Robust MEWMA-type Control Charts

for Monitoring the Covariance Matrix of

Multivariate Processes

Pei Xiao

Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Statistics

Marion R. Reynolds, Jr. (Chairman)
William H. Woodall
Dong-Yun Kim
Pang Du

January 25, 2013
Blacksburg, Virginia

Keywords: Multivariate control chart, non-normal processes, Robustness, Quadratic loss

Robust MEWMA-type Control Charts

for Monitoring the Covariance Matrix of

Multivariate Processes

Pei Xiao

Abstract

In multivariate statistical process control it is generally assumed that the process variables follow a multivariate normal distribution with mean vector μ and covariance matrix Σ, but this is rarely satisfied in practice. Some robust control charts have been developed to monitor the mean and variance of univariate processes, or the mean vector μ of multivariate processes, but the development of robust multivariate charts for monitoring Σ has not been adequately addressed. The control charts that are most affected by departures from normality are actually the charts for Σ not the charts for μ. In this article, the robust design of several MEWMA-type control charts for monitoring Σ is investigated. In particular, the robustness and efficiency of different MEWMA-type control charts are compared for the in-control and out-of-control cases over a variety of multivariate distributions. Additionally, the total extra quadratic loss is proposed to evaluate the overall performance of control charts for multivariate processes.

Acknowledgment

It is with immense gratitude that I acknowledge the continuous guidance, encouragement and support of my advisor, Dr. Marion R. Reynolds. His broad knowledge and his logical way of reasoning have been of great value for me. The attitude toward research and research methodologies that I learned from him can be applied to every aspects of life and will benefit me for the rest of my life.

I am also deeply grateful to my committee members, Dr. William H. Woodall, Dr. Dong-Yun Kim and Dr. Pang Du for their support, comments, suggestions and encouragement.

I would also like to thank my friends and colleagues, especially Wandi Huang, Ning Wang, Sai Wang, Liaosa Xu and Yiming Peng, for their valuable help during my research.

My heartfelt appreciation goes toward my parents, Zexun Xiao and Xiangyuan He, who have always provided support and encouragement throughout my life.

Finally, with deepest love, I would like to thank my husband, Dianbo Fu, who has been there for me no matter where I am, for all the unconditional support and encouragement.

Table of Contents

List of Illustrations

List of Tables

Chapter 1

Introduction

Control charts are fundamental techniques of statistical process control (SPC) which are widely used to monitor the quality characteristics in a process in order to maintain a desired level of quality. They were originally developed for quality control of manufacturing processes. But nowadays their applications have been extended to many type of processes. For instance, control charts can be used to monitor the mortality rate of surgeries in a hospital or the waiting time of customers in some service center.

Montgomery (2013) distinguished the source of variations in a quality characteristic as two types: common cause (inherent variation) and special cause (variation due to identifiable factors). Control charts are graphical tools designed to quickly detect any special cause in a process that may result in a change of the product quality. The effect of a special cause is usually assumed to be a change in a parameter associated with the distribution of the quality characteristic of interest.

The operation of a control chart includes taking samples from a process and plotting the control statistic, which is computed from the sample, in time order. If the statistic plots outside the predetermined control limits, the chart is said to give a signal and the process is considered to be out of control. An attempt should then be made to locate and remove the cause of the process change. A process having only common cause variation is considered to be in a state of statistical control.

However, a chart's signal can be triggered by either common causes or special causes. A signal aroused by common causes is called a false alarm. The ideal control chart should give a signal quickly when there is a change in the process due to a special cause and have a low false alarm rate when only inherent variation is present. The average time to signal (ATS) is a popular measure of chart performance which is defined as the expected length of time from the start of process monitoring until a signal is given. It is desirable that the ATS of a chart be small when the process is out of control and large when the process is in-control, which is equivalent to a low false alarm rate.

There are three major types of control charts in widespread use: the Shewhart chart, the cumulative sum (CUSUM) chart, and the exponentially weighted moving average (EWMA) chart. Among the three, the Shewhart chart has the longest history and is still very popular due to its simplicity. It uses only the current sample in calculating the control statistic thus it is relatively sensitive to large shifts in the process parameter being monitored. In order to detect small shifts quickly, EWMA charts introduced by Roberts (1959) and CUSUM charts introduced by Page (1954) were developed. Both charts calculate the control statistic from the current and past samples and have approximately the same performance. These three control charts can all be designed as one sided or two sided. A one-sided control chart is used when only one shift direction (increase or decrease) in a process parameter is of interest and two-sided when both directions are of interest.

Control charts are called variables control charts when they monitor continuous process variables and attributes control charts when they monitor discrete process variables. The above mentioned three types of control charts can all be applied to continuous and discrete process variables. In this dissertation, we only consider continuous variables, thus the control charts in the rest of this work refer to variables control charts. For continuous variables, a statistical controlled process usually has the process mean μ close to the target value and the process variance σ^2 within a certain level. In order to ensure that a process is under control, it is traditional to use two control charts to monitor μ and σ^2 separately and simultaneously.

It's common that control charts are used to monitor a single variable in a process. We call these charts univariate control charts. However, the overall quality of a product sometimes is determined by more than one quality characteristic. For instance, the quality of a certain part may be determined by its diameter, its weight and degree of hardness. In this situation, a set of univariate control charts might be considered to monitor the variables of interest one by one but do not take the correlation between those variables into account. As a result, using one control chart to handle multiple variables simultaneously is receiving increasing attention. These charts, described as multivariate control charts, are designed to use the correlation structure between the characteristics and simplify the monitoring work by only using one control statistics and one set of control limits. In the multivariate setting, the control charts usually monitor the mean vector $\boldsymbol{\mu}$ and the covariance matrix Σ of a process. Much work has been done to investigate multivariate control charts. Hotelling (1947) first proposed a multivariate Shewhart chart to monitor $\boldsymbol{\mu}$. After that, different forms of multivariate CUSUM (MCUSUM) and multivariate exponentially weighted moving average (MEWMA) charts have been developed. For example, Woodall and Ncube (1985) and Crosier (1988) proposed several designs of the MCUSUM chart; Lowry at al. (1992) and Reynolds and Cho (2006) investigated several MEWMA charts and compared their performance in different settings.

In the context of process control, the most common assumption when we determine the control limits of a control chart is that the process variable follows a normal distribution for univariate case or that the joint distribution of the variables is multivariate normal for the multivariate case. However, the underlying process is actually not normal in many situations, and consequently the performance of the standard control charts can be significantly affected. For instance, measurements from chemical processes are often skewed. Normal-like, but heavier-tailed distributions can also occur frequently in practice, giving a greater chance of obtaining extreme observations and leading to more false alarms (signals occur when the process is in control) for the standard charts. Moreover, when the production rate is low, many practical applications use small samples or even individual observations, thus the central limit theorem cannot be invoked and the distribution of the sample mean may be far from normal.

The normality assumption seems to be even more sensitive in the multivariate case. In the univariate case, one might still be able to plot the data and check how far the true distribution is away from the normal. But detecting non-normality is much more difficult when dealing with a high-dimensional space. In addition, the outliers tend to occur more frequently in the multivariate case. Brooks (1985) noted the increased chance for data errors in advanced manufacturing systems when multivariate data is collected. Moreover, the common used estimators of σ^2 or Σ actually amplify the impact of non-normality by taking the square of the distance between observations and the in-control value. Thus the control charts for variance are generally much less robust than charts for the mean. Even if a chart for the mean is relatively robust, when it is used with a non-robust chart for variance, the chart combination will not be robust. Crosier (1988) argued that robustness is an essential feature for a multivariate quality control scheme. The need for robust or distribution-free multivariate control charts has been noted in a number of articles (ref. e.g., Coleman (1997), Woodall and Montgomery (1999), Stoumbos et al. (2000)).

Although many multivariate control charts have been proposed since Hotelling (1947), how to design a robust multivariate chart has not been adequately addressed. For examples, Kapatou (1996) proposed a multivariate nonparametric control charts using a sign based statistic. Abu-Shawiesh and Abdullah (2001) designed a robust Shewhart chart in which Hotelling's T^2 statistic is based on the Hodges-Lehamnn and Shamos-Bickel-Lehmann estimator. Stoumbos and Sullivan (2002) discussed the robust design of the MEWMA chart. Nevertheless, these charts were only designed to monitor $\boldsymbol{\mu}$. Few robust multivariate charts to monitor Σ have been proposed. The existing proposed robust charts to monitor the variance were designed in the framework of univariate distributions or multivariate distributions with independent variables. Their robustness to heavy-tail and skewed distributions were discussed, but their performances with light-tailed distributions or correlated variables have not been addressed. And it is noticeable that robust control charts can be less efficient than the standard ones when used to detect shifts in μ or σ^2. However, the overall efficiency loss due to the robust design has not been investigated.

The main objective of this dissertation is to develop and investigate robust MEWMA-type control charts for monitoring Σ in multivariate processes. The study of robust designs for the MEWMA-type control charts is intended to obtain robust charts which give reasonably steady in-control performance over a variety of multivariate distributions, while remaining relatively high efficiency for the out-of-control state.

First, the dissertation is concerned with developing control charts for monitoring Σ that have robust in-control performance. We propose three MEWMA-type charts: one is based on a power of the absolute deviations of the observations from the target, one is based on the Winsorized observations, and one is a set of univariate EWMA-type charts. With appropriately chosen values of the control chart parameters, the proposed charts can be tuned to be robust over a wide range of distributions. Some indices, such as the percentage ATS deviation from the target in-control ATS and the root mean square of percentage ATS deviation over various non-normal distributions, are used to measure the robustness of some designs of these control charts.

Secondly, the efficiencies of robust designs for the proposed charts are also investigated when the process is out of control in order to find control charts which give quick detection for some shift sizes or even for a wide range of shift sizes. To compare the charts' overall performance for the out-of-control case, the total extra quadratic loss (TEQL) of a control chart in detecting a range of shifts is introduced to multivariate processes and evaluated by simulation. Based on the overall performance, some recommendations are given for the robust design.

To further improve the efficiency of the robust designs selected from the previous comparison for the out of control cases, the regression adjustment method (introduced by Hawkins (1991), see examples in Hawkins (1991), Hawkins (1993) and Reynolds and Cho (2006)) is used and applied to the robust designs. We find that the proposed control charts based on the regression adjusted variables can still be robust to various multivariate distributions while significantly improving the overall performance in terms of TEQL.

For this research problem, a process is assumed to have *p* continuous variables of interest. Both the bivariate and higher dimensional processes are considered. The target mean vector $\boldsymbol{\mu}_0$ and the in-control variance-covariance matrix Σ_0 are assumed to have been estimated with enough accuracy from historical data so that the estimation error can be neglected. The robust design of the proposed charts for the cases of highly correlated and independent variables are separately discussed. We assume here that the samples are independent from each other and the sample size is *n* at each sampling point. Due to the limited scope of the dissertation, here we only present numerical results for the case of $n = 1$, although the proposed control charts can also be applied in the case of $n > 1$. When we investigate the out of control performance of the proposed charts, we mostly assume that the variance of each variable does not decreases. The objective is to find a robust chart which can efficiently detect a wide range of increases in the variances. Given the multivariate setting, a change in $\boldsymbol{\mu}$ and/or Σ can be in numerous directions in p-dimensional space. The performance of control charts for monitoring Σ will depend on the direction of the shift. We compare the charts' efficiency performance mainly averaged over all random shift directions, but also consider shifts in some particular directions.

The rest of this dissertation is organized as follows. Chapter 2 contains a literature review of recent work in the area of robust control charts. Brief background information is described for various types of control charts. The simulation method and performance measures used in this dissertation for control charts are also introduced. Chapter 3 defines the proposed MEWMA-type control charts for monitoring Σ. The general form of the proposed charts is derived. The correlation coefficients and variances required to set up the proposed control statistics are demonstrated in a table, given different choices of parameter combinations. Chapter 4 assesses the performances of the proposed control charts when process is in control and out of control for some bivariate processes ($p = 2$). Two extreme cases are discussed separately: highly correlated variables and independent variables where the correlation coefficients are 0.9 and 0 respectively. In Chapter 5, the regression adjustment method is introduced to improve the efficiency of robust control charts. Their in-control and out-of-control performance is compared to previous robust designs for 5-variate processes. Chapter 6 demonstrates the application of regression adjustment based robust charts to a real dataset and gives guidance for practitioners on how to obtain the control limit for the recommended robust charts with a provided software package. Finally in Chapter 7, conclusions are made and some future work is discussed.

Chapter 2

Background

2.1 Definition of control charts

2.1.1 Notation and assumptions

In the multivariate setting, let $\mathbf{X} = (X_1, X_2, \ldots, X_p)$ represent the process variables of interest, which follow an unknown joint density function. In addition to the mean vector $\boldsymbol{\mu}$ and covariance matrix Σ which were defined earlier, we also define $\boldsymbol{\sigma}$ as the vector of standard deviations of the joint density function, which is the square roots of the diagonal elements of Σ. The in control value for these three are defined as $\boldsymbol{\mu_0}$, Σ_0 and $\boldsymbol{\sigma_0}$ respectively. $\boldsymbol{\mu_0}$, Σ_0 and $\boldsymbol{\sigma_0}$ could be target values of the process or estimated values from historical data.

Generally, independent samples with fixed size $n \geq 1$ are drawn from the process with the fixed sampling interval d. Let X_{kij} represent the j^{th} ($j = 1,2,\ldots,n$) observation of variable i ($i = 1,2,\ldots,p$) at sampling time k. When we only investigate the process where only one observation is taken at time k ($n = 1$), the notation of X_{kij} can be reduced to X_{ki}. Although numerical results are only give for the case of $n = 1$, the formulas given below are for $n \geq 1$.

To simplify the formulas, we use the standardized observations Z_{kij} instead of the original observations X_{kij} which are defined as

$$Z_{kij} = (X_{kij} - \mu_{0i}) / \sigma_{0i}$$

where the subscript i of μ_{0i} and σ_{0i} indicates the i^{th} component of $\boldsymbol{\mu}_\mathbf{0}$ and $\boldsymbol{\sigma}_\mathbf{0}$. Let $\mathbf{Z}$ = (Z_1, Z_2, ..., Z_p) represent the standardized form of $\mathbf{X}$ and $\sum_Z$ be the covariance matrix of $\mathbf{Z}$. Correspondingly, let $\boldsymbol{\mu}_{\mathbf{Z0}}$, $\boldsymbol{\sigma}_{\mathbf{Z0}}$ and $\sum_{Z0}$ be the in control values of $\boldsymbol{\mu}_\mathbf{Z}$, $\boldsymbol{\sigma}_\mathbf{Z}$ and $\sum_Z$. It is convenient that $\boldsymbol{\mu}_{\mathbf{Z0}}$ and $\boldsymbol{\sigma}_{\mathbf{Z0}}$ are vector of zeros and ones respectively. The $\sum_{Z0}$ is actually the in control correlation matrix of $\mathbf{X}$. When $n > 1$, let $\bar{Z}_{ki} = \sum_{j=1}^{n} Z_{kij} / n$ be the sample mean of i^{th} standardized variable at sampling point k.

In order to keep track of the different charts by some simple shorthand notation, we use "E" to represent the EWMA-type charts and "M" to represent the MEWMA-type charts, following the notation used by Reynolds and Cho (2006).

2.1.2 EWMA Charts

Roberts (1959) proposed the first EWMA control chart for monitoring the mean of a univariate process. In contrast to Shewhart-type charts, which only use the last sample, EWMA control statistics use all the past information with different weights and gain efficiency in detecting small and moderate shifts in the monitored parameter. See, for example, Montgomery (2013), Lucas and Saccucci (1990) for discussion of EWMA charts.

The control statistic of the EWMA chart to monitor the mean of the i^{th} variable at time k is defined as

$$E_{ki}^{Z} = (1-\lambda)E_{k-1,i}^{Z} + \lambda \bar{Z}_{ki}, \qquad i = 1,\ 2,\ \ldots,\ p,$$

where λ is a smoothing or weighting parameter and can be chosen from any value in (0, 1], and the staring value E_{0i}^{Z} is usually set to be 0, the in-control mean of the i^{th} standardized variable. This chart can be designed to detect different sizes of changes in the mean by changing the value of λ. When λ is large, less weight is given to past samples and the EWMA chart signals effectively for large shifts in the process mean;

when λ is small, more weight is assigned to past samples and the chart is designed to detect small shifts quickly. The EWMA chart based on E_{ki}^{Z} will be called the EZ chart.

To detect changes in σ^2 of one variable, a EWMA control chart designed for a normal process can be constructed based on the squared deviations from the target value (see, for example, Domangue and Patch (1991) or Reynolds and Stoumbos (2001)). The control statistic for detecting both increases and decreases in σ^2 at time k is

$$E_{ki}^{Z^2} = (1-\lambda)E_{k-1,i}^{Z^2} + \lambda(\sum_{j=1}^{n} Z_{kij}^2 / n), \quad i = 1, 2, \ldots, p,$$

where the starting value $E_{0i}^{Z^2} = 1$ and $0 < \lambda \le 1$. We call the EWMA chart based on squared deviations from target the EZ^2 chart.

In the univariate case, Reynolds and Stoumbos (2001, 2004, 2005) used a reset in the EWMA statistic to make a one-sided chart for detecting increases in process variability. The reset gives the EWMA chart more power to detect shifts which are increases. In this dissertation, we are interested in the capability of control charts to detect variance increases in any variable and thus one-sided charts are considered. At time k, the EWMA control statistic for detecting increases in σ^2 is

$$ER_{ki}^{Z^2} = (1-\lambda)\max(ER_{0,i}^{Z^2}, ER_{k-1,i}^{Z^2}) + \lambda(\sum_{j=1}^{n} Z_{kij}^2 / n), \qquad i = 1, 2, \ldots, p,$$

where $ER_{0i}^{Z^2} = 1$ and $0 < \lambda \le 1$. The control statistic is reset to the starting value if it drops below the starting point $E_{0i}^{Z^2}$. A signal is given when the control statistic $ER_{ki}^{Z^2}$ falls outside the upper control limit (UCL). We call the EWMA chart with the reset based on squared deviations from target the ERZ^2 chart.

A set of univariate EWMA charts can be used together to monitor a multivariate process. For the standardized variables, the set of EZ charts to monitor the mean of each variable separately can share the same set of control limits. So do the set of EZ^2 or ERZ^2 charts, which are used to monitor σ^2 for each variable. The combination of two sets of EWMA charts offers an option to monitor $\boldsymbol{\mu}$ and Σ simultaneously, though the number of control charts will be numerous if p is large. Reynolds and Cho (2006) investigated the performance of sets of EWMA charts in a multivariate normal setting. They found that the overall performance of sets of EWMA charts was not as good as the MEWMA-type charts, but there were some special cases where the performance of sets of EWMA charts was reasonably good.

Studies have shown that EWMA charts generally have similar detection performance to that of CUSUM charts. Hence CUSUM-type charts will not be explicitly compared to other charts in this dissertation.

2.1.3 MEWMA charts

In the last two decades, the study of MEWMA-type charts has received a lot of attention. Lowry et al. (1992) first extended the EWMA scheme to the multivariate setting to detect mean shifts. Prabhu and Runger (1997), Kramen and Schmid (1997), Reynolds and Kim (2005) also proposed designs of MEWMA charts to monitor the mean vector. They all assumed that $\boldsymbol{\mu}_0$ and Σ_0 are known. Comparisons between the MEWMA and the MCUSUM charts found that both charts had approximately the same performance, similar to what was found in the univariate case. However, only a few MEWMA-type charts to monitor Σ or $\boldsymbol{\mu}$ and Σ together have been investigated (see, e.g., Hawkins (1991), Yeh et al. (2003), Reynolds and Cho (2006)).

The MEWMA control charts proposed by Lowry et al. (1992) and investigated by Reynolds and Cho (2006) are the multivariate extension of EWMA charts defined in the previous section. The MEWMA control statistic for detecting the changes in $\boldsymbol{\mu}$ is

$$M_k^Z = nC_\infty^{-1}(E_{k1}^Z, E_{k2}^Z, \ldots E_{kp}^Z)\Sigma_{Z0}^{-1}(E_{k1}^Z, E_{k2}^Z, \ldots E_{kp}^Z)^{\mathrm{T}},$$

where $C_\infty = \lambda/(2-\lambda)$. It is easy to show that the variance of E_{ki}^Z is $C_k = \dfrac{\lambda[1-(1-\lambda)^{2k}]}{(2-\lambda)n}$. Thus C_∞ is the asymptotic variance of E_{ki}^Z when n = 1 and

$k \to \infty$. We can tell that the M_k^Z statistic is actually a quadratic form of $(E_{k1}^Z, E_{k2}^Z, \ldots E_{kp}^Z)$. For the standardized variables in **Z**, the target means are all zero. Any decrease or increase in the mean is a deviation from zero and leads to the increase of M_k^Z. Thus only an UCL is required for this chart. We call it the MZ chart.

To monitor Σ, the control statistic is established using the EZ^2 statistics for each variable. Thus we need the in-control covariance matrix of $(E_{k1}^{Z^2}, E_{k2}^{Z^2}, \ldots E_{kp}^{Z^2})$. Let Σ_{Z0}^2 represent the matrix with elements that are squares of the corresponding elements of Σ_{Z0}. Let $\rho_{ii'}$ be the correlation coefficient for variable *i* and *i'*, where $i \neq i'$. It is easy to show that $Cov(Z_{kij}^2, Z_{ki'j}^2) = 2\rho_{ii'}^2$. Then the asymptotic in-control covariance matrix of $(E_{k1}^{Z^2}, E_{k2}^{Z^2}, \ldots E_{kp}^{Z^2})$ is $2C_\infty \Sigma_{Z0}^2 / n$ (see Cho (1991)). The control statistic of the MEWMA chart based on the squared deviations from target is

$$M_k^{Z^2} = n(2C_\infty)^{-1}(E_{k1}^{Z^2}, E_{k2}^{Z^2}, \ldots E_{kp}^{Z^2})(\Sigma_{Z0}^2)^{-1}(E_{k1}^{Z^2}, E_{k2}^{Z^2}, \ldots E_{kp}^{Z^2})^{\mathrm{T}}.$$

Replacing the EZ^2 statistics by the ERZ^2 statistics, the one-side MEWMA chart based on the squared deviations from target is

$$MR_k^{Z^2} = n(2C_\infty)^{-1}(ER_{k1}^{Z^2}, ER_{k2}^{Z^2}, \ldots ER_{kp}^{Z^2})(\Sigma_{Z0}^2)^{-1}(ER_{k1}^{Z^2}, ER_{k2}^{Z^2}, \ldots ER_{kp}^{Z^2})^T.$$

Increases in variability result in larger values of $MR_k^{Z^2}$. Call this chart the MRZ^2 chart. When the reset is used in the EWMA statistics $2C_\infty \Sigma_{Z0}^2 / n$ is the approximate asymptotic in-control covariance matrix for $(ER_{k1}^{Z^2}, ER_{k2}^{Z^2}, \ldots ER_{kp}^{Z^2})$

The above MEWMA charts were introduced under the assumption that the process follows a *p*-variate normal distribution. The combination of the MZ and MZ^2 or the MZ and MRZ^2 charts has excellent performance in detecting shifts in the mean or in the variability (see Reynolds and Cho (2006) and Reynolds and Stoumbos (2008)).

Like univariate EWMA charts, univariate CUSUM charts can also be extended to multivariate CUSUM charts to monitor multi-dimensional processes, and these CUSUM charts will be generally referred as MCUSUM charts in the following part of the dissertation, although MCUSUM charts will not be explicitly investigated here.

2.2 Literature review for robust control charts

The following literature review is grouped according to univariate and multivariate robust control charts. Recall that the most common control charting schemes include the Shewhart, CUSUM, and EWMA charts. For both groups, we review robust charts of each type. In general, the development of a control chart includes: (1) finding control statistics for the process parameters (mean, variation or both) and (2) obtaining a proper control limit to monitor these parameters. When the normality assumption does not hold, control charts with the traditional estimators, say the sample mean and sample variance, and control limits obtained with normal observations can be severely affected. In particular the rate of false alarms can be much higher than expected. Thus approaches to designing a robust control chart are mainly to use robust estimators and make adjustments in the control limits, or use nonparametric methods.

2.2.1 Univariate robust charts

In the development of robust univariate control charts, nonparametric methods play a major role. Median and rank based control charts have been developed by many authors. For instance, the signed sequential ranks of observations or Wilcoxon signed rank statistics have been used in a CUSUM chart (Bakir and Reynolds (1979)), a EWMA-type chart (Amin and Searcy (1991), Hackl and Ledolter (1992)), and as well as a Shewhart-type chart (Bakir (2004, 2006), Jones et al. (2009)). The distribution of the process was assumed to be non-normal. Ranks were assigned to observations either at each sampling time or among some most recent samples. The control limits were calculated based on the order statistics. All of these charts were designed to monitor the process mean. The effects of autocorrelation, different sample sizes, mean drifts and shifts were discussed in some of these papers. Generally, rank based charts were slightly less efficient than parametric charts for normal data but can be considerably more efficient for non-normal data.

Park and Reynolds (1987) developed another nonparametric procedure to monitor the mean of a continuous process. The control chart is based on a two-sample nonparametric statistic which is called a “linear placement statistic” and which was introduced by Orban and Wolfe (1982). Shewhart and CUSUM versions of this design were both proposed. An initial standard sample was compared to the current sample at each sampling point and the asymptotic run length distribution was obtained to construct the control limits. Hackl and Ledolter (1991) extended Park and Reynolds’ work to the case of single observation and proposed an EWMA version of the linear placement statistic to monitor μ. This chart outperforms the observation-based EWMA in detecting small to moderate shifts in heavy-tailed distributions.

Alloway and Raghavachari (1991) proposed a univariate control chart for location based on the Hodges-Lehmann estimator, which is an estimator for the point of symmetry of a continuous distribution. This chart performs as well as the Shewhart $\overline{X}$ chart when the data are from a normal distribution. Its performance is better in the case of moderate sample sizes with observations from long-tailed symmetric distributions.

The above mentioned nonparametric designs mostly monitor the location of a process and require the sample size to be greater than one. The following robust charts are observation-based charts.

Hawkins (1981) proposed a CUSUM chart based on $|X/\sigma_0|^{1/2}$ to monitor the variance of a univariate process. The distribution of $|X/\sigma_0|^{1/2}$ was very close to the normal if X follows a N(0, σ_0^2) distribution. If the process had a nonzero unknown mean, the use of the difference between two successive observations instead of X still leads to the similar conclusion. By comparing the skewness and kurtosis of $|X/\sigma_0|^{1/2}$ to the normal distribution, he further showed that even if the distribution of X was heavy-tailed, the distribution of $| X / \sigma_0 |^{1/2}$ was still close to the normal, which made this chart robust. He concluded that the square root of the absolute value of observation can effectively normalize the heavy-tailed observations. However he only investigated the chart performance for the in-control state and did not address the robustness of his chart for skewed distributions.

Stoumbos and Reynolds (2000) studied the effects of non-normality and autocorrelation on the performances of various individuals control charts for monitoring the process mean and variance, which included the Shewhart X chart, the moving range (MR) chart, several EWMA charts, and combinations of these charts. Their study showed that the Shewhart X chart and MR chart designed under the assumption of normality would give a much larger number of false alarms, in other words, a significantly smaller in-control ATS when the process actually follows a *t*, gamma, or exponential distribution. Although some control charts for μ can be tuned to be robust to non-normal distributions, the more critical problem with non-robustness is with the control charts for σ^2. They found that the EMWA chart based on single observations to monitor μ and the EMWA chart based on the absolute deviations from target to monitor σ^2 could be designed to be robust to the normality assumption with respect to the issue of false alarms if the smoothing parameter (λ) was chosen between 0.03 and 0.05. By comparing the performance in the out-of-control state among these charts, they also concluded that the combination of these two EMWA charts could be very competitive at detecting small and moderate shifts in the μ and /or σ^2.

Hawkins and Olwell (1998) introduced a CUSUM chart based on the Winsorized observations instead of original observations for monitoring μ, which would not be affected by outliers that are more extreme than *w* standard deviations from target. They suggested a value of *w* around 2.0 to make charts robust.

Reynolds and Stoumbos (2009) investigated a CUSUM-type chart of absolute deviations from target for monitoring σ^2. They thoroughly discussed the in-control and out-of-control performance of this chart used together with a standard CUSUM for μ.

The cases of $n = 1$ and $n = 4$ were considered separately. The in-control ATS values of the proposed charts under various distributions, which included normal, Laplace, different gamma, *t*, and beta, were obtained and compared to the traditional Shewhart chart and standard CUSUM chart to monitor μ and σ^2 at the same time. The results showed that the proposed CUSUM scheme would have an in-control ATS that did not vary much across a variety of distributions if μ_1 and σ_1^2, the values that the CUSUM charts are designed to detect, were set to be not too large. The authors also extended the Winsorization approach to the CUSUM chart for monitoring σ^2. The efficiency for detecting a wide range of shifts for CUSUM combinations based on Winsorized observations were compared to those of CUSUM combinations based on absolute deviations from target as well as the Shewhart combinations and standard CUSUM combinations. The result showed that robust CUSUM chart combinations respond faster than the standard CUSUM combinations with the same μ_1 and σ_1^2 for very small shifts in μ and σ^2 while slower in detecting moderate and large shifts in μ and σ^2, when the observations were normal or from a heavy-tailed distribution. Moreover, the CUSUM combination based on absolute deviations from target usually outperformed the CUSUM combination based on Winsorized observations for detecting a wide range of shift sizes and for all the distributions considered in the paper.

2.2.2 Multivariate robust charts

Although multivariate process control methodology has progressed steadily, the robustness aspect of multivariate control charts has not been adequately investigated.

Shewhart-type robust charts

Everitt (1979), Tiku and Singh (1982), and Johnson (1987) pointed out that the Hotelling T^2 statistic was severely affected by the deviations from the multivariate normal distribution, especially by the skewness of the distribution.

In 1990, Alloway and Raghavachari proposed a Shewhart-type multivariate control chart for location using a trimmed mean in Hotelling's T^2 statistics. They indicated that, for small sample sizes, the multivariate central limit effect is even more difficult to achieve than in the univariate case. Thus the control charts based on trimmed means would be more appropriate in this situation. A numerical example of the bivariate case using telephone pole data was presented and comparisons was made with the control chart based on the standard Hotelling's T^2 statistic. The trimmed T^2 statistics showed some advantage by being less affected by the outliers.

Abu-Shawiesh and Abdullah (2001) introduced a new robust Shewhart-type control chart for monitoring the location of a bivariate process using both Hodges–Lehmann (HL) and Shamos–Bickel–Lehmann (SBL) estimators. The SBL estimator is the sample median of the pairwise distances for a random sample. They found that there was little difference in performance between the traditional Hotelling's T^2 chart and the proposed chart for slightly heavier-tailed distributions. The proposed robust chart showed more consistent performance than the traditional Hotelling's T^2 chart as the tail weight was increased in the in-control state. Upper control limits were provided for the robust chart given different sample sizes. However they did not investigate the performance of these charts for out-of-control cases.

Liu (1995) proposed a Shewhart-type multivariate nonparametric control chart based on the concept of simplicial data depth (Tukey (1975) and Liu (1990)), which simultaneously monitors the process location and scale parameters. Stoumbos and Jones (2000) and Stoumbos et al. (2001) investigated the effect of the size of the reference sample and the size of the subgroups on the statistical properties of control charts based on simplicial data depth. Although this Shewhart chart has potential for addressing the hard problem of multivariate nonparametric quality control, the limitation of this chart is its requirement of a large size for the reference sample to achieve sufficiently large in-control average run lengths when sample size is 1.

MCUSUM-type and MEWMA-type robust charts

Qiu and Hawkins (2001) suggested a rank-based nonparametric MCUSUM procedure to monitor the mean vector of a process which is constructed based on the cross-sectional antirank vector: the vector of the indices of the order statistics of the individual measurements at each sampling time point. This chart is computationally trivial compared to other nonparametric multivariate charts. However, the problem with this method is that it could not detect shifts in the direction in which all shift components are the same. In Qiu and Hawkins (2003), they proposed a new procedure as an improvement on the previous Qiu-Hawkins method in order to overcome the limitation. Antirank vectors would be determined by the order information among the observations and also the order information between the observations and their in-control means. They performed a simulation study to compare these two antirank-based MCUSUM charts for monitoring the mean of 4 independent standard normal variables. They also applied the improved CUSUM procedure to non-normal multiple dimensional data. Though the results showed the chart had good performance in detecting changes in the process, the relative efficiency in detecting shifts compared to other control charts was not addressed.

Chang (2007) introduced an MCUSUM control chart for skewed distributions using the weighted standard deviation, which is determined by the probability that the observed value was less than the target value. The performance of the proposed chart, which monitored the mean of the process, was investigated among variables with various correlation coefficients and for some fixed shift directions. The in-control performance was discussed for multivariate lognormal and Weibull distributions. Compared to conventional MCUSUM and MEWMA charts, simulation results for the proposed charts showed considerable improvements when the underlying distribution was skewed. However, this method requires knowing the underlying distribution in order to calculate the weights, which seems to be not very feasible in high dimensional processes.

Stoumbos and Sullivan (2002) investigated the effects of non-normality on the statistical performance of the MZ control chart and Hotelling's T^2 chart for sample size $n=1$. The latter chart is actually the special case of the MEWMA chart (λ = 1). They discussed the charts' performance under various multivariate t and gamma distributions when $p = 2$, 5, and 10 for both in-control and out-of-control cases. The parameters of non-normal distributions were picked to present distributions with various degree of skewness or different heaviness of the tails. The simulation results showed that the Hotelling's T^2 chart for individual observations had a much smaller in-control ATS with non-normal observations compared to the one with multivariate normal observations. The out-of-control performance also deteriorated as the average time to detect the same size of shift was increased when the process had a multivariate non-normal distribution. Compared to Hotelling's T^2 chart, the MZ chart offered much better performance in detecting sustained shifts in the mean vector. It could also be designed to be robust to non-normality for both in-control and out-of-control cases across a wide range of shift sizes. For robust MZ charts, Stoumbos and Sullivan recommended a smoothing parameter (λ) between 0.02 and 0.05. A λ value below 0.02 would ensure robust performance when the underlying distribution was highly skewed or extremely heavy-tailed. Moreover, λ could be larger to give the same robustness if the sample size can be increased. However, presumably $\lambda < 0.02$ will give slow detection of large shifts in $\boldsymbol{\mu}$. The authors did not address the issue of lower efficiency for the robust MZ chart.

2.2.3 Summary

In practice, the normality assumption is usually difficult to justify and often not appropriate, especially for multivariate processes. The development of robust control charts has received a considerable amount of attention from researchers. Nonparametric control charts are distribution-free but usually less powerful for detecting shifts compared to conventional charts. The nonparametric multivariate charts can be obscure for practitioners. And they may not apply to skewed distributions and may require a sample size great than one. Other observation-based robust designs generally try to reduce the impact of highly skewed distributions or distributions with heavy tails by adding weights or making some adjustment to the original form. The issue of robustness is even more complex for multivariate charts compared to univariate ones. Though there are many methods in the literature that have been proposed for constructing multivariate control charts, only a few robust multivariate charts were developed, especially for monitoring Σ. The existing robust multivariate charts are mostly evaluated with independent variables and thus the performance of robust control charts with correlated variables has not been adequately investigated.

2.3 The setup for performance evaluation, charts comparison and sampling plan

In this dissertation, we will use ATS as the measure of control chart performance. In order to make fair comparisons among different charts or chart combinations, the control limits for all of the charts or chart combinations are adjusted to give the same in-control ATS of 200 time units. All the results are based on 100,000 simulation runs so that the simulation error can be reduced to a very low level.

The ATS is counted from the start of a control chart. However, in applications a shift mostly happens at some random time after monitoring has started, not at the starting point of monitoring. A fair measure of the efficiency of control charts to detect a shift should be the time to give a signal after the shift occurs. Thus the ATS measure could be misleading and does not present the true expected time from the random point that the shift occurs to the time that a signal is generated. Moreover, EWMA or MEWMA statistics accumulate information over time. The expected length of time from the time point of the shift to time point of a signal depends on the values of the control statistics at the time point of the shift.

Therefore, the steady-state ATS (SSATS) is used as the measure of how efficiently the charts or chart combinations detect sustained shifts in process parameters. The SSATS is computed assuming that the control statistic has arrived at its steady-state distribution by the random time point when the shift occurs. The SSATS also allows for the possibility that a shift occurs randomly within the time interval between two consecutive samples. In other words, it is assumed that the change point within the interval is uniformly distributed and thus the expected change point is the midpoint of that interval. In this dissertation, the SSATS is evaluated in a scenario where the sustained shift in a parameter is introduced after 100 in-control observations are obtained.

In later chapters, the robust design for the proposed MEWMA-type charts will be discussed for bivariate processes and higher-dimensional processes. When $p=2$, $\rho_{ij}=0$ and $\rho_{ij}=0.9$ are considered to represent the cases of independence and high correlation respectively. When $p>2$, more complicated correlation structures will be discussed.

To investigate the out-of control performance of the proposed charts, we mainly consider random shifts in $\boldsymbol{\sigma}$ with the correlation between variables unchanged. But we also consider several cases where the correlation and variances change at the same time. Additionally, we investigate the charts' performances in detecting shifts in some special directions without any correlation change.

When comparing the performance in detecting shifts in $\boldsymbol{\mu}$, the shift size is specified in terms of the standardized mean shift vector, defined as $\boldsymbol{\omega}=(\omega_1,\omega_2,\ldots\omega_p)^T$, where

$$\omega_i=(\mu_i-\mu_{0i})/\sigma_{0i}, \qquad i=1,2,\ldots,p.$$

Lowry et al. (1992) pointed out that the out of control performance of the MZ chart depends on the mean vector only through the noncentrality parameter

$$\delta=\sqrt{\omega^T\,\Sigma_{Z0}^{-1}\,\omega}\,.$$

This property is true for the MZ chart, but a corresponding property does not hold for charts for monitoring Σ such as the MRZ^2 chart. The SSATS values of the MRZ^2 chart actually vary across different shift directions, thus an average SSATS is given over all shift directions by generating random shift directions in 100,000 iterations. A random shift direction is generated by the following steps. First, define a vector

$\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots \beta_p)^T$, with $\sum_{i=1}^{p} \beta_i^2 = 1$, where $\boldsymbol{\beta}$ is a randomly selected point on the unit sphere in p-dimensional space. Then a shift of $\boldsymbol{\mu}$ in the direction $\boldsymbol{\beta}$ corresponding to a specific value of the noncentrality parameter is obtained by letting $\boldsymbol{\omega} = \delta\boldsymbol{\beta} / \sqrt{\boldsymbol{\beta}^T \Sigma_{Z0}^{-1} \boldsymbol{\beta}}$.

