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Solution of the multigroup transport equation in L^p spaces

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The isotropic multigroup transport equation is solved in L^p, $p>1$, for both half range and full range problems, using resolvent integration techniques. The connection between these techniques and a spectral decomposition of the transport operator is indicated.

I. INTRODUCTION

Since Larsen and Habetler introduced a resolvent integral technique to solve the one-dimensional one-speed isotropic linear transport equation,[1] this method has been extended to study a variety of problems. In particular, Bowden, Sancaktar, and Zweifel have obtained a solution of the multigroup problem in Hilbert space,[2,3] and Larsen, Sancaktar, and Zweifel have extended the one-group results to L^p spaces.[4]

The purpose of this note is to indicate how these ideas can be combined to obtain a solution of the isotropic multigroup equation in L^p, $p>1$, for both half range and full range problems. The analysis demonstrates that the problem is reduced largely to estimating some relevant operator norms in the solution space $L^p(\mathbf{I})$ and in the spectral decomposition space $L^p(N,\sigma)$. These estimates are carried out in Lemmas 2–8, and lead to the representation theorem, Theorem 9.

We may point out that the elegant spectral analysis of Hangelbroek[5] to this problem does not appear to afford an alternate approach, except for the two-group, since, with the exception noted, it is not possible to symmetrize the production matrix C and simultaneously maintain the scattering matrix Σ diagonal. In Theorem 10 and the discussion preceding it, we indicate the connection between the von Neumann spectral theory utilized by Hangelbroek and the resolution of the identity obtained from the resolvent integrations.

Finally, Theorem 11 deals with the application of these results to half space theory.

II. THE MULTIGROUP PROBLEM

Let us define the Banach space $X_p(\mathbf{I})$ to be the space of (equivalence classes of) Lebesque measurable vector valued functions $\mathbf{f}$ from the real interval $\mathbf{I}=[-1,1]$ to $\mathbb{C}^n$ with norm

$$\|\mathbf{f}\|_p=\left\{\sum_{i=1}^{n}\int_{-1}^{1}d\mu\,|\mu\mathbf{f}_i(\mu)|^p\right\}^{1/p}.$$

We distinguish the subspace of constant vectors X_p^c to be functions $\mathbf{f}\in X_p$ such that, for each i, $1\le i\le n$, $\mathbf{f}_i(u)$ is independent of μ. In particular let

$$(\mathbf{e}_{(j)})_i(\mu)=\delta_{ij}$$

for $1\le j\le n$. Then $\mathbf{f}\in X_p^c$ precisely if there are constants a_j, $1\le j\le n$, such that

$$\mathbf{f}=\sum_{j=1}^{n}a_j\mathbf{e}_{(j)}.$$

On X_p^c an inner product may be defined:

$$[\mathbf{f},\mathbf{g}]=\sum_{i=1}^{n}\mathbf{f}_i\bar{\mathbf{g}}_i,\quad \mathbf{f},\mathbf{g}\in X_p^c$$

By a solution of the (full range) multigroup transport equation is meant a differentiable function $\psi:\mathbb{R}\to X_p(\mathbf{I})$ satisfying

$$\mu\frac{\partial}{\partial x}\psi(x)=-\Sigma\psi(x)+C\int_{-1}^{1}d\mu'\,\psi(x,\mu')+\mathbf{q}(x)\qquad(1)$$

where Σ is an $n\times n$ diagonal matrix with positive entries, C is any $n\times n$ matrix with nonnegative entries, μ indicates multiplication by the independent variable in $X_p(\mathbf{I})$,

$$(\mu\mathbf{f})_i(\mu)=\mu\mathbf{f}_i(\mu),$$

and $\mathbf{q}$ is the inhomogeneous source term, which we assume to be a Hölder continuous function $\mathbf{q}:\mathbb{R}\to X_p(\mathbf{I})$. We have written $\psi(x,\mu')$ for $\psi(x)$ evaluated at μ', and in the remainder, we will omit the x dependence altogether, writing $\psi(\mu')$. The solution of Eq. (1) is also understood to satisfy specified boundary conditions, typically $\|\psi(x)\|_p\to 0$ as $x\to\pm\infty$.

The transport operator, or more correctly, the reduced transport operator, K, is the bounded linear transformation on $X_p(\mathbf{I})$,

$$Kf=\Sigma^{-1}\mu f+\Sigma^{-1}C(\Sigma-2C)^{-1}\int_{-1}^{1}d\mu'\,\mu' f(\mu')$$

and its (unbounded) inverse is

$$(K^{-1}\mathbf{f})(\mu)=(1/\mu)\Sigma\mathbf{f}(\mu)-(1/\mu)C\int_{-1}^{1}d\mu'\,\mathbf{f}(\mu').$$

We may assume, without loss of generality, that $\Sigma_{ii}\ge 1$, $1\le i\le n$, and $\|\Sigma^{-1}\|=1$, and we shall do so. It is also necessary to make the noncriticality assumption[10]

$$\det(\Sigma-2C)\ne 0.$$

The spectrum N of K as an operator on $X_p(\mathbf{I})$ consists of the interval $\mathbf{I}$, which is continuous spectrum, and of point spectrum $\sigma_p(K)$.

