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The interaction function and lattice duals

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An interaction function is defined for lattice models in statistical mechanics. A correlation function expansion is derived, giving a direct proof of the duality relations for correlation functions.

A general theory of duality transformations between pairs of classical spin-$\frac{1}{2}$ lattice models has been developed by Gruber and Merlini[1] and independently by Wegner.[2] The theory of Gruber and Merlini is constructive, providing explicitly a family of "dual" lattices and Hamiltonians for any given spin-$\frac{1}{2}$ system. These duals are exact, all requisite boundary terms being provided for, which is necessary in considerations of correlation functions below criticality.

We define in this article the interaction functions $u_{H^*}(A,B)$ of lattice duals G and G^*, and express them in terms of correlation functions. This gives an easy derivation of the relationship between correlation functions of a lattice and its duals. The notation in this article, while somewhat different from Ref. 1 and some current usage, has the advantage, in addition to simplifying the derivations, of generalizing to higher spin lattices.[3] The reader is referred to Ref. 1 for details on the construction of dual spin-$\frac{1}{2}$ lattices.

1. DUAL LATTICES

We suppose we are given a finite set Λ of lattice sites in a ν-dimensional space, along with a Hamiltonian H defined on the configuration of Λ. It is convenient to take as the configuration space the group $P_2(\Lambda)$ of functions from Λ to Z_2, the integers modulo 2, with group multiplication

$$fg(\lambda)=f(\lambda)+g(\lambda) \bmod 2 .$$

Considering H as a function $H:P_2(\Lambda)\to C$, its Fourier decomposition

$$H(g)=\sum_{\sigma\in\hat{G}} H_\sigma \sigma(g), \quad g\in P_2(\Lambda)$$

in terms of the elements of the character group $\hat{G}$ of $G=P_2(\Lambda)$ is just the usual decomposition of H into a sum of products of spin matrices, since the characters of G are products of characters of Z_2. Define the set of nonzero interactions

$$B=\{\sigma\in G \mid H_\sigma\neq 0\} .$$

Dual lattices are constructed with the set B. Defining $P_2(B)$ as the group of functions from B to Z_2, let p be the group homomorphism

$$p:P_2(B)\to G \quad \text{by} \quad p(f)=\prod_{\sigma\in B}\sigma^{f(\sigma)}$$

and denote its kernel by K_p. Suppose X is any set which generates K_p as a group. Then X defines a dual of Λ, with configuration space $G^*=P_2(X)$ and dual Hamiltonian H^* defined as follows. Let

$$q:B\to B^*\subset G^* \quad \text{by} \quad q(\sigma):h\to h(\sigma)$$

for $\sigma\in B$ and $h\in X\subset P_2(B)$. $q(\sigma)$ is indeed a character on G^*, and these $q(\sigma)$ are to be the nonzero interactions of the dual. The coefficients $H_{q(\sigma)}$ are given by

$$H_{q(\sigma)}=\tfrac{1}{2}\beta\log\prod_{\substack{\sigma'\in B\\ q(\sigma')=q(\sigma)}}\tanh\beta H_{\sigma'}, \tag{1}$$

and

$$H^*=\sum_{q(\sigma)} H_{q(\sigma)}\, q(\sigma) .$$

In most models of physical interest, q is one–one, except perhaps near the boundary. Thus

$$H_{q(\sigma)}=\frac{1}{2\beta}\log\tanh\beta H_\sigma$$

except near the boundary, where (1) must be used.

The partition functions $Z(\beta H)=\sum_{g\in G}\exp(-\beta H(g))$ of G and $Z(\beta H^*)=\sum_{g\in G}\exp(-\beta H^*(g))$ of its duals G^* are related then by

$$Z(\beta H)=\frac{N(K_t^*)}{N(G)}\prod_{\sigma\in B}[\sinh(-\beta H_\sigma)\cosh(-\beta H_\sigma)]^{1/2}Z(\beta H^*)$$

where $N(S)$ is the cardinality of S, and K_t^* is defined after Eq. (2).