To detect changes in Σ, the shifts in $\boldsymbol{\sigma}$ are standardized as $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots \gamma_p)^T$, where

$$\gamma_i = \sigma_i / \sigma_{0i}, \qquad i = 1, 2, \ldots, p.$$

And the size of a shift in $\boldsymbol{\sigma}$ is expressed in terms of

$$\psi = 1 + \sqrt{(\gamma - 1)^T (\gamma - 1)} = 1 + \left(\sum_{i=1}^{p} (\gamma_i - 1)^2\right)^{1/2}.$$

Thus we have $\boldsymbol{\gamma}$ = **1** and ψ = 1 when $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0$ (in control case) and $\boldsymbol{\gamma}$ = **1**+(ψ-1) $\boldsymbol{\beta}$ when the shift in $\boldsymbol{\gamma}$ is from **1** in the direction of **1**+$\boldsymbol{\beta}$. Here we initially look at the case of increases in $\boldsymbol{\sigma}$, as this is usually what is of the most interest. Thus we use values of $\boldsymbol{\beta}$ with nonnegative components.

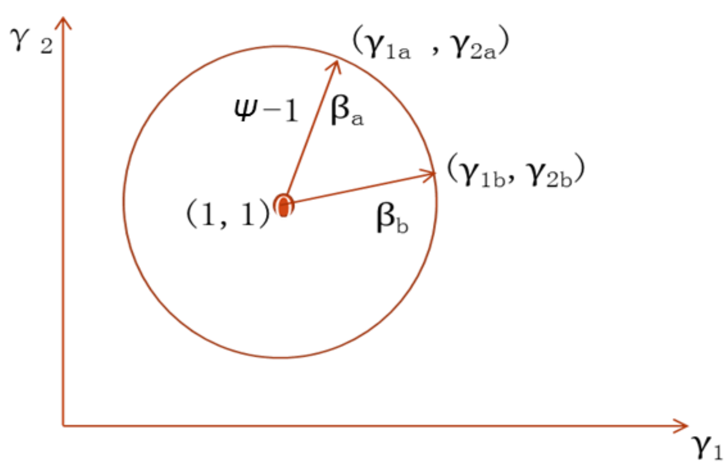

Figure 2.1. The shift size and shift directions

Figure 2.1 demonstrates the relationship among the total shift size ψ, the individual shift size γ_i, and the random shift directions $\boldsymbol{\beta}$ for a bivariate process. As we can see from Figure 2.1, following two random shift directions $\boldsymbol{\beta_a}$ and $\boldsymbol{\beta_b}$, (γ_1, γ_2) moves from (1, 1) to $(\gamma_{1a}, \gamma_{2a})$ and $(\gamma_{1b}, \gamma_{2b})$, respectively, where their total shift sizes are the same ψ. Please notice that γ is only at the upper right quarter of the circle since we focus on the case of increases in $\boldsymbol{\sigma}$.

When we compare the efficiency of control charts in detecting a range of shift sizes, the SSATS of one chart can be smaller than other charts for some shift sizes but larger for the other shift sizes. A one unit difference in two charts' SSATS values when it takes both charts around 10 time units to detect a shift is more significant than a one unit difference in SSATS value when it takes both charts around 50 time units to detect a shift. So it is not completely clear which chart has the best overall performance. Some measure of the overall performance will be useful when we compare charts' performance over a wide range of shift sizes. Thus the total quadratic loss associated with the operation of control charts is used here to compare different charts or chart combinations across a range of shift sizes.

Quadratic loss has been used to evaluate control charts which monitor the mean and/or the variance of a univariate process (see Domangue and Patch (1991), Reynolds and Stoumbos (2004), Reynolds and Lou (2010) and Wu et al. (2008)). Mohebbi and Hayre (1989) and Tsui and Woodall (1993) proposed multivariate control charts based on loss function and showed the use of loss functions could be meaningful for comparison of the performance of multivariate control charts. Here we extend the concept of total quadratic loss associated with multivariate control charts and propose a measurement for evaluating multivariate control charts' overall performance when detecting a wide range of shift sizes.

Let $\mathbf{X}(t)$ represent an observation taken at time t, where $\mathbf{X}(t)$ is p-dimension vector. The quadratic loss is defined based on the Mahalanobis distance (Mahalanobis (1936)) between $\mathbf{X(t)}$ and $\boldsymbol{\mu_0}$. Let QL represent the expected quadratic loss at time t, so

$$\mathrm{QL} = \mathrm{E}((\mathbf{X}(\mathrm{t}) - \boldsymbol{\mu}_{\mathbf{0}})^{\mathrm{T}} \Sigma_0^{-1} (\mathbf{X}(\mathrm{t}) - \boldsymbol{\mu}_{\mathbf{0}})) = \mathrm{trace}(\Sigma_0^{-1} \Sigma) + (\boldsymbol{\mu} - \boldsymbol{\mu}_{\mathbf{0}})^{\mathrm{T}} \Sigma_0^{-1} (\boldsymbol{\mu} - \boldsymbol{\mu}_{\mathbf{0}}),$$

where $\boldsymbol{\mu}$ and Σ are the mean and covariance matrix at time *t*.

When a process is in control, the quadratic loss is the trace of a *p*-dimension identity matrix, which is equal to *p*. If a mean shift of size δ occurs, the extra quadratic loss is $p + \delta^2 - p = \delta^2$. This extra loss is incurred until the shift is detected, so the total extra loss due to this shift, named as EQL(δ), is

$$\mathrm{EQL}(\delta) = \mathrm{E}\left(\int_{\tau^*}^{\mathrm{T}} \delta^2 \mathrm{dt}\right) = \delta^2 \mathrm{E}(\mathrm{T} - \tau^*) = \delta^2 \mathrm{SSATS},$$

where *T* is the signal time and τ^* is the shift time.

If a variance shift of size ψ occurs while the mean of a process is unchanged, the extra quadratic loss is just $trace(\Sigma_0^{-1}\Sigma) - p$, which will depend on $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots \gamma_p)^T$, the vector of standard deviations for the *p* standardized variables in the shift direction $\beta + 1$ and constrained to a total shift size of $\psi = 1 + \sqrt{(\boldsymbol{\gamma} - 1)^{\mathrm{T}}(\boldsymbol{\gamma} - 1)}$. Generally,

$$trace(\Sigma_0^{-1}\Sigma) - p =$$
$$trace\left(\begin{bmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,p} \\ \rho_{1,2} & \ddots & & \vdots \\ \vdots & & \ddots & \rho_{p-1,p} \\ \rho_{1,p} & \cdots & \rho_{p-1,p} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_1^2 & \rho_{1,2}\gamma_1\gamma_2 & \cdots & \rho_{1,p}\gamma_1\gamma_p \\ \rho_{1,2}\gamma_1\gamma_2 & \ddots & & \vdots \\ \vdots & & \ddots & \rho_{p-1,p}\gamma_{p-1}\gamma_p \\ \rho_{1,p}\gamma_1\gamma_p & \cdots & \rho_{p-1,p}\gamma_{p-1}\gamma_p & \gamma_p^2 \end{bmatrix}\right) - p$$

For the case of $p = 2$, it reduces to

$$trace(\Sigma_0^{-1}\Sigma) - p = trace\left(\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}^{-1} \begin{bmatrix} \gamma_1^2 & \rho\gamma_1\gamma_2 \\ \rho\gamma_1\gamma_2 & \gamma_2^2 \end{bmatrix}\right) - p = \frac{\gamma_1^2 + \gamma_2^2 - 2\rho^2\gamma_1\gamma_2}{1 - \rho^2} - 2.$$

The total extra quadratic loss at a shift size of ψ is defined as

$$EQL(\psi) = E\left(\int_{\tau^*}^{T} (trace(\Sigma_0^{-1}\Sigma) - p)dt\right) = E\left(\int_{\tau^*}^{T} trace(\Sigma_0^{-1}\Sigma)dt\right) - p \times SSATS$$

Unlike the mean shift case, the distribution of T depends on Σ (or the shift direction) so that the integral cannot be written as $(trace(\Sigma_0^{-1}\Sigma) - p) \times SSATS$. An average extra quadratic loss is evaluated over all shift directions for a given shift size ψ by generating random shift directions in 100,000 simulation iterations.

For both mean shifts and variance shifts, we will consider a wide range of shift sizes. In order to get the total EQL averaged over all shift sizes, we need some prior distributions for δ and ψ, say $\pi(\delta)$ and $\pi(\psi)$. Thus the total EQL over a range of shift sizes, represented as TEQL, is

$$TEQL = \int EQL(\delta)\pi(\delta)d\delta$$

for mean shifts and

$$TEQL = \int EQL(\psi)\pi(\psi)d\psi$$

for variance shifts, where the integral is taken over the range of shift sizes of interest and can be approximated by Gaussian quadrature. In this dissertation, ψ will be some values taken from the interval [1.1, 20]. The upper limit of 20 is chosen to represent a very large shift and generally will be detected in only 1 or 2 observations. Three prior distributions for ψ are considered (see Figure 2.2). A uniform prior on the interval [1.1, 20] is used to represent equal probability for each shift size. However, small or median shifts may be more likely than large shifts in practice. Two beta distributions on the interval [1.1, 20] which are skewed to the right are chosen as prior distributions to represent these situations. From Figure 2.2, we can tell that the beta(2,8) and beta(2,18) priors give more weight to the moderate and small shift sizes, respectively.

2.4 Non-normal multivariate distributions considered

In general, robust control charts should present stable performance over a variety of distributions. Technically, the control limits set to achieve the desired in-control ATS for the normal distribution should give a similar false alarm rate for other non-normally distributed processes. We simulate data from several non-normal multivariate distributions to assess the robust performance of control charts.

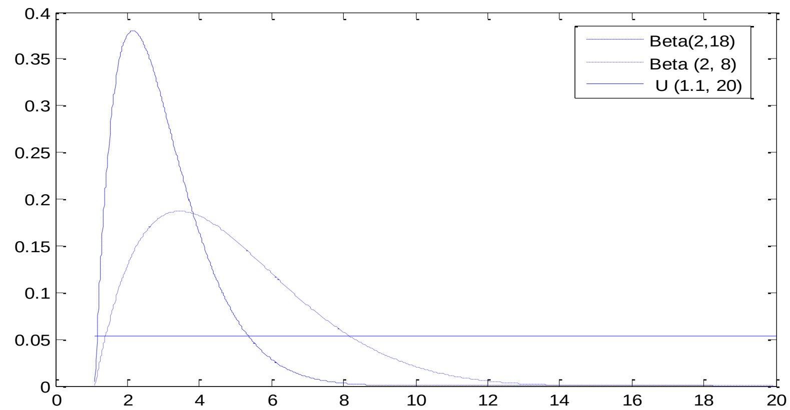

Figure 2.2. PDF plot for 3 priors for ψ on the interval [1.1, 20].

The first non-normal distribution considered is the multivariate central *t* distribution, which is symmetric around the zero mean vector in the in-control state, and has heavier tails compared to multivariate normal. The multivariate *t* distribution that we consider in this dissertation is often called "the general multivariate *t* distribution". If $\mathbf{Z}$ is a standardized *p*-variate normal distribution with zero mean vector and covariance matrix Σ, which is also the correlation matrix, and S^2 is an independent chi-squared distribution with ν degrees of freedom, then $\mathbf{Y} = \boldsymbol{\mu} + \mathbf{Z} / \sqrt{S^2 / \nu}$ is a *p*-variate *t* distribution with ν degrees of freedom, $E(\mathbf{Y}) = \boldsymbol{\mu}$ and $Var(\mathbf{Y}) = \nu \Sigma / (\nu - 2)$ when $\nu > 2$ (see Anderson,1984), expressed as $T_p(\nu)$. When we increase the degrees of freedom ν, the $T_p(\nu)$ distribution approaches the multivariate standard normal, expressed as $N_p(0, \Sigma_{Z0})$. Thus we chose ν to be 4 and 10 in order to simulate one multivariate *t* distribution which is quite distinct from the $N_p(\boldsymbol{0}, \Sigma_{Z0})$, and one which is closer to the normal.

Multivariate gamma and beta distributions are also considered here. The former is skewed and the latter is symmetric when the two shape parameters are equal but is lighter-tailed compared to the multivariate normal. We represent the *p*-variate gamma distributions as $G_p(r,s)$ where *r* is the shape parameter and *s* is the scale parameter. *P*-

variate beta distributions are represented as $\mathrm{B_p}(\alpha,\beta)$, where α and β are the two shape parameters. The covariance matrix is not included in the notation for simplicity. Their in-control correlation matrixes are the same as $\sum_{Z0}$. Generally, we consider the in-control covariance matrix ($\sum_{Z0}$) for all the p-variate distributions as

$$\begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & \ddots & & \vdots \\ \vdots & & \ddots & \rho \\ \rho & \cdots & \rho & 1 \end{bmatrix}_{p\times p}$$

which assumes that correlations between any two pairs of variables are equal.

When the p variables are independent, we just simulate multiple variables from univariate gamma or beta distributions independently. For $\rho \neq 0$, several forms of multivariate gamma and beta distributions have been proposed (see, e.g., Wagle (1968), Johnson and Kotz (1972), Mathal and Moschopoulos (1992), and Jones (2002)). In this dissertation, the multivariate gamma and beta data are obtained from the mixture simulation mechanism for p-variate distributions proposed by Minhajuddin et al. (2004).

To obtain observations from the $\mathrm{G_p}(r,s)$ distribution with correlation coefficient $\rho \neq 0$, we simulate a negative binomial observation L with parameters r and π, where $\pi = \mathrm{s}/(\mathrm{s}+\theta)$ and $\theta = s\rho\ /(1-\rho)$. Conditional on $L = l$, we independently generate $X_1, X_2, \ldots, X_p$ from the $\mathrm{G_1}\ (r+l, s+\theta)$ distribution with probability density function (PDF):

$$p(x \mid l\ ;\ \ r,s,\theta) = \frac{(s+\theta)^{r+l}}{\Gamma(r+l)} x^{r+l-1} e^{-(s+\theta)x}\ ,\ x \geq 0.$$

The joint distribution of $X_1, X_2, \ldots, X_p$ unconditional on L is a general $G_p(r,s)$ distribution. It can be shown that the marginal distribution of X_i is $\mathrm{G_1}(r,s)$ and that the correlation coefficient between two X_i and X_j for $i \neq j$ is ρ. The two multivariate gamma distributions considered here are $G_p(4,1)$ and $G_p(8,1)$. Without loss of generality, the scale parameters are set to be one. If we want to generate observations from the $G_2(4,1)$ distribution with $\rho = 0.9$, we will take $r = 4$, $\pi = 0.1$, $s = 1$, and $\theta = 9$.

To simulate data from a multivariate $B_p(\alpha, \beta)$ distribution with correlation coefficient $\rho \neq 0$, let M be distributed as a beta-binomial (Casella and Berger, 2001) variable with parameters v, α, and β, where v is a function of α, β, and ρ. Conditional on $M = m$, we independently generate p random variables $X_1, X_2, \ldots, X_p$ from a $B_1(\alpha + m, v + \beta - 1)$ distribution with PDF

$$p(x|m\ ;\ \alpha, \beta, v) = \frac{x^{\alpha+m-1}(1-x)^{v+\beta-m-1}}{\mathrm{B}(\alpha+m, v+\beta-m)}.$$

The joint distribution of $X_1, X_2, \ldots, X_p$ unconditional on M is $B_p(\alpha, \beta)$ .The correlation coefficient between any two X_i and X_j for $i \neq j$ is ρ. The multivariate beta distribution involved in our discussion is $B_p(4,4)$. Given $\alpha = 4$, $\beta = 4$, and $\rho = 0.9$, we have $v = 72$ (see Minhajuddin et al. (2004) for more details).

Figure 2.3 shows 10,000 standardized observations (**Z**) with $\rho = 0.9$ from the $N_2(\boldsymbol{0}, \Sigma_{Z0})$, $B_2(4,4)$, $T_2(4)$, $T_2(10)$, $G_2(4,1)$, and $G_2(8,1)$ distributions, respectively. When two variables are highly correlated, bivariate normal, *t*, and beta observations exhibit an elliptical shape, while gamma observations form a cometic shape. Looking at the scale of the horizontal and vertical axes, it is not surprising that the *t* observations present some extreme outliers in the right upper and left lower directions, while the gamma observations have some mild outliers in the right upper direction. The beta observations seem to be even more compact than the normal ones.

Figure 2.4 presents independent bivariate observations from the same distributions which are also standardized. The observations in 6 subplots spread out more in the off-diagonal direction compared to Figure 2.3. The two subplots with gamma data look like a quarter of a circle while the other four change the shape from an ellipse to a circle.

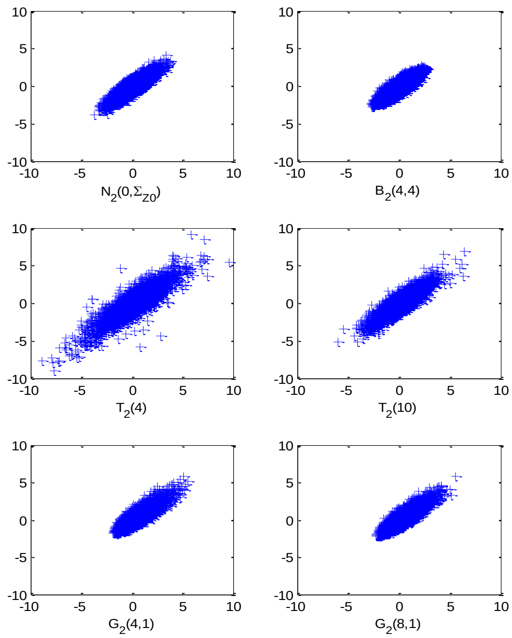

Figure 2.3. Standardized observations from different bivariate distributions with $\rho = 0.9$.

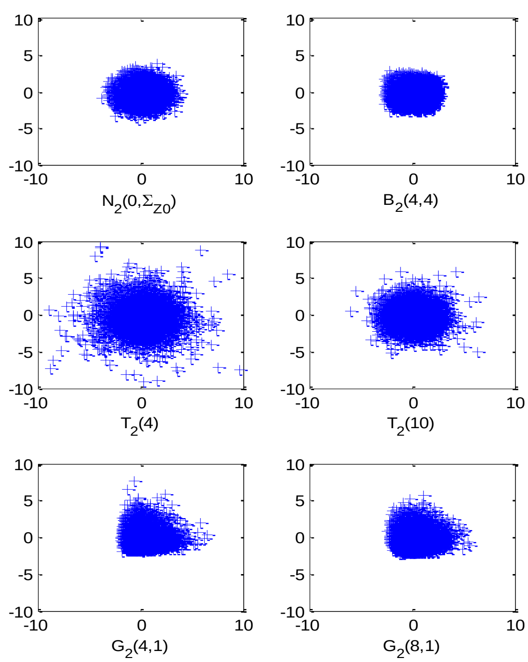

Figure 2.4. Standardized observations from different bivariate distributions with $\rho = 0$.

For high-dimensional processes, we additionally consider the following AR(1) correlation structure to discuss control chart performance when correlations between variables can range from high to low.

$$\begin{bmatrix} 1 & \rho & \cdots & \rho^{p-1} \\ \rho & \ddots & & \vdots \\ \vdots & & \ddots & \rho \\ \rho^{p-1} & \cdots & \rho & 1 \end{bmatrix}_{p\times p}$$

Simulation methods for the multivariate normal and *t* distributions with the above correlation matrix are available. However, how to generate multi-gamma and beta observations with such a correlation structure is still under study. Here we refer to the multivariate gamma distribution proposed by Krishnaiah (1985) and manage to obtain multi-gamma observations with a correlation structure approximately equal to the specified correlation matrix. This multivariate gamma distribution is given as one-half the diagonal elements of a matrix having a Wishart distribution. A symmetric matrix has a Wishart distribution with m degrees of freedom and scale matrix Σ if it can be written as $\mathbf{X}^{\mathbf{T}}\mathbf{X}$, where $\mathbf{X}$ is a data matrix of m observations from the multivariate normal distribution with zero mean vector and covariance matrix Σ.

Chapter 3

The Proposed Control Charts

From the literature view, we notice that EWMA-type control charts can be designed to be robust by using a small value of λ, which means the impact of the current observation on the control statistic will be largely averaged by previous ones. The square root of the absolute deviation or the absolute deviation from target will suffer less than the squared deviation from the highly skewed or heavy-tailed observations. Additionally, the Winsorization method can significantly improve the robustness of univariate CUSUM charts. Inspired by these ideas, we propose several MEWMA-type control charts, which can be tuned to be robust for various multivariate processes.

3.1 The MEWMA control chart based on a power of the absolute deviation from target

Suppose we need to monitor a multivariate process using standardized random variables, where the correlation coefficient between any two variables is ρ_{ij}, $i \neq j$. For the i^{th} variable at sampling time k, define

$$ER_{ki}^{|Z|^q} = (1-\lambda)Max\{ER_{0,i}^{|Z|^q}, ER_{k-1,i}^{|Z|^q}\} + \lambda\left(\sum_{j=1}^{n} | Z_{kij} |^q / n\right),$$
$$i = 1,2,\ldots p;\ \ j = 1,2\ldots,n,$$

where n is the sample size. The power parameter q can be any positive number. When q is 2, the ER|Z|^q chart becomes the ERZ^2 chart. The one-sided MEWMA-type control chart based on a power of the absolute deviation from target value is defined for sampling time k as

$$MR_k^{|Z|^q} = (ER_{k1}^{|Z|^q}, ER_{k2}^{|Z|^q}, \dots ER_{kp}^{|Z|^q})(\Sigma_{ER|Z_0|^q})^{-1}(ER_{k1}^{|Z|^q}, ER_{k2}^{|Z|^q}, \dots ER_{kp}^{|Z|^q})^T.$$

where $\Sigma_{ER|Z_0|^q}$ is the in-control correlation matrix of $E_{k1}^{|Z|^q}, E_{k2}^{|Z|^q}, \dots E_{kp}^{|Z|^q}$ and will be derived below.

For the purpose of robust design, we consider positive values of q less than 2. Since we generally don't know the true distribution for a process, we design the proposed chart based on multivariate normal observations. Then we adjust the parameter values to make the charts robust to non-normal observations.

In order to construct the MEWMW statistics based on the $ER|Z|^q$ statistics, we need to find the starting point $ER_{0,i}^{|Z|^q}$, the expected value of $|Z_{kij}|^q$ when Z_{kij} ~ N(0,1), and the in-control covariance matrix of $E_{k1}^{|Z|^q}, E_{k2}^{|Z|^q}, \dots E_{kp}^{|Z|^q}$, which is used as the approximate in-control covariance matrix of $ER_{k1}^{|Z|^q}, ER_{k2}^{|Z|^q}, \dots ER_{kp}^{|Z|^q}$. The derivations are as follows, with the subscripts k and j dropped from Z_{kij} for simplicity.

$$E(|Z_i|^q) = \int_{-\infty}^{\infty} \frac{|z|^q}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} dz = 2\int_0^{\infty} \frac{z^q}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} dz = 2^{\frac{q}{2}} \Gamma(\frac{q+1}{2})/\sqrt{\pi}. \quad (1)$$

$$E(|Z_i|^{2q}) = \int_{-\infty}^{\infty} \frac{|z|^{2q}}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} dz = 2\int_0^{\infty} \frac{z^{2q}}{\sqrt{2\pi}} \exp(-\frac{1}{2}z^2) dz = 2^{\frac{q}{2}} \Gamma(\frac{2q+1}{2})/\sqrt{\pi}$$

Thus

$$Var(|Z_i|^q) = E(|Z_i|^{2q}) - E(|Z_i|^q)^2 = 2^q \Gamma(\frac{2q+1}{2})/\sqrt{\pi} - 2^q \Gamma(\frac{q+1}{2})^2/\pi. \quad (2)$$

We rewrite $E_{ki}^{|Z|^q}$ as

$$E_{ki}^{|Z|^q} = (1-\lambda)^k E_{0,i}^{|Z|^q} + \frac{1}{n} \sum_{t=0}^{k-1} \lambda (1-\lambda)^i (\sum_{j=1}^{n} |Z_{k-t,ij}|^q), \quad i = 1,2,\dots,p.$$

Then

$$Var(E_{ki}^{|Z|^q}) = \frac{1}{n^2}\sum_{t=0}^{k-1}\lambda^2(1-\lambda)^{2i}Var(\sum_{j=1}^{n}|Z_{k-t,ij}|^q) = Var(|Z_i|^q)\frac{\lambda[1-(1-\lambda)^{2k}]}{(2-\lambda)n}$$

The asymptotic variance of $E_{ki}^{|Z|^q}$ is obtained by letting $k \to \infty$, which gives

$$Var(E_{ki}^{|Z|^q}) = \frac{C_\infty}{n}Var(|Z_i|^q),$$

where $C_\infty = \lambda/(2-\lambda)$.

Next, we calculate the covariance between any two $E_{ki}^{|Z|^q}$:

$$\begin{aligned} Cov(E_{ki}^{|Z|^q}, E_{ki'}^{|Z|^q}) &= Cov(\frac{1}{n}\sum_{t=0}^{k-1}\lambda(1-\lambda)^i\sum_{j=1}^{n}|Z_{k-t,ij}|^q, \frac{1}{n}\sum_{t=0}^{k-1}\lambda(1-\lambda)^i\sum_{j=1}^{n}|Z_{k-t,i'j}|^q) \\ &= \frac{1}{n}\sum_{t=0}^{k-1}\lambda^2(1-\lambda)^{2i}Cov(|Z_{k-t,ij}|^q, |Z_{k-t,i'j}|^q) \\ &\xrightarrow{k\to\infty} \frac{C_\infty}{n}Cov(|Z_i|^q, |Z_{i'}|^q) \end{aligned}$$

And

$$Cov(|Z_i|^q, |Z_{i'}|^q) = E(|Z_i|^q|Z_{i'}|^q) - E(|Z_i|^q)E(|Z_{i'}|^q) = E(|Z_iZ_{i'}|^q) - 2^q\Gamma(\frac{q+1}{2})^2/\pi. \quad (3)$$

$$\begin{aligned} E(|Z_iZ_{i'}|^q) &= \iint_R \frac{|z_iz_{i'}|^q}{2\pi\sqrt{1-\rho_{ij}^2}}\exp(-\frac{z_i^2+z_{i'}^2-2\rho_{ij}z_iz_{i'}}{2(1-\rho_{ij}^2)})dz_idz_{i'} \\ &= \frac{[2(1-\rho_{ij}^2)]^{\frac{2q+1}{2}}}{\sqrt{2\pi}}\Gamma^2(\frac{q+1}{2})\,{}_2F_1(\frac{1}{2}(q+1),\frac{1}{2}(q+1);\frac{1}{2};\rho_{ij}^2) \end{aligned} \quad (4)$$

where F is the generalized hypergeometric function. A generalized hypergeometric function is a function which can be defined in the form of a hypergeometric series (see Abramowitz and Stegun (1972)):

$${}_2F_1(a,b;c;X) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}$$

Let's name the in-control correlation matrix of $|Z_1|^q, |Z_2|^q, \dots |Z_p|^q$ as $\Sigma_{|Z0|^q}$. The off-diagonal elements of this correlation matrix are

$$\rho_{|z_i|^q,|z_{i'}|^q} = Cov(|Z_i|^q, |Z_{i'}|^q) / Var(|Z_i|^q),$$

the correlation coefficient between $|Z_i|^q$ *and* $|Z_{i'}|^q, i \neq i'$, which can be obtained by plugging in equations (2), (3), and (4). Thus the in-control covariance matrix of $ER_{k1}^{|Z|^q}, ER_{k2}^{|Z|^q}, \dots ER_{kp}^{|Z|^q}$ is approximate $C_\infty Var(|Z_i|^q) \Sigma_{|Z_0|^q} / n,$ and the control statistic of the proposed chart is

$$MR_k^{|Z|^q} = \frac{n}{c_\infty Var(|Z|^q)} (ER_{k1}^{|Z|^q}, ER_{k2}^{|Z|^q}, \dots ER_{kp}^{|Z|^q}) (\Sigma_{|Z_0|^q})^{-1} (ER_{k1}^{|Z|^q}, ER_{k2}^{|Z|^q}, \dots ER_{kp}^{|Z|^q})^{\mathrm{T}}.$$

We call this chart the $\mathrm{MR}|\mathrm{Z}|^q$ chart. When $q = 2$, the proposed chart actually becomes the MRZ^2 chart. In order to make the control limit of the $\mathrm{MR}|\mathrm{Z}|^q$ chart in the same scale as the control limit of the set of EWMA-type charts introduced later, let the $\mathrm{MR}|\mathrm{Z}|^q$ chart signals if

$$\mathrm{C}_\infty \mathrm{Var}(|\mathrm{Z}|^q)\mathrm{MR}_k^{|Z|^q}/\mathrm{n} > \mathrm{UCL}_{\mathrm{MR}|\mathrm{Z}|^q}$$

Although the $\mathrm{MR}|\mathrm{Z}|^q$ chart is designed based on expectations of standard normal observations, the $\mathrm{MR}|\mathrm{Z}|^q$ chart can be designed to approximately give a desired in-control ATS across a large range of multivariate distributions by decreasing the power q from 2 and adjusting the smoothing parameter λ correspondingly.

By decreasing the power of the absolute deviation, we decrease the impact of extreme observations on our control statistic but also lose some efficiency in detecting shifts in $\boldsymbol{\sigma}$ when the observations are multivariate normal. So it is reasonable to not go too far below 2 and choose a value of q between 0.5 and 2 to achieve robustness for the in-control case and still maintain a relatively high efficiency for the out-of-control case. Table 3.1 lists the values of $E(|Z_i|^q)$, $Var(|Z_i|^q)$, and $\rho_{|z_i|^q,|z_{i'}|^q}$ for q starting from 0.5 to 2 by an increment of 0.1, which can be calculated from equations (1)-(4). We will look into two cases, one where variables are highly correlated ($\rho_{ij} = 0.9$), and one where the variables are independent ($\rho_{ij} = 0$). For the later, the value of $\rho_{|z_i|^q,|z_{i'}|^q}$ is also 0 so that we provide the $\rho_{|z_i|^q,|z_{i'}|^q}$ value for $\rho_{ij} = \pm 0.9$ in Table 3.1. It is not surprising that $\rho_{ij} = 0.9$ and $\rho_{ij}' = -0.9$ result in the same $\rho_{|z_i|^q,|z_{i'}|^q}$, since we are taking the absolute value of the original standardized variables.

Table 3.1. The expected value and variance of $|Z_i|^q$ and $\rho_{|z_i|^q,|z_{i'}|^q}$ for different value of q

Power(q)	$E(\|Z_i\|^q)$	$Var(\|Z_i\|^q)$	ρ_{ij}	$\rho_{\|z_i\|^q,\|z_{i'}\|^q}$
0.5	0.8222	0.1219	±0.9	0.7027
0.6	0.8087	0.1596	±0.9	0.7239
0.7	0.8000	0.2016	±0.9	0.7413
0.8	0.7955	0.2487	±0.9	0.7558
0.9	0.7950	0.3022	±0.9	0.7676
1	0.7979	0.3634	±0.9	0.7773
1.1	0.8041	0.4340	±0.9	0.7852
1.2	0.8136	0.5159	±0.9	0.7916
1.3	0.8260	0.6116	±0.9	0.7968
1.4	0.8415	0.7238	±0.9	0.8008
1.5	0.8600	0.8561	±0.9	0.8040
1.6	0.8816	1.0126	±0.9	0.8064
1.7	0.9063	1.1984	±0.9	0.8081
1.8	0.9341	1.4196	±0.9	0.8092
1.9	0.9653	1.6837	±0.9	0.8098
2	1.0000	2.0000	±0.9	0.8100

Finding the expected value, variance, and correlation coefficient is the first step in the construction of the MR$|Z|^q$ chart; we need to further find an appropriate combination of q and λ to achieve robustness under different scenarios, which is discussed in detail in Chapter 4.

3.2 The MRZ^2 control chart based on Winsorized observations

Hawkins and Olwell (1998) recommended obtaining robust CUSUM charts by replacing the original observations with Winsorized observations for the univariate process. In particular, we define the j^{th} Winsorized observation at sampling time k as

$$W_{kij} = \begin{cases} \mu_0 - w\sigma_0 & \text{if } Z_{kij} < \mu_0 - w\sigma_0 \\ Z_{kij} & \text{if } \mu_0 - w\sigma_0 \le Z_{kij} \le \mu_0 + w\sigma_0 \quad i = 1,2,\dots p \\ \mu_0 + w\sigma_0 & \text{if } Z_{kij} > \mu_0 + w\sigma_0 \end{cases}$$

where $w > 0$ is the parameter that can be adjusted to limit the effect of any extreme observations outside of $\mu_0 \pm w\sigma_0$. We then monitor $\mathbf{W} = (W_1, W_2, \dots, W_p)$ instead of $\mathbf{Z}$. In the univariate case, Hawkins and Olwell (1998) suggested a value of w around 2 to monitor μ while Reynolds and Stoumbos (2009) suggested w = 2.5 to monitor σ.

Similar to the statistics based on the original observations, we define the one-sided EWMA control statistic based on the individual Winsorized observations as

$$ER_{ki}^{W^2} = (1-\lambda)\max(ER_{0,i}^{W^2}, ER_{k-1,i}^{W^2}) + \lambda(\sum_{j=1}^{n} W_{kij}^2 / n), \quad i = 1,2,\cdots,p.$$

Then the control statistic for multivariate observations is

$$MR_k^{W^2} = \frac{n}{C_\infty Var(W^2)}(ER_{k1}^{W^2}, ER_{k2}^{W^2}, \dots ER_{kp}^{W^2})(\Sigma_{w0}^2)^{-1}(ER_{k1}^{W^2}, ER_{k2}^{W^2}, \dots ER_{kp}^{W^2})^T$$

where Σ_{W0}^2 is the in-control correlation matrix of $W_1^2, W_2^2, \dots W_p^2$. This chart is named the MRW^2 chart and signals if

$$\mathrm{C_\infty Var(W^2) MR_k^{W^2}/n > UCL_{MRW^2}}.$$

For the MRW^2 control statistics, we also need to find the starting point $ER_{0,i}^{W^2}$, which is E(W_i^2), the value of Var(W_i^2), and the in-control correlation matrix of Σ_{w0}^2.

The derivations are as follows, with the subscripts k and j dropped from W_{kij} for simplicity.

$$
\begin{aligned}
E(W_i^2) &= \int_{-w}^{w} \frac{z^2}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + w^2 \int_{w}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + w^2 \int_{-\infty}^{-w} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= w^2 - (w^2 - 1) erf(w/\sqrt{2}) + (2w/\sqrt{2\pi}) e^{-\frac{1}{2}w^2}.
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
E(W_i^4) &= \int_{-w}^{w} \frac{z^4}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + w^4 \int_{w}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz + w^4 \int_{-\infty}^{-w} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\
&= -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}w^2} (2w^3 + 6w) + w^4 + (3 - w^4) erf(w/\sqrt{2})
\end{aligned} \tag{6}
$$

By plugging (5) and (6) into the equation $Var(W_i^2) = E(W_i^4) - E(W_i^2)^2$, we can obtain the variance of W_i^2.

However, we do not get a closed form expression for the correlation coefficient between any two $E_{ki}^{W^2}$. Although the sample correlation based on 1,000,000 observations of two Winsorized variables has small variation in the fourth decimal place, simulation results show that it does not affect the in-control and out-of-control performance of the MRW^2 chart. Therefore the correlation estimate based on simulation will be used to construct the MRW^2 chart.

3.3 The set of EWMA charts

A set of univariate EWMA charts can be used together to monitor a multivariate process. Reynolds and Cho (2006) have shown that the performance of sets of EWMA charts in a multivariate normal setting are better than MEWMA-type charts in some cases.

Domangue and Patch (1991) proposed a univariate EWMA chart based on powers of the absolute deviations from target and discussed the performance of EWMA charts with powers of 0.5 and 2 for the normal distribution. Hawkins (1981) suggested that a

CUSUM chart based on the square roots of the absolute deviations from target will suffer less than the squared deviation for highly skewed or heavy-tailed observations in a univariate process. Stoumbos and Reynolds (2000) proposed a robust design for a one-sided EWMA chart based on absolute deviations from target.

To obtain a robust set of EWMA charts over a variety of multivariate distributions, we consider the set of ER$|Z|^q$ control charts and call it the SER$|Z|^q$ chart. The robust design of the SER$|Z|^q$ charts is discussed in the following chapters. The SER$|Z|^q$ chart signals if any of the ER$|Z|^q$ charts in the set signals.

For the SER$|Z|^q$ chart, we only need to derive the starting value of the ER$|Z|^q$ statistic, which can refer to the set-up value of the MR$|Z|^q$ chart in Table 3.1.

Chapter 4

The Robust Design of the Proposed Charts

In this chapter, we investigate the robust design of the proposed charts for bivariate processes with independent or highly correlated variables. Though in practice we don't know the true underlying distribution for the process, we need to obtain the control limits by some method. Multivariate normal observations are used to determine the control limits of the MR$|Z|^q$, MRW^2, and SER$|Z|^q$ charts. Simulation studies will then show that the control limits work well within a range of multivariate non-normal distributions if the combination of parameters is chosen appropriately. The out-of-control performance is also compared across the various robust designs of the three charts under multivariate normal and non-normal distributions.

4.1 Robust designs for bivariate process

To find robust designs, we start our investigate with bivariate processes. We mainly focus on two situations: highly correlated variables and independent variables. For each situation, we tried various combinations of parameters for each of three charts. Some combinations that seem to work reasonably well when the process is in control are listed in Table 4.1-4.2.