In Ref. 2, the Case transform $\mathbf{F}:\mathbf{f}\to\mathbf{A}$ is derived for $\mathbf{f}$ Hölder continuous, where

$$F(\mathbf{f})(\nu)=\begin{cases}\dfrac{1}{2\pi i\nu}\left\{\Lambda^{-1}(\nu)\displaystyle\int_{-1}^{1}ds\, s(\nu I-s\Sigma^{-1})^{-1}\mathbf{f}(s)\right\}^{+} \\ -\left\{\Lambda^{-1}(\nu)\displaystyle\int_{-1}^{1}ds\, s(\nu I-s\Sigma^{-1})^{-1}\mathbf{f}(s)\right\}^{-}, \quad \nu\in \mathbf{I}, & (2a) \\ \dfrac{1}{\Omega'(\nu)}\left[\displaystyle\int_{-1}^{1}ds\; s(\nu I-s\Sigma^{-1})^{-1}\mathbf{f}(s),\boldsymbol{\alpha}\right]\boldsymbol{\beta}_\nu,\ \nu\in\sigma_p(K). & (2b)\end{cases}$$

Here the dispersion matrix $\Lambda(z)=B+T(z)$ and its determinant $\Omega(z)=\det\Lambda(z)$ are given by

$$B=(\Sigma-2C)C^{-1}\Sigma,\quad T(z)=-\int_{-1}^{1}ds\; s(zI-s\Sigma^{-1})^{-1}, \tag{3}$$

and the superscripts $\pm$ indicate boundary values obtained as z converges to Rez from above (below) the real axis. The vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}_\nu$ are defined as follows. Since $\nu\in\sigma_p(K)$ if and only if $\Omega(\nu)=0$, let us take $\boldsymbol{\beta}_\nu$ to satisfy

$$\Lambda(\nu)\boldsymbol{\beta}_\nu=0 \tag{4}$$

for each $\nu\in\sigma_p(K)$. Then for any $\nu\in\sigma_p(K)$, we may define $\boldsymbol{\alpha}$ by

$$\boldsymbol{\alpha}=\Lambda_c(\nu)\boldsymbol{\beta}_\nu$$

where $(\Lambda_c)_{ij}=\mathrm{cof}_{ij}\Lambda$ differs from the notation in Ref. 2 by a transpose.

In the above it has been assumed that $\Omega^{\pm}(\nu)$ does not vanish on the interval $\mathbf{I}=[-1,1]$ and that $\nu\in\sigma_p(K)$ has multiplicity one. We shall also assume that

$$\Gamma(\nu)=\begin{cases}\tfrac{1}{2}(\Lambda(\nu)^{+}+\Lambda(\nu)^{-}), & \nu\in\mathbf{I},\\ 1, & \nu\in\sigma_p(K)\end{cases} \tag{5}$$

does not vanish on $\mathbf{I}$, although we do believe that all of these restrictions could be removed without difficulty (see, for example, the treatment of a similar problem in Ref. 6.)

The importance of the Case transform lies in the completeness theorem and in its "spectral" behavior under K. Namely, if

$$\Phi(\nu,\mu)=\quad P\!\!\!\!\int \nu(\nu I-\mu\Sigma^{-1})^{-1}+\Delta(\Sigma\nu-\mu)\Sigma^{-1}\Gamma(\nu),\quad \nu\in\mathbf{I}, \tag{6a}$$

$$\Phi(\nu,\mu)=\quad(\nu I-\mu\Sigma^{-1})^{-1},\qquad \nu\in\sigma_p(K), \tag{6b}$$

where $P\!\!\!\!\int$ indicates a principal value integral is to be taken and

$$\Delta(\Sigma\nu-\mu)_{jk}=\delta_{jk}\delta(\sigma_j\nu-\mu),$$

then

$$\mathbf{f}=\int_N\Phi(\nu,\mu)\mathbf{A}(\nu)\,d\sigma(\nu) \tag{7}$$

for $\mathbf{A}=F(\mathbf{f})$ defined in Eqs. (2), and $\sigma(\nu)$ Lebesque measure on $\mathbf{I}$, $\sigma(\nu)=1$ for $\nu\in\sigma_p(K)$. Moreover,

$$K\mathbf{f}=\int_N\nu\Phi(\nu,\mu)\mathbf{A}(\nu)\,d\sigma(\nu).$$

Thus if we define F': $\mathbf{A}\to\mathbf{f}$ by

$$F'(\mathbf{A})(\mu)=\int_N\Phi(\nu,\mu)\mathbf{A}(\nu)\,d\sigma(\nu)$$

for $\mathbf{A}$ Hölder continuous, then Ref. 2 proves the following theorem.

Theorem 1: On Hölder continuous functions in $X_p(\mathbf{I})$,

$F'F=I$ and $F'\nu F=K$.

III. OPERATOR BOUNDS

Equation (2a) makes sense pointwise if f is Hölder continuous; in order to extend F to all of $X_p(\mathbf{I})$, we introduce the Banach space $X_p(N)$, where $\mathbf{A}\in X_p(N)$ if $\mathbf{A}$ is Lebesque measurable on $I\subset N$, $\mathbf{A}$ is proportional to $\boldsymbol{\beta}_\nu$ at each $\nu\in\sigma_p(K)$, and

$$\|\mathbf{A}\|_{p,\Gamma}\equiv\left\{\sum_{i=1}^{n}\int_N|\nu\Gamma(\nu)\mathbf{A}_i(\nu)|^p\,d\nu\right\}^{1/p}<\infty.$$

In other words,

$$\|\mathbf{A}\|_{p,\Gamma}=\left\{\sum_{\nu\in\sigma_p(K)}\sum_{i=1}^{n}|\mathbf{A}_i(\nu)|^p+\|\Gamma\mathbf{A}\|_p\right\}^{1/p}.$$

For a proper extension to all of $X_p(\mathbf{I})$ then, it is sufficient to prove:

(i) F:$H_p(\mathbf{I})\to X_p(N)$ is a bounded, densely defined operator, where

$$H_p(\mathbf{I})=\{f\in X_p(\mathbf{I})f \text{ Hölder continuous on } \mathbf{I}\}; \tag{8a}$$

(ii) F':$H_p(N)\to X_p(\mathbf{I})$ is bounded, where

$$H_p(N)=\{A\in X_p(N)\,|\,f \text{ Hölder continuous on } \mathbf{I}\} \tag{8b}$$

(iii) Ran$F$ is dense in $X_p(N)$.

