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Universality in the percolation problem—Anomalous dimensions of ϕ^4 operators*†

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We consider critical systems, such as the percolation problem, whose symmetry permits an invariant interaction of third order in the fluctuating fields ϕ. In the renormalization-group approach one is naturally led to look for infrared-stable fixed points which yield ϵ expansions in $6-\epsilon$ dimensions, with $\epsilon = 3$ as the physical value. Since the Gaussian fixed point becomes unstable to ϕ^4 interactions for $\epsilon > 2$, it is important to check that the fixed point obtained in the ϵ expansion remains stable to such perturbations. We report the calculation to first order in ϵ of the corrections to scaling induced by (stability with respect to) ϕ^4 interactions in a general class of such theories. The results indicate that ϕ^4 interactions remain irrelevant in the percolation problem.

I. INTRODUCTION

The basic philosophy underpinning most uses of the renormalization group (RG) is that universality of critical behavior will arise from the existence of a suitably stable fixed point (or set of fixed points) of the RG transformation. This philosophy is implemented by studying the RG in simple models which have the correct qualitative features to describe the physics. One should then verify that the fixed point obtained in the simple model is stable to perturbations of all other permissible interactions, so that universality of behavior is ensured.

Field-theory models are particularly suited for realizing these ideas[1] because the various types of interactions can be ordered according to the number of powers of the field ϕ and the total number of derivatives, and this ordering corresponds to their likely importance for critical behavior. One is therefore led to consider simple models in which only the lowest power of non-Gaussian (anharmonic) interaction is considered, e.g., the usual Landau-Ginzburg-Wilson Hamiltonian

$$H=\int d^d x\left(\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}r_0\phi^2+\frac{1}{4!}g_0(\phi^2)^2\right) \tag{1.1}$$

for short-range, $O(n)$-invariant problems (here ϕ_i is an n-component field, $\phi^2\equiv\phi_i\phi_i$, and d is the dimension of space).

Within the framework of the Hamiltonian (1.1), a RG transformation[2] can be set up which yields, for example, an expansion for critical behavior as a power series in ϵ, in $4-\epsilon$ dimensions. The critical behavior is independent of the precise value of g_0 because the fixed point is stable to ϕ^4 perturbations.

The signal of the appearance of an ϵ expansion around four dimensions is given by naive dimensional analysis. Taking (1.1) as the reduced dimensionless Hamiltonian, one readily checks that an interaction

$$g^{(l,m)}\int d^d x\, O(\partial^l)O(\phi^m) \tag{1.2}$$

is dimensionless provided that

$$[g^{(l,m)}]=[\Lambda]^{d-l-m(d-2)/2}, \tag{1.3}$$

where Λ denotes the wave-vector cutoff in the theory, with dimension $(\text{length})^{-1}$. In particular in Eq. (1.1)

$$[g_0]=[\Lambda]^{4-d}.$$

The appearance of a positive power of Λ when $d<4$ is reflected in the instability of the Gaussian (free field, all $g^*=0$) fixed point to ϕ^4 interaction, and the appearance of a nontrivial Heisenberg, fixed point which can be controlled in the ϵ expansion.

More generally, the Gaussian fixed point is stable to perturbations of the type (1.2) provided that $d-l-\frac{1}{2}m(d-2)<0$, i.e., $d(\frac{1}{2}m-1)>m-l$. Therefore, for critical behavior in three dimensions the additional interaction which is most likely to be dangerous is ϕ^6, which is marginal for the Gaussian fixed point. Hence, one should check that the Heisenberg fixed point *is* stable to all interactions (1.2), with particular attention paid to ϕ^6.

The program of evaluating the stability properties of the Heisenberg fixed point is fairly well developed in field theories with four main approximation schemes — straightforward ϵ expansion,[3] the approximate recursion formula,[4] $1/n$ expansion,[5] and expansion in ϵ', in $2+\epsilon'$ dimensions[6] using the nonlinear Hamiltonian of the Goldstone modes ($n \neq 1$).

In this article we consider this program for theories with ϕ^3 invariants, which dominate critical behavior (in the same sense that ϕ^4 dominates in $d=4-\epsilon$ above). The Gaussian fixed point is unstable to ϕ^3 interactions below six dimensions, and so one naturally searches for an ϵ expansion in $6-\epsilon$ dimensions with $\epsilon=3$ as the physical value. (Henceforth ϵ will refer to $6-d$, unless otherwise stated.) Apart from the problems of convergence when $\epsilon=3$, and of thermodynamic stability in such ϕ^3 dominated theories, the question of the stability of the nontrivial fixed point to perturbations of ϕ^4, ϕ^5, and ϕ^6 interactions is rather pressing. There are also fewer tools to attack the problem than in ϕ^4 dominated theories: no simple approximate recursion formula is known (since η is now nonvanishing at lowest nontrivial order in ϵ — note the paper by Golner in Ref. 4, however), simple $1/n$ expansions do not exist, in general, and the symmetry group is frequently discrete, so that there are no Goldstone modes. We appear to be left only with the possibility of ϵ expansion.

We shall use renormalized perturbation theory of composite operators and apply the RG to calculate the anomalous dimensions (critical exponents) associated with ϕ^4 perturbations of ϕ^3 theories to first order in ϵ, in $6-\epsilon$ dimensions.

The outline of the paper is as follows. In Sec. II we calculate these anomalous dimensions in a class of theories with a single ϕ^3 and two independent ϕ^4 invariants of some symmetry group of which the fields ϕ_i form the basis of an irreducible representation (i.e., there is only one quadratic invariant $\phi_i\phi_i$). The usefulness of the equations of motion is particularly clear in this general case. In Sec. III we shall review the percolation problem as providing an example of the above, and show that our calculations do support the stability of the nontrivial fixed point to ϕ^4 perturbations. The reduction of the general case to the situation where only one ϕ^4 invariant exists is given in the Appendix.

II. ANOMALOUS DIMENSIONS OF ϕ^4 OPERATORS IN ϕ^3 DOMINATED THEORIES

A. Critical behavior of ϕ^3 interactions

We start by reviewing briefly previous work on the critical behavior of systems described by a Hamiltonian of the form

$$H=\int d^d x\left(\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}m_0^2\phi^2+\frac{1}{3!}\,g_0 Q_{ijk}\phi_i\phi_j\phi_k\right). \quad (2.1)$$

Our notation is generally that of Ref. 7; ϕ_i is an n-component real field which transforms according to an irreducible representation of some symmetry group, so that $\phi^2=\phi_i\phi_i$ is an invariant. It is assumed that there is only a single invariant trilinear in the fields,[8] and the corresponding tensor is Q_{ijk} (symmetric in interchange of any pair of indices). Repeated index summation convention is always understood. Special cases of this Hamiltonian can be used to describe the isotropic to nematic phase transition in liquid crystals,[9,10] the percolation problem[7,10-12] (where the dimension of space $d=3$), and possibly as models for strong interactions of elementary particles[13] in $d=4$. In each of these cases we are interested in the critical behavior of (2.1), i.e., the behavior when wave vectors q and the "physical mass" m (inverse correlation length) are very much less than the cutoff Λ, which is implied in (2.1) (e.g., as a bound on q).

