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E. W. Larsen, S. Sancaktar, and P. F. Zweifel

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Extension of the Case formulas to L_p. Application to half and full space problems

E. W. Larsen

Courant Institute of Mathematical Sciences, New York University, New York, New York 10011

S. Sancaktar and P. F. Zweifel

Department of Physics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061
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The singular eigenfunction expansions originally applied by Case to solutions of the transport equation are extended from the space of Hölder-continuous functions to the function spaces $X_p = \{f \mid \mu f(\mu) \in L_p\}$, where the expansions are now to be interpreted in the X_p norm. The spectral family for the transport operator is then obtained explicitly, and is used to solve transport problems with X_p sources and incident distributions.

I. INTRODUCTION

In 1960, Case[1] introduced a method of solving one-speed, one-dimensional neutron transport problems by expanding the solution in terms of "normal modes." The expansion coefficients were obtained from certain singular integral equations determined by the boundary conditions of the problem. Although the completeness theorems proved by Case were not rigorous from a mathematical point of view, his method achieved great popularity because of its close analogy with the classical method of solving boundary value problems by eigenfunction expansions.

Recently, a rigorous derivation of the Case solutions has been obtained[2,3] by considering the spectral resolution of an operator K, which, for isotropic scattering, is defined by

$$Kf(x,\mu)=\mu f(x,\mu)+\frac{c}{2(1-c)}\int_{-1}^{1} sf(x,s)\,ds. \tag{1}$$

(The independendent variables x and μ represent, respectively, the neutron position and the cosine of the angle between the neutron velocity vector and the x axis, and c is a positive constant $\neq 1$.) In terms of K, which acts only on μ, the transport equation can be written as

$$\frac{\partial}{\partial x}\psi(x,\mu)+K^{-1}\psi(x,\mu)=h(x,\mu), \tag{2}$$

where $h=\mu^{-1}g$ and g represents the neutron source.

A different approach to the rigorous determination of the Case formalism has been given by Hangelbroek,[4] who has shown for $c<1$ that the operator K ($=A^{-1}[\mu]$ in Hangelbroek notation) is topologically equivalent to a self-adjoint operator on the Hilbert space $L_2(-1,1)$. By well-known spectral theorems,[5] this guarantees the existence of a spectral family for K. (This family is abstractly obtained by application of the Gel'fand—Naimark theorem after a commutative C^*-algebra is generated from K and the identity I.)

In all of these works, i.e., Refs. 1,2,4, final attention is restricted to Hölder-continuous (or piecewise Hölder-continuous) functions since the explicit formulas involve principle value integrals and boundary values of Cauchy integrals. The purpose of the present note is to show how these results can be extended to a much larger class of functions, namely, the spaces $X_p = \{f \mid \mu f \in L_p,\ p>1\}$. We do this by extending the results of Ref. 1 to these spaces. We also show that the expansion formulas of that reference can be used to construct the spectral family for the operator K. Finally, we indicate how this spectral family may be used to solve typical transport problems and suggest possible applications in other areas.

II. EXTENSION OF THE CASE FORMULAS

Our first step is to quote a theorem which will guarantee that an integral operator $A: L_p \to L_p$ of the form

$$Af(x)=\int_{-\infty}^{\infty}\frac{f(t)}{t-x}\,dt\equiv g(x) \tag{3}$$

is a bijection. This theorem is crucial for all the subsequent analysis.

Theorem 0[6]: Let $f(x)\in L_p(-\infty,\infty)$, $p>1$. Then the formula

$$g(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f(t)}{t-x}\,dt \tag{4}$$

defines almost everywhere a function $g(x)$ also belonging to $L_p(-\infty,\infty)$. The reciprocal formula

$$f(x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{g(t)}{t-x}\,dt \tag{5}$$

also holds almost everywhere and

$$\int_{-\infty}^{\infty}|g(x)|^p\,dx \le (M_p)^p\int_{-\infty}^{\infty}|f(x)|^p\,dx, \tag{6}$$

where M_p depends on p only. If $p=2$, then

$$\int_{-\infty}^{\infty}|g(x)|^2\,dx=\int_{-\infty}^{\infty}|f(x)|^2\,dx.$$

Now let us consider the formulas[1,2]

$$f'(\mu)=\int_{-1}^{1}A(\nu)\phi(\nu,\mu)\,d\nu+A(\nu_0)\phi(\nu_0,\mu)+A(-\nu_0)\phi(-\nu_0,\mu), \tag{7}$$

