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Non-equilibrium behavior at a liquid-gas critical point

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Abstract. Second-order phase transitions in a non-equilibrium liquid-gas model with reversible mode couplings, i.e., model H for binary-fluid critical dynamics, are studied using dynamic field theory and the renormalization group. The system is driven out of equilibrium either by considering different values for the noise strengths in the Langevin equations describing the evolution of the dynamic variables (effectively placing these at different temperatures), or more generally by allowing for anisotropic noise strengths, i.e., by constraining the dynamics to be at different temperatures in $d_\parallel$- and $d_\perp$-dimensional subspaces, respectively. In the first, *isotropic* case, we find one infrared-stable and one unstable renormalization group fixed point. At the stable fixed point, detailed balance is dynamically restored, with the two noise strengths becoming asymptotically equal. The ensuing critical behavior is that of the standard equilibrium model H. At the novel unstable fixed point, the temperature ratio for the dynamic variables is renormalized to infinity, resulting in an effective decoupling between the two modes. We compute the critical exponents at this new fixed point to one-loop order. For model H with spatially *anisotropic* noise, we observe a critical softening only in the $d_\perp$-dimensional sector in wave vector space with lower noise temperature. The ensuing effective two-temperature model H does not have any stable fixed point in any physical dimension, at least to one-loop order. We obtain formal expressions for the novel critical exponents in a double expansion about the upper critical dimension $d_c = 4 - d_\parallel$ and with respect to $d_\parallel$, i.e., about the equilibrium theory.

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1 Introduction

The theory of equilibrium dynamics for critical phenomena has identified a set of universality classes which describe the long-time, long-wavelength behavior of systems in the vicinity of critical points [1]. The situation is in this respect similar to that found in the study of static critical phenomena. However, whereas in the static case the nature of the universality classes is determined solely by the nature of the interactions that fixes the dimensionality n of the order parameter space as well as the effective spatial dimension d for its fluctuations, in the case of dynamical critical phenomena the corresponding universality classes are not simply characterized by the static or dynamic interactions which exist between the relevant system variables, but they also crucially depend on the conservation laws which are implemented by the dynamics and on the very existence of the detailed balance constraint that guarentees relaxation to thermal equilibrium in the long-time limit.

At the quantitative level, the equilibrium dynamics of critical systems is usually formulated in terms of coupled non-linear Langevin equations. Such equations typically include a dissipative term which involves the Hamiltonian describing the static critical properties and a Gaussian white noise term, which mimics thermal fluctuations and the random forces originating from couplings to fast, non-critical non-conserved modes. Furthermore, the equations of motion may also include purely reversible mode-coupling terms which represent dynamical interactions between the relevant slow system variables. In this framework, the conditions for the existence of detailed balance are (i) the Einstein relation between the relaxation constants and the noise strengths, and (ii) the condition that the probability current associated with reversible mode-coupling terms be divergence-free in the space of slow dynamic variables [2]. The existence of conservation laws in turn fixes the precise form of the relaxation terms [3]. If the coupled system of Langevin equations obeys these two conditions, it can be shown, e.g., by means of the associated Fokker-Planck equation, that the long-time steady state of the dynamics is indeed characterized by a Gibbsian probability distribution, with precisely the Hamiltonian that describes the static critical properties of the system. These conditions also insure that the dynamic susceptibilities reduce to the static ones in the limit of zero frequency, and imply the validity of fluctuation-dissipation relations between dynamical correlation functions and the dissipative parts of the response.

On a more general footing, one can also consider the study of dynamical phenomena which are truly non-equilibrium in the sense that they do not possess a steady state described by a Gibbs distribution, of which the best-known example is perhaps the Kardar-Parisi-Zhang equation (for $d > 1$) [4], which describes the curvature-driven epitaxial growth of a surface. Other prominent examples are driven diffusive systems [5], models of driven interfaces and growing surfaces [6], depinning transitions [7], and phase transitions from active to absorbing states [8], e.g., in diffusion-limited chemical reactions.