For interesting values of d one cannot obtain the critical behavior by straightforward perturbation in the bare coupling g_0 because it has a natural length scale given in terms of Λ. We write

$$g_0=u_0\Lambda^{3-d/2}, \quad (2.2)$$

according to Eq. (1.3), where u_0 is dimensionless and $O(1)$. For $d<6$, the power of Λ is positive, and its dimension is cancelled by the corresponding power of q or m, giving an arbitrarily large dimensionless expansion parameter $u_0(\Lambda/m)^{3-d/2}$ or $u_0(\Lambda/q)^{3-d/2}$.

This is a typical situation for setting up an ϵ expansion in $6-\epsilon$ dimensions, and has been studied for specific cases by several authors,[10,12] and for the general case (2.1) in Ref. 7. We shall use the conventional RG equations for the *massless* ($m=0$) vertex functions, which are dimensionally regularized and renormalized at a symmetric momentum point of magnitude μ.[14] They are

$$\left(\mu\frac{\partial}{\partial\mu}+\beta(u)\frac{\partial}{\partial u}-\tfrac{1}{2}N\eta(u)\right)\Gamma_R^{(N)}(q;u,\mu)=0\,. \quad (2.3)$$

Here the tensor indices are dropped for clarity. The parameter u is the dimensionless renormalized coupling constant, defined by the normalization condition

$$\Gamma_R^{(3)}(q;u,\mu)\big|_{q_i^2=\mu^2}=u\,\mu^{(6-d)/2}. \quad (2.4)$$

The functions $\beta(u)$, $\nu(u)$, and $\eta(u)$ have been calculated[7] up to two loops using the 't Hooft renormalization scheme for the general model (2.1). We are going to calculate the anomalous dimensions of ϕ^4 operators only to order ϵ, and hence quote only the one-loop results[15]

$$\beta(u) = u\left[-\tfrac{1}{2}\epsilon + (\tfrac{1}{4}\alpha - \beta)u^2 + O(u^4, \epsilon u^2)\right], \tag{2.5}$$

$$\eta(u) = \tfrac{1}{6}\alpha u^2 + O(u^4, \epsilon u^2), \tag{2.6}$$

$$(1/\nu)(u) - 2 = \tfrac{5}{6}\alpha u^2 + O(u^4, \epsilon u^2). \tag{2.7}$$

Here $\epsilon = 6 - d$, and α and β are the results of the tensor contractions

$$Q_{i_1 jk} Q_{jk i_2} = \alpha \delta_{i_1 i_2} \tag{2.8}$$

and

$$Q_{i_1 jk} Q_{i_2 kl} Q_{i_3 lj} = \beta Q_{i_1 i_2 i_3}. \tag{2.9}$$

The tensor forms of the right-hand sides of these equations are dictated by our assumption that δ_{ij} and Q_{ijk} are the only invariant (symmetric) two and three tensors — indeed this is a crucial requirement for the existence of the RG equation (2.3).

In this formalism the critical behavior can be obtained as a power series in ϵ if there exists an infrared stable fixed point, i.e., there exists a u^* such that $\beta(u^*) = 0$, and $\beta'(u^*) > 0$. From (2.5) we see that

$$u^{*2} = 2\epsilon/(\alpha - 4\beta) + O(\epsilon^2) \tag{2.10}$$

and

$$\beta'(u^*) = \epsilon + O(\epsilon^2). \tag{2.11}$$

Thus, to first order in ϵ, a real fixed point exists if $\alpha - 4\beta > 0$, and if it exists, it is infrared stable.

The critical behavior will then be characterized by the exponents $\eta = \eta(u^*)$, $\nu = \nu(u^*)$, etc. For example, the μ dependence of the massless (critical) vertex functions is given by

$$\Gamma_R^{(N)}(q; u, \mu) \sim \mu^{N\eta/2} f(q) \tag{2.12}$$

for $q \ll \mu$.

The low-q dependence of $\Gamma_R^{(N)}(q; u, \mu)$ is now obtained by dimensional analysis. There are additional powers $q^{-\eta(u^*)/2}$ for each external field of the vertex function over and above what one would obtain in a naive dimensional analysis which neglected μ.

The remainder of this section is concerned with the calculation of the new anomalous dimensions associated with ϕ^4 interactions. These will enable us to tell whether or not such interactions are likely to change, e.g., the result (2.12).

B. Renormalization of ϕ^4 operators

We shall consider here the case where the symmetry group of the Hamiltonian permits two ϕ^4 invariants, an O(n) invariant

$$S\phi^4 \equiv S_{ijkl}\phi_i\phi_j\phi_k\phi_l = (\phi^2)^2, \tag{2.13a}$$

where

$$S_{ijkl} = \tfrac{1}{3}(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}), \tag{2.13b}$$

and a second invariant denoted by

$$F\phi^4 \equiv F_{ijkl}\phi_i\phi_j\phi_k\phi_l. \tag{2.14}$$

This assumption is valid for a wide range of models; in special cases there is only one ϕ^4 invariant, $S\phi^4$, and the reduction to this simpler case is considered in the appendix.

We consider the Hamiltonian (2.1) with the additional interactions

$$\frac{1}{4!}\Lambda^{4-d}\int d^d x\,(s_0 S\phi^4 + f_0 F\phi^4) \tag{2.15}$$

[s_0 and f_0 are dimensionless, cf. Eq. (1.3)].

The effect of these new interactions is obtained by expanding the Boltzmann factor e^{-H} in powers of the new interactions. At zeroth order one obtains the vertex functions $\Gamma^{(N)}$ of the original Hamiltonian (2.1). At first order in the new interactions, the additional term corresponds to an N-point vertex function with an insertion of the operator (2.15), to be calculated with the original Hamiltonian (2.1). Some low-order graphs for these new vertex functions are shown in Fig. 1.

