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Uniqueness of solutions to the linearized Boltzmann equation

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Uniqueness theorems are proved for the linearized Boltzmann equation for both the "exterior" and "interior" problems under generalized Maxwell boundary conditions. The solution space is a weighted L_p space, and agrees with the space in which solutions have previously been constructed.

I. INTRODUCTION

Although the Boltzmann equation is more than 100 years old, only recently have rigorous mathematical treatments of the equation and other types of irreversible statistical mechanics been developed which would parallel corresponding rigorous treatments of equilibrium statistical mechanics as summarized for example in Ruelle's book.[1] Along such lines are investigations of uniqueness and existence of solutions to various forms of the linearized Boltzmann equation, which are still active areas in the mathematical physics literature. A quasirigorous approach to the neutron transport equation, including constructive methods of existence proofs, was developed in 1960 by Case,[2] and is reviewed extensively in a later book.[3] These same techniques were even earlier applied to the linearized Vlasov equation describing plasma oscillations[4,5] and later to the kinetic theory of gases[6–8] and radiative transport in stellar atmospheres.[9]

In the early 1970's strictly rigorous methods for solving these equations were introduced independently by Hangelbroek[10] and Larsen and Habetler.[11] These approaches have been described and compared with one another and the Case method in a review paper.[12]

The purpose of the present study is to prove uniqueness to supplement the rigorous constructive existence proofs mentioned above. Except for a brief remark near the end of the paper, we restrict our attention to the linearized Boltzmann equation describing gas kinetics. Our technique is based on Case's treatment[13] with two major differences. First, we consider the "exterior problem" instead of the "interior problem" studied by Case, although our results can easily be extended to the interior problem as well. More importantly, we believe that the (Hilbert) space used in Case's work is not the appropriate solution space. In particular, rigorous constructive solutions have been obtained[14,15] in a different space [the space $X_p^m(\mathbb{R}^n)$ defined below], and so we will prove uniqueness in that space. Further, the existence of a certain integral, which is crucial to our proof, can be inferred in $X_p^m(\mathbb{R}^n)$, and not in Case's Hilbert space.

For these reasons, let us define the space $X_p^m(\mathbb{R}^n)$, $p>1$, by

$$X_p^m(\mathbb{R}^n)=\bigoplus_{i=1}^{m} X_p(\mathbb{R}^n), \tag{1a}$$

$$X_p(\mathbb{R}^n)=\{f: c_i f\in L_p(\mathbb{R}^n,\mu),\ 1\leq i\leq n\}, \tag{1b}$$

where $L_p(\mathbb{R}^n,\mu)$ is the weighted Banach space with norm

$$\|f\|=\int_{\mathbb{R}^n}|f|^p\,d\mu(\mathbf{c})=\int_{\mathbb{R}^n}|f|^p\exp(-\mathbf{c}\cdot\mathbf{c})\,d^nc, \tag{1c}$$

[$L_p^m(\mathbb{R}^n,\mu)$ is related to $L_p(\mathbb{R}^n,\mu)$ in analogy with Eq. (1a)].

We call attention to two other attempts to develop uniqueness in a rigorous context. The first is presented in a series of papers by Cercignani and Pao (a bibliography appears on p. 154 of Ref. 7; cf. pp. 140ff of the same reference for a description). Unfortunately, the existence proofs are not constructive. Furthermore, the weighted Banach space necessary for the demonstration of existence encountered in Refs. 14 and 15 does not appear to be a convenient space in which to work (leaving aside questions of physical relevance). The second attempt is due to Giraud,[16] but his techniques are considerably more cumbersome than our simple methods based on Case's work.

II. THE TIME-INDEPENDENT, EXTERIOR PROBLEM

The time-dependent equation is considerably easier to treat than the time-independent, as a quick reading of Ref. 13 indicates, and so we will deal only with the latter. The interior problem, discussed in Ref. 13, is a straightforward modification of the exterior problem, and the appropriate uniqueness theorem for that case will be stated without proof.