4.1.1 The robust design for the MR$|Z|^q$ chart

Table 4.1 presents the in-control performance of robust MR$|Z|^q$ charts with values of q ranging from 0.5 to 1.5. Two different values of λ are paired with each q to show how the robustness is achieved. For a given combination of q and λ, the UCL is obtained to give an in-control ATS of 200 (see the column labeled [1]) for the standardized bivariate normal observations. Under the same UCL, in-control ATS values simulated with the $T_2(4)$, $T_2(10)$, $G_2(4,1)$, $G_2(8,1)$, and $B_2(4,4)$ observations are shown in the columns labeled [2]-[6], respectively.

To make comparisons, in-control ATS values of the MRZ^2 chart with $q = 2$, which is designed for normal observations, are also included in Table 4.1. For the MRZ^2 chart, we set λ to be 0.025 and 0.1, which are usually efficient for detecting small and moderately large shifts, respectively, with multivariate normal observations. With both values of λ, Table 4.1 shows that the in-control ATS value of the MRZ^2 charts for heavy-tailed or skewed distributions are much smaller than the target ATS of 200, implying a higher frequency of false alarms. However, the in-control ATS for the light-tailed beta distribution is very large, which may result in an unnecessarily slow response to shifts. We can also notice that the in-control ATS value of the MRZ^2 chart improves as λ decreases from 0.1 to 0.025.

Compared to the MRZ^2 charts, the MR$|Z|^q$ charts with q less than 2 and selected λ have significantly improved the robustness performance, though some of them are not quite satisfactory. Columns [2]-[6] in Table 4.1 show that the in-control performance of the MR$|Z|^q$ charts with non-normal observations get closer to the performance with normal observations as the skewness and kurtosis of the underlying distribution gets closer to the normal. For instance, the ATS for $T_2(10)$ observations is usually closer to 200 than that for $T_2(4)$ observations, except for the cases where ATS for the $T_2(4)$ is already approximately 200. Thus we conclude that the MR$|Z|^q$ charts with the combinations of q and λ shown in Table 4.1 can ensure the similar robustness as what we present in Table 4.1 or even better robustness if the charts are applied to observations from T_2 distributions with the degrees of freedom higher than 10 or from gamma distributions less skewed than the $G_2(8,1)$ distribution.

When we change the λ value, the response of T_2 observations is very similar to G_2 observations while opposite to that of $B_2(4,4)$ observations. As λ decreases, the in-control ATS under the T_2 and G_2 distributions increases while the ATS under the $B_2(4,4)$ distribution decreases. This phenomenon may be explained by the averaging effect of λ. A small value of λ gives more weight to previous observations and reduces the impact of current extreme observations. From the multivariate point of view, the extremity of observations can be measured by their Mahalanobis distances from the target. Thus, the recommended value of λ for a given q should be moderately small to obtain robustness for both the heavy-tailed or skewed distributions and the light-tailed distributions.

We now consider finding some combinations of q and λ which give relatively good robust performance based on Table 4.1. The choices are not quite straightforward since the in-control ATS values vary among distributions. Generally, we want a robust chart to maintain an in-control ATS close to the specified value among various distributions. Lower ATS implies a higher false alarm rate while larger ATS implies losing some efficiency for detecting shifts. A "good" robust chart should be well balanced between robustness and sensitivity. Although the sensitivity of the proposed control chart will be evaluated in a later section, here we preliminarily narrow down our choice of robust designs based on the following 2 indices:

1. The percentage ATS deviation from the target ATS (the in-control ATS of 200 achieved for normal observations).

The percentage ATS deviation is calculated as the difference between the ATS for the normal and non-normal distributions divided by 200, and the values are shown in columns [7]-[11]. Generally we would like a percentage deviation close to 0. A negative deviation implies an in-control ATS less than 200. To avoid the designs that result in more than a 25% increase in the false alarm rate, we would prefer a q and λ combination whose percentage deviations for 5 non-normal distributions are approximately not smaller than -20%.

2. The root mean square of percentage deviation among non-normal distributions (RMS).

In Table 4.1 two root mean squares (RMS(1) and RMS(2)) of the percentage ATS deviations from target ATS are provided. RMS(1) is based on the percentage deviation listed in columns [7]-[11], while RMS(2) is based on columns [7]-[10], which does not involve the ATS deviation with beta observations. The investigation of the charts' performance with beta observations are for completeness and theoretical purposes. RMS(1) helps us find a combination that gives a relatively steady performance across a variety of distributions. Nevertheless, practitioners are mostly concerned about the impact of heavy-tailed or skewed distributions on control charts, and thus RMS(2) may be more practical in this sense.

We prefer a robust chart with a small RMS(2) value for practical purposes and it will be even better if both RMS(1) and RMS(2) values are small. For a given q, we would like to choose a λ which gives the smallest RMS(2) (or RMS(1) and RMS(2)) value among charts which meet our expectation for the percentage ATS deviation from the target ATS. It excludes some values of λ that yield extremely large in-control ATS values for some distributions. Thus the combinations of q and λ when $q < 1.5$ listed in the Table 4.1 generally give moderately large in-control ATS values for one or two distributions.

Since we consider both the $\rho = 0.9$ and $\rho = 0$ cases for bivariate processes, an acceptable value of λ for a given q should give relative small RMS values for both cases. For example, we prefer 0.03 to be the value of λ when $q = 1.0$. Table 4.1 shows that the acceptable value of λ for a robust design can be relatively large given a small value of q. For instance, λ is as large as 0.13 when $q = 0.5$ while as small as 0.001 when $q = 1.5$. According to Table 4.1, each q can be paired with a λ to achieve robustness. The search over q and λ combinations stops at $q = 1.5$ because λ is down to 0.001 to achieve robustness and a reasonable number of robust designs have already been achieved.

Table 4.1. In control ATS of MR|Z|^q control charts for various distributions with selected q and λ combination when p=2

ρ	q	λ	UCL	**In-control ATS**						**Percentage deviation from Normal ATS(%)**					**RMS(1) of percentage deviation (%)**	**RMS(2) of percentage deviation (%)**
				$N_2(0, \sum_{z0})$ [1]	**$T_2(4)$ [2]**	**$T_2(10)$ [3]**	**$G_2(4,1)$ [4]**	**$G_2(8,1)$ [5]**	**$B_2(4,4)$ [6]**	**$T_2(4)$ [7]**	**$T_2(10)$ [8]**	**$G_2(4,1)$ [9]**	**$G_2(8,1)$ [10]**	**$B_2(4,4)$ [11]**		
0.9	0.5	0.14	1.29475	200	202.7	197.2	196.9	192.9	203.5	1.3	-1.4	-1.6	-3.6	1.8	2.1	2.2
		0.13	1.26906	200	213.5	201.3	197.8	199.1	198.3	6.8	0.7	-1.1	-0.5	-0.8	3.1	3.4
	0.7	0.09	1.219752	200	195.7	195.8	189.9	194.5	201.1	-2.2	-2.1	-5.1	-2.8	0.5	2.9	3.2
		0.08	1.178167	200	209.7	201.5	195.7	197.6	194.1	4.8	0.8	-2.2	-1.2	-3.0	2.8	2.7
	1.0	0.04	1.081256	200	189.5	195.3	186.7	192.4	197.9	-5.3	-2.3	-6.7	-3.8	-1.1	4.3	4.8
		0.03	1.004633	200	213.5	205.0	196.9	197.8	189.5	6.8	2.5	-1.6	-1.1	-5.3	4.1	3.7
	1.2	0.02	0.99321	200	187.3	194.5	185.6	192.4	197.5	-6.3	-2.8	-7.2	-3.8	-1.3	4.8	5.3
		0.01	0.87853	200	226.4	210.7	203	202.2	182.0	13.2	5.3	1.5	1.1	-9.0	7.6	7.2
	1.5	0.002	0.86280	200	192.3	196	189.1	194.9	197.1	-3.8	-2.0	-5.5	-2.6	-1.5	3.4	3.7
		0.001	0.84181	200	198.4	198.7	192.3	196.9	194.8	-0.8	-0.7	-3.8	-1.6	-2.6	2.2	2.1
	2.0	0.1	4.36537	200	66.3	103.2	83.6	109.1	826.2	-66.9	-48.4	-58.2	-45.5	313.1	148.5	55.4
		0.025	1.91820	200	90.9	129.3	113.2	139.6	391.0	-54.6	-35.4	-43.4	-30.2	95.5	56.8	41.9
0	0.5	0.14	2.01542	200	154.4	172.6	214.8	207.6	175.5	-22.8	-13.7	7.4	3.8	-12.3	13.6	13.9
		0.13	1.98354	200	163.4	177.3	219.3	210.0	171.3	-18.3	-11.4	9.7	5.0	-14.4	12.6	12.1
	0.7	0.09	1.93679	200	156.1	174.9	205.4	202.5	178.6	-22.0	-12.6	2.7	1.3	-10.7	12.4	12.7
		0.08	1.88444	200	169.1	181.8	211.8	205.9	172.9	-15.5	-9.1	5.9	3.0	-13.6	10.5	9.6
	1.0	0.04	1.79122	200	162.0	180.1	194.7	197.2	182.4	-19.0	-10.0	-2.7	-1.4	-8.8	10.5	10.8
		0.03	1.69014	200	191.1	194.2	207.0	203.9	172.5	-6.0	-2.9	3.5	2.0	-13.8	6.8	3.3
	1.2	0.02	1.70144	200	168.3	185.5	191.0	196.0	186.0	-15.9	-7.3	-4.5	-2.0	-7.0	8.7	9.1
		0.01	1.53952	200	215.4	208.2	210.1	206.5	170.6	7.7	4.1	5.1	3.3	-14.7	8.1	5.3
	1.5	0.002	1.54999	200	187.4	195.7	190.9	196.0	189.0	-6.3	-2.15	-4.6	-2	-5.5	4.5	4.2
		0.001	1.51577	200	196.4	199.7	194.4	198.0	186.8	-1.8	-0.15	-2.8	-1	-6.6	3.3	1.7
	2.0	0.1	6.11295	200	55.0	88.0	73.8	98.9	940.5	-72.5	-56.0	-63.1	-50.6	370.3	174.4	61.1
		0.025	3.17367	200	76.8	114.3	102.9	129.6	427.9	-61.6	-42.9	-48.6	-35.2	114.0	66.7	48.0

Table 4.2. In control ATS of some SER|Z| and MRW^2 control charts for various distributions when p=2

ρ	Control Chart	λ	w	UCL	**In-control ATS**						**Percentage deviation from Normal ATS(%)**					**RMS(1) of percentage deviation (%)**	**RMS(2) of percentage deviation (%)**
					$N_2(0, \sum_{z0})$ [1]	**$T_2(4)$ [2]**	**$T_2(10)$ [3]**	**$G_2(4,1)$ [4]**	**$G_2(8,1)$ [5]**	**$B_2(4,4)$ [6]**	**$T_2(4)$ [7]**	**$T_2(10)$ [8]**	**$G_2(4,1)$ [9]**	**$G_2(8,1)$ [10]**	**$B_2(4,4)$ [11]**		
0.9	MRW^2	0.025	2.7	1.83454	200	269.3	190.9	236.4	226.5	285.0	34.7	-4.6	18.2	13.3	42.5	26.6	20.8
			2.8	1.85415	200	237.9	181.6	215.2	213.6	304.9	19.0	-9.2	7.6	6.8	52.5	25.7	11.7
		0.1	2.4	3.59476	200	258.2	191.3	255.0	243.6	286.5	29.1	-4.3	27.5	21.8	43.3	28.2	22.9
			2.5	3.72687	200	223.0	180.1	225.5	227.0	325.4	11.5	-10.0	12.8	13.5	62.7	30.0	12.0
		0.2	2.2	5.57019	200	258.1	194.5	275.2	256.5	258.8	29.1	-2.8	37.6	28.3	29.4	28.0	27.7
			2.3	5.94274	200	228.9	184.7	242.7	246.3	295.0	14.5	-7.7	21.4	23.2	47.5	26.5	17.7
	SER|Z|	0.04		1.00171	200	188.9	196.2	186.7	193.0	198.2	-5.6	-1.9	-6.7	-3.5	-0.9	4.3	4.8
		0.03		0.96310	200	214.1	207.0	197.8	198.6	186.7	7.1	3.5	-1.1	-0.7	-6.7	4.6	4.0
0	MRW^2	0.025	2.7	3.03831	200	238.7	177.6	260.8	241.3	290.7	19.4	-11.2	30.4	20.7	45.4	28.0	21.5
			2.8	3.07051	200	209.3	168.5	233.3	224.5	316.5	4.7	-15.8	16.7	12.3	58.3	28.6	13.2
		0.1	2.4	5.14075	200	191.4	166.2	296.1	266.3	282.5	-4.3	-16.9	48.1	33.2	41.3	32.9	30.5
			2.5	5.30801	200	168.3	156.0	256.6	244.8	326.9	-15.9	-22.0	28.3	22.4	63.5	34.8	22.6
		0.2	2.2	7.26634	200	178.4	162.9	340.0	284.9	244.7	-10.8	-18.6	70.0	42.5	22.4	39.1	42.3
			2.3	7.70537	200	159.3	153.3	291.0	264.8	285.4	-20.4	-23.4	45.5	32.4	42.7	34.4	31.9
	SER|Z|	0.04		1.01483	200	172.4	189.6	172.9	185.0	203.8	-13.8	-5.2	-13.6	-7.5	1.9	9.6	10.7
		0.03		0.97494	200	200.1	203.5	186.1	193.3	190.0	0.0	1.8	-7.0	-3.3	-5.0	4.2	4.0

Although the value of λ could be further adjusted with very small modifications to obtain even better robust performance for the given value of q in Table 4.1, this table helps us to get some sense about how small λ should be and about the range of acceptable values of λ. And we may also be aware of the inertial problem of the MEWMA-type of control charts caused by the very small value of λ, say $\lambda = 0.001$. The inertial problem for the MEWMA-type control chart gets more serious as λ decreases and p increases (Woodall and Mahmoud (2005)).

4.1.2 The robust design for the SER|Z| and MRW^2 charts

In Chapter 3, we have proposed the set of ER$|Z|^q$ charts for monitoring the multivariate processes. The effect of q on the robust design for the SER$|Z|^q$ chart is very similar to that of the MR$|Z|^q$ chart. In order to avoid redundant comparisons, we focus on the robust design of the SER|Z| chart ($q = 1$) for the rest of the discussion. For the MRW^2 chart, we preselected three value of λ, and then modify w to achieve satisfying robustness.

Table 4.2 presents the in-control ATS values of the SER|Z| and MRW^2 charts for the same multivariate distributions shown in Table 4.1. Similar to the design of the MR$|Z|^q$ chart, the control limits are obtained to ensure an ATS of 200 with bivariate normal observations while the process is in control. The ATS deviations are recorded under non-normal distributions to measure the stability of the proposed control charts.

From Table 4.2, we can find that the robust design of the SER|Z| chart is quite similar to that of the MR|Z| chart. Based on the RMS values for both the $\rho = 0.9$ and $\rho = 0$ cases, an acceptable value of λ is also 0.03. Compared to the MR|Z| chart, the ATS deviations of SER|Z| chart are less sensitive to the varying of ρ in the processes.

For the design of the MRW^2 chart, robustness can be achieved with a large value of w when λ is small or a smaller value of w when λ is relative large. This is not surprising since a smaller w implies more extreme observations are replaced by the predetermined boundary, which allows more weights to be given to the current observations. Compared to the MR|Z|^q and SER|Z| charts, the in-control performance of the MRW^2 chart is less stable when ρ changes from 0.9 to 0, according to the ATS deviations with non-normal distributions (columns [7]-[11]). Additionally, the RMS values of the MRW^2 chart are generally much larger over all non-normal distributions.

One interesting thing about the in-control performance of the MRW^2 chart is that the in-control ATS decreases as the degrees of freedom of the multivariate T distribution increases or the value of shape parameter for the multi-gamma distribution increases. Thus we cannot necessarily expect a closer ATS value for the MRW^2 chart if the true distribution of a process gets closer to a multivariate normal distribution as we can expect for the MR|Z|^q and SER|Z| charts.

4.1.3 The effect of ρ on the robustness of the proposed charts

From previous sections, we find that the performance of some MR|Z|^q and MRW^2 charts might deteriorate as the correlation among variables decreases from 0.9 to 0 when p = 2. It will be interesting to further investigate the effect of ρ on the robustness of the three proposed charts. We selected one robust design of each chart and investigated their in-control performance for different values of ρ.

Table 4.3 contains in-control ATS values for the MR|Z| chart with λ = 0.03, the SER|Z| chart with λ= 0.03 and the MRW^2 chart with λ = 0.2 and w = 2.2 under various distributions with different ρ when p = 2. The in-control ATS values of the MR|Z| chart under T_2 distributions and the $B_2(4,4)$ distribution decrease as ρ decreases, while the in-control ATS under G_2 distributions did not show a consistent pattern. The in-control ATS values of the SER|Z| chart only show a decreasing pattern under T_2 distributions as ρ decreases. As for the MRW^2 chart, no pattern can be found for all distributions.

Generally speaking, the robustness of the MR|Z| and MRW^2 charts in terms of the RMS(1) and RMS(2) values are better when there is at least some correlation among the variables. The SER|Z| chart stays in a stable robustness as ρ varies from 0.9 to 0. Its RMS values seem larger when there is moderate correlation among variables only because its

ATS values for the G_2 distributions are relative large when ρ is 0.5. The MRW^2 chart still gives much larger RMS values compared to the other charts as ρ varies.

So far we have discussed the performance of the proposed charts when the process variables are positively correlated or independent. It will also be interesting to see the charts' performance with negatively correlated variables. However, the simulation mechanism for generating multivariate gamma observations when $\rho < 0$ has not been developed. So we did some limited investigation for the performance of the three robust charts under the bivariate *t* and beta distributions when $\rho = -0.9$ and $\rho = -0.5$.

Table 4.3. In control ATS of three robust charts for various distributions with different ρ when p=2

Control Chart	**λ**	**ρ**	**UCL**	**In-control ATS**						**RMS(1) of percentage deviation (%)**	**RMS(2) of percentage deviation (%)**
				$N_2(0,\sum_{z0})$ [1]	**$T_2(4)$ [2]**	**$T_2(10)$ [3]**	**$G_2(4,1)$ [4]**	**$G_2(8,1)$ [5]**	**$B_2(4,4)$ [6]**		
MR|Z|	0.03	0.9	1.00463	200.0	213.5	205.0	196.9	197.8	189.5	4.1	3.7
		0.7	1.20608	200.0	206.0	202.4	193.9	196.4	186.3	3.8	2.4
		0.5	1.40603	200.0	198.2	199.2	193.0	197.7	183.8	4.0	1.9
		0.2	1.63751	200.0	189.7	195.1	199.3	199.8	177.8	5.6	2.9
		0	1.69014	200.0	188.1	194.2	207.0	203.9	172.5	7.1	3.9
SER|Z|	0.03	0.9	0.96310	200.0	214.1	207.0	197.8	198.6	186.7	4.6	4.0
		0.7	0.96980	200.0	208.1	205.7	193.6	196.8	187.0	4.0	3.0
		0.5	0.97276	200.0	204.0	204.5	206.4	210.6	188.3	7.4	7.7
		0.2	0.97453	200.0	200.7	203.6	187.1	193.6	189.9	4.0	3.7
		0	0.97494	200.0	200.1	203.5	186.1	193.3	190.0	4.2	4.0
MRW^2 *w*=2.2	0.2	0.9	5.57019	200.0	258.1	194.5	275.2	256.5	258.8	28.0	27.7
		0.7	6.09337	200.0	264.2	195.0	284.7	260.7	258.1	30.3	30.6
		0.5	6.63064	200.0	233.0	185.8	272.6	253.1	267.8	26.4	24.2
		0.2	7.18087	200.0	188.9	167.2	276.7	253.9	260.5	26.1	25.0
		0	7.26634	200.0	178.4	162.9	340.0	284.9	244.7	39.1	42.3

Table 4.4 shows that the MR|Z|, SER|Z|, and MRW^2 charts designed for the case of $\rho = 0.9$ work quite well for the case of $\rho = -0.9$. For each chart, the in-control ATS difference for the bivariate normal, *t*, and beta distributions are very small between the

two cases, and can be considered to be due to the simulation error. A similar result is obtained for the cases of ρ = 0.5 and ρ = - 0.5. The result is reasonable since the distributions considered here are symmetric about the origin and the set-up values of the three robust charts are invariant to the sign of ρ (refer to Chapter 3). We conclude that the robust designs of the proposed charts obtained in this dissertation for the case of ρ = 0.9 can also apply to the case of ρ = - 0.9 if the underlying distribution is symmetric.

Table 4.4. In-control ATS of three robust charts for various distributions for $\rho = \pm 0.9/0.5$ when p=2

Control Chart	***w/q***	***λ***	**UCL**	***ρ***	**In-control ATS**			
					$N_2(0,\sum_{z0})$ [1]	**$T_2(4)$ [2]**	**$T_2(10)$ [3]**	**$B_2(4,4)$ [4]**
MR|Z|	1.0	0.03	1.00463	0.9	200.0	213.5	205.0	189.5
				- 0.9	199.6	213.1	204.9	187.2
			1.40603	0.5	200.0	198.2	199.2	183.8
				-0.5	199.8	198.2	198.5	182.9
SER|Z|		0.03	0.9631	0.9	200.0	214.1	207.0	186.7
				- 0.9	200.0	213.1	206.9	186.6
			0.97276	0.5	200.0	204.0	204.5	188.3
				-0.5	199.9	204.4	204.2	188.3
MRW^2	2.2	0.2	5.57019	0.9	200.0	258.1	194.5	258.8
				- 0.9	199.6	256.1	194.4	258.9
			6.63064	0.5	200.0	233.0	185.8	267.8
				-0.5	199.4	232.7	185.8	267.7

4.2 Efficiency comparison among selected robust charts

We now perform a systematic investigation of some robust designs of the three proposed charts for the out-of-control situation under both normal and non-normal distributions. Since heavier-tailed and skewed distributions are more likely in practice, the non-normal distributions considered for efficiency comparison are the $T_2(4)$ and $G_2(4,1)$ distributions. Based on the RMS values, we select several robust designs which give good robustness with both highly correlated and independent variables and try to find out how the choice of parameters affects the detection efficiency of the proposed control charts.

4.2.1 Efficiency comparison for random shifts in $\boldsymbol{\sigma}$ with $\boldsymbol{\rho}$ unchanged

For out-of-control processes, we first consider increases in $\boldsymbol{\sigma}$ in random shift directions with the correlation structure unchanged.

Tables 4.5 and 4.6 contain SSATS, EQL, and TEQL values and TEQL ratios of some selected $\mathrm{MR|Z|^q}$, SER|Z|, and $\mathrm{MRW^2}$ charts with bivariate normal observations for ρ = 0.9 and ρ = 0, respectively. The performance of these robust control charts in detecting a wide range of shifts in $\boldsymbol{\sigma}$ is compared to the $\mathrm{MRZ^2}$ chart with λ = 0.025. The value λ = 0.025 is chosen instead of 0.1, a relative large λ value, to give some robustness to the standard $\mathrm{MRZ^2}$ chart.

Although the SSATS values and TEQL values of a given chart are generally smaller for the independent case, the overall patterns in control chart comparisons for both cases are quite similar. The SSATS values in columns [1]-[9] of Tables 4.5 and 4.6 show that none of the robust charts can beat all the other charts in detecting all sizes of shifts. Among the robust $\mathrm{MR|Z|^q}$ charts (columns [5]-[9]), high power (say q = 1.5) with very small λ makes a chart respond quickest for very small shifts, while a low power with larger λ makes a $\mathrm{MR|Z|^q}$ chart gain some advantage in detecting median and large shifts. This property is consistent with what we can expect for the design of $\mathrm{MRZ^2}$ chart with normal observations. However, it is interesting to see that the $\mathrm{MR|Z|^q}$ chart with very small q or very small λ may be beaten by some other charts in detecting all shift sizes. For instance, the $\mathrm{MR|Z|^{0.5}}$ chart with λ = 0.13 (column [5] in Tables 4.5 and 4.6) is much worse than other $\mathrm{MR|Z|^q}$ charts for small shifts and also worse than the $\mathrm{MR|Z|^q}$ charts with q = 0.7 and 1.0 for moderate and large shifts. The $\mathrm{M|Z|^{1.5}}$ chart (column [9] in Table 4.5) is generally slower than the $\mathrm{M|Z|^{1.2}}$ chart (column [8]). Based on Tables 4.5 and 4.6, the $\mathrm{MR|Z|^q}$ chart with q = 1 works generally better than the other $\mathrm{MR|Z|^q}$ charts over a large range of shift sizes except for very small shift sizes.

The performance of the SER|Z| chart (column [4]) is uniformly better than the MR|Z| chart for all shift sizes when ρ = 0.9, but its performance is generally worse when ρ = 0. The MRW^2 chart with a larger value of λ and a smaller value of w signals faster for median and large shifts but slower for small shifts compared to the MRW^2 chart with a smaller λ and a larger w. Compared to some MR|Z|^q charts, the three MRW^2 charts (columns [1]-[3]) are slightly better for intermediate shifts while less sensitive for detecting large shifts (ψ > 4). However, the SER|Z| and MR|Z| charts beat the three MRW^2 charts for most shift sizes for both the ρ = 0.9 and ρ = 0 cases.

The TEQL value measures the overall performance of a control chart to detect shifts between 1.1 to 20. The TEQL ratio given for each prior distribution is the TEQL of a robust chart divided by TEQL of the standard MRZ^2 chart. It's not surprising that the TEQL ratios of robust charts are all above 1 for the three prior distributions in Tables 4.5 and 4.6, which indicates the percentage efficiency lost due to their gain in robustness. For all of the robust charts, the TEQL ratios are smaller when we assume that all shifts in the interval [1.1, 20] are equally likely (the uniform prior) and become larger when we assume small or moderate shifts would be more plausible (with the $B_1(2,18)$ and $B_1(2,8)$ priors). The TEQL ratios of MRW^2 charts are pretty large (greater than 2) when the uniform prior is assumed because of their insensitivity to very large shifts. Generally speaking, the robust charts lose at least 20% efficiency to the non-robust chart when applied to bivariate normal processes.

For the bivariate normal process, the SER|Z| chart has the best performance with highly correlated variables, but its loss is larger than all of the MR|Z|^q charts when the variables are independent. Considering both cases ρ = 0.9 and ρ = 0, the MR|Z| chart has outstanding performance among all the MR|Z|^q charts because its TEQL is either the smallest or second smallest over the three priors. Based on the TEQL values, the three MRW^2 charts cannot beat the MR|Z|^q and SER|Z| charts in the bivariate normal setting. The TEQL values of the MRW^2 chart with w = 2.2 (column [1]) are always smaller than the TEQL values for the other two MRW^2 charts.

Table 4.5. SSATS, EQL and TEQL values for some robust control charts with $N_2(0, \sum_{z0})$ observations for shifts in $\boldsymbol{\sigma}$ averaged over shift directions when $p = 2$, $\rho = 0.9$

	MRW^2			**SER|Z|**	**MR|Z|^q**					
w/q	**2.2**	**2.4**	**2.8**	**1.0**	**0.5**	**0.7**	**1.0**	**1.2**	**1.5**	**2.0**
λ	***0.2***	***0.1***	***0.025***	***0.03***	***0.13***	***0.08***	***0.03***	***0.01***	***0.001***	***0.025***
ψ	**[1]**	**[2]**	**[3]**	**[4]**	**[5]**	**[6]**	**[7]**	**[8]**	**[9]**	**[10]**
1.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0
1.1	102.0	95.1	87.5	79.4	101.8	95.0	84.2	77.5	72.9	80.2
1.2	57.2	51.9	48.2	43.3	59.7	53.9	46.6	43.8	43.0	42.5
1.4	24.9	22.6	23.1	19.6	27.5	24.5	22.0	22.1	22.9	19.0
1.6	14.6	13.8	14.9	12.4	16.4	14.8	14.0	14.5	15.3	11.7
1.8	10.2	9.9	11.0	9.1	11.3	10.4	10.2	10.8	11.4	8.3
2	7.9	7.7	8.8	7.1	8.6	8.0	8.0	8.5	9.0	6.4
2.5	5.2	5.3	6.1	4.7	5.4	5.2	5.2	5.6	5.9	4.0
3	4.1	4.2	4.9	3.5	4.0	3.9	3.9	4.2	4.3	2.9
4	3.1	3.2	3.7	2.4	2.7	2.6	2.7	2.8	2.9	1.9
5	2.6	2.7	3.2	1.9	2.1	2.1	2.1	2.2	2.2	1.5
7	2.2	2.3	2.7	1.3	1.6	1.5	1.5	1.5	1.5	1.1
10	1.9	2.0	2.4	1.0	1.2	1.1	1.1	1.1	1.1	0.8
20	1.7	1.8	2.0	0.7	0.8	0.7	0.7	0.7	0.7	0.6
					EQL(ψ) values					
1.1	28.5	26.6	24.5	22.2	28.5	26.6	23.5	21.7	20.4	22.4
1.2	34.9	31.7	29.5	25.9	36.5	32.9	28.5	26.7	26.3	26.0
1.4	35.4	32.1	32.9	33.2	39.4	35.1	31.4	31.6	32.8	27.0
1.6	35.3	33.3	36.2	30.1	40.3	36.3	34.2	35.7	37.7	28.3
1.8	36.7	35.7	40.2	32.9	41.8	38.3	37.3	39.7	42.2	30.1
2	39.3	38.9	44.8	35.9	44.2	41.0	40.8	43.8	46.6	32.0
2.5	48.7	49.6	58.5	44.2	52.3	49.6	50.3	54.4	57.1	37.6
3	61.0	63.2	75.0	53.3	62.6	59.7	60.6	65.2	67.6	43.9
4	92.6	96.9	115.9	72.8	85.5	81.7	82.4	87.8	89.3	58.4
5	132.3	139.3	166.9	94.1	111.9	106.2	106.1	112.3	112.8	75.2
7	235.0	248.9	297.9	142.8	170.3	161.3	159.0	166.1	165.5	116.0
10	446.2	473.1	564.4	232.2	275.3	260.2	254.8	264.2	259.9	194.5
20	1645.7	1741.3	2026.3	681.5	765.4	733.8	716.5	731.4	721.0	611.3
					TEQL values and ratio to the TEQL of MRZ^2 chart					
beta(2,18)	68.3	70.2	82.8	54.9	65.7	62.1	62.0	66.0	68.0	45.8
ratio	1.49	1.53	1.81	1.20	1.44	1.36	1.35	1.44	1.49	-
beta(2,8)	147.9	155.4	185.5	96.8	115.0	109.0	108.2	113.8	114.5	79.3
ratio	1.86	1.96	2.34	1.22	1.45	1.37	1.36	1.44	1.44	-
uniform	606.3	642.1	755.3	285.0	329.6	313.5	306.8	315.5	311.5	246.6
ratio	2.46	2.60	3.06	1.16	1.34	1.27	1.24	1.28	1.26	-

Table 4.6. SSATS, EQL and TEQL values for some robust control charts with $N_2(0, \sum_{z0})$ observations for shifts in **σ** averaged over shift directions when p=2, ρ=0

	MRW²			**SER|Z|**	**MR|Z|^q**					
w/q	**2.2**	**2.4**	**2.8**	**1.0**	**0.5**	**0.7**	**1.0**	**1.2**	**1.5**	**2.0**
λ	***0.2***	***0.1***	***0.025***	***0.03***	***0.13***	***0.08***	***0.03***	***0.01***	***0.001***	***0.025***
ψ	**[1]**	**[2]**	**[3]**	**[4]**	**[5]**	**[6]**	**[7]**	**[8]**	**[9]**	**[10]**
1.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0
1.1	94.0	85.2	76.9	73.5	86.8	80.0	69.2	62.6	58.2	67.1
1.2	52.1	45.5	41.6	39.0	47.0	42.3	36.6	34.3	33.8	34.4
1.4	22.8	20.2	20.4	18.2	20.5	18.5	17.2	17.2	17.7	15.5
1.6	13.4	12.4	13.4	11.7	12.1	11.2	10.9	11.4	11.8	9.6
1.8	9.5	9.0	10.0	8.5	8.4	8.0	8.0	8.4	8.8	6.8
2	7.3	7.1	8.1	6.7	6.4	6.2	6.3	6.7	6.9	5.2
2.5	4.9	4.9	5.6	4.4	4.1	4.0	4.1	4.4	4.5	3.3
3	3.8	3.9	4.5	3.3	3.1	3.0	3.1	3.3	3.3	2.4
4	2.9	3.0	3.4	2.2	2.1	2.1	2.1	2.2	2.2	1.6
5	2.5	2.6	2.9	1.7	1.6	1.6	1.6	1.7	1.6	1.2
7	2.1	2.2	2.4	1.2	1.2	1.2	1.1	1.2	1.1	0.9
10	1.9	1.9	2.1	0.9	0.9	0.9	0.9	0.9	0.9	0.7
20	1.6	1.7	1.8	0.6	0.6	0.6	0.6	0.6	0.6	0.6
EQL(ψ) values										
1.1	24.7	22.4	20.2	19.4	22.8	21.0	18.2	16.5	15.3	17.7
1.2	28.4	24.8	22.7	21.4	25.6	23.0	19.9	18.7	18.4	18.8
1.4	26.7	23.6	23.9	21.4	23.9	21.7	20.1	20.1	20.7	18.2
1.6	25.2	23.2	25.1	22.0	22.6	21.0	20.5	21.3	22.2	18.0
1.8	25.1	23.9	26.6	22.8	22.3	21.2	21.2	22.4	23.3	18.1
2	25.8	25.0	28.5	23.8	22.6	21.8	22.1	23.5	24.4	18.5
2.5	29.3	29.4	34.0	26.8	24.7	24.2	24.9	26.4	27.2	19.8
3	34.4	35.2	40.6	30.1	27.7	27.2	27.9	29.6	30.1	21.6
4	47.8	49.4	56.4	37.1	34.7	34.1	34.7	36.3	36.2	26.1
5	64.4	66.7	75.6	44.6	42.5	41.7	42.1	43.6	42.9	31.7
7	106.8	110.5	122.8	61.8	60.4	58.9	58.6	60.0	58.7	45.7
10	191.9	198.0	216.0	93.6	92.9	90.4	89.7	91.3	89.2	74.0
20	661.5	678.9	718.1	258.2	258.5	254.2	253.5	254.6	250.6	231.9
TEQL values and ratio to the TEQL of MRZ² chart										
beta(2,18)	38.0	38.2	43.3	30.6	29.2	28.4	28.6	30.0	30.3	22.6
ratio	1.68	1.69	1.91	1.36	1.29	1.26	1.27	1.33	1.34	-
beta(2,8)	70.5	72.6	81.1	45.5	43.8	42.7	43.0	44.5	44.0	33.6
ratio	2.10	2.16	2.41	1.35	1.30	1.27	1.28	1.32	1.31	-
uniform	253.3	260.6	280.0	113.2	112.6	110.1	109.5	110.9	108.9	94.8
ratio	2.67	2.75	2.95	1.19	1.19	1.16	1.16	1.17	1.15	-

Tables 4.7-4.10 evaluate the out-of control performance of some selected robust charts for non-normal bivariate distributions with highly correlated or independent variables. Three robust $MR|Z|^q$ charts with q ranging from 0.5 to 1.5 are selected to compare with the robust MRW^2 and SER|Z| charts. Since we don't know the true distribution for a process in practice, it would be reasonable to initially check the charts' performance with the control limits unchanged. Thus the in-control ATS values of the seven selected control charts with non-normal observations vary from one chart to another. To obtain the robust charts' true efficiency with the same in-control ATS, we will additionally adjusted the control limits according to the simulation distributions and compare the relative efficiencies among these robust charts.

Tables 4.7 and 4.8 give SSATS, EQL, and TEQL values and TEQL ratios of the seven control charts with normal control limits for $T_2(4)$ and $G_2(4,1)$ observations when $\rho = 0.9$ and $\rho = 0$. The denominator of the TEQL ratio is the smallest TEQL among the seven control charts for each ρ. Compared to Tables 4.5 and 4.6, the SSATS values with $T_2(4)$ observations for detecting the same size shift are generally larger and consequently the TEQL values are also larger under the same prior distribution given the same chart and the same correlation coefficient. The SSATS and TEQL values for $G_2(4,1)$ observations are slightly smaller than those for the normal observations.

For the overall performance comparison based on TEQL values, Tables 4.7 and 4.8 present a similar pattern to the one we find in Tables 4.5 and 4.6. The SER|Z| chart with $\lambda = 0.03$ still works best among all the robust charts with $T_2(4)$ and $G_2(4,1)$ observations when $\rho = 0.9$. The MR|Z| chart with $\lambda = 0.03$ seems still the best choice for detecting a wide range of shifts in non-normal processes for always being one of the two best charts over all three priors. Among the three MRW^2 charts, $w = 2.2$ has the best overall efficiency. The MRW^2 charts still cannot compete with the other robust charts when the uniform prior is assumed because of its slower reaction to large shifts. However, their overall performance is close to or even better than other robust charts when small and median shifts are more likely if w is relatively small.