$$A(\nu)=\frac{1}{N(\nu)}\int_{-1}^{1}\mu f'(\mu)\phi(\nu,\mu)\,d\mu, \tag{8}$$

which hold for Hölder-continuous f' and which we shall refer to as Case transforms. We can simplify notation by defining

$$f(\mu)=f'(\mu)-A(\nu_0)\phi(\nu_0,\mu)-A(-\nu_0)\phi(-\nu_0,\mu); \tag{9}$$

then Eqs. (7) and (8) become

$$f(\mu)=\int_{-1}^{1} A(\nu)\phi(\nu,\mu)\,d\nu, \quad -1<\mu<1, \tag{10}$$

$$A(\nu)=\frac{1}{N(\nu)}\int_{-1}^{1} \mu f(\mu)\phi(\nu,\mu)\,d\mu, \quad -1<\nu<1. \tag{11}$$

We wish to show that Eqs. (7) and (8) are valid for $\mu f'(\mu)\in L_p(-1,1)$, $p>1$. Since the discrete parts $A(\pm\nu_0)\phi(\pm\nu_0,\mu)$ are in L_p, then it suffices to show that (10) and (11) hold for functions f (from which the contribution of the discrete modes has been subtracted out).

Lemma 1: For each

$$f\in X_p, X_p=\{f \mid \|f\|_p\equiv(\int_{-1}^{1}|\mu f(\mu)|^p\,d\mu)^{1/p}<\infty\}, \quad p>1, \tag{12}$$

there is a corresponding $A(\nu)\in X_p$ defined by Eq. (11), and $A(\nu)$ depends continuously on f.

Proof: Using the definition

$$\phi_\nu(\nu,\mu)\equiv\frac{c\nu}{2}\frac{1}{\nu-\mu}+\delta(\nu-\mu)\lambda(\nu) \tag{13}$$

and the Case transform, we obtain

$$\nu A(\nu)=\frac{\nu}{N(\nu)}\left(\frac{c\nu}{2}\int_{-1}^{1}\frac{\mu f(\mu)}{\nu-\mu}\,d\mu+\int_{-1}^{1}\mu f(\mu)\lambda(\nu)\delta(\nu-\mu)\,d\mu\right), \tag{14}$$

which we consider as a formal abbreviation for

$$\nu A(\nu)=\frac{\lambda(\nu)}{\Lambda^+(\nu)\Lambda^-(\nu)}\nu f(\nu)+\frac{c}{2}\frac{\nu}{\Lambda^+(\nu)\Lambda^-(\nu)}\int_{-1}^{1}\frac{\mu f(\mu)}{\nu-\mu}\,d\mu. \tag{15}$$

In these equations, the expression

$$N=\nu\Lambda^+(\nu)\Lambda^-(\nu) \tag{16}$$

has been utilized.

The functions $\lambda(\nu)/\Lambda^+(\nu)\Lambda^-(\nu)$ and $\nu/\Lambda^+(\nu)\Lambda^-(\nu)$ are continuous and bounded on $[-1,1]$.[1] Thus the first term in Eq. (15) is in $L_p(-1,1)$ and the second term is in $L_p(-1,1)$ if

$$g(\nu)=\frac{1}{\pi}\int_{-1}^{1}\frac{\mu f(\mu)}{\nu-\mu}\,d\mu \tag{17}$$

is in $L_p(-1,1)$. But by Theorem 0,

$$\int_{-1}^{1}|g(\nu)|^p\,d\nu\le\int_{-\infty}^{\infty}|g(\nu)|^p\,d\nu\le(M_p)^p\int_{-1}^{1}|\mu f(\mu)|^p\,d\mu. \tag{18}$$

Thus from Eq. (15), it is clear that

$$\int_{-1}^{1}|\nu A(\nu)|^p\,d\nu\le N_p^p\int_{-1}^{1}|\mu f(\mu)|^p\,d\mu, \tag{19}$$

so that

$$\|A\|_p\le N_p\|f\|_p. \tag{20}$$

Let us define the map $T: X_p\to X_p$ by

$$A(\nu)=(Tf)(\mu); \tag{21}$$

then

$$\|T\|_p\le N_p. \quad ∎ \tag{22}$$

Note: Using Eq. (15), one can show that if one multiplies by $\lambda(\nu)$, then

$$\int_{-1}^{1}|\nu\lambda(\nu)A(\nu)|^p\,d\nu\le\hat{N}_p^p\int_{-1}^{1}|\mu f(\mu)|^p\,d\mu, \tag{23}$$

where $\hat{N}_p$ is a constant depending only on p.

