\,\lambda^3\,\mu^2 - 30000\,\lambda\,\mu^3 + 5625\,\mu^3\,(-75 + 8\,\mu) + 4\,\lambda^4\,\mu\,(-1225 + 18\,\mu) - 50\,\lambda^2\,\mu^2\,(-1675 + 72\,\mu))\right)\Big/\left(450\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 - 135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17 + 2\,\mu) + 50625\,\mu^3\,(-25 + 4\,\mu) + 8\,\lambda^4\,\mu\,(-1100 + 81\,\mu))^2\right)$$

$\pi[3] \to$

$$-\left((52\lambda^4 - 96\lambda^3\mu + 1800\lambda\mu^2 - 675\mu^2(-25+4\mu) + 12\lambda^2\mu(-175+12\mu))^2 (80\lambda^6 - 48\lambda^5\mu + 2400\lambda^3\mu^2 - 30000\lambda\mu^3 + 5625\mu^3(-75+8\mu) + 4\lambda^4\mu(-1225+18\mu) - 50\lambda^2\mu^2(-1675+72\mu))\right) \Big/ \left(450(144\lambda^6 - 432\lambda^5\mu + 15000\lambda^3\mu^2 - 135000\lambda\mu^3 - 11250\lambda^2\mu^2(-17+2\mu) + 50625\mu^3(-25+4\mu) + 8\lambda^4\mu(-1100+81\mu))^2\right)$$

$$s\{1, 2\} == -(\lambda(64\lambda^6 + 48\lambda^5\mu - 3000\lambda^3\mu^2 + 45000\lambda\mu^3 - 16875\mu^3(-25+4\mu) + 750\lambda^2\mu^2(-25+6\mu) - 8\lambda^4\mu(275+9\mu))) / (5(144\lambda^6 - 432\lambda^5\mu + 15000\lambda^3\mu^2 - 135000\lambda\mu^3 - 11250\lambda^2\mu^2(-17+2\mu) + 50625\mu^3(-25+4\mu) + 8\lambda^4\mu(-1100+81\mu)))$$

$$s[1, 3] == -(\lambda(64\lambda^6 + 48\lambda^5\mu - 3000\lambda^3\mu^2 + 45000\lambda\mu^3 - 16875\mu^3(-25+4\mu) + 750\lambda^2\mu^2(-25+6\mu) - 8\lambda^4\mu(275+9\mu))) / (5(144\lambda^6 - 432\lambda^5\mu + 15000\lambda^3\mu^2 - 135000\lambda\mu^3 - 11250\lambda^2\mu^2(-17+2\mu) + 50625\mu^3(-25+4\mu) + 8\lambda^4\mu(-1100+81\mu)))$$

$$s[2, 1] == (\lambda(104\lambda^6 - 192\lambda^5\mu + 6000\lambda^3\mu^2 - 45000\lambda\mu^3 + 16875\mu^3(-25+4\mu) - 750\lambda^2\mu^2(-115+12\mu) + 4\lambda^4\mu(-1375+72\mu))) / (5(144\lambda^6 - 432\lambda^5\mu + 15000\lambda^3\mu^2 - 135000\lambda\mu^3 - 11250\lambda^2\mu^2(-17+2\mu) + 50625\mu^3(-25+4\mu) + 8\lambda^4\mu(-1100+81\mu)))$$

$$s[3, 1] == (\lambda(104\lambda^6 - 192\lambda^5\mu + 6000\lambda^3\mu^2 - 45000\lambda\mu^3 + 16875\mu^3(-25+4\mu) - 750\lambda^2\mu^2(-115+12\mu) + 4\lambda^4\mu(-1375+72\mu))) / (5(144\lambda^6 - 432\lambda^5\mu + 15000\lambda^3\mu^2 - 135000\lambda\mu^3 - 11250\lambda^2\mu^2(-17+2\mu) + 50625\mu^3(-25+4\mu) + 8\lambda^4\mu(-1100+81\mu)))$$

$$s[1] \to (256\lambda^7 - 576\lambda^6\mu + 101250\mu^4(-25+4\mu) + 144\lambda^4\mu^2(25+6\mu) - 5400\lambda^2\mu^3(-50+7\mu) - 33750\lambda\mu^3(-25+12\mu) - 32\lambda^5\mu(125+27\mu) + 600\lambda^3\mu^2(-100+63\mu)) / (15(144\lambda^6 - 432\lambda^5\mu + 15000\lambda^3\mu^2 - 135000\lambda\mu^3 - 11250\lambda^2\mu^2(-17+2\mu) + 50625\mu^3(-25+4\mu) + 8\lambda^4\mu(-1100+81\mu)))$$

$$s[2] \to -(2(\lambda - 3\mu)(52\lambda^6 - 96\lambda^5\mu + 4200\lambda^3\mu^2 - 45000\lambda\mu^3 + 16875\mu^3(-25+4\mu) + 8\lambda^4\mu(-425+18\mu) - 75\lambda^2\mu^2(-925+84\mu))) / (15(144\lambda^6 - 432\lambda^5\mu + 15000\lambda^3\mu^2 - 135000\lambda\mu^3 - 11250\lambda^2\mu^2(-17+2\mu) + 50625\mu^3(-25+4\mu) + 8\lambda^4\mu(-1100+81\mu)))$$

$$s[3] \to (2(-52\lambda^7 + 252\lambda^6\mu + 8\lambda^5(425-54\mu)\mu + 144\lambda^4\mu^2(-100+3\mu) + 50625\mu^4(-25+4\mu) - 16875\lambda\mu^3(-25+12\mu) - 675\lambda^2\mu^3(-375+28\mu) + 75\lambda^3\mu^2(-925+252\mu))) / (15(144\lambda^6 - 432\lambda^5\mu + 15000\lambda^3\mu^2 - 135000\lambda\mu^3 - 11250\lambda^2\mu^2(-17+2\mu) + 50625\mu^3(-25+4\mu) + 8\lambda^4\mu(-1100+81\mu)))$$

## PROOF OF EQUATIONS (1) & (2)

$$\pi[i]\ \text{case1} \to (-1424\lambda^6 + 768\lambda^5\mu - 28800\lambda^3\mu^2 + 270000\lambda\mu^3 - 50625\mu^3(-75+8\mu) + 1800\lambda^2\mu^2(-475+24\mu) - 12\lambda^4\mu(-5125+96\mu)) \Big/ \left(36450(2\lambda^2 - 25\mu)^2\right)$$

$$\pi[i]\ \text{case2} \to \frac{-8\lambda^2 + 675\mu + 48\lambda\mu - 72\mu^2}{4050},$$

$\pi[\mathbf{i}]$ **case3' →**

$$-\Big(25\,\mu^2\,(16\,\lambda^2-24\,\lambda\,\mu+9\,\mu\,(-25+4\,\mu))^2\,(32\,\lambda^{10}-192\,\lambda^9\,\mu+24000\,\lambda^7\,\mu^2-1110000\,\lambda^5\,\mu^3+22500000\,\lambda^3\,\mu^4-168750000\,\lambda\,\mu^5+31640625\,\mu^5\,(-75+8\,\mu)+32\,\lambda^8\,\mu\,(-125+9\,\mu)-1406250\,\lambda^2\,\mu^4\,(-245+24\,\mu)-1000\,\lambda^6\,\mu^2\,(-365+36\,\mu)+15000\,\lambda^4\,\mu^3\,(-1150+111\,\mu))\Big)\Big/\Big(2\,(4\,\lambda^2-75\,\mu)^2\,(8\,\lambda^6-48\,\lambda^5\,\mu+3000\,\lambda^3\,\mu^2-4500\,\lambda^2\,(-35+\mu)\,\mu^2-135000\,\lambda\,\mu^3+50625\,\mu^3\,(-25+4\,\mu)+4\,\lambda^4\,\mu\,(-1025+18\,\mu))^2\Big)$$

$\pi[\mathbf{i}]$ **case3 →**

$$\Big(\mu\,(48\,\lambda^4-2000\,\lambda^2\,\mu+1200\,\lambda\,\mu^2-225\,\mu^2\,(-75+8\,\mu))\,(8\,\lambda^6-48\,\lambda^5\,\mu+3000\,\lambda^3\,\mu^2-45000\,\lambda\,\mu^3-1500\,\lambda^2\,\mu^2\,(-65+3\,\mu)+16875\,\mu^3\,(-25+4\,\mu)+4\,\lambda^4\,\mu\,(-1025+18\,\mu))^2\Big)\Big/\Big(2\,(4\,\lambda^2-75\,\mu)^2\,(8\,\lambda^6-48\,\lambda^5\,\mu+3000\,\lambda^3\,\mu^2-4500\,\lambda^2\,(-35+\mu)\,\mu^2-135000\,\lambda\,\mu^3+50625\,\mu^3\,(-25+4\,\mu)+4\,\lambda^4\,\mu\,(-1025+18\,\mu))^2\Big)$$

$\pi[\mathbf{i}]$ **case4' →**

$$-\Big((32\,\lambda^4+24\,\lambda^3\,\mu-600\,\lambda\,\mu^2+225\,\mu^2\,(-25+4\,\mu)-4\,\lambda^2\,\mu\,(-25+9\,\mu))^2\,(272\,\lambda^6-768\,\lambda^5\,\mu+28800\,\lambda^3\,\mu^2-270000\,\lambda\,\mu^3+50625\,\mu^3\,(-75+8\,\mu)-3600\,\lambda^2\,\mu^2\,(-125+12\,\mu)+12\,\lambda^4\,\mu\,(-1525+96\,\mu))\Big)\Big/\Big(450\,(144\,\lambda^6-432\,\lambda^5\,\mu+15000\,\lambda^3\,\mu^2-135000\,\lambda\,\mu^3-11250\,\lambda^2\,\mu^2\,(-17+2\,\mu)+50625\,\mu^3\,(-25+4\,\mu)+8\,\lambda^4\,\mu\,(-1100+81\,\mu))^2\Big)$$

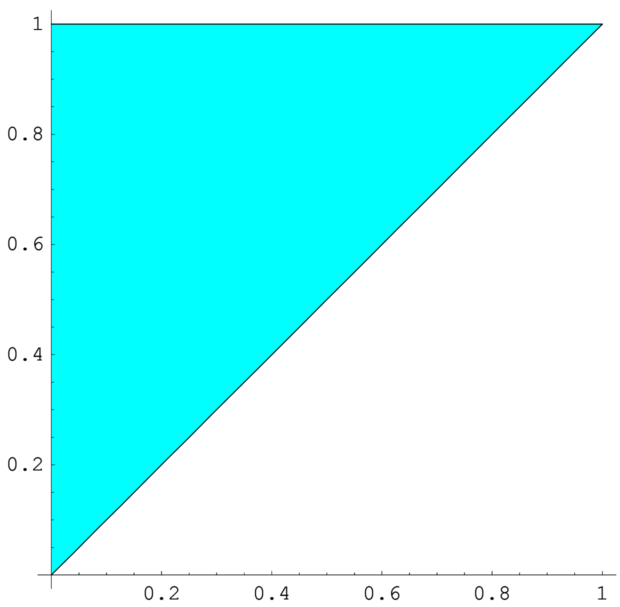
$\pi[\mathbf{i}]$ **case4 →**

$$-\Big((52\,\lambda^4-96\,\lambda^3\,\mu+1800\,\lambda\,\mu^2-675\,\mu^2\,(-25+4\,\mu)+12\,\lambda^2\,\mu\,(-175+12\,\mu))^2\,(80\,\lambda^6-48\,\lambda^5\,\mu+2400\,\lambda^3\,\mu^2-30000\,\lambda\,\mu^3+5625\,\mu^3\,(-75+8\,\mu)+4\,\lambda^4\,\mu\,(-1225+18\,\mu)-50\,\lambda^2\,\mu^2\,(-1675+72\,\mu))\Big)\Big/\Big(450\,(144\,\lambda^6-432\,\lambda^5\,\mu+15000\,\lambda^3\,\mu^2-135000\,\lambda\,\mu^3-11250\,\lambda^2\,\mu^2\,(-17+2\,\mu)+50625\,\mu^3\,(-25+4\,\mu)+8\,\lambda^4\,\mu\,(-1100+81\,\mu))^2\Big)$$

## CONJECTURE

$\pi[\mathbf{i}]$ **case4'** $>$ $\pi[\mathbf{i}]$ **case3'** $>$ $\pi[\mathbf{i}]$ **case2** $>$ $\pi[\mathbf{i}]$ **case1** $>$ $\pi[\mathbf{i}]$ **case4** $>$ $\pi[\mathbf{i}]$ **case3**

```
InequalityPlot[{-((32 λ^4 + 24 λ^3 μ - 600 λ μ^2 + 225 μ^2 (-25 + 4 μ) - 4 λ^2 μ (-25 + 9 μ))^2
      (272 λ^6 - 768 λ^5 μ + 28800 λ^3 μ^2 - 270000 λ μ^3 + 50625 μ^3 (-75 + 8 μ) -
        3600 λ^2 μ^2 (-125 + 12 μ) + 12 λ^4 μ (-1525 + 96 μ))) /
    (450 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
        50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))^2) >
   -(25 μ^2 (16 λ^2 - 24 λ μ + 9 μ (-25 + 4 μ))^2 (32 λ^10 - 192 λ^9 μ + 24000 λ^7 μ^2 - 1110000 λ^5 μ^3 +
        22500000 λ^3 μ^4 - 168750000 λ μ^5 + 31640625 μ^5 (-75 + 8 μ) + 32 λ^8 μ (-125 + 9 μ) -
        1406250 λ^2 μ^4 (-245 + 24 μ) - 1000 λ^6 μ^2 (-365 + 36 μ) + 15000 λ^4 μ^3 (-1150 + 111 μ))) /
    (2 (4 λ^2 - 75 μ)^2 (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 +
        50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))^2) > (-8 λ^2 + 675 μ + 48 λ μ - 72 μ^2)/4050 >
   (-1424 λ^6 + 768 λ^5 μ - 28800 λ^3 μ^2 + 270000 λ μ^3 - 50625 μ^3 (-75 + 8 μ) +
     1800 λ^2 μ^2 (-475 + 24 μ) - 12 λ^4 μ (-5125 + 96 μ)) / (36450 (2 λ^2 - 25 μ)^2) >
   -((52 λ^4 - 96 λ^3 μ + 1800 λ μ^2 - 675 μ^2 (-25 + 4 μ) + 12 λ^2 μ (-175 + 12 μ))^2 (80 λ^6 - 48 λ^5 μ + 2400
          λ^3 μ^2 - 30000 λ μ^3 + 5625 μ^3 (-75 + 8 μ) + 4 λ^4 μ (-1225 + 18 μ) - 50 λ^2 μ^2 (-1675 + 72 μ))) /
    (450 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
        50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))^2) >
   (μ (48 λ^4 - 2000 λ^2 μ + 1200 λ μ^2 - 225 μ^2 (-75 + 8 μ)) (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
        45000 λ μ^3 - 1500 λ^2 μ^2 (-65 + 3 μ) + 16875 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))^2) /
    (2 (4 λ^2 - 75 μ)^2 (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 +
        50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))^2), 0 < λ < μ ≤ 1}, {λ, 0, 1}, {μ, 0, 1}]
```



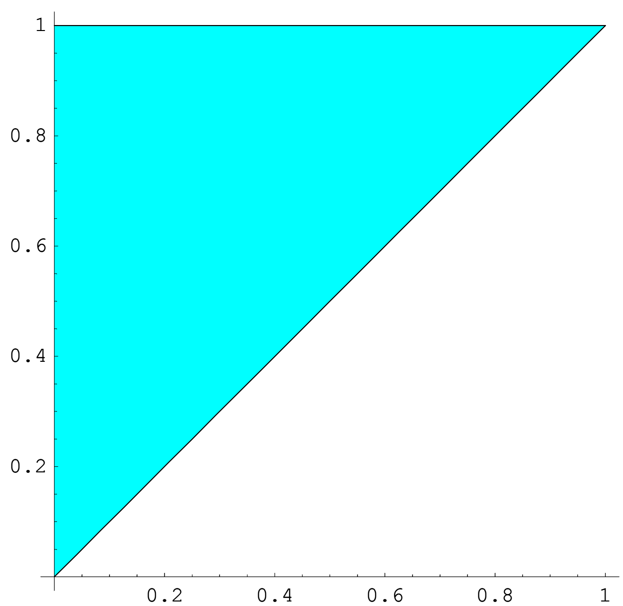

- Graphics -

## PROOF OF EQUATION (3)

**Define Joint Profits as $\sigma$**
**$\sigma$case1 = 3 * $\pi$[i] case1**
**$\sigma$case2 = 3 * $\pi$[i] case2**
**$\sigma$case3 = 2 * $\pi$[i] case3' + $\pi$[i] case3**
**$\sigma$case4 = $\pi$[i] case4' + 2 * $\pi$[i] case4**

### CONJECTURE : $\sigma$case2 > $\sigma$case3

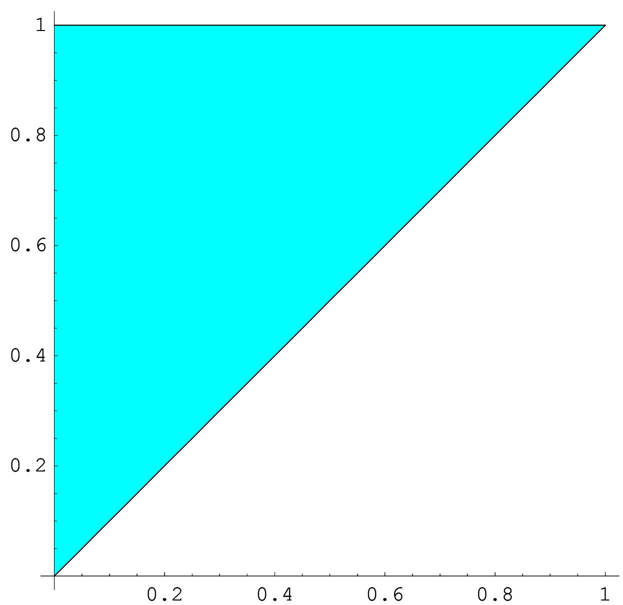
$$\text{InequalityPlot}\Big[\Big\{3*\left(\frac{-8\lambda^2+675\mu+48\lambda\mu-72\mu^2}{4050}\right) >$$
$$2*\Big(-\big(25\mu^2\,(16\lambda^2-24\lambda\mu+9\mu\,(-25+4\mu))^2\,(32\lambda^{10}-192\lambda^9\mu+24000\lambda^7\mu^2-1110000\lambda^5\mu^3+$$
$$22500000\lambda^3\mu^4-168750000\lambda\mu^5+31640625\mu^5\,(-75+8\mu)+32\lambda^8\mu\,(-125+9\mu)-$$
$$1406250\lambda^2\mu^4\,(-245+24\mu)-1000\lambda^6\mu^2\,(-365+36\mu)+15000\lambda^4\mu^3\,(-1150+111\mu))\big)\Big/$$
$$\big(2\,(4\lambda^2-75\mu)^2\,(8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-4500\lambda^2\,(-35+\mu)\,\mu^2-135000\lambda\mu^3+$$
$$50625\mu^3\,(-25+4\mu)+4\lambda^4\mu\,(-1025+18\mu))^2\big)\Big)+$$
$$\Big(\big(\mu\,(48\lambda^4-2000\lambda^2\mu+1200\lambda\mu^2-225\mu^2\,(-75+8\mu))\,(8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-$$
$$45000\lambda\mu^3-1500\lambda^2\mu^2\,(-65+3\mu)+16875\mu^3\,(-25+4\mu)+4\lambda^4\mu\,(-1025+18\mu))^2\big)\Big/$$
$$\big(2\,(4\lambda^2-75\mu)^2\,(8\lambda^6-48\lambda^5\mu+3000\lambda^3\mu^2-4500\lambda^2\,(-35+\mu)\,\mu^2-135000\lambda\mu^3+$$
$$50625\mu^3\,(-25+4\mu)+4\lambda^4\mu\,(-1025+18\mu))^2\big)\Big),\ 0<\lambda<\mu\leq 1\Big\},\ \{\lambda,0,1\},\ \{\mu,0,1\}\Big]$$




- Graphics -

### CONJECTURE : $\sigma$case3 > $\sigma$case4

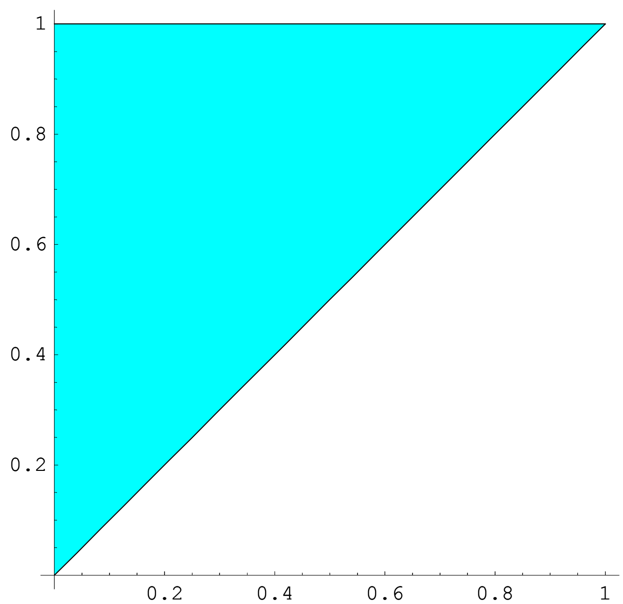
```
InequalityPlot[
 {2 * (-(25 μ^2 (16 λ^2 - 24 λ μ + 9 μ (-25 + 4 μ))^2 (32 λ^10 - 192 λ^9 μ + 24000 λ^7 μ^2 - 1110000 λ^5 μ^3 +
          22500000 λ^3 μ^4 - 168750000 λ μ^5 + 31640625 μ^5 (-75 + 8 μ) + 32 λ^8 μ (-125 + 9 μ) -
          1406250 λ^2 μ^4 (-245 + 24 μ) - 1000 λ^6 μ^2 (-365 + 36 μ) + 15000 λ^4 μ^3 (-1150 + 111 μ))) /
      (2 (4 λ^2 - 75 μ)^2 (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 +
          50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))^2)) +
   ((μ (48 λ^4 - 2000 λ^2 μ + 1200 λ μ^2 - 225 μ^2 (-75 + 8 μ)) (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 -
          45000 λ μ^3 - 1500 λ^2 μ^2 (-65 + 3 μ) + 16875 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))^2) /
     (2 (4 λ^2 - 75 μ)^2 (8 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 4500 λ^2 (-35 + μ) μ^2 - 135000 λ μ^3 +
          50625 μ^3 (-25 + 4 μ) + 4 λ^4 μ (-1025 + 18 μ))^2)) >
  (-((32 λ^4 + 24 λ^3 μ - 600 λ μ^2 + 225 μ^2 (-25 + 4 μ) - 4 λ^2 μ (-25 + 9 μ))^2
        (272 λ^6 - 768 λ^5 μ + 28800 λ^3 μ^2 - 270000 λ μ^3 + 50625 μ^3 (-75 + 8 μ) -
          3600 λ^2 μ^2 (-125 + 12 μ) + 12 λ^4 μ (-1525 + 96 μ))) /
     (450 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
          50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))^2)) +
   2 * (-((52 λ^4 - 96 λ^3 μ + 1800 λ μ^2 - 675 μ^2 (-25 + 4 μ) + 12 λ^2 μ (-175 + 12 μ))^2
        (80 λ^6 - 48 λ^5 μ + 2400 λ^3 μ^2 - 30000 λ μ^3 + 5625 μ^3 (-75 + 8 μ) +
          4 λ^4 μ (-1225 + 18 μ) - 50 λ^2 μ^2 (-1675 + 72 μ))) /
      (450 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
          50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))^2)), 0 < λ < μ ≤ 1}, {λ, 0, 1}, {μ, 0, 1}]
```




- Graphics -

**CONJECTURE : $\sigma$case4 > $\sigma$case1**

```
InequalityPlot[
 {(-((32 λ^4 + 24 λ^3 μ - 600 λ μ^2 + 225 μ^2 (-25 + 4 μ) - 4 λ^2 μ (-25 + 9 μ))^2 (272 λ^6 - 768 λ^5 μ +
          28800 λ^3 μ^2 - 270000 λ μ^3 + 50625 μ^3 (-75 + 8 μ) - 3600 λ^2 μ^2 (-125 + 12 μ) +
          12 λ^4 μ (-1525 + 96 μ))) / (450 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 -
          11250 λ^2 μ^2 (-17 + 2 μ) + 50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))^2)) +
    2 * (-((52 λ^4 - 96 λ^3 μ + 1800 λ μ^2 - 675 μ^2 (-25 + 4 μ) + 12 λ^2 μ (-175 + 12 μ))^2
          (80 λ^6 - 48 λ^5 μ + 2400 λ^3 μ^2 - 30000 λ μ^3 + 5625 μ^3 (-75 + 8 μ) +
           4 λ^4 μ (-1225 + 18 μ) - 50 λ^2 μ^2 (-1675 + 72 μ))) /
       (450 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
           50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))^2)) >
   3 * ((-1424 λ^6 + 768 λ^5 μ - 28800 λ^3 μ^2 + 270000 λ μ^3 - 50625 μ^3 (-75 + 8 μ) +
         1800 λ^2 μ^2 (-475 + 24 μ) - 12 λ^4 μ (-5125 + 96 μ)) /
      (36450 (2 λ^2 - 25 μ)^2)), 0 < λ < μ ≤ 1}, {λ, 0, 1}, {μ, 0, 1}]
```




- Graphics -

## COMPUTATIONS PERTAINING TO CASE 5

```
U[i] = V - μ * (F[i]) - p[i]

F[i] == d[i] - s[i]


c[1] == (s[1])^2

c[2] == (s[2])^2

c[3] == (s[3])^2
```

## FIND THE DEMANDS

```
Solve[{-μ * (d[1] - s[1]) - p[1] == -μ * (d[2] - s[2]) - p[2],
  -μ * (d[1] - s[1]) - p[1] == -μ * (d[3] - s[3]) - p[3],
  d[1] + d[2] + d[3] == 1}, {d[1], d[2], d[3]}]
```

$$\left\{\left\{d[2] \to -\frac{-\mu - p[1] + 2\,p[2] - p[3] + \mu\,s[1] - 2\,\mu\,s[2] + \mu\,s[3]}{3\,\mu},\right.\right.$$

$$d[1] \to -\frac{-\mu + 2\,p[1] - p[2] - p[3] - 2\,\mu\,s[1] + \mu\,s[2] + \mu\,s[3]}{3\,\mu},$$

$$\left.\left.d[3] \to -\frac{-\mu - p[1] - p[2] + 2\,p[3] + \mu\,s[1] + \mu\,s[2] - 2\,\mu\,s[3]}{3\,\mu}\right\}\right\}$$

$$\pi[1] == p[1] * \left(-\frac{-\mu + 2\,p[1] - p[2] - p[3] - 2\,\mu\,s[1] + \mu\,s[2] + \mu\,s[3]}{3\,\mu}\right) - c[1]$$

$$\pi[2] == p[2] * \left(-\frac{-\mu - p[1] + 2\,p[2] - p[3] + \mu\,s[1] - 2\,\mu\,s[2] + \mu\,s[3]}{3\,\mu}\right) - c[2]$$

$$\pi[3] == p[3] * \left(-\frac{-\mu - p[1] - p[2] + 2\,p[3] + \mu\,s[1] + \mu\,s[2] - 2\,\mu\,s[3]}{3\,\mu}\right) - c[3]$$

## SOLVE FOR THE LAST STAGE, FIND PRICES

$$D\left[p[1] * \left(-\frac{-\mu + 2\,p[1] - p[2] - p[3] - 2\,\mu\,s[1] + \mu\,s[2] + \mu\,s[3]}{3\,\mu}\right) - c[1],\ p[1]\right]$$

$$-\frac{2\,p[1]}{3\,\mu} - \frac{-\mu + 2\,p[1] - p[2] - p[3] - 2\,\mu\,s[1] + \mu\,s[2] + \mu\,s[3]}{3\,\mu}$$

$$D\left[p[2] * \left(-\frac{-\mu - p[1] + 2\,p[2] - p[3] + \mu\,s[1] - 2\,\mu\,s[2] + \mu\,s[3]}{3\,\mu}\right) - c[2],\ p[2]\right]$$

$$-\frac{2\,p[2]}{3\,\mu} - \frac{-\mu - p[1] + 2\,p[2] - p[3] + \mu\,s[1] - 2\,\mu\,s[2] + \mu\,s[3]}{3\,\mu}$$

$$D\left[p[3] * \left(-\frac{-\mu - p[1] - p[2] + 2\,p[3] + \mu\,s[1] + \mu\,s[2] - 2\,\mu\,s[3]}{3\,\mu}\right) - c[3],\ p[3]\right]$$

$$-\frac{2\,p[3]}{3\,\mu} - \frac{-\mu - p[1] - p[2] + 2\,p[3] + \mu\,s[1] + \mu\,s[2] - 2\,\mu\,s[3]}{3\,\mu}$$

$$\text{Solve}\left[\left\{-\frac{2\,p[1]}{3\,\mu} - \frac{-\mu + 2\,p[1] - p[2] - p[3] - 2\,\mu\,s[1] + \mu\,s[2] + \mu\,s[3]}{3\,\mu} == 0,\right.\right.$$

$$-\frac{2\,p[2]}{3\,\mu} - \frac{-\mu - p[1] + 2\,p[2] - p[3] + \mu\,s[1] - 2\,\mu\,s[2] + \mu\,s[3]}{3\,\mu} == 0,$$

$$\left.\left.-\frac{2\,p[3]}{3\,\mu} - \frac{-\mu - p[1] - p[2] + 2\,p[3] + \mu\,s[1] + \mu\,s[2] - 2\,\mu\,s[3]}{3\,\mu} == 0\right\},\ \{p[1],\ p[2],\ p[3]\}\right]$$

$$\left\{\left\{p[1] \to \frac{1}{10}\,(5\,\mu + 4\,\mu\,s[1] - 2\,\mu\,s[2] - 2\,\mu\,s[3]),\right.\right.$$

$$\left.\left.p[2] \to \frac{1}{10}\,(5\,\mu - 2\,\mu\,s[1] + 4\,\mu\,s[2] - 2\,\mu\,s[3]),\ p[3] \to -\frac{1}{10}\,\mu\,(-5 + 2\,s[1] + 2\,s[2] - 4\,s[3])\right\}\right\}$$

$$\text{Solve}\Big[\Big\{\pi[1] == p[1] * \left(-\frac{-\mu + 2\,p[1] - p[2] - p[3] - 2\,\mu\,s[1] + \mu\,s[2] + \mu\,s[3]}{3\,\mu}\right) - c[1],$$
$$\pi[2] == p[2] * \left(-\frac{-\mu - p[1] + 2\,p[2] - p[3] + \mu\,s[1] - 2\,\mu\,s[2] + \mu\,s[3]}{3\,\mu}\right) - c[2],$$
$$\pi[3] == p[3] * \left(-\frac{-\mu - p[1] - p[2] + 2\,p[3] + \mu\,s[1] + \mu\,s[2] - 2\,\mu\,s[3]}{3\,\mu}\right) - c[3],\ c[1] == (s[1])^2,$$
$$c[2] == (s[2])^2,\ c[3] == (s[3])^2,\ p[1] == \frac{1}{10}\,(5\,\mu + 4\,\mu\,s[1] - 2\,\mu\,s[2] - 2\,\mu\,s[3]),$$
$$p[2] == \frac{1}{10}\,(5\,\mu - 2\,\mu\,s[1] + 4\,\mu\,s[2] - 2\,\mu\,s[3]),\ p[3] == -\frac{1}{10}\,\mu\,(-5 + 2\,s[1] + 2\,s[2] - 4\,s[3])\Big\},$$
$$\{\pi[1], \pi[2], \pi[3]\}, \{p[1], p[2], p[3], c[1], c[2], c[3]\}\Big]$$

$$\Big\{\Big\{\pi[1] \to \frac{1}{150}\,(25\,\mu + 40\,\mu\,s[1] - 150\,s[1]^2 + 16\,\mu\,s[1]^2 - 20\,\mu\,s[2] -$$
$$16\,\mu\,s[1]\,s[2] + 4\,\mu\,s[2]^2 - 20\,\mu\,s[3] - 16\,\mu\,s[1]\,s[3] + 8\,\mu\,s[2]\,s[3] + 4\,\mu\,s[3]^2),$$
$$\pi[2] \to \frac{1}{150}\,(25\,\mu - 20\,\mu\,s[1] + 4\,\mu\,s[1]^2 + 40\,\mu\,s[2] - 16\,\mu\,s[1]\,s[2] - 150\,s[2]^2 +$$
$$16\,\mu\,s[2]^2 - 20\,\mu\,s[3] + 8\,\mu\,s[1]\,s[3] - 16\,\mu\,s[2]\,s[3] + 4\,\mu\,s[3]^2),$$
$$\pi[3] \to \frac{1}{150}\,(25\,\mu - 20\,\mu\,s[1] + 4\,\mu\,s[1]^2 - 20\,\mu\,s[2] + 8\,\mu\,s[1]\,s[2] + 4\,\mu\,s[2]^2 +$$
$$40\,\mu\,s[3] - 16\,\mu\,s[1]\,s[3] - 16\,\mu\,s[2]\,s[3] - 150\,s[3]^2 + 16\,\mu\,s[3]^2)\Big\}\Big\}$$

## SOLVE FOR NETWORK CAPACITIES

$$D\Big[\frac{1}{150}\,(25\,\mu + 40\,\mu\,s[1] - 150\,s[1]^2 + 16\,\mu\,s[1]^2 - 20\,\mu\,s[2] - 16\,\mu\,s[1]\,s[2] +$$
$$4\,\mu\,s[2]^2 - 20\,\mu\,s[3] - 16\,\mu\,s[1]\,s[3] + 8\,\mu\,s[2]\,s[3] + 4\,\mu\,s[3]^2),\ s[1]\Big]$$

$$\frac{1}{150}\,(40\,\mu - 300\,s[1] + 32\,\mu\,s[1] - 16\,\mu\,s[2] - 16\,\mu\,s[3])$$

$$D\Big[\frac{1}{150}\,(25\,\mu - 20\,\mu\,s[1] + 4\,\mu\,s[1]^2 + 40\,\mu\,s[2] - 16\,\mu\,s[1]\,s[2] - 150\,s[2]^2 +$$
$$16\,\mu\,s[2]^2 - 20\,\mu\,s[3] + 8\,\mu\,s[1]\,s[3] - 16\,\mu\,s[2]\,s[3] + 4\,\mu\,s[3]^2),\ s[2]\Big]$$

$$\frac{1}{150}\,(40\,\mu - 16\,\mu\,s[1] - 300\,s[2] + 32\,\mu\,s[2] - 16\,\mu\,s[3])$$

$$D\Big[\frac{1}{150}\,(25\,\mu - 20\,\mu\,s[1] + 4\,\mu\,s[1]^2 - 20\,\mu\,s[2] + 8\,\mu\,s[1]\,s[2] + 4\,\mu\,s[2]^2 +$$
$$40\,\mu\,s[3] - 16\,\mu\,s[1]\,s[3] - 16\,\mu\,s[2]\,s[3] - 150\,s[3]^2 + 16\,\mu\,s[3]^2),\ s[3]\Big]$$

$$\frac{1}{150}\,(40\,\mu - 16\,\mu\,s[1] - 16\,\mu\,s[2] - 300\,s[3] + 32\,\mu\,s[3])$$

$$\text{Solve}\Big[\Big\{\frac{1}{150}\,(40\,\mu - 300\,s[1] + 32\,\mu\,s[1] - 16\,\mu\,s[2] - 16\,\mu\,s[3]) == 0,$$
$$\frac{1}{150}\,(40\,\mu - 16\,\mu\,s[1] - 300\,s[2] + 32\,\mu\,s[2] - 16\,\mu\,s[3]) == 0,$$
$$\frac{1}{150}\,(40\,\mu - 16\,\mu\,s[1] - 16\,\mu\,s[2] - 300\,s[3] + 32\,\mu\,s[3]) == 0\Big\},\ \{s[1], s[2], s[3]\}\Big]$$

$$\Big\{\Big\{s[1] \to \frac{2\,\mu}{15},\ s[2] \to \frac{2\,\mu}{15},\ s[3] \to \frac{2\,\mu}{15}\Big\}\Big\}$$

$$\texttt{Solve}\Big[\Big\{p[1] == \frac{1}{10}\,(5\,\mu + 4\,\mu\, s[1] - 2\,\mu\, s[2] - 2\,\mu\, s[3]),$$
$$p[2] == \frac{1}{10}\,(5\,\mu - 2\,\mu\, s[1] + 4\,\mu\, s[2] - 2\,\mu\, s[3]),\ p[3] == -\frac{1}{10}\,\mu\,(-5 + 2\, s[1] + 2\, s[2] - 4\, s[3]),$$
$$s[1] == \frac{2\,\mu}{15},\ s[2] == \frac{2\,\mu}{15},\ s[3] == \frac{2\,\mu}{15}\Big\},\ \{p[1], p[2], p[3]\},\ \{s[1], s[2], s[3]\}\Big]$$

$$\Big\{\Big\{p[1] \to \frac{\mu}{2},\ p[2] \to \frac{\mu}{2},\ p[3] \to \frac{\mu}{2}\Big\}\Big\}$$

$$\texttt{Solve}\Big[\Big\{p[1] == \frac{\mu}{2},\ p[2] == \frac{\mu}{2},\ p[3] == \frac{\mu}{2},\ s[1] == \frac{2\,\mu}{15},\ s[2] == \frac{2\,\mu}{15},$$
$$s[3] == \frac{2\,\mu}{15},\ d[2] == -\frac{-\mu - p[1] + 2\, p[2] - p[3] + \mu\, s[1] - 2\,\mu\, s[2] + \mu\, s[3]}{3\,\mu},$$
$$d[1] == -\frac{-\mu + 2\, p[1] - p[2] - p[3] - 2\,\mu\, s[1] + \mu\, s[2] + \mu\, s[3]}{3\,\mu},$$
$$d[3] == -\frac{-\mu - p[1] - p[2] + 2\, p[3] + \mu\, s[1] + \mu\, s[2] - 2\,\mu\, s[3]}{3\,\mu}\Big\},$$
$$\{d[1], d[2], d[3]\},\ \{s[1], s[2], s[3], p[1].p[2], p[3]\}\Big]$$

$$\Big\{\Big\{d[1] \to \frac{1}{3},\ d[2] \to \frac{1}{3},\ d[3] \to \frac{1}{3}\Big\}\Big\}$$

$$\texttt{Solve}\Big[\Big\{d[2] == \frac{1}{3},\ d[1] == \frac{1}{3},\ d[3] == \frac{1}{3},\ s[1] == \frac{2\,\mu}{15},\ s[2] == \frac{2\,\mu}{15},$$
$$s[3] == \frac{2\,\mu}{15},\ p[1] == \frac{\mu}{2},\ p[2] == \frac{\mu}{2},\ p[3] == \frac{\mu}{2},\ c[1] == (s[1])^2,\ c[2] == (s[2])^2,$$
$$c[3] == (s[3])^2,\ \pi[1] == p[1] * d[1] - c[1],\ \pi[2] == p[2] * d[2] - c[2],$$
$$\pi[3] == p[3] * d[3] - c[3]\Big\},\ \{\pi[1], \pi[2], \pi[3], c[1], c[2], c[3]\},$$
$$\{s[1], s[2], s[3], p[1], p[2], p[3], d[1], d[1], d[3]\}\Big]$$

$$\Big\{\Big\{\pi[1] \to -\frac{1}{450}\,\mu\,(-75 + 8\,\mu),\ \pi[2] \to -\frac{1}{450}\,\mu\,(-75 + 8\,\mu),$$
$$\pi[3] \to -\frac{1}{450}\,\mu\,(-75 + 8\,\mu),\ c[1] \to \frac{4\,\mu^2}{225},\ c[2] \to \frac{4\,\mu^2}{225},\ c[3] \to \frac{4\,\mu^2}{225}\Big\}\Big\}$$

### AT THE EQUILIBRIUM

$$s[1] \to \frac{2\,\mu}{15},$$
$$s[2] \to \frac{2\,\mu}{15},$$
$$s[3] \to \frac{2\,\mu}{15},$$

$p[1] \to \frac{\mu}{2}$,

$p[2] \to \frac{\mu}{2}$,

$p[3] \to \frac{\mu}{2}$,

$d[1] \to \frac{1}{3}$,

$d[2] \to \frac{1}{3}$,

$d[3] \to \frac{1}{3}$,

$\pi[1] \to -\frac{1}{450}\,\mu\,(-75 + 8\,\mu)$,

$\pi[2] \to -\frac{1}{450}\,\mu\,(-75 + 8\,\mu)$,

$\pi[3] \to -\frac{1}{450}\,\mu\,(-75 + 8\,\mu)$,

$c[1] \to \frac{4\,\mu^2}{225}$,

$c[2] \to \frac{4\,\mu^2}{225}$, $c[3] \to \frac{4\,\mu^2}{225}$

## PROOF OF PROPOSITION 4

### case 1

$$\pi[i] \to (-1424\,\lambda^6 + 768\,\lambda^5\,\mu - 28800\,\lambda^3\,\mu^2 + 270000\,\lambda\,\mu^3 - 50625\,\mu^3\,(-75 + 8\,\mu) + 1800\,\lambda^2\,\mu^2\,(-475 + 24\,\mu) - 12\,\lambda^4\,\mu\,(-5125 + 96\,\mu)) \Big/ \left(36450\,(2\,\lambda^2 - 25\,\mu)^2\right)$$