The fact that the insertions (2.15) have a negative power of Λ for $d > 4$ indicates that they are likely to give a behavior which is no more singular than that of the original vertex functions. However this naive expectation is clouded by the fact that the graphs in Fig. 1 represent an expansion in a large coupling, effectively $\bar{u}_0(\Lambda/m)^{3-d/2}$ as before. Therefore we must set up RG equations for the vertex functions with operator insertions, in order to establish that they are indeed very much smaller than the original vertex functions. Further, in order to establish the behavior in ϵ expansion, we must renormalize these new vertex functions in six dimensions.

The theory of renormalization of composite operators such as $S\phi^4$ and $F\phi^4$ is well established.[16] The central theorem is that one must simultaneously renormalize all operators which have equal or lower naive dimensions, and the same transformation properties under the symmetry group of the Hamiltonian. The theory has been applied mainly in two circumstances: (a) for operators of dimension less than or equal to those in the Hamiltonian itself, where the number of operators to be considered is rather small[17]; and (b) for operators of arbitrarily high dimension, which belong to an irreducible representation which is different from

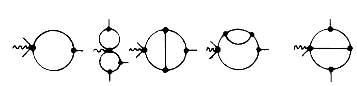

FIG. 1. Low-order graphs contributing to a three-point vertex function with insertion (2.15).

that of any others operators of the same naive dimension, so that no mixing of operators occurs.[18]

The only exception to these cases known to us is the calculation of the correction to scaling exponent of Heisenberg models in $2+\epsilon$ dimensions[19] where five operators mix and in fact there are technical complications over and above those which we face here.

The eight linearly independent operators of the same naive dimension which we must consider can be written:

$$A_1(x)=(1/4!)S\phi^4(x)\,, \tag{2.16a}$$

$$A_2(x)=(1/4!)F\phi^4(x)\,, \tag{2.16b}$$

$$A_3(x)=-(1/3!)\mu^{-\epsilon/2}Q_{ijk}\phi_i(x)\phi_j(x)\Box\phi_k(x)\,, \tag{2.16c}$$

$$A_4(x)=(1/2!)\mu^{-\epsilon}[\Box\phi(x)]^2\,, \tag{2.16d}$$

$$A_5(x)=(1/2!)\mu^{-\epsilon}\phi_i(x)\Box^2\phi_i(x)\,, \tag{2.16e}$$

$$A_6(x)=(1/3!)\mu^{-\epsilon/2}\Box[Q_{ijk}\phi_i(x)\phi_j(x)\phi_k(x)]\,, \tag{2.16f}$$

$$A_7(x)=(1/2!)\mu^{-\epsilon}\Box[\phi_i(x)\Box\phi_i(x)]\,, \tag{2.16g}$$

$$A_8(x)=(1/2!)\mu^{-\epsilon}\Box^2[\phi^2(x)]\,. \tag{2.16h}$$

Here we have introduced explicit powers of μ to make all eight operators of the same naive dimension in $6-\epsilon$ dimensions. $\Box$ denotes $\nabla_i\nabla_i$. The N-point vertex function with an insertion of operator A_a is denoted by $\Gamma_a^{(N)}(p;q;u_0\mu)$, where p is the external momentum flowing in at the insertion. Since we are considering the massless theory the bare mass m_0^2 is determined by the other parameters. The dependence on μ comes from two sources. First, from the definition of the bare dimensionless coupling constant, namely:

$$g_0=u_0\mu^{(6-d)/2}\,,$$

and second from the explicit μ dependence of the inserted operators.

The vertex functions $\Gamma_a^{(N)}$ are not multiplicatively renormalizable. The fact that the operators A_a couple to operators of lower dimension gives rise to terms which behave as Λ^4 and Λ^2 in the perturbation expansion of these vertex functions. Only the part which is logarithmically divergent can be renormalized multiplicatively. The higher divergences have to be subtracted as in Ref. 17. Or, alternatively, one can define new vertex functions, which are derivatives with respect to the external momenta of the original ones, such that only logarithmic divergences remain. This is often done when one considers $\partial\Gamma^{(2)}/\partial k^2$ in the usual ϕ^4 theory. The Λ^2 term in $\Gamma^{(2)}$ is independent of momentum, and thus $\partial\Gamma^{(2)}/\partial k^2$ is multiplicatively renormalizable.

In our case we define

$$S_{j_1j_2j_3j_4}\Gamma_a^{\{1\}}+F_{j_1j_2j_3j_4}\Gamma_a^{\{2\}}=\Gamma^{(4)}_{aj_1j_2j_3j_4}\,, \tag{2.17a}$$

$$\Gamma_a^{\{3\}}Q_{j_1j_2j_3}=\left(\frac{\partial}{\partial q_1^2}+\frac{\partial}{\partial q_2^2}+\frac{\partial}{\partial q_3^2}\right)\Gamma^{(3)}_{aj_1j_2j_3}\,, \tag{2.17b}$$

$$\Gamma_a^{\{4\}}\delta_{j_1j_2}=\frac{\partial}{\partial q_1^2}\frac{\partial}{\partial q_2^2}\Gamma^{(2)}_{aj_1j_2}\,, \tag{2.17c}$$

$$\Gamma_a^{\{5\}}\delta_{j_1j_2}=\left(\frac{\partial}{\partial q_1^4}+\frac{\partial}{\partial q_2^4}\right)\Gamma^{(2)}_{aj_1j_2}\,, \tag{2.17d}$$

etc. Note that the decomposition in (2.17a) is unique. The combination of derivatives is determined by the combinations which appear in the operators. We have displayed explicitly the tensor indices of the original vertex functions $\Gamma^{(N)}$.

These new vertices have logarithmic integrals only. To see this one has to perform the power counting for a term of mth order in the perturbation expansion of $\Gamma_a^{\{b\}}$. One first notes that A_a has l_a fields and $8-2l_a$ powers of momentum, where l_a is the number of factors of the field in A_a. Correspondingly, there are $8-2l_b$ momentum derivatives applied to $\Gamma_a^{(b)}$, in order to obtain $\Gamma_a^{\{b\}}$, with l_b the number of external legs of the vertex.

If a graph has L loops and I internal lines, then power counting gives

$$\begin{aligned}\delta&=Ld-2I+(8-2l_a)-(8-2l_b)\\&=Ld-2I+2(l_b-l_a)\end{aligned}$$

for the difference between the number of momentum powers in the numerator and in the denominator. In terms of m we have

$$L=I-m\,,\quad I=\tfrac{1}{2}(3m+l_a-l_b)\,,$$

hence

$$\delta=\tfrac{1}{2}(m+l_a-l_b)(d-6)=L(d-6)\,. \tag{2.18}$$

At $d=6$ all terms of all the new vertices diverge logarithmically.