Thus, if

$$\lambda(\nu)A(\nu)=(\hat{T}f)(\mu), \tag{24}$$

then

$$\|\hat{T}\|_p\le\hat{N}_p. \tag{25}$$

Lemma 2: For each $A(\nu)$ such that $A(\nu)$ and $\lambda(\nu)A(\nu)\in X_p$, there exists an $f\in X_p$ defined by Eq. (10).

Proof: By definition

$$\int_{-1}^{1}|\nu A(\nu)|^p\,d\nu<\infty \tag{26}$$

and

$$\int_{-1}^{1}|\nu\lambda(\nu)A(\nu)|^p\,d\nu<\infty. \tag{27}$$

By Eq. (10) we define the function f by

$$f(\mu)\equiv\lambda(\nu)A(\nu)+\frac{c}{2}\int_{-1}^{1}\frac{\nu A(\nu)}{\nu-\mu}\,d\nu. \tag{28}$$

Clearly the first term is in X_p, and so is the second by application of Theorem 0. ∎

Also, if we have a sequence $\{A_n\}$ such that $\|A_n-A\|_p\to0$ and $\|\lambda(\nu)A_n(\nu)-\lambda(\nu)A(\nu)\|_p\to0$, then it is obvious that $\|f_n-f\|_p\to0$.

It is known[2,7] that for $\mu f(\mu)$ Hölder-continuous [and f of the form of Eq. (9)], Eqs. (10) and (11) hold simultaneously. Since the Hölder-continuous functions are dense in L_p, let us choose a sequence $\{f_n\}$ such that $\mu f_n(\mu)$ is Hölder-continuous and $f_n\to f\in X_p$. Then $A_n\to A$ and $\lambda(\nu)A_n\to\lambda(\nu)A$ by Eqs. (22)–(25). Thus by the above paragraph, Eq. (10) holds in the limit.

The above results can be summarized by a lemma:

Lemma 3: Eqs. (10) and (11) hold for any $f\in X_p$ which is of the form of Eq. (9). We can combine Lemmas 1, 2, and 3 as the following theorem:

Theorem I: The domain of the reduced transport operator K may be extended to the spaces X_p, $p>1$, and the Case transforms (7), (8) hold for each f' such that $\mu f'(\mu)\in L_p(-1,1)$.

3. RESOLUTION OF IDENTITY OF K

For $-1<\omega<1$, we define the operator $E(\omega)$ as

$$E(\omega)f(\mu)=\int_{-1}^{\omega}A(\nu)\phi(\nu,\mu)\,d\nu \tag{29}$$

$$=\begin{cases}A(\mu)\lambda(\mu)+\dfrac{c}{2}\displaystyle\int_{-1}^{\omega}\frac{\nu A(\nu)}{\nu-\mu}\,d\nu, & -1<\mu\le\omega,\\[2ex] \dfrac{c}{2}\displaystyle\int_{-1}^{\omega}\frac{\nu A(\nu)}{\nu-\mu}\,d\nu, & \omega<\mu<1.\end{cases} \tag{30}$$

From the above analysis, it is clear that the terms in Eq. (30) represent bounded operators on X_p, acting on f. Thus for $-1<\omega<1$, $E(\omega)$ is a bounded operator. In this section we shall show that the family $E(\omega)$ forms part of the spectral family of K.

First, for $\epsilon>0$ we have

$$[E(\omega+\epsilon)-E(\omega)]f(\mu)=\int_{\omega}^{\omega+\epsilon}A(\nu)\phi(\nu,\mu)\,d\nu \tag{31}$$

$$=\begin{cases}\lambda(\mu)A(\mu), & \omega\le\mu\le\omega+\epsilon,\\ 0, & \text{otherwise},\end{cases}+\frac{c}{2}\int_\omega^{\omega+\epsilon}\frac{\nu A(\nu)}{\nu-\mu}d\nu. \quad (32)$$