### case 2

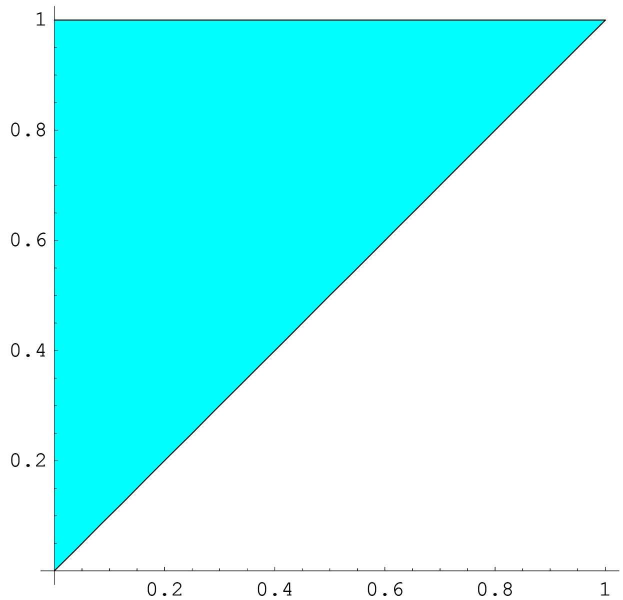
$$\pi[i] \to \frac{-8\,\lambda^2 + 675\,\mu + 48\,\lambda\,\mu - 72\,\mu^2}{4050}$$

### case N

$$\pi[i] \to -\frac{1}{450}\,\mu\,(-75 + 8\,\mu)$$

### CONJECTURE $\pi[i]$ _case2 $\geq$ $\pi[i]$ _case1 $\geq$ $\pi[i]$ _caseN

$$\text{InequalityPlot}\Big[\Big\{\frac{-8\lambda^2+675\mu+48\lambda\mu-72\mu^2}{4050}\geq (-1424\lambda^6+768\lambda^5\mu-28800\lambda^3\mu^2+270000\lambda\mu^3-50625\mu^3(-75+8\mu)+1800\lambda^2\mu^2(-475+24\mu)-12\lambda^4\mu(-5125+96\mu))\Big/\left(36450\,(2\lambda^2-25\mu)^2\right)\geq -\frac{1}{450}\mu(-75+8\mu),\ 0<\lambda<\mu\leq 1\Big\},\{\lambda,0,1\},\{\mu,0,1\}\Big]$$




- Graphics -

## COMPARISON OF CONSUMER WELFARE

## CASE 1

$$U[i] == V-\mu*(d[i]-s[i])-\lambda*\left(\frac{4*s[i]}{3}-4\,(s[i,j])-k\right)-p[i]$$

$$== V-\mu*\left(\frac{1}{3}-\left(-\frac{2\,(4\lambda^2-75\mu)\,(\lambda-3\mu)}{135\,(2\lambda^2-25\mu)}\right)\right)-\lambda*\left(\frac{4*\left(-\frac{2\,(4\lambda^2-75\mu)\,(\lambda-3\mu)}{135\,(2\lambda^2-25\mu)}\right)}{3}-4\left(\frac{\lambda}{15}\right)-k\right)-\left(\frac{\mu}{2}\right)$$

$$== V+\frac{496\lambda^4-240\lambda^3\mu+4500\lambda\mu^2-675\mu^2(-25+4\mu)+6\lambda^2\mu(-1325+24\mu)+810\,k\,(2\lambda^3-25\lambda\mu)}{810\,(2\lambda^2-25\mu)}$$

## CASE 2

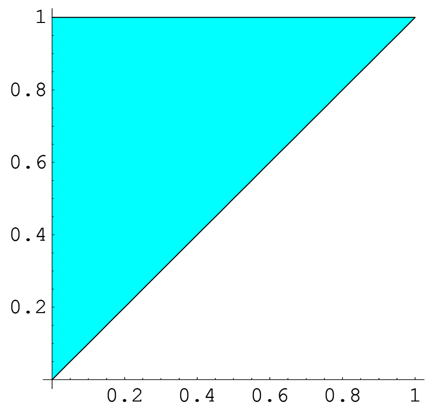
$$U[i] == V-\mu*(d[i]-s[i])-\lambda*\left(\frac{4*s[i]}{3}-k\right)-p[i]$$

$$== V+k\lambda+\frac{8\lambda^2}{135}-\frac{5\mu}{6}-\frac{2\lambda\mu}{9}+\frac{2\mu^2}{15}$$

**CONJECTURE : UTILITY IN CASE 1 > UTILITY IN CASE 2**

```
InequalityPlot[
```

$$\left\{V + \frac{496\lambda^4 - 240\lambda^3\mu + 4500\lambda\mu^2 - 675\mu^2(-25+4\mu) + 6\lambda^2\mu(-1325+24\mu) + 810k(2\lambda^3 - 25\lambda\mu)}{810(2\lambda^2 - 25\mu)} > V + k\lambda + \frac{8\lambda^2}{135} - \frac{5\mu}{6} - \frac{2\lambda\mu}{9} + \frac{2\mu^2}{15},\ 1 \geq \mu > \lambda > 0\right\}, \{\lambda, 0, 1\}, \{\mu, 0, 1\}]$$

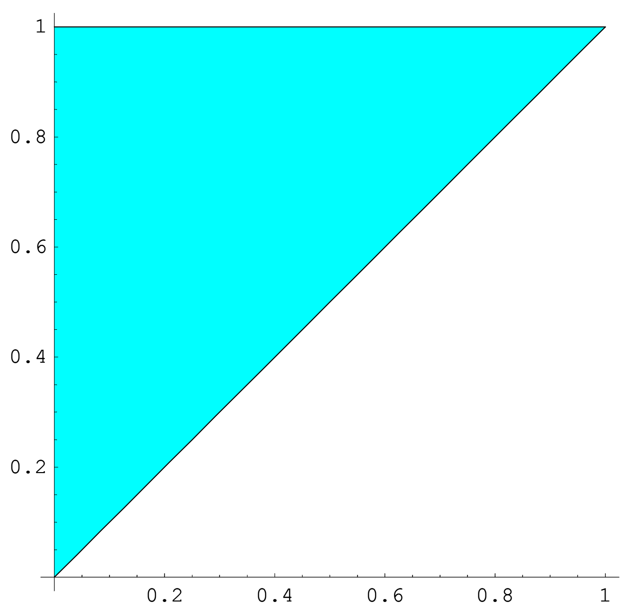



- Graphics -

## COMPARISON OF CASE 1 AND CASE N

**CONJECTURE INVESTMENT IN NETWORK CAPACITY IN CASE N> INVESTMENT IN NETWORK CAPACITY IN CASE 1**

$$\texttt{InequalityPlot}\left[\left\{-\frac{2(4\lambda^2 - 75\mu)(\lambda - 3\mu)}{135(2\lambda^2 - 25\mu)} \leq \frac{2\mu}{15},\ 1 \geq \mu > \lambda > 0\right\}, \{\lambda, 0, 1\}, \{\mu, 0, 1\}\right]$$




- Graphics -

## PARAMETER CONSTRAINTS

### Case 1 parameter constraints

$$d[i] > s[i] > 0$$

$$\frac{s[i]}{3} - s[i,j] - \frac{k}{4} > 0$$

**Hence**

$$\frac{1}{3} > -\frac{2\,(4\,\lambda^2 - 75\,\mu)\,(\lambda - 3\,\mu)}{135\,(2\,\lambda^2 - 25\,\mu)} > 0$$

$$-\frac{2\,(4\,\lambda^2 - 75\,\mu)\,(\lambda - 3\,\mu)}{3 * 135\,(2\,\lambda^2 - 25\,\mu)} - \frac{\lambda}{15} - \frac{k}{4} > 0$$

### Case 2 parameter constraints

$$d[i] > s[i] > 0$$

$$\frac{s[i]}{3} - \frac{k}{4} > 0$$

**Hence**

$$\frac{1}{3} > -\frac{2}{45}\,(\lambda - 3\,\mu) > 0$$

$$-\frac{2}{135}\,(\lambda - 3\,\mu) - \frac{k}{4} > 0$$

## Case 3 parameter constraints

**d[i] > s[i] > 0 where i = 1, 2**

**Hence**

$$(1875 \mu^2 (16 \lambda^2 - 24 \lambda \mu + 9 \mu (-25 + 4 \mu))) / (8 \lambda^6 - 48 \lambda^5 \mu + 3000 \lambda^3 \mu^2 - 4500 \lambda^2 (-35 + \mu) \mu^2 - 135000 \lambda \mu^3 + 50625 \mu^3 (-25 + 4 \mu) + 4 \lambda^4 \mu (-1025 + 18 \mu)) > (10 (\lambda - 3 \mu) \mu (32 \lambda^6 - 48 \lambda^5 \mu + 3000 \lambda^3 \mu^2 - 45000 \lambda \mu^3 + 16875 \mu^3 (-25 + 4 \mu) - 375 \lambda^2 \mu^2 (-155 + 12 \mu) + 2 \lambda^4 \mu (-1225 + 36 \mu))) / ((4 \lambda^2 - 75 \mu) (8 \lambda^6 - 48 \lambda^5 \mu + 3000 \lambda^3 \mu^2 - 4500 \lambda^2 (-35 + \mu) \mu^2 - 135000 \lambda \mu^3 + 50625 \mu^3 (-25 + 4 \mu) + 4 \lambda^4 \mu (-1025 + 18 \mu))) > 0$$

**d[3] > s[3] > 0**

**Hence**

$$(8 \lambda^6 - 48 \lambda^5 \mu + 3000 \lambda^3 \mu^2 - 45000 \lambda \mu^3 - 1500 \lambda^2 \mu^2 (-65 + 3 \mu) + 16875 \mu^3 (-25 + 4 \mu) + 4 \lambda^4 \mu (-1025 + 18 \mu)) / (8 \lambda^6 - 48 \lambda^5 \mu + 3000 \lambda^3 \mu^2 - 4500 \lambda^2 (-35 + \mu) \mu^2 - 135000 \lambda \mu^3 + 50625 \mu^3 (-25 + 4 \mu) + 4 \lambda^4 \mu (-1025 + 18 \mu)) > (10 (\lambda - 3 \mu) \mu (8 \lambda^6 - 48 \lambda^5 \mu + 3000 \lambda^3 \mu^2 - 45000 \lambda \mu^3 - 1500 \lambda^2 \mu^2 (-65 + 3 \mu) + 16875 \mu^3 (-25 + 4 \mu) + 4 \lambda^4 \mu (-1025 + 18 \mu))) / ((4 \lambda^2 - 75 \mu) (8 \lambda^6 - 48 \lambda^5 \mu + 3000 \lambda^3 \mu^2 - 4500 \lambda^2 (-35 + \mu) \mu^2 - 135000 \lambda \mu^3 + 50625 \mu^3 (-25 + 4 \mu) + 4 \lambda^4 \mu (-1025 + 18 \mu))) > 0$$

$$\left(\frac{s[1]}{3} + \frac{s[2]}{3} - \frac{k}{2}\right) > (s[1, 2] + s[2, 1]) > 0$$

**Hence**

$$(10 (\lambda - 3 \mu) \mu (32 \lambda^6 - 48 \lambda^5 \mu + 3000 \lambda^3 \mu^2 - 45000 \lambda \mu^3 + 16875 \mu^3 (-25 + 4 \mu) - 375 \lambda^2 \mu^2 (-155 + 12 \mu) + 2 \lambda^4 \mu (-1225 + 36 \mu))) / ((4 \lambda^2 - 75 \mu) (8 \lambda^6 - 48 \lambda^5 \mu + 3000 \lambda^3 \mu^2 - 4500 \lambda^2 (-35 + \mu) \mu^2 - 135000 \lambda \mu^3 + 50625 \mu^3 (-25 + 4 \mu) + 4 \lambda^4 \mu (-1025 + 18 \mu))) - \frac{k}{4} > (375 \lambda \mu^2 (16 \lambda^2 - 24 \lambda \mu + 9 \mu (-25 + 4 \mu))) / (8 \lambda^6 - 48 \lambda^5 \mu + 3000 \lambda^3 \mu^2 - 4500 \lambda^2 (-35 + \mu) \mu^2 - 135000 \lambda \mu^3 + 50625 \mu^3 (-25 + 4 \mu) + 4 \lambda^4 \mu (-1025 + 18 \mu)) > 0$$

$$\left(\frac{s[i]}{3} + \frac{s[3]}{3} - \frac{k}{2}\right) > 0, \ i = 1, 2$$

**Hence**

$$(10 (\lambda - 3 \mu) \mu (32 \lambda^6 - 48 \lambda^5 \mu + 3000 \lambda^3 \mu^2 - 45000 \lambda \mu^3 + 16875 \mu^3 (-25 + 4 \mu) - 375 \lambda^2 \mu^2 (-155 + 12 \mu) + 2 \lambda^4 \mu (-1225 + 36 \mu))) / ((4 \lambda^2 - 75 \mu) (8 \lambda^6 - 48 \lambda^5 \mu + 3000 \lambda^3 \mu^2 - 4500 \lambda^2 (-35 + \mu) \mu^2 - 135000 \lambda \mu^3 + 50625 \mu^3 (-25 + 4 \mu) + 4 \lambda^4 \mu (-1025 + 18 \mu))) + (10 (\lambda - 3 \mu) \mu (8 \lambda^6 - 48 \lambda^5 \mu + 3000 \lambda^3 \mu^2 - 45000 \lambda \mu^3 - 1500 \lambda^2 \mu^2 (-65 + 3 \mu) + 16875 \mu^3 (-25 + 4 \mu) + 4 \lambda^4 \mu (-1025 + 18 \mu))) / ((4 \lambda^2 - 75 \mu) (8 \lambda^6 - 48 \lambda^5 \mu + 3000 \lambda^3 \mu^2 - 4500 \lambda^2 (-35 + \mu) \mu^2 - 135000 \lambda \mu^3 + 50625 \mu^3 (-25 + 4 \mu) + 4 \lambda^4 \mu (-1025 + 18 \mu))) - \frac{3 * k}{2} > 0$$

**Null**

## Case 4 parameter constraints

**d[1] > s[1] > 0**

**Hence**

$$(-64 \lambda^6 - 48 \lambda^5 \mu + 3000 \lambda^3 \mu^2 - 45000 \lambda \mu^3 + 16875 \mu^3 (-25 + 4 \mu) -$$

$750\,\lambda^2\,\mu^2\,(-25+6\,\mu) + 8\,\lambda^4\,\mu\,(275+9\,\mu))\,/\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 -$
$135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17+2\,\mu) + 50625\,\mu^3\,(-25+4\,\mu) + 8\,\lambda^4\,\mu\,(-1100+81\,\mu)) >$
$(256\,\lambda^7 - 576\,\lambda^6\,\mu + 101250\,\mu^4\,(-25+4\,\mu) + 144\,\lambda^4\,\mu^2\,(25+6\,\mu) - 5400\,\lambda^2\,\mu^3\,(-50+7\,\mu) -$
$33750\,\lambda\,\mu^3\,(-25+12\,\mu) - 32\,\lambda^5\,\mu\,(125+27\,\mu) + 600\,\lambda^3\,\mu^2\,(-100+63\,\mu))\,/$
$(15\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 - 135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17+2\,\mu) +$
$50625\,\mu^3\,(-25+4\,\mu) + 8\,\lambda^4\,\mu\,(-1100+81\,\mu))) > 0$

**d[i] > s[i] > 0 where i = 2, 3**

**Hence**

$(104\,\lambda^6 - 192\,\lambda^5\,\mu + 6000\,\lambda^3\,\mu^2 - 45000\,\lambda\,\mu^3 + 16875\,\mu^3\,(-25+4\,\mu) -$
$750\,\lambda^2\,\mu^2\,(-115+12\,\mu) + 4\,\lambda^4\,\mu\,(-1375+72\,\mu))\,/\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 -$
$135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17+2\,\mu) + 50625\,\mu^3\,(-25+4\,\mu) + 8\,\lambda^4\,\mu\,(-1100+81\,\mu)) >$
$(2\,(-52\,\lambda^7 + 252\,\lambda^6\,\mu + 8\,\lambda^5\,(425-54\,\mu)\,\mu + 144\,\lambda^4\,\mu^2\,(-100+3\,\mu) + 50625\,\mu^4\,(-25+4\,\mu) -$
$16875\,\lambda\,\mu^3\,(-25+12\,\mu) - 675\,\lambda^2\,\mu^3\,(-375+28\,\mu) + 75\,\lambda^3\,\mu^2\,(-925+252\,\mu)))\,/$
$(15\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 - 135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17+2\,\mu) +$
$50625\,\mu^3\,(-25+4\,\mu) + 8\,\lambda^4\,\mu\,(-1100+81\,\mu))) > 0$

$\left(\frac{s[1]}{3} + \frac{s[i]}{3} - \frac{k}{2}\right) - (s[1,i] + s[i,1]) > 0,\ i = 2, 3$

**Hence**

$\frac{1}{3} * (256\,\lambda^7 - 576\,\lambda^6\,\mu + 101250\,\mu^4\,(-25+4\,\mu) + 144\,\lambda^4\,\mu^2\,(25+6\,\mu) - 5400\,\lambda^2\,\mu^3\,(-50+7\,\mu) -$
$33750\,\lambda\,\mu^3\,(-25+12\,\mu) - 32\,\lambda^5\,\mu\,(125+27\,\mu) + 600\,\lambda^3\,\mu^2\,(-100+63\,\mu))\,/$
$(15\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 - 135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17+2\,\mu) +$
$50625\,\mu^3\,(-25+4\,\mu) + 8\,\lambda^4\,\mu\,(-1100+81\,\mu))) +$
$\frac{1}{3} * (-(2\,(\lambda-3\,\mu)\,(52\,\lambda^6 - 96\,\lambda^5\,\mu + 4200\,\lambda^3\,\mu^2 - 45000\,\lambda\,\mu^3 + 16875\,\mu^3\,(-25+4\,\mu) +$
$8\,\lambda^4\,\mu\,(-425+18\,\mu) - 75\,\lambda^2\,\mu^2\,(-925+84\,\mu)))\,/\,(15\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 -$
$135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17+2\,\mu) + 50625\,\mu^3\,(-25+4\,\mu) + 8\,\lambda^4\,\mu\,(-1100+81\,\mu)))) -$
$\frac{k}{2} - (-(\lambda\,(64\,\lambda^6 + 48\,\lambda^5\,\mu - 3000\,\lambda^3\,\mu^2 + 45000\,\lambda\,\mu^3 - 16875\,\mu^3\,(-25+4\,\mu) +$
$750\,\lambda^2\,\mu^2\,(-25+6\,\mu) - 8\,\lambda^4\,\mu\,(275+9\,\mu)))\,/$
$(5\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 - 135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17+2\,\mu) +$
$50625\,\mu^3\,(-25+4\,\mu) + 8\,\lambda^4\,\mu\,(-1100+81\,\mu))) -$
$((\lambda\,(104\,\lambda^6 - 192\,\lambda^5\,\mu + 6000\,\lambda^3\,\mu^2 - 45000\,\lambda\,\mu^3 + 16875\,\mu^3\,(-25+4\,\mu) -$
$750\,\lambda^2\,\mu^2\,(-115+12\,\mu) + 4\,\lambda^4\,\mu\,(-1375+72\,\mu)))\,/$
$(5\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 - 135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17+2\,\mu) +$
$50625\,\mu^3\,(-25+4\,\mu) + 8\,\lambda^4\,\mu\,(-1100+81\,\mu)))) > 0$

$\frac{s[2]}{3} + \frac{s[3]}{3} - \frac{k}{2} > 0$

$\frac{2}{3} * (-(2\,(\lambda-3\,\mu)\,(52\,\lambda^6 - 96\,\lambda^5\,\mu + 4200\,\lambda^3\,\mu^2 - 45000\,\lambda\,\mu^3 +$
$16875\,\mu^3\,(-25+4\,\mu) + 8\,\lambda^4\,\mu\,(-425+18\,\mu) - 75\,\lambda^2\,\mu^2\,(-925+84\,\mu)))\,/$
$(15\,(144\,\lambda^6 - 432\,\lambda^5\,\mu + 15000\,\lambda^3\,\mu^2 - 135000\,\lambda\,\mu^3 - 11250\,\lambda^2\,\mu^2\,(-17+2\,\mu) +$
$50625\,\mu^3\,(-25+4\,\mu) + 8\,\lambda^4\,\mu\,(-1100+81\,\mu)))) - \frac{k}{2} > 0$

$s[1,i] > 0,\ i = 2, 3$

**Hence**

$-(\lambda\,(64\,\lambda^6 + 48\,\lambda^5\,\mu - 3000\,\lambda^3\,\mu^2 + 45000\,\lambda\,\mu^3 -$

$$16875\,\mu^3\,(-25+4\,\mu)+750\,\lambda^2\,\mu^2\,(-25+6\,\mu)-8\,\lambda^4\,\mu\,(275+9\,\mu)))\,/$$
$$(5\,(144\,\lambda^6-432\,\lambda^5\,\mu+15000\,\lambda^3\,\mu^2-135000\,\lambda\,\mu^3-11250\,\lambda^2\,\mu^2\,(-17+2\,\mu)+$$
$$50625\,\mu^3\,(-25+4\,\mu)+8\,\lambda^4\,\mu\,(-1100+81\,\mu)))>0$$

`s[i, 1] > 0, i = 2, 3`

**Hence**

$$(\lambda\,(104\,\lambda^6-192\,\lambda^5\,\mu+6000\,\lambda^3\,\mu^2-45000\,\lambda\,\mu^3+$$
$$16875\,\mu^3\,(-25+4\,\mu)-750\,\lambda^2\,\mu^2\,(-115+12\,\mu)+4\,\lambda^4\,\mu\,(-1375+72\,\mu)))\,/$$
$$(5\,(144\,\lambda^6-432\,\lambda^5\,\mu+15000\,\lambda^3\,\mu^2-135000\,\lambda\,\mu^3-11250\,\lambda^2\,\mu^2\,(-17+2\,\mu)+$$
$$50625\,\mu^3\,(-25+4\,\mu)+8\,\lambda^4\,\mu\,(-1100+81\,\mu)))>0$$

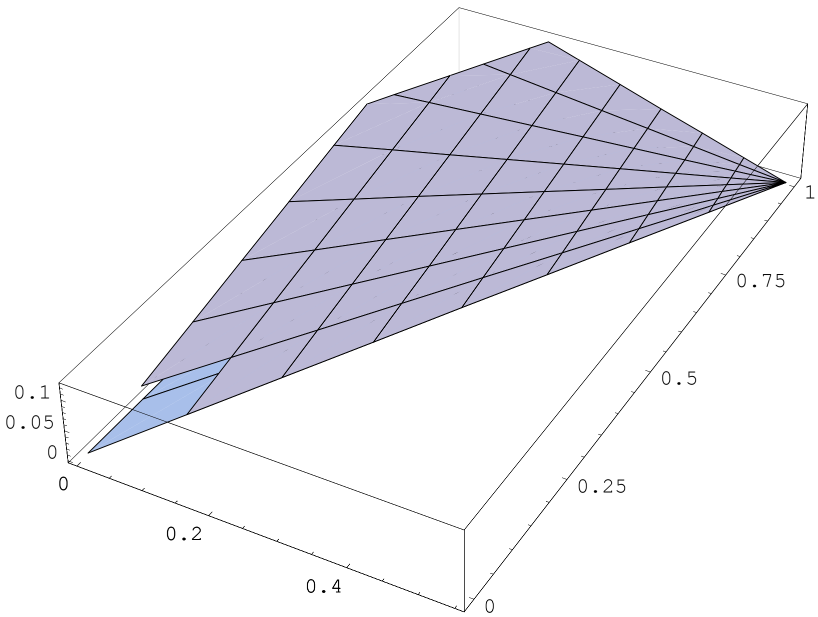
## Case N parameter constraints

`d[i] > s[i] > 0`

**Hence**

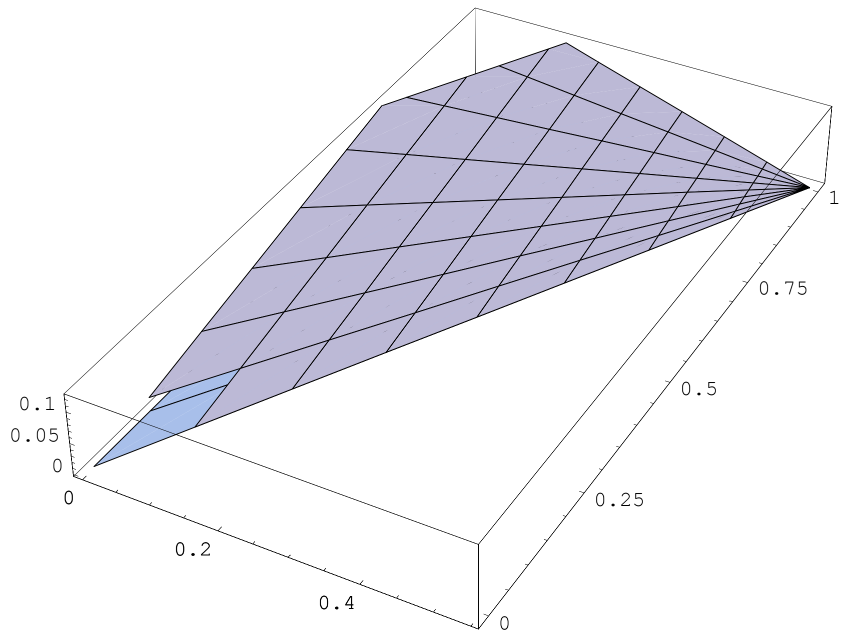
$$\frac{1}{3}>\frac{2\,\mu}{15}$$

$$\texttt{InequalityPlot3D}\Big[\Big\{\frac{1}{3}>-\frac{2\,(4\,\lambda^2-75\,\mu)\,(\lambda-3\,\mu)}{135\,(2\,\lambda^2-25\,\mu)}>0,$$
$$-\frac{2\,(4\,\lambda^2-75\,\mu)\,(\lambda-3\,\mu)}{3*135\,(2\,\lambda^2-25\,\mu)}-\frac{\lambda}{15}-\frac{k}{4}>0,\ \frac{1}{3}>-\frac{2}{45}\,(\lambda-3\,\mu)>0,\ -\frac{2}{135}\,(\lambda-3\,\mu)-\frac{k}{4}>0\,,$$
$$(1875\,\mu^2\,(16\,\lambda^2-24\,\lambda\,\mu+9\,\mu\,(-25+4\,\mu)))\,/\,(8\,\lambda^6-48\,\lambda^5\,\mu+3000\,\lambda^3\,\mu^2-$$
$$4500\,\lambda^2\,(-35+\mu)\,\mu^2-135000\,\lambda\,\mu^3+50625\,\mu^3\,(-25+4\,\mu)+4\,\lambda^4\,\mu\,(-1025+18\,\mu))>$$
$$(10\,(\lambda-3\,\mu)\,\mu\,(32\,\lambda^6-48\,\lambda^5\,\mu+3000\,\lambda^3\,\mu^2-45000\,\lambda\,\mu^3+16875\,\mu^3\,(-25+4\,\mu)-$$
$$375\,\lambda^2\,\mu^2\,(-155+12\,\mu)+2\,\lambda^4\,\mu\,(-1225+36\,\mu)))\,/\,((4\,\lambda^2-75\,\mu)\,(8\,\lambda^6-48\,\lambda^5\,\mu+3000\,\lambda^3\,\mu^2-$$
$$4500\,\lambda^2\,(-35+\mu)\,\mu^2-135000\,\lambda\,\mu^3+50625\,\mu^3\,(-25+4\,\mu)+4\,\lambda^4\,\mu\,(-1025+18\,\mu)))>0,$$
$$(8\,\lambda^6-48\,\lambda^5\,\mu+3000\,\lambda^3\,\mu^2-45000\,\lambda\,\mu^3-1500\,\lambda^2\,\mu^2\,(-65+3\,\mu)+16875\,\mu^3\,(-25+4\,\mu)+$$
$$4\,\lambda^4\,\mu\,(-1025+18\,\mu))\,/\,(8\,\lambda^6-48\,\lambda^5\,\mu+3000\,\lambda^3\,\mu^2-4500\,\lambda^2\,(-35+\mu)\,\mu^2-$$
$$135000\,\lambda\,\mu^3+50625\,\mu^3\,(-25+4\,\mu)+4\,\lambda^4\,\mu\,(-1025+18\,\mu))>$$
$$(10\,(\lambda-3\,\mu)\,\mu\,(8\,\lambda^6-48\,\lambda^5\,\mu+3000\,\lambda^3\,\mu^2-45000\,\lambda\,\mu^3-1500\,\lambda^2\,\mu^2\,(-65+3\,\mu)+$$
$$16875\,\mu^3\,(-25+4\,\mu)+4\,\lambda^4\,\mu\,(-1025+18\,\mu)))\,/\,((4\,\lambda^2-75\,\mu)\,(8\,\lambda^6-48\,\lambda^5\,\mu+3000\,\lambda^3\,\mu^2-$$
$$4500\,\lambda^2\,(-35+\mu)\,\mu^2-135000\,\lambda\,\mu^3+50625\,\mu^3\,(-25+4\,\mu)+4\,\lambda^4\,\mu\,(-1025+18\,\mu)))>0,$$
$$(10\,(\lambda-3\,\mu)\,\mu\,(32\,\lambda^6-48\,\lambda^5\,\mu+3000\,\lambda^3\,\mu^2-45000\,\lambda\,\mu^3+16875\,\mu^3\,(-25+4\,\mu)-$$
$$375\,\lambda^2\,\mu^2\,(-155+12\,\mu)+2\,\lambda^4\,\mu\,(-1225+36\,\mu)))\,/\,((4\,\lambda^2-75\,\mu)\,(8\,\lambda^6-48\,\lambda^5\,\mu+3000\,\lambda^3\,\mu^2-$$
$$4500\,\lambda^2\,(-35+\mu)\,\mu^2-135000\,\lambda\,\mu^3+50625\,\mu^3\,(-25+4\,\mu)+4\,\lambda^4\,\mu\,(-1025+18\,\mu)))-\frac{k}{4}>$$
$$(375\,\lambda\,\mu^2\,(16\,\lambda^2-24\,\lambda\,\mu+9\,\mu\,(-25+4\,\mu)))\,/\,(8\,\lambda^6-48\,\lambda^5\,\mu+3000\,\lambda^3\,\mu^2-$$
$$4500\,\lambda^2\,(-35+\mu)\,\mu^2-135000\,\lambda\,\mu^3+50625\,\mu^3\,(-25+4\,\mu)+4\,\lambda^4\,\mu\,(-1025+18\,\mu))>0,$$
$$(10\,(\lambda-3\,\mu)\,\mu\,(32\,\lambda^6-48\,\lambda^5\,\mu+3000\,\lambda^3\,\mu^2-45000\,\lambda\,\mu^3+16875\,\mu^3\,(-25+4\,\mu)-$$
$$375\,\lambda^2\,\mu^2\,(-155+12\,\mu)+2\,\lambda^4\,\mu\,(-1225+36\,\mu)))\,/\,((4\,\lambda^2-75\,\mu)\,(8\,\lambda^6-48\,\lambda^5\,\mu+3000\,\lambda^3\,\mu^2-$$
$$4500\,\lambda^2\,(-35+\mu)\,\mu^2-135000\,\lambda\,\mu^3+50625\,\mu^3\,(-25+4\,\mu)+4\,\lambda^4\,\mu\,(-1025+18\,\mu)))+$$
$$(10\,(\lambda-3\,\mu)\,\mu\,(8\,\lambda^6-48\,\lambda^5\,\mu+3000\,\lambda^3\,\mu^2-45000\,\lambda\,\mu^3-1500\,\lambda^2\,\mu^2\,(-65+3\,\mu)+$$
$$16875\,\mu^3\,(-25+4\,\mu)+4\,\lambda^4\,\mu\,(-1025+18\,\mu)))\,/$$
$$((4\,\lambda^2-75\,\mu)\,(8\,\lambda^6-48\,\lambda^5\,\mu+3000\,\lambda^3\,\mu^2-4500\,\lambda^2\,(-35+\mu)\,\mu^2-135000\,\lambda\,\mu^3+$$
$$50625\,\mu^3\,(-25+4\,\mu)+4\,\lambda^4\,\mu\,(-1025+18\,\mu)))-\frac{3*k}{2}>0,$$
$$1\geq\mu>\lambda>0\Big\},\ \{\lambda,\,0,\,1\},\ \{\mu,\,0,\,1\},\ \{k,\,0,\,10\}\Big]$$




*Out[2]=* - Graphics3D -

```
In[3]:= InequalityPlot3D[
  {(-64 λ^6 - 48 λ^5 μ + 3000 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) - 750 λ^2 μ^2 (-25 + 6 μ) +
       8 λ^4 μ (275 + 9 μ)) / (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 -
       11250 λ^2 μ^2 (-17 + 2 μ) + 50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ)) >
    (256 λ^7 - 576 λ^6 μ + 101250 μ^4 (-25 + 4 μ) + 144 λ^4 μ^2 (25 + 6 μ) - 5400 λ^2 μ^3 (-50 + 7 μ) -
       33750 λ μ^3 (-25 + 12 μ) - 32 λ^5 μ (125 + 27 μ) + 600 λ^3 μ^2 (-100 + 63 μ)) /
     (15 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
         50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))) > 0
  , (104 λ^6 - 192 λ^5 μ + 6000 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) - 750 λ^2 μ^2 (-115 + 12 μ) +
       4 λ^4 μ (-1375 + 72 μ)) / (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 -
       11250 λ^2 μ^2 (-17 + 2 μ) + 50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ)) >
    (2 (-52 λ^7 + 252 λ^6 μ + 8 λ^5 (425 - 54 μ) μ + 144 λ^4 μ^2 (-100 + 3 μ) + 50625 μ^4 (-25 + 4 μ) -
         16875 λ μ^3 (-25 + 12 μ) - 675 λ^2 μ^3 (-375 + 28 μ) + 75 λ^3 μ^2 (-925 + 252 μ))) /
     (15 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
         50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))) > 0,
  1/3 * (256 λ^7 - 576 λ^6 μ + 101250 μ^4 (-25 + 4 μ) + 144 λ^4 μ^2 (25 + 6 μ) - 5400 λ^2 μ^3 (-50 + 7 μ) -
         33750 λ μ^3 (-25 + 12 μ) - 32 λ^5 μ (125 + 27 μ) + 600 λ^3 μ^2 (-100 + 63 μ)) /
       (15 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
           50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))) +
     1/3 * (-(2 (λ - 3 μ) (52 λ^6 - 96 λ^5 μ + 4200 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) +
              8 λ^4 μ (-425 + 18 μ) - 75 λ^2 μ^2 (-925 + 84 μ))) / (15 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 -
             135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) + 50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ)))) -
     k/2 - (-(λ (64 λ^6 + 48 λ^5 μ - 3000 λ^3 μ^2 + 45000 λ μ^3 - 16875 μ^3 (-25 + 4 μ) +
              750 λ^2 μ^2 (-25 + 6 μ) - 8 λ^4 μ (275 + 9 μ))) / (5 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 -
             135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) + 50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ)))) -
     ((λ (104 λ^6 - 192 λ^5 μ + 6000 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) - 750 λ^2 μ^2 (-115 + 12 μ) +
            4 λ^4 μ (-1375 + 72 μ))) / (5 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 -
            11250 λ^2 μ^2 (-17 + 2 μ) + 50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ)))) > 0,
  2/3 * (-(2 (λ - 3 μ) (52 λ^6 - 96 λ^5 μ + 4200 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) +
              8 λ^4 μ (-425 + 18 μ) - 75 λ^2 μ^2 (-925 + 84 μ))) /
        (15 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
              50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ)))) - k/2 > 0,
  -(λ (64 λ^6 + 48 λ^5 μ - 3000 λ^3 μ^2 + 45000 λ μ^3 - 16875 μ^3 (-25 + 4 μ) +
          750 λ^2 μ^2 (-25 + 6 μ) - 8 λ^4 μ (275 + 9 μ))) /
     (5 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
         50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))) > 0,
  (λ (104 λ^6 - 192 λ^5 μ + 6000 λ^3 μ^2 - 45000 λ μ^3 + 16875 μ^3 (-25 + 4 μ) -
         750 λ^2 μ^2 (-115 + 12 μ) + 4 λ^4 μ (-1375 + 72 μ))) /
     (5 (144 λ^6 - 432 λ^5 μ + 15000 λ^3 μ^2 - 135000 λ μ^3 - 11250 λ^2 μ^2 (-17 + 2 μ) +
         50625 μ^3 (-25 + 4 μ) + 8 λ^4 μ (-1100 + 81 μ))) > 0,
  1/3 > 2 μ/15, 1 ≥ μ > λ > 0}, {λ, 0, 1}, {μ, 0, 1}, {k,
  0,
  10}]
```




*Out[3]=* - Graphics3D -

## Appendix B

# Appendix to Chapter 3

## B.1 Parameter Constraints

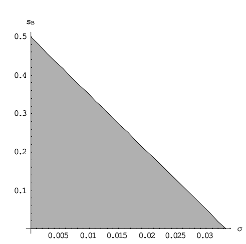


Figure B-1.1

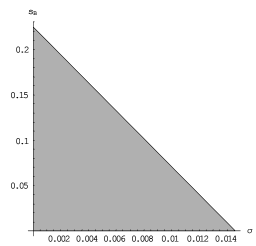


Figure B- 1. 2

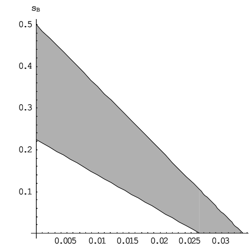


Figure B- 1. 3

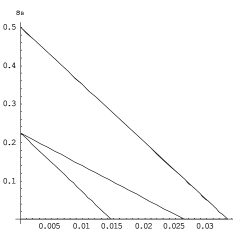


Figure B- 1. 4

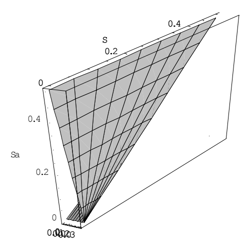


Figure B- 1. 5

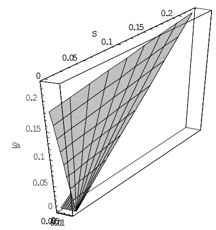


Figure B- 1. 6

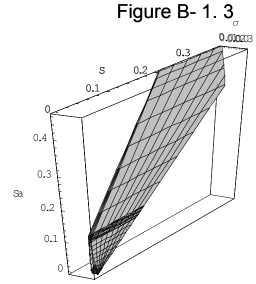


Figure B- 1. 7

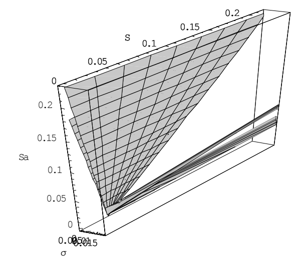


Figure B- 1. 8

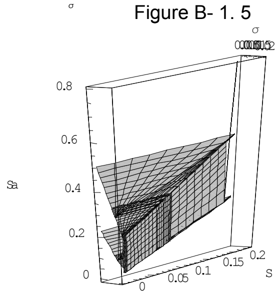


Figure B- 1. 9

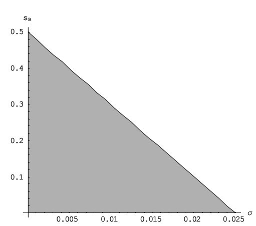


Figure B- 1. 10

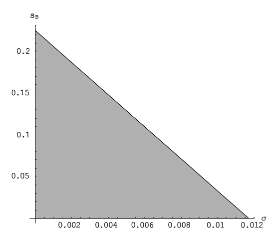


Figure B- 1. 11

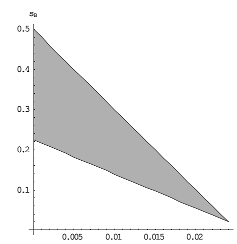


Figure B- 1. 12

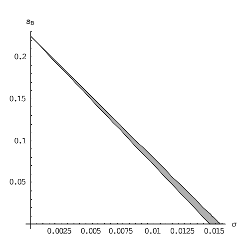


Figure B- 1. 13

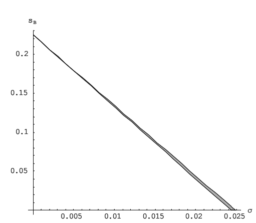


Figure B- 1. 14

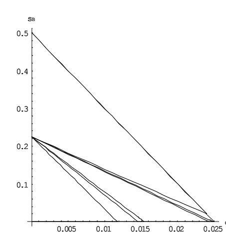


Figure B- 1. 15

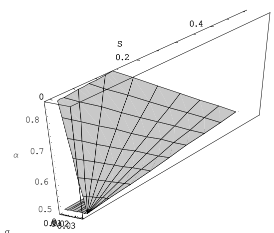


Figure B- 1. 16

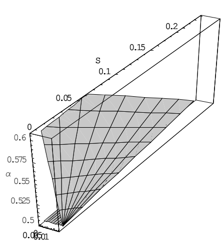


Figure B- 1. 17

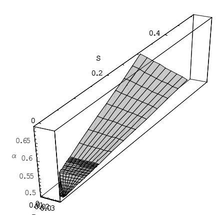


Figure B- 1. 18

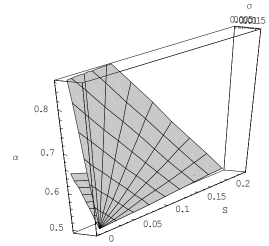


Figure B- 1. 19

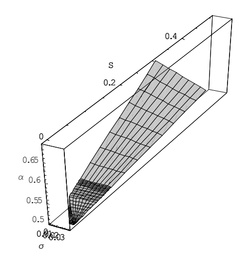


Figure B- 1. 20

Parameter Constraints

## B.2 Solving for Nash Equilibria

We will solve for the Nash equilibrium from the second stage. Given the network $g$, we know $n_{il}$ for $i \in N$, $l \in \{A, B\}$. Further, we know $n_{AA}, n_{AB}$ and $n_{BB}$.