As there has been, in our opinion, some continuing confusion in the literature over these rather simple observations, we reiterate the following. In the transformed space $X_p(N)$, the transport operator K acts simply as a multiplication operator. Hence transport problems can be related to problems involving the much simpler, and necessarily normal, multiplication operator. However, unless Ran$F$ is demonstrated to be dense in $X_p(N)$, there is no assurance that the solution of a transport problem solved in $X_p(N)$ will be the image under F of a vector in $X_p(\mathbf{I})$. This is, of course, equally true for the one group. If we consider, for example, the uniform slab problem, where the function A is given in Ref. 7 implicitly as the solution of a Fredholm integral equation, then unless A is known to be contained in Ran$F$, it cannot be assumed that $F'A=\phi$ satisfies $F\phi=A$, and hence that it is the desired solution of the slab problem. Note also that the boundedness of F and its invertibility on a dense set is not sufficient to deduce the invertibility of F on $X_p(\mathbf{I})$, unless it has been established that F' is bounded.

The analysis of Ref. 2 hinges on the following theorem concerning Hilbert transforms, which we quote in a form useful for our purposes.[8]

Lemma 2: Let $\mathbf{f}\in X_p(\mathbf{I})$. Then the formula

$$\mathbf{g}(\mu)=P\!\!\!\!\int_{-1}^{1}s(\mu I-s\Sigma^{-1})^{-1}\mathbf{f}(s)\,ds$$

defines almost everywhere a function $\mathbf{g}$ also belonging to $X_p(\mathbf{I})$, and for a constant M_p depending only upon p and $\|\Sigma^{-1}\|$,

$\|\mathbf{g}\|_p \leq M_p \|\mathbf{f}\|_p$.

Before proceeding to study F and F', we collect some important properties of the dispersion matrix.

Lemma 3:

(i) On $X_p(\mathbf{I})$,

$$\Lambda(\nu)^+ - \Lambda(\nu)^- = -2\pi i \nu \Sigma^2 \Delta_\Sigma(\nu), \tag{9a}$$

where $\Delta_\Sigma(\nu)_{ij} = 1$ if $i=j$ and $|\nu| \leq \sigma_i$, zero other-otherwise.

(ii) $\Gamma(\nu)$ is continuously differentiable on $\mathbf{I}/\mathbf{T}$, where $\mathbf{T} = \{\pm \sigma_i^{-1}\}_{i=1}^n$, and $\Gamma^{-1}(\nu)$ defines a bounded operator Γ^{-1} on $X_p(\mathbf{I})$.

(iii) On $X_p(\mathbf{I})$,

$$\Lambda^{-1}(\nu)^+ - \Lambda^{-1}(\nu)^- = 2\pi i \Lambda^{-1}(\nu)^+ \Sigma^2 \Delta_\Sigma(\nu) \Lambda^{-1}(\nu)^- \tag{9b}$$

is bounded, and on its range, Γ is bounded.

(iv) On $X_p(\mathbf{I})$,

$$\Lambda^{-1}(\nu)^+ - \Lambda^{-1}(\nu)^- = -2\Lambda^{-1}(\nu)^+ \Gamma(\nu) \Lambda^{-1}(\nu)^- \tag{9c}$$

is bounded, and on its range, Γ is bounded.

Proof: Let us consider (iii)—(iv) first. If $T: \mathbf{I} \to \mathbb{C}^n$ is a continuous map, then $\hat{T}: X_p(\mathbf{I}) \to X_n(\mathbf{I})$ given by

$$(\hat{T}f)(\mu) = T(\mu)\mathbf{f}(\mu)$$

is bounded. Therefore, the problem reduces to studying $\Lambda(\nu)$ on X_p^c for fixed ν in a neighborhood of the "endpoints" $\mathbf{T}$.

Let $b = +\sigma_i^{-1}$ or $-\sigma_i^{-1}$ be such an endpoint, and let $\lim_{\nu \to b\pm}$ indicate a limit taken along $\nu = b \pm i\epsilon$ with $\epsilon \to 0^+$. Suppose $M_b^\pm = N_b \cap \mathbb{C}_\pm$ with N_b a neighborhood of b such that $\overline{N_b}$ contains no other endpoints, and $\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \pm \operatorname{Im} z \geq 0\}$. Since $\lim_{\nu \to b\pm} \|\Lambda(\nu)e_i\| = \infty$, we claim that

$$\lim_{\nu \to b\pm} \Lambda^{-1}(\nu)e_i = 0.$$

For,

$$\Lambda(\nu)e_i = \boldsymbol{\alpha}_i(\nu)e_i + Be_i$$

$$= (\boldsymbol{\alpha}_i(\lambda) + \tfrac{1}{2}[Be_i, e_i])e_i + (I - P_i)Be_i$$

where $P_i: \psi \to \frac{1}{2}[\psi, e_i]e_i$ is the projection onto e_i and $\alpha_i(\nu) \to \pm\infty$, and therefore, since $\Lambda^{-1}(\nu)(I - P_i)$ is a continuous function of ν for $\nu \in M_b^\pm$, we see that

$$\Lambda^{-1}(\nu)e_i = (\boldsymbol{\alpha}_i(\lambda) + \tfrac{1}{2}[Be_i, e_i])^{-1}$$