The norm of the first term is just

$$[\int_\omega^{\omega+\epsilon}|\mu\lambda(\mu)A(\mu)|^p d\mu]^{1/p}$$

which tends to zero for a fixed A (i.e., fixed f) as $\epsilon\to 0$. Call

$$g(\mu,\epsilon)=\frac{1}{\pi}\int_\omega^{\omega+\epsilon}\frac{\nu A(\nu)}{\nu-\mu}d\nu. \quad (33)$$

Then by Theorem 0,

$$\|g(\mu,\epsilon)\|^p=\int_{-1}^1|\mu g(\mu,\epsilon)|^p d\mu\le\int_{-\infty}^\infty|g(\mu,\epsilon)|^p d\mu$$
$$\le M_p^p\int_\omega^{\omega+\epsilon}|\nu A(\nu)|^p d\nu, \quad (34)$$

and this term also approaches 0 for a fixed f as $\epsilon\to 0$. Thus we can state the result as a lemma.

Lemma 4: For each $f\in X_p$ and $-1\le\omega<1$,

$$\lim_{\substack{\epsilon\to 0\\ \epsilon>0}}\|E(\omega+\epsilon)-E(\omega)f\|_p=0. \quad (35)$$

That is, $E(\omega)$ is a continuous function of ω (in the strong operator topology). Now, we will verify the following lemma.

Lemma 5:

$$E(\omega_1)E(\omega_2)=E(\omega_2)E(\omega_1)=E(\omega) \quad (36)$$

where $\omega=\min\{\omega_1,\omega_2\}$.

Proof: For definiteness, we consider

$$\omega_1\le\omega_2,$$
$$E(\omega_2)f(\mu)=\int_{-1}^{\omega_2}A(\nu)\phi(\nu,\mu)d\nu. \quad (37)$$

Then the expansion coefficients of $E(\omega_2)f(\mu)$ are

$$B(\nu)=\begin{cases}A(\nu), & -1\le\nu\le\omega_2,\\ 0, & \omega_2<\nu\le 1.\end{cases} \quad (38)$$

Thus,

$$E(\omega_1)E(\omega_2)f(\mu)=\int_{-1}^{\omega_1}B(\nu)\phi(\nu,\mu)d\nu$$
$$=\int_{-1}^{\omega_1}A(\nu)\phi(\nu,\mu)d\nu$$
$$=E(\omega_1)f(\mu). \quad (39)$$

In a similar way, we obtain $E(\omega_2)E(\omega_1)f=E(\omega_1)f$. ∎

Before proving Lemma 7, we will prove the following lemma which is essential in proving Lemma 7.

Lemma 6:

$$Kf=\int_{-1}^1 zA(z)\phi(z,\mu)dz. \quad (40)$$

Proof: Applying the same method as in Ref. 2, we get for Γ a simple closed curve containing the line segment $[-1,1]$.

$$Kf(\mu)=\frac{1}{2\pi i}\oint_\Gamma K(z-K)^{-1}f(\mu)dz$$
$$=\frac{1}{2\pi i}\oint_\Gamma(K-z+z)(z-K)^{-1}f(\mu)dz$$
$$=\frac{1}{2\pi i}\oint_\Gamma(z(z-K)^{-1}-I)f(\mu)dz$$
$$=\frac{1}{2\pi i}\oint_\Gamma z(z-K)^{-1}f(\mu)d\mu. \quad (41)$$

We now compute the contour integral exactly as in Ref. 2 to obtain

$$Kf(\mu)=\int_{-1}^1 zA(z)\phi(z,\mu)dz. \ ∎ \quad (42)$$

Lemma 7: For each $f\in X_p$ of the form (9), $Kf=\int_{-1}^1\omega\, dE(\omega)f$.

Proof: Let

$$U(\omega)=\int_{-1}^1(E(\omega)f(\mu))g(\mu)d\mu, \quad (43)$$

where $f\in X_p$ and $g\in X_q$ (the dual space of X_p: $1/p+1/q=1$). Then

$$U(\omega)=\int_{-1}^\omega A(\mu)\lambda(\mu)g(\mu)d\mu+\int_{-1}^1 d\mu\int_{-1}^\omega d\nu\tfrac{1}{2}c\nu\frac{A(\nu)}{\nu-\mu}g(\mu)$$
$$=\int_{-1}^\omega A(\mu)\lambda(\mu)g(\mu)d\mu+\int_{-1}^\omega d\nu\tfrac{1}{2}c\nu A(\nu)\int_{-1}^1\frac{g(\mu)}{\nu-\mu}d\mu$$
$$\equiv\int_{-1}^\omega A(\mu)\lambda(\mu)g(\mu)d\mu+\int_{-1}^\omega\tfrac{1}{2}c\mu A(\mu)Lg(\mu)d\mu$$
$$=\int_{-1}^\omega A(\mu)[\lambda(\mu)g(\mu)+\tfrac{1}{2}c\mu Lg(\mu)]d\mu. \quad (44)$$