Table 4.7. SSATS, EQL and TEQL values for some robust control charts with $T_2(4)$ observations for sustained shifts in $\boldsymbol{\sigma}$ averaged over shift directions when p=2

	ρ = 0.9							ρ = 0						
	MRW^2			SER|Z|	MR|Z|^q			MRW^2			SER|Z|	MR|Z|^q		
w/q	*2.2*	*2.4*	*2.8*	*1.0*	*0.5*	*1.0*	*1.5*	*2.2*	*2.4*	*2.8*	*1.0*	*0.5*	*1.0*	*1.5*
λ	*0.2*	*0.1*	*0.025*	*0.03*	*0.13*	*0.03*	*0.001*	*0.2*	*0.1*	*0.025*	*0.03*	*0.13*	*0.03*	*0.001*
ψ	*[1]*	*[2]*	*[3]*	*[4]*	*[5]*	*[6]*	*[7]*	[8]	[9]	[10]	[11]	[12]	[13]	[14]
1.0	**258.1**	**258.2**	**237.9**	**214.1**	**213.5**	**213.5**	**198.4**	**178.4**	**191.4**	**209.3**	**200.1**	**163.4**	**188.1**	**187.4**
1.1	166.9	159.6	140.1	126.5	139.8	131.6	116.4	118.5	120.1	120.0	117.9	105.8	110.9	104.5
1.2	108.8	100.7	87.1	78.0	95.9	84.6	75.4	81.3	78.8	74.6	73.7	72.5	70.3	65.6
1.4	51.8	46.4	41.3	36.3	49.9	41.1	39.2	43.1	39.6	36.7	34.8	37.9	34.1	33.5
1.6	29.4	26.4	25.1	21.7	29.7	24.6	25.2	26.3	23.7	22.8	20.8	22.8	20.7	21.6
1.8	19.2	17.6	17.7	15.1	20.0	17.0	18.4	17.9	16.2	16.3	14.5	15.5	14.4	15.6
2	13.9	13.0	13.7	11.4	14.6	12.9	14.4	13.2	12.2	12.7	11.0	11.5	11.0	12.2
2.5	8.1	7.9	8.9	7.1	8.5	8.0	9.2	7.8	7.5	8.3	6.9	6.8	6.9	7.8
3	5.8	5.9	6.7	5.2	6.0	5.8	6.7	5.7	5.6	6.3	5.1	4.8	5.0	5.6
4	4.0	4.1	4.8	3.5	3.9	3.8	4.3	3.9	3.9	4.5	3.3	3.2	3.3	3.6
5	3.2	3.4	3.9	2.6	3.0	2.9	3.2	3.1	3.2	3.6	2.5	2.4	2.5	2.7
7	2.6	2.7	3.2	1.8	2.1	2.0	2.2	2.5	2.5	2.9	1.7	1.7	1.7	1.8
10	2.2	2.3	2.7	1.3	1.5	1.5	1.5	2.1	2.1	2.4	1.2	1.2	1.2	1.2
20	1.8	1.9	2.2	0.8	0.9	0.9	0.9	1.7	1.8	1.9	0.8	0.8	0.8	0.8
								EQL(ψ) values						
1.1	46.7	44.7	39.2	35.4	39.1	36.8	32.6	31.3	31.7	31.6	31.1	27.9	29.2	27.6
1.2	66.5	61.5	53.3	47.7	58.7	51.7	46.1	44.4	43.1	40.8	40.4	39.6	38.4	35.9
1.4	73.4	65.9	58.7	51.6	71.5	58.6	56.0	50.5	46.4	43.0	41.0	44.3	40.0	39.3
1.6	70.6	63.8	61.1	52.4	72.9	60.1	62.0	49.2	44.5	42.9	39.3	42.6	38.8	40.5
1.8	68.7	63.3	64.4	54.5	73.6	62.4	68.0	47.5	43.2	43.5	38.8	41.2	38.3	41.6
2	68.8	64.9	69.1	57.4	74.6	65.7	73.9	46.5	43.0	44.9	39.1	40.4	38.8	43.0
2.5	74.4	74.0	83.8	67.1	81.7	76.4	88.8	47.1	45.5	50.1	42.0	41.1	41.5	46.9
3	85.7	87.3	101.8	78.5	92.6	89.4	104.1	51.1	50.6	57.0	46.0	43.6	45.3	51.0
4	118.3	123.4	147.1	104.0	120.4	117.9	135.2	64.1	65.1	74.1	55.4	52.2	54.4	60.0
5	160.8	169.5	203.0	132.2	153.0	149.3	167.5	81.5	83.7	95.0	65.8	62.5	64.5	69.8
7	270.9	287.4	346.2	193.8	227.5	218.5	236.4	125.9	130.1	146.3	88.8	86.4	87.2	91.1
10	495.3	529.1	634.6	303.2	358.2	336.5	353.0	215.5	222.4	245.7	128.2	127.7	126.6	128.9
20	1740.2	1863.3	2196.2	810.5	938.6	874.9	887.0	701.6	720.4	771.7	310.9	316.4	308.3	307.5
							TEQL values and ratio to the smallest TEQL							
beta(2,18)	96.3	97.2	111.0	81.6	98.4	92.6	104.5	56.3	55.4	60.7	48.0	46.9	47.2	51.7
ratio	1.18	1.19	1.36	-	1.21	1.14	1.28	1.20	1.18	1.29	1.02	-	1.01	1.10
beta(2,8)	178.6	187.6	222.9	134.5	158.4	151.6	167.4	88.8	90.4	101.1	66.9	64.8	65.8	70.4
ratio	1.33	1.39	1.66	-	1.18	1.13	1.24	1.37	1.40	1.56	1.03	-	1.02	1.09
uniform	659.5	704.6	838.0	358.2	420.3	393.9	409.0	279.1	286.9	312.3	147.8	148.6	146.1	148.2
ratio	1.84	1.97	2.34	-	1.17	1.10	1.14	1.91	1.96	2.14	1.01	1.02	-	1.01

Table 4.8. SSATS, EQL and TEQL values for some robust control charts with G_2 (4,1) observations for sustained shifts in **σ** averaged over shift directions when *p*=2

	ρ = 0.9							**ρ = 0**						
	MRW²			**SER|Z|**	**MR|Z|^q**			**MRW²**			**SER|Z|**	**MR|Z|^q**		
w/q	***2.2***	***2.4***	***2.8***	**1.0**	***0.5***	***1.0***	***1.5***	***2.2***	***2.4***	***2.8***	**1.0**	***0.5***	***1.0***	***1.5***
λ	***0.2***	***0.1***	***0.025***	***0.03***	***0.13***	***0.03***	***0.001***	***0.2***	***0.1***	***0.025***	***0.03***	***0.13***	***0.03***	***0.001***
ψ	***[1]***	***[2]***	***[3]***	***[4]***	***[5]***	***[6]***	***[7]***	**[8]**	**[9]**	**[10]**	**[11]**	**[12]**	**[13]**	**[14]**
1.0	**275.2**	**255.0**	**215.2**	**197.8**	**197.8**	**196.9**	**192.3**	**340.0**	**296.1**	**233.3**	**186.1**	**219.3**	**207.0**	**194.4**
1.1	163.5	140.6	107.6	90.9	110.1	95.4	85.7	175.5	141.7	97.1	81.5	101.3	81.7	68.9
1.2	95.2	80.3	61.2	50.4	68.4	55.6	51.9	96.0	75.2	51.7	45.2	55.7	44.0	40.4
1.4	35.7	31.1	26.9	22.9	32.1	25.8	27.2	34.9	28.2	22.7	21.0	24.0	20.1	21.0
1.6	17.9	16.3	16.1	14.3	18.7	16.0	17.9	17.4	14.8	13.6	13.2	13.8	12.6	13.9
1.8	11.3	10.7	11.2	10.2	12.7	11.5	13.3	10.9	9.7	9.6	9.5	9.4	9.1	10.2
2	8.3	8.0	8.6	8.0	9.5	8.9	10.4	7.8	7.2	7.4	7.5	7.1	7.1	8.1
2.5	5.1	5.1	5.6	5.2	5.8	5.8	6.7	4.8	4.6	4.8	4.9	4.5	4.6	5.2
3	3.9	3.9	4.3	3.9	4.3	4.3	4.9	3.6	3.5	3.7	3.6	3.3	3.4	3.8
4	2.9	2.9	3.2	2.6	2.9	2.9	3.2	2.7	2.6	2.7	2.4	2.2	2.3	2.5
5	2.4	2.5	2.7	2.0	2.2	2.2	2.4	2.2	2.2	2.3	1.9	1.7	1.7	1.8
7	2.0	2.1	2.3	1.4	1.6	1.5	1.6	1.9	1.9	1.9	1.3	1.2	1.2	1.2
10	1.8	1.9	2.0	1.0	1.2	1.1	1.1	1.6	1.6	1.6	0.9	0.9	0.9	0.9
20	1.6	1.6	1.7	0.7	0.8	0.7	0.7	1.4	1.4	1.3	0.6	0.6	0.6	0.6
EQL(ψ) values														
1.1	45.8	39.4	30.1	25.5	30.8	26.7	24.0	46.2	37.3	25.6	21.5	26.7	21.5	18.1
1.2	58.2	49.1	37.4	30.8	41.8	34.0	31.7	52.4	41.0	28.2	24.7	30.4	24.0	22.1
1.4	50.5	44.0	38.3	32.6	46.1	36.9	38.9	40.9	33.1	26.6	24.7	28.1	23.6	24.6
1.6	43.0	39.2	39.0	34.6	46.1	39.2	44.2	32.5	27.7	25.5	24.8	25.8	23.6	26.0
1.8	40.6	38.5	40.9	37.1	47.1	42.2	49.1	28.9	25.9	25.5	25.5	25.0	24.2	27.2
2	40.9	39.8	43.6	40.1	48.8	45.4	53.9	27.6	25.5	26.0	26.5	25.0	25.0	28.4
2.5	47.3	47.4	53.3	48.8	56.4	55.2	65.2	28.8	27.7	29.0	29.5	26.8	27.8	31.4
3	57.8	58.7	66.0	58.1	66.2	65.8	76.8	32.7	32.0	33.4	33.0	29.7	31.0	34.5
4	86.1	88.6	98.9	78.3	89.5	88.4	99.7	43.9	43.5	45.1	40.6	36.7	37.9	40.9
5	122.8	126.5	140.7	99.9	116.3	112.5	122.8	58.2	58.1	59.5	48.5	44.8	45.5	47.6
7	218.7	226.1	248.5	148.1	175.8	165.2	173.9	94.7	94.6	95.8	65.7	63.0	62.3	63.0
10	416.7	431.4	467.7	236.7	278.7	259.5	266.1	167.6	167.2	167.2	97.2	95.7	92.7	92.7
20	1545.0	1595.1	1677.8	688.5	770.2	724.4	727.0	563.8	562.0	549.6	261.4	260.2	255.1	253.8
TEQL values and ratio to the smallest TEQL														
beta(2,18)	67.2	67.4	73.6	59.7	70.2	67.4	76.5	38.2	36.3	36.5	33.8	31.6	31.8	34.6
ratio	1.13	1.13	1.23	-	1.18	1.13	1.28	1.21	1.15	1.15	1.07	-	1.01	1.10
beta(2,8)	138.8	142.7	156.8	102.0	119.3	114.0	123.5	64.2	63.4	64.5	49.0	46.1	46.2	48.4
ratio	1.36	1.40	1.54	-	1.17	1.12	1.21	1.39	1.38	1.40	1.06	-	1.00	1.05
uniform	568.5	587.4	627.1	290.2	333.4	311.7	318.6	219.7	218.7	216.3	116.5	114.7	112.4	112.7
ratio	1.96	2.02	2.16	-	1.15	1.07	1.10	1.95	1.95	1.92	1.04	1.02	-	1.00

Table 4.9. SSATS, EQL and TEQL values for some robust control charts with $T_2(4)$ observations and adjusted control limit for sustained shifts in $\boldsymbol{\sigma}$ averaged over shift directions when p=2

	ρ = 0.9						**ρ = 0**					
	MRW²	**SER|Z|**	**MR|Z|^q**				**MRW²**	**SER|Z|**	**MR|Z|^q**			
w/q	***2.2***	***1.0***	***0.5***	***1.0***	***1.5***	***2***	***2.2***	***1.0***	***0.5***	***1.0***	***1.5***	***2***
λ	***0.2***	***0.03***	***0.13***	***0.03***	***0.001***	***0.025***	***0.2***	***0.03***	***0.13***	***0.03***	***0.001***	***0.025***
ψ	***[1]***	***[2]***	***[3]***	***[4]***	***[5]***	***[6]***	**[7]**	**[8]**	**[9]**	**[10]**	**[11]**	**[12]**
1.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0
1.1	132.3	118.9	130.4	123.2	116.6	134.7	130.7	117.4	127.2	117.3	106.0	130.7
1.2	88.2	73.9	90.0	80.4	75.5	91.7	89.0	73.3	85.3	73.9	66.2	89.1
1.4	43.7	34.9	47.3	39.5	39.4	46.0	46.5	34.7	43.5	35.5	33.9	45.6
1.6	25.5	21.0	28.4	23.8	25.4	27.4	28.0	20.8	25.7	21.3	21.8	27.5
1.8	17.0	14.6	19.2	16.6	18.5	18.8	18.8	14.4	17.1	14.8	15.8	18.9
2	12.5	11.2	14.1	12.6	14.5	14.1	13.8	11.0	12.5	11.3	12.3	14.1
2.5	7.4	7.0	8.3	7.8	9.2	8.3	8.1	6.9	7.3	7.0	7.8	8.3
3	5.4	5.1	5.9	5.7	6.7	5.8	5.8	5.0	5.2	5.1	5.7	5.8
4	3.7	3.4	3.8	3.8	4.3	3.6	4.0	3.3	3.3	3.4	3.6	3.5
5	3.0	2.6	2.9	2.9	3.2	2.6	3.2	2.5	2.5	2.5	2.7	2.5
7	2.4	1.8	2.0	2.0	2.2	1.7	2.5	1.7	1.8	1.7	1.8	1.6
10	2.0	1.3	1.5	1.4	1.5	1.2	2.1	1.2	1.3	1.2	1.3	1.1
20	1.7	0.8	0.9	0.9	0.9	0.8	1.7	0.8	0.8	0.8	0.8	0.7
EQL(ψ) values												
1.1	37.1	33.3	36.5	34.5	32.6	37.7	34.5	31.0	33.5	30.9	28.0	34.5
1.2	53.9	45.2	55.0	49.1	46.2	56.1	48.6	40.2	46.6	40.4	36.2	48.8
1.4	62.0	49.5	67.8	56.4	56.4	65.4	54.4	40.9	50.9	41.6	39.7	53.5
1.6	61.5	50.9	69.9	58.1	62.5	66.2	52.4	39.3	48.1	40.0	40.9	51.7
1.8	61.1	53.0	70.8	60.7	68.5	67.8	50.1	38.6	45.4	39.4	42.0	50.3
2	62.0	55.9	72.4	64.1	74.3	70.1	48.8	39.0	44.0	39.8	43.3	49.7
2.5	68.6	65.5	79.6	74.8	89.3	77.6	49.0	41.8	44.2	42.5	47.1	50.4
3	80.0	76.8	90.3	87.7	104.4	86.9	52.8	45.8	46.7	46.3	51.3	52.5
4	111.6	102.0	117.9	115.8	135.6	107.6	65.9	55.3	55.4	55.6	60.5	58.7
5	151.5	129.7	149.8	146.6	167.7	130.6	83.4	65.7	66.2	66.1	70.4	66.0
7	255.9	190.5	222.9	214.3	236.9	183.8	128.5	88.5	90.9	89.2	92.0	84.2
10	468.5	298.9	351.2	330.8	353.2	280.6	218.9	127.9	133.9	128.9	129.6	117.9
20	1660.2	803.2	924.4	862.6	885.1	754.5	709.6	310.5	326.4	312.2	308.2	288.9
TEQL values and ratio to the TEQL of MRZ² chart												
beta(2,18)	89.0	79.7	95.7	90.7	104.9	89.5	58.4	47.8	50.6	48.4	52.0	54.8
ratio	0.99	0.89	1.07	1.01	1.17	-	1.07	0.87	0.92	0.88	0.95	-
beta(2,8)	168.1	132.0	155.0	148.8	167.7	134.1	91.0	66.7	68.7	67.3	71.0	68.3
ratio	1.25	0.98	1.16	1.11	1.25	-	1.33	0.98	1.01	0.99	1.04	-
uniform	626.5	353.7	412.5	387.2	408.4	335.0	283.3	147.5	155.0	148.6	149.0	138.9
ratio	1.87	1.06	1.23	1.16	1.22	-	2.04	1.06	1.12	1.07	1.07	-

Table 4.10. SSATS, EQL and TEQL values for some robust control charts with $G_2\,(4,1)$ observations and adjusted control limit for sustained shifts in $\boldsymbol{\sigma}$ averaged over shift directions when p=2

	ρ = 0.9						ρ = 0					
	MRW^2	**SER|Z|**	**MR|Z|^q**				**MRW^2**	**SER|Z|**	**MR|Z|^q**			
w/q	***2.2***	***1.0***	***0.5***	***1.0***	***1.5***	***2***	***2.2***	***1.0***	***0.5***	***1.0***	***1.5***	***2***
λ	***0.2***	***0.03***	***0.13***	***0.03***	***0.001***	***0.025***	***0.2***	***0.03***	***0.13***	***0.03***	***0.001***	***0.025***
ψ	***[1]***	***[2]***	***[3]***	***[4]***	***[5]***	***[6]***	**[7]**	**[8]**	**[9]**	**[10]**	**[11]**	**[12]**
1.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0
1.1	120.8	91.7	111.5	96.9	87.9	109.8	109.2	86.3	93.9	80.4	70.4	96.8
1.2	72.8	50.7	69.1	56.0	52.9	66.2	63.4	47.3	52.7	43.4	41.1	55.6
1.4	29.3	23.0	32.3	26.0	27.6	30.5	25.8	21.7	23.0	19.9	21.2	25.5
1.6	15.3	14.3	18.9	16.1	18.3	18.1	13.8	13.6	13.3	12.4	14.0	15.4
1.8	10.0	10.3	12.9	11.6	13.5	12.4	8.9	9.8	9.1	9.0	10.3	10.7
2	7.4	8.0	9.6	9.0	10.6	9.3	6.6	7.7	6.9	7.0	8.1	8.1
2.5	4.6	5.2	5.9	5.8	6.8	5.6	4.2	5.0	4.3	4.6	5.2	4.9
3	3.6	3.9	4.3	4.3	5.0	3.9	3.2	3.7	3.2	3.4	3.8	3.4
4	2.6	2.6	2.9	2.9	3.2	2.4	2.4	2.5	2.2	2.3	2.5	2.1
5	2.3	2.0	2.3	2.2	2.4	1.8	2.0	1.9	1.7	1.7	1.8	1.5
7	1.9	1.4	1.6	1.5	1.6	1.2	1.7	1.3	1.2	1.2	1.2	1.0
10	1.7	1.0	1.2	1.1	1.1	0.9	1.4	0.9	0.9	0.9	0.9	0.8
20	1.5	0.7	0.8	0.7	0.7	0.6	1.2	0.6	0.6	0.6	0.6	0.6
EQL(ψ) values												
1.1	33.8	25.7	31.2	27.1	24.6	30.8	28.8	22.8	24.7	21.2	18.6	25.5
1.2	44.5	31.0	42.3	34.2	32.3	40.4	34.6	25.9	28.7	23.7	22.4	30.4
1.4	41.5	32.6	46.3	37.2	39.6	43.3	30.2	25.6	26.9	23.3	24.9	29.9
1.6	36.9	34.6	46.5	39.5	45.0	43.9	25.8	25.6	24.9	23.3	26.2	29.0
1.8	35.8	37.2	47.5	42.4	50.0	45.0	23.8	26.2	24.2	23.9	27.4	28.5
2	36.8	40.3	49.3	45.8	54.8	46.8	23.3	27.2	24.3	24.7	28.6	28.5
2.5	43.2	48.9	56.8	55.6	66.5	52.3	25.0	30.3	26.1	27.4	31.6	29.3
3	53.3	58.2	66.6	66.3	78.1	58.2	28.8	33.8	29.0	30.6	34.7	30.7
4	80.1	78.5	90.0	89.0	101.3	71.8	39.0	41.4	36.0	37.6	41.2	34.4
5	114.5	100.1	116.9	113.3	124.8	88.0	52.0	49.4	44.1	45.1	47.9	39.5
7	205.1	148.4	176.6	165.9	175.7	128.7	84.8	66.8	62.2	61.8	63.4	53.1
10	391.7	237.0	281.1	260.7	268.5	208.7	150.1	98.5	94.4	92.6	93.2	81.5
20	1464.2	689.2	771.0	726.1	732.8	637.4	504.6	262.5	258.6	254.7	254.2	239.8
TEQL values and ratio to the TEQL of MRZ^2 chart												
beta(2,18)	61.3	59.8	70.7	67.9	77.8	60.2	32.7	34.6	30.8	31.4	34.9	32.0
ratio	1.02	0.99	1.17	1.13	1.29	-	1.02	1.08	0.96	0.98	1.09	-
beta(2,8)	129.4	102.2	120.2	114.7	125.2	92.9	56.9	49.9	45.3	45.9	48.7	42.0
ratio	1.39	1.10	1.29	1.23	1.35	-	1.36	1.19	1.08	1.09	1.16	-
uniform	536.4	290.4	334.7	313.2	321.5	263.5	196.4	117.8	113.8	112.1	113.1	102.5
ratio	2.04	1.10	1.27	1.19	1.22	-	1.92	1.15	1.11	1.09	1.10	-

Although it would not be done in practice, we adjust the control limits of five robust charts which give good overall performance to have an in-control ATS of 200 for all the non-normal processes considered. The resulting SSATS, EQL, and TEQL values are shown in Tables 4.9 and 4.10. The UCL of the MRZ^2 chart is also adjusted to give the in-control ATS of 200 for each non-normal bivariate distribution to make a fair comparison with the robust designs. To evaluate the relative efficiency of robust designs to non-robust design, the denominator of the TEQL ratio is the TEQL value of the MRZ^2 chart.

Although the TEQL values differ from these in the previous tables, and especially the performance of the MRW^2 chart with $w = 2.2$ is significantly improved because its unadjusted in-control ATS was far above 200, the general trends we found in Tables 4.7 and 4.8 are about the same after the adjustment of the control limits. Thus the unequal in-control ATS in Tables 4.7 and 4.8 will not make a big change in the overall performance comparison among robust charts for the out-of-control processes. The TEQL ratios in Tables 4.9 and 4.10 indicate that the SER|Z| chart with $\lambda = 0.03$ has a better overall efficiency than the other robust charts and the MRZ^2 charts when they are designed for the $T_2(4)$ distribution. The $MR|Z|^{0.5}$ and MR|Z| are better than the other robust charts with independent gamma variables. Additionally, the TEQL ratio of robust charts to the MRZ^2 chart show that the robust charts can be more efficient when the small and median shifts are more likely.

Generally speaking, the SER|Z| chart is the best among all the robust designs when ρ is high. The MR|Z| chart is the better choice when ρ is 0 or close to 0. Due to the limit content of the dissertation, we focus on the general comparisons among robust charts for various distributions with different dimensions (in following chapters). The cutoff value of ρ for bivariate processes when the SER|Z| chart is recommended over the MR|Z| chart would not be investigated here.

4.2.2 Efficiency comparison for some special shift directions in $\boldsymbol{\sigma}$ with $\boldsymbol{\rho}$ unchanged

The SSATS and TEQL values given in Tables 4.5-4.10 are averaged over all shift directions corresponding to an increase in $\boldsymbol{\sigma}$. It may be interesting in practice to investigate the control charts' performance for some specific shift directions. Tables 4.11-4.15 give SSATS values of control charts for some shift directions in $\boldsymbol{\sigma}$ with ρ unchanged for the bivariate normal, $T_2(4)$, and $G_2(4,1)$ distributions. Robust designs of the $MR|Z|^q$ chart with q = 0.5, 1, and 1.5 are selected to compare to the SRE|Z| chart and three MRW^2 charts. The standard MRZ^2 chart is also included in Table 4.11 (for bivariate normal observations).

In Tables 4.11-4.15, the first set of shift directions corresponds to the two components of $\boldsymbol{\sigma}$ increasing by the same amount. The second set of shift directions corresponds to one of the two components of $\boldsymbol{\sigma}$ increasing and the other remaining unchanged. The last set of shift directions implies that one component of $\boldsymbol{\sigma}$ increases while the other component decreases by the same amount.

Although the SSATS values in Tables 4.11-4.15 do not present a consistent pattern over different distributions, we are able to reach some general conclusions. The MRZ^2 chart is still faster than other robust charts in detecting the shifts in the first and second set of specific directions with bivariate normal observations when ρ is either 0.9 or 0. For both bivariate normal and non-normal distributions, the MR|Z| chart works best for detecting all the fixed shift directions among the three $MR|Z|^q$ charts. The $MR|Z|^{0.5}$ chart is consistently slower and its ability in detecting the third set of shifts is very poor. Although the SER|Z| chart signals slower than the MR|Z| chart for first set of shift directions when ρ = 0, it shows outstanding performance for the second and third sets among all the robust control charts. It is not surprising that the SER|Z| chart works much better than the $MR|Z|^q$ charts in detecting the third set of shifts since the effect of increasing variance in one variable will not be cancelled out by the effect of decreasing variance in the other variable.

The MRW^2 charts are slower than the MR|Z| chart in detecting the first set of shift directions, but they show some advantages over the robust $MR|Z|^q$ charts in detecting shifts in the second and third set of shift directions when ρ = 0.9. In particular, the MRW^2 chart with w = 2.2 signals the fastest among robust control charts in detecting the third set of shifts when ρ = 0.9.

Table 4.11. SSATS values for some robust control charts with $N_2(0, \sum_{z0})$ observations for shifts in **σ** in some specific directions

		ρ = 0.9								**ρ = 0**							
		MRW²			**SER|Z|**	**MR|Z|^q**				**MRW²**			**SER|Z|**	**MR|Z|^q**			
	w/q	***2.2***	***2.4***	***2.8***	***1.0***	**0.5**	**1.0**	**1.5**	***2.0***	***2.2***	***2.4***	**2.8**	**1.0**	**0.5**	**1.0**	**1.5**	**2.0**
	λ	***0.2***	***0.1***	***0.025***	***0.03***	***0.13***	***0.03***	***0.001***	***0.025***	***0.2***	***0.1***	***0.025***	***0.03***	***0.13***	***0.03***	***0.001***	**0.025**
γ_1	γ_2	*[1]*	*[2]*	*[3]*	*[4]*	*[5]*	*6]*	*[7]*	**[8]**	**[9]**	**[10]**	**[11]**	**[12]**	**[13]**	**[14]**	**[15]**	**[16]**
1	**1**	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0
1.2	**1.2**	42.1	37.6	32.2	31.4	39.0	31.8	29.9	30.0	32.0	28.2	27.3	26.1	28.3	23.0	22.7	21.5
1.4	**1.4**	18.0	16.5	13.2	14.6	16.8	14.8	15.1	13.4	12.9	12.0	13.0	12.1	11.5	10.5	11.4	9.5
1.6	**1.6**	10.8	10.2	7.9	9.3	10.1	9.4	10.0	8.3	7.8	7.5	8.6	7.8	6.8	6.7	7.5	5.8
2	**2**	6.0	6.0	4.6	5.4	5.6	5.5	5.8	4.6	4.5	4.5	5.3	4.5	3.8	3.9	4.3	3.2
3	**3**	3.3	3.4	2.7	2.7	2.8	2.8	2.8	2.1	2.6	2.7	3.1	2.2	1.9	2.0	2.1	1.5
5	**5**	2.2	2.3	1.9	1.4	1.5	1.5	1.5	1.1	1.9	1.9	2.1	1.2	1.1	1.1	1.1	0.8
1.2	**1**	54.9	50.1	62.2	40.3	68.6	50.1	47.6	42.0	61.9	53.8	47.8	41.9	57.2	43.1	39.0	39.9
1.4	**1**	21.3	20.2	28.2	18.1	31.3	22.9	25.7	18.0	27.9	24.0	23.5	19.1	25.9	20.5	20.9	18.0
1.6	**1**	12.1	12.0	16.9	11.4	18.1	14.4	17.3	10.9	16.7	14.8	15.5	12.1	15.4	13.2	14.1	11.2
2	**1**	6.5	6.8	9.4	6.6	9.3	8.2	10.2	5.9	9.2	8.6	9.6	7.0	8.3	7.6	8.3	6.1
3	**1**	3.5	3.7	5.1	3.3	4.4	4.1	4.9	2.7	5.0	5.0	5.6	3.5	4.0	3.8	4.0	2.8
5	**1**	2.4	2.5	3.6	1.8	2.3	2.1	2.4	1.4	3.5	3.5	3.9	1.9	2.2	2.0	2.0	1.5
0.8	**1.2**	43.1	43.6	107.1	40.7	112.7	61.1	61.9	42.8	106.8	87.3	68.8	46.4	124.6	71.1	57.6	58.6
0.7	**1.3**	22.9	24.5	68.0	25.4	69.5	36.6	43.1	25.5	67.8	52.2	42.8	28.2	85.2	44.1	40.0	35.2
0.6	**1.4**	14.8	16.5	45.4	18.2	44.4	25.4	32.7	17.7	45.2	34.6	30.2	19.9	59.2	30.9	30.3	24.1
0.4	**1.6**	8.6	9.9	24.6	11.5	22.6	15.5	21.7	10.7	24.4	19.5	18.9	12.4	32.9	19.0	20.0	14.1
0.2	**1.8**	6.2	7.2	16.2	8.4	14.3	11.1	16.0	7.5	16.0	13.5	13.9	9.0	22.0	13.8	14.6	9.7

Table 4.12. SSATS values for some robust control charts with $T_2(4)$ observations for shifts in $\boldsymbol{\sigma}$ in some specific directions

		ρ = 0.9							ρ = 0						
		MRW^2			SER|Z|	MR|Z|^q			MRW^2			SER|Z|	MR|Z|^q		
	w/q	*2.2*	*2.4*	*2.8*	*1.0*	**0.5**	**1.0**	**1.5**	*2.2*	*2.4*	*2.8*	*1.0*	**0.5**	**1.0**	**1.5**
	λ	*0.2*	*0.1*	*0.025*	*0.03*	*0.13*	*0.03*	*0.001*	*0.2*	*0.1*	*0.025*	*0.03*	*0.13*	*0.03*	*0.001*
γ_1	γ_2	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]	[14]
1	**1**	258.1	258.2	237.9	214.1	213.5	213.5	198.4	178.4	191.4	209.3	200.1	163.4	188.1	187.4
1.2	**1.2**	82.5	74.1	62.9	59.2	68.0	59.9	54.3	57.0	53.9	50.4	51.9	49.8	47.5	45.4
1.4	**1.4**	37.0	32.6	28.9	26.5	30.6	26.9	26.1	25.7	23.6	22.9	22.7	22.1	20.7	21.5
1.6	**1.6**	21.0	18.8	17.7	15.9	17.7	16.1	16.5	14.7	13.7	14.2	13.6	12.7	12.4	13.6
2	**2**	10.2	9.6	9.9	8.5	8.9	8.7	9.3	7.4	7.2	8.0	7.4	6.4	6.7	7.6
3	**3**	4.6	4.6	5.0	4.0	4.0	4.1	4.4	3.5	3.6	4.1	3.5	2.9	3.1	3.5
5	**5**	2.7	2.8	3.1	2.0	2.1	2.1	2.2	2.2	2.3	2.5	1.7	1.6	1.6	1.7
1.2	**1**	108.2	100.3	88.9	75.6	107.0	90.3	81.2	92.1	89.2	83.4	77.4	83.0	79.6	72.2
1.4	**1**	45.1	41.6	39.7	33.0	56.5	42.0	42.2	51.6	46.3	41.5	35.1	45.6	39.1	37.7
1.6	**1**	23.6	22.5	23.6	19.4	32.8	24.5	27.5	31.8	27.8	25.8	20.7	28.2	23.9	24.6
2	**1**	10.7	10.8	12.8	10.3	15.5	12.8	15.8	16.0	14.3	14.6	11.0	14.3	12.9	14.1
3	**1**	4.7	5.0	6.4	4.8	6.3	5.8	7.4	7.1	6.8	7.6	5.1	6.2	6.0	6.6
5	**1**	2.8	3.0	4.0	2.5	3.1	3.0	3.5	4.2	4.2	4.7	2.6	3.2	3.1	3.2
0.8	**1.2**	91.9	93.7	97.1	77.2	157.7	112.5	101.9	147.2	139.6	121.7	87.6	137.8	117.8	95.6
0.7	**1.3**	49.2	52.2	58.8	48.0	115.4	70.4	68.8	120.0	103.3	80.6	54.3	114.7	81.0	65.3
0.6	**1.4**	30.3	33.3	39.7	33.1	81.1	46.9	50.5	94.0	74.3	56.1	37.0	92.2	56.6	48.2
0.4	**1.6**	15.7	17.8	23.1	19.4	41.9	26.1	32.1	55.6	41.0	32.7	21.3	58.4	32.3	30.7
0.2	**1.8**	10.3	11.9	16.1	13.5	25.2	17.6	23.2	34.3	25.8	22.6	14.6	40.0	22.0	22.2

Table 4.13. SSATS values for some robust control charts with $G_2(4,1)$ observations for shifts in **σ** in some specific directions

		ρ = 0.9							ρ = 0						
		MRW^2			SER|Z|	MR|Z|^q			MRW^2			SER|Z|	MR|Z|^q		
	w/q	***2.2***	***2.4***	***2.8***	***1.0***	**0.5**	**1.0**	**1.5**	***2.2***	***2.4***	***2.8***	***1.0***	**0.5**	**1.0**	**1.5**
	λ	***0.2***	***0.1***	***0.025***	***0.03***	***0.13***	***0.03***	***0.001***	***0.2***	***0.1***	***0.025***	***0.03***	***0.13***	***0.03***	***0.001***
γ_1	γ_2	**[1]**	**[2]**	**[3]**	**[4]**	**[5]**	**[6]**	**[7]**	**[8]**	**[9]**	**[10]**	**[11]**	**[12]**	**[13]**	**[14]**
1	**1**	275.2	255.0	215.2	197.8	197.8	196.9	192.3	340.0	296.1	233.3	186.1	219.3	207.0	194.4
1.2	**1.2**	68.5	56.6	43.0	37.2	45.1	38.0	36.1	55.2	43.6	32.2	30.1	33.7	27.8	27.4
1.4	**1.4**	24.3	21.2	18.6	17.0	19.3	17.2	18.0	17.4	15.0	13.6	13.3	13.2	12.3	13.5
1.6	**1.6**	12.6	11.4	11.2	10.6	11.4	10.8	11.7	8.7	8.1	8.2	8.8	7.6	7.6	8.7
2	**2**	6.1	5.9	6.1	6.1	6.1	6.1	6.7	4.4	4.3	4.5	5.0	4.1	4.4	5.0
3	**3**	3.1	3.1	3.2	3.0	2.9	3.0	3.2	2.3	2.3	2.4	2.4	2.0	2.2	2.4
5	**5**	2.0	2.1	2.1	1.5	1.6	1.6	1.6	1.6	1.6	1.6	1.2	1.1	1.1	1.2
1.2	**1**	88.9	76.7	60.8	47.6	77.5	58.7	56.1	110.5	86.7	58.7	51.8	68.3	51.5	46.2
1.4	**1**	28.6	26.3	25.6	21.0	36.2	26.6	30.0	41.5	32.8	26.3	21.1	30.6	24.0	24.5
1.6	**1**	13.8	13.4	15.0	13.0	20.7	16.3	20.1	21.0	17.4	15.9	13.3	17.7	15.2	16.4
2	**1**	6.5	6.7	8.1	7.3	10.3	9.1	11.8	9.7	8.7	8.8	7.4	9.2	8.6	9.6
3	**1**	3.3	3.5	4.3	3.6	4.6	4.4	5.6	4.8	4.5	4.7	3.7	4.3	4.2	4.6
5	**1**	2.2	2.3	2.9	1.9	2.4	2.3	2.6	3.2	3.1	3.1	1.9	2.3	2.2	2.2
0.8	**1.2**	71.6	68.8	64.9	48.1	121.0	71.3	70.5	189.2	143.4	88.6	55.2	143.1	82.9	65.4
0.7	**1.3**	33.5	35.1	37.7	29.8	78.2	42.7	48.7	112.4	81.4	53.1	32.5	99.6	51.5	45.3
0.6	**1.4**	19.4	21.4	25.5	21.1	51.0	29.2	36.7	67.6	49.0	36.1	22.2	69.5	35.7	34.2
0.4	**1.6**	9.7	11.1	14.7	13.0	25.6	17.4	24.2	30.5	23.4	20.7	14.2	37.8	21.4	22.4
0.2	**1.8**	6.5	7.5	10.3	9.4	15.9	12.3	17.7	17.9	14.7	14.2	10.1	24.9	15.2	16.4

Table 4.14. SSATS values for some robust control charts with $T_2(4)$ observations and adjusted control limit for shifts in $\boldsymbol{\sigma}$ in some specific directions

		ρ = 0.9							ρ = 0						
		MRW^2			**SER|Z|**	**MR|Z|^q**			**MRW^2**			**SER|Z|**	**MR|Z|^q**		
	w/q	***2.2***	***2.4***	***2.8***	***1.0***	**0.5**	**1.0**	**1.5**	***2.2***	***2.4***	***2.8***	***1.0***	**0.5**	**1.0**	**1.5**
	λ	***0.2***	***0.1***	***0.025***	***0.03***	***0.13***	***0.03***	***0.001***	***0.2***	***0.1***	***0.025***	***0.03***	***0.13***	***0.03***	***0.001***
γ_1	γ_2	**[1]**	**[2]**	**[3]**	**[4]**	**[5]**	**[6]**	**[7]**	**[8]**	**[9]**	**[10]**	**[11]**	**[12]**	**[13]**	**[14]**
1	**1**	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0
1.2	**1.2**	67.5	62.0	56.5	56.4	64.1	57.3	54.2	61.6	55.3	49.0	51.7	57.9	49.4	45.7
1.4	**1.4**	31.5	28.4	26.7	25.5	29.2	26.0	26.1	27.3	24.0	22.4	22.7	24.7	21.3	21.6
1.6	**1.6**	18.3	16.8	16.6	15.4	17.1	15.7	16.5	15.5	13.9	13.9	13.6	13.9	12.6	13.6
2	**2**	9.3	8.8	9.3	8.3	8.7	8.5	9.3	7.7	7.3	7.9	7.4	6.9	6.8	7.6
3	**3**	4.3	4.3	4.8	3.9	3.9	4.0	4.4	3.6	3.6	4.1	3.5	3.1	3.2	3.5
5	**5**	2.5	2.6	2.9	2.0	2.1	2.1	2.2	2.2	2.3	2.5	1.7	1.6	1.6	1.7
1.2	**1**	87.0	82.6	79.3	72.0	100.3	86.0	81.1	101.0	91.8	80.5	77.1	98.2	83.5	73.0
1.4	**1**	38.5	36.4	36.8	31.8	53.4	40.5	42.1	55.5	47.5	40.4	35.0	52.8	40.5	37.9
1.6	**1**	20.9	20.3	22.2	18.8	31.3	23.8	27.5	33.8	28.2	25.3	20.7	31.7	24.5	24.7
2	**1**	9.9	10.1	12.2	10.0	15.0	12.5	15.8	16.9	14.5	14.3	10.9	15.7	13.2	14.2
3	**1**	4.5	4.8	6.2	4.7	6.2	5.7	7.4	7.4	6.9	7.5	5.1	6.6	6.1	6.7
5	**1**	2.7	2.8	3.8	2.4	3.1	2.9	3.5	4.3	4.2	4.6	2.6	3.3	3.1	3.2
0.8	**1.2**	76.6	79.0	87.4	73.5	146.9	107.2	101.8	162.9	144.6	117.3	87.2	165.7	124.0	96.2
0.7	**1.3**	42.9	45.8	54.1	46.1	108.2	67.7	68.8	131.9	106.6	78.4	54.0	136.9	84.6	65.7
0.6	**1.4**	27.1	30.0	37.2	31.9	76.7	45.4	50.5	103.1	76.4	54.9	36.8	109.1	58.6	48.5
0.4	**1.6**	14.5	16.5	21.9	18.8	40.1	25.5	32.1	60.1	42.1	32.2	21.2	67.8	33.3	30.9
0.2	**1.8**	9.7	11.2	15.4	13.1	24.4	17.2	23.1	36.6	26.3	22.3	14.6	45.3	22.5	22.3

Table 4.15. SSATS values for some robust control charts with $G_2(4,1)$ observations and adjusted control limit for shifts in **σ** in some specific directions

		ρ = 0.9							ρ = 0						
		MRW^2			SER|Z|	MR|Z|^q			MRW^2			SER|Z|	MR|Z|^q		
	w/q	*2.2*	*2.4*	*2.8*	*1.0*	0.5	1.0	1.5	*2.2*	*2.4*	*2.8*	*1.0*	0.5	1.0	1.5
	λ	*0.2*	*0.1*	*0.025*	*0.03*	*0.13*	*0.03*	*0.001*	*0.2*	*0.1*	*0.025*	*0.03*	*0.13*	*0.03*	*0.001*
γ_1	γ_2	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]	[14]
1	**1**	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0
1.2	**1.2**	53.2	48.3	41.4	37.5	45.5	38.3	36.8	39.0	34.7	29.8	31.6	31.7	27.3	27.7
1.4	**1.4**	20.2	18.9	18.1	17.0	19.4	17.3	18.2	13.6	12.8	12.8	14.4	12.7	12.1	13.6
1.6	**1.6**	10.9	10.5	10.9	10.7	11.4	10.8	11.8	7.3	7.1	7.7	9.1	7.4	7.6	8.8
2	**2**	5.5	5.5	6.0	6.1	6.1	6.2	6.8	3.8	3.9	4.3	5.2	4.0	4.3	5.1
3	**3**	2.8	2.9	3.1	3.0	2.9	3.0	3.2	2.1	2.1	2.3	2.5	2.0	2.1	2.4
5	**5**	1.9	1.9	2.1	1.5	1.6	1.6	1.6	1.5	1.5	1.6	1.3	1.1	1.1	1.2
1.2	**1**	69.4	65.3	58.6	48.0	78.1	59.3	57.3	73.2	65.0	53.8	50.4	63.5	50.7	46.8
1.4	**1**	24.1	23.7	24.9	21.1	36.6	26.8	30.5	30.6	26.8	24.5	22.6	29.0	23.7	24.9
1.6	**1**	12.2	12.5	14.7	13.0	20.9	16.4	20.4	16.4	15.0	15.0	14.0	17.0	14.9	16.6
2	**1**	6.0	6.3	8.0	7.3	10.3	9.1	12.0	8.2	7.8	8.3	7.9	8.9	8.5	9.7
3	**1**	3.1	3.3	4.2	3.6	4.6	4.4	5.7	4.2	4.1	4.4	3.8	4.2	4.2	4.6
5	**1**	2.1	2.2	2.8	1.9	2.4	2.3	2.7	2.8	2.8	3.0	2.0	2.3	2.2	2.3
0.8	**1.2**	58.1	59.8	62.4	48.5	122.1	71.8	71.8	119.6	104.3	80.4	57.1	131.1	81.2	66.3
0.7	**1.3**	28.6	31.6	36.7	29.9	79.0	43.0	49.5	75.6	63.3	49.6	34.4	92.3	50.6	45.9
0.6	**1.4**	17.2	19.7	24.9	21.1	51.4	29.4	37.3	48.5	40.3	33.9	23.8	65.0	35.1	34.6
0.4	**1.6**	8.9	10.5	14.5	13.0	25.7	17.5	24.5	23.9	20.3	19.6	14.4	36.0	21.2	22.7
0.2	**1.8**	6.1	7.2	10.1	9.4	16.0	12.4	18.0	14.7	13.0	13.5	10.3	24.0	15.1	16.6

4.2.3 Efficiency comparison for random shifts in σ with ρ changed

It could be possible that **σ** changes while the correlation between variables changes at the same time. We select the robust design which gives the best overall performance for each of the three proposed multivariate control charts and investigate their performance for detecting random shifts in **σ** with ρ changing.