$$\times (e_i - \Lambda^{-1}(\nu)(I - P_i)Be_i) \to 0.$$

In other words,

$$\mathrm{Sp}\{e_i\} \subset \operatorname{Ker} \Lambda^{-1}(b)^\pm. \tag{10}$$

$\Lambda(\nu)^\pm$ is invertible and, as we have noted, a continuous function of $\nu \in M_b^\pm$ on the subspace $(I - P_i)X_p^c$. Since X_p^c is finite dimensional, $X_p^c = \operatorname{Ran}\Lambda(b) + \mathrm{Sp}\{e_i\}$, and this along with Eq. (10) enables us to conclude that $\Lambda^{-1}(b) = \lim_{\nu \to b\pm} \Lambda^{-1}(\nu)$ on X_p^c. Then, Eq. (10) and the continuity and boundedness of $\Lambda(\nu)^+ + \Lambda(\nu)^-$ on $(I - P_i)X_p$ for $\nu \in M_b^\pm$ gives us directly that

$$(\Lambda(\nu)^+ + \Lambda(\nu)^-)\Lambda^{-1}(\nu)^+(\Lambda(\nu)^+ \pm (\Lambda(\nu)^-)\Lambda^{-1}(\nu)^-$$

is also a bounded operator on X_p.

Equation (9a) results immediately from the Plemelj formulas applied componentwise to Eq. (3). Since $\Lambda(\nu)^\pm$ is analytic off the real axis and continuous on $[-1,1]\setminus\mathbf{T}$, the integration along $[-1,1]$ between neighboring endpoints $b_1, b_2 \in \mathbf{T}$ may be replaced by integration along the complex contours $\Gamma(\theta) = \frac{1}{2}(b_1 + b_2 + (b_1 - b_2)e^{i\theta})$, $\theta \in [0, \pm\pi]$. From this the continuity properties of Γ may be deduced. Finally, we note that the analysis of $\Lambda^{-1}(\nu)$ may also be applied to $\Gamma(\nu) = \Lambda(\nu)^+ + \Lambda(\nu)^-$ to obtain the existence of $\Gamma(\nu)$.

Corollary 4: $H_p(N)$ is dense in $X_p(N)$.

Lemma 5: $F | H_p(\mathbf{I})$ is bounded.

Proof: Using the Plemelj formulas for $\nu \in \mathbf{I}$,

$$\nu F(f)(\nu) = \frac{1}{2\pi i}(\Lambda^{-1}(\nu)^+ - \Lambda^{-1}(\nu)^-) \int_{-1}^{1} ds\, s(\nu I - s\Sigma^{-1})^{-1}\mathbf{f}(s)$$
$$+ \tfrac{1}{2}(\Lambda^{-1}(\nu)^+ + \Lambda^{-1}(\nu)^-)\Sigma^2 \nu (V_\Sigma \mathbf{f})(\nu), \tag{11}$$

where V_Σ is defined by

$$(V_\Sigma \mathbf{f})_i(\mu) = \begin{cases} \mathbf{f}_i(\mu\sigma_i), & |\mu| \leq 1/\sigma_i, \\ 0, & |\mu| > 1/\sigma_i. \end{cases}$$

By applying Γ to both sides of the equation, and integrating pth powers of each term over $\mathbf{I}$, $\|F(\mathbf{f})\|_{p,\Gamma}$ may be estimated by a sum of norms. Thus, the norm obtained from the first term on the right-hand side of the equation is bounded by $\|\Gamma(\Lambda^{-1})^+\Sigma^2\Delta_\Sigma(\Lambda^{-1})^-\| M_p \|\mathbf{f}\|_p$, and the second by $\|\Sigma\|^{1-1/p}\|\Gamma(\Lambda^{-1})^+\Gamma(\Lambda^{-1})^-\|\,\|\mathbf{f}\|_p$ since $\|\Delta_\Sigma\| = 1$ and $\|V_\Sigma\| = \|\Sigma^{-1-1/p}\|$. Then the contribution to $\|F(\mathbf{f})\|_{p,\Gamma}$ of the continuous spectrum is

$$(\|F(\mathbf{f})\|_{p,\Gamma})_{\sigma_i} \leq \{\|\Gamma(\Lambda^{-1})^+\Sigma^2\Delta_\Sigma(\Lambda^{-1})^-\| M_p$$
$$+ \|\Sigma\|^{1-1/p}\|\Gamma(\Lambda^{-1})^+\Gamma(\Lambda^{-1})^-\|\}\|\mathbf{f}\|_p.$$

If $\nu \in \sigma_p(K)$, then for q satisfying $1/p + 1/q = 1$, the Hölder inequality gives

$$\left|\left[\int_{-1}^{1} ds\, s(\nu I - s\Sigma^{-1})^{-1}\mathbf{f}(s), \boldsymbol{\alpha}\right]\right| \leq \|\mathbf{f}\|_p$$

$$\times \left\{\sum_i \int_{-1}^{1} ds \left|\frac{\alpha_i}{\nu - s\sigma_{ii} - 1}\right|^q\right\}^{1/q} \leq 2^{1/q} \sup_{\nu \in \sigma_p(K)} \frac{1}{d(\nu)} \|\boldsymbol{\alpha}\|_{[q]},$$

where

$$d(\nu) = \inf_{s \in \mathbf{I}} |\nu - s|$$

and $\|\ \|_{[q]}$ is the q-norm on $\mathbb{C}^n$,

$$\|\boldsymbol{\xi}\|_{[q]} = \left\{\sum_{i=1}^{n} |\boldsymbol{\xi}_i|^q\right\}^{1/q}, \quad \boldsymbol{\xi} \in \mathbb{C}^n.$$

Thus, the contribution to $\|F(\mathbf{f})\|_{p,\Gamma}$ of the point spectrum is

$$(\|F(f)\|_{p,\Gamma})_{\sigma_p} \leq \sup_{\nu' \in \sigma_p(K)} \frac{1}{d(\nu')} 2^{1-1/p} \|\boldsymbol{\alpha}\|_{[q]}$$

$$\times \Big(\sum_{\nu \in \sigma_p(K)} \|\boldsymbol{\beta}_\nu\|^p_{[q]} \Big)^{1/p} \|\mathbf{f}\|_p .$$

This completes the proof.