Thus $U(\omega)$ is differentiable a.e. and

$$U'(\omega)=A(\omega)[\lambda(\omega)g(\omega)+\tfrac{1}{2}c\omega Lg(\omega)]. \quad (45)$$

Now we get

$$\int_{-1}^1\omega\, dU(\omega)=\int_{-1}^1\omega U'(\omega)d\omega$$
$$=\int_{-1}^1\omega A(\omega)[\lambda(\omega)g(\omega)+\tfrac{1}{2}c\omega Lg(\omega)]d\omega$$
$$=\int_{-1}^1\omega A(\omega)\lambda(\omega)g(\omega)$$
$$+\int_{-1}^1\omega A(\omega)\frac{c\omega}{2}\int_{-1}^1\frac{g(\mu)}{\omega-\mu}d\mu\, d\omega$$
$$=\int_{-1}^1\left[\omega A(\omega)\lambda(\omega)+\int_{-1}^1\nu A(\nu)\frac{c\nu}{2}\frac{d\nu}{\nu-\mu}\right]g(\omega)d\omega. \quad (46)$$

By the previous lemma,

$$\int_{-1}^1\omega\, dU(\omega)=\int_{-1}^1 Kf(\omega)g(\omega)d\omega.$$

Thus

$$\int_{-1}^1 Kf(\mu)g(\mu)d\mu=\int_{-1}^1\omega\, dU(\omega) \quad (47)$$
$$=\int_{\omega=-1}^1\omega d\int_{\mu=-1}^1[E(\omega)f](\mu)g(\mu)d\mu$$
$$=\int_{-1}^1 d\mu g(\mu)\int_{\omega=-1}^1\omega\, d[E(\omega)f](\mu), \quad (48)$$

where the interchange of limits is justified because ω and $E(\omega)f$ are continuous in ω. The above equation implies

$$Kf(\mu)=\int_{-1}^{1}\omega\, d[E(\omega)f](\mu), \tag{49}$$

or

$$K=\int_{-1}^{1}\omega\, dE(\omega),$$

where the integral is defined in the weak sense. ∎

Note:

$$\int_{-1}^{1}\omega\, dE(\omega)f=\int_{-1}^{1}\omega\frac{dE(\omega)}{d\omega}d\omega f$$

$$=\int_{-1}^{1}\omega A(\omega)\phi(\omega,\mu)\,d\omega=Kf \tag{50}$$

gives the result directly, but only formally.

Lemma 8: $KE(\omega)=E(\omega)K.$ (51)

Proof: By Lemma 6,

$$Kf=\int_{-1}^{1}zA(z)\phi(z,\mu)\,dz. \tag{52}$$

Thus by Eq. (29),

$$E(\omega)Kf(\mu)=\int_{-1}^{\omega}\nu A(\nu)\phi(\nu,\mu)\,d\nu \tag{53}$$

In a similar way we obtain

$$K(Ef)=\int_{-1}^{\omega}zA(z)\phi(z,\mu)\,dz. \tag{54}$$

Equality of Eqs. (53) and (54) proves the lemma. ∎

Also, the following identities hold by definition:

$$E(-1)=0, \tag{55}$$

$$E(1)=1. \tag{56}$$

Lemmas 4, 5, 7, and 8, together with Eqs. (55) and (56) complete the proof of the following theorem[5]:

Theorem II: For $\mu f'(\mu)\in L_p(-1,1)$, we define

$$E(\pm\nu_0)f'(\mu)=\frac{1}{N(\pm\nu_0)}\int_{-1}^{1}sf'(s)\phi(\pm\nu_0,s)\,ds\,\phi(\pm\nu_0,\mu).$$

For $\nu\in[-1,1]$, we let $E(\nu)$ be defined by Eq. (29). Then

$$K^n f'(\mu)=\nu_0^n E(\nu_0)f'(\mu)+(-\nu_0)^n E(-\nu_0)f'(\mu)+\int_{-1}^{1}\nu^n\, dE(\nu)f(\mu),$$

and $E(\nu)$ is the spectral family of projection operators for the operator K.