A question which lies at the interface between these two subjects, i.e., between the study of systems with equilibrium and far-from-equilibrium dynamics, concerns the asymptotic scaling properties of a critical system originally in thermal equilibrium, which is however subjected to a perturbation that violates the detailed balance conditions. This issue is also relevant in an experimental context, because the maintenance of thermodynamic equilibrium in a critical system during an experiment is a non-trivial task, as due to critical slowing-down the relaxation times become very long. For the interpretation of the experimental results, it might thus be important to know whether the dynamical system will eventually be driven to a genuine non-equilibrium fixed point which is characterized by scaling exponents distinct from the original equilibrium critical behavior. A priori, a perturbation from thermal equilibrium may either imply a violation of condition (i) or of condition (ii) above. This latter case includes, for example, driven lattice gases in which the terms added to the Langevin equation modelling the equilibrium diffusive dynamics stem from a global gradient such as an energy or mass current through the system [5]. The violation of condition (i) corresponds, on the other hand, to the coupling of the system to a *local* energy gradient. This type of perturbation was the subject of study in Refs. [9,10,11,12], where the consequences of the violation of the Einstein relations in several equilibrium dynamical models, largely describing magnetic systems, were investigated within the framework of the dynamical renormalization group. Specifically, two generic types of detailed-balance violations were studied there, namely (a) coupling the order parameter and additional conserved quantities to heat baths with different temperatures, and (b) allowing for spatially anisotropic noise correlations for conserved variables.

In this paper, we extend the above studies [9,10,11,12] to a model which describes the dynamics of a liquid-gas phase transition, or the phase separation of a binary liquid, at its critical point. This is the so-called model H in the terminology of Ref. [1], which incorporates the interaction of the conserved scalar order parameter, a linear combination of the mass and energy densities, namely essentially the free energy density, with the conserved transverse mass current vector via a reversible mode-coupling term [13]. These two fields are sufficient to describe the critical dynamics although critical effects can also be seen in other modes, such as the sound mode [14]. Furthermore, a renormalization group analysis was applied to investigate non-universal properties and crossover behavior in the equilibrium dynamics [15]. The original approaches to the equilibrium critical dynamics of model H utilized self-consistent mode coupling theory [16]. More recently, Patashinski performed a linear analysis of the equations of motion to study the effect of perturbations which induce a non-uniform non-equilibrium stationary state [17].

We use the response functional formalism [18] to express the dynamical equations as a path integral which represents a very convenient form to develop the perturbation expansion and subsequent renormalization group analysis. We compute the beta functions of the theory to one-loop order in the isotropic case where the two dynamical variables are coupled to heat baths at different temperatures and also in the case where we allow for spatially anisotropic noise correlations for these conserved fields. In the first situation, we obtain one stable and one unstable fixed point. The stable fixed point is just the ordinary equilibrium fixed point, for which detailed balance is dynamically restored, and in whose vicinity the two noise temperatures become equal. On the other hand, at the genuinely non-equilibrium, but unstable fixed point, the ratio between the temperature of the noise coupled to the order parameter and that of the noise coupled to the conserved current is renormalized to infinity. In this limiting case, the critical exponents of the conserved order parameter are simply those of the decoupled diffusive model B. This is due to the existence of a unidirectional random heat flow from the order parameter heat bath to the mass current thermal reservoir which renders the effect of the mode-coupling terms in the dynamics of the order parameter negligible. This situation is quite analogous to that found in the earlier study of the non-equilibrium Sásvari–Schwabl–Szépfalusy (SSS) model [10], although there the order parameter in not conserved, and hence follows model A dynamics in the corresponding limiting case. Moreover, owing to the fact that a conserved critical field always relaxes much slower than a diffusive mode, here we find no fixed point that would describe the above temperature ratio scaling to zero.

When allowing for spatially anisotropic noise, we observe a softening of the dynamics only in the $d_\perp$-dimensional sector in wave vector space with lower noise temperature. We show that the renormalized coupling constants diverge as the dimension of the subspace with higher noise temperature approaches $d_\| = 0.838454$; thus, at least to one-loop order, there exists no finite fixed point for any physical value of $d_\|$. We then obtain formal expressions for the novel critical exponents in a double expansion about the upper critical dimension $d_c = 4 - d_\|$ and with respect to $d_\|$, i.e., about the equilibrium theory. Here again the results are similar to those we have previously obtained in the study of the two-temperature non-equilibrium model J [11]. These conclusions are thus in line with our earlier studies and with previous observations that whereas models with a non-conserved order parameter are quite robust against violations of the detailed balance conditions, models with conserved dynamics seem to be extremely sensitive to this type of perturbations provided they are rendered *anisotropic* [12].