Lemma 6: $F' | H_p(N)$ is bounded.

Proof: We have from Eqs. (6) and (7):

$$\mu(F^{-1}\mathbf{A})(\mu) = \mu \oint_{-1}^{1} \nu(\nu I - \mu\Sigma^{-1})^{-1}\mathbf{A}(\nu)\, d\nu + \mu\Sigma^{-2}(V_{\Sigma^{-1}}\Gamma\mathbf{A})(\mu)$$

$$+ \sum_{\nu \in \sigma_p(K)} \mu(\nu I - \mu\Sigma^{-1})^{-1}\mathbf{A}(\nu). \qquad (12)$$

The norm of the first term on the right-hand side is bounded by

$$M_p \|\mathbf{A}\|_p \leq M_p \|\Gamma^{-1}\| \; \|\Gamma\mathbf{A}\|_p = M_p \|\Gamma^{-1}\| \; \|A\|_{p,\Gamma}$$

and the second term by

$$\|\Sigma^{1/p-1}\| \; \|\mathbf{A}\|_{p,\Gamma}.$$

The third term may be estimated by

$$\left\{ \sum_{\nu \in \sigma_p(K)} \sum_{i=1}^{n} \int_{-1}^{1} d\mu \left| \frac{\mu}{\nu^2 - \mu\nu\sigma_i^{-1}} \right|^p |\nu \mathbf{A}_i(\nu)|^p \right\}^{1/p}$$

$$\leq \left(\frac{2}{p+1} \right)^{1/p} \sup_{\nu \in \sigma_p(K)} \frac{1}{\nu\, d(\nu)} \|\mathbf{A}\|_{p,\Gamma}.$$

Lemma 7: Let $J_p(N) = \{A \in H_p(N) \,|\, \Gamma A \in H_p(N)\}$. Then $F: H_p(\mathbf{I}) \to H_p(N)$ and $F': J_p(N) \to H_p(\mathbf{I})$.

Proof: Since the Cauchy integral of a Hölder continuous function is Hölder continuous on the interior of a Liapunov contour,[9] the only potential difficulty is at the boundary points $\pm 1 \backslash \sigma_i$. A typical term in the expression for $F(f)$ is

$$\sum_{k=1}^{n} \Lambda^{-1}(\nu)^{\pm} \int_{-1}^{1} ds (\nu I - s\Sigma_{kk}^{-1})^{-1} f_k(s) \mathbf{e}_k,$$

which is explicitly Hölder continous at $\nu = 1 \backslash \sigma_i$ except possibly for $k = i$. From Eq. (10), however,

$$\Lambda^{-1}(1/\sigma_i)^{\pm}\mathbf{e}_i = 0.$$

The second part of the lemma may be proved immediately from Eq. (12).

Lemma 8: Ran$F$ is dense in $H_p(N)$.

Proof: We first wish to reduce the transformation F between Banach spaces $X_p(N)$ and $X_p(\mathbf{I})$ by subspaces corresponding to the eigenspace of K and appropriate topological supplements. Thus, let us define the bounded linear forms ρ_ν on $H_p(\mathbf{I})$ by

$$\rho_\nu(f) = \Big[\int_{-1}^{1} ds\, s(\nu I - s\Sigma^{-1})^{-1}\mathbf{f}(s), \boldsymbol{\alpha} \Big]$$

for $\nu \in \sigma_p(K)$, and let $H_p(\mathbf{I})$ be the submanifold

$$H_p(\mathbf{I}) = \{ f \in H_p(\mathbf{I}) \,|\, \rho_\nu(f) = 0, \quad \nu \in \sigma_p(K) \}.$$

If $\boldsymbol{\gamma}_\nu$ is an eigenvector of K with eigenvalue ν, then

$$\rho_\nu(\boldsymbol{\gamma}_{\nu'}) = \delta_{\nu\nu'}\rho_\nu(\alpha_\nu).$$

Since we may take

$$\boldsymbol{\gamma}_\nu(\mu) = (\nu I - \Sigma^{-1}\mu)^{-1}\boldsymbol{\beta}_\nu,$$

whence, applying the identity

$$(\nu I - s\Sigma^{-1})^{-1}(\nu' I - s\Sigma^{-1})^{-1} = [1/(\nu' - \nu)](\nu - s\Sigma^{-1})^{-1} \; (\nu' - s\Sigma^{-1})^{-1} \qquad (13)$$

for $\nu \neq \nu'$ and Eq. (3), the form on $\boldsymbol{\gamma}_{\nu'}$ becomes

$$\rho_\nu(\boldsymbol{\gamma}_{\nu'}) = \int_{-1}^{1} ds \frac{s}{\nu' - \nu} [\{(\nu I - s\Sigma^{-1})^{-1} - B + \Lambda(\nu')\}\boldsymbol{\beta}_{\nu'}, \boldsymbol{\alpha}]$$

$$= -\frac{1}{\nu' - \nu}[\Lambda(\nu)\boldsymbol{\beta}_{\nu'}, \boldsymbol{\alpha}] = -\frac{1}{\nu' - \nu} \sum_i (\Lambda(\nu)\Lambda_c^t(\nu))_{ii} = 0. \qquad (14)$$

We have used the fact that $\Lambda\Lambda_c^t = 0$.