The linearized Boltzmann equation can be written in the form

$$\mathbf{c}\cdot\nabla h(\mathbf{c},\mathbf{r})=J(h), \tag{2}$$

where $\mathbf{c}\in\mathbb{R}^n$ is the (dimensionless) gas velocity, the gradient operator is with respect to the position variable $\mathbf{r}$, and the collision integral $J(h)$ is dissipative. This means that

$$\int_{\mathbb{R}^n} h(\mathbf{c},\mathbf{r})J(h(\mathbf{c},\mathbf{r}))\,d\mu(\mathbf{c})\leq 0, \tag{3}$$

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with equality iff $J(h)=0$. (This condition is necessary for the existence of an H-theorem for the Boltzmann equation). Specifically, the unbounded linear transformation $J: X_p^m(\mathbb{R}^n) \to L_1^m(\mathbb{R}^n, \mu)$ is given by

$$J(h)=\int_{\mathbb{R}^n} K(\mathbf{c}, \mathbf{c}') h(\mathbf{c}', \mathbf{r}) d\mu(\mathbf{c}') - \nu(\mathbf{c})h, \tag{4}$$

where $K(\mathbf{c}, \mathbf{c}')$ is the collision kernel and $\nu(\mathbf{c})$ is known as the scattering rate

$$\nu(\mathbf{c})=\int_{\mathbb{R}^n} K(\mathbf{c}', \mathbf{c}) d\mu(\mathbf{c}'). \tag{5}$$

The domain of J is the dense subset of $X_p^m(\mathbb{R}^n)$ for which $J(h_\mathbf{r}) \in L_1^m(\mathbb{R}^n, d\mu)$. [Where the spatial variable is held fixed, we shall write $h(\mathbf{c}, \mathbf{r}) = h_\mathbf{r}(\mathbf{c})$; e.g., $h_\mathbf{r} \in X_p^m(\mathbb{R}^n)$.]

In the model considered in Ref. 14, $m=1$, $n=1$, and $K(\mathbf{c}, \mathbf{c}')=1/\sqrt{\pi}$. The constructive solution of Eq. (2) obtained in that reference requires the space $X_p(\mathbb{R})$. This work has suggested our choice for the domain and range spaces of J in the more general case treated here. As a result of these choices the integral in Eq. (3) may exist as an extended real number.

Time reversal invariance actually requires that $K(\mathbf{c}, \mathbf{c}')$ be a real symmetric function of $\mathbf{c}$ and $\mathbf{c}'$; rotational invariance requires further that it depends only on $\mathbf{c} \cdot \mathbf{c}'$. These facts are well known[7,17]; however, we shall not make use of these conditions in our uniqueness theorems. The following useful result is readily obtained from Hölder's inequality.

Proposition 1: If $K(\mathbf{c}, \mathbf{c}')$ is a polynomial in $\mathbf{c}' \cdot \mathbf{c}$ with no constant term, then $J(h)$ is continuous.

Note also that the left-hand side of Eq. (2) defines a function in $L_1^m(\mathbb{R}^n, \mu)$ if the components of $\nabla h_\mathbf{r}$ are contained in $X_p^m(\mathbb{R}^n)$.

We now consider solutions of Eq. (2) in the exterior of a bounded set $V \subseteq \mathbb{R}^n$ with connected complement and piecewise smooth, orientable boundary. Appropriate boundary conditions will be imposed on ∂V and at the point ∞. We define a solution of Eq. (2) to be a map $h: \mathbb{R}^n \to X_p^m(\mathbb{R}^n)$ with continuous spatial first partial derivatives such that the components of $\nabla h_\mathbf{r} \in X_p^m(\mathbb{R}^n)$. (The continuity of the spatial partial derivatives is used only for the application of Gauss's theorem and so can be weakened.) The boundary conditions which are generally adopted on ∂V are the so-called linearized Maxwell conditions, namely

$$h(\mathbf{c}, \mathbf{r}_s) = (1-\alpha) h(c_\parallel, -\mathbf{c}_\perp, \mathbf{r}_s) + (2\alpha/\pi) \int_{\mathbf{n}_s \cdot \mathbf{c}' > 0} \mathbf{n}_s \cdot \mathbf{c}' h(\mathbf{c}', \mathbf{r}_s) d\mu(\mathbf{c}') + h_0(\mathbf{c}, \mathbf{r}_s), \tag{6}$$