Table 4.16 gives the SSATS values for three robust charts to detect random shift in **σ** with ρ decreasing when the in-control ρ is 0.9. When there is only a change in ρ, the MR|Z| chart gives SSATS value larger than 200 and in this case, an "-" is used in place of the SSATS. The MRW^2 and SER|Z| chart can detect the decreases in ρ when there is no random shift in **σ** but their SSATS are quite large when the decrease in ρ is small, say the ρ decreases from 0.9 to 0.7.

The SER|Z| chart is generally better than the MR|Z| chart in detecting the change in both ρ and **σ** for bivariate normal and non-normal distributions when the in-control ρ is 0.9. The MRW^2 chart with w =2.2 and λ = 0.2 beats the other two charts for detecting decreases in ρ when there is no shift or small to median shifts in **σ**. It loses its advantage to the other two charts when the shift size in **σ** is large.

Table 4.16. SSATS values for some robust control charts with control limits adjusted for the corresponding distribution for random shifts in σ when ρ decreases from the in-control value ρ=0.9

		$N_2(0, \sum_{z0})$			$T_2(4)$			$G_2(4,1)$		
		MRW^2	**SER|Z|**	**MR|Z|^q**	**MRW^2**	**SER|Z|**	**MR|Z|^q**	**MRW^2**	**SER|Z|**	**MR|Z|^q**
	w/q	***2.2***	***1.0***	**1.0**	***2.2***	***1.0***	**1.0**	***2.2***	***1.0***	**1.0**
	λ	***0.2***	***0.03***	***0.03***	***0.2***	***0.03***	***0.03***	***0.2***	***0.03***	***0.03***
ρ	***ψ***	**[1]**	**[2]**	**[3]**	**[4]**	**[5]**	**[6]**	**[7]**	**[8]**	**[9]**
0.9	1	200	200	200	200	200	200	200	200	200
0.2	1	61.1	137.8	-	86.0	146.1	-	70.1	137.9	-
0.5	1	81.4	144.5	-	103.8	154.4	-	92.2	146.0	-
0.7	1	118.4	156.8	-	134.4	166.1	-	126.4	160.4	-
0.2	1.2	25.8	34.3	47.6	44.4	59.5	83.1	33.1	39.9	56.2
0.5	1.2	30.9	36.1	47.5	51.7	62.2	82.6	40.1	42.9	56.9
0.7	1.2	39.4	38.5	47.4	62.9	66.4	81.9	50.9	45.4	56.3
0.2	1.6	9.4	10.9	13.1	16.4	18.4	22.6	10.4	12.4	15.1
0.5	1.6	10.2	11.3	13.4	17.9	19.0	23.0	11.3	12.9	15.5
0.7	1.6	11.5	11.8	13.7	20.2	19.8	23.4	12.7	13.4	15.8
0.2	2	5.7	6.4	7.4	9.0	10.0	11.8	5.8	7.2	8.4
0.5	2	6.0	6.6	7.6	9.6	10.3	12.0	6.1	7.4	8.6
0.7	2	6.5	6.8	7.8	10.4	10.6	12.3	6.5	7.6	8.8
0.2	3	3.2	3.2	3.6	4.3	4.6	5.3	3.1	3.5	4.0
0.5	3	3.3	3.3	3.7	4.4	4.7	5.4	3.1	3.6	4.1
0.7	3	3.4	3.4	3.8	4.7	4.9	5.5	3.3	3.7	4.2
0.2	5	2.2	1.6	1.9	2.5	2.3	2.6	2.1	1.8	2.0
0.5	5	2.2	1.7	1.9	2.6	2.4	2.7	2.1	1.8	2.1
0.7	5	2.2	1.8	2.0	2.7	2.5	2.8	2.1	1.9	2.1

Table 4.17 gives the SSATS values for the three robust charts to detect random shift in **σ** with ρ increasing when the in-control ρ is 0. The SER|Z| chart hardly detects the increases in ρ when **σ** is unchanged. The MRW^2 chart still gives the best performance among the three charts for this situation. When both ρ and **σ** change at the same time, the MR|Z| chart generally gives the best detection, except for detecting some median shifts in **σ** with heavy-tailed observations.

Table 4.17. SSATS values for some robust control charts with control limits adjusted to corresponding distributions for random shifts in σ when ρ increases from the in-control value ρ=0

		$N_2(0, \Sigma_{z0})$			$T_2(4)$			$G_2(4,1)$		
		MRW^2	SER|Z|	MR|Z|^q	MRW^2	SER|Z|	MR|Z|^q	MRW^2	SER|Z|	MR|Z|^q
	w/q	*2.2*	*1.0*	1.0	*2.2*	*1.0*	1.0	*2.2*	*1.0*	1.0
	λ	*0.2*	*0.03*	*0.03*	*0.2*	*0.03*	*0.03*	*0.2*	*0.03*	*0.03*
ρ	*ψ*	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]
0	1	200	200	200	200	200	200	200	200	200
0.2	1	176.0	184.0	174.0	187.9	-	189.7	153.3	-	170.6
0.5	1	116.3	-	141.1	152.3	-	170.1	106.1	-	139.7
0.7	1	86.3	-	117.7	125.8	-	151.8	83.2	-	120.0
0.2	1.2	48.6	39.5	36.6	84.8	74.1	73.0	57.2	48.1	43.2
0.5	1.2	39.9	41.5	35.4	73.2	77.2	69.6	46.4	51.2	42.2
0.7	1.2	33.6	44.1	33.9	63.4	82.5	65.9	39.0	54.3	40.0
0.2	1.6	12.6	11.7	10.9	26.6	21.0	21.4	13.6	13.8	12.5
0.5	1.6	12.0	12.1	11.1	24.9	21.6	21.4	13.0	14.3	12.8
0.7	1.6	11.4	12.6	11.2	23.1	22.4	21.3	12.3	14.8	12.9
0.2	2	6.7	6.8	6.3	13.2	11.1	11.3	6.6	7.8	7.1
0.5	2	6.6	6.9	6.4	12.7	11.4	11.4	6.6	8.0	7.2
0.7	2	6.5	7.2	6.5	12.3	11.7	11.5	6.5	8.2	7.4
0.2	3	3.4	3.3	3.1	5.5	5.0	5.1	3.2	3.7	3.4
0.5	3	3.4	3.4	3.2	5.4	5.2	5.2	3.2	3.8	3.5
0.7	3	3.4	3.5	3.2	5.4	5.3	5.3	3.2	3.9	3.6
0.2	5	2.1	1.7	1.6	2.9	2.5	2.5	2.0	1.9	1.7
0.5	5	2.1	1.8	1.7	3.0	2.6	2.6	2.0	1.9	1.8
0.7	5	2.2	1.8	1.7	3.0	2.6	2.7	2.0	2.0	1.8

4.3 Conclusion

Simulation results have shown that the three proposed multivariate control charts can be designed to be robust to non-normality. By choosing the proper combination of q and

λ, the MR|Z|^q chart can offer a good number of robust designs over a variety of bivariate distributions with different correlation matrices. When a process is more prone to producing values that fall far from its mean, the absolute deviation with a small value of q gives some robustness to control charts by reducing the impact of extreme observations. By the same principle, it also makes charts respond to light-tailed distributions closer to the normal.

Generally, for a given q, the in-control ATS increases as λ decreases when the MR|Z|^q chart is applied to multivariate t and gamma observations, and decreases as λ decreases when applied to multivariate beta observations. Thus a moderately small value of λ is advisable, which provides balanced robustness over various distributions for a given q in the MR|Z|^q chart.

The robust design of the SER|Z| chart is quite similar to that of the MR|Z|^q chart. Using a set of ER|Z| charts with λ around 0.03 is a simple way to obtain a robust chart to monitor bivariate processes. The MRW^2 charts provide an additional way to obtain robust performance over various bivariate distributions. A larger value of λ paired with a relative small value of w make the MRW^2 charts robust.

For the robust MR|Z|^q charts, $q = 1$ with $\lambda = 0.03$ is a combination that provides better overall performance to detect a wide range of random shifts based on the TEQL values. The overall performance of the SER|Z| chart is close to the MR|Z| chart for various distributions based on the TEQL values. Averaged over all shift directions, the SER|Z| chart is better than the MR|Z| chart when $\rho = 0.9$ and worse when $\rho = 0$ for most of the bivariate distributions. Additionally, for the shift direction in which one component of $\boldsymbol{\sigma}$ increases while the other component decreases by the same amount, the SER|Z| chart has a much quicker detection.

Due to the Winsorization method, the robust MRW^2 charts are not very sensitive to large shifts. Their TEQL values are much worse than the other two types of robust charts when medium and large random shifts are more likely. For the three MRW^2 charts considered here, $w = 2.2$ with $\lambda = 0.2$ gives better overall performance. It has advantages in in some situations over the other two type of control charts, especially in detecting random shifts with ρ changed when the in control $\rho = 0.9$. However, generally speaking, the overall performance of the MRW^2 charts cannot compete with the SER|Z| and MR|Z| charts.

Chapter 5

The Proposed Control Charts Based on the Regression Adjusted Variables

5.1 Control charts based on regression adjusted variables

Hawkins (1991, 1993) proposed multivariate Shewhart and CUSUM charts based on regression adjusted variables. A vector shift in the mean or variance of the original variables would translate into shifts in the regression adjusted variables. He found that the charts' performance in detecting shifts in $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ was improved compared to charts based on the original variables. Following the same transformation, Reynolds and Cho (2006) showed that the performance of MEWMA-type of control charts for $\sum$ could also be improved significantly by using the regression adjustment for multivariate normal observations.

Now, we consider using the regression adjusted variables instead of the original variables with the robust design of the proposed control charts to improve their efficiency for detecting out-of-control situations. Define the vector of regression-adjusted variables for the j^{th} observation at sampling time k as

$$\mathbf{A}_{kj} = (A_{k1j}, A_{k2j}, \cdots, A_{kpj})^{\mathrm{T}}$$

which can be obtained by taking the following transformation of the original variable Z_{kij}:

$$A_{kij} = (diag \textstyle\sum_{Z0}^{-1})^{-1/2} \textstyle\sum_{Z0}^{-1} Z_{kij}; \quad j = 1,2,\cdots,n$$

The matrix $diag\,\Sigma_{Z0}^{-1}$ is a diagonal matrix with the diagonal elements in Σ_{Z0}^{-1}, and $(diag\,\Sigma_{Z0}^{-1})^{-1/2}\,\Sigma_{Z0}^{-1}$ is the transformation matrix. When the process is in control, $\mathbf{A}_{kj}$ has mean $\boldsymbol{\mu}_{\mathbf{A}} = (diag\,\Sigma_{Z0}^{-1})^{-1/2}\,\Sigma_{Z0}^{-1}\,\boldsymbol{\mu}_{\mathbf{Z}}$ and covariance matrix

$$\Sigma_A = (diag\,\Sigma_{Z0}^{-1})^{-1/2}\,\Sigma_{Z0}^{-1}(diag\,\Sigma_{Z0}^{-1})^{-1/2}$$

The one-sided ER|Z|^q statistic is turned into the ER|A|^q statistic by simply replacing the Z_{kij} variables by the regression adjusted variables A_{kij}. Correspondingly, we name the SER|Z| and MR|Z|^q charts based on regression adjustment as the SER|A| and MR|A|^q charts. It should be noticed that the transformation doesn't change the original variable when the variables are independent. Thus the SER|A| and MR|A|^q charts are equivalent to the SER|Z| and MR|Z|^q charts when the original variables are independent.

The MRW^2 chart is based on the truncated variables. When we apply the regression adjustment on the truncated variables W_{kij}, the above formula for the mean vector and covariance of regression adjustment variables does not apply. We can estimate the mean vector and covariance after regression adjustment by simulation. However, simulation results show that the MRW^2 chart to monitor the regression adjusted variables instead of original variables cannot keep the original satisfying level of robustness even after we adjust the parameters' values. Therefore, we focus on the robust design of the MR|A|^q and SER|A| charts in this chapter. Their in-control and out-of-control performance will be compared to the MR|Z|^q and SER|Z| charts when the variables are not independent.

5.2 The robust design for the MR|A|^q and SER|A| charts based on regression adjusted variables

In the previous chapter, we have investigated the robust design and efficiency for the MR|Z|^q and SER|Z| charts in the bivariate setting. To avoid redundant discussion, we here focus on the charts' performance with higher dimensional processes. We look into the robust designs of these two type of charts for 5-variate and 8-variate processes. Unlike a bivariate processes, where there is only one correlation coefficient, we may have much more complicated correlation structures with higher dimensional processes. For the higher dimensional processes, we consider three correlation structures. Highly correlated variables with correlation coefficients all equal to 0.9, and independent variables with correlation coefficients all equal to 0 are taken into consideration. Additionally, an AR(1) correlation structure arouses our interest. We define $\rho_{ij} = 0.7^{|i-j|}$ to include pairwise variables with high, median and low correlation. Due to the constraint of the simulation algorithm, the ATS values of control charts with multi-beta observations with the AR(1) correlation structure are not available.

Tables 5.1 and 5.2 contain the in-control ATS values of the SER|A|, SER|Z|, MR|A|^q, and MR|Z|^q charts for 5-variate processes with the three correlation structures. For the MR|Z|^q robust designs, we select the MR|Z| chart, which give the best performance for the bivariate processes, as well as the MR|Z|^q chart with a higher q = 1.5. For the robust design of the MR|A|^q chart, the q is also selected between 0.5 and 1.5. The in-control ATS values of the MRA^2 chart with λ = 0.025 and 0.001 are also included for comparison.

The q and λ combinations for the MR|A|^q chart when ρ = 0.9 and ρ_{ij} = 0.7$^{|i-j|}$ show that the regression adjusted variables don't change the general property that the MR|Z|^q and SER|Z| charts can be tuned to be robust. For the same q and λ combination, the MR|A|^q and SER|A| charts give a smaller ATS value with multivariate t observations but a larger ATS value with the multivariate gamma observations compared to the MR|Z|^q and SER|Z| charts, respectively. To achieve a satisfying level of robustness, the advisable value of λ may be a little smaller. For instance, we prefer λ = 0.02 for the MR|A| chart, but λ = 0.03 for the MR|Z| chart when $\rho \neq 0$. Based on the RMS values, we find that the MR|A|^q and SER|A| charts retain a good level of robustness if q and λ are properly selected.

Comparing the in-control ATS values of the MR|Z|^q chart in Tables 5.1-5.2 to Tables 4.1-4.2, the same combinations of q and λ still give small RMS values when the variables are highly correlated although we have increased p from 2 to 5. However, the ATS values for non-normal observations gets further below 200 for some of the combinations when $\rho = 0$ and $p = 5$. The ATS deviations with non-normal observations when $\rho_{ij} = 0.7^{|i-j|}$ are somewhere between the other two cases. We notice that the recommended value of λ is around 0.06 when $q = 0.5$ and $\rho = 0$ in the 5-variate processes while the recommended value of λ is around 0.13 in the bivariate processes. The robustness of the MR|Z| chart gets hurt slightly when the correlation decreases from 0.9 to 0, but the MR$|Z|^{1.5}$ chart with $\lambda = 0.001$ shows a good resistance to the changes of p and ρ. Thus we conclude that the robust design of the MR$|Z|^q$ chart with a relatively large q and small λ gives a more steady robustness over different dimensions and correlation structure. Although the ATS deviation with non-normal observations of the MR|Z| chart deteriorates when $\rho = 0$, a λ as small as 0.02 can give a satisfying robustness for the higher dimensional processes.

Table 5.3 shows the in-control performance of the acceptable combinations of q and λ for the robust designs of the MR$|A|^q$ and SER|A| charts when $p = 8$. The acceptable value of λ has to get down to 0.05 for the MR$|A|^{0.5}$ chart. But the MR|A| chart with $\lambda = 0.02$ and MR$|A|^{1.5}$ with $\lambda = 0.001$, which work well when $p = 5$, still show good robustness when $p = 8$.

Based on Tables 5.1-5.3, the robustness of the SER|A| and SER|Z| charts is generally less affected by the increase in p and decrease in ρ compared to the MR|A| and MR|Z| charts. The choice of $\lambda = 0.03$ for bivariate processes still give a reasonable robustness even for $p = 5$ or 8. The in-control performance of the MRA^2 chart gets even worse as p increases from 5 to 8. Its robustness is not satisfactory even when λ is adjusted to 0.001.

5.3 The efficiency comparison for the MR$|Z|^q$ and SER|Z| charts based on regression adjusted variables

The comparison results for out-of-control performance among the robust designs generally follow a similar pattern even though the process dimension varies. We here consider processes with $p = 5$, as representative of other processes. Some robust designs which work well with three correlation structures for different p in the in-control case are selected to investigate their detection efficiency under $N_5(0, \Sigma_{Z0})$, $T_5(4)$, and $G_5(4,1)$ distributions. Only *t* and gamma distributions are considered since heavier-tailed and skewed distributions are more common from a practical point of view.

Table 5.1. In control ATS of control charts for various distributions with ρ=0.9 and ρ=0 when p=5

ρ	Control Chart	q	λ	UCL	In-control ATS $N_5(0,\Sigma_{z0})$ [1]	$T_5(4)$ [2]	$T_5(10)$ [3]	$G_5(4,1)$ [4]	$G_5(8,1)$ [5]	$B_5(4,4)$ [6]	RMS(1) (%)	RMS(2) (%)
0.9	MR|Z|^q	1	0.03	1.09142	200	208.3	205.3	195.2	199.3	188.5	3.5	2.7
			0.02	1.00008	200	247.2	220.3	209.0	207.3	176.7	12.9	13.2
		1.5	0.002	0.92300	200	191.1	196.2	190.1	196.7	197.5	3.2	3.6
			0.001	0.90044	200	198.5	199.5	193.3	199.0	195.0	1.9	1.7
	MR|A|^q	0.5	0.07	3.60383	200	190.4	182.3	186.6	191.5	186.7	6.5	6.4
			0.06	3.53837	200	219.5	196.5	198.9	199.0	187.1	5.3	5.0
		1	0.03	3.33876	200	150.9	171.1	176.5	185.9	191.6	14.5	16.1
			0.02	3.16133	200	192.4	196.0	199.9	199.7	192.2	2.6	2.2
		1.5	0.002	3.13873	200	182.1	195.6	195.8	198.5	198.6	4.2	4.7
			0.001	3.07526	200	195.4	201.7	201.1	201.2	198.7	1.2	1.3
	SER|Z|		0.03	0.97981	200	207.5	206.4	194.1	198.8	187.8	3.8	2.9
			0.02	0.93304	200	251.3	223.1	209.6	207.3	175.7	14.0	14.4
	SRE|A|		0.03	1.00591	200	181.1	197.1	197.2	198.0	199.3	4.8	5.3
			0.02	0.95512	200	232.0	222.5	220.4	211.7	200.8	10.2	11.4
	MRA^2	2	0.025	5.99853	200	58.5	92.7	95.8	124.9	195.2	49.0	54.8
			0.001	4.16627	200	107.4	146.4	147.1	167.8	199.5	27.6	30.9
0	MR|Z|^q	0.5	0.07	4.16320	200	174.5	172.7	300.1	244.2	118.4	31.6	28.9
			0.06	4.09163	200	202.9	186.3	306.1	246.4	114.9	32.3	29.2
		1	0.03	3.99096	200	150.1	170.3	231.9	215.5	148.7	19.1	17.0
			0.02	3.79253	200	183.1	191.1	251.4	225.0	144.0	18.4	15.1
		1.5	0.002	3.85885	200	180.6	195.2	198.5	200.1	181.5	6.1	5.0
			0.001	3.78251	200	194.3	201.7	202.21	202.13	179.88	4.7	1.7
		2	0.025	7.26253	200	55.7	88.1	84.2	113.1	554.1	94.8	58.3
			0.001	5.14886	200	105.1	144.8	134.8	159.2	273.0	36.5	34.1
	SER|Z|		0.03	1.00689	200	175.1	196.0	165.3	179.5	201.1	10.6	11.9
			0.02	0.95600	200	226.5	221.2	186.8	192.8	179.6	9.5	9.3

Table 5.2. In control ATS of some control charts for various distributions with the AR(1) correlation matrix when p=5

Control Chart	q	λ	UCL	**In-control ATS** $N_5(0,\sum_{z0})$ [1]	$T_5(4)$ [2]	$T_5(10)$ [3]	$G_5(4,1)$ [4]	$G_5(8,1)$ [5]	**RMS(2) (%)**
$MR\|Z\|^q$	1	0.03	2.07468	200	182.2	191.2	190.4	192.4	5.8
		0.02	1.94219	200	220.1	211.5	207.4	204.7	6.2
	1.5	0.002	1.85137	200	188.2	196.8	190.2	196.3	4.0
		0.001	1.81136	200	198.9	201.9	195.6	199.1	1.2
$MR\|A\|^q$	0.5	0.07	3.22717	200	206.0	192.0	270.5	266.1	24.3
		0.06	3.16493	200	236.7	205.3	285.7	275.4	30.0
	1	0.03	2.94782	200	156.4	176.0	246.7	255.7	22.0
		0.02	2.78339	200	198.7	200.3	277.3	272.8	26.6
	1.5	0.002	2.73500	200	180.0	194.2	250.6	258.2	20.0
		0.001	2.67891	200	196.6	200.9	254.0	259.5	20.1
SER|Z|		0.03	0.99998	200	190.1	201.4	182.4	192.1	5.4
		0.02	0.94978	200	240.3	222.1	202.6	203.0	11.5
SER|A|		0.03	1.00430	200	183.3	197.4	226.8	235.6	11.9
		0.02	0.95360	200	232.3	220.7	257.3	253.6	21.8
MRA^2	2	0.025	5.29947	200	61.5	96.7	117.7	160.8	48.8
		0.001	3.62504	200	108.7	146.7	178.6	211.0	50.9

Table 5.3. In control ATS of selected control charts for various distributions with three type of correlation matrices when p=8

					In-control ATS							
ρ_{ij}	**Control Chart**	q	λ	**UCL**	**$N_8(0,\sum_{z0})$ [1]**	**$T_8(4)$ [2]**	**$T_8(10)$ [3]**	**$G_8(4,1)$ [4]**	**$G_8(8,1)$ [5]**	**$B_8(4,4)$ [6]**	**RMS(1) (%)**	**RMS(2) (%)**
0.9	MR$\|A\|^q$	0.5	0.05	5.77778	200.0	202.1	181.5	187.6	190.3	183.0	6.7	6.1
			0.04	5.66987	200.0	258.0	206.0	210.3	204.0	184.8	13.7	14.8
		1	0.03	5.60724	200.0	123.9	147.0	154.9	170.5	185.7	24.2	26.8
			0.02	5.35283	200.0	165.3	177.1	183.6	189.7	188.7	10.6	11.5
		1.5	0.002	5.43661	200.0	178.0	193.2	195.0	198.5	198.1	5.3	29.5
			0.001	5.33255	200.0	193.8	201.2	202.2	202.4	198.5	1.6	2.2
		2.0	0.025	9.97190	200.0	49.7	78.6	81.4	110.7	189.6	54.5	60.9
			0.001	7.23113	200.0	101.6	141.4	143.2	165.6	198.9	29.6	33.1
	SRE$\|A\|$	1	0.03	1.02137	200.0	166.0	191.6	191.5	195.3	199.0	8.1	9.1
			0.02	0.96788	200.0	219.9	218.7	216.7	209.9	199.8	7.5	8.4
$(0.7)^{\|i-j\|}$	MR$\|A\|^q$	0.5	0.05	4.85078	200.0	223.7	193.6	348.5	318.9	-	-	47.9
			0.04	3.86082	200.0	281.3	216.0	382.7	336.8	-	-	60.7
		1	0.03	4.58616	200.0	134.1	157.0	272.9	284.5	-	-	34.1
			0.02	4.36217	200.0	177.2	185.2	314.7	309.1	-	-	40.2
		1.5	0.002	4.34600	200.0	179.6	194.1	278.1	284.5	-	-	29.3
			0.001	4.26105	200.0	194.5	201.5	282.3	284.3	-	-	29.5
		2.0	0.025	8.10962	200.0	52.7	83.6	113.7	162.7	-	-	52.5
			0.001	5.76261	200.0	103.6	143.4	186.4	223.8	-	-	28.8
	SRE$\|A\|$	1	0.03	1.01991	200.0	168.7	192.2	228.8	194.6	-	-	10.9
			0.02	0.96654	200.0	222.4	218.6	263.1	265.2	-	-	23.8
0	MR$\|Z\|^q$	0.5	0.05	6.31448	200.0	193.0	175.0	361.4	267.5	97.6	45.7	44.2
			0.04	5.02581	200.0	245.8	199.9	370.7	270.3	94.9	48.6	47.5
		1	0.03	6.25432	200.0	121.3	141.9	253.3	228.5	135.1	14.9	16.6
			0.02	5.97994	200.0	167.9	175.4	278.7	233.4	127.0	26.3	23.6
		1.5	0.002	6.16510	200.0	177.0	193.0	202.4	202.0	177.5	26.8	29.5
			0.001	6.04830	200.0	195.2	201.1	206.0	203.8	180.6	4.8	2.2
		2.0	0.025	11.23797	200.0	48.3	75.8	76.4	105.5	660.4	117.2	62.6
			0.001	8.23178	200.0	100.7	141.0	132.1	157.3	275.6	35.7	35.2
	SRE$\|Z\|$	1	0.03	1.02194	200.0	163.6	190.0	153.7	171.1	206.7	14.9	16.6
			0.02	0.96836	200.0	218.3	217.8	178.0	186.4	181.0	9.2	9.1

5.3.1 Efficiency comparison for random shifts with ρ unchanged

Tables 5.4-5.10 contain the SSATS and TEQL values and TEQL ratios of robust MR|Z|^q, MR|A|^q, SER|Z|, and SER|A| charts with 5-variate normal, $T_5(4)$, and $G_5(4,1)$ observations for ρ_{ij} = 0.9, 0, and $0.7^{|i-j|}$, respectively. The control limit for each chart is adjusted to give an in-control ATS of 200 when the process is multivariate normal. Although the in-control ATS values for the non-normal distributions will only be approximately equal to 200, rough comparisons between charts can still be made.

The MRA^2 chart with λ = 0.025 is still included for comparison when the process is multivariate normal. It is not surprising that the TEQL value for the same control chart under the same distribution increases as p increases from 2 to 5. However, the parameters' effects on detecting different sizes of shifts for the robust charts follow the general rule as what we saw for the bivariate processes.

Based on the TEQL values, the SER|Z| chart with λ = 0.03 detects shifts faster than the MR|Z| chart with λ = 0.02 when ρ_{ij} = 0.9 and $0.7^{|i-j|}$ for all the distributions considered. But MR|Z| chart is quicker in detection when ρ_{ij} = 0, which is similar to what we found for the bivariate processes. As we expect, the proposed charts based on the regression adjusted variables have better detection for shifts than the charts based on the original variables. Their SSATS to detect each shift size is smaller and correspondingly their TEQL value are much less.

Among the three MR|A|^q charts, the MR|A| chart with λ = 0.02 is the best choice for most circumstances, although its loss for normal observations still cannot beat the MRA^2 chart. In Tables 5.4 and 5.7, the TEQL ratios are obtained by dividing the TEQL values of robust charts by the TEQL value of the MRA^2 chart, which are all greater than 1. The TEQL ratios show that the percentage of efficiency loss due to the robust designs for normal observations are smaller when all shift sizes are equally likely and increase as small and median shift sizes become more likely, where the robust designs will lose at least 30% efficiency to the non-robust design.

The general performance of the SER|A| chart cannot compete with the MR|A| chart considering normal and non-normal distributions with different correlation structure. With approximately equal in-control ATS values, the SER|A| chart is more efficient only when the true process is $T_5(4)$ compared to the MR|A| chart. For other processes, its overall performance is close to the other two MR|A|^q charts.

Since the regression adjustment based charts exhibit much quicker detection than the regular charts for a wide range of shifts, we only adjust the control limits of the regression adjustment based charts corresponding to the non-normal distributions, and compare their efficiency when designed under the $T_5(4)$ and $G_5(4,1)$ distributions. The SSATS and TEQL value with adjusted control limits are given in Tables 5.11 and 5.12. The TEQL values of the robust charts are divided by the TEQL value of the MRA^2 chart to obtain the TEQL ratios in these two tables.

Compared to TEQL values for multi-normal processes, the TEQL values of the same control chart with $T_5(4)$ and $G_5(4,1)$ observations are generally larger. The non-robust MRA^2 chart is sensitive to extreme values and thus it has smaller SSATS values for large shifts than the robust charts. Its TEQL is the still the smallest among the 5 control charts for the three correlation structures when the uniform prior is assumed. However, when small and median shifts are more likely, the robust charts show some advantage over the MRA^2 chart. In Table 5.11, the TEQL ratios with the B(2,18) prior indicate that a robust design can reduce the loss for the MRA^2 chart by about 10%-20% when applied to the heavier-tailed processes. Although the robust designs are still less efficient than the MRA^2 chart for most of cases with $G_5(4,1)$ observations (Table 5.12) , the TEQL ratios are generally smaller than those in Table 5.4 for the same chart.

In Table 5.11 and 5.12, there is no robust chart that is generally better than other charts for $T_5(4)$ or $G_5(4,1)$ observations. Each robust chart may have a smaller loss than the others for certain scenarios. From the TEQL values in Table 5.11, we notice that the MR|A| chart works better when variables are highly correlated. And the SER|A| chart is the best for independent *t* variables or *t* variables with the AR(1) correlation structure.

Switching to Table 5.12, the MR|A|$^{0.5}$ chart works better for independent variables, while the MR|A| chart is the best choice for other situations.

5.3.2 Efficiency comparison for some specific directions with $\boldsymbol{\rho}$ unchanged

The SSATS and TEQL values given in the last section are averaged over random shift directions corresponding to an increase in $\boldsymbol{\sigma}$. It may be useful in practice to investigate the control charts' performance for some specific shift directions, especially between the MEWMA-type control charts and the set of individual charts. Tables 5.13-5.15 contain SSATS values of robust SER|A| and MR|A| charts for some shift directions in $\boldsymbol{\sigma}$ with 5-variate normal, *t*, and gamma distributions. Similar to what we considered for the bivariate processes, the first set of special shift directions corresponds to each component of $\boldsymbol{\sigma}$ increasing by the same amount. The second set of shift directions corresponds to one of the five components of $\boldsymbol{\sigma}$ increasing and the others remaining unchanged. The last set of shift directions implies that three or two components of $\boldsymbol{\sigma}$ increase by the same amount while two decrease by the same amount.

For both multivariate normal and non-normal distributions, the MR|A| chart with λ = 0.02 signals faster for the first type of shifts direction while the SER|A| chart with λ = 0.03 is consistently faster in detecting the second type of shift direction for all three correlation structures. When ρ_{ij} = 0.9, the MR|A| chart signals faster for the third type of shift direction. The SER|A| chart outperforms the MR|A| chart for the other two correlation structures for detecting the third type of shift direction. It is a little surprising to see that the regression adjustment method reduces the efficiency difference between the MR|Z| and SER|Z| charts in detecting the third type of shift direction.

5.3.3 Efficiency comparisons for random shifts in $\boldsymbol{\sigma}$ with $\boldsymbol{\rho}$ changed

Tables 5.16 and 5.17 contain the SSATS values of robust SER|A| and MR|A| charts for random shifts in $\boldsymbol{\sigma}$ with correlation changes for 5-variate normal, *t*, and gamma distributions. The control limit of each chart has been adjusted to give in-control ATS of 200 corresponding to the process distributions.

When the in-control ρ_{ij} = 0.9 or ρ_{ij} = $0.7^{|i-j|}$, the regression-adjusted-variable-based control charts are very sensitive to the correlation decreases for all three 5-variate distributions considered here, even when there is no random shift in $\boldsymbol{\sigma}$. When the in-control ρ_{ij} = 0, the SER|Z| chart cannot detect the correlation change if there is no variances change. It also takes more than 100 samples for the MR|Z| chart to detect a 0.2 increment in ρ. As the increment gets larger, the detection of the MR|Z| chart gets quicker. With the in-control ρ_{ij} = $0.7^{|i-j|}$, both the MR|A| and SER|A| charts barely detect the increment in ρ_{ij} when there is no variances increase or just small variances increases.

Based on Tables 5.16 and 5.17, the MR|A| chart generally shows faster detection for changes in both ρ and $\boldsymbol{\sigma}$ than the SER|A| chart, except for medium and large shifts with $T_5(4)$ observations when the in-control ρ_{ij} = 0.

5.4 Summary

To sum up, the SER|A| and MR|A|^q charts can be robust if an appropriate value of λ is chosen for a given q. From our study, The MR|A| chart with λ = 0.02 (or the MR|Z| chart when ρ_{ij} = 0), which shows good robustness over various multivariate distributions, generally has a better overall efficiency over the robust SER|A| and SER|Z| charts and other robust MR|A|^q and MR|Z|^q charts with $q \neq 1$ for random shifts in $\boldsymbol{\sigma}$. The SER|A| chart with λ = 0.03 is the best choice if there is a priority to detect shifts in one or two particular variables.

When the variables are not independent, the SER|A| and MR|A|^q chart based on the regression adjusted variables are much more efficient to detect a wide range of random shifts than the SER|Z| and MR|Z|^q charts. They also present good detection for both change in ρ and $\boldsymbol{\sigma}$. However, we may keep in mind that the SER|A| and MR|A|^q chart are monitoring the residuals for the regression model fitted on original variables. A shift in the $\boldsymbol{\sigma}$ of original variables is translated to the behavior of the residual variables by the transformation matrix.