If the projection P is defined on $X_p(\mathbf{I})$ by

$$P\mathbf{f} = \sum_{\nu \in \sigma_p(K)} (\rho_\nu(\mathbf{f})/\rho_\nu(\boldsymbol{\gamma}_\nu))\boldsymbol{\gamma}_\nu$$

and if P denotes the projection onto the subspace

$$X_p(N)_0 = \{ \mathbf{A} \in X_p(N) \,|\, \mathbf{A}(\nu) = 0, \quad \nu \in \mathbf{I} \}$$

of $X_p(N)$ along the subspace

$$X_p(N)_1 = \{ \mathbf{A} \in X_p(N) \,|\, \mathbf{A}(\nu) = 0, \quad \nu \in \sigma_p(K) \}$$

then we assert that

$$(I - P)F' = F'(I - \hat{P}).$$

To demonstrate this, we must compute the integrals

$$\int_{-1}^{1} ds\, s(\nu I - s\Sigma^{-1})^{-1} \oint_{-1}^{1} dt\, (tI - s\Sigma^{-1})^{-1}\mathbf{A}(t)\, dt +$$

$$+ \int_{-\|\Sigma\|}^{\|\Sigma\|} ds\, s(\nu I - s\Sigma^{-1})^{-1}\Sigma^{-2}V_{\Sigma^{-1}}\mathbf{A}(s)$$

for $\nu \in \sigma_p(K)$ and $\mathbf{A} \in X_p(N)_1 \cap H_p(N)$. With the identity (13) and some rearrangements, these become

$$\int_{-1}^{1} dt (t - \nu)^{-1}(-\Sigma^{-1}\Lambda(\nu) + \Sigma^{-1}\Gamma(t))\mathbf{A}(t)$$

$$- \int_{-1}^{1} dt\, t(t - \nu)^{-1}\Sigma^{-1}\Gamma(t)\mathbf{A}(t).$$

Hence, for $\mathbf{f} \in X_p(N)_1$,

$$\rho_\nu(F'\mathbf{f}) = \int_{-1}^{1} dt (t - \nu)^{-1}[\Sigma^{-1}\Lambda(\nu)\mathbf{A}(t), \boldsymbol{\alpha}],$$

which vanishes by the same reasoning as for Eq. (14).

Now we are prepared to determine Ran$F$. Since $F': \boldsymbol{\beta}_\nu \to PX_p(\mathbf{I})$, $\nu \in \sigma_p(k)$, and $F': X_p(N)_1 \to (I - P)X_p(\mathbf{I})$, it is sufficient to prove F' is one–one on $X_p(N)_1$. Thus, let us suppose $F'(\mathbf{A}) = 0$, $\mathbf{A} \in X_p(N)_1 \cap H_p(N)$, and define

$$(V_{\Sigma^{-1}}N)(z) = \int_{-1}^{1} \nu(\nu I - z\Sigma^{-1})^{-1}\mathbf{A}(\nu)\, d\nu.$$

Then the Plemelj formulas give

$$\tfrac{1}{2} V_{\Sigma^{-1}}(\mathbf{N}^+ + \mathbf{N}^-)(z) = \oint_{-1}^{1} \nu(\nu - z\Sigma^{-1})^{-1}\mathbf{A}(\nu)\, d\nu$$

and

$$\frac{1}{2\pi i \mu}(\mathbf{N}^+ - \mathbf{N}^-)(\mu) = \mathbf{A}(\mu), \quad \mu \in \mathbf{I}.$$

Substituting these expressions into

$$\oint_{-1}^{1} \nu(\nu - \mu\Sigma^{-1})^{-1}\mathbf{A}(\nu)\, d\nu + \Sigma^{-2}(V_{\Sigma^{-1}}\Gamma\mathbf{A})(\mu) = 0,$$

we obtain

$$\pi i \Sigma^2 V_{\Sigma^{-1}}\mu(\mathbf{N}^+ + \mathbf{N}^-) + V_{\Sigma^{-1}}\Gamma(\mathbf{N}^+ - \mathbf{N}^-) = 0,$$

which, with Eq. (9a) and appropriate cancellations, becomes

$$V_{\Sigma^{-1}}(\Lambda^+\mathbf{N}^+ - \Lambda^-\mathbf{N}^-) = 0.$$

Hence, by Liouville's Theorem applied to $\mathbf{J}(z) = \Lambda(z)\mathbf{N}(z)$, we conclude that $\mathbf{A}(\nu) \equiv 0$.

IV. SPECTRAL THEOREM

Lemmas 5, 6, and 8, along with the results of Theorem 1, have as an immediate consequence the following theorem.

Theorem 9: The transformation $F: X_p(\mathbf{I}) \to X_p(N)$ is an invertible bounded linear transformation, and $F^{-1} = F'$. Moreover,

$$FK = \nu F$$

is valid on $X_p(\mathbf{I})$.

We emphasize that Theorem 9, by diagonalizing the bounded operator K, provides effectively a spectral representation of K. This is most transparent in Hilbert space language ($p = 2$), where a new inner product may be introduced on $X_2(\mathbf{I})$,

$$\{f, g\} = (Ff, Fg)_{2,\Gamma},$$

Here, $(\cdot,\cdot)_{2,\Gamma}$ indicates the inner product on $X_2(N)$ derived from the norm $\|\ \|_{2,\Gamma}$. Then if $N \subset \mathbb{R}$,

$$\{Kf, g\} = (FKf, Fg)_{p,\Gamma} = (\nu Ff, Fg)_{p,\Gamma} = (Ff, \nu Fg)_{p,\Gamma} = \{f, Kg\}$$

whence K is self-adjoint, and a similar calculation shows K is normal for $N \subset \mathbb{C}$. Furthermore,

$$FK^n = \nu FK^{n-1} = \circ\circ\circ = \nu^n F,$$

so, since N is necessarily compact, the map

$$\kappa: K^n \to \nu^n$$

extends to the Gelfand transformation from the C^* algebra generated by K and K^* to the algebra of continuous functions on N with uniform norm. (Actually, by Mergelyan's Theorem, C^* algebra is generated by K alone, even when K is not self-adjoint.[11])