The structure of this paper is as follows: In section 2 we present the coupled non-linear Langevin equations which define our two non-equilibrium versions of model H, namely the isotropic case where the two dynamic variables are at different temperatures, and the situation where we allow for anisotropic noise correlations, and we show how these models can be mapped to a dynamic field theory. The action of these functionals is then separated into a Gaussian part and a non-linear contribution which is to be treated in a perturbation expansion. In section 3, we present the renormalization group analysis of the isotropic non-equilibrium model H to one-loop order and discuss its behavior near the renormalization group fixed points. In section 4, after introducing spatially anisotropic noise correlations, we derive the renormalization group flow equations for the resulting effective two-temperature model H, and we compute the values of the critical exponents in a formal double expansion about the equilibrium theory, also to one-loop order. We demonstrate that there exists no one-loop renormalizationg group fixed point in any physical dimension, however. Finally, in section 5 we present our conclusions.

2 Model equations

In this section, we briefly outline the basic model equations for our isotropic and anisotropic non-equilibrium generalizations of model H. Following the equilibrium theory [13], we consider a second-order phase transition for the scalar order parameter $\psi_0(\mathbf{x},t) = e(\mathbf{x},t) + (\bar{\mu} - T\bar{s})\,\rho(\mathbf{x},t)$ at a liquid-gas (or binary-fluid) critical point, where $e(\mathbf{x},t)$ is the energy density, $\rho(\mathbf{x},t)$ the mass density and $\bar{\mu}$, T and $\bar{s}$ are, respectively, the chemical potential, the temperature and the entropy at equilibrium (we denote unrenormalized quantities by the subscript '0'). This order parameter is dynamically coupled to the transverse mass current $\mathbf{j}_0(\mathbf{x},t)$, which satisfies $\nabla\cdot\mathbf{j}_0 = 0$. Notice that the longitudinal current $\mathbf{j}_0^{\|}$ is related to the mass density through the continuity equation $\partial_t\rho + \nabla\cdot\mathbf{j}_0^{\|} = 0$. The two dynamic fields ψ_0 and $\mathbf{j}_0$ yield three hydrodynamic modes, namely thermal and viscous diffusion. A fourth hydrodynamic mode, i.e., the sound mode, is also present in a fluid, but for low momenta its characteristic frequency is much higher than the frequencies of the shear and energy modes and one may thus disregard it to first approximation.

Next, the static critical properties of the system are described by the following Landau–Ginzburg–Wilson free energy functional in d space dimensions,

$$H_0[\psi_0,\mathbf{j}_0] = \int d^d x \,\left\{\frac{r_0}{2}\,\psi_0^2(\mathbf{x}) + \frac{1}{2}\,[\nabla\psi_0(\mathbf{x})]^2 + \frac{u_0}{4!}\,\psi_0^4(\mathbf{x}) + \frac{1}{2}\,\mathbf{j}_0^2(\mathbf{x})\right\}\,, \quad (2.1)$$

where $r_0 = (T - T_c^0)/T_c^0$ denotes the relative deviation from the mean-field critical temperature T_c^0. Note that the quadratic term in $\mathbf{j}_0$ simply represents the kinetic energy of a stationary mass current. Since $\mathbf{j}_0$ itself is a non-critical variable, the coefficient in front of this term will be weakly temperature-dependent only, and can thus be set constant in the vicinity of T_c, and eventually absorbed into $\mathbf{j}_0$.