2. THE INTERACTION FUNCTION

The correlation functions $\rho(\sigma)$ of G are defined by

$$\rho(\sigma)=Z(\beta H)^{-1}\sum_{g\in G}\exp(-\beta H(g))\sigma(g), \quad \sigma\in\hat{G}$$

with H^* replacing H for the correlation functions $\rho(\sigma^*)$ of G^*, $\sigma^*\in\hat{G}^*$. Note that $\rho(\sigma)=0$ if σ is not a product of elements of B.[1]

Define the characteristic projection $t:G^*\to P_2(B^*)$ by

$$t(g^*):\sigma\to\tfrac{1}{2}(1-\sigma(g^*)), \quad \sigma\in B^* . \tag{2}$$

The support of $t(g^*)$ is precisely those characters $\sigma\in B^*$ whose value at g^* is -1. Now if the kernel and range of t are denoted, respectively, by K_t^* and R_t^*, then the map $Q:K_p\to P_2(B^*)$ given by

$$Q(f)(q(\sigma))=f(\sigma), \quad \sigma\in B ,$$

is a group isomorphism $K_p\to R_t^*$. In particular, $f\in K_p$,

$$f(\sigma)=\begin{cases}1, & \sigma\in S,\\ 0, & \sigma\in Y-S,\end{cases} \quad \text{if and only if } Q(f)\in R_t^*,$$

$$Q(f)(q(\sigma))=\begin{cases}1, & \sigma\in S,\\ 0, & \sigma\in Y-S,\end{cases}$$

and then

$$\prod_{\sigma\in f^{-1}(1)} \tanh(-\beta H_\sigma) = \prod_{\sigma\in Q(f)^{-1}(1)} \exp(2\beta H_\sigma).$$

Let the symbol $\sum_f(S,T)$ with $S, T\subset B$ indicate that the summation [over $f\in P_2(B)$, say] is to be restricted to f satisfying $f(\sigma)=0$ if $\sigma\in S$, $f(\sigma)=1$ if $\sigma\in T$. Then the interaction function $u_{H^*}(A,C)$ is given by

$$u_{H^*}(A,C) = \sum_{f\in R_t^*}^{(A,C)} \prod_{\sigma\in f^{-1}(1)} \exp(2\beta H_\sigma)$$

for $A, C\subset B^*$.

We wish to evaluate u_{H^*} in terms of the correlation functions of G^*. Thus, suppose Y and W are any disjoint subsets of B^*. Writing $\overline{Y}$ for $\pi\sigma$, $\sigma\in Y$, etc., obtain from (2):

$$\left(\prod_{\sigma\in B^*} e^{-\beta H_\sigma}\right)^{-1} Z^*\rho(\overline{Y\cup W})$$

$$= \sum_{g\in G^*}\prod_{\sigma\in B^*} e^{-\beta H_\sigma(\sigma(g)-1)} \prod_{\sigma'\in Y\cup W}\sigma'(g)$$

$$= N(K_t^*)\sum_{f\in R_t^*}\prod_{\sigma\in f^{-1}(1)} e^{2\beta H_\sigma}\prod_{\sigma'\in Y}(-2f(\sigma')+1)$$

$$\times\prod_{\sigma''\in W}(2-2f(\sigma'')-1).$$

Now, expanding the product

$$\prod_{\sigma'\in Y}(-2f(\sigma')+1) = \sum_{L\subset f^{-1}(1)\cap Y}(-2)^{N(L)},$$

and similarly with $\Pi_{\sigma''\in W}(2-2f(\sigma'')-1)$, this becomes

$$N(K_t^*)\sum_{\substack{L\subset Y\\ M\subset W}} u_{H^*}(M,L)(-2)^{N(L)+N(M)}(-1)^{N(W)}.$$

Therefore, with a change in summation variable,

$$\left(\prod_{\sigma\in B^*} e^{-\beta H_\sigma}\right)^{-1} Z^*\sum_{\substack{Y\subset C\\ W\subset A}}(-1)^{N(Y)}\rho(\overline{Y\cup W})$$

$$= N(K_t^*)\sum_{\substack{L\subset C\\ M\subset A}}\sum_{\substack{Z\subset C-L\\ V\subset A-M}} u_{H^*}(M,L)2^{N(L)+N(M)}(-1)^{N(Z)+N(V)}$$

$$= 2^{N(A)+N(C)}N(K_t^*)u_{H^*}(A,C), \tag{3}$$

which gives the desired expression.