The renormalization can be performed in the following way: First the vertices are rendered dimensionless via

$$\hat\Gamma^{\{b\}}=\mu^{-\epsilon(l_b-4)/2}\Gamma^{\{b\}}\,. \tag{2.19}$$

Then the renormalization matrices are defined as

$$\hat\Gamma^{\{c\}}_{Ra}(p;q;u,\mu)=Z_{ab}Z^{l_c/2}\hat\Gamma_b^{\{c\}}(p;q;u_0;\mu)\,, \tag{2.20}$$

where the bare functions have been dimensionally regularized. Z is the conventional wave function renormalization constant.

The matrix Z_{ab} is determined by imposing the normalization conditions

$$\hat\Gamma^{\{b\}}_{Ra}(p;q;u,\mu)\big|_\mu=\delta_a^b\,, \tag{2.21}$$

where $|_\mu$ means that the vertex function is evaluated at the symmetry point μ appropriate to the particular vertex type. For example, $\Gamma_1^{\{1\}}$ contains

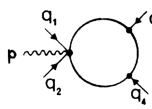

FIG. 2. Graph contributing to $\Gamma_1^{\{1\}}$. The momentum p is carried by the vertex insertion.

graphs like Fig. 2; the symmetry point here means $q_1^2=q_2^2=q_3^2=q_4^2=p^2=\mu^2$ and $(q_1+q_2)^2=(q_1+q_3)^2=$ etc. $=\frac{3}{2}\mu^2$, satisfying the constraint

$$3(q_1^2+q_2^2+q_3^2+q_4^2+p^2)=\sum_{\text{pairs } q,p}(q_1+q_2)^2,$$

obtained from momentum conservation.

With the normalization conditions (2.21), the dimensionless Z_{ab} can depend only on the dimensionless u and on ϵ, since the theory is dimensionally regularized. The RG equations obeyed by $\Gamma_{R,b}^{\{d\}}$ are obtained by taking $\mu\partial/\partial\mu$ of Eq. (2.20), holding g_0 constant. The result is

$$\left[\left(\mu\frac{\partial}{\partial\mu}+\beta(u)\frac{\partial}{\partial u}-\tfrac{1}{2}l_d\eta(u)-\tfrac{1}{2}\epsilon(4-l_d)\right)\delta_{ab}-\gamma_{ab}(u)\right]\hat{\Gamma}_{Rb}^{\{d\}}=0\,, \qquad (2.22)$$

where

$$\gamma_{ab}=-Z_{ac}\mu\frac{\partial}{\partial\mu}\bigg|_{g_0}Z_{cb}^{-1}+\tfrac{1}{2}\epsilon\sum_c Z_{ac}(l_c-4)Z_{cb}^{-1}. \qquad (2.23)$$

The second term on the right-hand side comes from the explicit μ dependence of A_a and $\hat{\Gamma}$. Z_{ab} is obtained as a power series in the bare coupling constant g_0 by combining Eqs. (2.19)–(2.21). We find

$$Z_{ab}^{-1}=\hat{\Gamma}_a^{\{b\}}\big|_\mu Z^{l_b/2}. \qquad (2.24)$$

The possibility of inverting Eq. (2.21) makes the normalization conditions a somewhat more elegant approach in this case, than that of minimal subtraction of ϵ poles.

In Secs. II C and II D we shall give some details on the calculation of the one loop contribution to γ_{ab}, and discuss the diagonalization of this matrix to obtain the anomalous dimensions and the explicit solution of (2.22).

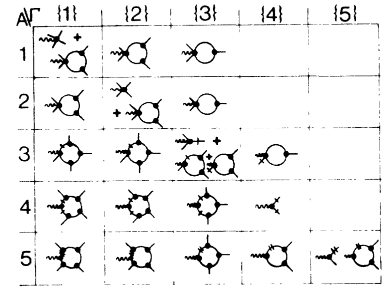

FIG. 3. Matrix of graphs contributing to $\hat{\Gamma}_a^{\{b\}}$ in Eq. (2.25). An extra stroke on a leg indicates a factor q^2. Empty entries indicate that either at this level the corresponding vertex is not generated, or that it has no pole in ϵ.

C. Calculation of γ_{ab} to order ϵ

We start this section by making a preliminary simplifying remark. Operators such as (2.16f)–(2.16h) which are total derivatives can simply be neglected. They do indeed require renormalization amongst their *own* vertex types (and this is just the renormalization of the lower-dimensional operators without total derivatives), but are *not* overall divergent when inserted into the other vertex types (2.19). Therefore only the five operators (2.16a)–(2.16e) and vertex types (2.19) need be considered.

The bare vertex functions $\hat{\Gamma}_a^{\{b\}}$ have a δ_a^b contribution from tree diagrams. The one loop graphs are shown in the matrix of graphs Fig. 3. An extra stroke on a leg at a vertex indicates a factor of q^2 from the derivatives in Eqs. (2.16). This factor of q^2 exactly cancels the massless propagator $1/q^2$. Some contributions in the third and fourth columns are therefore independent of external momenta and give zero when the differentiations in Eqs. (2.17) are done. In particular the momentum dependence of the graphs in the last two columns is such that *all* off-diagonal elements in the last column are zero. The form of Z_{ab} means that we need consider only the first four operators and vertex types. Explicit calculation gives[20]

$$\hat{\Gamma}_a^b\big|_\mu=\begin{bmatrix}1+6g_0^2\mu^{-\epsilon}\rho_{ss}/\epsilon & 6g_0^2\mu^{-\epsilon}\rho_{sf}/\epsilon & \frac{1}{3}g_0\mu^{-\epsilon/2}/\epsilon & 0\\ 6g_0^2\mu^{-\epsilon}\rho_{fs}/\epsilon & 1+6g_0^2\mu^{-\epsilon}\rho_{ff}/\epsilon & \frac{1}{2}g_0\mu^{-\epsilon/2}\kappa/\epsilon & 0\\ -8g_0^2\mu^{-3\epsilon/2}\gamma_s/\epsilon & -8g_0^3\mu^{-3\epsilon/2}\gamma_f/\epsilon & 1+\frac{1}{3}g_0^2\mu^{-\epsilon}\beta/\epsilon & \frac{1}{9}g_0\mu^{-\epsilon/2}\alpha/\epsilon\\ 12g_0^4\mu^{-2\epsilon}\gamma_s/\epsilon & 12g_0^4\mu^{-2\epsilon}\gamma_f/\epsilon & g_0^3\mu^{-3\epsilon/2}\beta/\epsilon & 1\end{bmatrix}. \qquad (2.25)$$