This effective free energy (2.1) determines the equilibrium probability distribution for the vector order parameter ψ_0 and for the mass current $\mathbf{j}_0$,

$$P_{\rm eq}[\psi_0,\mathbf{j}_0] = \frac{e^{-H_0[\psi_0,\mathbf{j}_0]/k_{\rm B}T}}{\int \mathcal{D}[\psi_0]\,\mathcal{D}[\mathbf{j}_0]\,e^{-H_0[\psi_0,\mathbf{j}_0]/k_{\rm B}T}}\,. \quad (2.2)$$

Note that it follows from (2.2) that at the purely static level the order parameter $\psi_0(\mathbf{x})$ and the mass current $\mathbf{j}_0(\mathbf{x})$ are completely decoupled; moreover, $\mathbf{j}_0(\mathbf{x})$ is a Gaussian variable, whose contribution to the free energy can be readily factored out of the functional integral. The task is then reduced to the computation of two independent critical exponents, e.g., η and ν, by means of perturbation theory with respect to the static non-linear coupling u_0 and by employing the renormalization group procedure, within a systematic expansion in terms of $\epsilon = 4 - d$ about the static upper critical dimension $d_c = 4$. Here, η describes the power-law decay of the order parameter correlation function at criticality, $\langle\psi(\mathbf{x})\psi(\mathbf{x}')\rangle \propto 1/|\mathbf{x}-\mathbf{x}'|^{d-2+\eta}$, or, equivalently, of the static susceptibility, $\chi(\mathbf{q}) \propto 1/q^{2-\eta}$, and the exponent ν characterizes the divergence of the correlation length as T_c is approached, $\xi \propto |T - T_c|^{-\nu}$. Notice that fluctuations also shift the true transition temperature T_c downwards as compared to the mean-field critical temperature T_c^0, i.e., $r_{0c} = T_c - T_c^0 < 0$. Since we will need to consider (time-dependent) correlation functions of the dynamical variables, it is convenient to add to the free energy functional a term involving external sources h and $\mathbf{A}$, whereupon the (generating) functional reads $H = H_0 - \int d^d x\,[h(\mathbf{x})\psi_0(\mathbf{x}) + \mathbf{j}_0(\mathbf{x})\cdot\mathbf{A}(\mathbf{x})]$.

With this free energy functional H, the coupled non-linear Langevin equations defining model H read [13]

$$\frac{\partial\psi_0}{\partial t} = \lambda_0\nabla^2\frac{\delta H}{\delta\psi_0} - g_0\nabla\psi_0\cdot\frac{\delta H}{\delta\mathbf{j}_0} + \eta \quad (2.3)$$

$$= \lambda_0\nabla^2(r_0 - \nabla^2)\psi_0 + \frac{\lambda_0 u_0}{6}\nabla^2\psi_0^3 - \lambda_0\nabla^2 h - g_0\mathbf{j}_0\cdot\nabla\psi_0 + g_0\,\nabla\psi_0\cdot\mathbf{A} + \eta \quad (2.4)$$

and

$$\frac{\partial\mathbf{j}_0}{\partial t} = \mathcal{T}\left[D_0\nabla^2\frac{\delta H}{\delta\mathbf{j}_0} + g_0\nabla\psi_0\frac{\delta H}{\delta\psi_0} + \zeta\right] \quad (2.5)$$

$$= \mathcal{T}\left[D_0\nabla^2\mathbf{j}_0 - D_0\nabla^2\mathbf{A} + g_0\nabla\psi_0(r_0 - \nabla^2)\psi_0 - g_0\nabla\psi_0 h + \zeta\right]. \quad (2.6)$$

Here, g_0 denotes the strength of the reversible mode-coupling terms, and η and ζ represent fluctuating forces with zero mean, $\langle\eta(\mathbf{x},t)\rangle = 0$, $\langle\zeta^\alpha(\mathbf{x},t)\rangle = 0$. The couplings λ_0 and D_0 are, respectively, the thermal conductivity and shear viscosity (in appropriate units). $\mathcal{T}[\dots]$ denotes a projection operator which selects the transverse part of the vector in brackets; in Fourier space $\mathcal{T}^{\alpha\beta}(\mathbf{k}) = \delta^{\alpha\beta} - k^\alpha k^\beta / k^2$. The above Langevin-type equations of motion are invariant under Galilean transformations, a symmetry which can be utilized in the renormalization procedure (see below and the appendix).