Let ,

$$\begin{aligned}
\phi_i &= \frac{\sigma\left(n_{iA} \cdot \theta_A + n_{iB} \cdot \theta_B\right)}{2} \text{ for all } i \in N \\
\widetilde{\theta} &= \alpha \cdot \theta_A + (1-\alpha) \cdot \theta_B \\
\theta_A &= \alpha - s_A - \sigma\left(n_{AA} + \frac{n_{AB}}{2}\right) \\
\theta_B &= (1-\alpha) - s_B - \sigma\left(n_{BB} + \frac{n_{AB}}{2}\right)
\end{aligned}$$

We know that if $i \in N_A$,

$$\delta_i = \theta_A \cdot (\alpha \cdot d_i - n_{iA} \cdot (\sigma/2)) + \theta_B \cdot ((1-\alpha) \cdot d_i - n_{iB} \cdot (\sigma/2))$$

If $i \in N_B$,

$$\delta_i = \theta_B \cdot ((1-\alpha) \cdot d_i - n_{iB} \cdot (\sigma/2)) + \theta_A \cdot (\alpha \cdot d_i - n_{iA} \cdot (\sigma/2))$$

Hence,

$$\delta_i = d_i \cdot \widetilde{\theta} - \phi_i \text{ for all } i \in N$$

Equating utilities across websites, for all $i \neq j$

$$\begin{aligned}
U_i &= U_j \\
&\Rightarrow \delta_i + p_i = \delta_j + p_j \\
&\Rightarrow d_i \cdot \widetilde{\theta} - \phi_i + p_i = d_j \cdot \widetilde{\theta} - \phi_j + p_j = \lambda \text{ (say)} \\
&\Rightarrow d_i = \frac{\lambda + \phi_i - p_i}{\widetilde{\theta}} \text{ for all } i \in N
\end{aligned}$$

Now,

$$
\begin{aligned}
\sum_{i=1}^{n} d_i &= 1 \\
&\Rightarrow \frac{n \cdot \lambda - \sum_{i=1}^{n} p_i + \sum_{i=1}^{n} \phi_i}{\widetilde{\theta}} = 1 \\
&\Rightarrow \lambda = \frac{\widetilde{\theta} + \sum_{i=1}^{n} p_i - \sum_{i=1}^{n} \phi_i}{n}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
d_i &= \left(\frac{1}{\widetilde{\theta}}\right) \cdot \left[ \left( \frac{\widetilde{\theta} + \sum_{j=1}^{n} p_i - \sum_{j=1}^{n} \phi_i}{n} \right) + \phi_i - p_i \right] \\
&= \frac{1}{n} + \left(\frac{1}{n \cdot \widetilde{\theta}}\right) \left( \sum_{j=1}^{n} (p_j - \phi_j) - n \cdot (p_i - \phi_i) \right) \\
&= \frac{1}{n} + \left(\frac{1}{n \cdot \widetilde{\theta}}\right) \left( \sum_{j \neq i} (p_j - \phi_j) - (n-1) \cdot (p_i - \phi_i) \right) \qquad \text{(B.1)}
\end{aligned}
$$

The above equation gives us equilibrium demands for all ISPs. Next we will solve for equilibrium prices. Profits for ISP $i$ are given by

$$\pi_i = p_i \cdot d_i - C_i$$

where costs of ISP $i$ connected to backbone $l$ are given by

$$C_i = \left(\frac{\sigma^2}{2}\right) \cdot n_i + c_l$$

Since $C_i$ does not depend on prices, it can be treated as a constant.

Hence,

$$
\begin{aligned}
p_i \cdot d_i &= \frac{p_i}{n} + \left(\frac{p_i}{n \cdot \widetilde{\theta}}\right) \left(\sum_{j \neq i} (p_j - \phi_j) - (n-1) \cdot (p_i - \phi_i)\right) \\
&= \frac{p_i}{n} + \left(\frac{p_i}{n \cdot \widetilde{\theta}}\right) \left(\sum_{j \neq i} (p_j - \phi_j)\right) - \left(\frac{n-1}{n \cdot \widetilde{\theta}}\right) \cdot (p_i^2 - \phi_i \cdot p_i) \\
\frac{\partial \pi_i}{\partial p_i} &= \frac{1}{n} + \left(\frac{1}{n \cdot \widetilde{\theta}}\right) \left(\sum_{j \neq i} (p_j - \phi_j)\right) - \left(\frac{n-1}{n \cdot \widetilde{\theta}}\right) \cdot (2 \cdot p_i - \phi_i) = 0 \\
&\Rightarrow \; p_i = \left(\frac{1}{2 \cdot n - 1}\right) \cdot \left[\widetilde{\theta} + \sum_{j=1}^{n} p_j - \sum_{j=1}^{n} \phi_j + n \cdot \phi_i\right]
\end{aligned}
$$

Let

$$
\sum_{j=1}^{n} p_j = \widetilde{p} \text{ and } \sum_{j=1}^{n} \phi_j = \widetilde{\phi}
$$

Then

$$
p_i = \left(\frac{1}{2 \cdot n - 1}\right) \cdot \left[\widetilde{\theta} + \widetilde{p} - \widetilde{\phi} + n \cdot \phi_i\right]
$$

Summing up for all $i$,

$$
\begin{aligned}
\widetilde{p} &= \left(\frac{n}{2 \cdot n - 1}\right) \cdot \left[\widetilde{\theta} + \widetilde{p} - \widetilde{\phi}] + \left(\frac{n}{2 \cdot n - 1}\right) \cdot \widetilde{\phi}\right] \\
&= \left(\frac{n}{2 \cdot n - 1}\right) \cdot (\widetilde{\theta} + \widetilde{p}) \\
&\Rightarrow \; \widetilde{p} = \frac{\widetilde{\theta} \cdot n}{n-1}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
p_i &= \left(\frac{1}{2 \cdot n - 1}\right) \cdot \left[\widetilde{\theta} + \frac{\widetilde{\theta} \cdot n}{n-1} - \sum_{j=1}^{n} \phi_j + n \cdot \phi_i\right] \\
&= \frac{\widetilde{\theta}}{n-1} + \left(\frac{1}{2n-1}\right) \left(n \cdot \phi_i - \sum_{j=1}^{n} \phi_j\right) \qquad \text{(B.2)}
\end{aligned}
$$

The above equation gives us equilibrium prices. From the above two equations, after some manipulation, we get equilibrium demand of ISP $i$ :

$$d_i = \left(\frac{n-1}{n * \widetilde{\theta}}\right) \left[\frac{\widetilde{\theta}}{n-1} + \left(\frac{1}{2n-1}\right) \left(n \cdot \phi_i - \sum_{j=1}^{n} \phi_j\right)\right] \tag{B.3}$$

Hence the profits are given by:

$$\pi_i = \left(\frac{n-1}{n * \widetilde{\theta}}\right) \left[\frac{\widetilde{\theta}}{n-1} + \left(\frac{1}{2n-1}\right) \left(n \cdot \phi_i - \sum_{j=1}^{n} \phi_j\right)\right]^2 - \left(\frac{\sigma^2}{2}\right) \cdot n_i - c_l \tag{B.4}$$

where ISP $i$ is connected to backbone $l$.

## B.3 Computations

### "VERIFYING GENERAL FORMULA WITH MATHEMATICA"

```
"Solving with Mathematica"
n = 5;
?n
m = 3;
?m
```

$$\text{Case} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix};$$

```
var = ( m - 1  n - m  n - m - 1  m );
M = (Case.var);
nA = M[[1, 1]]
nab = M[[2, 2]]
nB = M[[3, 3]]
nba = M[[4, 4]]
```

$$\text{nAA} = \frac{m * \text{nA}}{2};$$

$$\text{nBB} = \frac{(n - m) * \text{nB}}{2};$$

```
nAB = nba * nab;
tA = α;
tB = 1 - α;
```

$$\text{wA} = \text{nAA} * \sigma + \frac{\text{nAB}}{2} * \sigma;$$

$$\text{wB} = \text{nBB} * \sigma + \frac{\text{nAB}}{2} * \sigma;$$

```
θA = tA - wA - Sa;
θB = tB - wB - Sb;
```

$$\text{Do}\left[\delta[\text{i}] = \text{d}[\text{i}] * (\alpha * \theta\text{A} + (1 - \alpha) * \theta\text{B}) - \sigma * \left(\frac{\text{nA} * \theta\text{A}}{2} + \frac{\text{nab}}{2} * \theta\text{B}\right), \{\text{i}, 1, \text{m}, 1\}\right];$$

$$\text{Do}\left[\delta[\text{i}] = \text{d}[\text{i}] * (\alpha * \theta\text{A} + (1 - \alpha) * \theta\text{B}) - \sigma * \left(\frac{\text{nba}}{2} * \theta\text{A} + \frac{\text{nB} * \theta\text{B}}{2}\right), \{\text{i}, \text{m} + 1, \text{n}, 1\}\right];$$

```
Do[U[i] = V - δ[i] - p[i], {i, 1, n, 1}];
"Solving for demands";
Simplify[Solve[
```

$$\text{Append}\left[\text{Table}[\text{U}[\text{i}] == \text{U}[\text{i} + 1], \{\text{i}, 1, \text{n} - 1, 1\}], \sum_{\text{i}=1}^{\text{n}} \text{d}[\text{i}] == 1\right], \text{Table}[\text{d}[\text{i}], \{\text{i}, 1, \text{n}, 1\}]\Big]\Big];$$

```
Do[d[i] = d[i] /. %, {i, 1, n, 1}];
```

```
demand[a] = d[1];
demand[b] = d[m + 1];
```

$$\mathbf{Do}\left[\phi[i] = p[i] * d[i] - FA - \frac{\sigma^2}{2} * (nA + nab),\ \{i, 1, m, 1\}\right];$$

$$\mathbf{Do}\left[\phi[i] = p[i] * d[i] - FB - \frac{\sigma^2}{2} * (nB + nba),\ \{i, m + 1, n, 1\}\right];$$

```
"Solving for prices";
Simplify[Solve[Table[D[ϕ[i], p[i]] == 0, {i, 1, n, 1}] , Table[p[i], {i, 1, n, 1}]]];
Do[p[i] = p[i] /. %, {i, 1, n, 1}];
"Solving for profits";
```

$$\mathbf{Do}\left[\phi[i] = p[i] * d[i] - FA - \frac{\sigma^2}{2} * (nA + nab),\ \{i, 1, m, 1\}\right];$$

$$\mathbf{Do}\left[\phi[i] = p[i] * d[i] - FB - \frac{\sigma^2}{2} * (nB + nba),\ \{i, m + 1, n, 1\}\right];$$

```
"Solving for demands";
Simplify[Do[d[i] = d[i], {i, 1, n, 1}]];
price[a] = p[1];
price[b] = p[m + 1];
"--------------------------------------------------------------------------------"
"General Formula"
ξ = α * θA + (1 - α) * θB;
```

$$\mathbf{Do}\left[\gamma[i] = \sigma\left(\frac{nA * \theta A}{2} + \frac{nab * \theta B}{2}\right),\ \{i, 1, m, 1\}\right];$$

$$\mathbf{Do}\left[\gamma[i] = \sigma\left(\frac{nba * \theta A}{2} + \frac{nB * \theta B}{2}\right),\ \{i, m + 1, n, 1\}\right];$$

$$\mathbf{Do}\left[\kappa[i] = n * \gamma[i] - \left(\sum_{j=1}^{n} \gamma[j]\right),\ \{i, 1, n, 1\}\right];$$

$$\mathbf{Do}\left[\rho[i] = \left(\frac{\xi}{n - 1} + \frac{\kappa[i]}{(2 * n - 1)}\right),\ \{i, 1, n, 1\}\right];$$

$$\mathbf{Do}\left[\beta[i] = \left(\frac{n - 1}{n * \xi}\right) * \left(\frac{\xi}{n - 1} + \frac{\kappa[i]}{(2 * n - 1)}\right),\ \{i, 1, n, 1\}\right];$$

$$\mathbf{Do}\left[\nu[i] = \left(\left(\frac{n - 1}{n * \xi}\right) * \left(\frac{\xi}{n - 1} + \frac{\kappa[i]}{(2 * n - 1)}\right)^2 - FA - \frac{\sigma^2}{2} * (nA + nab)\right),\ \{i, 1, m, 1\}\right];$$

$$\mathbf{Do}\left[\nu[i] = \left(\left(\frac{n - 1}{n * \xi}\right) * \left(\frac{\xi}{n - 1} + \frac{\kappa[i]}{(2 * n - 1)}\right)^2 - FB - \frac{\sigma^2}{2} * (nB + nba)\right),\ \{i, m + 1, n, 1\}\right];$$

```
Do[φ[i] = Simplify[p[i] - ρ[i]], {i, 1, n, 1}]
Do[Γ[i] = Simplify[ϕ[i] - ν[i]], {i, 1, n, 1}]
Do[ϵ[i] = Simplify[d[i] - β[i]], {i, 1, n, 1}]
?d
?p
?ϕ
?ρ
?ν
?β
?φ
?Γ
?ϵ
Clear[d, p, ϕ, ρ, ν, φ, Γ, ξ, κ, n, m, nA, nB, nab, nba]
Null
```

Verifying general formula with Mathematica

Solving with Mathematica

Global`n

n = 5

Global`m

m = 3

0

0

0

0

------------------------------------------------------------------------------

General Formula

Global`d

$d[1] = \{\{\frac{1}{5}\}\}$

$d[2] = \{\{\frac{1}{5}\}\}$

$d[3] = \{\{\frac{1}{5}\}\}$

$d[4] = \{\{\frac{1}{5}\}\}$

$d[5] = \{\{\frac{1}{5}\}\}$

Global`p

$p[1] = \{\frac{1}{4}\ (1 + Sb\ (-1 + \alpha) - (2 + Sa)\ \alpha + 2\ \alpha^2)\}$

$p[2] = \{\frac{1}{4}\ (1 + Sb\ (-1 + \alpha) - (2 + Sa)\ \alpha + 2\ \alpha^2)\}$

$p[3] = \{\frac{1}{4}\ (1 + Sb\ (-1 + \alpha) - (2 + Sa)\ \alpha + 2\ \alpha^2)\}$

$p[4] = \{\frac{1}{4}\ (1 + Sb\ (-1 + \alpha) - (2 + Sa)\ \alpha + 2\ \alpha^2)\}$

$p[5] = \{\frac{1}{4}\ (1 + Sb\ (-1 + \alpha) - (2 + Sa)\ \alpha + 2\ \alpha^2)\}$

Global`$\phi$

$\phi[1] = \{\{-FA + \frac{1}{20}\ (1 + Sb\ (-1 + \alpha) - (2 + Sa)\ \alpha + 2\ \alpha^2)\}\}$

$\phi[2] = \{\{-FA + \frac{1}{20}\ (1 + Sb\ (-1 + \alpha) - (2 + Sa)\ \alpha + 2\ \alpha^2)\}\}$

$\phi[3] = \{\{-FA + \frac{1}{20}\ (1 + Sb\ (-1 + \alpha) - (2 + Sa)\ \alpha + 2\ \alpha^2)\}\}$

$\phi[4] = \{\{-FB + \frac{1}{20}\ (1 + Sb\ (-1 + \alpha) - (2 + Sa)\ \alpha + 2\ \alpha^2)\}\}$

$\phi[5] = \{\{-FB + \frac{1}{20}\ (1 + Sb\ (-1 + \alpha) - (2 + Sa)\ \alpha + 2\ \alpha^2)\}\}$

Global`$\rho$

$\rho[1] = \frac{1}{4}\ ((1 - \alpha)\ (1 - Sb - \alpha) + \alpha\ (-Sa + \alpha))$

$\rho[2] = \frac{1}{4}\ ((1 - \alpha)\ (1 - Sb - \alpha) + \alpha\ (-Sa + \alpha))$

$\rho[3] = \frac{1}{4}\ ((1 - \alpha)\ (1 - Sb - \alpha) + \alpha\ (-Sa + \alpha))$

$\rho[4] = \frac{1}{4}\ ((1 - \alpha)\ (1 - Sb - \alpha) + \alpha\ (-Sa + \alpha))$

$\rho[5] = \frac{1}{4}\ ((1 - \alpha)\ (1 - Sb - \alpha) + \alpha\ (-Sa + \alpha))$

Global`$\nu$

$\nu[1] = -FA + \frac{1}{20}\ ((1 - \alpha)\ (1 - Sb - \alpha) + \alpha\ (-Sa + \alpha))$

$\nu[2] = -FA + \frac{1}{20}\ ((1 - \alpha)\ (1 - Sb - \alpha) + \alpha\ (-Sa + \alpha))$

$\nu[3] = -FA + \frac{1}{20}\ ((1 - \alpha)\ (1 - Sb - \alpha) + \alpha\ (-Sa + \alpha))$

$\nu[4] = -FB + \frac{1}{20}\ ((1 - \alpha)\ (1 - Sb - \alpha) + \alpha\ (-Sa + \alpha))$

$\nu[5] = -FB + \frac{1}{20}\ ((1 - \alpha)\ (1 - Sb - \alpha) + \alpha\ (-Sa + \alpha))$

Global`$\beta$

$\beta[1] = \frac{1}{5}$

$\beta[2] = \frac{1}{5}$

$\beta[3] = \frac{1}{5}$

$\beta[4] = \frac{1}{5}$

$\beta[5] = \frac{1}{5}$

Global`$\varphi$

$\varphi[1] = \{0\}$

$\varphi[2] = \{0\}$

$\varphi[3] = \{0\}$

$\varphi[4] = \{0\}$

$\varphi[5] = \{0\}$

```
Global`Γ
Γ[1] = {{0}}
Γ[2] = {{0}}
Γ[3] = {{0}}
Γ[4] = {{0}}
Γ[5] = {{0}}
Global`ϵ
ϵ[1] = {{0}}
ϵ[2] = {{0}}
ϵ[3] = {{0}}
ϵ[4] = {{0}}
ϵ[5] = {{0}}
```

**"THE GENERAL FORMULA FOR ARBITRARY m AND n AND ARBITRARY NETWORKS"**

```
Clear[d, p, ϕ, ξ, κ, γ, θA, θB, wA, wB, tA, tB, n, m, niA, niB,
 nA, nB, nab, nba, nAA, nAB, nBB, njA, njB, σ, α]
"i belongs to backbone A and j to backbone B"
tA = α;
tB = 1 - α;
```

$$wA = nAA * \sigma + \frac{nAB}{2} * \sigma;$$

$$wB = nBB * \sigma + \frac{nAB}{2} * \sigma;$$

$$\theta A = tA - wA - Sa;$$

$$\theta B = tB - wB - Sb;$$

$$\xi = \alpha * \theta A + (1 - \alpha) * \theta B;$$

$$\gamma[i] = \sigma\left(\frac{niA * \theta A}{2} + \frac{niB * \theta B}{2}\right);$$

$$\gamma[j] = \sigma\left(\frac{njA * \theta A}{2} + \frac{njB * \theta B}{2}\right);$$

$$\kappa[i] = n * \gamma[i] - \sigma\left(nAA * \theta A + nBB * \theta B + \frac{nAB * \theta A}{2} + \frac{nAB * \theta B}{2}\right);$$

$$\kappa[j] = n * \gamma[j] - \sigma\left(nAA * \theta A + nBB * \theta B + \frac{nAB * \theta A}{2} + \frac{nAB * \theta B}{2}\right);$$

$$p[i] = \text{Simplify}\left[\left(\frac{\xi}{n - 1} + \frac{\kappa[i]}{(2 * n - 1)}\right)\right];$$

$$p[j] = \text{Simplify}\left[\left(\frac{\xi}{n - 1} + \frac{\kappa[j]}{(2 * n - 1)}\right)\right];$$

$$d[i] = \text{Simplify}\left[\left(\frac{n - 1}{n * \xi}\right) * \left(\frac{\xi}{n - 1} + \frac{\kappa[i]}{(2 * n - 1)}\right)\right];$$

$$d[j] = \text{Simplify}\left[\left(\frac{n - 1}{n * \xi}\right) * \left(\frac{\xi}{n - 1} + \frac{\kappa[j]}{(2 * n - 1)}\right)\right];$$

$$\phi[i] = \text{Simplify}\left[\left(\left(\frac{n - 1}{n * \xi}\right) * \left(\frac{\xi}{n - 1} + \frac{\kappa[i]}{(2 * n - 1)}\right)^2 - FA - \frac{\sigma^2}{2} * (niA + niB)\right)\right];$$

$$\phi[j] = \text{Simplify}\left[\left(\left(\frac{n - 1}{n * \xi}\right) * \left(\frac{\xi}{n - 1} + \frac{\kappa[j]}{(2 * n - 1)}\right)^2 - FB - \frac{\sigma^2}{2} * (njB + njA)\right)\right];$$

```
?d
?p
?ϕ
Clear[d, p, ϕ, ξ, κ, γ, θA, θB, wA, wB, tA, tB, n,
 m, niA, niB, nA, nB, nab, nba, nAA, nAB, nBB, njA, njB, σ, α]
Null
```

```
THE GENERAL FORMULA FOR ARBITRARY m AND n AND ARBITRARY NETWORKS

i belongs to backbone A and j to backbone B

Global`d
```

$$d[i] = \left(2\,(-1+n)\left(\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\,\alpha\,(2+Sa+nAA\,\sigma-nBB\,\sigma)}{2\,(-1+n)}+\frac{1}{-4+8\,n}\,(\sigma\,(2\,nAB^2\,\sigma+nAB\,(-2+2\,Sa+2\,Sb+4\,nAA\,\sigma+4\,nBB\,\sigma-n\,niA\,\sigma-n\,niB\,\sigma)-2\,(-2\,nBB^2\,\sigma-2\,nAA\,(Sa-\alpha+nAA\,\sigma)+nBB\,(2-2\,Sb-2\,\alpha+n\,niB\,\sigma)+n\,(niB\,(-1+Sb+\alpha)+niA\,(Sa-\alpha+nAA\,\sigma))))))\right)\right) \Big/ (n\,(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\,\alpha\,(2+Sa+nAA\,\sigma-nBB\,\sigma)))$$

$$d[j] = \left(2\,(-1+n)\left(\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\,\alpha\,(2+Sa+nAA\,\sigma-nBB\,\sigma)}{2\,(-1+n)}+\frac{1}{-4+8\,n}\,(\sigma\,(2\,nAB^2\,\sigma+nAB\,(-2+2\,Sa+2\,Sb+4\,nAA\,\sigma+4\,nBB\,\sigma-n\,njA\,\sigma-n\,njB\,\sigma)-2\,(-2\,nBB^2\,\sigma-2\,nAA\,(Sa-\alpha+nAA\,\sigma)+nBB\,(2-2\,Sb-2\,\alpha+n\,njB\,\sigma)+n\,(njB\,(-1+Sb+\alpha)+njA\,(Sa-\alpha+nAA\,\sigma))))))\right)\right) \Big/ (n\,(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\,\alpha\,(2+Sa+nAA\,\sigma-nBB\,\sigma)))$$

```
Global`p
```

$$p[i] = \frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\,\alpha\,(2+Sa+nAA\,\sigma-nBB\,\sigma)}{2\,(-1+n)}+\frac{1}{-4+8\,n}\,(\sigma\,(2\,nAB^2\,\sigma+nAB\,(-2+2\,Sa+2\,Sb+4\,nAA\,\sigma+4\,nBB\,\sigma-n\,niA\,\sigma-n\,niB\,\sigma)-2\,(-2\,nBB^2\,\sigma-2\,nAA\,(Sa-\alpha+nAA\,\sigma)+nBB\,(2-2\,Sb-2\,\alpha+n\,niB\,\sigma)+n\,(niB\,(-1+Sb+\alpha)+niA\,(Sa-\alpha+nAA\,\sigma)))))$$

$$p[j] = \frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\,\alpha\,(2+Sa+nAA\,\sigma-nBB\,\sigma)}{2\,(-1+n)}+\frac{1}{-4+8\,n}\,(\sigma\,(2\,nAB^2\,\sigma+nAB\,(-2+2\,Sa+2\,Sb+4\,nAA\,\sigma+4\,nBB\,\sigma-n\,njA\,\sigma-n\,njB\,\sigma)-2\,(-2\,nBB^2\,\sigma-2\,nAA\,(Sa-\alpha+nAA\,\sigma)+nBB\,(2-2\,Sb-2\,\alpha+n\,njB\,\sigma)+n\,(njB\,(-1+Sb+\alpha)+njA\,(Sa-\alpha+nAA\,\sigma)))))$$

```
Global`ϕ
```

$$\phi[i] = -FA-\frac{1}{2}\,(niA+niB)\,\sigma^2+\left(2\,(-1+n)\left(\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\,\alpha\,(2+Sa+nAA\,\sigma-nBB\,\sigma)}{2\,(-1+n)}+\frac{1}{-4+8\,n}\,(\sigma\,(2\,nAB^2\,\sigma+nAB\,(-2+2\,Sa+2\,Sb+4\,nAA\,\sigma+4\,nBB\,\sigma-n\,niA\,\sigma-n\,niB\,\sigma)-2\,(-2\,nBB^2\,\sigma-2\,nAA\,(Sa-\alpha+nAA\,\sigma)+nBB\,(2-2\,Sb-2\,\alpha+n\,niB\,\sigma)+n\,(niB\,(-1+Sb+\alpha)+niA\,(Sa-\alpha+nAA\,\sigma)))))\right)^2\right) \Big/ (n\,(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\,\alpha\,(2+Sa+nAA\,\sigma-nBB\,\sigma)))$$

$$\phi[j] = -FB-\frac{1}{2}\,(njA+njB)\,\sigma^2+\left(2\,(-1+n)\left(\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\,\alpha\,(2+Sa+nAA\,\sigma-nBB\,\sigma)}{2\,(-1+n)}+\frac{1}{-4+8\,n}\,(\sigma\,(2\,nAB^2\,\sigma+nAB\,(-2+2\,Sa+2\,Sb+4\,nAA\,\sigma+4\,nBB\,\sigma-n\,njA\,\sigma-n\,njB\,\sigma)-2\,(-2\,nBB^2\,\sigma-2\,nAA\,(Sa-\alpha+nAA\,\sigma)+nBB\,(2-2\,Sb-2\,\alpha+n\,njB\,\sigma)+n\,(njB\,(-1+Sb+\alpha)+njA\,(Sa-\alpha+nAA\,\sigma)))))\right)^2\right) \Big/ (n\,(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\,\alpha\,(2+Sa+nAA\,\sigma-nBB\,\sigma)))$$

```
Null

Global`p

Global`ϕ
```

## "VERIFYING THE GENERAL FORMULA FOR ARBITRARY m AND n"

```
Clear[d, p, ϕ, M, κa, κb, χa, χb, ξa, ξb, n, m,
 niA, niB, njA, njB, nA, nAA, nBB, nAB, nB, nab, nba, σ, α]
n = 5;
?n
m = 2;
?m
```

$$\text{Case} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix};$$

```
var = ( m - 1  n - m  n - m - 1  m );
M = (Case.var);
```

```
nA = M[[1, 1]];
nab = M[[2, 2]];
nB = M[[3, 3]];
nba = M[[4, 4]];

nAA = (m * nA)/2;
nBB = ((n - m) * nB)/2;
nAB = nba * nab;
tA = α;
tB = 1 - α;
wA = nAA * σ + nAB/2 * σ;
wB = nBB * σ + nAB/2 * σ;
θA = tA - wA - Sa;
θB = tB - wB - Sb;
Do[δ[i] = d[i] * (α * θA + (1 - α) * θB) - σ * ((nA * θA)/2 + nab/2 * θB), {i, 1, m, 1}];
Do[δ[i] = d[i] * (α * θA + (1 - α) * θB) - σ * (nba/2 * θA + (nB * θB)/2), {i, m + 1, n, 1}];
Do[U[i] = V - δ[i] - p[i], {i, 1, n, 1}];
"Solving for demands";
Simplify[Solve[
   Append[Table[U[i] == U[i + 1], {i, 1, n - 1, 1}], Sum[d[i], {i, 1, n}] == 1], Table[d[i], {i, 1, n, 1}]]];
Do[d[i] = d[i] /. %, {i, 1, n, 1}];
Do[ϕ[i] = p[i] * d[i] - FA - σ^2/2 * (nA + nab), {i, 1, m, 1}];
Do[ϕ[i] = p[i] * d[i] - FB - σ^2/2 * (nB + nba), {i, m + 1, n, 1}];
"Solving for prices";
Simplify[Solve[Table[D[ϕ[i], p[i]] == 0, {i, 1, n, 1}] , Table[p[i], {i, 1, n, 1}]]];
Do[p[i] = p[i] /. %, {i, 1, n, 1}];
"Solving for profits";
Do[ϕ[i] = p[i] * d[i] - FA - σ^2/2 * (nA + nab), {i, 1, m, 1}];
Do[ϕ[i] = p[i] * d[i] - FB - σ^2/2 * (nB + nba), {i, m + 1, n, 1}];
"Solving for demands";
Simplify[Do[d[i] = d[i], {i, 1, n, 1}]];
price[a] = p[1];
price[b] = p[m + 1];
demand[a] = d[1];
demand[b] = d[m + 1];
profit[a] = ϕ[1];
profit[b] = ϕ[m + 1];
niA = nA;
niB = nab;
njA = nba;
```

$$\mathrm{njB} = \mathrm{nB};$$

$$\mathrm{dA} =$$

$$\mathrm{d[i]} = \left(2\,(-1+n)\left(\frac{2 + 2\,\mathrm{Sb}\,(-1+\alpha) + 4\,\alpha^2 - \mathrm{nAB}\,\sigma - 2\,\mathrm{nBB}\,\sigma - 2\,\alpha\,(2 + \mathrm{Sa} + \mathrm{nAA}\,\sigma - \mathrm{nBB}\,\sigma)}{2\,(-1+n)} + \frac{1}{-4+8\,n}\right.\right.$$
$$(\sigma\,(2\,\mathrm{nAB}^2\,\sigma + \mathrm{nAB}\,(-2 + 2\,\mathrm{Sa} + 2\,\mathrm{Sb} + 4\,\mathrm{nAA}\,\sigma + 4\,\mathrm{nBB}\,\sigma - n\,\mathrm{niA}\,\sigma - n\,\mathrm{niB}\,\sigma) -$$
$$2\,(-2\,\mathrm{nBB}^2\,\sigma - 2\,\mathrm{nAA}\,(\mathrm{Sa} - \alpha + \mathrm{nAA}\,\sigma) + \mathrm{nBB}\,(2 - 2\,\mathrm{Sb} - 2\,\alpha + n\,\mathrm{niB}\,\sigma) +$$
$$\left.\left. n\,(\mathrm{niB}\,(-1 + \mathrm{Sb} + \alpha) + \mathrm{niA}\,(\mathrm{Sa} - \alpha + \mathrm{nAA}\,\sigma)))))\right)\right) \Big/$$
$$(n\,(2 + 2\,\mathrm{Sb}\,(-1+\alpha) + 4\,\alpha^2 - \mathrm{nAB}\,\sigma - 2\,\mathrm{nBB}\,\sigma - 2\,\alpha\,(2 + \mathrm{Sa} + \mathrm{nAA}\,\sigma - \mathrm{nBB}\,\sigma)));$$

$$\mathrm{dB} = \left(2\,(-1+n)\left(\frac{2 + 2\,\mathrm{Sb}\,(-1+\alpha) + 4\,\alpha^2 - \mathrm{nAB}\,\sigma - 2\,\mathrm{nBB}\,\sigma - 2\,\alpha\,(2 + \mathrm{Sa} + \mathrm{nAA}\,\sigma - \mathrm{nBB}\,\sigma)}{2\,(-1+n)} +\right.\right.$$
$$\frac{1}{-4+8\,n}\,(\sigma\,(2\,\mathrm{nAB}^2\,\sigma + \mathrm{nAB}\,(-2 + 2\,\mathrm{Sa} + 2\,\mathrm{Sb} + 4\,\mathrm{nAA}\,\sigma + 4\,\mathrm{nBB}\,\sigma - n\,\mathrm{njA}\,\sigma - n\,\mathrm{njB}\,\sigma) -$$
$$2\,(-2\,\mathrm{nBB}^2\,\sigma - 2\,\mathrm{nAA}\,(\mathrm{Sa} - \alpha + \mathrm{nAA}\,\sigma) + \mathrm{nBB}\,(2 - 2\,\mathrm{Sb} - 2\,\alpha + n\,\mathrm{njB}\,\sigma) + n$$
$$\left.\left.(\mathrm{njB}\,(-1 + \mathrm{Sb} + \alpha) + \mathrm{njA}\,(\mathrm{Sa} - \alpha + \mathrm{nAA}\,\sigma)))))\right)\right) \Big/$$
$$(n\,(2 + 2\,\mathrm{Sb}\,(-1+\alpha) + 4\,\alpha^2 - \mathrm{nAB}\,\sigma - 2\,\mathrm{nBB}\,\sigma - 2\,\alpha\,(2 + \mathrm{Sa} + \mathrm{nAA}\,\sigma - \mathrm{nBB}\,\sigma)));$$

$$\mathrm{pA} = \frac{2 + 2\,\mathrm{Sb}\,(-1+\alpha) + 4\,\alpha^2 - \mathrm{nAB}\,\sigma - 2\,\mathrm{nBB}\,\sigma - 2\,\alpha\,(2 + \mathrm{Sa} + \mathrm{nAA}\,\sigma - \mathrm{nBB}\,\sigma)}{2\,(-1+n)} +$$
$$\frac{1}{-4+8\,n}\,(\sigma\,(2\,\mathrm{nAB}^2\,\sigma + \mathrm{nAB}\,(-2 + 2\,\mathrm{Sa} + 2\,\mathrm{Sb} + 4\,\mathrm{nAA}\,\sigma + 4\,\mathrm{nBB}\,\sigma - n\,\mathrm{niA}\,\sigma - n\,\mathrm{niB}\,\sigma) -$$
$$2\,(-2\,\mathrm{nBB}^2\,\sigma - 2\,\mathrm{nAA}\,(\mathrm{Sa} - \alpha + \mathrm{nAA}\,\sigma) + \mathrm{nBB}\,(2 - 2\,\mathrm{Sb} - 2\,\alpha + n\,\mathrm{niB}\,\sigma) +$$
$$n\,(\mathrm{niB}\,(-1 + \mathrm{Sb} + \alpha) + \mathrm{niA}\,(\mathrm{Sa} - \alpha + \mathrm{nAA}\,\sigma)))));$$

$$\mathrm{pB} = \frac{2 + 2\,\mathrm{Sb}\,(-1+\alpha) + 4\,\alpha^2 - \mathrm{nAB}\,\sigma - 2\,\mathrm{nBB}\,\sigma - 2\,\alpha\,(2 + \mathrm{Sa} + \mathrm{nAA}\,\sigma - \mathrm{nBB}\,\sigma)}{2\,(-1+n)} +$$
$$\frac{1}{-4+8\,n}\,(\sigma\,(2\,\mathrm{nAB}^2\,\sigma + \mathrm{nAB}\,(-2 + 2\,\mathrm{Sa} + 2\,\mathrm{Sb} + 4\,\mathrm{nAA}\,\sigma + 4\,\mathrm{nBB}\,\sigma - n\,\mathrm{njA}\,\sigma - n\,\mathrm{njB}\,\sigma) -$$
$$2\,(-2\,\mathrm{nBB}^2\,\sigma - 2\,\mathrm{nAA}\,(\mathrm{Sa} - \alpha + \mathrm{nAA}\,\sigma) + \mathrm{nBB}\,(2 - 2\,\mathrm{Sb} - 2\,\alpha + n\,\mathrm{njB}\,\sigma) +$$
$$n\,(\mathrm{njB}\,(-1 + \mathrm{Sb} + \alpha) + \mathrm{njA}\,(\mathrm{Sa} - \alpha + \mathrm{nAA}\,\sigma)))));$$

$$\phi\mathrm{A} = -\mathrm{FA} - \frac{1}{2}\,(\mathrm{niA} + \mathrm{niB})\,\sigma^2 + \left(2\,(-1+n)\right.$$
$$\left(\frac{2 + 2\,\mathrm{Sb}\,(-1+\alpha) + 4\,\alpha^2 - \mathrm{nAB}\,\sigma - 2\,\mathrm{nBB}\,\sigma - 2\,\alpha\,(2 + \mathrm{Sa} + \mathrm{nAA}\,\sigma - \mathrm{nBB}\,\sigma)}{2\,(-1+n)} +\right.$$
$$\frac{1}{-4+8\,n}\,(\sigma\,(2\,\mathrm{nAB}^2\,\sigma + \mathrm{nAB}\,(-2 + 2\,\mathrm{Sa} + 2\,\mathrm{Sb} + 4\,\mathrm{nAA}\,\sigma + 4\,\mathrm{nBB}\,\sigma - n\,\mathrm{niA}\,\sigma - n\,\mathrm{niB}\,\sigma) - 2$$
$$(-2\,\mathrm{nBB}^2\,\sigma - 2\,\mathrm{nAA}\,(\mathrm{Sa} - \alpha + \mathrm{nAA}\,\sigma) + \mathrm{nBB}\,(2 - 2\,\mathrm{Sb} - 2\,\alpha + n\,\mathrm{niB}\,\sigma) +$$
$$\left.\left. n\,(\mathrm{niB}\,(-1 + \mathrm{Sb} + \alpha) + \mathrm{niA}\,(\mathrm{Sa} - \alpha + \mathrm{nAA}\,\sigma)))))\right)^2\right) \Big/$$
$$(n\,(2 + 2\,\mathrm{Sb}\,(-1+\alpha) + 4\,\alpha^2 - \mathrm{nAB}\,\sigma - 2\,\mathrm{nBB}\,\sigma - 2\,\alpha\,(2 + \mathrm{Sa} + \mathrm{nAA}\,\sigma - \mathrm{nBB}\,\sigma)));$$

$$\phi\mathrm{B} = -\mathrm{FB} - \frac{1}{2}\,(\mathrm{njA} + \mathrm{njB})\,\sigma^2 +$$
$$\left(2\,(-1+n)\left(\frac{2 + 2\,\mathrm{Sb}\,(-1+\alpha) + 4\,\alpha^2 - \mathrm{nAB}\,\sigma - 2\,\mathrm{nBB}\,\sigma - 2\,\alpha\,(2 + \mathrm{Sa} + \mathrm{nAA}\,\sigma - \mathrm{nBB}\,\sigma)}{2\,(-1+n)} +\right.\right.$$
$$\frac{1}{-4+8\,n}\,(\sigma\,(2\,\mathrm{nAB}^2\,\sigma + \mathrm{nAB}\,(-2 + 2\,\mathrm{Sa} + 2\,\mathrm{Sb} + 4\,\mathrm{nAA}\,\sigma + 4\,\mathrm{nBB}\,\sigma - n\,\mathrm{njA}\,\sigma - n\,\mathrm{njB}\,\sigma) - 2$$
$$(-2\,\mathrm{nBB}^2\,\sigma - 2\,\mathrm{nAA}\,(\mathrm{Sa} - \alpha + \mathrm{nAA}\,\sigma) + \mathrm{nBB}\,(2 - 2\,\mathrm{Sb} - 2\,\alpha + n\,\mathrm{njB}\,\sigma) +$$

```
                    n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))) ^2 ) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)));

χa = Simplify[dA - demand[a]]
χb = Simplify[dB - demand[b]]
κa = Simplify[pA - price[a]]
κb = Simplify[pB - price[b]]
ξa = Simplify[ϕA - profit[a]]
ξb = Simplify[ϕB - profit[b]]
Clear[d, p, ϕ, M, κa, κb, χa, χb, ξa, ξb, n, m,
 niA, niB, njA, njB, nAA, nAB, nBB, nA, nB, nab, nba, σ, α]
Null
```

VERIFYING THE GENERAL FORMULA FOR ARBITRARY m AND n

```
Global`n

n = 5

Global`m

m = 2

{{0}}

{{0}}

{0}

{0}

{{0}}

{{0}}
```