Only the divergent $(1/\epsilon)$ parts of the graphs are exhibited. Finite parts do not contribute to γ_{ab} at this order. Note that all elements are indeed dimensionless. The coefficients ρ_{ss}, etc., arise from the tensor contractions in each graph, and are defined as follows:

$$(Q_{i_1ij}Q_{i_2jk}S_{iki_3i_4})_S=\rho_{ss}S_{i_1i_2i_3i_4}+\rho_{sf}F_{i_1i_2i_3i_4}\,, \qquad (2.26)$$

$$(Q_{i_1ij}Q_{i_2jk}F_{iki_3i_4})_S = \rho_{fs}S_{i_1i_2i_3i_4}+\rho_{ff}F_{i_1i_2i_3i_4}, \tag{2.27}$$

$$(Q_{i_1ij}F_{iji_3i_4})_S = \kappa Q_{i_1i_3i_4}, \tag{2.28}$$

$$(Q_{i_1ij}Q_{i_2jk}Q_{i_3kl}Q_{i_4li})_S = \gamma_s S_{i_1i_2i_3i_4}+\gamma_f F_{i_1i_2i_3i_4}, \tag{2.29}$$

where S means the tensor is symmetrized, as in, e.g., Eq. (2.13b).

It is now straightforward to use Eqs. (2.23) and (2.24), the known form[7] of Z,

$$Z=1-\tfrac{1}{6}g_0^2\mu^{-\epsilon}\alpha/\epsilon+O(g_0^4)$$

and $u^2=g_0^2\mu^{-\epsilon}+O(g_0^4)$ [from Eq. (2.4)] to obtain

$$\gamma_{ab}=\begin{bmatrix} 6u^2\rho_{ss}-\frac{1}{3}\alpha u^2 & 6u^2\rho_{sf} & \frac{1}{3}u & 0\\ 6u^2\rho_{fs} & 6u^2\rho_{ff}-\frac{1}{3}\alpha u^2 & \frac{1}{2}\kappa u & 0\\ -8u^3\gamma_s & -8u^3\gamma_f & \frac{1}{3}u^2\beta-\frac{1}{4}u^2\alpha-\frac{1}{2}\epsilon & -\frac{1}{9}\alpha u\\ 12u^4\gamma_s & 12u^4\gamma_f & u^3\beta & -\frac{1}{6}u^2\alpha-\epsilon \end{bmatrix}. \tag{2.30}$$

Terms of higher order in ϵ and u^2 are of course omitted in this matrix.

D. Eigenvalues of γ_{ab}

In the remainder of this section we set u equal to its fixed-point value (2.10). The diagonalization of γ_{ab} is then simplified because two of the eigenvectors can be obtained by using the equations of motion. The first use is easy to see: in the determinant of $\gamma_{ab}-\lambda\delta_{ab}$, add $\frac{3}{2}u$ of the 3rd row to the 4th. The fourth row then becomes

$$0,\ 0,\ \tfrac{3}{2}u^3\beta-\tfrac{3}{8}u^3\alpha-\tfrac{3}{4}\epsilon u-\tfrac{3}{2}\lambda u,\ \ -\epsilon-\lambda.$$

The third element is $-\frac{3}{2}u(\epsilon+\lambda)$ when u has its fixed-point value (2.10). Therefore, one of the eigenvalues is $\lambda=-\epsilon$. The particular combination of rows to obtain this result corresponds to

$$A_4+\tfrac{3}{2}uA_3,$$

i.e.,

$$(1/2!)\mu^{-\epsilon}\Box\phi_i(\Box\phi_i-\tfrac{1}{2}u\mu^{+\epsilon/2}Q_{ijk}\phi_j\phi_k).$$

The combination in parentheses vanishes according to the classical equations of motion, corresponding to the Lagrangian (2.1), with u_0 replaced by u and the mass set to zero. At the order of one loop these replacements can be made. In general one has to use the implications of these equations on the renormalized vertices with the insertions of the operators which enter the equations of motion. (See, e.g., Refs. 17 and 14.)

Guided by this remark we look for an eigenvector for a remaining 3×3 determinant obtained by a similarity transformation aligning the "4th axis" with the first eigenvector:

$$\begin{vmatrix} 6u^2\rho_{ss}-\frac{1}{3}\alpha u^2-\lambda & 6u^2\rho_{sf} & \frac{1}{3}u\\ 6u^2\rho_{fs} & 6u^2\rho_{ff}-\frac{1}{3}\alpha u^2-\lambda & \frac{1}{3}\kappa u\\ -8u^3\gamma_s & -8u^3\gamma_f & \frac{1}{3}u^2\beta-\frac{5}{12}\alpha u^2-\frac{1}{2}\epsilon-\lambda \end{vmatrix}. \tag{2.31}$$

The corresponding operators are $(1/4!)S\phi^4$, $(1/4!)F\phi^4$, $-(1/3!)\mu^{-\epsilon/2}Q\phi^2\Box\phi$ [Eq. (2.16)]. They are linked by a naive equation of motion

$$\begin{aligned} 0&=Q_{ijk}\phi_i\phi_j(\Box\phi_k-\tfrac{1}{2}u\mu^{+\epsilon/2}Q_{klm}\phi_l\phi_m)\\ &=Q_{ijk}\phi_i\phi_j\Box\phi_k-\tfrac{1}{2}u\mu^{\epsilon/2}(Q_{ijk}Q_{klm})\phi_i\phi_j\phi_l\phi_m.\\ &\propto -A_3-2\beta_s uA_1-2\beta_f uA_2, \end{aligned}$$

where the β's are defined by

$$(Q_{ijk}Q_{klm})_S=\beta_s S_{ijlm}+\beta_f F_{ijlm}. \tag{2.32}$$

Now take therefore the combinations $(-2\beta_s u)\times$ (1st row) $-(2\beta_f u)\times$ (2nd row)$-$(3rd row). It is straightforward to show that the three terms thus obtained have a common factor $\frac{1}{6}\alpha u^2-\epsilon-\lambda$, using the algebraic identities

$$\tfrac{1}{3}\beta_s+\tfrac{1}{2}\kappa\beta_f=\tfrac{1}{6}(\alpha+2\beta), \tag{2.33a}$$

$$4\gamma_f=6\rho_{sf}\beta_s+(6\rho_{ff}-2\beta)\beta_f, \tag{2.33b}$$

$$4\gamma_s=(6\rho_{ss}-2\beta)\beta_s+6\rho_{fs}\beta_f, \tag{2.33c}$$

and the fixed-point equation (2.10).

These identities, as well as

$$\rho_{ss}=\tfrac{1}{3}(\alpha+2\beta_s),\quad \rho_{sf}=\tfrac{2}{3}\beta_f, \tag{2.34}$$

can be obtained by manipulations amongst the tensor definitions (2.26)–(2.28) and (2.32). For example, putting (2.13b) into the left-hand side of (2.26) and using (2.32) on the result, one arrives at (2.34).