An advantage of the SER|Z| chart is that it provides p separate statistics corresponding to the original p variables. Though we still have p statistics with the SER|A| chart, the interpretation of shifts in the original variables will be more difficult based on the behavior of the residual variables. Moreover, the direct interpretation will also be difficult for the MR|A|^q and MR|Z|^q charts since they are integrated forms of a complicated function of the original variables. Thus, some additional diagnostic plots may be needed for the SER|A|, MR|A|^q, and MR|Z|^q charts after a signal occurs.

Table 5.4. SSATS, EQL, and TEQL values for some control charts with $N_5(0, \Sigma z_0)$ observations for random shifts in $\boldsymbol{\sigma}$ when p = 5 and ρ = 0.9

	SER|Z|	**MR|Z|^q**		**SER|A|**	**MR|A|^q**			
q	**1**	**1**	**1.5**	**1**	**0.5**	**1**	**1.5**	**2**
λ	***0.03***	***0.03***	***0.001***	***0.03***	***0.06***	***0.02***	***0.001***	***0.025***
ψ	**[1]**	**[2]**	**[3]**	**[4]**	**[5]**	**[6]**	**[7]**	**[8]**
1.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0
1.1	101.2	108.3	94.2	82.5	84.3	68.9	58.5	70.3
1.2	58.0	66.0	61.7	42.7	43.1	35.4	34.2	34.5
1.4	27.2	32.0	34.8	18.2	17.1	15.3	16.8	13.8
1.6	17.1	20.0	23.9	10.9	9.7	9.2	10.5	7.8
1.8	12.3	14.3	18.0	7.6	6.6	6.4	7.4	5.2
2	9.6	11.1	14.3	5.8	5.0	4.9	5.6	3.9
2.5	6.2	7.2	9.4	3.6	3.1	3.0	3.4	2.3
3	4.6	5.3	6.9	2.6	2.3	2.2	2.4	1.6
4	3.1	3.5	4.5	1.7	1.5	1.4	1.5	1.0
5	2.3	2.6	3.3	1.3	1.2	1.1	1.1	0.8
7	1.6	1.8	2.2	0.9	0.9	0.8	0.8	0.6
10	1.1	1.3	1.5	0.7	0.7	0.6	0.6	0.6
20	0.7	0.8	0.8	0.5	0.5	0.5	0.5	0.5
			EQL(ψ) values					
1.1	41.6	44.3	38.5	33.9	34.6	28.2	23.9	28.8
1.2	52.0	58.9	54.6	38.2	38.4	31.6	30.4	30.8
1.4	56.2	66.2	71.8	37.6	35.2	31.8	34.8	28.5
1.6	60.2	70.4	83.9	38.2	33.9	32.3	36.9	27.5
1.8	64.7	75.3	94.7	39.9	34.4	33.7	38.7	27.4
2	69.8	80.9	104.8	41.9	35.9	35.3	40.6	27.8
2.5	84.3	96.8	128.8	48.3	41.5	40.6	45.6	30.3
3	99.3	114.1	151.7	55.4	48.7	46.8	51.2	33.9
4	131.5	150.6	196.3	71.4	65.6	61.5	64.5	44.1
5	164.9	188.7	240.7	89.9	85.3	78.3	80.4	57.9
7	239.2	272.2	332.9	133.1	132.1	120.3	121.1	95.5
10	367.6	413.5	486.7	218.6	222.1	205.0	204.1	178.4
20	960.3	1041.3	1157.4	722.3	728.3	709.4	707.9	687.8
			TEQL values and ratio to the TEQL of MRA^2 chart					
beta(2,18)	101.3	116.4	149.7	57.5	52.0	49.1	53.0	37.1
ratio	2.73	3.14	4.03	1.55	1.40	1.32	1.43	-
beta(2,8)	166.7	190.1	237.5	93.7	89.8	83.3	85.9	65.2
ratio	2.56	2.92	3.64	1.44	1.38	1.28	1.32	-
uniform	429.6	477.5	549.7	284.6	286.9	272.4	272.7	250.3
ratio	1.72	1.91	2.20	1.14	1.15	1.09	1.09	-

Table 5.5. SSATS, EQL, and TEQL values for some robust charts with $T_5(4)$ observations for random shifts in $\boldsymbol{\sigma}$ when p = 5 and ρ = 0.9

	SER|Z|	**MR|Z|^q**		**SER|A|**	**MR|A|^q**		
q	**1**	**1**	**1.5**	**1**	**0.5**	**1**	**1.5**
λ	***0.03***	***0.03***	***0.001***	***0.03***	***0.06***	***0.02***	***0.001***
ψ	**[1]**	**[2]**	**[3]**	**[4]**	**[5]**	**[6]**	**[7]**
1.0	207.5	208.3	198.5	181.1	219.5	192.2	195.4
1.1	145.8	151.2	139.0	121.3	146.7	124.7	115.3
1.2	100.8	110.7	102.2	79.8	97.8	81.5	74.7
1.4	51.4	60.5	59.8	35.6	44.3	37.2	36.9
1.6	30.8	36.8	40.2	19.9	23.6	21.1	22.4
1.8	21.1	25.0	29.6	13.2	14.8	14.0	15.4
2	15.8	18.7	23.2	9.8	10.5	10.2	11.5
2.5	9.6	11.2	14.9	5.8	5.9	5.9	6.7
3	6.9	8.0	10.8	4.1	4.1	4.1	4.7
4	4.5	5.1	6.9	2.6	2.6	2.6	2.8
5	3.3	3.8	5.1	1.9	2.0	1.9	2.0
7	2.3	2.5	3.3	1.3	1.4	1.3	1.3
10	1.6	1.8	2.2	0.9	1.0	0.9	0.9
20	0.9	1.0	1.1	0.6	0.7	0.6	0.6
			EQL(ψ) values				
1.1	59.9	62.0	57.0	49.8	60.1	51.3	47.3
1.2	90.0	98.7	91.3	71.4	87.2	72.9	66.7
1.4	106.1	125.2	124.0	73.5	91.8	77.0	76.4
1.6	108.3	129.5	142.1	69.7	82.7	74.2	79.1
1.8	110.5	131.7	157.3	68.9	77.6	73.1	81.1
2	114.4	135.6	171.2	70.0	75.8	74.0	83.3
2.5	129.8	151.8	204.0	77.0	79.4	79.8	90.4
3	148.7	172.3	236.8	86.6	88.3	88.2	99.1
4	190.3	219.3	301.5	109.6	112.5	110.3	120.1
5	236.8	270.6	365.6	135.2	141.5	135.8	143.8
7	335.8	381.4	495.6	193.6	209.0	194.6	199.5
10	500.4	565.3	703.5	297.8	329.0	300.3	300.4
20	1207.1	1330.8	1534.2	825.1	900.7	831.5	823.5
			TEQL values and ratio to the smallest TEQL				
beta(2,18)	153.6	178.2	234.4	91.5	96.9	93.7	102.2
ratio	1.68	1.95	2.56	-	1.06	1.02	1.12
beta(2,8)	238.6	273.1	359.2	139.1	148.1	140.6	148.1
ratio	1.72	1.96	2.58	-	1.06	1.01	1.06
uniform	569.3	636.9	771.1	359.0	394.8	362.6	362.6
ratio	1.59	1.77	2.15	-	1.10	1.01	1.01

Table 5.6. SSATS, EQL, and TEQL values for some robust charts with $G_5(4,1)$ observations for random shifts in $\boldsymbol{\sigma}$ when p = 5 and ρ = 0.9

	SER|Z|	**MR|Z|^q**		**SER|A|**	**MR|A|^q**		
q	**1**	**1**	**1.5**	**1**	**0.5**	**1**	**1.5**
λ	***0.03***	***0.03***	***0.001***	***0.03***	***0.06***	***0.02***	***0.001***
ψ	**[1]**	**[2]**	**[3]**	**[4]**	**[5]**	**[6]**	**[7]**
1.0	194.1	195.2	193.3	197.2	198.9	199.8	201.1
1.1	111.4	119.1	108.0	93.6	98.1	84.7	71.6
1.2	68.1	77.4	72.8	50.8	54.4	45.7	42.3
1.4	32.0	38.0	41.6	21.5	22.2	19.9	20.8
1.6	19.7	23.3	28.2	12.7	12.3	11.7	13.0
1.8	14.0	16.4	21.1	8.7	8.2	8.1	9.1
2	10.8	12.6	16.8	6.6	6.1	6.1	6.9
2.5	6.9	7.9	10.9	4.1	3.7	3.7	4.1
3	5.1	5.8	8.0	2.9	2.7	2.6	2.9
4	3.4	3.8	5.1	1.9	1.8	1.7	1.8
5	2.5	2.9	3.7	1.4	1.4	1.3	1.3
7	1.7	1.9	2.4	0.9	1.0	0.9	0.9
10	1.2	1.3	1.6	0.7	0.7	0.7	0.7
20	0.7	0.8	0.9	0.5	0.5	0.5	0.5
			EQL(ψ) values				
1.1	45.8	48.8	44.3	38.5	40.2	34.7	29.4
1.2	60.8	69.0	64.9	45.5	48.6	40.8	37.7
1.4	66.3	78.4	85.8	44.4	45.8	41.1	43.1
1.6	69.1	81.7	99.9	44.3	43.3	41.3	45.7
1.8	73.3	86.1	111.7	45.5	43.1	42.4	47.9
2	78.5	91.7	122.6	47.7	44.1	44.0	50.0
2.5	92.6	107.3	149.3	54.2	49.4	49.5	55.5
3	108.6	125.1	174.6	62.2	56.9	56.3	61.4
4	142.6	163.9	223.0	79.1	75.3	72.0	75.0
5	178.9	204.8	271.2	97.5	96.3	89.5	90.8
7	254.9	290.8	366.3	140.0	145.5	131.2	130.6
10	381.9	431.1	518.8	224.5	238.1	215.8	212.8
20	972.2	1052.8	1183.9	727.4	743.5	719.3	715.3
			TEQL values and ratio to the smallest TEQL				
beta(2,18)	111.0	128.3	171.6	64.1	61.1	58.6	62.8
ratio	1.90	2.19	2.93	1.09	1.04	-	1.07
beta(2,8)	178.7	204.6	264.8	100.7	100.8	93.6	95.7
ratio	1.91	2.19	2.83	1.08	1.08	-	1.02
uniform	442.8	491.6	578.4	290.5	300.7	282.6	281.2
ratio	1.57	1.75	2.06	1.03	1.07	1.00	-

Table 5.7. SSATS, EQL, and TEQL values for some control charts with $N_5(0, \Sigma z_0)$ observations for random shifts in **σ** when p = 5 and $\rho_{ij} = 0.7^{|i-j|}$

	SER|Z|	**MR|Z|^q**		**SER|A|**	**MR|A|^q**			
q	**1**	**1**	**1.5**	**1**	**0.5**	**1**	**1.5**	**2**
λ	***0.03***	***0.03***	***0.001***	***0.03***	***0.06***	***0.02***	***0.001***	***0.025***
ψ	**[1]**	**[2]**	**[3]**	**[4]**	**[5]**	**[6]**	**[7]**	**[8]**
1.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0	200.0
1.1	93.0	93.5	74.4	86.4	91.7	75.5	63.2	73.7
1.2	53.1	55.4	47.5	48.2	51.7	42.0	39.0	39.7
1.4	25.0	27.0	26.8	22.0	22.8	19.8	20.8	17.6
1.6	15.7	17.0	18.3	13.6	13.5	12.4	13.7	10.5
1.8	11.4	12.2	13.7	9.8	9.4	8.9	10.0	7.3
2	8.9	9.5	10.9	7.6	7.1	6.9	7.8	5.5
2.5	5.8	6.1	7.1	4.8	4.5	4.3	4.9	3.2
3	4.3	4.5	5.2	3.5	3.3	3.1	3.5	2.3
4	2.8	3.0	3.4	2.3	2.2	2.0	2.2	1.4
5	2.1	2.2	2.5	1.7	1.7	1.5	1.6	1.0
7	1.4	1.5	1.6	1.1	1.2	1.0	1.0	0.7
10	1.0	1.0	1.1	0.8	0.9	0.8	0.7	0.6
20	0.6	0.6	0.6	0.6	0.6	0.5	0.5	0.5
				EQL(ψ) values				
1.1	36.3	36.4	29.0	33.7	35.7	29.4	24.6	30.1
1.2	43.1	44.7	38.4	38.9	41.8	33.9	31.5	33.4
1.4	43.8	46.9	46.4	38.4	39.8	34.6	36.3	31.7
1.6	44.2	47.6	51.0	38.3	38.0	34.8	38.3	30.4
1.8	45.5	48.7	54.7	38.9	37.5	35.5	39.7	29.7
2	47.3	50.4	57.8	40.1	37.9	36.5	41.0	29.5
2.5	52.5	55.8	65.0	43.8	40.7	39.6	44.2	30.0
3	58.4	62.0	71.9	47.8	44.8	43.1	47.2	31.3
4	70.5	75.2	85.3	56.6	54.6	51.4	54.3	35.8
5	83.5	89.2	98.5	66.3	66.0	60.4	62.5	41.8
7	110.7	118.8	126.4	88.0	91.7	81.7	81.6	58.7
10	156.2	168.0	173.5	128.0	137.6	121.2	119.4	95.8
20	385.0	400.1	402.4	351.4	367.5	347.0	344.1	325.2
				TEQL values and ratio to the TEQL of MRA^2 chart				
beta(2,18)	59.2	63.1	71.1	48.8	47.1	44.3	47.7	33.3
ratio	1.78	1.90	2.13	1.47	1.41	1.33	1.43	-
beta(2,8)	83.5	89.2	97.8	67.4	67.8	62.0	64.1	45.4
ratio	1.84	1.97	2.15	1.49	1.49	1.37	1.41	-
uniform	181.2	191.9	197.3	155.4	163.7	149.8	149.2	128.6
ratio	1.41	1.49	1.53	1.21	1.27	1.16	1.16	-

Table 5.8. SSATS, EQL, and TEQL values for some robust charts with $T_5(4)$ observations for random shifts in **σ** when p = 5 and $\rho_{ij} = 0.7^{|i-j|}$

	SER|Z|	**MR|Z|^q**		**SER|A|**	**MR|A|^q**		
q	**1**	**1**	**1.5**	**1**	**0.5**	**1**	**1.5**
λ	***0.03***	***0.03***	***0.001***	***0.03***	***0.06***	***0.02***	***0.001***
ψ	**[1]**	**[2]**	**[3]**	**[4]**	**[5]**	**[6]**	**[7]**
1.0	190.1	182.2	198.9	183.3	236.7	198.7	195.6
1.1	131.4	125.7	126.8	125.6	161.6	132.7	119.6
1.2	93.0	93.2	90.6	86.7	113.1	91.8	82.1
1.4	48.7	53.5	51.9	43.4	58.0	46.9	44.3
1.6	29.4	34.0	34.7	25.6	33.3	28.2	28.6
1.8	20.2	23.7	25.7	17.4	21.8	19.3	20.5
2	15.2	17.9	20.2	13.0	15.6	14.4	15.7
2.5	9.3	10.8	12.9	7.8	8.8	8.6	9.5
3	6.7	7.7	9.3	5.6	6.1	6.0	6.7
4	4.3	4.9	5.9	3.5	3.8	3.7	4.1
5	3.2	3.6	4.3	2.6	2.8	2.7	2.9
7	2.1	2.4	2.7	1.7	1.9	1.8	1.9
10	1.4	1.6	1.8	1.2	1.4	1.2	1.2
20	0.8	0.9	0.9	0.7	0.8	0.7	0.7
			EQL(ψ) values				
1.1	51.4	49.1	49.5	49.0	62.9	51.8	46.6
1.2	75.6	75.7	73.2	70.5	91.5	74.6	66.6
1.4	85.2	93.5	90.8	75.9	101.2	82.1	77.3
1.6	82.9	95.2	97.7	72.2	93.7	79.2	80.3
1.8	81.0	94.6	103.0	69.5	86.7	76.9	81.8
2	81.1	94.7	107.5	68.8	82.9	76.2	83.1
2.5	85.0	99.3	118.0	71.1	80.4	77.8	87.1
3	91.4	106.7	128.7	76.2	83.5	82.0	91.8
4	107.9	124.9	149.9	88.6	95.8	93.9	103.1
5	126.2	145.3	172.0	102.5	111.8	108.0	115.6
7	165.3	189.2	217.4	133.5	150.3	140.3	144.5
10	229.3	262.2	288.7	187.2	218.0	196.1	195.7
20	499.8	559.0	578.2	431.1	507.2	447.2	436.4
			TEQL values and ratio to the smallest TEQL				
beta(2,18)	95.4	110.0	127.9	79.8	91.0	86.0	93.0
ratio	1.20	1.38	1.60	-	1.14	1.08	1.17
beta(2,8)	127.5	146.7	170.4	104.4	117.2	110.7	117.3
ratio	1.22	1.40	1.63	-	1.12	1.06	1.12
uniform	255.2	289.5	312.8	213.0	249.7	222.8	221.6
ratio	1.20	1.36	1.47	-	1.17	1.05	1.04

Table 5.9. SSATS, EQL, and TEQL values for some multivariate control charts with $G_5(4,1)$ observations for random shifts in $\boldsymbol{\sigma}$ when p = 5 and $\rho_{ij} = 0.7^{|i-j|}$

	SER|Z|	**MR|Z|^q**		**SER|A|**	**MR|A|^q**		
q	**1**	**1**	**1.5**	**1**	**0.5**	**1**	**1.5**
λ	***0.03***	***0.03***	***0.001***	***0.03***	***0.06***	***0.02***	***0.001***
ψ	**[1]**	**[2]**	**[3]**	**[4]**	**[5]**	**[6]**	**[7]**
1.0	182.4	190.4	195.6	226.8	285.7	277.3	255
1.1	99.9	104.3	89.4	111.8	139.6	116.2	93.1
1.2	61.4	66.5	58.7	63.0	78.2	62.3	54.7
1.4	29.4	33.0	32.9	27.6	32.6	27.5	27.9
1.6	18.2	20.6	22.3	16.5	18.2	16.5	17.9
1.8	13.0	14.6	16.7	11.6	12.2	11.6	12.9
2	10.1	11.2	13.2	8.9	9.1	8.8	10.0
2.5	6.5	7.1	8.5	5.6	5.5	5.5	6.1
3	4.8	5.2	6.2	4.1	4.0	3.9	4.4
4	3.1	3.4	4.0	2.6	2.6	2.5	2.7
5	2.3	2.5	2.9	1.9	1.9	1.8	1.9
7	1.5	1.7	1.8	1.3	1.4	1.2	1.2
10	1.0	1.1	1.2	0.9	1.0	0.9	0.8
20	0.6	0.6	0.7	0.6	0.6	0.6	0.6
			EQL(ψ) values				
1.1	39.1	40.6	34.8	43.6	54.4	45.1	36.1
1.2	49.8	54.0	47.5	51.1	63.2	50.5	44.2
1.4	51.4	57.6	57.5	48.5	56.8	48.1	48.6
1.6	51.3	58.0	62.6	46.6	51.2	46.4	50.3
1.8	52.2	58.4	66.6	46.4	48.6	46.2	51.5
2	53.6	59.5	70.1	47.1	48.0	46.7	52.7
2.5	59.1	64.8	78.2	50.7	49.9	49.7	56.0
3	65.4	71.1	85.9	55.1	54.0	53.6	59.6
4	78.7	85.2	100.9	65.2	64.7	62.7	67.4
5	92.6	100.2	115.8	75.8	77.3	72.9	76.2
7	121.4	131.7	145.6	98.6	106.0	95.5	95.6
10	167.2	182.7	192.6	138.7	155.3	136.3	133.4
20	392.6	411.0	417.5	360.5	387.6	358.9	354.2
			TEQL values and ratio to the smallest TEQL				
beta(2,18)	66.7	72.8	84.9	57.0	58.0	55.4	60.1
ratio	1.20	1.31	1.53	1.03	1.05	-	1.09
beta(2,8)	92.2	100.3	114.0	76.6	79.7	74.2	77.3
ratio	1.24	1.35	1.54	1.03	1.07	-	1.04
uniform	189.9	204.2	214.0	165.1	180.5	163.0	161.6
ratio	1.18	1.26	1.32	1.02	1.12	1.01	-

Table 5.10. SSATS and TEQL values for some multivariate control charts under various distributions for random shifts in **σ** when p = 5, ρ = 0

Distribution	**$N_5(0,\sum_{z0})$**					**$T_5(4)$**				**$G_5(4,1)$**			
Chart type	**SER|Z|**	**MR|Z|^q**				**SER|Z|**	**MR|Z|^q**			**SER|Z|**	**MR|Z|^q**		
p	1	0.5	1	1.5	2.0	1	0.5	1	1.5	1	0.5	1	1.5
λ	0.03	*0.06*	*0.02*	*0.001*	*0.025*	0.03	*0.06*	*0.02*	*0.001*	0.03	*0.06*	*0.02*	*0.001*
ψ	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[11]	[12]	[13]	[14]
1.0	*200*	*200*	*200*	*200*	*200*	*175.1*	*202.9*	*183.1*	*194.3*	*165.3*	*306.1*	*251.4*	*202.2*
1.1	85.3	84.0	68.0	57.5	69.1	121.7	138.6	121.8	116.0	83.8	126.4	92.9	67.9
1.2	47.4	46.6	37.4	35.8	36.3	86.0	99.4	85.7	80.4	51.0	65.8	49.5	42.4
1.4	22.5	21.1	18.2	19.7	16.4	45.5	54.7	45.9	45.1	25.2	27.0	22.8	23.2
1.6	14.3	12.9	11.7	13.4	10.1	27.7	33.4	28.8	29.9	16.0	15.8	14.3	15.6
1.8	10.5	9.2	8.6	10.0	7.1	19.2	22.7	20.2	22.0	11.6	10.9	10.3	11.6
2	8.2	7.1	6.8	7.9	5.4	14.5	16.8	15.5	17.2	9.1	8.3	8.1	9.2
2.5	5.3	4.6	4.4	5.2	3.3	8.9	9.9	9.6	11.0	5.8	5.2	5.2	5.9
3	3.9	3.5	3.3	3.8	2.3	6.4	7.0	6.9	7.9	4.3	3.9	3.8	4.3
4	2.6	2.4	2.2	2.4	1.5	4.1	4.5	4.4	5.0	2.8	2.6	2.5	2.7
5	1.9	1.8	1.7	1.8	1.1	3.0	3.4	3.3	3.6	2.1	2.0	1.9	2.0
7	1.3	1.3	1.1	1.2	0.8	2.0	2.3	2.2	2.3	1.4	1.4	1.3	1.3
10	0.9	0.9	0.8	0.8	0.6	1.3	1.6	1.5	1.5	0.9	1.0	0.9	0.9
20	0.6	0.6	0.5	0.5	0.5	0.7	0.9	0.8	0.8	0.6	0.6	0.6	0.6
TEQL and TEQL ratio to the smallest TEQL													
beta(2,18)	45.7	41.1	38.2	43.0	28.1	76.8	85.7	81.5	90.6	50.2	46.9	44.6	49.2
ratio	1.63	1.46	1.36	1.53	-	-	1.12	1.06	1.18	1.13	1.05	-	1.10
beta(2,8)	59.2	57.0	51.2	54.9	35.9	94.8	107.4	101.8	110.9	64.8	63.1	58.2	61.7
ratio	1.65	1.59	1.43	1.53	-	-	1.13	1.07	1.17	1.11	1.08	-	1.06
uniform	117.4	122.9	108.8	110.0	90.9	171.0	209.9	184.3	185.9	123.1	131.6	116.2	116.2
ratio	1.29	1.35	1.20	1.21	-	-	1.23	1.08	1.09	1.06	1.13	1.00	-

Table 5.11. SSATS and TEQL values of selected robust control charts with control limit adjusted to $T_5(4)$ observations for random shifts in $\boldsymbol{\sigma}$

ρ_{ij}	**0.9**					**$(0.7)^{\|i-j\|}$**					**0**				
Chart type	**SER|A|**	**MR|A|^q**				**SER|A|**	**MR|A|^q**				**SER|Z|**	**MR|Z|^q**			
p	1	0.5	1	1.5	2.0	1	0.5	1	1.5	2.0	1	0.5	1	1.5	2.0
λ	0.03	*0.06*	*0.02*	*0.001*	*0.025*	0.03	*0.06*	*0.02*	*0.001*	*0.025*	0.03	*0.06*	*0.02*	*0.001*	*0.025*
ψ	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]	[14]	[15]
1.0	*200*	*200*	*200*	*200*	*200*	*200*	*200*	*200*	*200*	*200*	*200*	*200*	*200*	*200*	*200*
1.1	133.6	133.2	128.7	116.3	147.1	136.8	137.5	132.7	121.1	150.0	138.1	136.8	131.8	118.7	148.5
1.2	86.7	89.6	83.8	75.4	106.5	93.4	97.5	91.6	83.1	113.9	96.9	98.1	91.7	82.1	114.1
1.4	38.0	41.4	38.1	37.3	53.8	45.8	51.1	46.9	44.7	64.9	49.7	54.0	48.9	45.7	67.4
1.6	21.0	22.4	21.5	22.7	29.9	26.9	30.1	28.2	28.8	39.3	29.8	33.1	30.2	30.4	42.5
1.8	13.8	14.2	14.1	15.6	19.2	18.1	20.0	19.3	20.7	26.3	20.4	22.5	21.1	22.3	29.3
2	10.2	10.1	10.4	11.6	13.5	13.5	14.5	14.4	15.8	19.1	15.3	16.7	16.1	17.5	21.7
2.5	6.0	5.7	6.0	6.8	7.3	8.1	8.3	8.5	9.6	10.7	9.4	9.8	9.9	11.1	12.5
3	4.2	4.0	4.2	4.7	4.8	5.8	5.8	6.0	6.7	7.1	6.7	7.0	7.1	8.0	8.5
4	2.7	2.6	2.6	2.9	2.8	3.6	3.6	3.7	4.1	4.1	4.3	4.5	4.5	5.0	4.9
5	2.0	1.9	2.0	2.1	1.9	2.7	2.7	2.7	2.9	2.8	3.1	3.3	3.3	3.6	3.4
7	1.4	1.4	1.3	1.4	1.2	1.8	1.8	1.8	1.9	1.7	2.1	2.3	2.2	2.3	2.1
10	1.0	1.0	1.0	1.0	0.9	1.2	1.3	1.2	1.2	1.1	1.4	1.6	1.5	1.5	1.3
20	0.6	0.7	0.6	0.6	0.6	0.7	0.8	0.7	0.7	0.6	0.8	0.9	0.8	0.8	0.7
						TEQL and TEQL ratio to the TEQL of MRA^2 /MRZ^2 chart									
beta(2,18)	94.8	93.7	94.8	102.8	108.2	82.7	85.2	86.0	93.7	102.3	81.0	85.2	84.6	91.6	101.7
ratio	0.88	0.87	0.88	0.95	-	0.81	0.83	0.84	0.92	-	0.80	0.84	0.83	0.90	-
beta(2,8)	143.3	144.5	141.9	148.8	141.8	107.5	111.5	110.8	118.2	115.3	99.0	106.9	104.9	112.1	109.4
ratio	1.01	1.02	1.00	1.05	-	0.93	0.97	0.96	1.03	-	0.90	0.98	0.96	1.02	-
uniform	365.9	387.5	364.6	363.5	339.1	217.4	238.9	222.7	222.7	204.3	176.7	209.2	188.8	187.4	167.3
ratio	1.08	1.14	1.08	1.07	-	1.06	1.17	1.09	1.09	-	1.06	1.25	1.13	1.12	-

Table 5.12. SSATS and TEQL values of selected robust control charts with control limit adjusted to the $G_5(4,1)$ observations for random shifts in $\boldsymbol{\sigma}$

ρ_{ij}	0.9					$(0.7)^{\vert i-j \vert}$					0				
Chart type	**SER|A|**	**MR|A|^q**				**SER|A|**	**MR|A|^q**				**SER|Z|**	**MR|Z|^q**			
p	1	0.5	1	1.5	2.0	1	0.5	1	1.5	2.0	1	0.5	1	1.5	2.0
λ	0.03	*0.06*	*0.02*	*0.001*	*0.025*	0.03	*0.06*	*0.02*	*0.001*	*0.025*	0.03	*0.06*	*0.02*	*0.001*	*0.025*
ψ	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]	[12]	[13]	[14]	[15]
1.0	*200*	*200*	*200*	*200*	*200*	*200*	*200*	*200*	*200*	*200*	*200*	*200*	*200*	*200*	*200*
1.1	94.3	98.2	84.0	69.7	90.4	100.0	101.8	88.7	76.4	98.3	99.6	88.2	77.3	67.1	98.0
1.2	51.0	54.5	45.4	41.3	49.2	57.1	60.3	50.8	46.6	57.2	58.7	49.5	43.2	42.0	58.2
1.4	21.5	22.2	19.8	20.5	20.9	26.0	27.1	23.8	24.3	26.0	27.7	22.2	20.7	23.0	27.6
1.6	12.7	12.3	11.7	12.8	11.9	15.8	15.8	14.6	15.8	15.4	17.3	13.5	13.2	15.5	16.7
1.8	8.8	8.2	8.1	9.0	7.9	11.1	10.8	10.3	11.4	10.6	12.4	9.5	9.6	11.5	11.7
2	6.6	6.1	6.1	6.8	5.8	8.5	8.1	7.9	8.8	7.9	9.7	7.3	7.5	9.1	8.8
2.5	4.1	3.7	3.7	4.1	3.2	5.4	5.0	5.0	5.5	4.6	6.2	4.7	4.8	5.9	5.2
3	2.9	2.7	2.6	2.8	2.2	3.9	3.6	3.6	3.9	3.1	4.5	3.5	3.6	4.3	3.6
4	1.9	1.8	1.7	1.8	1.3	2.5	2.4	2.3	2.4	1.9	3.0	2.4	2.4	2.7	2.2
5	1.4	1.4	1.3	1.3	1.0	1.9	1.8	1.7	1.7	1.3	2.2	1.8	1.8	2.0	1.5
7	0.9	1.0	0.9	0.9	0.7	1.2	1.3	1.1	1.1	0.9	1.5	1.3	1.2	1.3	1.0
10	0.7	0.7	0.7	0.7	0.6	0.9	0.9	0.8	0.8	0.6	1.0	0.9	0.8	0.9	0.7
20	0.5	0.5	0.5	0.5	0.5	0.6	0.6	0.6	0.5	0.5	0.6	0.6	0.6	0.6	0.5
TEQL and TEQL ratio to the TEQL of MRA^2 /MRZ^2 chart															
beta(2,18)	64.4	61.1	58.5	61.9	49.7	54.8	52.3	50.2	53.7	45.0	53.4	41.8	41.5	48.9	43.0
ratio	1.30	1.23	1.18	1.25	-	1.22	1.16	1.12	1.19	-	1.24	0.97	0.97	1.14	-
beta(2,8)	101.2	100.8	93.4	94.6	75.9	74.0	73.5	68.4	70.2	55.8	68.3	57.4	54.8	61.4	49.2
ratio	1.34	1.33	1.23	1.25	-	1.33	1.32	1.23	1.26	-	1.39	1.17	1.11	1.25	-
uniform	290.9	300.6	282.4	280.1	258.4	162.1	170.7	155.9	154.2	136.2	127.5	122.4	112.0	115.7	100.1
ratio	1.13	1.16	1.09	1.08	-	1.19	1.25	1.14	1.13	-	1.27	1.22	1.12	1.16	-

Table 5.13. The SSATS values of the robust MR|A| and SER|A| charts for some specific shift direction with $N_5(0,\sum_{z0})$ observations

		ρ_{ij}			0.9		$0.7^{\|i-j\|}$		0	
		Control chart			**MR|A|**	**SER|A|**	**MR|A|**	**SER|A|**	**MR|Z|**	**SER|Z|**
γ_1	γ_2	γ_3	γ_4	γ_5	**0.02**	**0.03**	**0.02**	**0.03**	**0.02**	**0.03**
1	1	1	1	1	200.0	200.0	200.0	200.0	200.0	200.0
1.2	1.2	1.2	1.2	1.2	15.1	20.5	16.5	21.1	13.7	19.7
1.4	1.4	1.4	1.4	1.4	7.1	9.7	7.7	9.9	6.4	9.3
1.6	1.6	1.6	1.6	1.6	4.6	6.3	5.0	6.4	4.2	6.0
2	2	2	2	2	2.7	3.6	2.9	3.7	2.4	3.5
3	3	3	3	3	1.4	1.7	1.5	1.8	1.2	1.6
5	5	5	5	5	0.7	0.8	0.8	0.9	0.7	0.8
1.2	1	1	1	1	37.2	31.7	52.8	48.9	55.8	53.3
1.4	1	1	1	1	14.5	11.3	25.0	20.3	28.9	23.8
1.6	1	1	1	1	8.4	6.6	15.7	12.2	19.3	14.8
2	1	1	1	1	4.3	3.6	8.7	6.7	11.5	8.4
3	1	1	1	1	2.0	1.8	4.0	3.2	5.8	4.1
5	1	1	1	1	1.1	1.0	2.0	1.7	3.0	2.2
0.8	0.8	1.2	1.2	1.2	12.3	15.6	33.6	27.4	34.1	28.1
0.6	0.6	1.4	1.4	1.4	4.0	5.6	14.2	11.5	15.9	12.6
0.2	0.2	1.8	1.8	1.8	1.7	2.3	5.7	5.0	7.6	5.8
0.8	0.8	1.2	1.2	1	16.9	16.9	48.9	32.7	54.7	36.9
0.6	0.6	1.4	1.4	1	5.2	6.1	19.0	12.8	25.0	16.0
0.2	0.2	1.8	1.8	1	2.0	2.5	6.9	5.3	11.8	7.3

Table 5.14. The SSATS values of the robust MR|A| and SER|A| charts for some specific shift direction with $T_5(4)$ observations

Control limit					**Normal control limit**						**Adjusted control limit**					
ρ_{ij}					**0.9**		**$0.7^{\|i-j\|}$**		**0**		**0.9**		**$0.7^{\|i-j\|}$**		**0**	
Control chart					**MR|A|**	**SER|A|**	**MR|A|**	**SER|A|**	**MR|Z|**	**SER|Z|**	**MR|A|**	**SER|A|**	**MR|A|**	**SER|A|**	**MR|Z|**	**SER|Z|**
γ_1	γ_2	γ_3	γ_4	γ_5	**0.02**	**0.03**	**0.02**	**0.03**	**0.02**	**0.03**	**0.02**	**0.03**	**0.02**	**0.03**	**0.02**	**0.03**
1	1	1	1	1	192.4	181.1	198.7	183.3	183.1	175.1	200.0	200.0	200.0	200.0	200.0	200.0
1.2	1.2	1.2	1.2	1.2	38.8	42.8	40.8	43.8	36.4	41.5	39.7	45.9	40.8	46.3	38.3	45.4
1.4	1.4	1.4	1.4	1.4	16.6	18.8	17.4	19.2	15.6	18.2	16.8	19.8	17.4	20.0	16.2	19.5
1.6	1.6	1.6	1.6	1.6	10.0	11.4	10.5	11.6	9.4	11.1	10.1	11.9	10.5	12.1	9.7	11.7
2	2	2	2	2	5.5	6.2	5.7	6.3	5.2	6.0	5.5	6.5	5.7	6.5	5.3	6.3
3	3	3	3	3	2.6	2.9	2.7	2.9	2.4	2.8	2.6	3.0	2.7	3.0	2.5	2.9
5	5	5	5	5	1.3	1.4	1.4	1.4	1.2	1.3	1.3	1.4	1.4	1.4	1.3	1.4
1.2	1	1	1	1	78.6	60.2	105.5	88.0	107.4	92.8	80.5	64.4	105.4	94.3	116.0	103.7
1.4	1	1	1	1	29.4	18.7	51.8	36.9	61.4	43.9	29.8	19.4	51.8	38.5	65.2	47.6
1.6	1	1	1	1	16.2	9.9	30.7	20.3	39.5	25.4	16.4	10.3	30.7	21.0	41.5	27.0
2	1	1	1	1	8.0	5.1	16.0	10.2	22.3	13.2	8.1	5.2	16.0	10.5	23.2	13.8
3	1	1	1	1	3.4	2.4	7.0	4.5	10.8	6.0	3.5	2.5	7.0	4.6	11.1	6.2
5	1	1	1	1	1.7	1.3	3.3	2.3	5.4	3.0	1.7	1.4	3.3	2.3	5.5	3.1
0.8	0.8	1.2	1.2	1.2	29.6	31.3	73.8	55.5	76.2	57.7	30.0	33.3	73.8	58.9	81.2	63.3
0.6	0.6	1.4	1.4	1.4	8.1	9.9	28.2	20.9	32.7	24.0	8.2	10.3	28.2	21.8	34.1	25.7
0.2	0.2	1.8	1.8	1.8	2.9	3.8	10.3	8.1	13.6	9.9	3.0	3.9	10.3	8.4	14.0	10.4
0.8	0.8	1.2	1.2	1	38.4	32.3	99.5	64.6	108.4	72.9	39.1	34.2	99.4	69.0	116.5	80.5
0.6	0.6	1.4	1.4	1	10.4	10.0	36.9	23.0	50.2	30.3	10.5	10.4	36.8	24.0	52.8	32.4
0.2	0.2	1.8	1.8	1	3.6	3.9	12.3	8.5	20.5	12.1	3.6	4.0	12.3	8.7	21.2	12.7

Table 5.15. The SSATS values of the robust MR|A| and SER|A| charts for some specific shift direction with $G_5(4,1)$ observations