IV. HALF-RANGE THEORY

Let $\mu\psi_0(\mu)$ be defined and Hölder-continuous on $0\leq\mu\leq 1$. Define[3]

$$\psi_e(\mu)=\begin{cases}\psi_0(\mu), & 0<\mu\leq 1,\\ \int_0^1 J(\mu,s)\psi_0(s)\,ds, & -1\leq\mu<0,\end{cases} \tag{57}$$

where

$$J(\mu,s)=\frac{1}{X(\mu)X(-s)}\frac{cs}{2}\frac{1}{s-\mu}, \tag{58}$$

and

$$X(z)=\Lambda^{1/2}(\infty)(z-\nu_0)\exp\left(\frac{1}{2\pi i}\int_0^1\ln\frac{\Lambda^+(s)}{\Lambda^-(s)}\frac{ds}{s-z}\right), \tag{59}$$

$$\Lambda(z)=X(z)X(-z). \tag{60}$$

Then the full range coefficients $A(\nu)$ of $\psi_e(\mu)$ are zero for $\nu<0$;

$$\psi_e(\mu)=\int_0^1 A(\nu)\phi(\nu,\mu)\,d\nu+A(\nu_0)\phi(\nu_0,\mu). \tag{61}$$

Extending $\mu\psi_0$ to $L_p(0,1)$, we obtain by continuity the extension ψ_e of ψ_0, and $\psi_e\in X_p$, also.

By definition of $E(\omega)$, we have for $-1\leq\omega\leq 1$,

$$E(\omega)\psi_e(\mu)=\int_{-1}^{\omega}A(\nu)\phi(\nu,\mu)\,d\nu$$

$$=\begin{cases}\int_0^{\omega}A(\nu)\phi(\nu,\mu)\,d\nu, & 0<\omega\leq 1,\\ 0, & -1\leq\omega\leq 0.\end{cases} \tag{62}$$

Thus

$$\psi_e=\int_0^1 dE(\omega)\psi_e(\mu)+E(\nu_0)\phi(\nu_0,\mu). \tag{63}$$

For $0<\mu<1$, this reduces to

$$\psi_0=\int_0^1 dE(\omega)\psi_e(\mu)+E(\nu_0)\phi(\nu_0,\mu), \tag{64}$$

which is just a statement of the half-range completeness theorem.

V. SOLUTIONS OF TRANSPORT PROBLEMS

Consider the problem

$$\begin{aligned}&\frac{\partial\psi}{\partial x}+K^{-1}\psi=0, && x>0,\\ &\psi(0,\mu)=\psi_0(\mu),\quad \mu\psi_0\in L_p(0,1), && 0<\mu\leq 1.\end{aligned} \tag{65}$$

Letting $\psi_e(\mu)$ be the extension of ψ_0 described in Eq. (57), we claim that the solution of this problem is

$$\psi(x,\mu)=\int_0^1\exp(-x/\nu)\,d[E(\nu)\psi_e(\mu)]. \tag{66}$$

This function satisfies the boundary conditions, and it also satisfies the transport equation, as can be seen by inspection.

Consider next the problem

$$\begin{aligned}&\frac{\partial\psi}{\partial x}+K^{-1}\psi=q_0(x,\mu), && x_0<x<x_1,\\ &q_0(x,\mu)=q(x,\mu)/\mu\in X_p, && p>1.\end{aligned} \tag{67}$$

We will look only for a particular solution. Boundary conditions can be met by using solutions of the homogeneous equation.

We look for a particular solution of the form

$$\psi(x,\mu)=\int_{-1}^{1}d[E(\nu)\psi(x,\mu,\nu)]. \tag{68}$$

We have the identity

$$q_0(x,\mu)=\int_{-1}^{1}d[E(\nu)q_0(x,\mu)]. \tag{69}$$

Inserting Eqs. (67) and (69) in Eq. (70), we obtain

$$\int_{-1}^{1}d[E(\nu)q_0(x,\mu)]=\int_{-1}^{1}dE(\nu)\left[\frac{\partial\psi}{\partial x}(x,\mu,\nu)+\frac{1}{\nu}\psi(x,\mu,\nu)\right]. \tag{70}$$