Summarizing these results, there are two eigenvalues

$$\lambda = -\epsilon\,, \tag{2.35a}$$

$$\lambda = -\epsilon + \tfrac{1}{6}\alpha u^2\,, \tag{2.35b}$$

while the remaining eigenvalues are solutions of the equation

$$0 = \det \begin{vmatrix} 5u^2\rho_{ss} - \lambda & 5u^2\rho_{sf} \\ u^2(6\rho_{fs} - \kappa\beta_s) & u^2(6\rho_{ff} - \tfrac{1}{3}\alpha - \kappa\beta_f) - \lambda \end{vmatrix}\,, \tag{2.35c}$$

which is obtained from (2.31) using the identities (2.33) and (2.34). Although in general, no further simplification occurs in the solution to this quadratic, it can be shown that the roots are always real.

A proper analysis of the equations of motion would be required to establish if (2.35a) and (2.35b), in the form $-\epsilon$ and $-\epsilon+\eta$ [cf. Eq. (2.6)], remain eigenvalues of γ_{ab} at higher order in perturbation theory.

Given the set of eigenvalues λ_α, it remains to write down the conditions for irrelevance of the ϕ^4 interactions. Of all the types $\Gamma^{(b)}$ which are multiplicatively renormalizable, $\Gamma^{(4)}$ (i.e., $b=1,2$) alone is a simple vertex function. So, for clarity, we consider $\Gamma^{(4)}_\alpha$, where α denotes an insertion of that linear combination of operators A_a which corresponds to the eigenvector numbered α.

Consider these functions at the critical point (zero mass). Eq. (2.12) implies

$$\Gamma^{(4)}(q_i;u^*,\mu) = \mu^{2\eta}F(q_i)\,. \tag{2.36}$$

Let k be a scale of the momenta q_i, then dimensional analysis prescribes the dependence of $\Gamma^{(4)}$ on k. It is

$$\Gamma^{(4)}(q_i;u^*,\mu) = \mu^{2\eta}k^{-2+\epsilon-2\eta}f(q_i/k)\,. \tag{2.37}$$

On the other hand the solution of the renormalization-group equation for $\Gamma^{(4)}$, at the fixed point, gives

$$\Gamma^{(4)}_\alpha(q_i;p;u^*,\mu) = \mu^{2\eta+\lambda_\alpha}G(q_i;p)\,. \tag{2.38}$$

If the insertion is at zero momentum, dimensional analysis gives

$$\Gamma^{(4)}_\alpha(q_i;0,u^*,\mu) = \mu^{2\eta+\lambda_\alpha}k^{-2\eta-\lambda_\alpha}g(q_i/k)\,. \tag{2.39}$$

But it is just this vertex which gives the coefficient of g_α in $\Gamma^{(4)}$, where g_α is the coupling constant with which A_α enters the Lagrangian. Hence irrelevance is the condition that as $k\to 0$, for q_i/k finite, (2.39) vanish faster than (2.37). That is,

$$2 - \epsilon - \lambda_\alpha > 0\,. \tag{2.40}$$

In particular the ϕ^4 interactions are irrelevant for the nontrivial fixed point in the dimension in which they are marginal for the Gaussian fixed point ($d=4, \epsilon=2$) if all the λ_α's are negative. Thus, a positive λ would be a warning, indicating the possible existence of a dangerous relevant operator.

III. SPECIFIC EXAMPLE–ASHKIN-TELLER-POTTS MODEL

Particular examples of the class of Hamiltonians (2.1) yield many interesting models. The five-component field belonging to the $L=2$ representation of O(3) is the order parameter in the model for the nematic to isotropic transition in liquid crystals[9]; an ϵ expansion exists in $6-\epsilon$ dimensions, although it is perhaps unlikely that it controls a second-order phase transition in three dimensions.[10] The (N^2-1)-dimensional adjoint representation of SU(N) can also be studied[13] as a model for the strong interactions of elementary particles (with $d=4, \epsilon=2$ as the physical value); it is remarkable that to first order in ϵ, a real infrared stable fixed point exists only for SU(3) and SU(4).

The strongest evidence for a model whose ϵ expansion in $6-\epsilon$ dimensions does control physics is found in the percolation problem.[11] This can be formulated as the $n=0$ limit of the generalized Ashkin-Teller-Potts model,[21] which consists of an n-component field, with a symmetry group S_{n+1} (the permutation group of $n+1$ objects). The field theory version of this model[22] contains trilinear couplings, with an invariant tensor which can be written

$$Q_{ijk} = \sum_{\alpha=1}^{n+1} e_i^\alpha e_j^\alpha e_k^\alpha\,, \tag{3.1}$$

where e_i^α ($\alpha = 1,2,\ldots,n+1$) are a set of $n+1$ vectors, of length $(n)^{1/2}$ to the $n+1$ vertices of the "tetrahedron" in n dimensions. They are normalized to

$$\sum_{i=1}^{n} e_i^\alpha e_i^\beta = (n+1)\delta_{\alpha\beta} - 1\,, \tag{3.2}$$

and satisfy

$$\sum_{\alpha=1}^{n+1} e_i^\alpha e_j^\alpha = (n+1)\delta_{ij}\,, \tag{3.3}$$

All tensor contractions in graphs can be evaluated using (3.2) and (3.3). The ϵ expansion of this model has received much attention; a real infrared stable fixed point exists[12] for $n<\frac{7}{3}$. There is evidence that for $n=0$ this fixed point may indeed control behavior in the percolation problem in three dimensions.[7,10,12] It seems unlikely that the existence of a fixed point in ϵ expansion for $n=2$ has anything to do with the continuous transition predicted for $n\le 3$ in two dimensions.[23]

In all of these field-theory models, dominated by ϕ^3 interactions, there is always the problem of the thermodynamic stability of the ground state. This has been much discussed in the above references (see also Ref. 24) but we know of no convincing treatment of it.