## "GENERAL FORMULA FOR THE EIGHT FOCAL NETWORKS"

```
"i belongs to A and j belongs to B"
Clear[d, ϕ, p, niA, niB, njA, njB, nAA, nBB, nAB, profit]
"Case1: Complete network"
niA = (m - 1);
niB = (n - m);
njA = m;
njB = (n - m - 1);
```

$$nAA = \frac{m\ (m-1)}{2};$$

$$nBB = \frac{(n-m) * (n-m-1)}{2};$$

```
nAB = m * (n - m);

d[i] = Simplify[
```

$$\Bigg(2\ (-1+n)\ \Bigg(\frac{2 + 2\ Sb\ (-1+\alpha) + 4\ \alpha^2 - nAB\ \sigma - 2\ nBB\ \sigma - 2\ \alpha\ (2 + Sa + nAA\ \sigma - nBB\ \sigma)}{2\ (-1+n)} + \frac{1}{-4+8\ n}$$

$$(\sigma\ (2\ nAB^2\ \sigma + nAB\ (-2 + 2\ Sa + 2\ Sb + 4\ nAA\ \sigma + 4\ nBB\ \sigma - n\ niA\ \sigma - n\ niB\ \sigma) - 2\ (-2\ nBB^2\ \sigma -$$

```
          2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
          n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ)))))))) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
d[j] = Simplify[(2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)) / (2 (-1 + n)) +
       1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
          2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
          n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))))) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
p[i] = Simplify[(2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)) / (2 (-1 + n)) +
    1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) -
       2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
         n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ)))))];
p[j] = Simplify[(2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)) / (2 (-1 + n)) +
    1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
       2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
         n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))];
ϕ[i] = Simplify[-FA - 1/2 (niA + niB) σ^2 + (2 (-1 + n)
       ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)) / (2 (-1 + n)) +
         1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) -
            2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
               n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ))))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
ϕ[j] = Simplify[-FB - 1/2 (njA + njB) σ^2 +
    (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)) / (2 (-1 + n)) +
         1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
            2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
               n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ))))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
?d
?p
?ϕ
```

```
profit[AgAB1] = ϕ[i];
profit[BgAB1] = ϕ[j];
Clear[d, ϕ, p, niA, niB, njA, njB, nAA, nBB, nAB]
"------------------------------------------------------------"

"Case2: Empty network"
niA = 0;
niB = 0;
njA = 0;
njB = 0;
nAA = 0;
nBB = 0;
nAB = 0;

d[i] = Simplify[
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) + 1/(-4 + 8 n)
        (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) - 2 (-2 nBB^2 σ -
              2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
              n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ))))))) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
d[j] = Simplify[(2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
          1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
             2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
              n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ))))))) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
p[i] = Simplify[(2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
    1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) -
          2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
             n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ)))))];
p[j] = Simplify[(2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
    1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
          2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
             n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))];
ϕ[i] = Simplify[-FA - 1/2 (niA + niB) σ^2 + (2 (-1 + n)
      ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
        1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) -
```

```
2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
      n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ)))))^2) /
  (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
ϕ[j] = Simplify[-FB - 1/2 (njA + njB) σ^2 +
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
        1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
          2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
            n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))^2) /
  (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
?d
?p
?ϕ
profit[Ag000] = ϕ[i];
profit[Bg000] = ϕ[j];
Clear[d, ϕ, p, niA, niB, njA, njB, nAA, nBB, nAB]
"-------------------------------------------------------------"

"Case3: A and B peer but no interpeering"
niA = (m - 1);
niB = 0;
njA = 0;
njB = (n - m - 1);
nAA = (m * (m - 1))/2;
nBB = ((n - m) * (n - m - 1))/2;
nAB = 0;

d[i] = Simplify[
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) + 1/(-4 + 8 n)
       (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) - 2 (-2 nBB^2 σ -
          2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
          n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ)))))) /
  (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
d[j] = Simplify[(2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
     1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
       2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
         n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))) /
```

```
(n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
p[i] = Simplify[(2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
    1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) -
        2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
          n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ)))))];
p[j] = Simplify[(2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
    1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
        2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
          n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))];
ϕ[i] = Simplify[-FA - 1/2 (niA + niB) σ^2 + (2 (-1 + n)
      ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
        1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) -
            2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
              n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ))))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
ϕ[j] = Simplify[-FB - 1/2 (njA + njB) σ^2 +
    (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
        1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
            2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
              n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ))))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
?d
?p
?ϕ
profit[AgAB0] = ϕ[i];
profit[BgAB0] = ϕ[j];
Clear[d, ϕ, p, niA, niB, njA, njB, nAA, nBB, nAB]
"------------------------------------------------------------"

"Case 4: A peers and B does not peer but no interpeering"
niA = (m - 1);
niB = 0;
njA = 0;
njB = 0;
nAA = (m * (m - 1))/2;
```

$$\mathrm{nBB} = 0;$$
$$\mathrm{nAB} = 0;$$

$$\mathrm{d}[\mathrm{i}] = \mathrm{Simplify}\Big[\Big(2\,(-1+\mathrm{n})\,\Big(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)}{2\,(-1+\mathrm{n})}+\frac{1}{-4+8\,\mathrm{n}}\,(\sigma\,(2\,\mathrm{nAB}^2\,\sigma+\mathrm{nAB}\,(-2+2\,\mathrm{Sa}+2\,\mathrm{Sb}+4\,\mathrm{nAA}\,\sigma+4\,\mathrm{nBB}\,\sigma-\mathrm{n}\,\mathrm{niA}\,\sigma-\mathrm{n}\,\mathrm{niB}\,\sigma)-2\,(-2\,\mathrm{nBB}^2\,\sigma-2\,\mathrm{nAA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)+\mathrm{nBB}\,(2-2\,\mathrm{Sb}-2\,\alpha+\mathrm{n}\,\mathrm{niB}\,\sigma)+\mathrm{n}\,(\mathrm{niB}\,(-1+\mathrm{Sb}+\alpha)+\mathrm{niA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)))))\Big)\Big)\Big/(\mathrm{n}\,(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)))\Big];$$

$$\mathrm{d}[\mathrm{j}] = \mathrm{Simplify}\Big[\Big(2\,(-1+\mathrm{n})\,\Big(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)}{2\,(-1+\mathrm{n})}+\frac{1}{-4+8\,\mathrm{n}}\,(\sigma\,(2\,\mathrm{nAB}^2\,\sigma+\mathrm{nAB}\,(-2+2\,\mathrm{Sa}+2\,\mathrm{Sb}+4\,\mathrm{nAA}\,\sigma+4\,\mathrm{nBB}\,\sigma-\mathrm{n}\,\mathrm{njA}\,\sigma-\mathrm{n}\,\mathrm{njB}\,\sigma)-2\,(-2\,\mathrm{nBB}^2\,\sigma-2\,\mathrm{nAA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)+\mathrm{nBB}\,(2-2\,\mathrm{Sb}-2\,\alpha+\mathrm{n}\,\mathrm{njB}\,\sigma)+\mathrm{n}\,(\mathrm{njB}\,(-1+\mathrm{Sb}+\alpha)+\mathrm{njA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)))))\Big)\Big)\Big/(\mathrm{n}\,(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)))\Big];$$

$$\mathrm{p}[\mathrm{i}] = \mathrm{Simplify}\Big[\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)}{2\,(-1+\mathrm{n})}+\frac{1}{-4+8\,\mathrm{n}}\,(\sigma\,(2\,\mathrm{nAB}^2\,\sigma+\mathrm{nAB}\,(-2+2\,\mathrm{Sa}+2\,\mathrm{Sb}+4\,\mathrm{nAA}\,\sigma+4\,\mathrm{nBB}\,\sigma-\mathrm{n}\,\mathrm{niA}\,\sigma-\mathrm{n}\,\mathrm{niB}\,\sigma)-2\,(-2\,\mathrm{nBB}^2\,\sigma-2\,\mathrm{nAA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)+\mathrm{nBB}\,(2-2\,\mathrm{Sb}-2\,\alpha+\mathrm{n}\,\mathrm{niB}\,\sigma)+\mathrm{n}\,(\mathrm{niB}\,(-1+\mathrm{Sb}+\alpha)+\mathrm{niA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)))))\Big];$$

$$\mathrm{p}[\mathrm{j}] = \mathrm{Simplify}\Big[\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)}{2\,(-1+\mathrm{n})}+\frac{1}{-4+8\,\mathrm{n}}\,(\sigma\,(2\,\mathrm{nAB}^2\,\sigma+\mathrm{nAB}\,(-2+2\,\mathrm{Sa}+2\,\mathrm{Sb}+4\,\mathrm{nAA}\,\sigma+4\,\mathrm{nBB}\,\sigma-\mathrm{n}\,\mathrm{njA}\,\sigma-\mathrm{n}\,\mathrm{njB}\,\sigma)-2\,(-2\,\mathrm{nBB}^2\,\sigma-2\,\mathrm{nAA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)+\mathrm{nBB}\,(2-2\,\mathrm{Sb}-2\,\alpha+\mathrm{n}\,\mathrm{njB}\,\sigma)+\mathrm{n}\,(\mathrm{njB}\,(-1+\mathrm{Sb}+\alpha)+\mathrm{njA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)))))\Big];$$

$$\phi[\mathrm{i}] = \mathrm{Simplify}\Big[-\mathrm{FA}-\frac{1}{2}\,(\mathrm{niA}+\mathrm{niB})\,\sigma^2+\Big(2\,(-1+\mathrm{n})\,\Big(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)}{2\,(-1+\mathrm{n})}+\frac{1}{-4+8\,\mathrm{n}}\,(\sigma\,(2\,\mathrm{nAB}^2\,\sigma+\mathrm{nAB}\,(-2+2\,\mathrm{Sa}+2\,\mathrm{Sb}+4\,\mathrm{nAA}\,\sigma+4\,\mathrm{nBB}\,\sigma-\mathrm{n}\,\mathrm{niA}\,\sigma-\mathrm{n}\,\mathrm{niB}\,\sigma)-2\,(-2\,\mathrm{nBB}^2\,\sigma-2\,\mathrm{nAA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)+\mathrm{nBB}\,(2-2\,\mathrm{Sb}-2\,\alpha+\mathrm{n}\,\mathrm{niB}\,\sigma)+\mathrm{n}\,(\mathrm{niB}\,(-1+\mathrm{Sb}+\alpha)+\mathrm{niA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)))))\Big)^2\Big)\Big/(\mathrm{n}\,(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)))\Big];$$

$$\phi[\mathrm{j}] = \mathrm{Simplify}\Big[-\mathrm{FB}-\frac{1}{2}\,(\mathrm{njA}+\mathrm{njB})\,\sigma^2+\Big(2\,(-1+\mathrm{n})\,\Big(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)}{2\,(-1+\mathrm{n})}+\frac{1}{-4+8\,\mathrm{n}}\,(\sigma\,(2\,\mathrm{nAB}^2\,\sigma+\mathrm{nAB}\,(-2+2\,\mathrm{Sa}+2\,\mathrm{Sb}+4\,\mathrm{nAA}\,\sigma+4\,\mathrm{nBB}\,\sigma-\mathrm{n}\,\mathrm{njA}\,\sigma-\mathrm{n}\,\mathrm{njB}\,\sigma)-$$

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          2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
               n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
?d
?p
?ϕ
profit[AgA00] = ϕ[i];
profit[BgA00] = ϕ[j];
Clear[d, ϕ, p, niA, niB, njA, njB, nAA, nBB, nAB]

"----------------------------------------------------------"

"Case 5: B peers and A does not peer but no interpeering"
niA = 0;
niB = 0;
njA = 0;
njB = (n - m - 1);
nAA = 0;
nBB = ((n - m) * (n - m - 1))/2;
nAB = 0;

d[i] = Simplify[
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) + 1/(-4 + 8 n)
        (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) - 2 (-2 nBB^2 σ -
             2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
             n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ)))))) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
d[j] = Simplify[(2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
        1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
           2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
             n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
p[i] = Simplify[(2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
    1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) -
       2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
          n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ)))))];
p[j] = Simplify[(2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
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      1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
          2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
             n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))];
ϕ[i] = Simplify[-FA - 1/2 (niA + niB) σ^2 + (2 (-1 + n)
       ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
        1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) -
            2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
               n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ)))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
ϕ[j] = Simplify[-FB - 1/2 (njA + njB) σ^2 +
    (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
        1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
            2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
               n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
?d
?p
?ϕ
profit[Ag0B0] = ϕ[i];
profit[Bg0B0] = ϕ[j];
Clear[d, ϕ, p, niA, niB, njA, njB, nAA, nBB, nAB]

"-----------------------------------------------------------"

"Case 6: A peers and B does not peer but there is interpeering"
niA = (m - 1);
niB = (n - m);
njA = m;
njB = 0;
nAA = (m * (m - 1))/2;
nBB = 0;
nAB = m * (n - m);

d[i] = Simplify[
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) + 1/(-4 + 8 n)
       (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) - 2 (-2 nBB^2 σ -
             2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
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        n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ))))))) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
d[j] = Simplify[(2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)) / (2 (-1 + n)) +
      1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
          2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
            n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ))))))) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
p[i] = Simplify[(2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)) / (2 (-1 + n)) +
    1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) -
        2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
          n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ)))))];
p[j] = Simplify[(2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)) / (2 (-1 + n)) +
    1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
        2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
          n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))];
ϕ[i] = Simplify[-FA - 1/2 (niA + niB) σ^2 + (2 (-1 + n)
      ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)) / (2 (-1 + n)) +
        1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) -
            2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
              n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ))))))^2) /
      (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
ϕ[j] = Simplify[-FB - 1/2 (njA + njB) σ^2 +
    (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)) / (2 (-1 + n)) +
        1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
            2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
              n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ))))))^2) /
      (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
?d
?p
?ϕ
profit[AgA01] = ϕ[i];
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profit[BgA01] = ϕ[j];
Clear[d, ϕ, p, niA, niB, njA, njB, nAA, nBB, nAB]
"-------------------------------------------------------------------------------"

"Case 7: B peers and A does not peer but there is interpeering"
niA = 0;
niB = (n - m);
njA = m;
njB = (n - m - 1);
nAA = 0;
nBB = (n - m) * (n - m - 1) / 2;
nAB = m * (n - m);

d[i] = Simplify[
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)) / (2 (-1 + n)) + 1/(-4 + 8 n)
        (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) - 2 (-2 nBB^2 σ -
              2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
              n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ)))))) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
d[j] = Simplify[(2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)) / (2 (-1 + n)) +
        1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
           2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
              n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
p[i] = Simplify[(2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)) / (2 (-1 + n)) +
    1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) -
        2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
           n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ)))))];
p[j] = Simplify[(2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)) / (2 (-1 + n)) +
    1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
        2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
           n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))];
ϕ[i] = Simplify[-FA - 1/2 (niA + niB) σ^2 + (2 (-1 + n)
       ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)) / (2 (-1 + n)) +
         1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) -
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        2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
            n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ))))))^2)) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
ϕ[j] = Simplify[-FB - 1/2 (njA + njB) σ^2 +
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
        1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
        2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
            n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ))))))^2)) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
?d
?p
?ϕ
profit[Ag0B1] = ϕ[i];
profit[Bg0B1] = ϕ[j];
Clear[d, ϕ, p, niA, niB, njA, njB, nAA, nBB, nAB]
"----------------------------------------------------------------"
"Case 8: No intrapeering but there is interpeering"
niA = 0;
niB = (n - m);
njA = m;
njB = 0;
nAA = 0;
nBB = 0;
nAB = m * (n - m);

d[i] = Simplify[
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) + 1/(-4 + 8 n)
        (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) - 2 (-2 nBB^2 σ -
            2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
            n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ))))))) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
d[j] = Simplify[(2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
        1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
        2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
            n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ))))))) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
```

```
p[i] = Simplify[(2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
    1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) -
        2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
          n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ)))))];
p[j] = Simplify[(2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
    1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
        2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
          n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))];
ϕ[i] = Simplify[-FA - 1/2 (niA + niB) σ^2 + (2 (-1 + n)
      ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
        1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) -
            2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
              n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ))))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
ϕ[j] = Simplify[-FB - 1/2 (njA + njB) σ^2 +
    (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
        1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
            2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
              n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ))))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))];
?d
?p
?ϕ
profit[Ag001] = ϕ[i];
profit[Bg001] = ϕ[j];
Clear[d, ϕ, p, niA, niB, njA, njB, nAA, nBB, nAB]
"----------------------------------------------------------------------------"
```

```
General formula for the eight focal networks

i belongs to A and j belongs to B

Case1: Complete network

Global`d
```

$$d[i] = (-n^4 \sigma^2 + n^3 \sigma (-4 + 4 \alpha + (2 + 3 m) \sigma) - n^2 \sigma (-8 - 2 Sa + 2 Sb + 10 \alpha + \sigma + 2 m^2 \sigma + m (-4 + 8 \alpha + 6 \sigma)) - 2 (2 + 4 \alpha^2 - 2 m \sigma - m Sa \sigma + m^2 \sigma^2 - 2 \alpha (2 + Sa - 2 m \sigma) + Sb (-2 + 2 \alpha + m \sigma)) + n (8 + 16 \alpha^2 - 4 \sigma - 8 m \sigma - 2 Sa \sigma - 2 m Sa \sigma + 3 m \sigma^2 + 4 m^2 \sigma^2 - 2 \alpha (8 + 4 Sa - 3 \sigma - 8 m \sigma) + 2 Sb (-4 + 4 \alpha + \sigma + m \sigma))) / (2 n (-1 + 2 n) (2 + 2 Sb (-1 + \alpha) + 4 \alpha^2 - m \sigma + n \sigma + m n \sigma - n^2 \sigma - \alpha (4 + 2 Sa + 2 m (-1 + n) \sigma + n \sigma - n^2 \sigma)))$$

$$d[j] = (n^3 \sigma (-4 + 4 \alpha + m \sigma) - 2 n^2 \sigma (3 (-1 + \alpha) + m^2 \sigma + m (-2 + 4 \alpha + \sigma)) - 2 (2 + 4 \alpha^2 - 2 m \sigma - m Sa \sigma + m^2 \sigma^2 - 2 \alpha (2 + Sa - 2 m \sigma) + Sb (-2 + 2 \alpha + m \sigma)) + n (8 + 16 \alpha^2 - 2 \sigma - 8 m \sigma - 2 m Sa \sigma + m \sigma^2 + 4 m^2 \sigma^2 - 2 \alpha (8 + 4 Sa - \sigma - 8 m \sigma) + 2 Sb (-4 + 4 \alpha + m \sigma))) / (2 n (-1 + 2 n) (2 + 2 Sb (-1 + \alpha) + 4 \alpha^2 - m \sigma + n \sigma + m n \sigma - n^2 \sigma - \alpha (4 + 2 Sa + 2 m (-1 + n) \sigma + n \sigma - n^2 \sigma)))$$

Global`p

$$p[i] = \frac{1}{4 (-1+n) (-1+2 n)} (-n^4 \sigma^2 + n^3 \sigma (-4 + 4 \alpha + (2 + 3 m) \sigma) - n^2 \sigma (-8 - 2 Sa + 2 Sb + 10 \alpha + \sigma + 2 m^2 \sigma + m (-4 + 8 \alpha + 6 \sigma)) - 2 (2 + 4 \alpha^2 - 2 m \sigma - m Sa \sigma + m^2 \sigma^2 - 2 \alpha (2 + Sa - 2 m \sigma) + Sb (-2 + 2 \alpha + m \sigma)) + n (8 + 16 \alpha^2 - 4 \sigma - 8 m \sigma - 2 Sa \sigma - 2 m Sa \sigma + 3 m \sigma^2 + 4 m^2 \sigma^2 - 2 \alpha (8 + 4 Sa - 3 \sigma - 8 m \sigma) + 2 Sb (-4 + 4 \alpha + \sigma + m \sigma)))$$

$$p[j] = \frac{1}{4 (-1+n) (-1+2 n)} (n^3 \sigma (-4 + 4 \alpha + m \sigma) - 2 n^2 \sigma (3 (-1 + \alpha) + m^2 \sigma + m (-2 + 4 \alpha + \sigma)) - 2 (2 + 4 \alpha^2 - 2 m \sigma - m Sa \sigma + m^2 \sigma^2 - 2 \alpha (2 + Sa - 2 m \sigma) + Sb (-2 + 2 \alpha + m \sigma)) + n (8 + 16 \alpha^2 - 2 \sigma - 8 m \sigma - 2 m Sa \sigma + m \sigma^2 + 4 m^2 \sigma^2 - 2 \alpha (8 + 4 Sa - \sigma - 8 m \sigma) + 2 Sb (-4 + 4 \alpha + m \sigma)))$$

Global`$\phi$

$$\phi[i] = \frac{1}{8} \Big(-8 FA - 4 (-1 + n) \sigma^2 + (n^4 \sigma^2 - n^3 \sigma (-4 + 4 \alpha + (2 + 3 m) \sigma) + n^2 \sigma (-8 - 2 Sa + 2 Sb + 10 \alpha + \sigma + 2 m^2 \sigma + m (-4 + 8 \alpha + 6 \sigma)) + 2 (2 + 4 \alpha^2 - 2 m \sigma - m Sa \sigma + m^2 \sigma^2 - 2 \alpha (2 + Sa - 2 m \sigma) + Sb (-2 + 2 \alpha + m \sigma)) - n (8 + 16 \alpha^2 - 4 \sigma - 8 m \sigma - 2 Sa \sigma - 2 m Sa \sigma + 3 m \sigma^2 + 4 m^2 \sigma^2 - 2 \alpha (8 + 4 Sa - 3 \sigma - 8 m \sigma) + 2 Sb (-4 + 4 \alpha + \sigma + m \sigma)))^2 \Big/ ((1 - 2 n)^2 (-1 + n) n (2 + 2 Sb (-1 + \alpha) + 4 \alpha^2 - m \sigma + n \sigma + m n \sigma - n^2 \sigma - \alpha (4 + 2 Sa + 2 m (-1 + n) \sigma + n \sigma - n^2 \sigma)))\Big)$$

$$\phi[j] = \frac{1}{8} \Big(-8 FB - 4 (-1 + n) \sigma^2 + (n^3 \sigma (-4 + 4 \alpha + m \sigma) - 2 n^2 \sigma (3 (-1 + \alpha) + m^2 \sigma + m (-2 + 4 \alpha + \sigma)) - 2 (2 + 4 \alpha^2 - 2 m \sigma - m Sa \sigma + m^2 \sigma^2 - 2 \alpha (2 + Sa - 2 m \sigma) + Sb (-2 + 2 \alpha + m \sigma)) + n (8 + 16 \alpha^2 - 2 \sigma - 8 m \sigma - 2 m Sa \sigma + m \sigma^2 + 4 m^2 \sigma^2 - 2 \alpha (8 + 4 Sa - \sigma - 8 m \sigma) + 2 Sb (-4 + 4 \alpha + m \sigma)))^2 \Big/ ((1 - 2 n)^2 (-1 + n) n (2 + 2 Sb (-1 + \alpha) + 4 \alpha^2 - m \sigma + n \sigma + m n \sigma - n^2 \sigma - \alpha (4 + 2 Sa + 2 m (-1 + n) \sigma + n \sigma - n^2 \sigma)))\Big)$$

-----------------------------------------------------------

Case2: Empty network

Global`d

$$d[i] = \frac{1}{n}$$

$$d[j] = \frac{1}{n}$$

Global`p

$$p[i] = \frac{1+Sb (-1+\alpha)-(2+Sa) \alpha+2 \alpha^2}{-1+n}$$

$$p[j] = \frac{1+Sb (-1+\alpha)-(2+Sa) \alpha+2 \alpha^2}{-1+n}$$

Global`$\phi$

$$\phi[i] = -FA + \frac{1+Sb (-1+\alpha)-(2+Sa) \alpha+2 \alpha^2}{(-1+n) n}$$

$$\phi[j] = -FB + \frac{1+Sb (-1+\alpha)-(2+Sa) \alpha+2 \alpha^2}{(-1+n) n}$$

---

Case3: A and B peer but no interpeering

Global`d

$$d[i] = \left(2\,(-1+n)\left(-\frac{-2-2\,Sb\,(-1+\alpha)-4\,\alpha^2+m\,\sigma+m^2\,\sigma-n\,\sigma-2\,m\,n\,\sigma+n^2\,\sigma+\alpha\,(4+2\,Sa+2\,m\,(-1+n)\,\sigma+n\,\sigma-n^2\,\sigma)}{2\,(-1+n)}+\frac{(m-n)\,\sigma\,(-2-2\,Sa+2\,Sb+4\,\alpha+2\,m^3\,\sigma-3\,m^2\,n\,\sigma+2\,n^2\,\sigma-n^3\,\sigma-n\,(-2+2\,Sb+2\,\alpha+\sigma)+m\,(-2+2\,Sa+2\,Sb+2\,\sigma-4\,n\,\sigma+3\,n^2\,\sigma))}{-4+8\,n}\right)\right) \Big/ \left(n\,(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-(m-n)\,(1+m-n)\,\sigma-\alpha\,(4+2\,Sa+2\,m\,(-1+n)\,\sigma+n\,\sigma-n^2\,\sigma))\right)$$

$$d[j] = \left(2\,(-1+n)\left(-\frac{-2-2\,Sb\,(-1+\alpha)-4\,\alpha^2+m\,\sigma+m^2\,\sigma-n\,\sigma-2\,m\,n\,\sigma+n^2\,\sigma+\alpha\,(4+2\,Sa+2\,m\,(-1+n)\,\sigma+n\,\sigma-n^2\,\sigma)}{2\,(-1+n)}+\frac{m\,\sigma\,(-2-2\,Sa+2\,Sb+4\,\alpha+2\,m^3\,\sigma-3\,m^2\,n\,\sigma+2\,n^2\,\sigma-n^3\,\sigma-n\,(-2+2\,Sb+2\,\alpha+\sigma)+m\,(-2+2\,Sa+2\,Sb+2\,\sigma-4\,n\,\sigma+3\,n^2\,\sigma))}{-4+8\,n}\right)\right) \Big/ \left(n\,(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-(m-n)\,(1+m-n)\,\sigma-\alpha\,(4+2\,Sa+2\,m\,(-1+n)\,\sigma+n\,\sigma-n^2\,\sigma))\right)$$

Global`p

$$p[i] = -\frac{-2-2\,Sb\,(-1+\alpha)-4\,\alpha^2+m\,\sigma+m^2\,\sigma-n\,\sigma-2\,m\,n\,\sigma+n^2\,\sigma+\alpha\,(4+2\,Sa+2\,m\,(-1+n)\,\sigma+n\,\sigma-n^2\,\sigma)}{2\,(-1+n)}+\frac{(m-n)\,\sigma\,(-2-2\,Sa+2\,Sb+4\,\alpha+2\,m^3\,\sigma-3\,m^2\,n\,\sigma+2\,n^2\,\sigma-n^3\,\sigma-n\,(-2+2\,Sb+2\,\alpha+\sigma)+m\,(-2+2\,Sa+2\,Sb+2\,\sigma-4\,n\,\sigma+3\,n^2\,\sigma))}{-4+8\,n}$$

$$p[j] = -\frac{-2-2\,Sb\,(-1+\alpha)-4\,\alpha^2+m\,\sigma+m^2\,\sigma-n\,\sigma-2\,m\,n\,\sigma+n^2\,\sigma+\alpha\,(4+2\,Sa+2\,m\,(-1+n)\,\sigma+n\,\sigma-n^2\,\sigma)}{2\,(-1+n)}+\frac{m\,\sigma\,(-2-2\,Sa+2\,Sb+4\,\alpha+2\,m^3\,\sigma-3\,m^2\,n\,\sigma+2\,n^2\,\sigma-n^3\,\sigma-n\,(-2+2\,Sb+2\,\alpha+\sigma)+m\,(-2+2\,Sa+2\,Sb+2\,\sigma-4\,n\,\sigma+3\,n^2\,\sigma))}{-4+8\,n}$$

Global`$\phi$

$$\phi[i] = -FA-\frac{1}{2}\,(-1+m)\,\sigma^2+\left(2\,(-1+n)\left(-\frac{-2-2\,Sb\,(-1+\alpha)-4\,\alpha^2+m\,\sigma+m^2\,\sigma-n\,\sigma-2\,m\,n\,\sigma+n^2\,\sigma+\alpha\,(4+2\,Sa+2\,m\,(-1+n)\,\sigma+n\,\sigma-n^2\,\sigma)}{2\,(-1+n)}+\frac{(m-n)\,\sigma\,(-2-2\,Sa+2\,Sb+4\,\alpha+2\,m^3\,\sigma-3\,m^2\,n\,\sigma+2\,n^2\,\sigma-n^3\,\sigma-n\,(-2+2\,Sb+2\,\alpha+\sigma)+m\,(-2+2\,Sa+2\,Sb+2\,\sigma-4\,n\,\sigma+3\,n^2\,\sigma))}{-4+8\,n}\right)^2\right) \Big/ \left(n\,(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-(m-n)\,(1+m-n)\,\sigma-\alpha\,(4+2\,Sa+2\,m\,(-1+n)\,\sigma+n\,\sigma-n^2\,\sigma))\right)$$

$$\phi[j] = -FB+\frac{1}{2}\,(1+m-n)\,\sigma^2+\left(2\,(-1+n)\left(-\frac{-2-2\,Sb\,(-1+\alpha)-4\,\alpha^2+m\,\sigma+m^2\,\sigma-n\,\sigma-2\,m\,n\,\sigma+n^2\,\sigma+\alpha\,(4+2\,Sa+2\,m\,(-1+n)\,\sigma+n\,\sigma-n^2\,\sigma)}{2\,(-1+n)}+\frac{m\,\sigma\,(-2-2\,Sa+2\,Sb+4\,\alpha+2\,m^3\,\sigma-3\,m^2\,n\,\sigma+2\,n^2\,\sigma-n^3\,\sigma-n\,(-2+2\,Sb+2\,\alpha+\sigma)+m\,(-2+2\,Sa+2\,Sb+2\,\sigma-4\,n\,\sigma+3\,n^2\,\sigma))}{-4+8\,n}\right)^2\right) \Big/ \left(n\,(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-(m-n)\,(1+m-n)\,\sigma-\alpha\,(4+2\,Sa+2\,m\,(-1+n)\,\sigma+n\,\sigma-n^2\,\sigma))\right)$$

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Case 4: A peers and B does not peer but no interpeering

Global`d

$$d[i] = \frac{2\,(-1+n)\left(\frac{(-1+m)\,(m-n)\,\sigma\,(2\,Sa-2\,\alpha+(-1+m)\,m\,\sigma)}{-4+8\,n}+\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-2\,\alpha\left(2+Sa+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)}{2\,(-1+n)}\right)}{n\left(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-2\,\alpha\left(2+Sa+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)\right)}$$

$$d[j] = \frac{2\,(-1+n)\left(\frac{(-1+m)\,m\,\sigma\,(2\,Sa-2\,\alpha+(-1+m)\,m\,\sigma)}{-4+8\,n}+\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-2\,\alpha\left(2+Sa+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)}{2\,(-1+n)}\right)}{n\left(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-2\,\alpha\left(2+Sa+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)\right)}$$

Global`p

$$p[i] = \frac{(-1+m)\,(m-n)\,\sigma\,(2\,Sa-2\,\alpha+(-1+m)\,m\,\sigma)}{-4+8\,n}+\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-2\,\alpha\left(2+Sa+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)}{2\,(-1+n)}$$

$$p[j] = \frac{(-1+m)\,m\,\sigma\,(2\,Sa-2\,\alpha+(-1+m)\,m\,\sigma)}{-4+8\,n}+\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-2\,\alpha\left(2+Sa+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)}{2\,(-1+n)}$$

Global`$\phi$

$$\phi[i] = -FA - \frac{1}{2}\,(-1+m)\,\sigma^2 + \frac{2\,(-1+n)\left(\frac{(-1+m)\,(m-n)\,\sigma\,(2\,Sa-2\,\alpha+(-1+m)\,m\,\sigma)}{-4+8\,n}+\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-2\,\alpha\,\left(2+Sa+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)}{2\,(-1+n)}\right)^2}{n\,\left(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-2\,\alpha\,\left(2+Sa+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)\right)}$$

$$\phi[j] = -FB + \frac{2\,(-1+n)\left(\frac{(-1+m)\,m\,\sigma\,(2\,Sa-2\,\alpha+(-1+m)\,m\,\sigma)}{-4+8\,n}+\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-2\,\alpha\,\left(2+Sa+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)}{2\,(-1+n)}\right)^2}{n\,\left(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-2\,\alpha\,\left(2+Sa+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)\right)}$$

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Case 5: B peers and A does not peer but no interpeering

Global`d

$$d[i] = \frac{2\,(-1+n)\left(-\frac{(-1-m+n)\,(-m+n)\,\sigma\,(2-2\,Sb-2\,\alpha-(m-n)\,(1+m-n)\,\sigma)}{-4+8\,n}+\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-(m-n)\,(1+m-n)\,\sigma-2\,\alpha\,\left(2+Sa-\frac{1}{2}\,(m-n)\,(1+m-n)\,\sigma\right)}{2\,(-1+n)}\right)}{n\,\left(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-(m-n)\,(1+m-n)\,\sigma-2\,\alpha\,\left(2+Sa-\frac{1}{2}\,(m-n)\,(1+m-n)\,\sigma\right)\right)}$$

$$d[j] = \frac{2\,(-1+n)\left(\frac{m\,(1+m-n)\,\sigma\,\left(-2+2\,Sb+2\,\alpha+m\,\sigma+m^2\,\sigma-n\,\sigma-2\,m\,n\,\sigma+n^2\,\sigma\right)}{-4+8\,n}+\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-(m-n)\,(1+m-n)\,\sigma-2\,\alpha\,\left(2+Sa-\frac{1}{2}\,(m-n)\,(1+m-n)\,\sigma\right)}{2\,(-1+n)}\right)}{n\,\left(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-(m-n)\,(1+m-n)\,\sigma-2\,\alpha\,\left(2+Sa-\frac{1}{2}\,(m-n)\,(1+m-n)\,\sigma\right)\right)}$$

Global`p

$$p[i] = -\frac{(-1-m+n)\,(-m+n)\,\sigma\,(2-2\,Sb-2\,\alpha-(m-n)\,(1+m-n)\,\sigma)}{-4+8\,n} + \frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-(m-n)\,(1+m-n)\,\sigma-2\,\alpha\,\left(2+Sa-\frac{1}{2}\,(m-n)\,(1+m-n)\,\sigma\right)}{2\,(-1+n)}$$

$$p[j] = \frac{m\,(1+m-n)\,\sigma\,\left(-2+2\,Sb+2\,\alpha+m\,\sigma+m^2\,\sigma-n\,\sigma-2\,m\,n\,\sigma+n^2\,\sigma\right)}{-4+8\,n} + \frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-(m-n)\,(1+m-n)\,\sigma-2\,\alpha\,\left(2+Sa-\frac{1}{2}\,(m-n)\,(1+m-n)\,\sigma\right)}{2\,(-1+n)}$$

Global`$\phi$

$$\phi[i] = -FA + \frac{2\,(-1+n)\left(\frac{(-1-m+n)\,(-m+n)\,\sigma\,(2-2\,Sb-2\,\alpha-(m-n)\,(1+m-n)\,\sigma)}{-4+8\,n}-\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-(m-n)\,(1+m-n)\,\sigma-2\,\alpha\,\left(2+Sa-\frac{1}{2}\,(m-n)\,(1+m-n)\,\sigma\right)}{2\,(-1+n)}\right)^2}{n\,\left(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-(m-n)\,(1+m-n)\,\sigma-2\,\alpha\,\left(2+Sa-\frac{1}{2}\,(m-n)\,(1+m-n)\,\sigma\right)\right)}$$

$$\phi[j] = -FB + \frac{1}{2}\,(1+m-n)\,\sigma^2 + \frac{2\,(-1+n)\left(\frac{m\,(1+m-n)\,\sigma\,\left(-2+2\,Sb+2\,\alpha+m\,\sigma+m^2\,\sigma-n\,\sigma-2\,m\,n\,\sigma+n^2\,\sigma\right)}{-4+8\,n}+\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-(m-n)\,(1+m-n)\,\sigma-2\,\alpha\,\left(2+Sa-\frac{1}{2}\,(m-n)\,(1+m-n)\,\sigma\right)}{2\,(-1+n)}\right)^2}{n\,\left(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-(m-n)\,(1+m-n)\,\sigma-2\,\alpha\,\left(2+Sa-\frac{1}{2}\,(m-n)\,(1+m-n)\,\sigma\right)\right)}$$

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Case 6: A peers and B does not peer but there is interpeering

Global`d

$$d[i] = \frac{2\,(-1+n)\left(\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2+m\,(m-n)\,\sigma-2\,\alpha\,\left(2+Sa+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)}{2\,(-1+n)}+\frac{(m-n)\,\sigma\,\left(2\,(-Sa+\alpha+n\,(-1+Sb+\alpha))+m^3\,\sigma-2\,m^2\,n\,\sigma+m\,\left(2-2\,Sb-2\,\alpha+\sigma-n\,\sigma+n^2\,\sigma\right)\right)}{-4+8\,n}\right)}{n\,\left(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2+m\,(m-n)\,\sigma-2\,\alpha\,\left(2+Sa+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)\right)}$$

$$d[j] = \frac{2\,(-1+n)\left(\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2+m\,(m-n)\,\sigma-2\,\alpha\,\left(2+Sa+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)}{2\,(-1+n)}+\frac{m\,\sigma\,\left(2\,(-Sa+\alpha+n\,(-1+Sb+\alpha))+m^3\,\sigma-2\,m^2\,n\,\sigma+m\,\left(2-2\,Sb-2\,\alpha+\sigma-n\,\sigma+n^2\,\sigma\right)\right)}{-4+8\,n}\right)}{n\,\left(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2+m\,(m-n)\,\sigma-2\,\alpha\,\left(2+Sa+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)\right)}$$

Global`p

$$p[i] = \frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2+m\,(m-n)\,\sigma-2\,\alpha\,\left(2+Sa+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)}{2\,(-1+n)} + \frac{(m-n)\,\sigma\,\left(2\,(-Sa+\alpha+n\,(-1+Sb+\alpha))+m^3\,\sigma-2\,m^2\,n\,\sigma+m\,\left(2-2\,Sb-2\,\alpha+\sigma-n\,\sigma+n^2\,\sigma\right)\right)}{-4+8\,n}$$

$$p[j] = \frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2+m\,(m-n)\,\sigma-2\,\alpha\,\left(2+Sa+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)}{2\,(-1+n)} + \frac{m\,\sigma\,\left(2\,(-Sa+\alpha+n\,(-1+Sb+\alpha))+m^3\,\sigma-2\,m^2\,n\,\sigma+m\,\left(2-2\,Sb-2\,\alpha+\sigma-n\,\sigma+n^2\,\sigma\right)\right)}{-4+8\,n}$$

Global`$\phi$

$$\phi[i] = -\text{FA} - \frac{1}{2}(-1+n)\,\sigma^2 + \frac{2(-1+n)\left(\frac{2+2\,\text{Sb}(-1+\alpha)+4\alpha^2+m(m-n)\sigma-2\alpha\left(2+\text{Sa}+\frac{1}{2}(-1+m)m\sigma\right)}{2(-1+n)}+\frac{(m-n)\sigma\left(2(-\text{Sa}+\alpha+n(-1+\text{Sb}+\alpha))+m^3\sigma-2m^2n\sigma+m\left(2-2\,\text{Sb}-2\alpha+\sigma-n\sigma+n^2\sigma\right)\right)}{-4+8n}\right)^2}{n\left(2+2\,\text{Sb}(-1+\alpha)+4\alpha^2+m(m-n)\sigma-2\alpha\left(2+\text{Sa}+\frac{1}{2}(-1+m)m\sigma\right)\right)}$$

$$\phi[j] = -\text{FB} - \frac{m\sigma^2}{2} + \frac{2(-1+n)\left(\frac{2+2\,\text{Sb}(-1+\alpha)+4\alpha^2+m(m-n)\sigma-2\alpha\left(2+\text{Sa}+\frac{1}{2}(-1+m)m\sigma\right)}{2(-1+n)}+\frac{m\sigma\left(2(-\text{Sa}+\alpha+n(-1+\text{Sb}+\alpha))+m^3\sigma-2m^2n\sigma+m\left(2-2\,\text{Sb}-2\alpha+\sigma-n\sigma+n^2\sigma\right)\right)}{-4+8n}\right)^2}{n\left(2+2\,\text{Sb}(-1+\alpha)+4\alpha^2+m(m-n)\sigma-2\alpha\left(2+\text{Sa}+\frac{1}{2}(-1+m)m\sigma\right)\right)}$$

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Case 7: B peers and A does not peer but there is interpeering

Global`d

$$d[i] = \frac{2(-1+n)\left(\frac{(m-n)\sigma\left(-2+2\,\text{Sb}+2\alpha+m^3\sigma-n\sigma-m^2n\sigma+n^2\sigma+m(-2\,\text{Sa}+2\alpha+\sigma-n\sigma)\right)}{-4+8n}+\frac{2+2\,\text{Sb}(-1+\alpha)+4\alpha^2-m\sigma+n\sigma+mn\sigma-n^2\sigma+\alpha\left(-4-2\,\text{Sa}+m\sigma+m^2\sigma-n\sigma-2mn\sigma+n^2\sigma\right)}{2(-1+n)}\right)}{n\left(2+2\,\text{Sb}(-1+\alpha)+4\alpha^2-m\sigma+n\sigma+mn\sigma-n^2\sigma+\alpha(-4-2\,\text{Sa}+m\sigma+m^2\sigma-n\sigma-2mn\sigma+n^2\sigma)\right)}$$

$$d[j] = -\big(4-4\,\text{Sb}-8\alpha-4\,\text{Sa}\,\alpha+4\,\text{Sb}\,\alpha+8\alpha^2-4m\sigma-2m^2\,\text{Sa}\,\sigma+2m\,\text{Sb}\,\sigma+4m\alpha\sigma+4m^2\alpha\sigma+m^2\sigma^2+m^4\sigma^2- n^3\sigma(-4+4\alpha+m\sigma)+n^2\sigma(6(-1+\alpha)+m^2\sigma+m^3\sigma+2m(-2+4\alpha+\sigma))-n(8+16\alpha^2-2\sigma-8m\sigma- 2m^2\,\text{Sa}\,\sigma+m\sigma^2+2m^2\sigma^2+m^3\sigma^2+m^4\sigma^2+2\,\text{Sb}(-4+4\alpha+m\sigma)+2\alpha(-8-4\,\text{Sa}+\sigma+5m\sigma+3m^2\sigma))\big)\,/ \big(2n(-1+2n)(2+2\,\text{Sb}(-1+\alpha)+4\alpha^2-m\sigma+n\sigma+mn\sigma-n^2\sigma+\alpha(-4-2\,\text{Sa}+m\sigma+m^2\sigma-n\sigma-2mn\sigma+n^2\sigma))\big)$$

Global`p

$$p[i] = \frac{(m-n)\sigma(-2+2\,\text{Sb}+2\alpha+m^3\sigma-n\sigma-m^2n\sigma+n^2\sigma+m(-2\,\text{Sa}+2\alpha+\sigma-n\sigma))}{-4+8n} + \frac{2+2\,\text{Sb}(-1+\alpha)+4\alpha^2-m\sigma+n\sigma+mn\sigma-n^2\sigma+\alpha(-4-2\,\text{Sa}+m\sigma+m^2\sigma-n\sigma-2mn\sigma+n^2\sigma)}{2(-1+n)}$$

$$p[j] = -\frac{1}{4(-1+n)(-1+2n)}\big(4-4\,\text{Sb}-8\alpha-4\,\text{Sa}\,\alpha+4\,\text{Sb}\,\alpha+8\alpha^2-4m\sigma-2m^2\,\text{Sa}\,\sigma+2m\,\text{Sb}\,\sigma+4m\alpha\sigma+4m^2\alpha\sigma+m^2\sigma^2+m^4\sigma^2- n^3\sigma(-4+4\alpha+m\sigma)+n^2\sigma(6(-1+\alpha)+m^2\sigma+m^3\sigma+2m(-2+4\alpha+\sigma))-n(8+16\alpha^2-2\sigma-8m\sigma- 2m^2\,\text{Sa}\,\sigma+m\sigma^2+2m^2\sigma^2+m^3\sigma^2+m^4\sigma^2+2\,\text{Sb}(-4+4\alpha+m\sigma)+2\alpha(-8-4\,\text{Sa}+\sigma+5m\sigma+3m^2\sigma))\big)$$

Global`$\phi$

$$\phi[i] = -\text{FA} + \frac{1}{2}(m-n)\,\sigma^2 + \frac{2(-1+n)\left(\frac{(m-n)\sigma\left(-2+2\,\text{Sb}+2\alpha+m^3\sigma-n\sigma-m^2n\sigma+n^2\sigma+m(-2\,\text{Sa}+2\alpha+\sigma-n\sigma)\right)}{-4+8n}+\frac{2+2\,\text{Sb}(-1+\alpha)+4\alpha^2-m\sigma+n\sigma+mn\sigma-n^2\sigma+\alpha\left(-4-2\,\text{Sa}+m\sigma+m^2\sigma-n\sigma-2mn\sigma+n^2\sigma\right)}{2(-1+n)}\right)^2}{n\left(2+2\,\text{Sb}(-1+\alpha)+4\alpha^2-m\sigma+n\sigma+mn\sigma-n^2\sigma+\alpha(-4-2\,\text{Sa}+m\sigma+m^2\sigma-n\sigma-2mn\sigma+n^2\sigma)\right)}$$

$$\phi[j] = \frac{1}{8}\Big(-8\,\text{FB}-4(-1+n)\sigma^2+\big(4-4\,\text{Sb}-8\alpha-4\,\text{Sa}\,\alpha+4\,\text{Sb}\,\alpha+8\alpha^2-4m\sigma-2m^2\,\text{Sa}\,\sigma+2m\,\text{Sb}\,\sigma+4m\alpha\sigma+4m^2\alpha\sigma+ m^2\sigma^2+m^4\sigma^2-n^3\sigma(-4+4\alpha+m\sigma)+n^2\sigma(6(-1+\alpha)+m^2\sigma+m^3\sigma+2m(-2+4\alpha+\sigma))- n(8+16\alpha^2-2\sigma-8m\sigma-2m^2\,\text{Sa}\,\sigma+m\sigma^2+2m^2\sigma^2+m^3\sigma^2+m^4\sigma^2+ 2\,\text{Sb}(-4+4\alpha+m\sigma)+2\alpha(-8-4\,\text{Sa}+\sigma+5m\sigma+3m^2\sigma))\big)^2\Big/\big((1-2n)^2(-1+n)n (2+2\,\text{Sb}(-1+\alpha)+4\alpha^2-m\sigma+n\sigma+mn\sigma-n^2\sigma+\alpha(-4-2\,\text{Sa}+m\sigma+m^2\sigma-n\sigma-2mn\sigma+n^2\sigma))\big)\Big)$$

-----------------------------------------------------------------

Case 8: No intrapeering but there is interpeering

Global`d

$$d[i] = \frac{2(-1+n)\left(\frac{2+2\,\text{Sb}(-1+\alpha)-2(2+\text{Sa})\alpha+4\alpha^2+m(m-n)\sigma}{2(-1+n)}+\frac{(m-n)\sigma\left(2n(-1+\text{Sb}+\alpha)+2m^3\sigma-3m^2n\sigma+m\left(2-2\,\text{Sa}-2\,\text{Sb}+n^2\sigma\right)\right)}{-4+8n}\right)}{n\left(2+2\,\text{Sb}(-1+\alpha)-2(2+\text{Sa})\alpha+4\alpha^2+m(m-n)\sigma\right)}$$

$$d[j] = \frac{2(-1+n)\left(\frac{2+2\,\text{Sb}(-1+\alpha)-2(2+\text{Sa})\alpha+4\alpha^2+m(m-n)\sigma}{2(-1+n)}+\frac{m\sigma\left(2n(-1+\text{Sb}+\alpha)+2m^3\sigma-3m^2n\sigma+m\left(2-2\,\text{Sa}-2\,\text{Sb}+n^2\sigma\right)\right)}{-4+8n}\right)}{n\left(2+2\,\text{Sb}(-1+\alpha)-2(2+\text{Sa})\alpha+4\alpha^2+m(m-n)\sigma\right)}$$

Global`p

$$p[i] = \frac{2+2\,\mathrm{Sb}\,(-1+\alpha)-2\,(2+\mathrm{Sa})\,\alpha+4\,\alpha^2+m\,(m-n)\,\sigma}{2\,(-1+n)} + \frac{(m-n)\,\sigma\,(2\,n\,(-1+\mathrm{Sb}+\alpha)+2\,m^3\,\sigma-3\,m^2\,n\,\sigma+m\,(2-2\,\mathrm{Sa}-2\,\mathrm{Sb}+n^2\,\sigma))}{-4+8\,n}$$

$$p[j] = \frac{2+2\,\mathrm{Sb}\,(-1+\alpha)-2\,(2+\mathrm{Sa})\,\alpha+4\,\alpha^2+m\,(m-n)\,\sigma}{2\,(-1+n)} + \frac{m\,\sigma\,(2\,n\,(-1+\mathrm{Sb}+\alpha)+2\,m^3\,\sigma-3\,m^2\,n\,\sigma+m\,(2-2\,\mathrm{Sa}-2\,\mathrm{Sb}+n^2\,\sigma))}{-4+8\,n}$$

Global`$\phi$

$$\phi[i] = -\mathrm{FA}+\frac{1}{2}\,(m-n)\,\sigma^2+\frac{2\,(-1+n)\left(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)-2\,(2+\mathrm{Sa})\,\alpha+4\,\alpha^2+m\,(m-n)\,\sigma}{2\,(-1+n)}+\frac{(m-n)\,\sigma\,\left(2\,n\,(-1+\mathrm{Sb}+\alpha)+2\,m^3\,\sigma-3\,m^2\,n\,\sigma+m\,\left(2-2\,\mathrm{Sa}-2\,\mathrm{Sb}+n^2\,\sigma\right)\right)}{-4+8\,n}\right)^2}{n\,(2+2\,\mathrm{Sb}\,(-1+\alpha)-2\,(2+\mathrm{Sa})\,\alpha+4\,\alpha^2+m\,(m-n)\,\sigma)}$$

$$\phi[j] = -\mathrm{FB}-\frac{m\,\sigma^2}{2}+\frac{2\,(-1+n)\left(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)-2\,(2+\mathrm{Sa})\,\alpha+4\,\alpha^2+m\,(m-n)\,\sigma}{2\,(-1+n)}+\frac{m\,\sigma\,\left(2\,n\,(-1+\mathrm{Sb}+\alpha)+2\,m^3\,\sigma-3\,m^2\,n\,\sigma+m\,\left(2-2\,\mathrm{Sa}-2\,\mathrm{Sb}+n^2\,\sigma\right)\right)}{-4+8\,n}\right)^2}{n\,(2+2\,\mathrm{Sb}\,(-1+\alpha)-2\,(2+\mathrm{Sa})\,\alpha+4\,\alpha^2+m\,(m-n)\,\sigma)}$$

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**?profit**

Global`profit

$$\mathrm{profit}[\mathrm{Ag000}] = -\mathrm{FA}+\frac{1+\mathrm{Sb}\,(-1+\alpha)-(2+\mathrm{Sa})\,\alpha+2\,\alpha^2}{(-1+n)\,n}$$

$$\mathrm{profit}[\mathrm{Ag001}] = -\mathrm{FA}+\frac{1}{2}\,(m-n)\,\sigma^2+\frac{2\,(-1+n)\left(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)-2\,(2+\mathrm{Sa})\,\alpha+4\,\alpha^2+m\,(m-n)\,\sigma}{2\,(-1+n)}+\frac{(m-n)\,\sigma\,\left(2\,n\,(-1+\mathrm{Sb}+\alpha)+2\,m^3\,\sigma-3\,m^2\,n\,\sigma+m\,\left(2-2\,\mathrm{Sa}-2\,\mathrm{Sb}+n^2\,\sigma\right)\right)}{-4+8\,n}\right)^2}{n\,(2+2\,\mathrm{Sb}\,(-1+\alpha)-2\,(2+\mathrm{Sa})\,\alpha+4\,\alpha^2+m\,(m-n)\,\sigma)}$$

$$\mathrm{profit}[\mathrm{Ag0B0}] = -\mathrm{FA}+\frac{2\,(-1+n)\left(\frac{(-1-m+n)\,(-m+n)\,\sigma\,(2-2\,\mathrm{Sb}-2\,\alpha-(m-n)\,(1+m-n)\,\sigma)}{-4+8\,n}-\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-(m-n)\,(1+m-n)\,\sigma-2\,\alpha\,\left(2+\mathrm{Sa}-\frac{1}{2}\,(m-n)\,(1+m-n)\,\sigma\right)}{2\,(-1+n)}\right)^2}{n\,\left(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-(m-n)\,(1+m-n)\,\sigma-2\,\alpha\,\left(2+\mathrm{Sa}-\frac{1}{2}\,(m-n)\,(1+m-n)\,\sigma\right)\right)}$$

$$\mathrm{profit}[\mathrm{Ag0B1}] = -\mathrm{FA}+\frac{1}{2}\,(m-n)\,\sigma^2+\frac{2\,(-1+n)\left(\frac{(m-n)\,\sigma\,\left(-2+2\,\mathrm{Sb}+2\,\alpha+m^3\,\sigma-n\,\sigma-m^2\,n\,\sigma+n^2\,\sigma+m\,(-2\,\mathrm{Sa}+2\,\alpha+\sigma-n\,\sigma)\right)}{-4+8\,n}+\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-m\,\sigma+n\,\sigma+m\,n\,\sigma-n^2\,\sigma+\alpha\,\left(-4-2\,\mathrm{Sa}+m\,\sigma+m^2\,\sigma-n\,\sigma-2\,m\,n\,\sigma+n^2\,\sigma\right)}{2\,(-1+n)}\right)^2}{n\,(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-m\,\sigma+n\,\sigma+m\,n\,\sigma-n^2\,\sigma+\alpha\,(-4-2\,\mathrm{Sa}+m\,\sigma+m^2\,\sigma-n\,\sigma-2\,m\,n\,\sigma+n^2\,\sigma))}$$

$$\mathrm{profit}[\mathrm{AgA00}] = -\mathrm{FA}-\frac{1}{2}\,(-1+m)\,\sigma^2+\frac{2\,(-1+n)\left(\frac{(-1+m)\,(m-n)\,\sigma\,(2\,\mathrm{Sa}-2\,\alpha+(-1+m)\,m\,\sigma)}{-4+8\,n}+\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-2\,\alpha\,\left(2+\mathrm{Sa}+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)}{2\,(-1+n)}\right)^2}{n\,\left(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-2\,\alpha\,\left(2+\mathrm{Sa}+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)\right)}$$

$$\mathrm{profit}[\mathrm{AgA01}] = -\mathrm{FA}-\frac{1}{2}\,(-1+n)\,\sigma^2+\frac{2\,(-1+n)\left(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2+m\,(m-n)\,\sigma-2\,\alpha\,\left(2+\mathrm{Sa}+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)}{2\,(-1+n)}+\frac{(m-n)\,\sigma\,\left(2\,(-\mathrm{Sa}+\alpha+n\,(-1+\mathrm{Sb}+\alpha))+m^3\,\sigma-2\,m^2\,n\,\sigma+m\,\left(2-2\,\mathrm{Sb}-2\,\alpha+\sigma-n\,\sigma+n^2\,\sigma\right)\right)}{-4+8\,n}\right)^2}{n\,\left(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2+m\,(m-n)\,\sigma-2\,\alpha\,\left(2+\mathrm{Sa}+\frac{1}{2}\,(-1+m)\,m\,\sigma\right)\right)}$$

$$\mathrm{profit}[\mathrm{AgAB0}] = -\mathrm{FA}-\frac{1}{2}\,(-1+m)\,\sigma^2+\left(2\,(-1+n)\,\left(-\frac{-2-2\,\mathrm{Sb}\,(-1+\alpha)-4\,\alpha^2+m\,\sigma+m^2\,\sigma-n\,\sigma-2\,m\,n\,\sigma+n^2\,\sigma+\alpha\,(4+2\,\mathrm{Sa}+2\,m\,(-1+n)\,\sigma+n\,\sigma-n^2\,\sigma)}{2\,(-1+n)}+\frac{(m-n)\,\sigma\,(-2-2\,\mathrm{Sa}+2\,\mathrm{Sb}+4\,\alpha+2\,m^3\,\sigma-3\,m^2\,n\,\sigma+2\,n^2\,\sigma-n^3\,\sigma-n\,(-2+2\,\mathrm{Sb}+2\,\alpha+\sigma)+m\,(-2+2\,\mathrm{Sa}+2\,\mathrm{Sb}+2\,\sigma-4\,n\,\sigma+3\,n^2\,\sigma))}{-4+8\,n}\right)^2\right)\Big/ (n\,(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-(m-n)\,(1+m-n)\,\sigma-\alpha\,(4+2\,\mathrm{Sa}+2\,m\,(-1+n)\,\sigma+n\,\sigma-n^2\,\sigma)))$$

$$\mathrm{profit}[\mathrm{AgAB1}] = \frac{1}{8}\,\Big(-8\,\mathrm{FA}-4\,(-1+n)\,\sigma^2+(n^4\,\sigma^2-n^3\,\sigma\,(-4+4\,\alpha+(2+3\,m)\,\sigma)+n^2\,\sigma\,(-8-2\,\mathrm{Sa}+2\,\mathrm{Sb}+10\,\alpha+\sigma+2\,m^2\,\sigma+m\,(-4+8\,\alpha+6\,\sigma))+2\,(2+4\,\alpha^2-2\,m\,\sigma-m\,\mathrm{Sa}\,\sigma+m^2\,\sigma^2-2\,\alpha\,(2+\mathrm{Sa}-2\,m\,\sigma)+\mathrm{Sb}\,(-2+2\,\alpha+m\,\sigma))-n\,(8+16\,\alpha^2-4\,\sigma-8\,m\,\sigma-2\,\mathrm{Sa}\,\sigma-2\,m\,\mathrm{Sa}\,\sigma+3\,m\,\sigma^2+4\,m^2\,\sigma^2-2\,\alpha\,(8+4\,\mathrm{Sa}-3\,\sigma-8\,m\,\sigma)+2\,\mathrm{Sb}\,(-4+4\,\alpha+\sigma+m\,\sigma)))^2\Big/((1-2\,n)^2\,(-1+n)\,n\,(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-m\,\sigma+n\,\sigma+m\,n\,\sigma-n^2\,\sigma-\alpha\,(4+2\,\mathrm{Sa}+2\,m\,(-1+n)\,\sigma+n\,\sigma-n^2\,\sigma)))\Big)$$

$$\mathrm{profit}[\mathrm{Bg000}] = -\mathrm{FB}+\frac{1+\mathrm{Sb}\,(-1+\alpha)-(2+\mathrm{Sa})\,\alpha+2\,\alpha^2}{(-1+n)\,n}$$

$$\mathrm{profit}[\mathrm{Bg001}] = -\mathrm{FB}-\frac{m\,\sigma^2}{2}+\frac{2\,(-1+n)\left(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)-2\,(2+\mathrm{Sa})\,\alpha+4\,\alpha^2+m\,(m-n)\,\sigma}{2\,(-1+n)}+\frac{m\,\sigma\,\left(2\,n\,(-1+\mathrm{Sb}+\alpha)+2\,m^3\,\sigma-3\,m^2\,n\,\sigma+m\,\left(2-2\,\mathrm{Sa}-2\,\mathrm{Sb}+n^2\,\sigma\right)\right)}{-4+8\,n}\right)^2}{n\,(2+2\,\mathrm{Sb}\,(-1+\alpha)-2\,(2+\mathrm{Sa})\,\alpha+4\,\alpha^2+m\,(m-n)\,\sigma)}$$

$$\text{profit}[\text{Bg0B0}] = -FB + \frac{1}{2}(1+m-n)\,\sigma^2 + \frac{2(-1+n)\left(\frac{m(1+m-n)\sigma\left(-2+2\,Sb+2\alpha+m\sigma+m^2\sigma-n\sigma-2mn\sigma+n^2\sigma\right)}{-4+8n}+\frac{2+2\,Sb(-1+\alpha)+4\alpha^2-(m-n)(1+m-n)\sigma-2\alpha\left(2+Sa-\frac{1}{2}(m-n)(1+m-n)\sigma\right)}{2(-1+n)}\right)^2}{n\left(2+2\,Sb(-1+\alpha)+4\alpha^2-(m-n)(1+m-n)\sigma-2\alpha\left(2+Sa-\frac{1}{2}(m-n)(1+m-n)\sigma\right)\right)}$$

$$\begin{aligned}\text{profit}[\text{Bg0B1}] = {} & \frac{1}{8}\Big(-8\,FB-4(-1+n)\sigma^2+(4-4\,Sb-8\alpha-4\,Sa\,\alpha+4\,Sb\,\alpha+8\alpha^2-4m\sigma-2m^2\,Sa\,\sigma+2m\,Sb\,\sigma+4m\alpha\sigma+4m^2\alpha\sigma+ \\ & m^2\sigma^2+m^4\sigma^2-n^3\sigma(-4+4\alpha+m\sigma)+n^2\sigma(6(-1+\alpha)+m^2\sigma+m^3\sigma+2m(-2+4\alpha+\sigma))- \\ & n(8+16\alpha^2-2\sigma-8m\sigma-2m^2\,Sa\,\sigma+m\sigma^2+2m^2\sigma^2+m^3\sigma^2+m^4\sigma^2+ \\ & 2\,Sb(-4+4\alpha+m\sigma)+2\alpha(-8-4\,Sa+\sigma+5m\sigma+3m^2\sigma)))^2\Big/\left((1-2n)^2(-1+n)\,n\right. \\ & \left.(2+2\,Sb(-1+\alpha)+4\alpha^2-m\sigma+n\sigma+mn\sigma-n^2\sigma+\alpha(-4-2\,Sa+m\sigma+m^2\sigma-n\sigma-2mn\sigma+n^2\sigma)))\right)\Big)\end{aligned}$$

$$\text{profit}[\text{BgA00}] = -FB + \frac{2(-1+n)\left(\frac{(-1+m)m\sigma(2\,Sa-2\alpha+(-1+m)m\sigma)}{-4+8n}+\frac{2+2\,Sb(-1+\alpha)+4\alpha^2-2\alpha\left(2+Sa+\frac{1}{2}(-1+m)m\sigma\right)}{2(-1+n)}\right)^2}{n\left(2+2\,Sb(-1+\alpha)+4\alpha^2-2\alpha\left(2+Sa+\frac{1}{2}(-1+m)m\sigma\right)\right)}$$

$$\begin{aligned}\text{profit}[\text{BgA01}] = {} & \\ & -FB-\frac{m\sigma^2}{2}+\frac{2(-1+n)\left(\frac{2+2\,Sb(-1+\alpha)+4\alpha^2+m(m-n)\sigma-2\alpha\left(2+Sa+\frac{1}{2}(-1+m)m\sigma\right)}{2(-1+n)}+\frac{m\sigma\left(2(-Sa+\alpha+n(-1+Sb+\alpha))+m^3\sigma-2m^2n\sigma+m\left(2-2\,Sb-2\alpha+\sigma-n\sigma+n^2\sigma\right)\right)}{-4+8n}\right)^2}{n\left(2+2\,Sb(-1+\alpha)+4\alpha^2+m(m-n)\sigma-2\alpha\left(2+Sa+\frac{1}{2}(-1+m)m\sigma\right)\right)}\end{aligned}$$

$$\begin{aligned}\text{profit}[\text{BgAB0}] = {} & \\ & -FB+\frac{1}{2}(1+m-n)\sigma^2+\Bigg(2(-1+n)\Bigg(-\frac{-2-2\,Sb(-1+\alpha)-4\alpha^2+m\sigma+m^2\sigma-n\sigma-2mn\sigma+n^2\sigma+\alpha(4+2\,Sa+2m(-1+n)\sigma+n\sigma-n^2\sigma)}{2(-1+n)}+ \\ & \frac{m\sigma(-2-2\,Sa+2\,Sb+4\alpha+2m^3\sigma-3m^2n\sigma+2n^2\sigma-n^3\sigma-n(-2+2\,Sb+2\alpha+\sigma)+m(-2+2\,Sa+2\,Sb+2\sigma-4n\sigma+3n^2\sigma))}{-4+8n}\Bigg)^2\Bigg)\Bigg/ \\ & (n(2+2\,Sb(-1+\alpha)+4\alpha^2-(m-n)(1+m-n)\sigma-\alpha(4+2\,Sa+2m(-1+n)\sigma+n\sigma-n^2\sigma)))\end{aligned}$$

$$\begin{aligned}\text{profit}[\text{BgAB1}] = {} & \\ & \frac{1}{8}\Big(-8\,FB-4(-1+n)\sigma^2+(n^3\sigma(-4+4\alpha+m\sigma)-2n^2\sigma(3(-1+\alpha)+m^2\sigma+m(-2+4\alpha+\sigma))-2(2+ \\ & 4\alpha^2-2m\sigma-m\,Sa\,\sigma+m^2\sigma^2-2\alpha(2+Sa-2m\sigma)+Sb(-2+2\alpha+m\sigma))+ \\ & n(8+16\alpha^2-2\sigma-8m\sigma-2m\,Sa\,\sigma+m\sigma^2+4m^2\sigma^2-2\alpha(8+4\,Sa-\sigma-8m\sigma)+2\,Sb(-4+4\alpha+m\sigma)))^2\Big/ \\ & ((1-2n)^2(-1+n)\,n\,(2+2\,Sb(-1+\alpha)+4\alpha^2-m\sigma+n\sigma+mn\sigma-n^2\sigma- \\ & \alpha(4+2\,Sa+2m(-1+n)\sigma+n\sigma-n^2\sigma)))\Big)\end{aligned}$$

## "PAIRWISE STABILITY"

**"i belongs to backbone A and j belongs to backbone B"**

**Clear[ηiA, ηiB, ηjA, ηjB, ηAA, ηBB, ηAB, ϕ, ν]**

$$\begin{aligned}\phi[i] = {} & -FA-\frac{1}{2}(niA+niB)\sigma^2+ \\ & \Bigg(2(-1+n)\Bigg(\frac{2+2\,Sb(-1+\alpha)+4\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\alpha(2+Sa+nAA\,\sigma-nBB\,\sigma)}{2(-1+n)}+ \\ & \frac{1}{-4+8n}(\sigma(2\,nAB^2\sigma+nAB(-2+2\,Sa+2\,Sb+4\,nAA\,\sigma+4\,nBB\,\sigma-n\,niA\,\sigma-n\,niB\,\sigma)-2 \\ & (-2\,nBB^2\sigma-2\,nAA(Sa-\alpha+nAA\,\sigma)+nBB(2-2\,Sb-2\alpha+n\,niB\,\sigma)+ \\ & n(niB(-1+Sb+\alpha)+niA(Sa-\alpha+nAA\,\sigma)))))\Bigg)^2\Bigg)\Bigg/ \\ & (n(2+2\,Sb(-1+\alpha)+4\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\alpha(2+Sa+nAA\,\sigma-nBB\,\sigma)));\end{aligned}$$

$$\begin{aligned}\phi[j] = {} & -FB-\frac{1}{2}(njA+njB)\sigma^2+ \\ & \Bigg(2(-1+n)\Bigg(\frac{2+2\,Sb(-1+\alpha)+4\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\alpha(2+Sa+nAA\,\sigma-nBB\,\sigma)}{2(-1+n)}+\end{aligned}$$