Control limit					Normal control limit						Adjusted control limit					
ρ_{ij}					0.9		$0.7^{\|i-j\|}$		0		0.9		$0.7^{\|i-j\|}$		0	
Control chart					MR|A|	SER|A|	MR|A|	SER|A|	MR|Z|	SER|Z|	MR|A|	SER|A|	MR|A|	SER|A|	MR|Z|	SER|Z|
γ_1	γ_2	γ_3	γ_4	γ_5	0.02	0.03	0.02	0.03	0.02	0.03	0.02	0.03	0.02	0.03	0.02	0.03
1	1	1	1	1	199.9	197.2	277.3	226.8	251.4	165.3	200.0	200.0	200.0	200.0	200.0	200.0
1.2	1.2	1.2	1.2	1.2	19.8	24.9	23.0	27.2	17.0	21.9	19.7	24.9	20.0	25.5	15.5	24.2
1.4	1.4	1.4	1.4	1.4	9.0	11.5	10.2	12.1	7.6	10.3	9.0	11.5	9.1	11.5	7.1	11.1
1.6	1.6	1.6	1.6	1.6	5.8	7.3	6.4	7.7	4.9	6.6	5.8	7.3	5.8	7.3	4.5	7.1
2	2	2	2	2	3.4	4.2	3.7	4.4	2.8	3.8	3.4	4.2	3.4	4.2	2.6	4.1
3	3	3	3	3	1.7	2.0	1.8	2.1	1.4	1.8	1.7	2.0	1.7	2.0	1.3	1.9
5	5	5	5	5	0.9	1.0	0.9	1.0	0.7	0.9	0.9	1.0	0.9	1.0	0.7	0.9
1.2	1	1	1	1	46.1	37.4	75.7	61.2	73.4	58.0	45.9	37.5	61.1	56.6	62.7	65.7
1.4	1	1	1	1	18.0	12.7	32.9	24.0	35.8	26.5	17.9	12.7	28.4	22.9	32.1	28.8
1.6	1	1	1	1	10.3	7.2	19.9	14.0	23.1	16.3	10.3	7.2	17.7	13.5	21.1	17.4
2	1	1	1	1	5.2	3.8	10.7	7.5	13.4	9.2	5.2	3.8	9.7	7.2	12.4	9.6
3	1	1	1	1	2.3	1.9	4.8	3.5	6.6	4.4	2.3	1.9	4.4	3.4	6.2	4.7
5	1	1	1	1	1.2	1.1	2.3	1.8	3.4	2.3	1.2	1.1	2.1	1.7	3.2	2.4
0.8	0.8	1.2	1.2	1.2	15.9	18.5	46.1	35.5	42.3	32.1	15.8	18.5	39.3	33.2	37.9	35.5
0.6	0.6	1.4	1.4	1.4	4.9	6.5	17.9	13.8	18.4	14.2	4.9	6.5	16.0	13.2	17.1	15.2
0.2	0.2	1.8	1.8	1.8	1.9	2.6	6.9	5.8	8.5	6.5	1.9	2.6	6.4	5.6	8.1	6.8
0.8	0.8	1.2	1.2	1	21.5	19.7	68.1	42.8	68.7	42.1	21.4	19.8	56.8	40.2	60.3	46.8
0.6	0.6	1.4	1.4	1	6.4	6.9	24.0	15.5	29.2	18.1	6.4	6.9	21.4	14.8	26.9	19.4
0.2	0.2	1.8	1.8	1	2.4	2.8	8.3	6.1	13.2	8.1	2.4	2.8	7.6	5.9	12.4	8.5

Table 5.16. The SSATS values of the robust MR|A| and SER|A| charts for random shifts in **σ** with ρ changing when the in-control ρ_{ij} is 0.9 or 0

In-control ρ_{ij}		**0.9**						**0**					
Distribution		**$N_5(0,\sum_{z0})$**		**$T_5(4)$**		**$G_5(4,1)$**		**$N_5(0,\sum_{z0})$**		**$T_5(4)$**		**$G_5(4,1)$**	
Control chart		**MR|A|**	**SER|A|**	**MR|A|**	**SER|A|**	**MR|A|**	**SER|A|**	**MR|Z|**	**SER|Z|**	**MR|Z|**	**SER|Z|**	**MR|Z|**	**SER|Z|**
In-control ATS		**200**	**200**	**200**	**200**	**200**	**200**	**200**	**200**	**200**	**200**	**200**	**200**
ρ_{ij}'	**ψ**	**0.02**	**0.03**	**0.02**	**0.03**	**0.02**	**0.03**	**0.02**	**0.03**	**0.02**	**0.03**	**0.02**	**0.03**
0.2	1	1.5	1.9	2.9	3.3	1.8	2.2	145.8	180.3	180.3	-	137.4	-
0.5	1	2.2	2.9	4.4	5.1	2.7	3.3	93.5	-	146.1	-	91.7	-
0.7	1	3.8	5.1	8.0	9.4	4.7	5.9	70.5	-	120.2	-	71.9	-
0.2	1.2	1.3	1.7	2.6	2.9	1.6	1.9	36.8	48.9	89.2	98.6	42.4	62.0
0.5	1.2	1.9	2.5	3.8	4.4	2.4	2.9	33.7	56.0	78.7	110.2	38.9	72.5
0.7	1.2	3.2	4.2	6.5	7.6	3.9	4.9	31.0	65.1	70.1	127.9	35.6	85.2
0.2	1.6	1.1	1.4	2.1	2.3	1.3	1.6	11.8	14.6	30.1	30.2	13.6	17.9
0.5	1.6	1.5	2.0	3.0	3.4	1.9	2.2	12.2	15.9	29.5	32.7	14.2	19.9
0.7	1.6	2.3	3.0	4.6	5.3	2.9	3.5	12.3	17.6	28.8	35.9	14.3	22.0
0.2	2	1.0	1.2	1.8	2.0	1.2	1.3	6.9	8.3	16.1	15.6	7.7	9.9
0.5	2	1.3	1.6	2.4	2.7	1.6	1.8	7.2	8.9	16.3	16.5	8.1	10.8
0.7	2	1.8	2.3	3.6	3.9	2.2	2.6	7.5	9.7	16.4	17.8	8.5	11.7
0.2	3	0.7	0.9	1.3	1.4	0.9	0.9	3.4	4.0	7.2	6.8	3.7	4.7
0.5	3	0.9	1.1	1.7	1.8	1.1	1.2	3.6	4.2	7.4	7.1	3.9	5.0
0.7	3	1.2	1.4	2.2	2.4	1.4	1.6	3.8	4.6	7.6	7.6	4.1	5.3
0.2	5	0.6	0.6	0.9	0.9	0.6	0.7	1.7	1.9	3.4	3.2	1.8	2.3
0.5	5	0.7	0.7	1.1	1.1	0.7	0.8	1.8	2.1	3.5	3.4	1.9	2.4
0.7	5	0.8	0.9	1.3	1.4	0.9	0.9	2.0	2.2	3.7	3.6	2.1	2.6

Table 5.17. The SSATS values of the robust MR|A| and SER|A| charts for random shifts in **σ** with correlations changing when the in-control $\rho_{ij} = \rho^{|i-j|}$ and ρ=0.7

Distribution		**$N_5(0,\sum_{z0})$**		**$T_5(4)$**		**$G_5(4,1)$**	
Control chart		**MR|A|**	**SER|A|**	**MR|A|**	**SER|A|**	**MR|A|**	**SER|A|**
In-control ATS		**200**	**200**	**200**	**200**	**200**	**200**
ρ'	ψ	**0.02**	**0.03**	**0.02**	**0.03**	**0.02**	**0.03**
0.2	1	4.5	5.8	9.2	10.5	5.0	6.4
0.5	1	10.8	13.8	25.3	28.8	12.1	15.2
0.9	1	-	-	-	-	-	-
0.2	1.2	3.8	4.8	7.5	8.6	4.2	5.3
0.5	1.2	7.9	10.0	17.5	19.8	8.8	11.0
0.9	1.2	-	-	-	-	-	-
0.2	1.6	2.8	3.5	5.4	6.1	3.1	3.9
0.5	1.6	5.0	6.2	10.3	11.1	5.6	6.7
0.9	1.6	-	-	-	-	-	-
0.2	2	2.3	2.8	4.2	4.7	2.5	3.0
0.5	2	3.6	4.4	7.1	7.5	4.0	4.8
0.9	2	-	88.4	-	93.1	-	71.3
0.2	3	1.5	1.8	2.7	2.9	1.7	2.0
0.5	3	2.1	2.5	3.9	4.1	2.4	2.7
0.9	3	9.1	7.4	17.5	12.6	9.9	8.1
0.2	5	1.0	1.1	1.6	1.7	1.0	1.2
0.5	5	1.2	1.3	2.1	2.1	1.3	1.5
0.9	5	2.6	2.6	4.7	4.1	2.8	2.8

Chapter 6

The Application of the Proposed Robust Charts

In the previous chapters, simulation results have shown that the SER|A| and MR|A| control charts can be designed to give a good robustness and relatively high efficiency over different processes with different dimensions. However, the designs of the proposed charts are not invariant to the covariance structure of a process. To make the robust SER|A| and MR|A| charts practical to practitioners, in this chapter we propose some methods to obtain the control limits for an arbitrary process and give real data examples to show how to apply the robust charts to an application.

6.1 Control limit chart and software package

Due to the complexity of the correlation matrix structure for high-dimensional processes, here we only present the control limits of the SER|A| chart with $\lambda = 0.03$ and the MR|A| chart with $\lambda = 0.02$ for processes when $p = 2$. Figures 6.1 and 6.2 are the control limit plots for the robust SER|A| and MR|A| charts with different ρ for bivariate processes. Control limits are obtained to give the in-control ATS from 100 to 800 in the bivariate normal setting. The values of ρ shown in the plots are 0.9, 0.7, 0.5, 0.3 and 0.1.

To find the control limit for a process, first select the curve with the appropriate ρ. Then select the target in-control ATS value. By finding the cross-over point in the curve, we can locate the corresponding control limit. These two control limit plots only provide control limit for limited cases. If a value of ρ is not listed in the plots, a simple way is to round it to the nearest value shown in plots or interpolate by eye in the plot to estimate the control limit accurately enough for practical applications. Or one can use the software package provided below to get a more accurate control limit, though it may take a much longer time to run the software package.

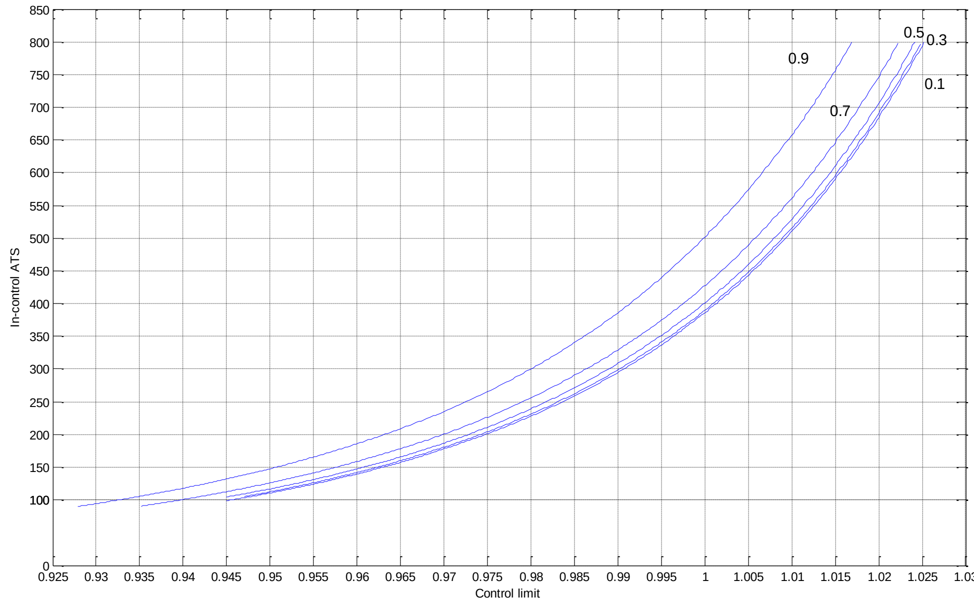

Figure 6.1. Control limit plot for the SER|A| chart with $\lambda = 0.03$ for different ρ when $p = 2$

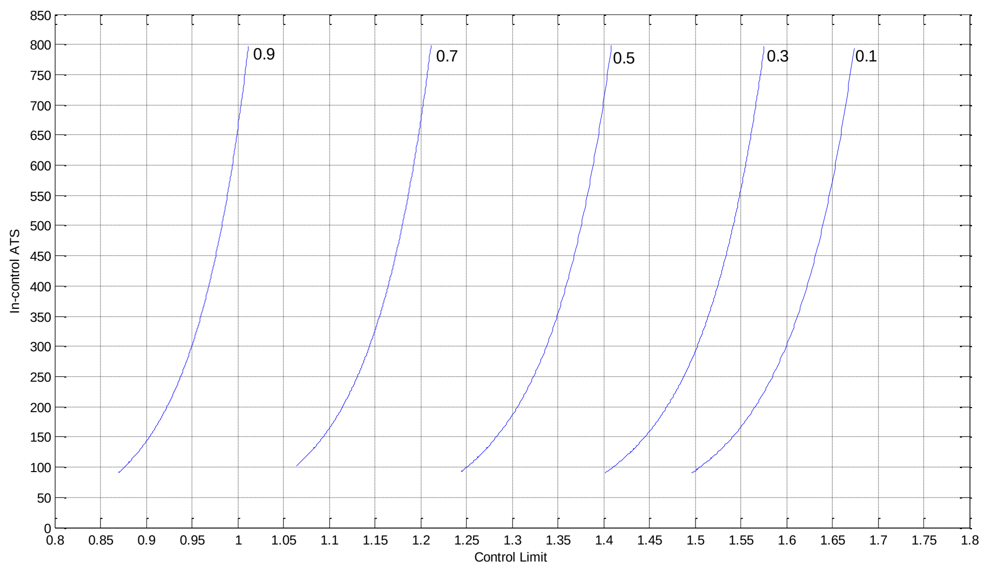

Figure 6.2. Control limit plot for the MR|A| chart $\lambda = 0.02$ for different ρ when $p = 2$

To make the robust charts more convenient for practical use, Matlab coding for obtaining the control limit for a user-specified in-control ATS value when $p \geq 2$ is also provided for the SER|A| chart with $\lambda = 0.03$ and the MR|A| chart with $\lambda = 0.02$. The packages are attached in appendix I and II and are recommended for obtaining control limits for multivariate processes when p is between 2 and 10. Once the user specifies the in-control correlation structure, process dimension, the desired in-control ATS value, and the number of simulation runs, the corresponding control limits of the robust chart will be generated.

6.2 Example

In this section, we apply the robust MR|A| chart with $\lambda = 0.02$ and the SER|A| chart with $\lambda = 0.03$ to real data sets. Their performance is compared to the MRA^2 chart with $\lambda = 0.025$.

The aluminium electrolytic capacitor data

The dataset is from an aluminium electrolytic capacitor (AEC) manufacturing process (kindly supplied to us by Fugee Tsung), which follows a sequence of operations to transform raw materials into AEC. The inspection step of the AEC process before packing includes measuring the capacitance, loss tangent, and leakage current level. These are the three variables of interest in the dataset.

Table 6.1. The estimated $\boldsymbol{\mu}_0$, $\boldsymbol{\sigma}_0$ and Σ_{Z0} for AEC data

Sample mean vector, $\hat{\boldsymbol{\mu}}_0$			**Sample correlation matrix, $\hat{\Sigma}_{Z0}$**		
448.82	4.6054	23.03	1	-0.2194	0.1578
			-0.2194	1	0.118
Sample standard deviation, $\hat{\boldsymbol{\sigma}}_0$			0.1578	0.118	1
9.5689	0.6287	5.7328			

The dataset includes 200 observations and the first 100 observations are used as the historical sample (phase I data) to estimate the in-control $\boldsymbol{\mu}_0$, $\boldsymbol{\sigma}_0$ and Σ_{Z0} (Table 6.1). The variables are standardized by subtracting the estimated in-control means and dividing by the estimated standard deviations. The three standardized variables are labeled as Z_1, Z_2

and Z_3 and their time series plots ((a)-(c)) and normal Q-Q plots ((d)-(f)) are shown in Figure 6.3. We can tell from the Q-Q plots that the marginal distributions of these three variables are not normal but right skewed.

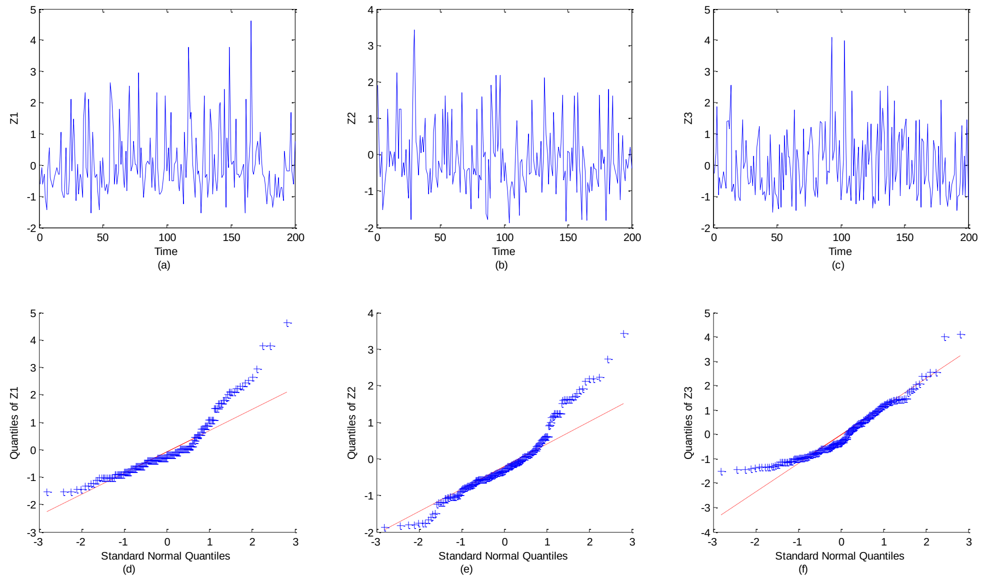

Figure 6.3. (a)-(c): The time series plot for AEC data; (d)-(f): the Q-Q plots for Z_1, Z_2 and Z_3

To use the Matlab functions CLMR_ABSA and CLSER_ABSA provided in Appendices I and II, we plug in the estimates of Σ_{Z0}, then specify the in-control ATS to be 200, the number of simulation runs to be 100,000 and p = 3. Running the code in Matlab, we can obtain the control limits of the MR|A| and SER|A| charts. Table 6.2 gives an example of the Matlab input and output to obtain the control limit of the MR|A| chart by calling the Matlab function CLMR_ABSA. Additionally, we find the control limit for the MRA^2 charts with a target in-control ATS of 200.

Multiplying $\mathbf{Z}$ = (Z_1, Z_2, Z_3) by the transformation matrix $(diag\,\Sigma_{Z0}^{-1})^{-1/2}\,\Sigma_{Z0}^{-1}$, we transform the original variables into the regression adjusted variables A_1, A_2 and A_3. Figure 6.4(a)-(c) shows the three control charts along with their control limits (the solid line in each subplot) for monitoring these three regression adjusted variables. All three control charts give no signal in the first 100 observations (Phase I), and therefore we focus on the charts' behavior with the remaining 100 observations (Phase II).

Table 6.2. The Matlab input and output for obtaining the control limit of the MR|A| chart

Input:	control_limit = CLMR_ABSA (200, 3, [1 -0.219 0.1578 -0.2194 1 0.118 0.1578 0.118 1], 100000)
Output:	control_limit = 2.1772

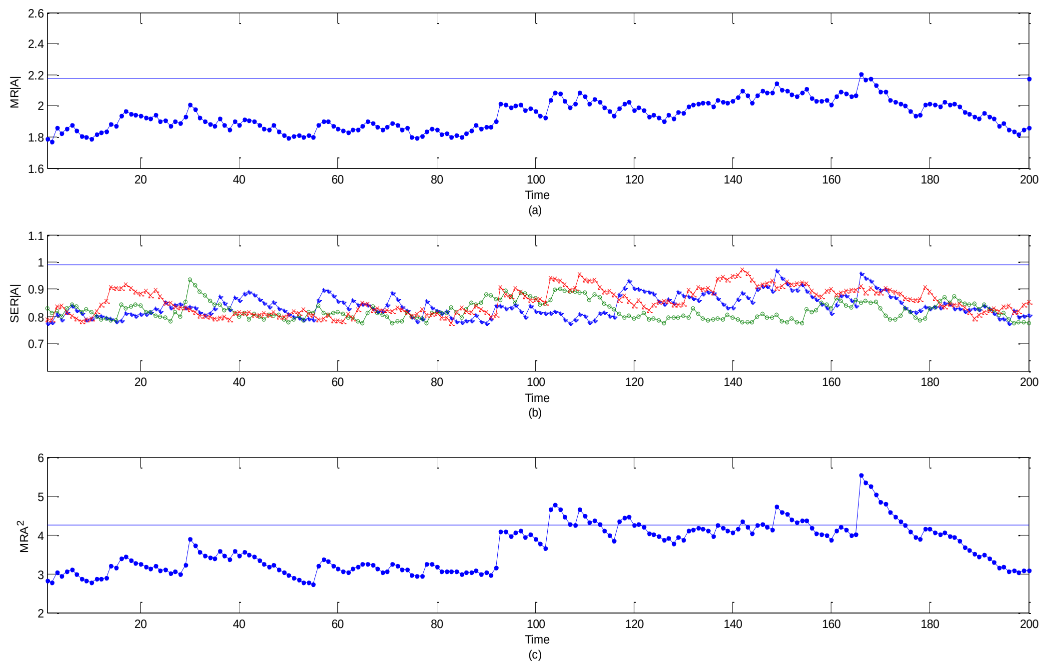

Figure 6.4. (a)-(c): The MR|A|, SER|A| and MRA^2 charts for monitoring the AEC process

The MR|A| chart (Figure 6.4(a)) signals at the 166th observation (the 66th observation in Phase II) while no signal occurs in the SER|A| chart (Figure 6.4(b)). The three lines with different marks in Figure 6.4(b) represent the three individual ER|A| charts. None of them exceed the control limit. Compared to the MR|A| chart, the SER|A| chart seems less sensitive to the possible process change while the MRA^2 chart unsurprisingly is much more sensitive to some extreme observations. The MRA^2 chart gives more than 25 signals after the 100th observation. It seems that the long right tails of Z_1 and Z_2 (Figure 6.3 (d) and (e)) may account for the many signals in the MRA^2 chart.

When applying control charts to the real data, the actual state of the process is unknown. We do not know whether a signal is a false alarm or there has been a change in the mean, a change in the variance, or a change in both. Reynolds and Cho (2006) found that the MEWMA-type control chart based on squared deviation are good at detecting increases in $\boldsymbol{\sigma}$ as well as large shifts in $\boldsymbol{\mu}$ for multivariate normal processes. This property may still apply to the robust designs of the proposed MEWMA-type control charts for non-normal processes. Thus additional diagnostic procedures would be needed after a signal to try to determine not only which variables have changed, but whether it is the mean vector, the covariance matrix, or both.

The aluminum smelter process data

The aluminum smelter dataset (kindly supplied to us by Peihua Qiu) comes from the process of extracting aluminum metal. Aluminum smelting refers to the electrolysis step of reducing the alumina into metallic aluminum. This dataset contains five variables, the content of SiO_2, Fe_2O_3, MgO, CaO, and Al_2O_3, and is comprised of 185 observations. Since this dataset has been used in several papers, we follow the general procedure used by other authors and use the first 95 observations as the Phase I observations to estimate the in-control values of the parameters for the process. We also prewhiten the dataset into residual vectors because of the autocorrelation in the original observations. The detailed prewhitening procedure may be obtained from Qiu (2008).

Figure 6.5 shows the normal Q-Q plots for standardized residuals (Z_1-Z_5) of the original 5 variables after the prewhitening process, respectively. The Z_2 and Z_3 variables have a shape close to the normal distribution except for a few outliers. The distributions of Z_1 and Z_4 are right skewed while the distribution of Z_5 is left skewed.

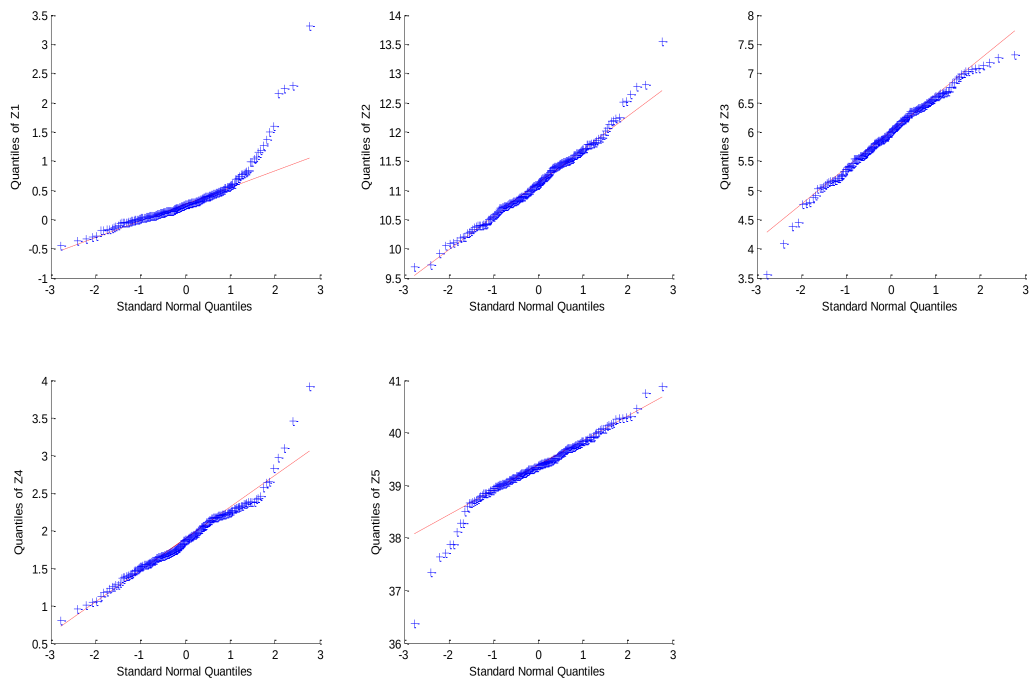

Figure 6.5.The normal Q-Q plots of Z_1-Z_5 (after prewhitening) for the aluminum smelter data

Following the same procedure explained in the first example, we obtain plots of the MR|A|, SER|A|, and MRA^2 charts for monitoring the aluminum smelter process (Figure 6.6 (a)-(c)). For the Phase I observations, the MRA^2 chart (Figure 6.6(c)) gives a few signals while the other two robust charts remain silent. For the Phase II observations, all three charts signal at the 179th observation. However, the MR|A| chart fails to signal at the 137th observation while the SER|A| and MRA^2 charts detect a change at this time point.

The in-control ATS of the MRA^2 chart with skewed variables is much less than the target value of 200. It’s not surprising that it signals in the Phase I data and catches every possible shift in the phase II data. By checking the individual curve of the SER|A| chart (Figure 6.6(b)), we notice that one or two of the 5 variables seems to have a moderate shift in variance. From the discussion in Chapter 5, we know that the SER|A| chart gives the best detection when only one or two variables shift. Thus, this may explain that why the SER|A| chart indicates a shift at the 137th observation while the MR|A| chart does not.

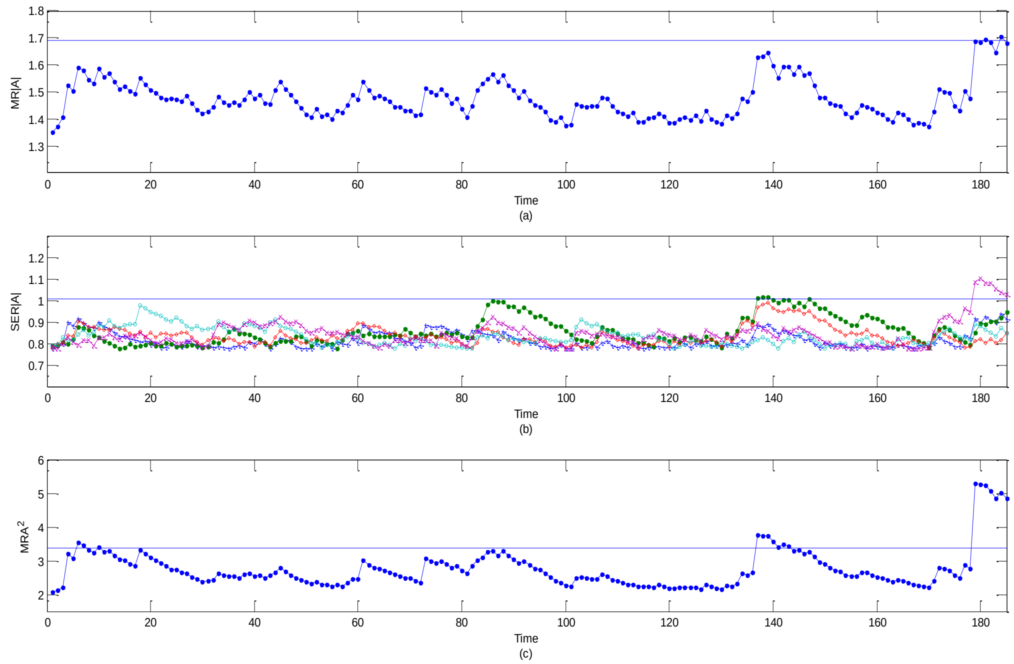

Figure 6.6.(a)-(c): The MR|A|, SER|A| and MRA^2 charts for monitoring aluminum smelter process

It may be interesting to investigate how the shifts in original variables translate into the shifts in the regression adjusted variables. According to Hawkins (1991), the charts for the original variables and the corresponding regression adjusted variables hardly differ if they are weakly correlated with the other variables. The improvement in performance for regression adjustment based control chart would be better if the variables are more strongly correlated.

Table 6.3 gives the in-control correlation matrix estimated from the first 95 observations. We can see that Z_2 and Z_3 are strongly correlated and Z_4 and Z_5 are moderately correlated. Z_1 is weakly correlated with other variables. We obtained the control limit for the SER|Z| chart with λ = 0.03 so that we could compare this chart to the SER|A| chart with λ = 0.03 for the aluminum smelter data.

Table 6.3. The estimated Σ_{Z0} for the aluminum smelter data

		Sample correlation matrix, $\hat{\Sigma}_{Z0}$		
1	0.20	-0.05	-0.02	-0.19
0.20	1	-0.64	-0.03	0.18
-0.05	-0.64	1	-0.30	0.19
-0.02	-0.03	-0.30	1	-0.41
-0.19	0.18	0.19	-0.41	1

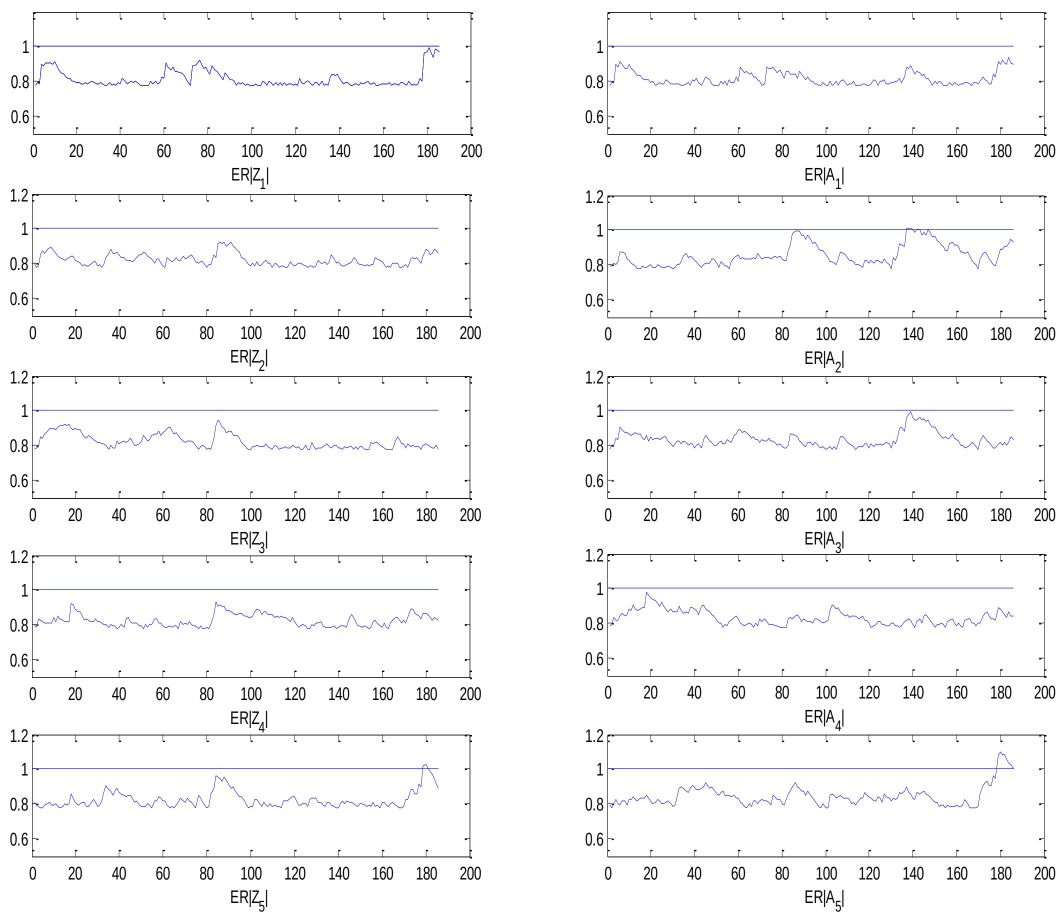

Figure 6.7. The individual statistic of the SER|Z| and SER|A| charts.

Figure 6.7 gives the individual EWMA statistics for each original variable and regression adjusted variable from the SER|Z| and SER|A| charts. We notice that both

$ER|A_5|$ and $ER|Z_5|$ subplots give a signal at the 179th observation. The $ER|A_2|$ subplot gives an additional signal at the 137th observation but the $ER|Z_2|$ subplot does not. General speaking, we find that there is no improvement for the $ER|A_1|$ statistic compared to the $ER|Z_1|$ statistic. The $ER|A_2|$, $ER|A_3|$, and $ER|A_5|$ statistics show a more obvious trend of shifts compared to their original variable based counterparts.

Chapter 7

Conclusion and Future Work

7.1 Conclusion

Simulation results have shown that the proposed MR$|Z|^q$, SER$|Z|$, and MRW^2 charts can be designed to be robust to non-normality. By choosing a proper combination of q and λ (or w and λ), the proposed charts obtain a relatively steady in-control ATS over a variety of multivariate distributions. The efficiencies of the three types of control charts with different combinations of parameters are evaluated for detecting a wide range of shifts in σ with multivariate normal, *t*, and gamma observations. Our conclusions from the investigation of their performance for in-control and out-of-control cases are summarized as follows:

Robust design

For the MR$|Z|^q$ and SER$|Z|^q$ charts, the absolute deviation from target with power $q < 2$ can be a more robust measurement of process variance, compared to the traditional squared deviation from target ($q = 2$). When an in-control process is more prone to producing values that fall far from its mean, the absolute deviation with a small value of q gives some robustness to control charts by reducing the impact of extreme observations. Generally, a small value of λ for a given q can make the proposed charts robust to non-normal distributions. However, given the same q and a control limit to give a target in-control ATS value with multivariate normal observations, the in-control ATS of the MR$|Z|^q$ or SER$|Z|^q$ chart increases as λ decreases when the robust chart is applied to the multivariate t and gamma observations, and decreases as λ decreases when applied to the multivariate beta observations. The in-control ATS can be very large for heavy-tailed or right-skewed distributions but be quite small for light-tailed distributions if λ is too small. Thus a moderately small value of λ is recommended for a given q, which provides balanced robustness over various distributions. A good value for λ tends to be smaller as q gets closer to 2.

Similarly, a large w should be paired with a relative small λ to make the MRW^2 chart robust over various distributions. Following a general property of the MRZ^2 chart, the MRW^2 chart with larger λ gives a signal faster for median and large shifts but slower for small shifts compared to the chart with relatively small λ.

The TEQL is introduced to measure the overall out-of-control performance of control charts. The average total loss to detect a range of shifts due to the robust designs may be as large as 30% compare to the MRZ^2 chart with λ = 0.025 when applied to multivariate normal processes. But the robust designs can be more efficient for the heavier-tailed or skewed processes. Based on the TEQL value for normal and non-normal processes, the MR|Z| chart with λ = 0.03 is a good choice for the MR|Z|^q chart design. The SER|Z| chart with λ = 0.03 has an efficiency advantage over the MR|Z| chart when the variables are highly correlated. But its performance is generally worse when variables are independent. A small w paired with a relatively large λ makes the design of the MRW^2 chart more efficient for detecting a wide range of shifts. Given the combinations of w and λ discussed in the dissertation, the MRW^2 chart with w = 2.2 and λ = 0.2 has the smallest TEQL value for different processes. However, the overall performance of the MRW^2 chart cannot compete with the performance of the MR|Z| and SER|Z| charts.

The above design scheme applies quite well to bivariate and higher dimensional processes considered here. The investigation for the effect of ρ on the robustness of the proposed charts indicates that the proposed charts generally have better robust in-control performance over various distributions if ρ is greater than 0. As the number of variables involved in a process increases significantly, the suggested value of λ for a given MR|Z|^q chart may be slightly smaller to ensure acceptable robustness.

Regression adjustment based robust charts

The discussion in Chapter 5 shows that the regression adjustment method is recommended to improve the performance of robust charts. It can improve the MR|Z|^q and SER|Z| charts' detection notably for a wide range of shifts. This conclusion applies to both normal and non-normal processes. The regression adjustment method will not significantly change the robustness property of the proposed MR|Z|^q and SER|Z| charts with appropriate adjustment of the parameter combinations. For the same q, it may require a slightly smaller λ compared to the parameter combinations of the original robust designs. A possible concern with this method is the interpretation of shifts when a signal is present because the charts based on the regression adjustment monitor the residuals of the regression model fitted to the original observations instead of the original variables.

Recommendation

Based on the efficiency comparisons for processes with different dimensions and different distributions, the MR|A| chart with λ = 0.02 has better overall performance among all the robust designs when there are some correlation among variables. When process variables are independent, the MR|Z| chart with λ = 0.02 is recommended. It has been shown that the MR|A| chart for the correlated variables or the MR|Z| chart for the independent variables gives a satisfying robustness over various processes with up to 8 variables and gives a more efficient performance for detecting a wide range of shift sizes.