when $\mathbf{n}_s \cdot \mathbf{c} < 0$. Here $\mathbf{r}_s \in \partial V$, $0 \leqslant \alpha \leqslant 1$, and $\mathbf{n}_s$ is the outward normal to ∂V at $\mathbf{r}_s$; $\mathbf{c}_\perp$ is the component of $\mathbf{c}$ perpendicular to ∂V at $\mathbf{r}_s$ and $c_\parallel$ is the parallel component. At infinity we require

$$\lim_{|\mathbf{r}| \to \infty} h(\mathbf{c}, \mathbf{r}) = h_\infty(\mathbf{c}), \tag{7a}$$

in the sense

$$\lim_{|\mathbf{r}| \to \infty} \int (\mathbf{n}_s \cdot \mathbf{c})[h(\mathbf{c}, \mathbf{r}) - h_\infty(\mathbf{c})]^2 dS = 0, \tag{7b}$$

where the integration is carried out over a sphere of fixed radius $|\mathbf{r}|$. We now state

Theorem 1: Subject to conditions (6), (7a), and (7b), Eq. (2) has at most one solution for $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{r} \in \mathbb{R}^n \backslash V$.

Proof: Assume that two solutions h_1 and h_2 exist. Then $h = h_1 - h_2$ obeys Eq. (2) subject to

$$h(\mathbf{c}, \mathbf{r}) = (1-\alpha) h(c_\parallel, -\mathbf{c}_\perp, \mathbf{r}_s) + (2\alpha/\pi) \int_{\mathbf{n}_s \cdot \mathbf{c}' > 0} \mathbf{n}_s \cdot \mathbf{c}' h(\mathbf{c}', \mathbf{r}_s) d\mu(\mathbf{c}') \tag{8a}$$

and

$$\lim_{|\mathbf{r}| \to \infty} h(\mathbf{c}, \mathbf{r}) = 0 \tag{8b}$$

[the limit being defined by (7b)].

We now proceed as in Ref. 13, i.e., multiply Eq. (2) by $\exp(-c^2) h(\mathbf{c}, \mathbf{r})$ and integrate over $d^n r$ and $d^n c$. The integral on the left-hand side can be converted into a surface integral over ∂V plus a large sphere of radius $|\mathbf{r}| \to \infty$, by application of Gauss's theorem after the identity $h \nabla h = \frac{1}{2} \nabla h^2$ is employed. By virtue of (8b), the integral over the surface of radius $|\mathbf{r}|$ vanishes as $|\mathbf{r}| \to \infty$.

We thus arrive at Eq. (13) of Ref. 13, except that the order of integration is reversed:

$$\frac{1}{2} \int_{\mathbb{R}^n} \int_{\partial V} \mathbf{n}_s \cdot \mathbf{c}\, h^2(\mathbf{c}, \mathbf{r}_s) dS\, d\mu(\mathbf{c}) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n \backslash V} h(\mathbf{c}, \mathbf{r}) J(h) d^n r\, d\mu(\mathbf{c}). \tag{9}$$

Because of the continuity of $h(\mathbf{c}, \mathbf{r})$ in $\mathbf{r}$, Fubini's theorem applies,[18] and we can carry out the integration over $\mathbf{c}$ first. Dissipativity shows that the right-hand side of Eq. (9) is nonpositive (it may equal $-\infty$). The left-hand side can be simplified, as in Ref. 13, by applying the boundary condition (8a), and we conclude [Eq. (8) of Ref. 13] that

$$\int_{\mathbb{R}^n} \mathbf{n}_s \cdot \mathbf{c} h^2(\mathbf{c}, \mathbf{r}_s) d\mu(\mathbf{c}) = [2\alpha(1-\alpha) + \alpha^2] \Big[\int_{\mathbf{n}_s \cdot \mathbf{c} > 0} \mathbf{n}_s \cdot \mathbf{c} h^2(\mathbf{c}, \mathbf{r}_s) d\mu(\mathbf{c}) - (2/\pi) \Big| \int_{\mathbf{n}_s \cdot \mathbf{c} > 0} \mathbf{n}_s \cdot \mathbf{c} h(\mathbf{c}, \mathbf{r}_s) d\mu(\mathbf{c}) \Big|^2 \Big]. \tag{10}$$

The second integral always exists via Hölder's inequality.