This is solved by taking

$$q_0(x,\mu)=\frac{\partial\psi}{\partial x}(x,\mu,\nu)+\frac{1}{\nu}\psi(x,\mu,\nu), \tag{71}$$

or

$$\frac{d}{dx}\exp(x/\nu)\psi(x,\mu,\nu)=\exp(x/\nu)q_0(x,\mu). \tag{72}$$

For $\nu > 0$, we integrate from x_0 to x to get

$$\exp(x/\nu)\psi(x,\mu,\nu) - \exp(x_0/\nu)\psi(x_0,\mu,\nu) = \int_{x_0}^{x} \exp(s/\nu) q_0(s,\mu)\, ds, \quad (73)$$

so that

$$\psi(x,\mu,\nu) = \exp[(x_0 - x)/\nu]\psi(x_0,\mu,\nu) + \int_{x_0}^{x} \exp[(s-x)/\nu] q_0(s,\mu)\, ds. \quad (74)$$

For $\nu < 0$, we integrate from x_1 to x to get a similar equation:

$$\psi(x,\mu,\nu) = \exp[(x_1 - x)/\nu]\psi(x_1,\mu,\nu) + \int_{x_1}^{x} \exp[(s-x)/\nu] q_0(s,\mu)\, ds \quad (75)$$

The general solution of Eq. (67) which is bounded is then given by Eq. (68), where ψ is defined in Eqs. (74) and (75), and $\psi(x_i,\mu,\nu)$ is arbitrary. A particular solution is obtained by setting $\psi(x_i,\mu,\nu) = 0$.

VI. DISCUSSION

The Case transforms, Eqs. (7) and (8) were originally derived for Hölder-continuous functions f', and we have shown that they can be extended to functions $f' \in X_p$. Furthermore, we have constructed the spectral family for K in each of the X_p spaces, $p > 1$, and we have shown how to use this spectral family to solve typical problems. Several aspects of these results seem worthy of further comment.

First, the conditions that $f'(\mu)$ be in L_p means that the Case formulas hold for functions f' which can be highly singular at $\mu = 0$. However, this feature was shown in Sec. V to be essential in solving problems with sources, since the modified source $q_0 = \mu^{-1} q$ had to be written as a "full-range" expansion. Second, the removal of the unphysical Hölder condition on f' constitutes an obvious generalization. We emphasize that for Hölder-continuous f', Eqs. (7) and (8) hold pointwise for each μ, while, for $f' \in X_p$, Eqs. (7) and (8) hold only in the (integral) norm of X_p.

The existence of a spectral family for K was established by Hangelbroek[4] for L_2 and $c < 1$ by showing that K is topologically equivalent to a self-adjoint operator. Our results show that this family exists not only for L_2 and $c < 1$, but for the much larger spaces X_p, $p > 1$, and for any value of $c > 0$.

This physically natural space for the transport operator is X_1, but in this space interesting mathematical difficulties occur. For $p = 1$, Theorem 0 is no longer true; in fact the principal value operator is definitely unbounded.[6] Also the projection operators $E(\omega)$ for $-1 < \omega < 1$ are unbounded, so formulas such as Eq. (42) cannot hold in the usual Stieltjes sense.

We shall not consider here the problem of generalizing the Case formulas to X_1. Instead, we refer the reader to Ref. 7, in which this problem has been solved by methods different from those introduced here.

Since the solutions of the transport equations obtained in Ref. 7 include those obtained in Sec. V of the present paper as a special case, we should comment on the applicability of the present results. First, convenient orthogonality relations such as Eq. (11) and a corresponding formula for the half-range expansion coefficient are available in the Case approach, but not in the method used in Ref. 7. Second, the completeness and spectral properties of the Case eigenfunctions or closely allied techniques have been applied to a number of problems independent of simply solving the transport equation. For example, the solution of the inverse problem,[8] the rigorous derivation of the equation of invariant imbedding,[9] the justification of the multiple scattering expansion,[10] the solution of the transport equation in cylindrical or spherical geometries,[11] as well as a whole host of problems involving the solution of equations which have arisen in a number of astronomical, physical or engineering applications.[11] It is hoped that generalizing the class of functions which may be expanded in Case eigenfunctions may increase the number of possible applications such as those enumerated above.

Finally, we point out that it is of some mathematical interest to demonstrate the spectral family for an operator (not normal) for which a general spectral theorem has not been proved. Other such operators may be treated by methods similar to those used here.

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