Modulo this problem, we can calculate the effect on the last model due to perturbations by the ϕ^4 interactions, $(\phi^2)^2$ and $F_{ijkl}\phi_i\phi_j\phi_k\phi_l$, where

$$F_{ijkl}=\sum_\alpha e_i^\alpha e_j^\alpha e_k^\alpha e_l^\alpha\,, \tag{3.4}$$

using the general results of Sec. II. The tensorial contractions in Eqs. (2.8), (2.9), (2.26), and (2.27) are calculated using Eqs. (3.2) and (3.3). The results are

$$\alpha=(n+1)^2(n-1)\,, \tag{3.5a}$$

$$\beta=(n+1)^2(n-2)\,, \tag{3.5b}$$

$$\rho_{ss}=\tfrac{1}{3}(n+1)^2(n-3)\,, \tag{3.5c}$$

$$\rho_{sf}=\tfrac{2}{3}(n+1)\,, \tag{3.5d}$$

$$\rho_{fs}=(n+1)^3\,, \tag{3.5e}$$

$$\rho_{ff}=(n+1)^2(n-2)\,. \tag{3.5f}$$

The explicit value of the fixed-point coupling [Eq. (2.10)] is

$$u^{*2}=2\epsilon/(n+1)^2(7-3n)+O(\epsilon^2)\,. \tag{3.5g}$$

(This is positive for $n<\frac{7}{3}$ as mentioned before).

Substitution into Eqs. (2.35) gives the eigenvalues

$$\lambda=-\epsilon+O(\epsilon^2)\,, \tag{3.6a}$$

$$\lambda=-\epsilon+\epsilon(n-1)/3(7-3n)+O(\epsilon^2)\,, \tag{3.6b}$$

$$\lambda_\pm=[\epsilon/3(7-3n)][19n-47 \pm(81n^2-186n+889)^{1/2}]+O(\epsilon^2)\,. \tag{3.6c}$$

For the percolation problem we set $n=0$ and obtain

$$\lambda=-\epsilon,\ -\tfrac{22}{21}\epsilon\,,\ [-\tfrac{47}{21}\pm\tfrac{1}{21}(889)^{1/2}]\epsilon. \tag{3.7}$$

We note first that all of these eigenvalues are negative, which is at least an indication that the ϕ^3 fixed point remains stable to ϕ^4 perturbations down to four dimensions, where the Gaussian fixed point becomes unstable [see the remarks after Eq. (2.37)]. Even for $\epsilon=3$ and 4, the stability condition (2.37) is satisfied for all the eigenvalues (3.7), although it would be remarkable if the ϵ expansion turned out to be a reliable guide for these large values of ϵ. Nevertheless taken at their face value these results do indicate that ϕ^4 interactions will not change the results obtained previously by neglecting them, and that the ϕ^3 fixed point is physically important.

By contrast let us consider the $n=2$ case briefly. There is still a fixed point in the ϵ expansion, according to Eq. (3.5g). When $n=2$ there is only one ϕ^4 invariant and we use the results of the appendix to obtain the roots

$$\lambda=-\epsilon\,,\ -\epsilon+\tfrac{1}{3}\epsilon\,,\ \tfrac{20}{3}\epsilon\,. \tag{3.8}$$

[The last root is also obtained as the root λ_+ in Eq. (3.6c) when $n=2$.]

The convergence of the ϵ expansion is of course even more unreliable here, because the typical expansion factor $\epsilon/(7-3n)$ is in no reasonable way small. However, one may note a qualitative difference between Eq. (3.8) and Eq. (3.7): In Eq. (3.8) there is a positive root which shows one of the eigenvalues even starting off in the wrong direction for stability.

More generally we remark that this is a feature of all the cases we know which satisfy the additional requirement of having only one ϕ^4 invariant. These correspond to $n=5$ [for the spin-2 representation of O(3)], $n=2$ (for the Ashkin-Teller-Potts model) and $n=8$ [for the adjoint representation of SU(3)]. All of these n are greater than $\frac{2}{3}$ so that the $(\phi^2)^2$ perturbations have critical exponents which start off for small ϵ with the sign which will promote instability in lower dimensions.

Finally we remark on the case $n=1$, which is of some theoretical interest. In the lattice version of the model, this corresponds to the Ising model. Critical exponents are therefore expected to be mean-field-like for $d>4$ ($\epsilon<2$). This is indeed the case for the Hamiltonian (2.1) because all self-energy graphs vanish when $n=1$ [by arguments similar to the $n=-2$ case of $O(n)$-invariant Heisenberg models[25]]. The model is not Gaussian of course, because the three-point vertex is still nontrivial. The true Ising-model nature is reflected however in the eigenvalue λ_+ in Eq. (3.6c), which vanishes when $n=1$.

It is reasonable to speculate that this result remains true to all orders in ϵ. Our interpretation is that the zero eigenvalue remains down to four dimensions, where the ϕ^4 operator therefore becomes marginal for both the Gaussian and nontrivial ϕ^3 fixed points, which may even coincide there. One may speculate further; given these indications that the radius of convergence for $n=1$ is $\epsilon=2$, and that the numerical convergence of the ϵ expansion becomes poorer as n increases, then it seems rather unlikely that the ϵ-expansion fixed point for $n=2$ has physical significance.

IV. CONCLUSION

We have calculated the anomalous dimension of ϕ^4 operators to first order in ϵ, for a class of theories which have infrared stable fixed points in $6-\epsilon$ dimensions. We have explained the calculations in some detail, because the renormalization theory of composite operators of high dimension, although well understood, has not been widely applied. The aim of these calculations (apart from providing us with a pedagogical exercise) was to check whether the ϕ^4 interactions were relevant (destabilizing) or irrelevant for the ϕ^3 fixed point.

The results were analyzed in detail for the generalized Ashkin-Teller-Potts model, and we believe that even the first order in ϵ has provided more insight than might have been anticipated. There is strong evidence that for $n=0$ the ϕ^3 fixed point may be the physical fixed point describing the percolation problem, being stable to ϕ^4 interactions. There is a physically interpretable zero anomalous dimension for the Ising case ($n=1$). Correspondingly, for $n=2$ the ϕ^3 fixed point, when it exists, would appear to become unstable to ϕ^4 interactions as d is lowered towards 4.

It would certainly be interesting to push these calculations further by, e.g., looking at next order in ϵ, or studying ϕ^5 or higher interactions although substantial labor would be required.