3. DUAL CORRELATION FUNCTIONS

The interaction functions can be used to derive directly the duality relations for correlation functions. Let $Y\subset B$. Then, using

$$\exp(-\beta H_\sigma(g)) = \cosh(-\beta H_\sigma) + \sigma(g)\sinh(-\beta H_\sigma)$$

and the orthonormality of the characters,

$$Z\rho(\overline{Y}) = N(G)\prod_{\sigma\in B}\cosh(-\beta H_\sigma)\sum_{\substack{f\in P_2(B)\\ p(f)=Y}}\prod_{\sigma'\in f^{-1}(1)}\tanh(-\beta H_{\sigma'}).$$

From the one-one correspondence between $f\in P_2(B)$ with $p(f)=\overline{Y}$ and $f'\in K_p$ with

$$f':\sigma\to\begin{cases} f(\sigma), & \text{if } \sigma\notin Y,\\ f(\sigma)+1, & \text{if } \sigma\in Y,\end{cases}$$

the expansion can be written as

$$Z\left(N(G)\prod_{\sigma\in B}\cosh(-\beta H_\sigma)\right)^{-1}\rho(\overline{Y})$$

$$= \sum_{S\subset Y}\sum_{\substack{f\in P_2(B)\\ p(f)=\overline{Y}}}^{(S,Y-S)}\prod_{\sigma'\in f^{-1}(1)}\tanh(-\beta H_{\sigma'})$$

$$= \sum_{S\subset Y}\sum_{f\in K_p}^{(Y-S,S)}\prod_{\sigma'\in f^{-1}(1)}\tanh(-\beta H_{\sigma'})\prod_{\sigma\in S}[\tanh(-\beta H_\sigma)]^{-1}$$

$$\times\prod_{\sigma\in Y-S}\tanh(-\beta H_\sigma)$$

$$= \sum_{\substack{S\subset Y\\ S^*\cap(Y-S)^*=\phi}} u_{H^*}(Y^*-S^*,S^*)\prod_{\sigma\in S}(\tanh(-\beta H_\sigma))^{-1}$$

$$\times\prod_{\sigma\in Y-S}\tanh(-\beta H_\sigma),$$

where it has been necessary to consider in the sum over S only sets $S\subset Y$ for which $S^*=\{q(\sigma)\,|\,\sigma\in S\}$ and $(Y-S)^*$ are disjoint. Thus, the interaction function expansion (3) gives the general relation between the correlation functions of G and the correlation functions of a dual G^*,

$$\rho_G(\overline{Y}) = \sum_{T^*\subset W^*}\rho_{G^*}(\overline{T}^*)K(W,T^*), \tag{4}$$

where

$$K(W,T^*) = 2^{-N(W^*)}\sum_{\substack{S\subset W\\ S^*\cap(W-S)^*=\phi}}(-1)^{N(S^*\cap T^*)}$$

$$\times\prod_{\sigma\in S}(\tanh(-\beta H_\sigma))^{-1}\prod_{\sigma\in W-S}\tanh(-\beta H_\sigma)$$

for any $W\subset B$ such that $\overline{W}=\overline{Y}$.

In the event that the duality map q is one-one, Eq. (4) simplifies to the path formula of Kadanoff and Ceva.[4] Injectivity of q is equivalent to requiring that the elements of K_p separate the bonds σ of B, and is satisfied, for example, by a hexagonal Ising lattice with periodic boundary conditions, or with an external field at the boundary, but is not satisfied by this lattice with open boundary conditions.

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