The first integral either exists, or is $+\infty$, since h is measurable. If the integral exists, we follow Case's reasoning[13] (using the Schwartz inequality) to conclude that Eq. (10) is nonnegative. If it is infinite, the same conclusion is immediate. In either case it follows that both sides of Eq. (9) must vanish. Thus [Eq. (3)]

$$J(h) = 0, \tag{11a}$$

and

$$\int_{\mathbf{n}_s \cdot \mathbf{c} > 0} \mathbf{n}_s \cdot \mathbf{c} h^2(\mathbf{c}, \mathbf{r}_s) d\mu(\mathbf{c}) = (2/\pi) \Big| \int_{\mathbf{n}_s \cdot \mathbf{c} > 0} \mathbf{n}_s \cdot \mathbf{c} h(\mathbf{c}, \mathbf{r}_s) d\mu(\mathbf{c}) \Big|^2. \tag{11b}$$

[These are Eqs. (14) and (15) of Ref. 13.] From the

Schwartz inequality and Eq. (8a) it follows that

$$h(\mathbf{c}, \mathbf{r}_s) = h(\mathbf{r}_s), \quad \text{everywhere for almost all } \mathbf{c}, \tag{12}$$

i.e., a function of $\mathbf{r}_s$ alone.

From Eqs. (11a) and (2) we conclude further that $h(\mathbf{c}, \mathbf{r})$ is independent of $\mathbf{r}$ along any ray originating at point $\mathbf{r}_s$ in the direction $\mathbf{c}$, for $\mathbf{n}_s \cdot \mathbf{c} > 0$. Consider two rays originating at points $\mathbf{r}_s$ and $\mathbf{r}_s'$ and intersecting at $\mathbf{r}$. It follows that $h(\mathbf{r}_s) = h(\mathbf{r}_s')$ and, in fact, that $h(\mathbf{c}, \mathbf{r}_s)$ is a constant also independent of $\mathbf{r}_s$. Thus

$$h(\mathbf{c}, \mathbf{r}) = \text{const}, \quad \text{everywhere for almost all } \mathbf{c} \in \mathbb{R}^n. \tag{13}$$

Finally, from (8b) we conclude

$$h(\mathbf{c}, \mathbf{r}) = 0, \quad \mathbf{r} \in \mathbb{R}^n \backslash V,$$
$$\text{everywhere for almost all } \mathbf{c} \in \mathbb{R}^n. \tag{14}$$

This completes the proof of Theorem 1.

Uniqueness for the initial-boundary value problem is slightly simpler to prove since it is possible to conclude, as in Ref. 13, that

$$\int_{\mathbb{R}^n \backslash V} d^n r \int d\mu(\mathbf{c})\, h^2(\mathbf{c}, \mathbf{r}, t) = 0, \tag{15}$$

which implies

$$h(\mathbf{r}, \mathbf{c}, t) = 0, \quad \mathbf{r} \in \mathbb{R}^n \backslash V, \; 0 < t < \infty,$$
$$\text{everywhere for almost all } \mathbf{c} \in \mathbb{R}^n. \tag{16}$$

Here we assume again that h is continuously differentiable in t, and one extra change in order of integration is required.

We may remark that the major portion of the above proof already appears in Ref. 13. However, it was felt necessary to justify certain of the mathematical manipulations in order to make Case's treatment "rigorous." Note in particular that Case chose to work in the space $L_2(\mathbb{R}^3, \mu)$; in fact, Kuščer[17] states specifically that this is the appropriate solution space. However, in this space there is no guarantee that

$$\int_{\mathbf{n}_s \cdot \mathbf{c} > 0} \mathbf{n}_s \cdot \mathbf{c}\, h(\mathbf{c}, \mathbf{r}_s)\, d\mu(\mathbf{c})$$

exists [see Eq. (10)]. The existence of this integral is crucial to the proof of Theorem 1. Furthermore, the work on the BKG model referred to earlier indicates that our choice of the X_p spaces is appropriate. It is interesting that, in studies of the neutron transport equation, the X_p spaces also entered in a natural way.[19]

We now state without proof

Theorem 2: Eq. (2), subject to condition (6), has, up to an additive constant, at most one solution for $\mathbf{c} \in \mathbb{R}^n$, $\mathbf{r} \in V$. Here $\mathbf{n}_s$ must be interpreted as the *inward* normal at $\mathbf{r}_s$.