In order to fully characterize the dynamics, we furthermore need to specify the correlations of the stochastic forces. Given that both the order parameter and the transverse current are conserved quantities, the strength of the random forces has to vanish at zero momentum. We assume that these random fields are Gaussian correlated and we write, in general, the second moment of the distributions in Fourier space as:

$$\langle \eta(\mathbf{k},\nu)\eta(\mathbf{k}',\nu')\rangle = 2\widetilde{\lambda}_0(\mathbf{k})\,\delta(\mathbf{k}+\mathbf{k}')\,\delta(\nu+\nu') \tag{2.7}$$

and

$$\langle \zeta^\alpha(\mathbf{k},\nu)\zeta^\beta(\mathbf{k}',\nu')\rangle = 2\widetilde{D}_0(\mathbf{k})\,\delta(\mathbf{k}+\mathbf{k}')\,\delta(\nu+\nu') \times \left(\delta^{\alpha\beta} - \frac{k^\alpha k^\beta}{k^2}\right), \tag{2.8}$$

where the transverse projector in (2.8) insures that the random force is in the transverse directions only. For the equilibrium model H, the functions $\widetilde{\lambda}_0(\mathbf{k})$ and $\widetilde{D}_0(\mathbf{k})$ are equal to $\widetilde{\lambda}_0(\mathbf{k}) = \lambda_0\, k_B T\, k^2$, $\widetilde{D}_0(\mathbf{k}) = D_0\, k_B T\, k^2$, i.e., the noise correlators satisfy the Einstein relations and the conservation conditions $\widetilde{\lambda}_0(\mathbf{0}) = \widetilde{D}_0(\mathbf{0}) = 0$.

As stated above, we will consider two choices for the noise correlators which do not satisfy the Einstein conditions and therefore describe *non-equilibrium* versions of model H. Both these choices can be viewed as generic perturbations from the equilibrium situation. In the first case, we take:

$$\widetilde{\lambda}_0(\mathbf{k}) = \widetilde{\lambda}_0\, k^2 \tag{2.9}$$

and

$$\widetilde{D}_0(\mathbf{k}) = \widetilde{D}_0\, k^2, \tag{2.10}$$

with $\widetilde{\lambda}_0/\lambda_0 \neq \widetilde{D}_0/D_0$. Since the equilibrium model H corresponds to the case in which $\widetilde{\lambda}_0/\lambda_0 = \widetilde{D}_0/D_0 = k_B T$, one may interpret Eqs. (2.9) and (2.10) as describing a situation in which the two dynamical variables, the order parameter and the transverse current, are effectively placed in contact with heat baths at *different* temperatures.

In the second case, we take more generally:

$$\widetilde{\lambda}_0(\mathbf{k}) = \widetilde{\lambda}_0^{\parallel}\, k_\parallel^2 + \widetilde{\lambda}_0^{\perp}\, k_\perp^2 \tag{2.11}$$

and

$$\widetilde{D}_0(\mathbf{k}) = \widetilde{D}_0^{\parallel}\, k_\parallel^2 + \widetilde{D}_0^{\perp}\, k_\perp^2\ . \tag{2.12}$$

This choice of the noise correlators corresponds to a situation where two spatial sectors of dimensions $d_\parallel$ and $d_\perp$ are placed at different temperatures, i.e., where real space isotropy is broken [19]. Without loss of generality, we set the effective noise temperature higher in the parallel subspace.

Once the noise distribution is specified, one can represent the Langevin equations (2.4) and (2.6), with (2.7) and (2.8), as a dynamic field theory, following standard procedures [18,10]. This results in a probability distribution for the dynamic fields ψ_0 and $\mathbf{j}_0$,

$$P[\{\psi_0, \mathbf{j}_0\}] \propto \int \mathcal{D}[\{i\widetilde{\psi}_0\}] \int \mathcal{D}[\{i\widetilde{\mathbf{j}}_0\}] \times \times\, e^{J[\{\widetilde{\psi}_0\},\{\psi_0\},\{\widetilde{\mathbf{j}}_0\},\{\mathbf{j}_0\}]}\ , \tag{2.13}$$

with the statistical weight given by the Janssen-De Dominicis functional $J = J_{\rm har} + J_{\rm rel} + J_{\rm mc} + J_{\rm sc}$, which we divide into an harmonic part $J_{\rm har}$, which one can integrate exactly; the purely relaxational (of static origin) and reversible dynamic non-linear terms $J_{\rm rel}$ and $J_{\rm mc}$, which are to be expanded in a power series, giving rise to the perturbation series in terms of Feynman diagrams; and a term $J_{\rm sc}$ which depends on the external sources.