```
      1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) - 2
          (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
            n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))^2) /
   (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)));
"-----------------------------------------------------------------------------"
"Intra-backbone link formation in backbone A"
Clear[ηiA, ηiB, ηjA, ηjB, ηAA, ηBB, ηAB, ϕ, ν]
ϕ[i] = -FA - 1/2 (niA + niB) σ^2 +
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
      1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) - 2
          (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
            n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ)))))^2) /
   (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)));
ϕ[j] = -FB - 1/2 (njA + njB) σ^2 +
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
      1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) - 2
          (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
            n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))^2) /
   (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)));
ηiA = niA + 1;
ηiB = niB;
ηjA = njA;
ηjB = njB;
ηAA = nAA + 1;
ηBB = nBB;
ηAB = nAB;
ϕ[i '] = -FA - 1/2 (ηiA + ηiB) σ^2 +
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ))/(2 (-1 + n)) +
      1/(-4 + 8 n) (σ (2 ηAB^2 σ + ηAB (-2 + 2 Sa + 2 Sb + 4 ηAA σ + 4 ηBB σ - n ηiA σ - n ηiB σ) - 2
          (-2 ηBB^2 σ - 2 ηAA (Sa - α + ηAA σ) + ηBB (2 - 2 Sb - 2 α + n ηiB σ) +
            n (ηiB (-1 + Sb + α) + ηiA (Sa - α + ηAA σ)))))^2) /
   (n (2 + 2 Sb (-1 + α) + 4 α^2 - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ)));
ϕ[j '] = -FB - 1/2 (ηjA + ηjB) σ^2 +
```

$$\left(2\ (-1+n)\ \left(\frac{2+2\ \mathbf{Sb}\ (-1+\alpha)+4\ \alpha^2-\eta\mathbf{AB}\ \sigma-2\ \eta\mathbf{BB}\ \sigma-2\ \alpha\ (2+\mathbf{Sa}+\eta\mathbf{AA}\ \sigma-\eta\mathbf{BB}\ \sigma)}{2\ (-1+n)}+\right.\right.$$
$$\frac{1}{-4+8\ n}\ (\sigma\ (2\ \eta\mathbf{AB}^2\ \sigma+\eta\mathbf{AB}\ (-2+2\ \mathbf{Sa}+2\ \mathbf{Sb}+4\ \eta\mathbf{AA}\ \sigma+4\ \eta\mathbf{BB}\ \sigma-n\ \eta\mathbf{jA}\ \sigma-n\ \eta\mathbf{jB}\ \sigma)-2$$
$$(-2\ \eta\mathbf{BB}^2\ \sigma-2\ \eta\mathbf{AA}\ (\mathbf{Sa}-\alpha+\eta\mathbf{AA}\ \sigma)+\eta\mathbf{BB}\ (2-2\ \mathbf{Sb}-2\ \alpha+n\ \eta\mathbf{jB}\ \sigma)+$$
$$\left.\left.n\ (\eta\mathbf{jB}\ (-1+\mathbf{Sb}+\alpha)+\eta\mathbf{jA}\ (\mathbf{Sa}-\alpha+\eta\mathbf{AA}\ \sigma))))\right)^2\right)\Big/$$
$$(n\ (2+2\ \mathbf{Sb}\ (-1+\alpha)+4\ \alpha^2-\eta\mathbf{AB}\ \sigma-2\ \eta\mathbf{BB}\ \sigma-2\ \alpha\ (2+\mathbf{Sa}+\eta\mathbf{AA}\ \sigma-\eta\mathbf{BB}\ \sigma)));$$

```
ν[i] = ϕ[i'] - ϕ[i];
ν[j] = ϕ[j'] - ϕ[j];
ΔprofitintraAform[i] = Simplify[ν[i]];
ΔprofitintraAform[j] = Simplify[ν[j]];
? ΔprofitintraAform
Clear[ηiA, ηiB, ηjA, ηjB, ηAA, ηBB, ηAB, ϕ, ν]
"-----------------------------------------------------------------"
"Intra-backbone link deletion in backbone A"
Clear[ηiA, ηiB, ηjA, ηjB, ηAA, ηBB, ηAB, ϕ, ν]
```

$$\phi[\mathbf{i}]=-\mathbf{FA}-\frac{1}{2}\ (\mathbf{niA}+\mathbf{niB})\ \sigma^2+$$
$$\left(2\ (-1+n)\ \left(\frac{2+2\ \mathbf{Sb}\ (-1+\alpha)+4\ \alpha^2-\mathbf{nAB}\ \sigma-2\ \mathbf{nBB}\ \sigma-2\ \alpha\ (2+\mathbf{Sa}+\mathbf{nAA}\ \sigma-\mathbf{nBB}\ \sigma)}{2\ (-1+n)}+\right.\right.$$
$$\frac{1}{-4+8\ n}\ (\sigma\ (2\ \mathbf{nAB}^2\ \sigma+\mathbf{nAB}\ (-2+2\ \mathbf{Sa}+2\ \mathbf{Sb}+4\ \mathbf{nAA}\ \sigma+4\ \mathbf{nBB}\ \sigma-n\ \mathbf{niA}\ \sigma-n\ \mathbf{niB}\ \sigma)-2$$
$$(-2\ \mathbf{nBB}^2\ \sigma-2\ \mathbf{nAA}\ (\mathbf{Sa}-\alpha+\mathbf{nAA}\ \sigma)+\mathbf{nBB}\ (2-2\ \mathbf{Sb}-2\ \alpha+n\ \mathbf{niB}\ \sigma)+$$
$$\left.\left.n\ (\mathbf{niB}\ (-1+\mathbf{Sb}+\alpha)+\mathbf{niA}\ (\mathbf{Sa}-\alpha+\mathbf{nAA}\ \sigma))))\right)^2\right)\Big/$$
$$(n\ (2+2\ \mathbf{Sb}\ (-1+\alpha)+4\ \alpha^2-\mathbf{nAB}\ \sigma-2\ \mathbf{nBB}\ \sigma-2\ \alpha\ (2+\mathbf{Sa}+\mathbf{nAA}\ \sigma-\mathbf{nBB}\ \sigma)));$$

$$\phi[\mathbf{j}]=-\mathbf{FB}-\frac{1}{2}\ (\mathbf{njA}+\mathbf{njB})\ \sigma^2+$$
$$\left(2\ (-1+n)\ \left(\frac{2+2\ \mathbf{Sb}\ (-1+\alpha)+4\ \alpha^2-\mathbf{nAB}\ \sigma-2\ \mathbf{nBB}\ \sigma-2\ \alpha\ (2+\mathbf{Sa}+\mathbf{nAA}\ \sigma-\mathbf{nBB}\ \sigma)}{2\ (-1+n)}+\right.\right.$$
$$\frac{1}{-4+8\ n}\ (\sigma\ (2\ \mathbf{nAB}^2\ \sigma+\mathbf{nAB}\ (-2+2\ \mathbf{Sa}+2\ \mathbf{Sb}+4\ \mathbf{nAA}\ \sigma+4\ \mathbf{nBB}\ \sigma-n\ \mathbf{njA}\ \sigma-n\ \mathbf{njB}\ \sigma)-2$$
$$(-2\ \mathbf{nBB}^2\ \sigma-2\ \mathbf{nAA}\ (\mathbf{Sa}-\alpha+\mathbf{nAA}\ \sigma)+\mathbf{nBB}\ (2-2\ \mathbf{Sb}-2\ \alpha+n\ \mathbf{njB}\ \sigma)+$$
$$\left.\left.n\ (\mathbf{njB}\ (-1+\mathbf{Sb}+\alpha)+\mathbf{njA}\ (\mathbf{Sa}-\alpha+\mathbf{nAA}\ \sigma))))\right)^2\right)\Big/$$
$$(n\ (2+2\ \mathbf{Sb}\ (-1+\alpha)+4\ \alpha^2-\mathbf{nAB}\ \sigma-2\ \mathbf{nBB}\ \sigma-2\ \alpha\ (2+\mathbf{Sa}+\mathbf{nAA}\ \sigma-\mathbf{nBB}\ \sigma)));$$

```
ηiA = niA - 1;
ηiB = niB;
ηjA = njA;
ηjB = njB;
ηAA = nAA - 1;
ηBB = nBB;
ηAB = nAB;
```

$$\phi[\mathbf{i'}]=-\mathbf{FA}-\frac{1}{2}\ (\eta\mathbf{iA}+\eta\mathbf{iB})\ \sigma^2+$$
$$\left(2\ (-1+n)\ \left(\frac{2+2\ \mathbf{Sb}\ (-1+\alpha)+4\ \alpha^2-\eta\mathbf{AB}\ \sigma-2\ \eta\mathbf{BB}\ \sigma-2\ \alpha\ (2+\mathbf{Sa}+\eta\mathbf{AA}\ \sigma-\eta\mathbf{BB}\ \sigma)}{2\ (-1+n)}+\right.\right.$$

```
        1/(-4 + 8 n) (σ (2 ηAB^2 σ + ηAB (-2 + 2 Sa + 2 Sb + 4 ηAA σ + 4 ηBB σ - n ηiA σ - n ηiB σ) - 2
              (-2 ηBB^2 σ - 2 ηAA (Sa - α + ηAA σ) + ηBB (2 - 2 Sb - 2 α + n ηiB σ) +
                 n (ηiB (-1 + Sb + α) + ηiA (Sa - α + ηAA σ))))))^2) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ)));
ϕ[j'] = -FB - 1/2 (ηjA + ηjB) σ^2 +
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ))/(2 (-1 + n)) +
        1/(-4 + 8 n) (σ (2 ηAB^2 σ + ηAB (-2 + 2 Sa + 2 Sb + 4 ηAA σ + 4 ηBB σ - n ηjA σ - n ηjB σ) - 2
              (-2 ηBB^2 σ - 2 ηAA (Sa - α + ηAA σ) + ηBB (2 - 2 Sb - 2 α + n ηjB σ) +
                 n (ηjB (-1 + Sb + α) + ηjA (Sa - α + ηAA σ))))))^2) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ)));
ν[i] = ϕ[i'] - ϕ[i];
ν[j] = ϕ[j'] - ϕ[j];
ΔprofitintraAdel[i] = Simplify[ν[i]];
ΔprofitintraAdel[j] = Simplify[ν[j]];
? ΔprofitintraAdel
Clear[ηiA, ηiB, ηjA, ηjB, ηAA, ηBB, ηAB, ϕ, ν]
"-----------------------------------------------------------------"
"Intra-backbone link formation in backbone B"
Clear[ηiA, ηiB, ηjA, ηjB, ηAA, ηBB, ηAB, ϕ, ν]
ϕ[i] = -FA - 1/2 (niA + niB) σ^2 +
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
        1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) - 2
              (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
                 n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ))))))^2) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)));
ϕ[j] = -FB - 1/2 (njA + njB) σ^2 +
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
        1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) - 2
              (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
                 n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ))))))^2) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)));
ηiA = niA;
ηiB = niB;
ηjA = njA;
```

```
ηjB = njB + 1;
ηAA = nAA;
ηBB = nBB + 1;
ηAB = nAB;
ϕ[i'] = -FA - 1/2 (ηiA + ηiB) σ^2 +
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ))/(2 (-1 + n)) +
          1/(-4 + 8 n) (σ (2 ηAB^2 σ + ηAB (-2 + 2 Sa + 2 Sb + 4 ηAA σ + 4 ηBB σ - n ηiA σ - n ηiB σ) - 2
                (-2 ηBB^2 σ - 2 ηAA (Sa - α + ηAA σ) + ηBB (2 - 2 Sb - 2 α + n ηiB σ) +
                   n (ηiB (-1 + Sb + α) + ηiA (Sa - α + ηAA σ)))))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ)));
ϕ[j'] = -FB - 1/2 (ηjA + ηjB) σ^2 +
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ))/(2 (-1 + n)) +
          1/(-4 + 8 n) (σ (2 ηAB^2 σ + ηAB (-2 + 2 Sa + 2 Sb + 4 ηAA σ + 4 ηBB σ - n ηjA σ - n ηjB σ) - 2
                (-2 ηBB^2 σ - 2 ηAA (Sa - α + ηAA σ) + ηBB (2 - 2 Sb - 2 α + n ηjB σ) +
                   n (ηjB (-1 + Sb + α) + ηjA (Sa - α + ηAA σ)))))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ)));
ν[i] = ϕ[i'] - ϕ[i];
ν[j] = ϕ[j'] - ϕ[j];
ΔprofitintraBform[i] = Simplify[ν[i]];
ΔprofitintraBform[j] = Simplify[ν[j]];
? ΔprofitintraBform
Clear[ηiA, ηiB, ηjA, ηjB, ηAA, ηBB, ηAB, ϕ, ν]
"----------------------------------------------------------------------"
"Intra-backbone link deletion in backbone B"
Clear[ηiA, ηiB, ηjA, ηjB, ηAA, ηBB, ηAB, ϕ, ν]
ϕ[i] = -FA - 1/2 (niA + niB) σ^2 +
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
          1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) - 2
                (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
                   n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ)))))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)));
ϕ[j] = -FB - 1/2 (njA + njB) σ^2 +
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
```

```
          1
      -------- (σ (2 nAB² σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) - 2
      -4 + 8 n
            (-2 nBB² σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
                                                                              2
              n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))) ) ) /
   (n (2 + 2 Sb (-1 + α) + 4 α² - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)));
ηiA = niA;
ηiB = niB;
ηjA = njA;
ηjB = njB - 1;
ηAA = nAA;
ηBB = nBB - 1;
ηAB = nAB;
                 1
ϕ[i'] = -FA - --- (ηiA + ηiB) σ² +
                 2
   (2 (-1 + n) ( 2 + 2 Sb (-1 + α) + 4 α² - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ)
                ------------------------------------------------------------------------- +
                                              2 (-1 + n)
          1
      -------- (σ (2 ηAB² σ + ηAB (-2 + 2 Sa + 2 Sb + 4 ηAA σ + 4 ηBB σ - n ηiA σ - n ηiB σ) - 2
      -4 + 8 n
            (-2 ηBB² σ - 2 ηAA (Sa - α + ηAA σ) + ηBB (2 - 2 Sb - 2 α + n ηiB σ) +
                                                                              2
              n (ηiB (-1 + Sb + α) + ηiA (Sa - α + ηAA σ)))))) ) ) /
   (n (2 + 2 Sb (-1 + α) + 4 α² - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ)));
                 1
ϕ[j'] = -FB - --- (ηjA + ηjB) σ² +
                 2
   (2 (-1 + n) ( 2 + 2 Sb (-1 + α) + 4 α² - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ)
                ------------------------------------------------------------------------- +
                                              2 (-1 + n)
          1
      -------- (σ (2 ηAB² σ + ηAB (-2 + 2 Sa + 2 Sb + 4 ηAA σ + 4 ηBB σ - n ηjA σ - n ηjB σ) - 2
      -4 + 8 n
            (-2 ηBB² σ - 2 ηAA (Sa - α + ηAA σ) + ηBB (2 - 2 Sb - 2 α + n ηjB σ) +
                                                                              2
              n (ηjB (-1 + Sb + α) + ηjA (Sa - α + ηAA σ)))))) ) ) /
   (n (2 + 2 Sb (-1 + α) + 4 α² - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ)));
ν[i] = ϕ[i'] - ϕ[i];
ν[j] = ϕ[j'] - ϕ[j];
ΔprofitintraBdel[i] = Simplify[ν[i]];
ΔprofitintraBdel[j] = Simplify[ν[j]];
? ΔprofitintraBdel
Clear[ηiA, ηiB, ηjA, ηjB, ηAA, ηBB, ηAB, ϕ, ν]
"-----------------------------------------------------------------"
"Inter-backbone link formation"
Clear[ηiA, ηiB, ηjA, ηjB, ηAA, ηBB, ηAB, ϕ, ν]
                1
ϕ[i] = -FA - --- (niA + niB) σ² +
                2
   (2 (-1 + n) ( 2 + 2 Sb (-1 + α) + 4 α² - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)
                ------------------------------------------------------------------------- +
                                              2 (-1 + n)
          1
      -------- (σ (2 nAB² σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) - 2
      -4 + 8 n
```