These remarks can equally well be expressed in terms of a spectral resolution for K. Recalling that the Dunford integral was used to obtain

$$\mathbf{f} = \frac{1}{2\pi i}\int_\Gamma dz\, R(z,K)\mathbf{f} = \lim_{\epsilon\to 0^+}\frac{1}{2\pi i}\int_{-1}^{1} d\mu\{R(\mu - i\epsilon, K)$$

$$- R(\mu + i\epsilon, K)\}\mathbf{f} + \frac{1}{2\pi i}\sum_{\nu\in\sigma_p(K)}\int_{\Gamma_\nu} dz\, R(z,K)\mathbf{f}$$

$$= \int_N d\sigma(\nu)\Phi(\mu,\nu)F(\mathbf{f}),$$

one expects that

$$(E([-1,\omega])\mathbf{f})(\mu) = \int_{-1}^{\omega} d\nu\, \Phi(\mu,\nu)F(\mathbf{f}), \quad \omega \in I, \tag{15a}$$

$$(E(\nu)\mathbf{f})(\mu) = \Phi(\mu,\nu)F(\mathbf{f})(\nu), \quad \nu \in \sigma_p(K), \tag{15b}$$

defines a resolution of the idenity for the normal operator K. This is indeed the case, the essential feature being the fact that in X_2 K is similar to the sum of a self-adjoint operator and a *finite dimensional* normal operator on the span of $\{\gamma_\nu \mid \nu \in \mathbb{C}\backslash\mathbb{R}\}$.

To see this, we recall that the spectral projections can be obtained by the formula[12]

$$E((a,b)) = \lim_{\delta\to 0^+}\lim_{\epsilon\to 0^+}\frac{1}{2\pi i}\int_{a+\delta}^{b+\delta}(R(\mu - \epsilon i, T) - R(\mu + \epsilon i, T))\, d\mu \tag{16}$$

in the strong operator topology, for T any bounded self-adjoint operator on a Hilbert space. It is not difficult to see that this formula extends to operators T which are similar to self-adjoint operators. Further, for any closed operator T, if N_1 is a subset of the spectrum $\sigma(T)$, and Γ is a rectifiable, simple closed curve containing N_1 in its interior and $\sigma(T)\backslash N_1$ in its exterior, then

$$E(N_1) = \frac{1}{2\pi i}\int_\Gamma R(z,T)\, dz \tag{17}$$

is the spectral projection corresponding to N_1. Thus the first of these formulas gives

$$(E([-1,\omega])\mathbf{f})(\mu) = \lim_{\epsilon\to 0^+}\frac{1}{2\pi i}\left(\int_{-1}^{\omega} d\mu\, R(\mu - i\epsilon, K)\mathbf{f} + \int_{\omega}^{-1} d\mu\, R(\mu + i\epsilon, K)\mathbf{f}\right)$$

which reduces to Eq. (15a) by precisely the same steps leading to Eq. (7), and the second formula gives Eq. (15)

This analysis—in particular Eq. (16)—is valid in $X_2(\mathbf{I})$. However, it may be extended to $X_p(\mathbf{I})$ by observing that $\mathbf{M} \equiv X_2(\mathbf{I}) \cap X_p(\mathbf{I})$ is dense in $X_p(\mathbf{I})$ for all $p > 1$. Then the boundedness of the projections E in p-norm follows from the analysis of Lemma 6, as is evident from Eqs. (15), and the algebraic properties of the spectral resolution are a consequence of the boundedness of the projections and the density of $\mathbf{M}$. In more detail, since

$$E([-1,\omega])E([-1,\omega']) = E([-1,\omega'])E([-1,\omega]) = E([-1,\omega])$$

on $\mathbf{M}$ for $\omega' \geq \omega$, and the projections are bounded operators, we immediately obtain this nondecreasing property on $\overline{\mathbf{M}} = X_p(\mathbf{I})$. Likewise, the validity of

$$KE = EK$$

on $\mathbf{M}$ and the boundedness of K on $X_p(\mathbf{I})$ extends the equality to all of $X_p(\mathbf{I})$. The identity

$$E([-1,1]) + \sum_{\nu\in\sigma_p(K)} E(\nu) = I$$

also results from these density arguments, or alternatively, directly from Eqs. (15). Finally, the extension of (strong) right continuity

$$\lim E([-1,\lambda + 0]) = E([-1,\lambda])$$

to $X_p(\mathbf{I})$ may be seen easily by using the uniform bound on the projections

$\|E\| \leq 1.$

We state these results in a theorem.

Theorem 10: The spectral decomposition of the transport operator A in $X_p(\mathbf{I})$ is given by

$$K=\int_{-1}^{1}\lambda\, dE(\lambda)+\sum_{\nu\in\sigma_p(K)}\nu E(\nu),$$

where $E(\lambda)$ is obtained from Eq. (15a) and $E(\nu)$ from Eq. (15b).