The proof proceeds in direct analogy with that of Theorem 1. Extension to the initial boundary value problem is also immediate. The additive constant ambiguity in the interior solution[13] does not exist for the exterior because in the latter the behavior at infinity is specified.

Siewert[20] has raised the question as to the uniqueness of the solution to the equation of radiative transport in a half-space, subject to reflecting boundary conditions at $x = 0$ as given by Eq. (6). The relevant equation is

$$\mu \frac{\partial \psi}{\partial x}(x, \mu) = J(\psi), \quad \mu \in [-1, 1], \; x \in \mathbb{R}^+, \tag{17a}$$

where

$$J(\psi) = -\psi + \int_{-1}^{1} \psi(x, \mu')\, d\mu', \tag{17b}$$

and solutions ψ are to be sought in $\widetilde{X}_p$ with norm

$$\| f \|_p = \{ \int_{-1}^{1} |\mu f(\mu)|^p d\mu \}^{1/p}.$$

It is trivial to show that $J(\psi)$ is dissipative, and the left-hand side may be treated analogously to the gas case. One concludes that the solution to Eq. (17) is unique. A similar result can also be shown to hold for the full three-dimensional equation

$$\mathbf{\Omega} \cdot \nabla \psi = J(\psi),$$

where

$$J(\psi) = -\psi + \int d\mathbf{\Omega}' f(\mathbf{\Omega} \cdot \mathbf{\Omega}')\, \psi(\mathbf{r}, \mathbf{\Omega}')$$

and

$$\int f(\mathbf{\Omega} \cdot \mathbf{\Omega}')\, d\mathbf{\Omega}' = 1.$$

The equation and the notation are identical to those of one-speed neutron transport with $c = 1$, as discussed in Ref. 3. The uniqueness proofs for the neutron transport equation, as described there, were primarily the inspiration for Ref. 13, and they may be made rigorous along the same lines as discussed in this paper for the Boltzmann equation.

III. THE ONE-DIMENSIONAL BGK MODEL

The present work was actually motivated by the constructive solutions obtained in Refs. 14 and 15. The BGK model equations considered there are not really linearized versions of the Boltzmann equation since the dependent variable h does not represent the deviation of the gas distribution function from equilibrium, but rather certain moments thereof. For this reason, it is probably necessary to show (although it is stated without proof in Ref. 7 and is presumably well known) that the collision operator is dissipative. The relevant equations can be written

$$c \frac{dh}{dx} = J(h), \quad x, c \in \mathbb{R}, \tag{18}$$

where for the scalar case

$$J_s(h) = \int_{-\infty}^{\infty} h(c', x)\, d\mu(c') - h(c, x), \tag{19}$$

and for the vector case

$$J_v(h) = Q(c) \int_{-\infty}^{\infty} Q^T(c')\, h(c', x)\, d\mu(c') - h(c', x), \tag{20a}$$

where Q is a 2×2 matrix

$$Q(c)=\begin{bmatrix}\sqrt{\frac{2}{3}}(c^2-\frac{1}{2}) & 1\\ \sqrt{\frac{2}{3}} & 0\end{bmatrix}. \tag{20b}$$

Furthermore, the boundary condition (6) is replaced, it turns out, by the simpler condition

$$h(c,0)=h_0(c)+\alpha h(-c,0), \quad c>0, \tag{21}$$

where $\alpha=0$ corresponds to diffuse reflection and $\alpha=1$ to specular reflection. (We are considering the half-space problem, $x\in\mathbb{R}^+$, $c\in\mathbb{R}$.)

The proof for J_s follows from the Schwartz inequality. For J_v, one proves that

$$Q(c)\int_{-\infty}^{\infty} Q^T(c')h(c',x)\,d\mu(c'),$$

is a projection, from which the result follows fairly easily. The solutions as constructed in Ref. 14 can be verified to be continuously differentiable (in fact C_∞) in x by application of the Lebesque monotone convergence theorem. The proof of Theorem 1 (and Theorem 2) is readily adapted to the semi-infinite case; in fact, the argument is somewhat simpler, since the gradient found in Eq. (2) is replaced by $\partial h/\partial x$, and one can simply integrate from zero to infinity. However, the continuity properties of these solutions suggest that the more general case treated here should satisfy the continuity conditions imposed on $h(\mathbf{c},\mathbf{r})$ which allow application of Gauss's theorem.

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