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APPENDIX: PARTICULAR CASE OF ONE ϕ^4 INVARIANT

In some special cases (e.g., the Potts model for $n=2$) there is only one ϕ^4 invariant, $S\phi^4$. The form of the RG equations is as before. At first order in the ϵ expansion the matrix γ_{ab} is obtained by deleting the second row and second column of expression (2.30):

$$\gamma_{ab}=\begin{pmatrix} 6u^2\rho_{ss}-\frac{1}{3}\alpha u^2 & \frac{1}{3}u & 0 \\ -8u^3\gamma_s & \frac{1}{3}u^2\beta-\frac{1}{4}u^2\alpha-\frac{1}{2}\epsilon & \frac{1}{9}\alpha u \\ 12u^4\gamma_s & u^3\beta & -\frac{1}{6}u^2\alpha-\epsilon \end{pmatrix}. \tag{A1}$$

Using the equation of motion eigenvectors we obtain the eigenvalues $-\epsilon$ and $-\epsilon+\frac{1}{6}\alpha u^2$ as before. [As in Sec. II D, u has its fixed-point value (2.10) in these equations.] The third eigenvalue is then

$$\lambda=\tfrac{1}{2}\epsilon-\tfrac{11}{12}\alpha u^2+6u^2\rho_{ss}+\tfrac{1}{3}u^2\beta\,. \tag{A2}$$

It is amusing that in this special case the values of critical exponents to order ϵ can be expressed simply in terms of the number of components n of the field, irrespective of the particular symmetry group. From (2.33a), with $\beta_f=0$, we have

$$2\beta_s=\alpha+2\beta\,. \tag{A3}$$

Further taking a trace of (2.32) gives

$$Q_{ijk}Q_{jkm}=\tfrac{1}{2}(n+2)\beta_s\delta_{im}\,,$$

i.e., from the definition (2.8)

$$\alpha=\tfrac{1}{2}(n+2)\beta_s\,. \tag{A4}$$

Combining (A3) and (A4) the fixed-point value (2.10) is given by

$$\alpha u^{*2}=[2(n+2)/(3n-2)]\epsilon+O(\epsilon^2)\,. \tag{A5}$$

Thus, provided a real fixed-point exists, the critical exponents are given from Eqs. (2.6) and (2.7):

$$\eta=[(n+2)/3(3n-2)]\epsilon+O(\epsilon^2) \tag{A6}$$

and

$$1/\nu-2=[5(n+2)/3(3n-2)]\epsilon+O(\epsilon^2)\,. \tag{A7}$$

Finally using (2.34a)

$$\rho_{ss}=\tfrac{1}{3}(\alpha+2\beta_s)\,,$$

the third eigenvalue (A2) is

$$\lambda=[10(n+6)/3(3n-2)]\epsilon+O(\epsilon^2)\,. \tag{A8}$$

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[1]For reviews of the renormalization group in field theories see, for example, K. G. Wilson and J. Kogut, Phys. Rep. 12, 75 (1974); and *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, New York, 1976), Vol. VI.

[2]We consider the RG of renormalized perturbation theory, which does not require the introduction of other couplings (unlike, e.g., the recursion formula).

[3]See, for example, F. J. Wegner, Phys. Rev. B 6, 1891 (1972); E. Brezin, J. C. le Guillou, and J. Zinn-Justin, *ibid.* 10, 892 (1974); D. J. Wallace and R. K. P. Zia, J. Phys. C 8, 839 (1975).

[4]K. G. Wilson and J. Kogut in Ref. 1; M. K. Grover, L. P. Kadanoff, and F. J. Wegner, Phys. Rev. B 6, 311 (1972); G. R. Golner, *ibid.* 8, 339 (1973).

[5]S. K. Ma, Phys. Rev. A 10, 1818 (1974); D. J. Wallace and R. K. P. Zia in Ref. 3.

[6]A. M. Polyakov, Phys. Lett. B 59, 79 (1975); A. A. Migdal, Zh. Eksp. Teor. Fiz. 69, 1457 (1975); E. Brezin and J. Zinn-Justin, Phys. Rev. Lett. 36, 691 (1976); and Phys. Rev. B 14, 3110 (1976).

[7]D. J. Amit, J. Phys. A 9, 1441 (1976).

[8]A nontrivial irreducible representation has no invariant one-tensor. Therefore Q_{ijk} is traceless, $\delta_{ij}Q_{ijk}=0$, and there is a phase of the system (2.1) in which $\langle\phi_i\rangle=0$.

[9]P. G. de Gennes, Phys. Lett. A 30, 454 (1969).

[10]R. G. Priest and T. C. Lubensky, Phys. Rev. B 13, 4159 (1976), and Erratum.

[11]P. W. Kasteleyn and C. M. Fortuin, Physica (Utr.) 57, 536 (1972).

[12]A. B. Harris, T. C. Lubensky, W. K. Holcomb, and C. Dasgupta, Phys. Rev. Lett. 35, 327 (1975).

[13]A. J. McKane, D. J. Wallace, and R. K. P. Zia, Phys. Lett. B 65, 171 (1976).

[14]See, for example, Brezin *et al.* in Ref. 1.

[15]The corrections of order ϵu^2 in Eqs. (2.5)–(2.7) are absent in the 't Hooft scheme. To the order of one loop the three functions are identical, whether calculated by minimal subtraction, or via the normalization conditions.

[16]W. Zimmerman, in *Lectures on Elementary Particles and Quantum Field Theory*, edited by S. Deser *et al.* (MIT, Cambridge, Mass., 1971); J. Lowenstein, Phys. Rev. D 4, 2281 (1970); P. K. Mitter, *ibid.* 7, 2927 (1973).

[17]E. Brezin, C. de Dominicis, and J. Zinn-Justin, Nuovo Cimento Lett. 9, 483 (1974); see also Ref. 14, Sec. VII. 3.

[18]K. G. Wilson and J. Kogut, Ref. 1; F. Wegner, Ref. 3; D. J. Wallace and R. K. P. Zia, Ref. 3; a 2×2 matrix problem occurs in analysis of deep inelastic structure functions in asymptotically free gauge theories [see H. D. Politzer, Phys. Rep. 14, 129 (1974) for a review] and in predictions of nonleptonic decay rates [F. A. Wilczek, A. Zee, R. L. Kingsley, and S. B. Treiman, Phys. Rev. D 12, 2768 (1975)].

[19]E. Brezin, J. Zinn-Justin, and J. C. le Guillou, Phys. Rev. D 14, 2615 (1976).

[20]The integrals to be evaluated can be found in Ref. 7. The symmetry points μ are slightly different in Ref. 7. but this does not affect the residues of the poles in ϵ in Eq. (2.25).

[21]J. Ashkin and E. Teller, Phys. Rev. 64, 178 (1943); R. B. Potts, Proc. Camb. Philos. Soc. 48, 106 (1952).

[22]R. J. Golner, Phys. Rev. B 8, 3419 (1973); D. J. Amit and A. Shcherbakov, J. Phys. C 7, L96 (1974); R. K. P. Zia and D. J. Wallace, J. Phys. A 8, 1495 (1975); J. Rudnick, *ibid.* 8, 1125 (1975).

[23]R. J. Baxter, J. Phys. C 6, L445 (1973).

[24]S. Alexander, Solid State Commun. 14, 1069 (1974).

[25]R. Balian and G. Toulouse, Phys. Rev. Lett. 30, 544 (1973); M. E. Fisher, *ibid.* 30, 679 (1973).