V. HALF SPACE PROBLEM

The multigroup half space problem consists of the transport equation (1) defined for all $x \geq 0$ along with the boundary conditions

$$\psi(x,\mu)=\mathbf{f}_0(\mu),\quad 0\leq\mu\leq 1,$$

$$\lim_{x\to\infty}\psi(x,\mu)=0,$$

for a given boundary (vector valued) function $\mathbf{f}_0$ on the real interval $\mathbf{J}=[0,1]\subset\mathbf{I}$. Let us define the subspace $X_p(\mathbf{J})\subset X_p(\mathbf{I})$ by $f(\mu)=0$ for $-1\leq\mu\leq 0$ if $\mathbf{f}\in X_p(\mathbf{I})$, so that

$$X_p(\mathbf{I})=X_p(\mathbf{J})\oplus X_p(\mathbf{I}\setminus\mathbf{J}).$$

It is assumed that the given function $\mathbf{f}_0\in X_p(\mathbf{J})$. Then by well-known arguments, the solution of the half space problem is equivalent to the construction of a (non-orthogonal) projection Q satisfying:

(i) $(Q\mathbf{f})(\mu)=\mathbf{f}(\mu),\quad 0\leq\mu\leq 1,$

(ii) $(zI-K)^{-1}Q\mathbf{f}$ analytic in z for $\mathrm{Re}z<0.$

The second condition implies that Q is a projection onto

$$X_p(\mathbf{N})=\left\{\int_0^1 dE(\lambda)+\sum_{\substack{\nu\in\sigma_p(K)\\ R(\nu)>0}}E(\nu)\right\}X_p(\mathbf{I})$$

and the first that Q is a projection along $X_p(\mathbf{I}\setminus\mathbf{J})$. The notation $R(\nu)>0$ signifies that either $\mathrm{Re}\nu>0$ else or $\mathrm{Re}\nu=0$, $\mathrm{Im}\nu>0$. In Caseology language, these conditions ensure that the negative frequency eigenvectors $\Phi(\mu,\nu)$, $\nu<0$ or $\mathrm{Re}\nu<0$ are absent from $Q\mathbf{f}$ for any $\mathbf{f}\in X_p(\mathbf{J})$.

In Ref. 3, some recent results of Mullikin[13] on a certain matrix Riemann problem are utilized to construct the projection Q on $X_2(\mathbf{I})$,

$$(V_\Sigma Q\mathbf{f})(-\mu)=\int_0^1\frac{s}{\mu-s}X^{-1}(\mu)Y^{-1}(-s)\Sigma^2(V_\Sigma\mathbf{f})(s)\,ds,\quad 0<\mu\leq 1, \tag{18a}$$

$$(Q\mathbf{f})(\mu)=\mathbf{f}(\mu),\quad -1\leq\mu\leq 0, \tag{18b}$$

where the matrices $X(z)$ and $Y(z)$ factor the dispersion matrix,

$$\Lambda(z)=Y(-z)X(z), \tag{19}$$

and satisfy some additional analyticity properties. In particular, X and Y are both continuous and invertible as functions from $[-1,0]$ to matrices on X_p^c, and therefore $X^{-1}(\mu)$ and $Y^{-1}(\mu)$ are bounded as operators on $X_p(\mathbf{J})$.

To extend Eq. (18) to $X_p(\mathbf{I})$ from $\mathbf{M}=X_2(\mathbf{I})\cap X_p(\mathbf{I})$, it is only necessary to observe that Q is a bounded operator on $X_p(\mathbf{I})$. In fact,

$$\|Q\mathbf{f}\|_p\leq\|\Sigma\|^2\|V_\Sigma Q\mathbf{f}\|_p$$
$$\leq M_p\|\Sigma\|^4\ \|X^{-1}\|\ \|\mu Y^{-1}\|\ \|\Sigma\|^4\|\mathbf{f}\|_p,$$

where Lemma 2 has been utilized, and $\|X^{-1}\|$, $\|Y^{-1}\|$ are computed only on $X_p(\mathbf{I}\setminus\mathbf{J})$. Thus the half space theory may be developed for $X_p(\mathbf{I})$. The factorization of Mullikin, Eq. (19), is presently only known for $\|\Sigma^{-1}C\|<\frac{1}{2}$, which, of course, limits these results.

From the viewpoint of expansion theorems, the operator of interest is the product FQ, which is bounded on $X_p(\mathbf{I})$ since each of the factors is. In Ref. 3, this operator is derived for Hölder continuous functions $\mathbf{f}$,

$$(FQ\mathbf{f})(\nu)=\frac{1}{2\pi i\nu}\,(X^{-1}(\nu)^+-X^{-1}(\nu)^-)\int_0^1 ds\,\frac{s}{\nu-s}\,Y^{-1}(-s)\Sigma^2$$
$$\times(V_\Sigma\mathbf{f})(s)+\tfrac{1}{2}(X^{-1}(\nu)^++X^{-1}(\nu)^-)Y^{-1}(-\nu)\Sigma^2(V_\Sigma\mathbf{f})(\nu)$$

for $0\leq\nu\leq 1$, and (20a)

$$(FQ\mathbf{f})(\nu)=\frac{1}{\Omega'(\nu)}\left[\int_0^1 ds\,\frac{s}{\nu-s}\,Y^{-1}(-s)\Sigma^2(V_\Sigma\mathbf{f})(s),\boldsymbol{\alpha}'\right]\boldsymbol{\beta}_\nu \tag{20b}$$

for $\nu\in\sigma_p(K)$, $R(\nu)>0$, where $\boldsymbol{\alpha}'$ is defined by

$$\boldsymbol{\alpha}'=X_c(\nu)\boldsymbol{\beta}_\nu,$$

and X_c is defined analogously to Λ_c.

Theorem 11: If $\|\Sigma^{-1}C\|<\frac{1}{2}$, then Eqs. (18) define a bounded projection on $X_p(N)$ along $X_p(\mathbf{I}\setminus\mathbf{J})$. K^{-1} is semibounded on $QX_p(\mathbf{I})$, and thus is the generator of a holomorphic semigroup. Equations (20) for FQ are valid (almost everywhere) for $\mathbf{f}\in X_p(\mathbf{I})$.

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