The SER|A| chart with λ = 0.03 offers a good alternative robust design for various processes considered here. Though its overall performance cannot beat the MR|A| chart for detecting random shifts, it is recommended when the variance of only one or two variables will increase.

7.2 Future Work

To extend current work, we suggest future work to be broken down into the following categories:

1) Further development of the MR|A|^q and SER|A| chart

The MR|A|^q and SER|A| charts discussed here are investigated for the case of monitoring a process with $n=1$. Actually they can also be applied to sampling schemes with multiple observations at each sampling point. Investigating the case of $n=4$ would be interesting to see whether the conclusions for n = 1 still hold. Generally, with more observations collected at one sampling time, we expect the recommended value of λ could be larger for a given q compared to the case of n = 1.

We have investigated the performance of the robust MR|A|^q and SER|A| charts for detecting increases in $\boldsymbol{\sigma}$ with correlation coefficients unchanged. Another possible future extension can be the investigation of the charts' performance for detecting decreases in $\boldsymbol{\sigma}$. A two-side robust design should be of interest in many applications. Additionally, we suggest future investigation of the robustness and efficiency of the proposed charts for more complicated situations where the variances and correlation structure change at the same time.

So far we have mainly discussed the robust design of control charts for bivariate and 5-variate processes. In practice, there are many high dimensional processes. Although we find that the proposed MR|A|^q and SER|A| chart can be tuned to be robust to non-normality for processes with p up to 8, it would be good to know whether they can further be applied to even higher-dimensional processes.

2) Robust chart combinations to monitor $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ simultaneously

Generally, we need to monitor the mean and variance of a process simultaneously to ensure good control of quality. It's not reasonable to monitor the covariance structure of a process alone. Thus a future work can be the investigation of charts' performance in monitoring $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ at the same time by using a combination of a robust mean chart and a robust covariance chart. The robust MR|A|^q charts or the robust SER|A| chart could be paired with a robust MZ chart recommended by Stoumbos and Sullivan (2002). The performance of these combinations can be compared to some non-robust chart combinations.

Appendix I

Matlab coding for the MR|A| chart

Function CLMR_ABSA|: Matlab coding for generating control limit of the MR|A| chart with $\lambda = 0.02$.

```
function control_limit=CLMR_ABSA(target,p,sig,simulation_run)
%%%%%imput%%%%%%%%%%%%%%%%%%
%target: enter the target in-control ATS
%p: enter the dimension of the process
%sig: enter the in-control correlation matrix
%simulation_run: enter an integer,10000 or larger is recommended
%%%%%end of imput%%%%%%%%%%%%

lamb=0.02;
q=1;
start=0.7978;var=0.3633;
isig=inv(sig);
RA=diag(diag(isig))^(-0.5)*isig;
RAV=RA*diag(diag(isig))^(-0.5);
if ~isempty(find( sig>1 | sig<-1, 1))
    error('invalid correlation matrix');
end;
if ~isequal(sig, sig')
    error('invalid correlation matrix, correlation matrix must be
symmetric');
end;
if length(sig)~= p
    error('p does not match the dimension of correlation matrix');
end;
abssig=zeros(p,p);
for j=1:p
   for jj=1:p
    rho=RAV(j,jj);
    if j~=jj
    k=1;
    ey=2^(0.5*q*k)*gamma(0.5*(q*k+1))/sqrt(pi);
    k=k+1;
    ey2=2^(0.5*q*k)*gamma(0.5*(q*k+1))/sqrt(pi);
    ey3=ey2-ey^2;
    a=q; b=q;
    a1=0.5*(a+1); b1=(b+1)*0.5; z=rho^2;
```

```
    eyy=((1-
rho^2)*2)^(0.5*(a+b+1))*gamma(a1)*gamma(b1)*hypergeometric2f1(a1,b1,0.5
,z,100)/(pi*sqrt(2));
    covyy=eyy-ey^2;
    coryy=covyy/ey3;
    abssig(j,jj)=coryy;
    else abssig(j,jj)=1;
    end;
    end;
end;
iabssig=inv(abssig);
mu=zeros(1,p);
if (length(p)~=1) || (fix(p) ~= p) || (p < 2)|| (p>10),
   error('invalid number of variables,...CLMR|A| requires that P must
be a integer between 2 and 10.')
end;
if p>8
    max_h_mz=7.4;
    min_h_mz=6.5;
 else if p>6
    max_h_mz=6;
    min_h_mz=5;
    else if  p>4
        max_h_mz=4.5;
        min_h_mz=3.5;
        else if p>2
        max_h_mz=3;
        min_h_mz=2;
         else
        max_h_mz=1.5;
        min_h_mz=0.85;
             end;
         end;
      end;
end;
length_mz=10;
RLJ=zeros(1,length_mz);%store the signal time
h_mz=linspace(min_h_mz,max_h_mz,length_mz);
iter=100;RL=zeros(iter,length_mz);


m=1; mean1=ones(p,m)*start;
%     matlabpool open local 7;
for i=1:iter
    rand('seed',i);randn('seed',i);
    index1=1; %initial signal mark
    index2=0;
    RLJ=zeros(1,length_mz);%store the signal time
     stat1=0;
    k1=0;flag1=0;
     ez=mean1;
      while(flag1==0)
              sample=abs(RA*mvnrnd(mu,sig,1)').^q;
               ez=(1-lamb)*max(mean1,ez)+lamb*sample;
```

```
               stat1=(ez')*iabssig*(ez);
               max_mz=stat1;
        if ( h_mz(index1)<=max_mz&& max_mz<max_h_mz)
           index2=find(h_mz>max_mz,1);
          for l=index1:(index2-1)
           temp1=find(stat1>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
                index1=index2;
          end
        end;
        if max_mz>=max_h_mz
            flag1=1;
           for l=index1:length_mz
           temp1=find(stat1>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
           end
        end
       k1=k1+1;
       if k1>target+3000;
           flag1=1;
           RLJ=k1;
       end;
      end
        RL(i,:)=RLJ;
end;
%     matlabpool close;
  ats=sum(RL)/iter;
  [h_mz',ats'];

while(max(ats)<target)
     min_h_mz= max_h_mz;
     max_h_mz= max_h_mz+0.3;
     RLJ=zeros(1,length_mz);
     h_mz=linspace(min_h_mz,max_h_mz,length_mz);
     RL=zeros(iter,length_mz);
    for i=1:iter
    rand('seed',i);randn('seed',i);
    index1=1; %initial signal mark
    index2=0;
    RLJ=zeros(1,length_mz);%store the signal time
     stat1=0;
    k1=0;flag1=0;
     ez=mean1;
      while(flag1==0)
              sample=abs(RA*mvnrnd(mu,sig,1)').^q;
               ez=(1-lamb)*max(mean1,ez)+lamb*sample;
               stat1=(ez')*iabssig*(ez);
               max_mz=stat1;
        if ( h_mz(index1)<=max_mz&& max_mz<max_h_mz)
           index2=find(h_mz>max_mz,1);
          for l=index1:(index2-1)
           temp1=find(stat1>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
                index1=index2;
```

```
          end
        end;
        if max_mz>=max_h_mz
            flag1=1;
           for l=index1:length_mz
           temp1=find(stat1>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
           end
        end
       k1=k1+1;
      end
        RL(i,:)=RLJ;
    end;
     ats=sum(RL)/iter;
end;

while(min(ats)>target)
     if min(ats)>target+3000
     max_h_mz= min_h_mz-0.5;
     min_h_mz= min_h_mz-0.7;
     else
     max_h_mz= min_h_mz;
     min_h_mz= min_h_mz-0.2;
     end;
     RLJ=zeros(1,length_mz);
     h_mz=linspace(min_h_mz,max_h_mz,length_mz);
     RL=zeros(iter,length_mz);
     for i=1:iter
    rand('seed',i);randn('seed',i);
    index1=1;
    index2=0;
    RLJ=zeros(1,length_mz);
     stat1=0;
    k1=0;flag1=0;
     ez=mean1;
      while(flag1==0)
              sample=abs(RA*mvnrnd(mu,sig,1)').^q;
               ez=(1-lamb)*max(mean1,ez)+lamb*sample;
               stat1=(ez')*iabssig*(ez);
               max_mz=stat1;
        if ( h_mz(index1)<=max_mz&& max_mz<max_h_mz)
           index2=find(h_mz>max_mz,1);
          for l=index1:(index2-1)
           temp1=find(stat1>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
                index1=index2;
          end
        end;
        if max_mz>=max_h_mz
            flag1=1;
           for l=index1:length_mz
           temp1=find(stat1>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
           end
```

```
        end
       k1=k1+1;
       if k1>target+3000;
           flag1=1;
           RLJ=k1;
       end;
      end
        RL(i,:)=RLJ;
    end;
     ats=sum(RL)/iter;
end;

if max(ats)>target && min(ats)<target
    if max(ats)>target+20
       max_h_mz=h_mz(find(ats>(target+20),1));
    end;
    if min(ats)<target-20
      min_h_mz=max(h_mz(ats<(target-20)));
    end;
    iter=1000;
    length_mz=50;
   RLJ=zeros(1,length_mz);
   h_mz=linspace(min_h_mz,max_h_mz,length_mz);
   RL=zeros(iter,length_mz);
   for i=1:iter
   rand('seed',i);randn('seed',i);
   index1=1;
   index2=0;
   RLJ=zeros(1,length_mz);
    stat1=0;
   k1=0;flag1=0;
    ez=mean1;
     while(flag1==0)
             sample=abs(RA*mvnrnd(mu,sig,1)').^q;
              ez=(1-lamb)*max(mean1,ez)+lamb*sample;
              stat1=(ez')*iabssig*(ez);
              max_mz=stat1;
        if ( h_mz(index1)<=max_mz&& max_mz<max_h_mz)
           index2=find(h_mz>max_mz,1);
          for l=index1:(index2-1)
           temp1=find(stat1>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
                index1=index2;
          end
        end;
        if max_mz>=max_h_mz
            flag1=1;
           for l=index1:length_mz
           temp1=find(stat1>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
           end
        end
       k1=k1+1;
      end
```

```
        RL(i,:)=RLJ;
    end;
     ats=sum(RL)/iter;
end;

 while(max(ats)<target)
     min_h_mz= max_h_mz;
     max_h_mz= max_h_mz+0.01;
     RLJ=zeros(1,length_mz);
     h_mz=linspace(min_h_mz,max_h_mz,length_mz);
     RL=zeros(iter,length_mz);
    for i=1:iter
    rand('seed',i);randn('seed',i);
    index1=1;
    index2=0;
    RLJ=zeros(1,length_mz);
     stat1=0;
    k1=0;flag1=0;
     ez=mean1;
      while(flag1==0)
              sample=abs(RA*mvnrnd(mu,sig,1)').^q;
               ez=(1-lamb)*max(mean1,ez)+lamb*sample;
               stat1=(ez')*iabssig*(ez);
               max_mz=stat1;
        if ( h_mz(index1)<=max_mz&& max_mz<max_h_mz)
           index2=find(h_mz>max_mz,1);
          for l=index1:(index2-1)
           temp1=find(stat1>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
                index1=index2;
          end
        end;
        if max_mz>=max_h_mz
            flag1=1;
           for l=index1:length_mz
           temp1=find(stat1>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
           end
        end
       k1=k1+1;
      end
        RL(i,:)=RLJ;
    end;
     ats=sum(RL)/iter;
 end;

 while(min(ats)>target)
     max_h_mz= min_h_mz;
     min_h_mz= min_h_mz-0.01;
     RLJ=zeros(1,length_mz);
     h_mz=linspace(min_h_mz,max_h_mz,length_mz);
     RL=zeros(iter,length_mz);
     for i=1:iter
    rand('seed',i);randn('seed',i);
```

```
    index1=1;
    index2=0;
    RLJ=zeros(1,length_mz);
     stat1=0;
    k1=0;flag1=0;
     ez=mean1;
      while(flag1==0)
              sample=abs(RA*mvnrnd(mu,sig,1)').^q;
               ez=(1-lamb)*max(mean1,ez)+lamb*sample;
               stat1=(ez')*iabssig*(ez);
               max_mz=stat1;
        if ( h_mz(index1)<=max_mz&& max_mz<max_h_mz)
           index2=find(h_mz>max_mz,1);
          for l=index1:(index2-1)
           temp1=find(stat1>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
                index1=index2;
          end
        end;
        if max_mz>=max_h_mz
            flag1=1;
           for l=index1:length_mz
           temp1=find(stat1>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
           end
        end
       k1=k1+1;
      end
        RL(i,:)=RLJ;
    end;
     ats=sum(RL)/iter;
 end;

if max(ats)>target && min(ats)<target
     if max(ats)>target+20
        max_h_mz=h_mz(find(ats>(target+20),1));
     end;
     if min(ats)<target-20
       min_h_mz=max(h_mz(ats<(target-20)));
     end;
     iter=simulation_run;
     length_mz=100;
    RLJ=zeros(1,length_mz);%store the signal time
    h_mz=linspace(min_h_mz,max_h_mz,length_mz);
    RL=zeros(iter,length_mz);
    for i=1:iter
    rand('seed',i);randn('seed',i);
    index1=1; %initial signal mark
    index2=0;
    RLJ=zeros(1,length_mz);%store the signal time
     stat1=0;
    k1=0;flag1=0;
     ez=mean1;
      while(flag1==0)
```

```
              sample=abs(RA*mvnrnd(mu,sig,1)').^q;
               ez=(1-lamb)*max(mean1,ez)+lamb*sample;
               stat1=(ez')*iabssig*(ez);
               max_mz=stat1;
        if ( h_mz(index1)<=max_mz&& max_mz<max_h_mz)
           index2=find(h_mz>max_mz,1);
          for l=index1:(index2-1)
           temp1=find(stat1>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
                index1=index2;
          end
        end;
        if max_mz>=max_h_mz
            flag1=1;
           for l=index1:length_mz
           temp1=find(stat1>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
           end
        end
       k1=k1+1;
      end
        RL(i,:)=RLJ;
    end;
     ats=sum(RL)/iter;
end;

pb=polyfit(h_mz',log(ats'),2);
a=pb(1);b=pb(2);c=pb(3)-log(target);
control_limit=(-b+(b^2-4*a*c)^0.5)/(2*a);%return the control limit
```

Appendix II

Matlab coding for the SER|A| chart

Function CLSER_ABSA: Matlab coding for generating control limit of the SER|A| chart with λ = 0.03.

```
function control_limit=CLSER_ABSA(target,p,sig,simulation_run)
%%%%%imput%%%%%%%%%%%%%%%%%%
%target: enter the target in-control ATS
%p: enter the dimension of the process
%sig: enter the in-control correlation matrix
%simulation_run: enter an integer;10000 or larger is recommended
%%%%%end of imput%%%%%%%%%%%


lamb=0.03;
q=1;
start=0.7978;
isig=inv(sig);
RA=diag(diag(isig))^(-0.5)*isig;
if ~isempty(find( sig>1 | sig<-1, 1))
    error('invalid correlation matrix');
end;
if ~isequal(sig, sig')
    error('invalid correlation matrix, correlation matrix must be
symmetric');
end;
if length(sig)~= p
    error('p does not match the dimension of correlation matrix');
end;
mu=zeros(1,p);
if (length(p)~=1) || (fix(p) ~= p) || (p < 2)|| (p>10),
   error('invalid number of variables,...CLMR|A| requires that P must
be a integer between 2 and 10.')
end;
if p>8
    max_h_mz=1.14;
    min_h_mz=1.04;
 else if p>6
    max_h_mz=1.11;
    min_h_mz=1.01;
```

```
    else if  p>4
        max_h_mz=1.08;
        min_h_mz=1.004;
        else if p>2
        max_h_mz=1.05;
        min_h_mz=0.98;
         else
        max_h_mz=1.02;
        min_h_mz=0.92;
             end;
         end;
      end;
end;
length_mz=10;
RLJ=zeros(1,length_mz);%store the signal time
h_mz=linspace(min_h_mz,max_h_mz,length_mz);
iter=100;RL=zeros(iter,length_mz);

m=1; mean1=ones(p,m)*start;
%     matlabpool open local 7;
for i=1:iter
    rand('seed',i);randn('seed',i);
    index1=1; %initial signal mark
    index2=0;
    RLJ=zeros(1,length_mz);%store the signal time
     stat1=0;
    k1=0;flag1=0;
     ez=mean1;
      while(flag1==0)
              sample=abs(RA*mvnrnd(mu,sig,1)').^q;
               ez=(1-lamb)*max(mean1,ez)+lamb*sample;
               max_mz= max(ez);
        if ( h_mz(index1)<=max_mz&& max_mz<max_h_mz)
            index2=find(h_mz>max_mz,1);
          for l=index1:(index2-1)
            temp1=find(max_mz>h_mz(l));
                 RLJ(l)=k1*m+temp1(1);
                 index1=index2;
          end
        end;
        if max_mz>=max_h_mz
             flag1=1;
            for l=index1:length_mz
            temp1=find(max_mz>h_mz(l));
                 RLJ(l)=k1*m+temp1(1);
            end
        end
       k1=k1+1;
       if k1>target+3000;
           flag1=1;
           RLJ=k1;
       end;
      end
        RL(i,:)=RLJ;
```

```
end;
%     matlabpool close;
ats=sum(RL)/iter;
  [h_mz',ats'];

 while(max(ats)<target)
     min_h_mz= max_h_mz;
     max_h_mz= max_h_mz+0.05;
     RLJ=zeros(1,length_mz);
     h_mz=linspace(min_h_mz,max_h_mz,length_mz);
     RL=zeros(iter,length_mz);
    for i=1:iter
    rand('seed',i);randn('seed',i);
    index1=1;
    index2=0;
    RLJ=zeros(1,length_mz);
     stat1=0;
    k1=0;flag1=0;
     ez=mean1;
      while(flag1==0)
                sample=abs(RA*mvnrnd(mu,sig,1)').^q;
                 ez=(1-lamb)*max(mean1,ez)+lamb*sample;
                 max_mz= max(ez);
        if ( h_mz(index1)<=max_mz&& max_mz<max_h_mz)
            index2=find(h_mz>max_mz,1);
          for l=index1:(index2-1)
            temp1=find(max_mz>h_mz(l));
                  RLJ(l)=k1*m+temp1(1);
                  index1=index2;
          end
        end;
        if max_mz>=max_h_mz
             flag1=1;
            for l=index1:length_mz
            temp1=find(max_mz>h_mz(l));
                  RLJ(l)=k1*m+temp1(1);
            end
        end
       k1=k1+1;
      end
        RL(i,:)=RLJ;
    end;
     ats=sum(RL)/iter;
 end;

 while(min(ats)>target)
     if min(ats)>target+3000
     max_h_mz= min_h_mz-0.05;
     min_h_mz= min_h_mz-0.08;
     else
     max_h_mz= min_h_mz;
     min_h_mz= min_h_mz-0.05;
     end;
     RLJ=zeros(1,length_mz);
```

```
     h_mz=linspace(min_h_mz,max_h_mz,length_mz);
     RL=zeros(iter,length_mz);
     for i=1:iter
    rand('seed',i);randn('seed',i);
    index1=1;
    index2=0;
    RLJ=zeros(1,length_mz);
     stat1=0;
    k1=0;flag1=0;
     ez=mean1;
      while(flag1==0)
              sample=abs(RA*mvnrnd(mu,sig,1)').^q;
               ez=(1-lamb)*max(mean1,ez)+lamb*sample;
               max_mz= max(ez);
        if ( h_mz(index1)<=max_mz&& max_mz<max_h_mz)
           index2=find(h_mz>max_mz,1);
          for l=index1:(index2-1)
           temp1=find(max_mz>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
                index1=index2;
          end
        end;
        if max_mz>=max_h_mz
            flag1=1;
           for l=index1:length_mz
           temp1=find(max_mz>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
           end
        end
       k1=k1+1;
      end
        RL(i,:)=RLJ;
    end;
     ats=sum(RL)/iter;
end;

if max(ats)>target && min(ats)<target
     if max(ats)>target+20
        max_h_mz=h_mz(find(ats>(target+20),1));
     end;
     if min(ats)<target-20
       min_h_mz=max(h_mz(ats<(target-20)));
     end;
     iter=1000;
     length_mz=50;
    RLJ=zeros(1,length_mz);
    h_mz=linspace(min_h_mz,max_h_mz,length_mz);
    RL=zeros(iter,length_mz);
    for i=1:iter
    rand('seed',i);randn('seed',i);
    index1=1;
    index2=0;
    RLJ=zeros(1,length_mz);
     stat1=0;
```

```
    k1=0;flag1=0;
     ez=mean1;
      while(flag1==0)
              sample=abs(RA*mvnrnd(mu,sig,1)').^q;
               ez=(1-lamb)*max(mean1,ez)+lamb*sample;
               max_mz= max(ez);
        if ( h_mz(index1)<=max_mz&& max_mz<max_h_mz)
           index2=find(h_mz>max_mz,1);
          for l=index1:(index2-1)
           temp1=find(max_mz>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
                index1=index2;
          end
        end;
        if max_mz>=max_h_mz
            flag1=1;
           for l=index1:length_mz
           temp1=find(max_mz>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
           end
        end
       k1=k1+1;
      end
        RL(i,:)=RLJ;
    end;
     ats=sum(RL)/iter;
end;

while(max(ats)<target)
     min_h_mz= max_h_mz;
     max_h_mz= max_h_mz+0.01;
     RLJ=zeros(1,length_mz);%store the signal time
     h_mz=linspace(min_h_mz,max_h_mz,length_mz);
     RL=zeros(iter,length_mz);
    for i=1:iter
    rand('seed',i);randn('seed',i);
    index1=1; %initial signal mark
    index2=0;
    RLJ=zeros(1,length_mz);%store the signal time
     stat1=0;
    k1=0;flag1=0;
     ez=mean1;
      while(flag1==0)
              sample=abs(RA*mvnrnd(mu,sig,1)').^q;
               ez=(1-lamb)*max(mean1,ez)+lamb*sample;
               max_mz= max(ez);
        if ( h_mz(index1)<=max_mz&& max_mz<max_h_mz)
           index2=find(h_mz>max_mz,1);
          for l=index1:(index2-1)
           temp1=find(max_mz>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
                index1=index2;
          end
        end;
```

```
        if max_mz>=max_h_mz
            flag1=1;
           for l=index1:length_mz
           temp1=find(max_mz>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
           end
        end
       k1=k1+1;
      end
        RL(i,:)=RLJ;
    end;
     ats=sum(RL)/iter;
end;

while(min(ats)>target)
     max_h_mz= min_h_mz;
     min_h_mz= min_h_mz-0.01;
     RLJ=zeros(1,length_mz);%store the signal time
     h_mz=linspace(min_h_mz,max_h_mz,length_mz);
     RL=zeros(iter,length_mz);
     for i=1:iter
    rand('seed',i);randn('seed',i);
    index1=1; %initial signal mark
    index2=0;
    RLJ=zeros(1,length_mz);%store the signal time
     stat1=0;
    k1=0;flag1=0;
     ez=mean1;
      while(flag1==0)
              sample=abs(RA*mvnrnd(mu,sig,1)').^q;
               ez=(1-lamb)*max(mean1,ez)+lamb*sample;
               max_mz= max(ez);
        if ( h_mz(index1)<=max_mz&& max_mz<max_h_mz)
           index2=find(h_mz>max_mz,1);
          for l=index1:(index2-1)
           temp1=find(max_mz>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
                index1=index2;
          end
        end;
        if max_mz>=max_h_mz
            flag1=1;
           for l=index1:length_mz
           temp1=find(max_mz>h_mz(l));
                RLJ(l)=k1*m+temp1(1);
           end
        end
       k1=k1+1;
      end
        RL(i,:)=RLJ;
    end;
     ats=sum(RL)/iter;
end;
```

```
if max(ats)>target && min(ats)<target
     if max(ats)>target+20
        max_h_mz=h_mz(find(ats>(target+20),1));
     end;
     if min(ats)<target-20
       min_h_mz=max(h_mz(ats<(target-20)));
     end;
     iter=simulation_run;
     length_mz=100;
    RLJ=zeros(1,length_mz);%store the signal time
    h_mz=linspace(min_h_mz,max_h_mz,length_mz);
    RL=zeros(iter,length_mz);
    for i=1:iter
    rand('seed',i);randn('seed',i);
    index1=1; %initial signal mark
    index2=0;
    RLJ=zeros(1,length_mz);%store the signal time
     stat1=0;
    k1=0;flag1=0;
     ez=mean1;
      while(flag1==0)
              sample=abs(RA*mvnrnd(mu,sig,1)').^q;
               ez=(1-lamb)*max(mean1,ez)+lamb*sample;
               max_mz= max(ez);
        if ( h_mz(index1)<=max_mz&& max_mz<max_h_mz)
            index2=find(h_mz>max_mz,1);
          for l=index1:(index2-1)
            temp1=find(max_mz>h_mz(l));
                 RLJ(l)=k1*m+temp1(1);
                 index1=index2;
          end
        end;
        if max_mz>=max_h_mz
             flag1=1;
            for l=index1:length_mz
            temp1=find(max_mz>h_mz(l));
                 RLJ(l)=k1*m+temp1(1);
            end
        end
       k1=k1+1;
      end
        RL(i,:)=RLJ;
    end;
     ats=sum(RL)/iter;
end;

pb=polyfit(h_mz',log(ats'),2);
a=pb(1);b=pb(2);c=pb(3)-log(target);
control_limit=(-b+(b^2-4*a*c)^0.5)/(2*a); %%% return the control limit
```

References

[1]Abramowitz, M. and Stegun, I. A. (1972). "Hypergeometric Functions." *Ch. 15 in Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing.* New York: Dover, pp. 555-566.

[2]Abu-Shawiesh, M. O. and Abdullah, B. M. (2001). " A New Robust Bivariate Control Chart for Location". *Communications in Statistics-Simulation and Computation* 30, pp. 513-529.

[3]Alloway, JR, J. A. and Raghavachari, M. (1990). "Multivariate Control Charts Based on Trimmed Means". *ASQC Quality Congress Transactions*. American Society for Quality Control, San Francisco, pp. 449-453.

[4]Amin, R. and Searcy, A. J. (1991). "A Nonparametric Exponentially Weighted Moving Average Control Scheme". *Communications in Statistics-Simulation and Computation* 20, pp. 1049-1072.

[5]Anderson. T.W. (1984). *An Introduction to Multivariate Statistical Analysis*. 2nd ed. Wiley: New York, NY.

[6]Bakir, S. T. and Reynolds, Jr., M. R. (1979). "A Non-parametric Procedure for Process Control Based on Within-group Ranking”. *Technometrics* 21, pp.175-183.

[7]Bakir, S. T. (2004). “A Distribution-Free Shewhart Quality Control Chart Based on Signed-Rank-Like Statistics”. *Quality Engineering* 16(4), pp. 613-623.

[8]Bakir, S. T. (2006). "Distribution –Free Quality Control Charts Based on Signed-Randk-Like Statistic". *Communications in Statistics: Theory and Methods* 35, pp. 743-757.

[9]Brooks, A. (1985). "Heat Treating Shows Why SPC Is No Cure-all for Manufacturers". *Production Engineering* 32, pp. 66-77.

[10]Casella, G. and Berger, R. L. (2001), *Statistical Inference*, Duxbury: Pacific Grove.

[11]Chang, Young Soon (2007). "Multivariate CUSUM and EWMA Control Charts for Skewed Populations Using Weighted Standard Deviation". *Communications in Statistics- Simulation and Computation* 36, pp. 921-936.

[12]Cho, G. Y.(1991). "Multivariate Control Charts for the Mean Vector and Variance-Covariance Matrix with Variable Sampling Intervals". *Ph.D. Dissertation*, Department of Statistics, Virginia Tech.

[13]Coleman, D.E. (1997). "A discussion on Statistically-based Process Monitoring and Control". Eds. D.C. Montgomery and W.H. Woodall, *Journal of Quality Technology* 29, pp. 148-149.

[14]Crosier, R. B. (1988). "Multivariate Generalizations of Cumulative Sum Quality Schemes". *Technometrics* 30, pp. 291-303.

[15]Domangue, R. and Patch, S. C. (1991). “Some Omnibus Exponentially Weighted Moving Average Statistical Process Monitoring Schemes”. *Technometrics* 33, pp.299-313.

[16]Everitt, B. S. (1979). "A Monte Carlo Investigation of Robustness of Hotelling's One and Two Sample T^2 tests". *Journal of the American Statistical Association* 74, pp. 48-51.

[17]Hackl, P. and Ledolter, J. (1991). "A Control Chart Based on Ranks". *Journal of Quality Technology* 23, pp. 117-124.

[18]Hackl, P. and Ledolter, J. (1992). "A New Nonparametric Quality Control Technique". *Communications in statistics- Simulation and Computation* 21, pp. 423-443.

[19]Hawkins, Douglas M. (1981). "A CUSUM for a Scale Parameter". *Journal of Quality Technology*, Vol. 13, No. 4, October 1981, pp. 228-231.

[20]Hawkins, Douglas M. (1991). "Multivariate Quality Control Based on Regression-Adjusted Variables". *Technometrics* 33, pp. 61-75.

[21]Hawkins, Douglas M. (1993). "Regression Adjustment for Variables in Multivariate Quality Control". *Journal of Quality Technology* 25, pp. 170-182.

[22]Hawkins, Douglas M. and Olwell D. H. (1998). *Cumulative Sum Control Charts and Charting for Quality Improvement*. Springer: New York, 1998.

[23]Hotelling, H. (1947). "Multivariate Quality Control". in *Techniques of Statistical Analysis*, eds. C. Eisenhart, M. W. Hasty, and W. A. Wallis, New York: McDraw Hill, pp. 111-184.

[24]Johnson, M. E. (1987). *Multivariate Statistical Simulation*. John Wiley and Sons, New York, NY

[25]Johnson. N. L and Kotz. S. (1972). *Distributions in Statistics: Continuous Multivariate Distributions*. John Wiley and Sons, New York, NY

[26]Jones-farmer, L. Allison; Jordan, Victoria and Champ, Charles W (2009). "Distribution-free Phase I Control Charts for Subgroup Location". *Journal of Quality Technology* 41, pp. 304-316.

[27]Jones M. C. (2002). "Multivariate T and Beta Distributions Associated with the Multivariate F Distribution". *Metrika*, Volume 54, Number 3, pp. 215-231.

[28]Kapatou, A. (1996). "Multivariate Nonparametric Control Charts Using Small Samples". Ph.D. dissertation, Virginia Tech.

[29]Kramen, H. and Schmid, W. (1997). "EWMA Charts for Multivariate Time Series". *Sequential Analysis* 16, pp. 131-154.

[30]Krishnaiah, P. R. (1985). "Multivariate Gamma Distributions". *Encyclopedia of Statistical Sciences*. Eds. N. L. Johnson and S. Kotz, New York: Wiley.

[31]Liu, R. Y. (1990). "On a Notion of Data Depth Based on Random Simplices". *The Annals of Statistics* 18, pp. 405-414.

[32]Liu, R. Y. (1995). "Control Charts for Multivariate Processes". *Journal of the American Statistical Association* 90, pp. 1380-1387.

[33]Lowry, C. A.; Woodall, W. H.; Champ, C. W.; and Rigdon, S. E. (1992). "A Multivariate Exponentially Weighted Moving Average Control Chart". *Technometrics* 34, pp. 46-53.

[34]Lucas, J. M. and Saccucci, M. S. (1990). "Exponentially Weighted Moving Average Control Schemes: Properties and Enhancements". *Technometrics* 32, pp. 1-12.

[35]Mahalanobis, P.C.(1936). "On the Generalized Distance in Statistics". *Proceeding of the National Academy India*, Vol. 12, pp. 49-55

[36]Mathal A. M. and Moschopoulos P. G. (1992). "A Form of Multivariate Gamma Distribution". *Annals of the Institute of Statistical Mathematics*, Volume 44, Number 1, pp. 97-106.

[37]Minhajuddin. A. T. M.; Harris. I. R. and Schucany. W. R. (2004). "Simulating Multivariate Distributions With Specific correlations". *Journal of Statistical Computation and Simulation*, Vol. 74, No. 8, pp. 599–607.

[38]Mohebbi, C. and Hayre, L. (1989). "Multivariate Control Charts : A Loss Function Approach". *Sequential Analvsis*, 8, pp.253-268.

[39]Montgomery D. C. (2013). *Introduction to Statistical Quality Control* (7th edition). John Wiley & Sons, New York, NY

[40]Orban, J. and Wolfe, D.A. (1982). "A Class of Distribution-Free Two-Sample Tests Based on Placements". *Journal of the American Statistical Association* 77, pp. 666-672.

[41]Page, E. S. (1954). "Contrinuous Inspection Schemes". *Biometrica* 41, pp. 100-114.

[42]Park, C. and Reynolds, Jr., M. R. (1987). "Nonparametric Procedures For Monitoring A Location Parameter Based on Linear Placement Statistics". *Sequential Analysis* 6, pp. 303-323.

[43]Prabhu, S. S. and Runger, G. C. (1997). "Designing a Multivariate EWMA Control Chart". *Journal of Quality Technology* 29, pp. 8-15.

[44]Qiu, P. and Hawkins, D. (2001). "A Rank-based Multivariate CUSUM Procedure". *Technometrics* 43, pp. 120-132.

[45]Qiu, P. and Hawkins, D. (2003). "A Nonparametric Multivariate Cumulative Sum Procedure for Detecting Shifts in All Directions". *Journal of the Royal Statistical Society*, D 52, pp. 151-164.

[46]Qiu, P. (2008). "Distribution-Free Multivariate Process Control Based on Log-linear Modeling". *IIE Transactions* 40, pp.664-677.

[47]Reynolds, Jr., M. R. and Cho, G. (2006). "Multivariate Control Charts for Monitoring the Mean Vector and Covariance Matrix ". *Journal of Quality Technology* 38, pp. 230-253.

[48]Reynolds, Jr., M. R. and Kim, K. (2005). "Multivariate Monitoring of the Process Mean Vector Using Sequential Sampling". *Journal of Quality Technology* 37, pp. 149-162.

[49]Reynolds, Jr., M. R. and Stoumbos, Z. G. (2001). "Monitoring the Process Mean and Variance Using Individual Observations and Variable Sampling Intervals". *Journal of Quality Technology* 33, pp. 181-205.

[50]Reynolds, Jr., M. R. and Stoumbos, Z. G. (2004). "Control Charts and the Efficient Allocation of Sampling Resources". *Technometrics* 46, pp. 200-214.

[51]Reynolds, Jr., M. R. and Stoumbos, Z. G. (2005). "Should Exponentially Weighted Average and Cumulative Sum Charts be Used with Shewhart Limits?". *Technometrics* 47, pp. 409-424.

[52]Reynolds, Jr., M. R. and Stoumbos, Z. G. (2008). "Combinations of Multivariate Shewhart and MEWMA Control Charts for Monitoring the Mean Vector and Covariance Matrix". *Journal of Quality Technology* 40, pp. 381-393.

[53]Reynolds, Jr., M. R. and Stoumbos, Z. G. (2009). "Robust CUSUM Charts for Monitoring the Process Mean and Variance". *Quality and Reliability Engineering International* 26, pp. 453-473.

[54]Reynolds, Jr., M. R. and Lou, J. (2010). "An Evaluation of a GLR Control Chart for Monitoring the Process Mean". *Journal of Quality Technology* 42, pp. 287-310.

[55]Roberts, S. W. (1959). "Control Chart Tests Based on Geometric Moving Averages". *Technometrics* 1, pp. 239-250.

[56]Stoumbos, Z. G. and Jones, L. A. (2000). "On the Properties and Design of Individuals Control Charts Based on Simplicial Depth". *Nonlinear Studies*, Vol.7, pp. 147-178.

[57]Stoumbos, Z. G.; Jones, L. A.; Woodall, W. H.; Reynolds, Jr., M. R. (2001). "On Nonparametric Multivariate Control Charts Based on Data Depth". *Frontiers in Statistical Quality Control* 6, pp. 207-227.

[58]Stoumbos, Z. G.; Reynolds, Jr., M. R.; Ryan, T.P.; and Woodall W. H. (2000)."The State of Statistical Process Control as We Proceed Into the 21st Century". *Journal of the American Statistical Association* 95, pp. 992-998.

[59]Stoumbos, Z. G. and Reynolds, Jr., M. R. (2000). "Robustness to Non-normality and Autocorrelation of Individuals Control Charts". *Journal of Statistical Computation and Simulation* 66, pp. 145-187.

[60]Stoumbos, Z. G. and Sullivan, J. H. (2002). "Robustness to Non-Normality of the Multivariate EWMA Control Chart". *Journal of Quality Technology* 34, pp. 260-276.

[61]Tiku, M. L. and Singh, M. (1982). "Robust Statistics for Testing Mean Vectors of Multivariate Distribution". *Communications in Statistics-Theory and Methods* 11, pp. 985-1001.

[62]Tsui, K. and Woodall, W. H.(1993). "Multivariate Control Charts based on Loss Functions". *Sequential Analysis* 12(1), pp. 79-92.

[63]Tukey, J. W. (1975). "Mathematics and the Picturing of Data". *Proceedings of the International Congress of Mathematicians*, Vancouver, Vol.2, pp.523-531.

[64]Wagle B. (1968). "Multivariate Beta Distribution and A Test for Multivariate Normality". *Journal of the Royal Statistical Society*. Series B (Methodological), vol. 30, pp. 511-516.

[65]Woodall, W. H. and Ncube, M. M. (1985). "Multivariate CUSUM Quality-Control Procedures". *Technometrics* 30, pp. 285-292.

[66]Woodall W. H. and Montgomery, D. C. (1999). "Research Issues and Ideas in Statistical Process Control". *Journal of Quality Technology* 31, pp. 376-386.

[67]Woodall W. H. and Mahmoud, M.A. (2005). "The Inertial Properties of Quality Control Charts". *Technometrics* 47, pp. 425-436.

[68]Wu, Z. ; Yang, M.; Jiang, W. and Khoo, M. B. C. (2008). "Optimization Designs of the Combined Shewhart-CUSUM Control Charts". *Computational Statistics and Data Analysis*, 53, pp. 496-506.

[69]Yeh, A. B.; Lin, D. K. J; Zhou, H.; and Venkataramani, C. (2003). "A Multivariate Exponentially Weighted Moving Average Control Chart for Monitoring Process Variability". *Journal of Applied Statistic* 20, pp. 507-536.