```
        (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
           n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ))))))^2) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)));
ϕ[j] = -FB - 1/2 (njA + njB) σ^2 +
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)) / (2 (-1 + n)) +
         1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) - 2
        (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
           n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ))))))^2) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)));
ηiA = niA;
ηiB = niB + 1;
ηjA = njA + 1;
ηjB = njB;
ηAA = nAA;
ηBB = nBB;
ηAB = nAB + 1;
ϕ[i '] = -FA - 1/2 (ηiA + ηiB) σ^2 +
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ)) / (2 (-1 + n)) +
         1/(-4 + 8 n) (σ (2 ηAB^2 σ + ηAB (-2 + 2 Sa + 2 Sb + 4 ηAA σ + 4 ηBB σ - n ηiA σ - n ηiB σ) - 2
        (-2 ηBB^2 σ - 2 ηAA (Sa - α + ηAA σ) + ηBB (2 - 2 Sb - 2 α + n ηiB σ) +
           n (ηiB (-1 + Sb + α) + ηiA (Sa - α + ηAA σ))))))^2) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ)));
ϕ[j '] = -FB - 1/2 (ηjA + ηjB) σ^2 +
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ)) / (2 (-1 + n)) +
         1/(-4 + 8 n) (σ (2 ηAB^2 σ + ηAB (-2 + 2 Sa + 2 Sb + 4 ηAA σ + 4 ηBB σ - n ηjA σ - n ηjB σ) - 2
        (-2 ηBB^2 σ - 2 ηAA (Sa - α + ηAA σ) + ηBB (2 - 2 Sb - 2 α + n ηjB σ) +
           n (ηjB (-1 + Sb + α) + ηjA (Sa - α + ηAA σ))))))^2) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ)));
ν[i] = ϕ[i '] - ϕ[i];
ν[j] = ϕ[j '] - ϕ[j];
Δprofitinterform[i] = Simplify[ν[i]];
Δprofitinterform[j] = Simplify[ν[j]];
? Δprofitinterform
Clear[ηiA, ηiB, ηjA, ηjB, ηAA, ηBB, ηAB, ϕ, ν]
```

```
"-----------------------------------------------------------------------------"
"Inter-backbone link deletion"
Clear[ηiA, ηiB, ηjA, ηjB, ηAA, ηBB, ηAB, ϕ, ν]
ϕ[i] = -FA - 1/2 (niA + niB) σ^2 +
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
          1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) - 2
                 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
                    n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ)))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)));
ϕ[j] = -FB - 1/2 (njA + njB) σ^2 +
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
          1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) - 2
                 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
                    n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)));
ηiA = niA;
ηiB = niB - 1;
ηjA = njA - 1;
ηjB = njB;
ηAA = nAA;
ηBB = nBB;
ηAB = nAB - 1;
ϕ[i '] = -FA - 1/2 (ηiA + ηiB) σ^2 +
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ))/(2 (-1 + n)) +
          1/(-4 + 8 n) (σ (2 ηAB^2 σ + ηAB (-2 + 2 Sa + 2 Sb + 4 ηAA σ + 4 ηBB σ - n ηiA σ - n ηiB σ) - 2
                 (-2 ηBB^2 σ - 2 ηAA (Sa - α + ηAA σ) + ηBB (2 - 2 Sb - 2 α + n ηiB σ) +
                    n (ηiB (-1 + Sb + α) + ηiA (Sa - α + ηAA σ)))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ)));
ϕ[j '] = -FB - 1/2 (ηjA + ηjB) σ^2 +
   (2 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ))/(2 (-1 + n)) +
          1/(-4 + 8 n) (σ (2 ηAB^2 σ + ηAB (-2 + 2 Sa + 2 Sb + 4 ηAA σ + 4 ηBB σ - n ηjA σ - n ηjB σ) - 2
                 (-2 ηBB^2 σ - 2 ηAA (Sa - α + ηAA σ) + ηBB (2 - 2 Sb - 2 α + n ηjB σ) +
```

```
                  n (ηjB (-1 + Sb + α) + ηjA (Sa - α + ηAA σ)))))^2) /
    (n (2 + 2 Sb (-1 + α) + 4 α^2 - ηAB σ - 2 ηBB σ - 2 α (2 + Sa + ηAA σ - ηBB σ)));
ν[i] = ϕ[i'] - ϕ[i];
ν[j] = ϕ[j'] - ϕ[j];
Δprofitinterdel[i] = Simplify[ν[i]];
Δprofitinterdel[j] = Simplify[ν[j]];
? Δprofitinterdel
Clear[ηiA, ηiB, ηjA, ηjB, ηAA, ηBB, ηAB, ϕ, ν]
```

PAIRWISE STABILITY

i belongs to backbone A and j belongs to backbone B

-----------------------------------------------------------------------------------

Intra-backbone link formation in backbone A

```
Global`ΔprofitintraAform
```

```
ΔprofitintraAform[i] =
 1/2 ((niA + niB) σ^2 - (1 + niA + niB) σ^2 - (4 (-1 + n) ((2+2 Sb (-1+α)+4 α^2-nAB σ-2 nBB σ-2 α (2+Sa+nAA σ-nBB σ))/(2 (-1+n)) +
          1/(-4+8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) -
              2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
                  n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ))))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))) +
    (4 (-1 + n) ((2+2 Sb (-1+α)+4 α^2-nAB σ-2 nBB σ-2 α (2+Sa+σ+nAA σ-nBB σ))/(2 (-1+n)) +
          1/(-4+8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 (1 + nAA) σ + 4 nBB σ - n (1 + niA) σ - n niB σ) -
              2 (-2 nBB^2 σ - 2 (1 + nAA) (Sa - α + σ + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
                  n (niB (-1 + Sb + α) + (1 + niA) (Sa - α + σ + nAA σ))))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + σ + nAA σ - nBB σ))))
```

```
ΔprofitintraAform[j] =
 1/n (2 (-1 + n) (-((2+2 Sb (-1+α)+4 α^2-nAB σ-2 nBB σ-2 α (2+Sa+nAA σ-nBB σ))/(2 (-1+n)) + 1/(-4+8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb +
                  4 nAA σ + 4 nBB σ - n njA σ - n njB σ) - 2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) +
                  nBB (2 - 2 Sb - 2 α + n njB σ) + n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ))))))^2 /
        (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)) +
      ((2+2 Sb (-1+α)+4 α^2-nAB σ-2 nBB σ-2 α (2+Sa+σ+nAA σ-nBB σ))/(2 (-1+n)) + 1/(-4+8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 σ +
                 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) - 2 (-2 nBB^2 σ - 2 (1 + nAA) (Sa - α + σ + nAA σ) +
                 nBB (2 - 2 Sb - 2 α + n njB σ) + n (njB (-1 + Sb + α) + njA (Sa - α + σ + nAA σ))))))^2 /
        (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + σ + nAA σ - nBB σ))))
```

------------------------------------------------------------------

Intra-backbone link deletion in backbone A

```
Global`ΔprofitintraAdel
```

$$\Delta profitintraAdel[i] = \frac{1}{2} \Bigg(-(-1+niA+niB)\,\sigma^2 + (niA+niB)\,\sigma^2 + \bigg(4\,(-1+n)\,\Big(\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\,\alpha\,(2+Sa+(-1+nAA-nBB)\,\sigma)}{2\,(-1+n)} + \frac{1}{-4+8\,n}\,(\sigma\,(2\,nAB^2\,\sigma+nAB\,(-2+2\,Sa+2\,Sb+4\,(-1+nAA)\,\sigma+4\,nBB\,\sigma-n\,(-1+niA)\,\sigma-n\,niB\,\sigma) - 2\,(-2\,nBB^2\,\sigma-2\,(-1+nAA)\,(Sa-\alpha+(-1+nAA)\,\sigma)+nBB\,(2-2\,Sb-2\,\alpha+n\,niB\,\sigma) + n\,(niB\,(-1+Sb+\alpha)+(-1+niA)\,(Sa-\alpha+(-1+nAA)\,\sigma)))))\Big)^2\bigg) \Big/ (n\,(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\,\alpha\,(2+Sa+(-1+nAA-nBB)\,\sigma))) - \bigg(4\,(-1+n)\,\Big(\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\,\alpha\,(2+Sa+nAA\,\sigma-nBB\,\sigma)}{2\,(-1+n)} + \frac{1}{-4+8\,n}\,(\sigma\,(2\,nAB^2\,\sigma+nAB\,(-2+2\,Sa+2\,Sb+4\,nAA\,\sigma+4\,nBB\,\sigma-n\,niA\,\sigma-n\,niB\,\sigma) - 2\,(-2\,nBB^2\,\sigma-2\,nAA\,(Sa-\alpha+nAA\,\sigma) + nBB\,(2-2\,Sb-2\,\alpha+n\,niB\,\sigma)+n\,(niB\,(-1+Sb+\alpha)+niA\,(Sa-\alpha+nAA\,\sigma)))))\Big)^2\bigg) \Big/ (n\,(2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\,\alpha\,(2+Sa+nAA\,\sigma-nBB\,\sigma)))\Bigg)$$

$$\Delta profitintraAdel[j] = \frac{1}{n} \Bigg(2\,(-1+n)\,\bigg(\Big(\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\,\alpha\,(2+Sa+(-1+nAA-nBB)\,\sigma)}{2\,(-1+n)} + \frac{1}{-4+8\,n}\,(\sigma\,(2\,nAB^2\,\sigma+nAB\,(-2+2\,Sa+2\,Sb+4\,(-1+nAA)\,\sigma+4\,nBB\,\sigma-n\,njA\,\sigma-n\,njB\,\sigma) - 2\,(-2\,nBB^2\,\sigma-2\,(-1+nAA)\,(Sa-\alpha+(-1+nAA)\,\sigma) + nBB\,(2-2\,Sb-2\,\alpha+n\,njB\,\sigma)+n\,(njB\,(-1+Sb+\alpha)+njA\,(Sa-\alpha+(-1+nAA)\,\sigma)))))\Big)^2 \Big/ (2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\,\alpha\,(2+Sa+(-1+nAA-nBB)\,\sigma)) - \Big(\frac{2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\,\alpha\,(2+Sa+nAA\,\sigma-nBB\,\sigma)}{2\,(-1+n)} + \frac{1}{-4+8\,n}\,(\sigma\,(2\,nAB^2\,\sigma+nAB\,(-2+2\,Sa+2\,Sb+4\,nAA\,\sigma+4\,nBB\,\sigma-n\,njA\,\sigma-n\,njB\,\sigma) - 2\,(-2\,nBB^2\,\sigma-2\,nAA\,(Sa-\alpha+nAA\,\sigma) + nBB\,(2-2\,Sb-2\,\alpha+n\,njB\,\sigma)+n\,(njB\,(-1+Sb+\alpha)+njA\,(Sa-\alpha+nAA\,\sigma)))))\Big)^2 \Big/ (2+2\,Sb\,(-1+\alpha)+4\,\alpha^2-nAB\,\sigma-2\,nBB\,\sigma-2\,\alpha\,(2+Sa+nAA\,\sigma-nBB\,\sigma))\bigg)\Bigg)$$

-----------------------------------------------------------------

Intra-backbone link formation in backbone B

Global`ΔprofitintraBform

$$
\begin{aligned}
&\Delta\text{profitintraBform}[i] = \\
&\frac{1}{n}\Bigg(2\,(-1+n)\,\Bigg(-\Bigg(\frac{2+2\,\text{Sb}\,(-1+\alpha)+4\,\alpha^{2}-\text{nAB}\,\sigma-2\,\text{nBB}\,\sigma-2\,\alpha\,(2+\text{Sa}+\text{nAA}\,\sigma-\text{nBB}\,\sigma)}{2\,(-1+n)}+\frac{1}{-4+8\,n}\,(\sigma\,(2\,\text{nAB}^{2}\,\sigma+\text{nAB}\,(-2+2\,\text{Sa}+2\,\text{Sb}+ \\
&\qquad 4\,\text{nAA}\,\sigma+4\,\text{nBB}\,\sigma-n\,\text{niA}\,\sigma-n\,\text{niB}\,\sigma)-2\,(-2\,\text{nBB}^{2}\,\sigma-2\,\text{nAA}\,(\text{Sa}-\alpha+\text{nAA}\,\sigma)+ \\
&\qquad \text{nBB}\,(2-2\,\text{Sb}-2\,\alpha+n\,\text{niB}\,\sigma)+n\,(\text{niB}\,(-1+\text{Sb}+\alpha)+\text{niA}\,(\text{Sa}-\alpha+\text{nAA}\,\sigma))))))\Bigg)^{2}\Bigg/ \\
&\quad (2+2\,\text{Sb}\,(-1+\alpha)+4\,\alpha^{2}-\text{nAB}\,\sigma-2\,\text{nBB}\,\sigma-2\,\alpha\,(2+\text{Sa}+\text{nAA}\,\sigma-\text{nBB}\,\sigma))+ \\
&\quad \Bigg(\frac{2+2\,\text{Sb}\,(-1+\alpha)+4\,\alpha^{2}-\text{nAB}\,\sigma-2\,(1+\text{nBB})\,\sigma-2\,\alpha\,(2+\text{Sa}+(-1+\text{nAA}-\text{nBB})\,\sigma)}{2\,(-1+n)}+\frac{1}{-4+8\,n}\,(\sigma\,(2\,\text{nAB}^{2}\,\sigma+\text{nAB}\,(-2+2\,\text{Sa}+2\,\text{Sb}+ \\
&\qquad 4\,\text{nAA}\,\sigma+4\,(1+\text{nBB})\,\sigma-n\,\text{niA}\,\sigma-n\,\text{niB}\,\sigma)-2\,(-2\,(1+\text{nBB})^{2}\,\sigma-2\,\text{nAA}\,(\text{Sa}-\alpha+\text{nAA}\,\sigma)+ \\
&\qquad (1+\text{nBB})\,(2-2\,\text{Sb}-2\,\alpha+n\,\text{niB}\,\sigma)+n\,(\text{niB}\,(-1+\text{Sb}+\alpha)+\text{niA}\,(\text{Sa}-\alpha+\text{nAA}\,\sigma))))))\Bigg)^{2}\Bigg/ \\
&\quad (2+2\,\text{Sb}\,(-1+\alpha)+4\,\alpha^{2}-\text{nAB}\,\sigma-2\,(1+\text{nBB})\,\sigma-2\,\alpha\,(2+\text{Sa}+(-1+\text{nAA}-\text{nBB})\,\sigma))\Bigg)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\Delta\text{profitintraBform}[j] = \\
&\frac{1}{2}\Bigg((\text{njA}+\text{njB})\,\sigma^{2}-(1+\text{njA}+\text{njB})\,\sigma^{2}-\Bigg(4\,(-1+n)\,\Bigg(\frac{2+2\,\text{Sb}\,(-1+\alpha)+4\,\alpha^{2}-\text{nAB}\,\sigma-2\,\text{nBB}\,\sigma-2\,\alpha\,(2+\text{Sa}+\text{nAA}\,\sigma-\text{nBB}\,\sigma)}{2\,(-1+n)}+ \\
&\qquad \frac{1}{-4+8\,n}\,(\sigma\,(2\,\text{nAB}^{2}\,\sigma+\text{nAB}\,(-2+2\,\text{Sa}+2\,\text{Sb}+4\,\text{nAA}\,\sigma+4\,\text{nBB}\,\sigma-n\,\text{njA}\,\sigma-n\,\text{njB}\,\sigma)- \\
&\qquad\quad 2\,(-2\,\text{nBB}^{2}\,\sigma-2\,\text{nAA}\,(\text{Sa}-\alpha+\text{nAA}\,\sigma)+\text{nBB}\,(2-2\,\text{Sb}-2\,\alpha+n\,\text{njB}\,\sigma)+ \\
&\qquad\qquad n\,(\text{njB}\,(-1+\text{Sb}+\alpha)+\text{njA}\,(\text{Sa}-\alpha+\text{nAA}\,\sigma))))))\Bigg)^{2}\Bigg)\Bigg/ \\
&\quad (n\,(2+2\,\text{Sb}\,(-1+\alpha)+4\,\alpha^{2}-\text{nAB}\,\sigma-2\,\text{nBB}\,\sigma-2\,\alpha\,(2+\text{Sa}+\text{nAA}\,\sigma-\text{nBB}\,\sigma)))+ \\
&\quad \Bigg(4\,(-1+n)\,\Bigg(\frac{2+2\,\text{Sb}\,(-1+\alpha)+4\,\alpha^{2}-\text{nAB}\,\sigma-2\,(1+\text{nBB})\,\sigma-2\,\alpha\,(2+\text{Sa}+(-1+\text{nAA}-\text{nBB})\,\sigma)}{2\,(-1+n)}+ \\
&\qquad \frac{1}{-4+8\,n}\,(\sigma\,(2\,\text{nAB}^{2}\,\sigma+\text{nAB}\,(-2+2\,\text{Sa}+2\,\text{Sb}+4\,\text{nAA}\,\sigma+4\,(1+\text{nBB})\,\sigma-n\,\text{njA}\,\sigma-n\,(1+\text{njB})\,\sigma)- \\
&\qquad\quad 2\,(-2\,(1+\text{nBB})^{2}\,\sigma-2\,\text{nAA}\,(\text{Sa}-\alpha+\text{nAA}\,\sigma)+(1+\text{nBB})\,(2-2\,\text{Sb}-2\,\alpha+n\,(1+\text{njB})\,\sigma)+ \\
&\qquad\qquad n\,((1+\text{njB})\,(-1+\text{Sb}+\alpha)+\text{njA}\,(\text{Sa}-\alpha+\text{nAA}\,\sigma))))))\Bigg)^{2}\Bigg)\Bigg/ \\
&\quad (n\,(2+2\,\text{Sb}\,(-1+\alpha)+4\,\alpha^{2}-\text{nAB}\,\sigma-2\,(1+\text{nBB})\,\sigma-2\,\alpha\,(2+\text{Sa}+(-1+\text{nAA}-\text{nBB})\,\sigma)))\Bigg)
\end{aligned}
$$

---------------------------------------------------------------------

Intra-backbone link deletion in backbone B

Global`ΔprofitintraBdel

$$\Delta\mathrm{profitintraBdel}[i] = \frac{1}{n}\Bigg(2\,(-1+n)\,\Bigg(\bigg(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2+2\,\sigma-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\sigma+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)}{2\,(-1+n)}+\frac{1}{-4+8\,n}\,(\sigma\,(2\,\mathrm{nAB}^2\,\sigma+\mathrm{nAB}\,(-2+2\,\mathrm{Sa}+2\,\mathrm{Sb}+4\,\mathrm{nAA}\,\sigma+4\,(-1+\mathrm{nBB})\,\sigma-n\,\mathrm{niA}\,\sigma-n\,\mathrm{niB}\,\sigma)-2\,(-2\,(-1+\mathrm{nBB})^2\,\sigma-2\,\mathrm{nAA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)+(-1+\mathrm{nBB})\,(2-2\,\mathrm{Sb}-2\,\alpha+n\,\mathrm{niB}\,\sigma)+n\,(\mathrm{niB}\,(-1+\mathrm{Sb}+\alpha)+\mathrm{niA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)))))\bigg)^2\Big/(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2+2\,\sigma-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\sigma+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma))-\bigg(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)}{2\,(-1+n)}+\frac{1}{-4+8\,n}\,(\sigma\,(2\,\mathrm{nAB}^2\,\sigma+\mathrm{nAB}\,(-2+2\,\mathrm{Sa}+2\,\mathrm{Sb}+4\,\mathrm{nAA}\,\sigma+4\,\mathrm{nBB}\,\sigma-n\,\mathrm{niA}\,\sigma-n\,\mathrm{niB}\,\sigma)-2\,(-2\,\mathrm{nBB}^2\,\sigma-2\,\mathrm{nAA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)+\mathrm{nBB}\,(2-2\,\mathrm{Sb}-2\,\alpha+n\,\mathrm{niB}\,\sigma)+n\,(\mathrm{niB}\,(-1+\mathrm{Sb}+\alpha)+\mathrm{niA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)))))\bigg)^2\Big/(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma))\Bigg)\Bigg)$$

$$\Delta\mathrm{profitintraBdel}[j] = \frac{1}{2}\Bigg(-(-1+\mathrm{njA}+\mathrm{njB})\,\sigma^2+(\mathrm{njA}+\mathrm{njB})\,\sigma^2+\Bigg(4\,(-1+n)\,\bigg(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2+2\,\sigma-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\sigma+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)}{2\,(-1+n)}+\frac{1}{-4+8\,n}\,(\sigma\,(2\,\mathrm{nAB}^2\,\sigma+\mathrm{nAB}\,(-2+2\,\mathrm{Sa}+2\,\mathrm{Sb}+4\,\mathrm{nAA}\,\sigma+4\,(-1+\mathrm{nBB})\,\sigma-n\,\mathrm{njA}\,\sigma-n\,(-1+\mathrm{njB})\,\sigma)-2\,(-2\,(-1+\mathrm{nBB})^2\,\sigma-2\,\mathrm{nAA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)+(-1+\mathrm{nBB})\,(2-2\,\mathrm{Sb}-2\,\alpha+n\,(-1+\mathrm{njB})\,\sigma)+n\,((-1+\mathrm{njB})\,(-1+\mathrm{Sb}+\alpha)+\mathrm{njA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)))))\bigg)^2\Bigg)\Big/(n\,(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2+2\,\sigma-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\sigma+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)))-\Bigg(4\,(-1+n)\,\bigg(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)}{2\,(-1+n)}+\frac{1}{-4+8\,n}\,(\sigma\,(2\,\mathrm{nAB}^2\,\sigma+\mathrm{nAB}\,(-2+2\,\mathrm{Sa}+2\,\mathrm{Sb}+4\,\mathrm{nAA}\,\sigma+4\,\mathrm{nBB}\,\sigma-n\,\mathrm{njA}\,\sigma-n\,\mathrm{njB}\,\sigma)-2\,(-2\,\mathrm{nBB}^2\,\sigma-2\,\mathrm{nAA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)+\mathrm{nBB}\,(2-2\,\mathrm{Sb}-2\,\alpha+n\,\mathrm{njB}\,\sigma)+n\,(\mathrm{njB}\,(-1+\mathrm{Sb}+\alpha)+\mathrm{njA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)))))\bigg)^2\Bigg)\Big/(n\,(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)))\Bigg)$$

---------------------------------------------------------------

Inter-backbone link formation

Global`Δprofitinterform

Δprofitinterform[i] =

$$
\begin{aligned}
&\frac{1}{2}\Bigg((\mathrm{niA}+\mathrm{niB})\,\sigma^2-(1+\mathrm{niA}+\mathrm{niB})\,\sigma^2-\Bigg(4\,(-1+n)\,\bigg(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)}{2\,(-1+n)}+\\
&\qquad\frac{1}{-4+8\,n}\,(\sigma\,(2\,\mathrm{nAB}^2\,\sigma+\mathrm{nAB}\,(-2+2\,\mathrm{Sa}+2\,\mathrm{Sb}+4\,\mathrm{nAA}\,\sigma+4\,\mathrm{nBB}\,\sigma-n\,\mathrm{niA}\,\sigma-n\,\mathrm{niB}\,\sigma)-\\
&\qquad\quad 2\,(-2\,\mathrm{nBB}^2\,\sigma-2\,\mathrm{nAA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)+\mathrm{nBB}\,(2-2\,\mathrm{Sb}-2\,\alpha+n\,\mathrm{niB}\,\sigma)+\\
&\qquad\qquad n\,(\mathrm{niB}\,(-1+\mathrm{Sb}+\alpha)+\mathrm{niA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma))))))\bigg)^2\Bigg)\Big/\\
&\quad(n\,(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)))+\\
&\quad\Bigg(4\,(-1+n)\,\bigg(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-(1+\mathrm{nAB})\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)}{2\,(-1+n)}+\\
&\qquad\frac{1}{-4+8\,n}\,(\sigma\,(2\,(1+\mathrm{nAB})^2\,\sigma+(1+\mathrm{nAB})\,(-2+2\,\mathrm{Sa}+2\,\mathrm{Sb}+4\,\mathrm{nAA}\,\sigma+4\,\mathrm{nBB}\,\sigma-n\,\mathrm{niA}\,\sigma-n\,(1+\mathrm{niB})\,\sigma)-\\
&\qquad\quad 2\,(-2\,(\mathrm{nBB}\,(-1+\mathrm{Sb}+\alpha)+\mathrm{nBB}^2\,\sigma+\mathrm{nAA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma))+\\
&\qquad\qquad n\,(-1+\mathrm{Sb}+\alpha+\mathrm{nBB}\,\sigma+\mathrm{niA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)+\mathrm{niB}\,(-1+\mathrm{Sb}+\alpha+\mathrm{nBB}\,\sigma))))))\bigg)^2\Bigg)\Big/\\
&\quad(n\,(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-(1+\mathrm{nAB})\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)))\Bigg)
\end{aligned}
$$

Δprofitinterform[j] =

$$
\begin{aligned}
&\frac{1}{2}\Bigg((\mathrm{njA}+\mathrm{njB})\,\sigma^2-(1+\mathrm{njA}+\mathrm{njB})\,\sigma^2-\Bigg(4\,(-1+n)\,\bigg(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)}{2\,(-1+n)}+\\
&\qquad\frac{1}{-4+8\,n}\,(\sigma\,(2\,\mathrm{nAB}^2\,\sigma+\mathrm{nAB}\,(-2+2\,\mathrm{Sa}+2\,\mathrm{Sb}+4\,\mathrm{nAA}\,\sigma+4\,\mathrm{nBB}\,\sigma-n\,\mathrm{njA}\,\sigma-n\,\mathrm{njB}\,\sigma)-\\
&\qquad\quad 2\,(-2\,\mathrm{nBB}^2\,\sigma-2\,\mathrm{nAA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)+\mathrm{nBB}\,(2-2\,\mathrm{Sb}-2\,\alpha+n\,\mathrm{njB}\,\sigma)+\\
&\qquad\qquad n\,(\mathrm{njB}\,(-1+\mathrm{Sb}+\alpha)+\mathrm{njA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma))))))\bigg)^2\Bigg)\Big/\\
&\quad(n\,(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)))+\\
&\quad\Bigg(4\,(-1+n)\,\bigg(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-(1+\mathrm{nAB})\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)}{2\,(-1+n)}+\frac{1}{-4+8\,n}\,(\sigma\,(2\,(1+\mathrm{nAB})^2\,\sigma+(1+\mathrm{nAB})\,(-2+2\,\mathrm{Sa}+\\
&\qquad\quad 2\,\mathrm{Sb}+4\,\mathrm{nAA}\,\sigma+4\,\mathrm{nBB}\,\sigma-n\,(1+\mathrm{njA})\,\sigma-n\,\mathrm{njB}\,\sigma)-2\,(-2\,\mathrm{nBB}^2\,\sigma-2\,\mathrm{nAA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)+\\
&\qquad\quad \mathrm{nBB}\,(2-2\,\mathrm{Sb}-2\,\alpha+n\,\mathrm{njB}\,\sigma)+n\,(\mathrm{njB}\,(-1+\mathrm{Sb}+\alpha)+(1+\mathrm{njA})\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma))))))\bigg)^2\Bigg)\Big/\\
&\quad(n\,(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^2-(1+\mathrm{nAB})\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)))\Bigg)
\end{aligned}
$$

----------------------------------------------------------------------------

Inter-backbone link deletion

Global`Δprofitinterdel

```
Δprofitinterdel[i] =
 1/2 (-(-1 + niA + niB) σ^2 + (niA + niB) σ^2 + (4 (-1 + n) ((2+2 Sb (-1+α)+4 α^2+σ-nAB σ-2 nBB σ-2 α (2+Sa+nAA σ-nBB σ))/(2 (-1+n)) +
        1/(-4+8 n) (σ (2 (-1 + nAB)^2 σ + (-1 + nAB) (-2 + 2 Sa + 2 Sb + n σ + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) -
              2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n (-1 + niB) σ) +
                 n ((-1 + niB) (-1 + Sb + α) + niA (Sa - α + nAA σ))))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 + σ - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))) -
    (4 (-1 + n) ((2+2 Sb (-1+α)+4 α^2-nAB σ-2 nBB σ-2 α (2+Sa+nAA σ-nBB σ))/(2 (-1+n)) + 1/(-4+8 n) (σ (2 nAB^2 σ + nAB
                (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) - 2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) +
                  nBB (2 - 2 Sb - 2 α + n niB σ) + n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ))))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))))
```

```
Δprofitinterdel[j] =
 1/2 (-(-1 + njA + njB) σ^2 + (njA + njB) σ^2 + (4 (-1 + n) ((2+2 Sb (-1+α)+4 α^2+σ-nAB σ-2 nBB σ-2 α (2+Sa+nAA σ-nBB σ))/(2 (-1+n)) +
        1/(-4+8 n) (σ (2 (-1 + nAB)^2 σ + (-1 + nAB) (-2 + 2 Sa + 2 Sb + n σ + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
              2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
                 n (njB (-1 + Sb + α) + (-1 + njA) (Sa - α + nAA σ))))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 + σ - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))) -
    (4 (-1 + n) ((2+2 Sb (-1+α)+4 α^2-nAB σ-2 nBB σ-2 α (2+Sa+nAA σ-nBB σ))/(2 (-1+n)) + 1/(-4+8 n) (σ (2 nAB^2 σ + nAB
                (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) - 2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) +
                  nBB (2 - 2 Sb - 2 α + n njB σ) + n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ))))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))))
```

## "PAIRWISE STABILITY FOR THE EIGHT FOCAL NETWORKS WITH SIX FIRMS"

```
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
Clear[Sa, Sb, α, m, n, FA, FB]
Clear[p1, p2, p3, p4, p5, p6, p7, p8, p9]
```

```
ΔprofitintraAform[i] = 1/2 ((niA + niB) σ^2 - (1 + niA + niB) σ^2 -
    (4 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))/(2 (-1 + n)) +
          1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n niA σ - n niB σ) -
               2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
                  n (niB (-1 + Sb + α) + niA (Sa - α + nAA σ))))))^2) /
     (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ))) +
    (4 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + σ + nAA σ - nBB σ))/(2 (-1 + n)) + 1/(-4 + 8 n)
          (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 (1 + nAA) σ + 4 nBB σ - n (1 + niA) σ - n niB σ) -
               2 (-2 nBB^2 σ - 2 (1 + nAA) (Sa - α + σ + nAA σ) + nBB (2 - 2 Sb - 2 α + n niB σ) +
                  n (niB (-1 + Sb + α) + (1 + niA) (Sa - α + σ + nAA σ))))))^2) /
```

$$
\left.\left(n\left(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^{2}-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\sigma+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)\right)\right)\right);
$$

$$
\begin{aligned}
&\Delta\mathrm{profitintraAdel}[i]=\frac{1}{2}\Bigg(-(-1+\mathrm{niA}+\mathrm{niB})\,\sigma^{2}+(\mathrm{niA}+\mathrm{niB})\,\sigma^{2}+\Bigg(4\,(-1+n)\\
&\quad\Bigg(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^{2}-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+(-1+\mathrm{nAA}-\mathrm{nBB})\,\sigma)}{2\,(-1+n)}+\frac{1}{-4+8\,n}\,(\sigma\\
&\qquad(2\,\mathrm{nAB}^{2}\,\sigma+\mathrm{nAB}\,(-2+2\,\mathrm{Sa}+2\,\mathrm{Sb}+4\,(-1+\mathrm{nAA})\,\sigma+4\,\mathrm{nBB}\,\sigma-n\,(-1+\mathrm{niA})\,\sigma-n\,\mathrm{niB}\,\sigma)-\\
&\qquad 2\,(-2\,\mathrm{nBB}^{2}\,\sigma-2\,(-1+\mathrm{nAA})\,(\mathrm{Sa}-\alpha+(-1+\mathrm{nAA})\,\sigma)+\mathrm{nBB}\,(2-2\,\mathrm{Sb}-2\,\alpha+n\,\mathrm{niB}\,\sigma)+\\
&\qquad n\,(\mathrm{niB}\,(-1+\mathrm{Sb}+\alpha)+(-1+\mathrm{niA})\,(\mathrm{Sa}-\alpha+(-1+\mathrm{nAA})\,\sigma))))\Bigg)^{2}\Bigg)\Big/\\
&\quad(n\,(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^{2}-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+(-1+\mathrm{nAA}-\mathrm{nBB})\,\sigma)))-\\
&\quad\Bigg(4\,(-1+n)\,\Bigg(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^{2}-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)}{2\,(-1+n)}+\\
&\qquad\frac{1}{-4+8\,n}\,(\sigma\,(2\,\mathrm{nAB}^{2}\,\sigma+\mathrm{nAB}\,(-2+2\,\mathrm{Sa}+2\,\mathrm{Sb}+4\,\mathrm{nAA}\,\sigma+4\,\mathrm{nBB}\,\sigma-n\,\mathrm{niA}\,\sigma-n\,\mathrm{niB}\,\sigma)-\\
&\qquad 2\,(-2\,\mathrm{nBB}^{2}\,\sigma-2\,\mathrm{nAA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)+\mathrm{nBB}\,(2-2\,\mathrm{Sb}-2\,\alpha+n\,\mathrm{niB}\,\sigma)+\\
&\qquad n\,(\mathrm{niB}\,(-1+\mathrm{Sb}+\alpha)+\mathrm{niA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)))))\Bigg)^{2}\Bigg)\Big/\\
&\quad(n\,(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^{2}-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)))\Bigg);
\end{aligned}
$$

$$
\begin{aligned}
&\Delta\mathrm{profitintraBform}[j]=\frac{1}{2}\Bigg((\mathrm{njA}+\mathrm{njB})\,\sigma^{2}-(1+\mathrm{njA}+\mathrm{njB})\,\sigma^{2}-\\
&\quad\Bigg(4\,(-1+n)\,\Bigg(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^{2}-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)}{2\,(-1+n)}+\\
&\qquad\frac{1}{-4+8\,n}\,(\sigma\,(2\,\mathrm{nAB}^{2}\,\sigma+\mathrm{nAB}\,(-2+2\,\mathrm{Sa}+2\,\mathrm{Sb}+4\,\mathrm{nAA}\,\sigma+4\,\mathrm{nBB}\,\sigma-n\,\mathrm{njA}\,\sigma-n\,\mathrm{njB}\,\sigma)-\\
&\qquad 2\,(-2\,\mathrm{nBB}^{2}\,\sigma-2\,\mathrm{nAA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)+\mathrm{nBB}\,(2-2\,\mathrm{Sb}-2\,\alpha+n\,\mathrm{njB}\,\sigma)+\\
&\qquad n\,(\mathrm{njB}\,(-1+\mathrm{Sb}+\alpha)+\mathrm{njA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)))))\Bigg)^{2}\Bigg)\Big/\\
&\quad(n\,(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^{2}-\mathrm{nAB}\,\sigma-2\,\mathrm{nBB}\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+\mathrm{nAA}\,\sigma-\mathrm{nBB}\,\sigma)))+\Bigg(4\,(-1+n)\\
&\quad\Bigg(\frac{2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^{2}-\mathrm{nAB}\,\sigma-2\,(1+\mathrm{nBB})\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+(-1+\mathrm{nAA}-\mathrm{nBB})\,\sigma)}{2\,(-1+n)}+\frac{1}{-4+8\,n}\\
&\qquad(\sigma\,(2\,\mathrm{nAB}^{2}\,\sigma+\mathrm{nAB}\,(-2+2\,\mathrm{Sa}+2\,\mathrm{Sb}+4\,\mathrm{nAA}\,\sigma+4\,(1+\mathrm{nBB})\,\sigma-n\,\mathrm{njA}\,\sigma-n\,(1+\mathrm{njB})\,\sigma)-\\
&\qquad 2\,(-2\,(1+\mathrm{nBB})^{2}\,\sigma-2\,\mathrm{nAA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)+(1+\mathrm{nBB})\,(2-2\,\mathrm{Sb}-2\,\alpha+\\
&\qquad n\,(1+\mathrm{njB})\,\sigma)+n\,((1+\mathrm{njB})\,(-1+\mathrm{Sb}+\alpha)+\mathrm{njA}\,(\mathrm{Sa}-\alpha+\mathrm{nAA}\,\sigma)))))\Bigg)^{2}\Bigg)\Big/\\
&\quad(n\,(2+2\,\mathrm{Sb}\,(-1+\alpha)+4\,\alpha^{2}-\mathrm{nAB}\,\sigma-2\,(1+\mathrm{nBB})\,\sigma-2\,\alpha\,(2+\mathrm{Sa}+(-1+\mathrm{nAA}-\mathrm{nBB})\,\sigma)))\Bigg);
\end{aligned}
$$

$$\Delta\text{profitintraBdel}[j] = \frac{1}{2}\Bigg(-(-1+\text{njA}+\text{njB})\,\sigma^2 + (\text{njA}+\text{njB})\,\sigma^2 +$$

$$\Bigg(4\,(-1+n)\,\Bigg(\frac{2+2\,\text{Sb}\,(-1+\alpha)+4\,\alpha^2+2\,\sigma-\text{nAB}\,\sigma-2\,\text{nBB}\,\sigma-2\,\alpha\,(2+\text{Sa}+\sigma+\text{nAA}\,\sigma-\text{nBB}\,\sigma)}{2\,(-1+n)} +$$

$$\frac{1}{-4+8\,n}\,(\sigma\,(2\,\text{nAB}^2\,\sigma+\text{nAB}\,(-2+2\,\text{Sa}+2\,\text{Sb}+4\,\text{nAA}\,\sigma+4\,(-1+\text{nBB})\,\sigma -$$
$$n\,\text{njA}\,\sigma-n\,(-1+\text{njB})\,\sigma)-2\,(-2\,(-1+\text{nBB})^2\,\sigma -$$
$$2\,\text{nAA}\,(\text{Sa}-\alpha+\text{nAA}\,\sigma)+(-1+\text{nBB})\,(2-2\,\text{Sb}-2\,\alpha+n\,(-1+\text{njB})\,\sigma) +$$
$$n\,((-1+\text{njB})\,(-1+\text{Sb}+\alpha)+\text{njA}\,(\text{Sa}-\alpha+\text{nAA}\,\sigma))))\Bigg)^2\Bigg)\Bigg/$$

$$(n\,(2+2\,\text{Sb}\,(-1+\alpha)+4\,\alpha^2+2\,\sigma-\text{nAB}\,\sigma-2\,\text{nBB}\,\sigma-2\,\alpha\,(2+\text{Sa}+\sigma+\text{nAA}\,\sigma-\text{nBB}\,\sigma))) -$$

$$\Bigg(4\,(-1+n)\,\Bigg(\frac{2+2\,\text{Sb}\,(-1+\alpha)+4\,\alpha^2-\text{nAB}\,\sigma-2\,\text{nBB}\,\sigma-2\,\alpha\,(2+\text{Sa}+\text{nAA}\,\sigma-\text{nBB}\,\sigma)}{2\,(-1+n)} +$$

$$\frac{1}{-4+8\,n}\,(\sigma\,(2\,\text{nAB}^2\,\sigma+\text{nAB}\,(-2+2\,\text{Sa}+2\,\text{Sb}+4\,\text{nAA}\,\sigma+4\,\text{nBB}\,\sigma-n\,\text{njA}\,\sigma-n\,\text{njB}\,\sigma) -$$
$$2\,(-2\,\text{nBB}^2\,\sigma-2\,\text{nAA}\,(\text{Sa}-\alpha+\text{nAA}\,\sigma)+\text{nBB}\,(2-2\,\text{Sb}-2\,\alpha+n\,\text{njB}\,\sigma) +$$
$$n\,(\text{njB}\,(-1+\text{Sb}+\alpha)+\text{njA}\,(\text{Sa}-\alpha+\text{nAA}\,\sigma)))))\Bigg)^2\Bigg)\Bigg/$$

$$(n\,(2+2\,\text{Sb}\,(-1+\alpha)+4\,\alpha^2-\text{nAB}\,\sigma-2\,\text{nBB}\,\sigma-2\,\alpha\,(2+\text{Sa}+\text{nAA}\,\sigma-\text{nBB}\,\sigma)))\Bigg);$$

$$\Delta\text{profitinterform}[i] = \frac{1}{2}\Bigg((\text{niA}+\text{niB})\,\sigma^2-(1+\text{niA}+\text{niB})\,\sigma^2 -$$

$$\Bigg(4\,(-1+n)\,\Bigg(\frac{2+2\,\text{Sb}\,(-1+\alpha)+4\,\alpha^2-\text{nAB}\,\sigma-2\,\text{nBB}\,\sigma-2\,\alpha\,(2+\text{Sa}+\text{nAA}\,\sigma-\text{nBB}\,\sigma)}{2\,(-1+n)} +$$

$$\frac{1}{-4+8\,n}\,(\sigma\,(2\,\text{nAB}^2\,\sigma+\text{nAB}\,(-2+2\,\text{Sa}+2\,\text{Sb}+4\,\text{nAA}\,\sigma+4\,\text{nBB}\,\sigma-n\,\text{niA}\,\sigma-n\,\text{niB}\,\sigma) -$$
$$2\,(-2\,\text{nBB}^2\,\sigma-2\,\text{nAA}\,(\text{Sa}-\alpha+\text{nAA}\,\sigma)+\text{nBB}\,(2-2\,\text{Sb}-2\,\alpha+n\,\text{niB}\,\sigma) +$$
$$n\,(\text{niB}\,(-1+\text{Sb}+\alpha)+\text{niA}\,(\text{Sa}-\alpha+\text{nAA}\,\sigma)))))\Bigg)^2\Bigg)\Bigg/$$

$$(n\,(2+2\,\text{Sb}\,(-1+\alpha)+4\,\alpha^2-\text{nAB}\,\sigma-2\,\text{nBB}\,\sigma-2\,\alpha\,(2+\text{Sa}+\text{nAA}\,\sigma-\text{nBB}\,\sigma))) +$$

$$\Bigg(4\,(-1+n)\,\Bigg(\frac{2+2\,\text{Sb}\,(-1+\alpha)+4\,\alpha^2-(1+\text{nAB})\,\sigma-2\,\text{nBB}\,\sigma-2\,\alpha\,(2+\text{Sa}+\text{nAA}\,\sigma-\text{nBB}\,\sigma)}{2\,(-1+n)} +$$

$$\frac{1}{-4+8\,n}\,(\sigma\,(2\,(1+\text{nAB})^2\,\sigma+(1+\text{nAB})\,(-2+2\,\text{Sa}+2\,\text{Sb}+4\,\text{nAA}\,\sigma+4\,\text{nBB}\,\sigma-n\,\text{niA}\,\sigma -$$
$$n\,(1+\text{niB})\,\sigma)-2\,(-2\,(\text{nBB}\,(-1+\text{Sb}+\alpha)+\text{nBB}^2\,\sigma+\text{nAA}\,(\text{Sa}-\alpha+\text{nAA}\,\sigma)) +$$
$$n\,(-1+\text{Sb}+\alpha+\text{nBB}\,\sigma+\text{niA}\,(\text{Sa}-\alpha+\text{nAA}\,\sigma)+\text{niB}\,(-1+\text{Sb}+\alpha+\text{nBB}\,\sigma)))))\Bigg)^2\Bigg)\Bigg/$$

$$(n\,(2+2\,\text{Sb}\,(-1+\alpha)+4\,\alpha^2-(1+\text{nAB})\,\sigma-2\,\text{nBB}\,\sigma-2\,\alpha\,(2+\text{Sa}+\text{nAA}\,\sigma-\text{nBB}\,\sigma)))\Bigg);$$

$$\Delta\text{profitinterform}[j] = \frac{1}{2}\Bigg((\text{njA}+\text{njB})\,\sigma^2-(1+\text{njA}+\text{njB})\,\sigma^2 -$$

$$\Bigg(4\,(-1+n)\,\Bigg(\frac{2+2\,\text{Sb}\,(-1+\alpha)+4\,\alpha^2-\text{nAB}\,\sigma-2\,\text{nBB}\,\sigma-2\,\alpha\,(2+\text{Sa}+\text{nAA}\,\sigma-\text{nBB}\,\sigma)}{2\,(-1+n)} +$$

$$\frac{1}{-4+8\,n}\;(\sigma\;(2\;\mathrm{nAB}^2\;\sigma+\mathrm{nAB}\;(-2+2\;\mathrm{Sa}+2\;\mathrm{Sb}+4\;\mathrm{nAA}\;\sigma+4\;\mathrm{nBB}\;\sigma-n\;\mathrm{njA}\;\sigma-n\;\mathrm{njB}\;\sigma)-$$

$$2\;(-2\;\mathrm{nBB}^2\;\sigma-2\;\mathrm{nAA}\;(\mathrm{Sa}-\alpha+\mathrm{nAA}\;\sigma)+\mathrm{nBB}\;(2-2\;\mathrm{Sb}-2\;\alpha+n\;\mathrm{njB}\;\sigma)+$$

$$n\;(\mathrm{njB}\;(-1+\mathrm{Sb}+\alpha)+\mathrm{njA}\;(\mathrm{Sa}-\alpha+\mathrm{nAA}\;\sigma))))))\Big)^2\Big)\Big/$$

$$(n\;(2+2\;\mathrm{Sb}\;(-1+\alpha)+4\;\alpha^2-\mathrm{nAB}\;\sigma-2\;\mathrm{nBB}\;\sigma-2\;\alpha\;(2+\mathrm{Sa}+\mathrm{nAA}\;\sigma-\mathrm{nBB}\;\sigma)))+$$

$$\Bigg(4\;(-1+n)\;\Bigg(\frac{2+2\;\mathrm{Sb}\;(-1+\alpha)+4\;\alpha^2-(1+\mathrm{nAB})\;\sigma-2\;\mathrm{nBB}\;\sigma-2\;\alpha\;(2+\mathrm{Sa}+\mathrm{nAA}\;\sigma-\mathrm{nBB}\;\sigma)}{2\;(-1+n)}+$$

$$\frac{1}{-4+8\,n}\;(\sigma\;(2\;(1+\mathrm{nAB})^2\;\sigma+(1+\mathrm{nAB})\;(-2+2\;\mathrm{Sa}+2\;\mathrm{Sb}+4\;\mathrm{nAA}\;\sigma+4\;\mathrm{nBB}\;\sigma-$$

$$n\;(1+\mathrm{njA})\;\sigma-n\;\mathrm{njB}\;\sigma)-2\;(-2\;\mathrm{nBB}^2\;\sigma-2\;\mathrm{nAA}\;(\mathrm{Sa}-\alpha+\mathrm{nAA}\;\sigma)+\mathrm{nBB}\;(2-2\;\mathrm{Sb}-$$

$$2\;\alpha+n\;\mathrm{njB}\;\sigma)+n\;(\mathrm{njB}\;(-1+\mathrm{Sb}+\alpha)+(1+\mathrm{njA})\;(\mathrm{Sa}-\alpha+\mathrm{nAA}\;\sigma))))))\Bigg)^2\Bigg)\Bigg/$$

$$(n\;(2+2\;\mathrm{Sb}\;(-1+\alpha)+4\;\alpha^2-(1+\mathrm{nAB})\;\sigma-2\;\mathrm{nBB}\;\sigma-2\;\alpha\;(2+\mathrm{Sa}+\mathrm{nAA}\;\sigma-\mathrm{nBB}\;\sigma)))\Bigg);$$

$$\Delta\mathrm{profitinterdel}[i]=\frac{1}{2}\;\Bigg(-(-1+\mathrm{niA}+\mathrm{niB})\;\sigma^2+(\mathrm{niA}+\mathrm{niB})\;\sigma^2+$$

$$\Bigg(4\;(-1+n)\;\Bigg(\frac{2+2\;\mathrm{Sb}\;(-1+\alpha)+4\;\alpha^2+\sigma-\mathrm{nAB}\;\sigma-2\;\mathrm{nBB}\;\sigma-2\;\alpha\;(2+\mathrm{Sa}+\mathrm{nAA}\;\sigma-\mathrm{nBB}\;\sigma)}{2\;(-1+n)}+$$

$$\frac{1}{-4+8\,n}\;(\sigma\;(2\;(-1+\mathrm{nAB})^2\;\sigma+(-1+\mathrm{nAB})\;(-2+2\;\mathrm{Sa}+2\;\mathrm{Sb}+n\;\sigma+4\;\mathrm{nAA}\;\sigma+4\;\mathrm{nBB}\;\sigma-$$

$$n\;\mathrm{niA}\;\sigma-n\;\mathrm{niB}\;\sigma)-2\;(-2\;\mathrm{nBB}^2\;\sigma-2\;\mathrm{nAA}\;(\mathrm{Sa}-\alpha+\mathrm{nAA}\;\sigma)+\mathrm{nBB}\;(2-2\;\mathrm{Sb}-2\;\alpha+n$$

$$(-1+\mathrm{niB})\;\sigma)+n\;((-1+\mathrm{niB})\;(-1+\mathrm{Sb}+\alpha)+\mathrm{niA}\;(\mathrm{Sa}-\alpha+\mathrm{nAA}\;\sigma))))))\Bigg)^2\Bigg)\Bigg/$$

$$(n\;(2+2\;\mathrm{Sb}\;(-1+\alpha)+4\;\alpha^2+\sigma-\mathrm{nAB}\;\sigma-2\;\mathrm{nBB}\;\sigma-2\;\alpha\;(2+\mathrm{Sa}+\mathrm{nAA}\;\sigma-\mathrm{nBB}\;\sigma)))-$$

$$\Bigg(4\;(-1+n)\;\Bigg(\frac{2+2\;\mathrm{Sb}\;(-1+\alpha)+4\;\alpha^2-\mathrm{nAB}\;\sigma-2\;\mathrm{nBB}\;\sigma-2\;\alpha\;(2+\mathrm{Sa}+\mathrm{nAA}\;\sigma-\mathrm{nBB}\;\sigma)}{2\;(-1+n)}+$$

$$\frac{1}{-4+8\,n}\;(\sigma\;(2\;\mathrm{nAB}^2\;\sigma+\mathrm{nAB}\;(-2+2\;\mathrm{Sa}+2\;\mathrm{Sb}+4\;\mathrm{nAA}\;\sigma+4\;\mathrm{nBB}\;\sigma-n\;\mathrm{niA}\;\sigma-n\;\mathrm{niB}\;\sigma)-$$

$$2\;(-2\;\mathrm{nBB}^2\;\sigma-2\;\mathrm{nAA}\;(\mathrm{Sa}-\alpha+\mathrm{nAA}\;\sigma)+\mathrm{nBB}\;(2-2\;\mathrm{Sb}-2\;\alpha+n\;\mathrm{niB}\;\sigma)+$$

$$n\;(\mathrm{niB}\;(-1+\mathrm{Sb}+\alpha)+\mathrm{niA}\;(\mathrm{Sa}-\alpha+\mathrm{nAA}\;\sigma))))))\Bigg)^2\Bigg)\Bigg/$$

$$(n\;(2+2\;\mathrm{Sb}\;(-1+\alpha)+4\;\alpha^2-\mathrm{nAB}\;\sigma-2\;\mathrm{nBB}\;\sigma-2\;\alpha\;(2+\mathrm{Sa}+\mathrm{nAA}\;\sigma-\mathrm{nBB}\;\sigma)))\Bigg);$$

$$\Delta\mathrm{profitinterdel}[j]=\frac{1}{2}\;\Bigg(-(-1+\mathrm{njA}+\mathrm{njB})\;\sigma^2+(\mathrm{njA}+\mathrm{njB})\;\sigma^2+$$

$$\Bigg(4\;(-1+n)\;\Bigg(\frac{2+2\;\mathrm{Sb}\;(-1+\alpha)+4\;\alpha^2+\sigma-\mathrm{nAB}\;\sigma-2\;\mathrm{nBB}\;\sigma-2\;\alpha\;(2+\mathrm{Sa}+\mathrm{nAA}\;\sigma-\mathrm{nBB}\;\sigma)}{2\;(-1+n)}+$$

$$\frac{1}{-4+8\,n}\;(\sigma\;(2\;(-1+\mathrm{nAB})^2\;\sigma+(-1+\mathrm{nAB})\;(-2+2\;\mathrm{Sa}+2\;\mathrm{Sb}+n\;\sigma+4\;\mathrm{nAA}\;\sigma+4\;\mathrm{nBB}\;\sigma-$$

$$n\;\mathrm{njA}\;\sigma-n\;\mathrm{njB}\;\sigma)-2\;(-2\;\mathrm{nBB}^2\;\sigma-2\;\mathrm{nAA}\;(\mathrm{Sa}-\alpha+\mathrm{nAA}\;\sigma)+\mathrm{nBB}\;(2-2\;\mathrm{Sb}-2\;\alpha+$$

$$n\;\mathrm{njB}\;\sigma)+n\;(\mathrm{njB}\;(-1+\mathrm{Sb}+\alpha)+(-1+\mathrm{njA})\;(\mathrm{Sa}-\alpha+\mathrm{nAA}\;\sigma))))))\Bigg)^2\Bigg)\Bigg/$$

$$(n\;(2+2\;\mathrm{Sb}\;(-1+\alpha)+4\;\alpha^2+\sigma-\mathrm{nAB}\;\sigma-2\;\mathrm{nBB}\;\sigma-2\;\alpha\;(2+\mathrm{Sa}+\mathrm{nAA}\;\sigma-\mathrm{nBB}\;\sigma)))-$$

```
(4 (-1 + n) ((2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)) / (2 (-1 + n)) +
      1/(-4 + 8 n) (σ (2 nAB^2 σ + nAB (-2 + 2 Sa + 2 Sb + 4 nAA σ + 4 nBB σ - n njA σ - n njB σ) -
          2 (-2 nBB^2 σ - 2 nAA (Sa - α + nAA σ) + nBB (2 - 2 Sb - 2 α + n njB σ) +
              n (njB (-1 + Sb + α) + njA (Sa - α + nAA σ)))))^2) /
   (n (2 + 2 Sb (-1 + α) + 4 α^2 - nAB σ - 2 nBB σ - 2 α (2 + Sa + nAA σ - nBB σ)))) ;

"Basic syntax for checking pairwise stability for focal networks"
"Area where parameter constraints are satisfied"
p1 = InequalityPlot[
  {0 < Sa < α < 1, 0 < Sb < 1 - α < 1, 0 < σ < (α - Sa)/(m * (n - 1)) < 1, 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1},
  {σ, 0, 1}, {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1, ColorOutput → GrayLevel]
"Area where complete network is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = (m - 1);
niB = (n - m);
njA = m;
njB = (n - m - 1);
nAA = m (m - 1)/2;
nBB = (n - m) * (n - m - 1)/2;
nAB = m * (n - m);
p2 = InequalityPlot[{Δprofitintra Adel[i] ≤ 0, ΔprofitintraBdel[j] ≤ 0,
   Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,
   0 < σ < (α - Sa)/(m * (n - 1)) < 1, 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1}, {σ, 0, 1},
  {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1, ColorOutput → GrayLevel]
"Area where empty network is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = 0;
njA = 0;
njB = 0;
nAA = 0;
nBB = 0;
nAB = 0;
p3 = InequalityPlot[{(ΔprofitintraAform[i] ≤ 0 && ΔprofitintraBform[j] ≤ 0) &&
    ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
      (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
      (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) && (0 < Sa < α < 1 &&
      0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa)/(m * (n - 1)) < 1 && 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1)},
  {σ, 0, 1}, {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1,
```

```
  ColorOutput → GrayLevel]
"Area where network gAB0 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = (m - 1);
niB = 0;
njA = 0;
njB = (n - m - 1);
nAA = m * (m - 1) / 2;
nBB = (n - m) * (n - m - 1) / 2;
nAB = 0;
p4 = InequalityPlot[{(ΔprofitintraAdel[i] ≤ 0 && ΔprofitintraBdel[j] ≤ 0) &&
    ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
      (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
      (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) && (0 < Sa < α < 1 &&
      0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa) / (m * (n - 1)) < 1 && 0 < σ < (1 - α - Sb) / ((n - m) * (2 * n - 2 * m - 1)) < 1)},
  {σ, 0, 1}, {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1,
  ColorOutput → GrayLevel]
"Area where gA00 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = (m - 1);
niB = 0;
njA = 0;
njB = 0;
nAA = m * (m - 1) / 2;
nBB = 0;
nAB = 0;
p5 = InequalityPlot[{(ΔprofitintraAdel[i] ≤ 0 && ΔprofitintraBform[j] ≤ 0) &&
    ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
      (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
      (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) && (0 < Sa < α < 1 &&
      0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa) / (m * (n - 1)) < 1 && 0 < σ < (1 - α - Sb) / ((n - m) * (2 * n - 2 * m - 1)) < 1)},
  {σ, 0, 1}, {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1,
  ColorOutput → GrayLevel]
"Area where g0B0 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = 0;
njA = 0;
njB = (n - m - 1);
nAA = 0;
nBB = (n - m) * (n - m - 1) / 2;
nAB = 0;
p6 = InequalityPlot[{(ΔprofitintraAform[i] ≤ 0 && ΔprofitintraBdel[j] ≤ 0) &&
```

```
    ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
      (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
      (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) && (0 < Sa < α < 1 &&
      0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa)/(m * (n - 1)) < 1 && 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1)},
  {σ, 0, 1}, {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1,
  ColorOutput → GrayLevel]
"Area where gA01 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = (m - 1);
niB = (n - m);
njA = m;
njB = 0;
nAA = (m * (m - 1))/2;
nBB = 0;
nAB = m * (n - m);
p7 = InequalityPlot[{ΔprofitintraAdel[i] ≤ 0, ΔprofitintraBform[j] ≤ 0,
   Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,
   0 < σ < (α - Sa)/(m * (n - 1)) < 1, 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1}, {σ, 0, 1},
  {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1, ColorOutput → GrayLevel]
"Area where g0B1 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = (n - m);
njA = m;
njB = (n - m - 1);
nAA = 0;
nBB = ((n - m) * (n - m - 1))/2;
nAB = m * (n - m);
p8 = InequalityPlot[{ΔprofitintraAform[i] ≤ 0, ΔprofitintraBdel[j] ≤ 0,
   Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,
   0 < σ < (α - Sa)/(m * (n - 1)) < 1, 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1}, {σ, 0, 1},
  {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1, ColorOutput → GrayLevel]
"Area where g001 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = (n - m);
njA = m;
njB = 0;
nAA = 0;
nBB = 0;
nAB = m * (n - m);
p9 = InequalityPlot[{ΔprofitintraAform[i] ≤ 0, ΔprofitintraBform[j] ≤ 0,
   Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,
```

```
    0 < σ < (α - Sa)/(m * (n - 1)) < 1, 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1}, {σ, 0, 1},
   {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1, ColorOutput → GrayLevel]
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
Show[p1, p2, p3, p4, p5, p6, p7, p8, p9, ColorOutput → None]
Clear[p1, p2, p3, p4, p5, p6, p7, p8, p9]


Now, we will apply this syntax to four different cases.

"Perfect Symmetry"
Sa = S
Sb = S
α = 0.5
m = 3
n = 6
FA = 0
FB = 0
"Area where parameter constraints are satisfied"
p1 = InequalityPlot[
   {0 < Sa < α < 1, 0 < Sb < 1 - α < 1, 0 < σ < (α - Sa)/(m * (n - 1)) < 1, 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1},
   {σ, 0, 1}, {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1, ColorOutput → GrayLevel]
"Area where complete network is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = (m - 1);
niB = (n - m);
njA = m;
njB = (n - m - 1);
nAA = (m (m - 1))/2;
nBB = ((n - m) * (n - m - 1))/2;
nAB = m * (n - m);
p2 = InequalityPlot[{ΔprofitintraAdel[i] ≤ 0, ΔprofitintraBdel[j] ≤ 0,
    Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,
    0 < σ < (α - Sa)/(m * (n - 1)) < 1, 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1}, {σ, 0, 1},
   {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1, ColorOutput → GrayLevel]
"Area where empty network is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = 0;
njA = 0;
njB = 0;
nAA = 0;
nBB = 0;
nAB = 0;
p3 = InequalityPlot[{(ΔprofitintraAform[i] ≤ 0 && ΔprofitintraBform[j] ≤ 0) &&
     ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
       (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
```

```
      (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) && (0 < Sa < α < 1 &&
      0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa)/(m * (n - 1)) < 1 && 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1)},
  {σ, 0, 1}, {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1,
  ColorOutput → GrayLevel]
"Area where network gAB0 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = (m - 1);
niB = 0;
njA = 0;
njB = (n - m - 1);
nAA = (m * (m - 1))/2;
nBB = ((n - m) * (n - m - 1))/2;
nAB = 0;
p4 = InequalityPlot[{(ΔprofitintraAdel[i] ≤ 0 && ΔprofitintraBdel[j] ≤ 0) &&
    ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
      (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
      (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) && (0 < Sa < α < 1 &&
      0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa)/(m * (n - 1)) < 1 && 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1)},
  {σ, 0, 1}, {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1,
  ColorOutput → GrayLevel]
"Area where gA00 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = (m - 1);
niB = 0;
njA = 0;
njB = 0;
nAA = (m * (m - 1))/2;
nBB = 0;
nAB = 0;
p5 = InequalityPlot[{(ΔprofitintraAdel[i] ≤ 0 && ΔprofitintraBform[j] ≤ 0) &&
    ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
      (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
      (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) && (0 < Sa < α < 1 &&
      0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa)/(m * (n - 1)) < 1 && 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1)},
  {σ, 0, 1}, {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1,
  ColorOutput → GrayLevel]
"Area where g0B0 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = 0;
njA = 0;
njB = (n - m - 1);
```

```
nAA = 0;
nBB = (n - m) * (n - m - 1) / 2;
nAB = 0;
p6 = InequalityPlot[{(ΔprofitintraAform[i] ≤ 0 && ΔprofitintraBdel[j] ≤ 0) &&
    ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
      (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
      (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) && (0 < Sa < α < 1 &&
      0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa) / (m * (n - 1)) < 1 && 0 < σ < (1 - α - Sb) / ((n - m) * (2 * n - 2 * m - 1)) < 1)},
  {σ, 0, 1}, {S, 0, 1}, AxesLabel → {σ, s_B}, AspectRatio → 1,
  ColorOutput → GrayLevel]
"Area where g_A01 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = (m - 1);
niB = (n - m);
njA = m;
njB = 0;
nAA = m * (m - 1) / 2;
nBB = 0;
nAB = m * (n - m);
p7 = InequalityPlot[{ΔprofitintraAdel[i] ≤ 0, ΔprofitintraBform[j] ≤ 0,
   Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,
   0 < σ < (α - Sa) / (m * (n - 1)) < 1, 0 < σ < (1 - α - Sb) / ((n - m) * (2 * n - 2 * m - 1)) < 1}, {σ, 0, 1},
  {S, 0, 1}, AxesLabel → {σ, s_B}, AspectRatio → 1, ColorOutput → GrayLevel]
"Area where g_0B1 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = (n - m);
njA = m;
njB = (n - m - 1);
nAA = 0;
nBB = (n - m) * (n - m - 1) / 2;
nAB = m * (n - m);
p8 = InequalityPlot[{ΔprofitintraAform[i] ≤ 0, ΔprofitintraBdel[j] ≤ 0,
   Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,
   0 < σ < (α - Sa) / (m * (n - 1)) < 1, 0 < σ < (1 - α - Sb) / ((n - m) * (2 * n - 2 * m - 1)) < 1}, {σ, 0, 1},
  {S, 0, 1}, AxesLabel → {σ, s_B}, AspectRatio → 1, ColorOutput → GrayLevel]
"Area where g_001 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = (n - m);
njA = m;
njB = 0;
nAA = 0;
```

```
nBB = 0;
nAB = m * (n - m);
p9 = InequalityPlot[{Δprofitintra Aform[i] ≤ 0, ΔprofitintraBform[j] ≤ 0,
   Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,
   0 < σ < (α - Sa)/(m * (n - 1)) < 1, 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1}, {σ, 0, 1},
  {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1, ColorOutput → GrayLevel]
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
Show[p1, p2, p3, p4, p5, p6, p7, p8, p9, ColorOutput → None]
Clear[p1, p2, p3, p4, p5, p6, p7, p8, p9]
```

Perfect Symmetry

S

S

0.5

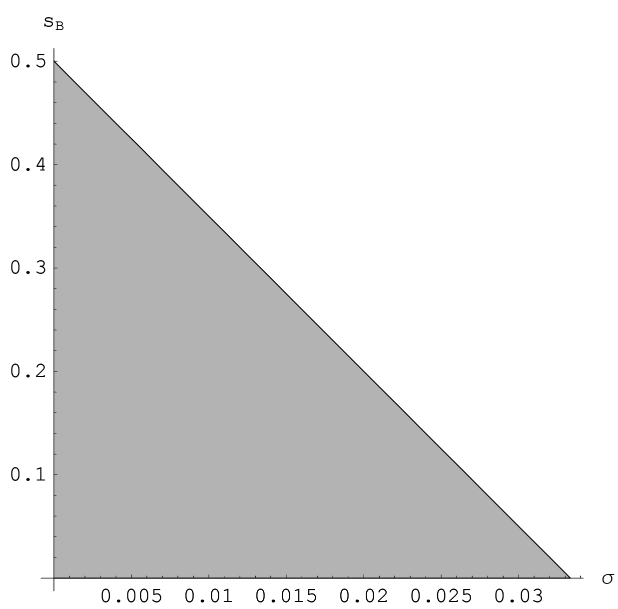
3

6

0

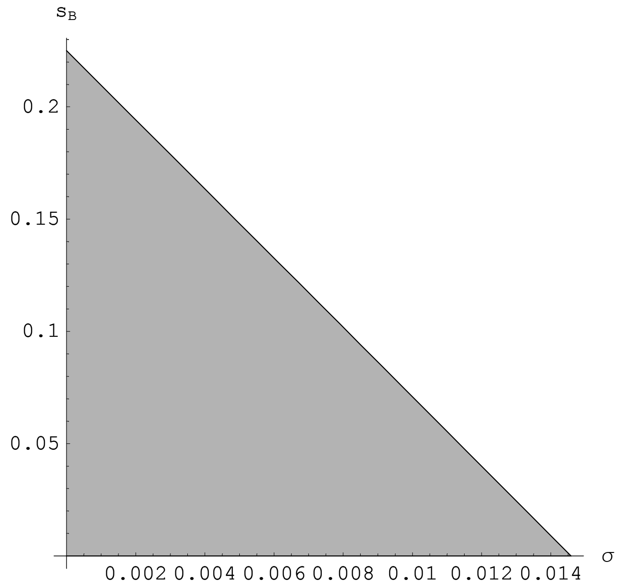
0

Area where parameter constraints are satisfied



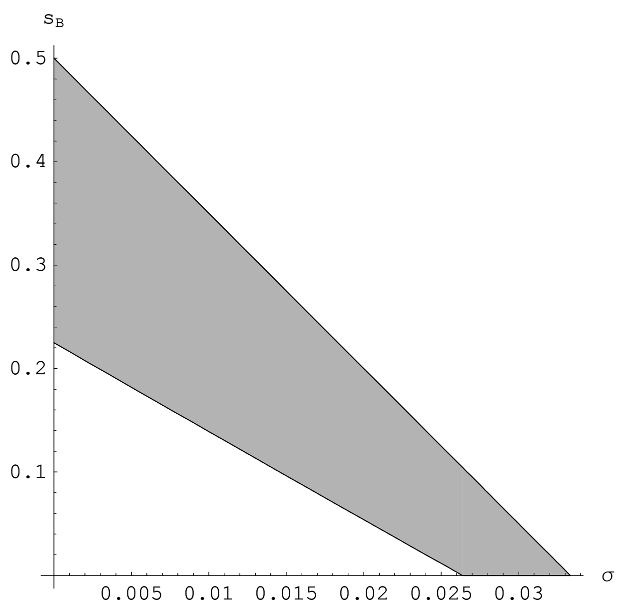

- Graphics -

Area where complete network is pairwise stable



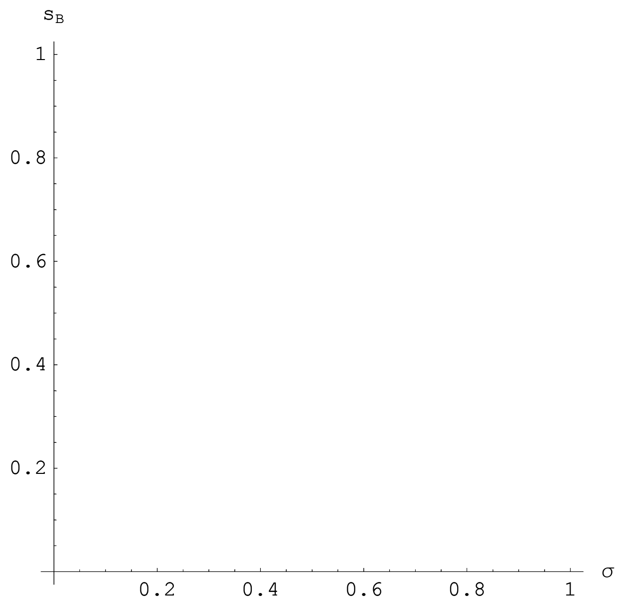

- Graphics -

Area where empty network is pairwise stable



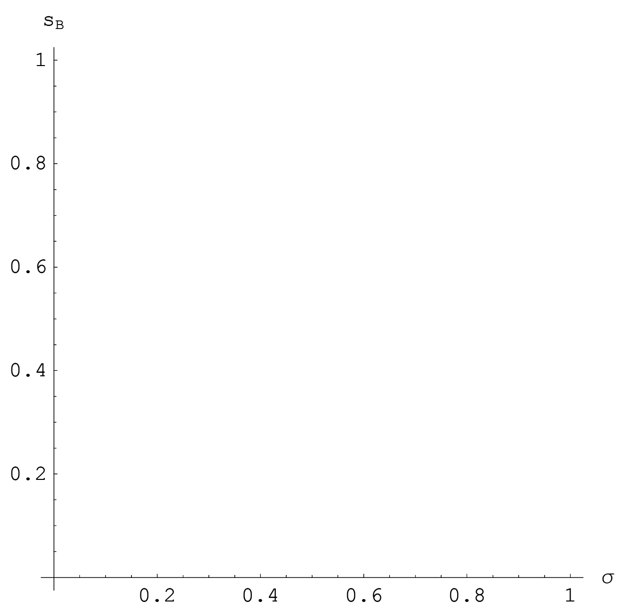

- Graphics -

Area where network $g_{AB0}$ is pairwise stable



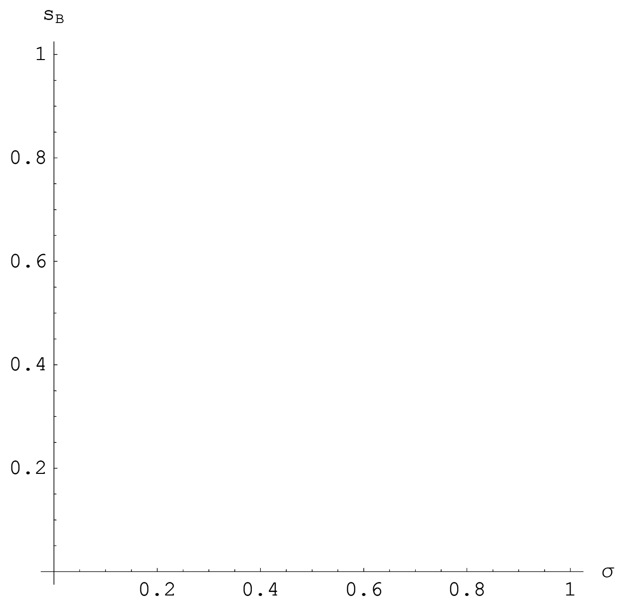

- Graphics -

Area where $g_{A00}$ is pairwise stable



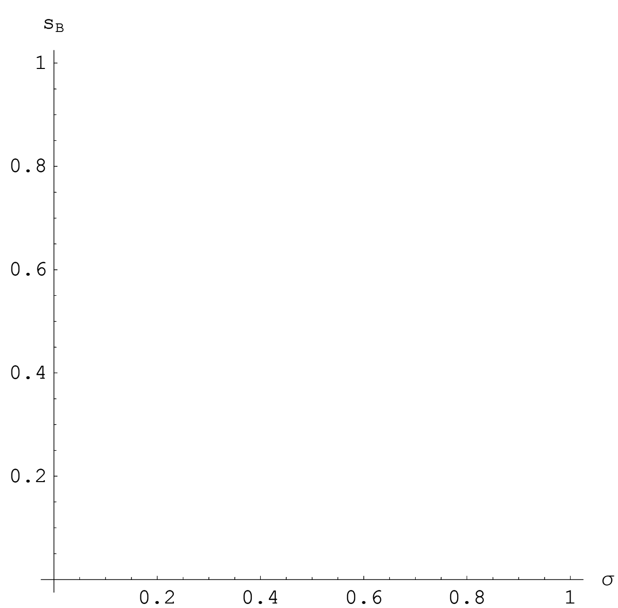

- Graphics -

Area where $g_{0B0}$ is pairwise stable



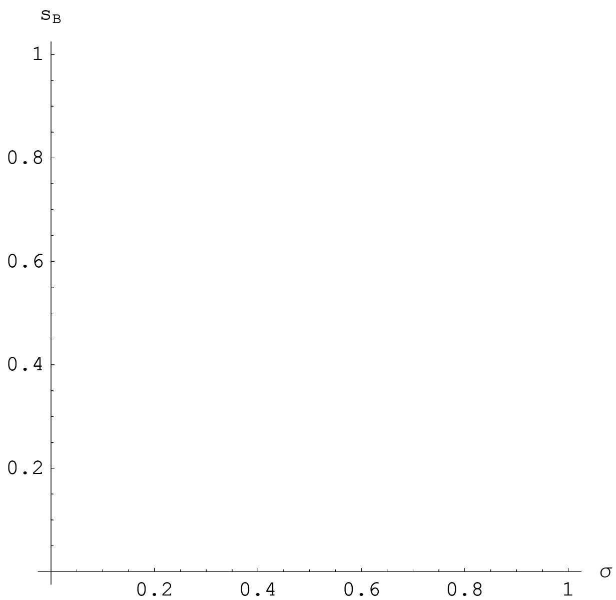

- Graphics -

Area where $g_{A01}$ is pairwise stable



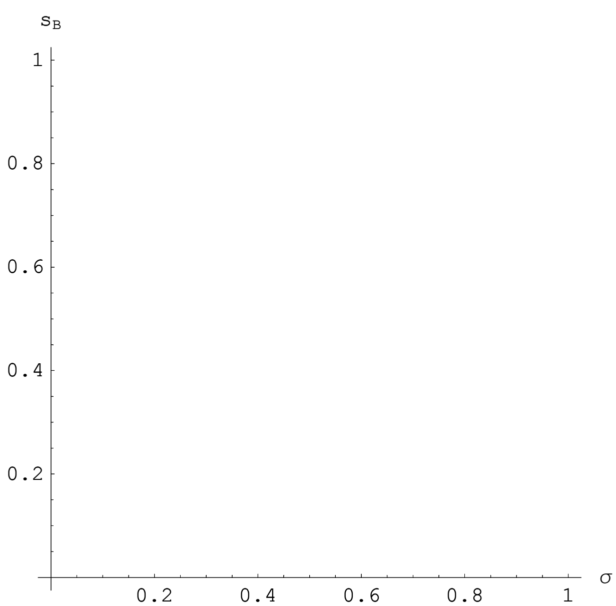

- Graphics -

Area where $g_{0B1}$ is pairwise stable

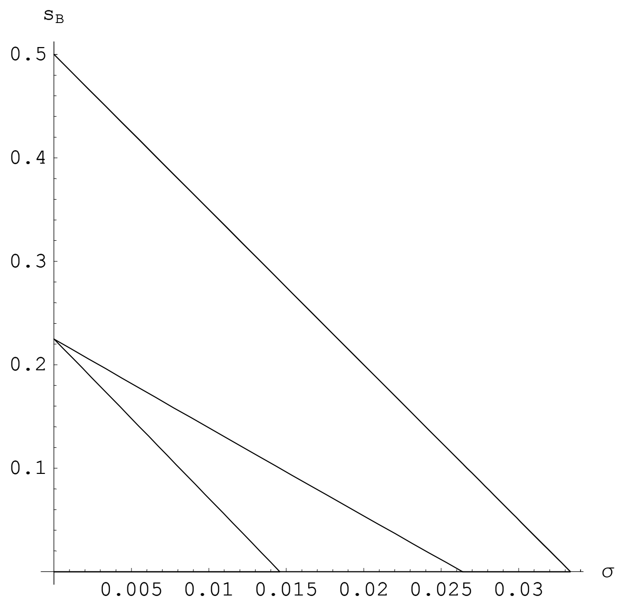



- Graphics -

Area where $g_{001}$ is pairwise stable




- Graphics -




```
- Graphics -

"Unequal network capacity"
Sb = S
α = 0.5
m = 3
n = 6
FA = 0
FB = 0
"Area where parameter constraints are satisfied"
p1 = InequalityPlot3D[{0 < Sa < α < 1, 0 < Sb < 1 - α < 1, 0 < σ < (α - Sa)/(m * (n - 1)) < 1,
    0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1, Sa > S}, {σ, 0, 0.1}, {S, 0, 1},
   {Sa, 0, 1}, AxesLabel → {σ, Sb, Sa}, AspectRatio → 1, ColorOutput → GrayLevel]
"Area where complete network is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = (m - 1);
niB = (n - m);
njA = m;
njB = (n - m - 1);
nAA = m (m - 1)/2;
nBB = (n - m) * (n - m - 1)/2;
nAB = m * (n - m);
p2 = InequalityPlot3D[{ΔprofitintraAdel[i] ≤ 0, ΔprofitintraBdel[j] ≤ 0,
    Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,
    0 < σ < (α - Sa)/(m * (n - 1)) < 1, 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1, Sa > S}, {σ, 0, 0.1}, {S, 0, 1},
   {Sa, 0, 1}, AxesLabel → {σ, Sb, Sa}, AspectRatio → 1, ColorOutput → GrayLevel]
"Area where empty network is pairwise stable"
```

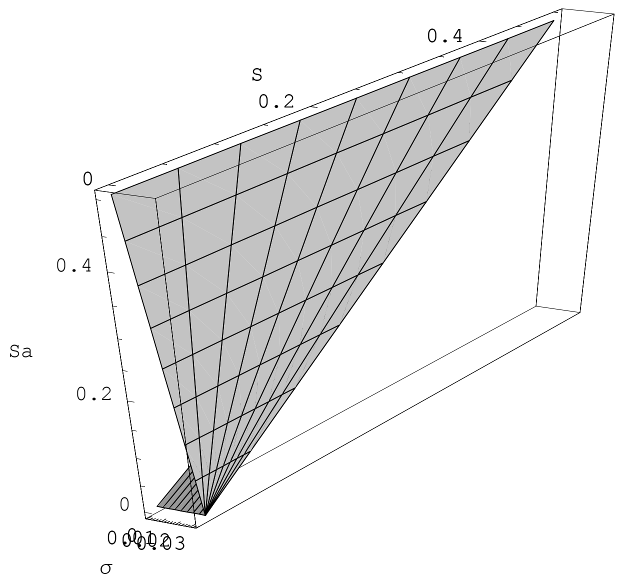
```
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = 0;
njA = 0;
njB = 0;
nAA = 0;
nBB = 0;
nAB = 0;
p3 = InequalityPlot3D[{(ΔprofitintraAform[i] ≤ 0 && ΔprofitintraBform[j] ≤ 0) &&
    ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
      (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
      (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) &&
    (0 < Sa < α < 1 && 0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa)/(m * (n - 1)) < 1 &&
       0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1 && Sa > S)},
  {σ, 0, 0.1}, {S, 0, 1}, {Sa, 0, 1}, AxesLabel → {σ, Sb, Sa},
  AspectRatio → 1,
  ColorOutput → GrayLevel]
"Area where network gAB0 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = (m - 1);
niB = 0;
njA = 0;
njB = (n - m - 1);
nAA = (m * (m - 1))/2;
nBB = ((n - m) * (n - m - 1))/2;
nAB = 0;
p4 = InequalityPlot3D[{(ΔprofitintraAdel[i] ≤ 0 && ΔprofitintraBdel[j] ≤ 0) &&
    ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
      (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
      (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) &&
    (0 < Sa < α < 1 && 0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa)/(m * (n - 1)) < 1 &&
       0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1 && Sa > S)},
  {σ, 0, 0.1}, {S, 0, 1}, {Sa, 0, 1}, AxesLabel → {σ, Sb, Sa},
  AspectRatio → 1,
  ColorOutput → GrayLevel]

"Area where gA00 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = (m - 1);
niB = 0;
njA = 0;
njB = 0;
nAA = (m * (m - 1))/2;
```

```
nBB = 0;
nAB = 0;
p5 = InequalityPlot3D[{(ΔprofitintraAdel[i] ≤ 0 && ΔprofitintraBform[j] ≤ 0) &&
    ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
      (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
      (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) &&
    (0 < Sa < α < 1 && 0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa)/(m * (n - 1)) < 1 &&
      0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1 && Sa > S)},
  {σ, 0, 0.1}, {S, 0, 1}, {Sa, 0, 1}, AxesLabel → {σ, Sb, Sa},
  AspectRatio → 1,
  ColorOutput → GrayLevel]

"Area where g0 B0 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = 0;
njA = 0;
njB = (n - m - 1);
nAA = 0;
nBB = (n - m) * (n - m - 1)/2;
nAB = 0;
p6 = InequalityPlot3D[{(ΔprofitintraAform[i] ≤ 0 && ΔprofitintraBdel[j] ≤ 0) &&
    ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
      (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
      (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) &&
    (0 < Sa < α < 1 && 0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa)/(m * (n - 1)) < 1 &&
      0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1 && Sa > S)},
  {σ, 0, 0.1}, {S, 0, 1}, {Sa, 0, 1}, AxesLabel → {σ, Sb, Sa},
  AspectRatio → 1,
  ColorOutput → GrayLevel]

"Area where gA01 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = (m - 1);
niB = (n - m);
njA = m;
njB = 0;
nAA = m * (m - 1)/2;
nBB = 0;
nAB = m * (n - m);
p7 = InequalityPlot3D[{ΔprofitintraAdel[i] ≤ 0, ΔprofitintraBform[j] ≤ 0,
   Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,
   0 < σ < (α - Sa)/(m * (n - 1)) < 1, 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1, Sa > S}, {σ, 0, 0.1}, {S, 0, 1},
```

```
  {Sa, 0, 1}, AxesLabel → {σ, Sb, Sa}, AspectRatio → 1, ColorOutput → GrayLevel]

"Area where g0B1 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = (n - m);
njA = m;
njB = (n - m - 1);
nAA = 0;
nBB = (n - m) * (n - m - 1) / 2;
nAB = m * (n - m);
p8 = InequalityPlot3D[{Δprofitintra Aform[i] ≤ 0, ΔprofitintraBdel[j] ≤ 0,
   Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,
   0 < σ < (α - Sa) / (m * (n - 1)) < 1, 0 < σ < (1 - α - Sb) / ((n - m) * (2 * n - 2 * m - 1)) < 1, Sa > S}, {σ, 0, 0.1}, {S, 0, 1},
  {Sa, 0, 1}, AxesLabel → {σ, Sb, Sa}, AspectRatio → 1, ColorOutput → GrayLevel]

"Area where g001 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = (n - m);
njA = m;
njB = 0;
nAA = 0;
nBB = 0;
nAB = m * (n - m);
p9 = InequalityPlot3D[{ΔprofitintraAform[i] ≤ 0, ΔprofitintraBform[j] ≤ 0,
   Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,
   0 < σ < (α - Sa) / (m * (n - 1)) < 1, 0 < σ < (1 - α - Sb) / ((n - m) * (2 * n - 2 * m - 1)) < 1, Sa > S}, {σ, 0, 0.1}, {S, 0, 1},
  {Sa, 0, 1}, AxesLabel → {σ, Sb, Sa}, AspectRatio → 1, ColorOutput → GrayLevel]
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
Clear[Sa, Sb, α, m, n, FA, FB]
Clear[p1, p2, p3, p4, p5, p6, p7, p8, p9]
```

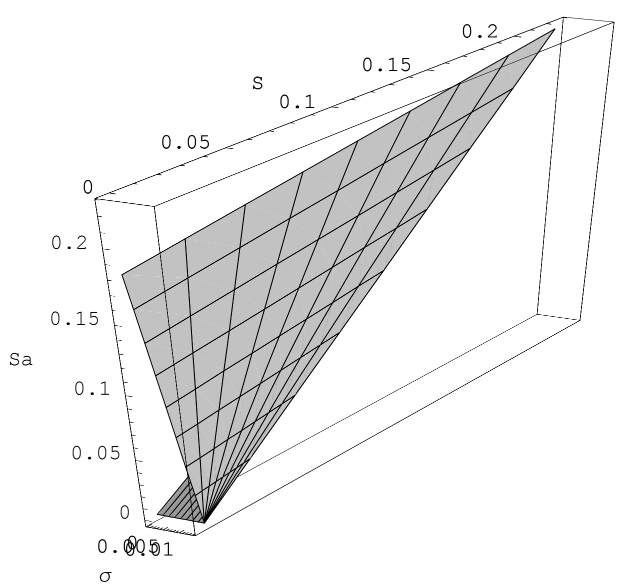
```
Unequal network capacity

S

0.5

3

6

0

0
```

Area where parameter constraints are satisfied



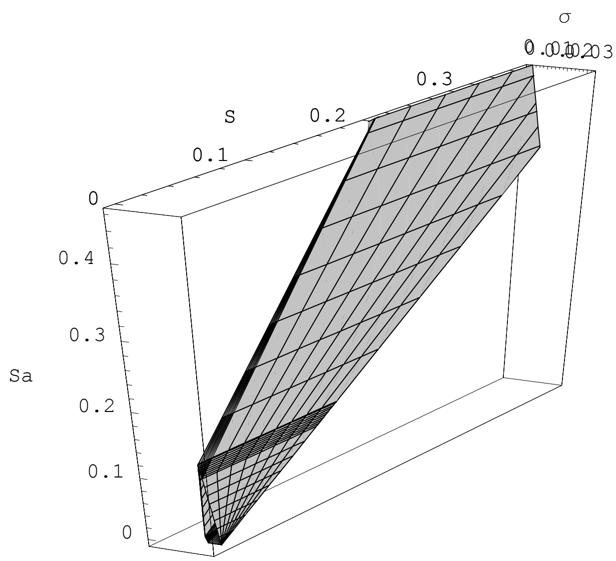

- Graphics3D -

Area where complete network is pairwise stable



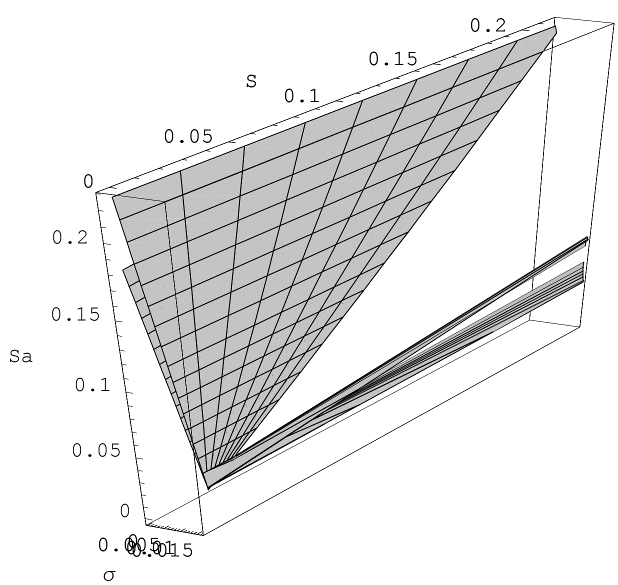

- Graphics3D -

Area where empty network is pairwise stable



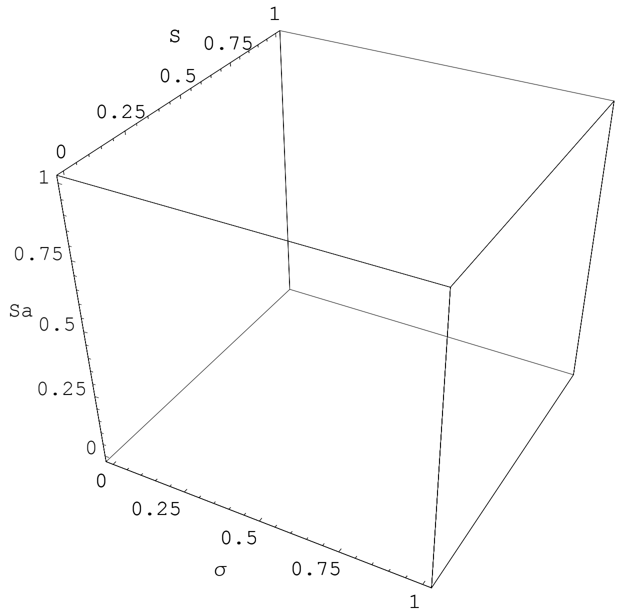

- Graphics3D -

Area where network $g_{AB0}$ is pairwise stable



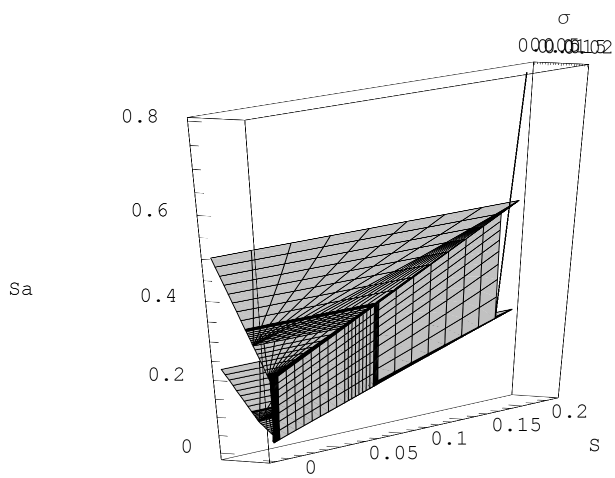

- Graphics3D -

Area where $g_{A00}$ is pairwise stable



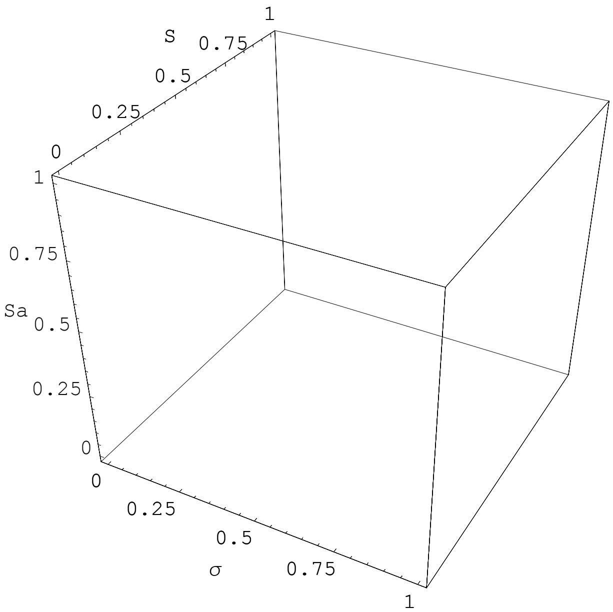

- Graphics3D -

Area where $g_{0B0}$ is pairwise stable



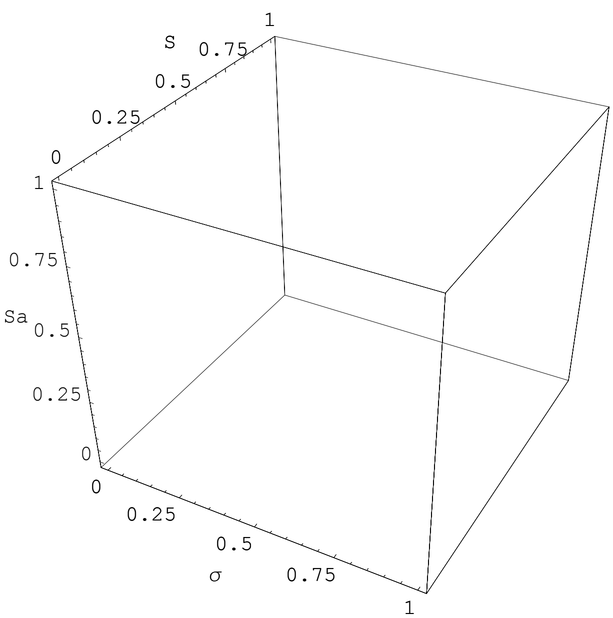

- Graphics3D -

Area where $g_{A01}$ is pairwise stable



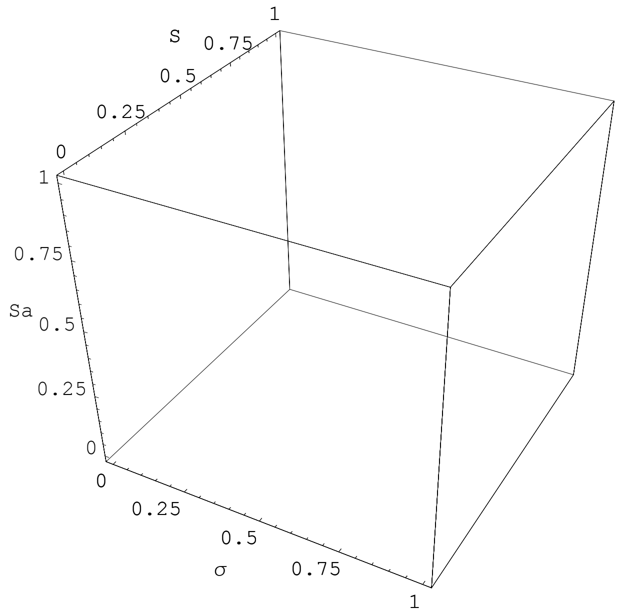

- Graphics3D -

Area where $g_{0B1}$ is pairwise stable




- Graphics3D -

Area where $g_{001}$ is pairwise stable




- Graphics3D -

**"Unequal number of firms"**
**Sa = S**
**Sb = S**
**α = 0.5**
**m = 4**
**n = 6**
**FA = 0**
**FB = 0**
**"Area where parameter constraints are satisfied"**
**p1 = InequalityPlot[**

$$\left\{0 < Sa < \alpha < 1,\ 0 < Sb < 1 - \alpha < 1,\ 0 < \sigma < \frac{(\alpha - Sa)}{m * (n-1)} < 1,\ 0 < \sigma < \frac{(1 - \alpha - Sb)}{(n-m) * (2 * n - 2 * m - 1)} < 1\right\},$$

**{σ, 0, 0.1}, {S, 0, 1}, AxesLabel → {σ, $s_B$}, AspectRatio → 1, ColorOutput → GrayLevel]**
**"Area where complete network is pairwise stable"**
**Clear[niA, niB, njA, njB, nAA, nBB, nAB]**
**niA = (m - 1);**
**niB = (n - m);**
**njA = m;**
**njB = (n - m - 1);**

$$nAA = \frac{m\,(m-1)}{2};$$

$$nBB = \frac{(n-m) * (n-m-1)}{2};$$

**nAB = m * (n - m);**
**p2 = InequalityPlot[{ΔprofitintraAdel[i] ≤ 0, ΔprofitintraBdel[j] ≤ 0,**
**Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,**

$$0 < \sigma < \frac{(\alpha - Sa)}{m * (n-1)} < 1,\ 0 < \sigma < \frac{(1 - \alpha - Sb)}{(n-m) * (2 * n - 2 * m - 1)} < 1\},\ \{\sigma, 0, 0.1\},$$

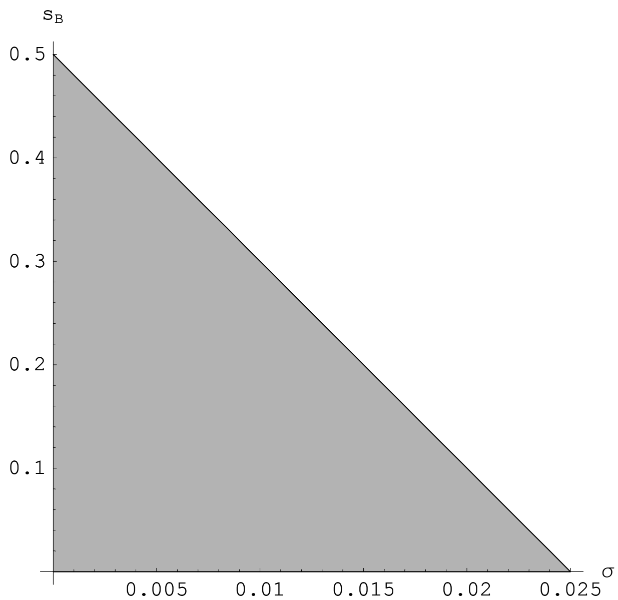
**{S, 0, 1}, AxesLabel → {σ, $s_B$}, AspectRatio → 1, ColorOutput → GrayLevel]**

```
"Area where empty network is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = 0;
njA = 0;
njB = 0;
nAA = 0;
nBB = 0;
nAB = 0;
p3 = InequalityPlot[{(ΔprofitintraAform[i] ≤ 0 && ΔprofitintraBform[j] ≤ 0) &&
    ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
      (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
      (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) && (0 < Sa < α < 1 &&
      0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa)/(m * (n - 1)) < 1 && 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1)},
  {σ, 0, 0.1}, {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1,
  ColorOutput → GrayLevel]
"Area where network gAB0 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = (m - 1);
niB = 0;
njA = 0;
njB = (n - m - 1);
nAA = (m * (m - 1))/2;
nBB = ((n - m) * (n - m - 1))/2;
nAB = 0;
p4 = InequalityPlot[{(ΔprofitintraAdel[i] ≤ 0 && ΔprofitintraBdel[j] ≤ 0) &&
    ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
      (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
      (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) && (0 < Sa < α < 1 &&
      0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa)/(m * (n - 1)) < 1 && 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1)},
  {σ, 0, 0.1}, {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1,
  ColorOutput → GrayLevel]
"Area where gA00 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = (m - 1);
niB = 0;
njA = 0;
njB = 0;
nAA = (m * (m - 1))/2;
nBB = 0;
nAB = 0;
p5 = InequalityPlot[{(ΔprofitintraAdel[i] ≤ 0 && ΔprofitintraBform[j] ≤ 0) &&
    ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
```

```
      (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
      (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) && (0 < Sa < α < 1 &&
      0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa)/(m * (n - 1)) < 1 && 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1)},
  {σ, 0, 0.1}, {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1,
  ColorOutput → GrayLevel]
"Area where g0B0 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = 0;
njA = 0;
njB = (n - m - 1);
nAA = 0;
nBB = ((n - m) * (n - m - 1))/2;
nAB = 0;
p6 = InequalityPlot[{(ΔprofitintraAform[i] ≤ 0 && ΔprofitintraBdel[j] ≤ 0) &&
    ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
      (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
      (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) && (0 < Sa < α < 1 &&
      0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa)/(m * (n - 1)) < 1 && 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1)},
  {σ, 0, 0.1}, {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1,
  ColorOutput → GrayLevel]
"Area where gA01 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = (m - 1);
niB = (n - m);
njA = m;
njB = 0;
nAA = (m * (m - 1))/2;
nBB = 0;
nAB = m * (n - m);
p7 = InequalityPlot[{ΔprofitintraAdel[i] ≤ 0, ΔprofitintraBform[j] ≤ 0,
   Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,
   0 < σ < (α - Sa)/(m * (n - 1)) < 1, 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1}, {σ, 0, 0.1},
  {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1, ColorOutput → GrayLevel]
"Area where g0B1 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = (n - m);
njA = m;
njB = (n - m - 1);
nAA = 0;
nBB = ((n - m) * (n - m - 1))/2;
nAB = m * (n - m);
```

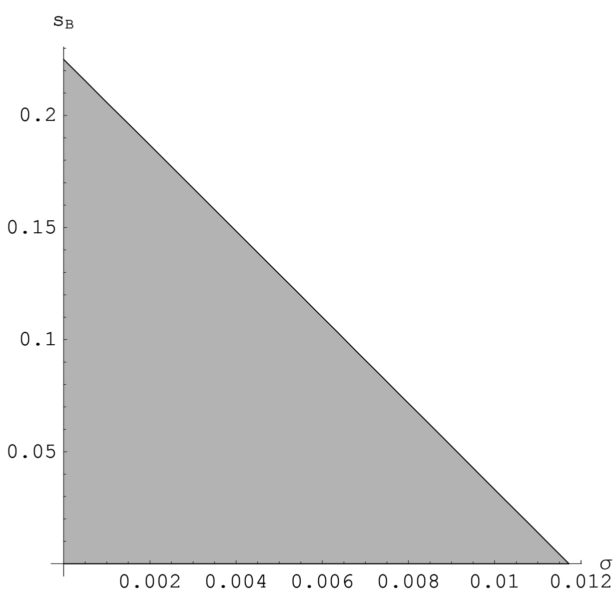
```
p8 = InequalityPlot[{Δprofitintra Aform[i] ≤ 0, ΔprofitintraBdel[j] ≤ 0,
   Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,
   0 < σ < (α - Sa)/(m * (n - 1)) < 1, 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1}, {σ, 0, 0.1},
  {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1, ColorOutput → GrayLevel]
"Area where g001 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = (n - m);
njA = m;
njB = 0;
nAA = 0;
nBB = 0;
nAB = m * (n - m);
p9 = InequalityPlot[{ΔprofitintraAform[i] ≤ 0, ΔprofitintraBform[j] ≤ 0,
   Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,
   0 < σ < (α - Sa)/(m * (n - 1)) < 1, 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1}, {σ, 0, 0.1},
  {S, 0, 1}, AxesLabel → {σ, sB}, AspectRatio → 1, ColorOutput → GrayLevel]
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
Clear[Sa, Sb, α, m, n, FA, FB]
Show[p1, p2, p3, p4, p5, p6, p7, p8, p9, ColorOutput → None]
```

```
Unequal number of firms

S

S

0.5

4

6

0

0

Area where parameter constraints are satisfied
```



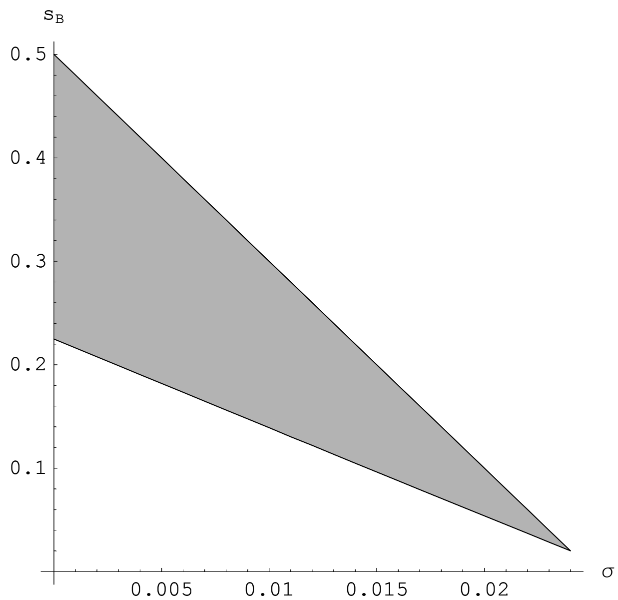

- Graphics -

Area where complete network is pairwise stable



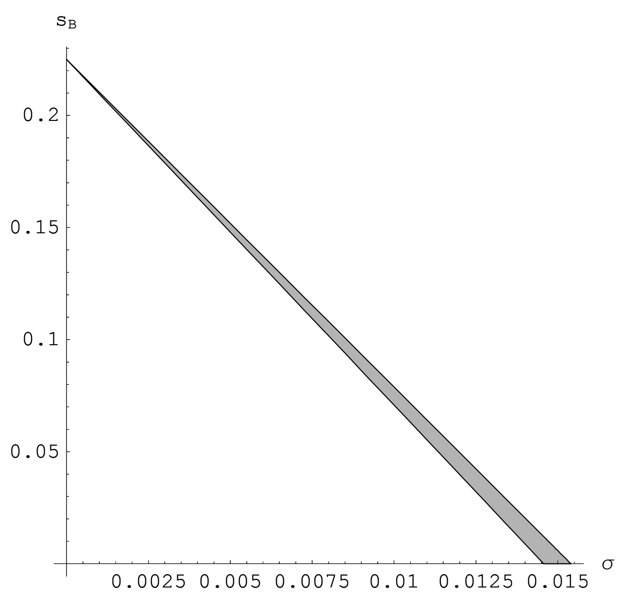

- Graphics -

Area where empty network is pairwise stable



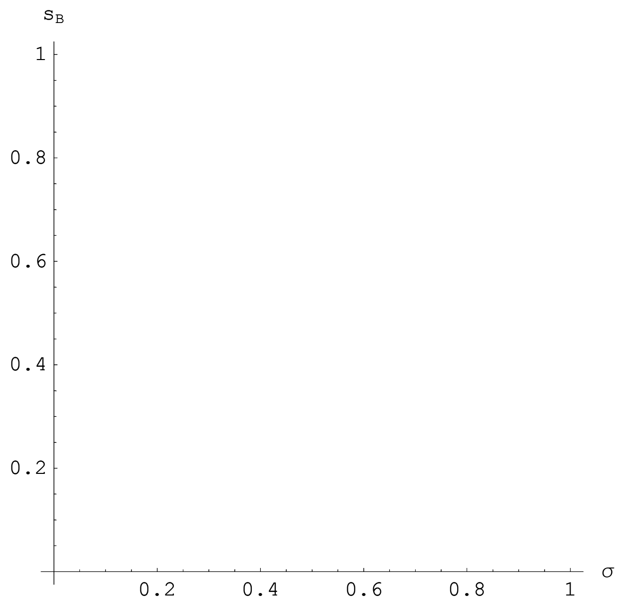

- Graphics -

Area where network $g_{AB0}$ is pairwise stable



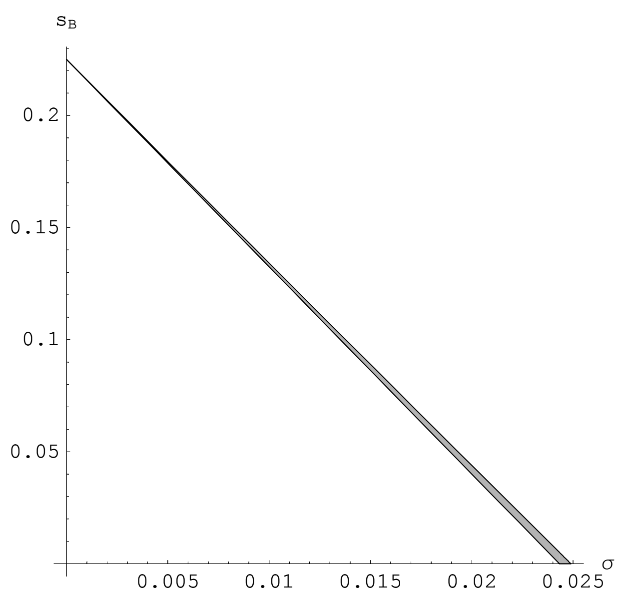

- Graphics -

Area where $g_{A00}$ is pairwise stable



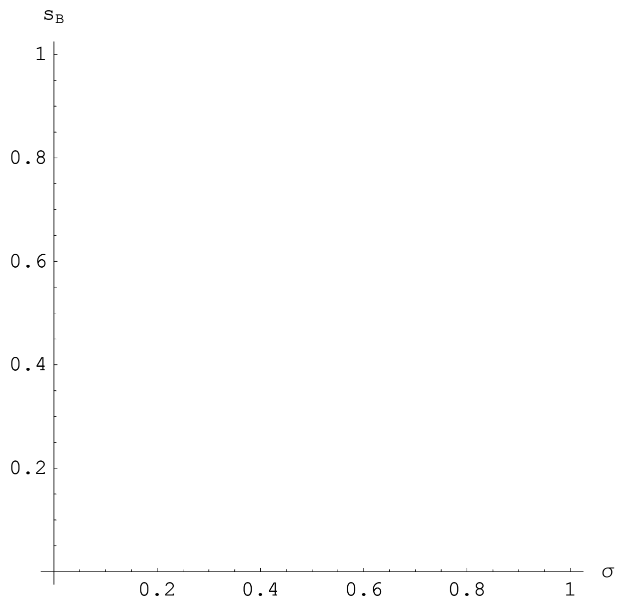

- Graphics -

Area where $g_{0\,B0}$ is pairwise stable



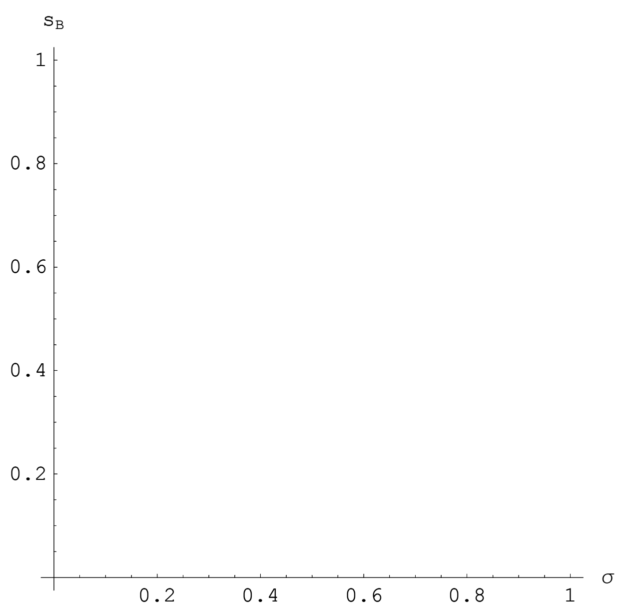

- Graphics -

Area where $g_{A01}$ is pairwise stable



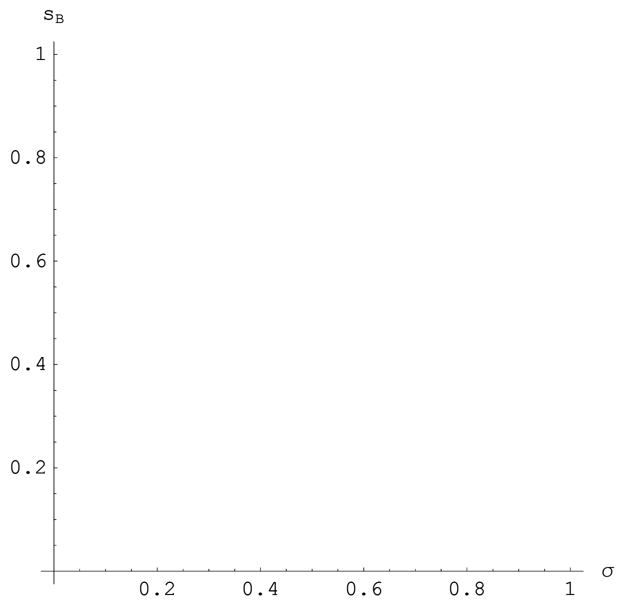

- Graphics -

Area where $g_{0B1}$ is pairwise stable

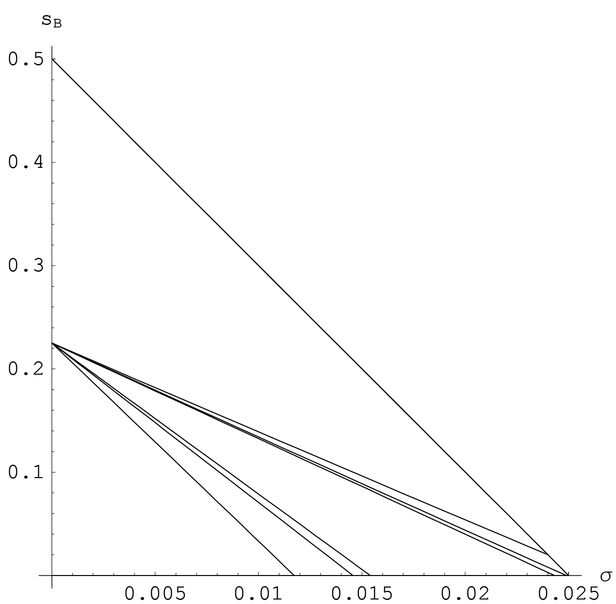



- Graphics -

Area where $g_{001}$ is pairwise stable




- Graphics -




- Graphics -

**"Unequal consumer base"**

**Sb = S**

**Sa = S**

**m = 3**

**n = 6**

**FA = 0**

**FB = 0**

**"Area where parameter constraints are satisfied"**

```
p1 = InequalityPlot3D[{0 < Sa < α < 1, 0 < Sb < 1 - α < 1, 0 < σ < (α - Sa)/(m * (n - 1)) < 1,
    0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1, α > 0.5}, {σ, 0, 1}, {S, 0, 1},
   {α, 0, 1}, AxesLabel → {σ, Sb, α}, AspectRatio → 1, ColorOutput → GrayLevel]
"Area where complete network is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = (m - 1);
niB = (n - m);
njA = m;
njB = (n - m - 1);
nAA = m (m - 1)/2;
nBB = (n - m) * (n - m - 1)/2;
nAB = m * (n - m);
p2 = InequalityPlot3D[{ΔprofitintraAdel[i] ≤ 0, ΔprofitintraBdel[j] ≤ 0,
    Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,
    0 < σ < (α - Sa)/(m * (n - 1)) < 1, 0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1, α > 0.5}, {σ, 0, 1}, {S, 0, 1},
   {α, 0, 1}, AxesLabel → {σ, Sb, α}, AspectRatio → 1, ColorOutput → GrayLevel]
"Area where empty network is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = 0;
njA = 0;
njB = 0;
nAA = 0;
nBB = 0;
nAB = 0;
p3 = InequalityPlot3D[{(ΔprofitintraAform[i] ≤ 0 && ΔprofitintraBform[j] ≤ 0) &&
     ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
       (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
       (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) &&
     (0 < Sa < α < 1 && 0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa)/(m * (n - 1)) < 1 &&
       0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1 && α > 0.5)},
   {σ, 0, 1}, {S, 0, 1}, {α, 0, 1}, AxesLabel → {σ, Sb, α}, AspectRatio → 1,
   ColorOutput → GrayLevel]
"Area where network gAB0 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = (m - 1);
niB = 0;
njA = 0;
njB = (n - m - 1);
nAA = m * (m - 1)/2;
nBB = (n - m) * (n - m - 1)/2;
```

```
nAB = 0;
p4 = InequalityPlot3D[{(Δprofitintra Adel[i] ≤ 0 && ΔprofitintraBdel[j] ≤ 0) &&
    ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
      (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
      (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) &&
    (0 < Sa < α < 1 && 0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa)/(m * (n - 1)) < 1 &&
      0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1 && α > 0.5)},
  {σ, 0, 1}, {S, 0, 1}, {α, 0, 1}, AxesLabel → {σ, Sb, α}, AspectRatio → 1,
  ColorOutput → GrayLevel]

"Area where g_A00 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = (m - 1);
niB = 0;
njA = 0;
njB = 0;
nAA = (m * (m - 1))/2;
nBB = 0;
nAB = 0;
p5 = InequalityPlot3D[{(ΔprofitintraAdel[i] ≤ 0 && ΔprofitintraBform[j] ≤ 0) &&
    ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
      (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
      (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) &&
    (0 < Sa < α < 1 && 0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa)/(m * (n - 1)) < 1 &&
      0 < σ < (1 - α - Sb)/((n - m) * (2 * n - 2 * m - 1)) < 1 && α > 0.5)},
  {σ, 0, 1}, {S, 0, 1}, {α, 0, 1}, AxesLabel → {σ, Sb, α}, AspectRatio → 1,
  ColorOutput → GrayLevel]

"Area where g_0B0 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = 0;
njA = 0;
njB = (n - m - 1);
nAA = 0;
nBB = ((n - m) * (n - m - 1))/2;
nAB = 0;
p6 = InequalityPlot3D[{(ΔprofitintraAform[i] ≤ 0 && ΔprofitintraBdel[j] ≤ 0) &&
    ((Δprofitinterform[i] ≤ 0 && Δprofitinterform[j] ≤ 0) ||
      (Δprofitinterform[i] > 0 && Δprofitinterform[j] < 0) ||
      (Δprofitinterform[i] < 0 && Δprofitinterform[j] > 0)) &&
    (0 < Sa < α < 1 && 0 < Sb < 1 - α < 1 && 0 < σ < (α - Sa)/(m * (n - 1)) < 1 &&
```

```
      0 < σ < (1 - α - Sb) / ((n - m) * (2 * n - 2 * m - 1)) < 1 && α > 0.5)},
  {σ, 0, 1}, {S, 0, 1}, {α, 0, 1}, AxesLabel → {σ, Sb, α}, AspectRatio → 1,
  ColorOutput → GrayLevel]

"Area where gA01 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = (m - 1);
niB = (n - m);
njA = m;
njB = 0;
nAA = m * (m - 1) / 2;
nBB = 0;
nAB = m * (n - m);
p7 = InequalityPlot3D[{ΔprofitintraAdel[i] ≤ 0, ΔprofitintraBform[j] ≤ 0,
   Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,
   0 < σ < (α - Sa) / (m * (n - 1)) < 1, 0 < σ < (1 - α - Sb) / ((n - m) * (2 * n - 2 * m - 1)) < 1, α > 0.5}, {σ, 0, 1}, {S, 0, 1},
  {α, 0, 1}, AxesLabel → {σ, Sb, α}, AspectRatio → 1, ColorOutput → GrayLevel]

"Area where g0B1 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = (n - m);
njA = m;
njB = (n - m - 1);
nAA = 0;
nBB = (n - m) * (n - m - 1) / 2;
nAB = m * (n - m);
p8 = InequalityPlot3D[{ΔprofitintraAform[i] ≤ 0, ΔprofitintraBdel[j] ≤ 0,
   Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,
   0 < σ < (α - Sa) / (m * (n - 1)) < 1, 0 < σ < (1 - α - Sb) / ((n - m) * (2 * n - 2 * m - 1)) < 1, α > 0.5}, {σ, 0, 1}, {S, 0, 1},
  {α, 0, 1}, AxesLabel → {σ, Sb, α}, AspectRatio → 1, ColorOutput → GrayLevel]

"Area where g001 is pairwise stable"
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
niA = 0;
niB = (n - m);
njA = m;
njB = 0;
nAA = 0;
nBB = 0;
nAB = m * (n - m);
p9 = InequalityPlot3D[{ΔprofitintraAform[i] ≤ 0, ΔprofitintraBform[j] ≤ 0,
   Δprofitinterdel[i] ≤ 0, Δprofitinterdel[j] ≤ 0, 0 < Sa < α < 1, 0 < Sb < 1 - α < 1,
   0 < σ < (α - Sa) / (m * (n - 1)) < 1, 0 < σ < (1 - α - Sb) / ((n - m) * (2 * n - 2 * m - 1)) < 1, α > 0.5}, {σ, 0, 1}, {S, 0, 1},
```

```
  {α, 0, 1}, AxesLabel → {σ, Sb, α}, AspectRatio → 1, ColorOutput → GrayLevel]
Clear[niA, niB, njA, njB, nAA, nBB, nAB]
Clear[Sa, Sb, α, m, n, FA, FB]
Show[p1, p2, p3, p4, p5, p6, p7, p8, p9]
```

Unequal consumer base

S

S

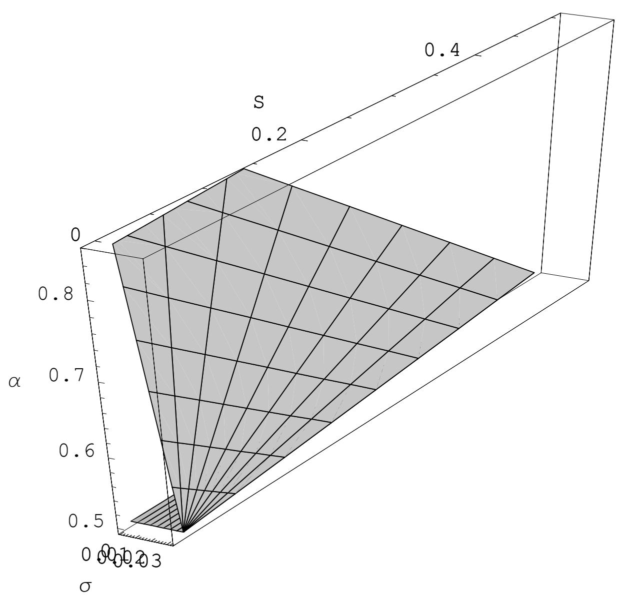
3

6

0

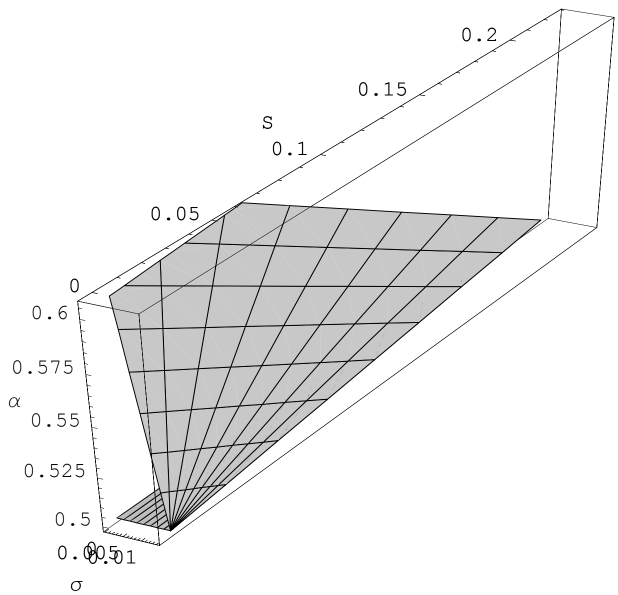
0

Area where parameter constraints are satisfied



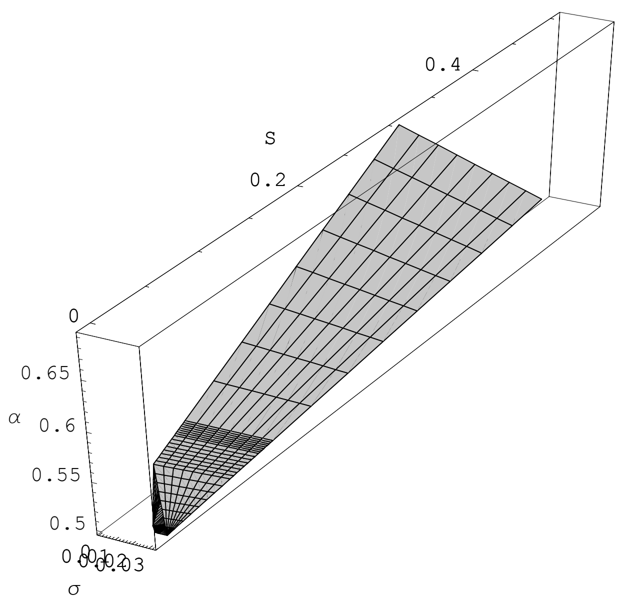

- Graphics3D -

Area where complete network is pairwise stable



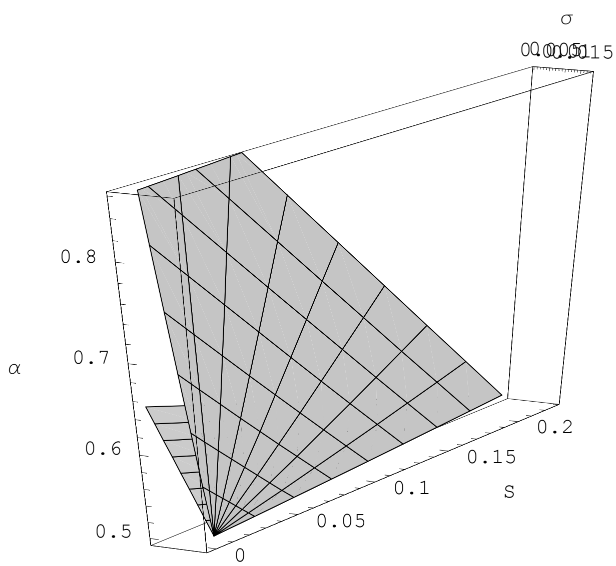

- Graphics3D -

Area where empty network is pairwise stable




- Graphics3D -

Area where network $g_{AB0}$ is pairwise stable


σ
0.8
0.7
α
0.6
0.5
0
0.05
0.1
0.15
0.2
s

## Vita

Narine Badasyan was born on 4th of July 1971 in Yerevan, Armenia. She graduated with honors from Yerevan State University in 1993, receiving a Bachelor of Science Degree in Applied Mathematics. While pursuing an MBA at the American University of Armenia, Narine worked as a Business Consultant at the Export Association of Machine Tool Enterprises. Narine joined the Economics Program of Virginia Polytechnic Institute and State University to pursue a Ph.D. in 1998. Narine completed the Ph.D. in June of 2004.