The harmonic part, in terms of the original dynamic fields $\psi_0(\mathbf{k},\nu)$, $\mathbf{j}_0(\mathbf{k},\nu)$ and the auxiliary fields $\widetilde{\psi}_0(\mathbf{k},\nu)$, $\widetilde{\mathbf{j}}_0(\mathbf{k},\nu)$ (we use, for convenience, the Fourier space representation) reads

$$\begin{aligned} &J_{\rm har}[\{\widetilde{\psi}_0\},\{\psi_0\},\{\widetilde{\mathbf{j}}_0\},\{\mathbf{j}_0\}] \\ &= \int \frac{d^dk_1}{(2\pi)^d}\frac{d\nu_1}{2\pi}\Big[\widetilde{\lambda}_0(\mathbf{k}_1)\widetilde{\psi}_0(-\mathbf{k}_1,-\nu_1)\,\widetilde{\psi}_0(\mathbf{k}_1,\nu_1) \\ &+ \sum_{\alpha,\beta}\widetilde{D}_0(\mathbf{k}_1)\,\widetilde{j}_0^\alpha(-\mathbf{k}_1,-\nu_1)\mathcal{T}^{\alpha\beta}(\mathbf{k}_1)\,\widetilde{j}_0^\beta(\mathbf{k}_1,\nu_1) \\ &- \widetilde{\psi}_0(-\mathbf{k}_1,-\nu_1)\left[-i\nu_1 + \lambda_0 k_1^2\left(r_0 + k_1^2\right)\right]\psi_0(\mathbf{k}_1,\nu_1) \\ &- \sum_{\alpha,\beta}\widetilde{j}_0^\alpha(-\mathbf{k}_1,-\nu_1)\big(-i\nu_1 + D_0 k_1^2\big)\mathcal{T}^{\alpha\beta}(\mathbf{k}_1)\, j_0^\beta(\mathbf{k}_1,\nu_1)\Big]\ , \end{aligned} \tag{2.14}$$

where the projector $\mathcal{T}^{\alpha\beta}(\mathbf{k}_1)$ insures that only the transverse components of the fields $\widetilde{j}_0^\alpha$, j_0^β contribute to the action. The longitudinal component, as mentioned before, does not represent independent fluctuations, and its contributions (though formally infinite) can be factored out of the functional integral.

The static non-linearity leads, in turn, to the relaxation vertex

$$\begin{aligned} &J_{\rm rel}[\{\widetilde{\psi}_0\},\{\psi_0\}] \\ &\quad= -\frac{\lambda_0 u_0}{6}\int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d}\frac{d\nu_1}{2\pi}\frac{d\nu_2}{2\pi}\frac{d\nu_3}{2\pi} \\ &\quad\times k^2\widetilde{\psi}_0(-\mathbf{k},-\nu)\,\psi_0(\mathbf{k}_1,\nu_1)\,\psi_0(\mathbf{k}_2,\nu_2)\,\psi_0(\mathbf{k}_3,\nu_3), \end{aligned} \tag{2.15}$$

where $(\mathbf{k},\nu) = (\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3, \nu_1+\nu_2+\nu_3)$. The purely dynamic couplings generate the mode-coupling vertices

$$\begin{aligned} &J_{\rm mc}[\{\widetilde{\psi}_0\},\{\psi_0\},\{\widetilde{\mathbf{j}}_0\},\{\mathbf{j}_0\}] \\ &= -ig_0\int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{d\nu_1}{2\pi}\frac{d\nu_2}{2\pi}\sum_{\alpha,\beta} \\ &\Big[k_1^\alpha\,\mathcal{T}^{\alpha\beta}(\mathbf{k}_2)\,\widetilde{\psi}(-\mathbf{k},-\nu)\psi(\mathbf{k}_1,\nu_1)\, j^\beta(\mathbf{k}_2,\nu_2) \\ &- \frac{1}{2}\left(k_1^\alpha k_2^2 + k_2^\alpha k_1^2\right)\mathcal{T}^{\alpha\beta}(\mathbf{k}) \\ &\times \widetilde{j}^\beta(-\mathbf{k},-\nu)\,\psi(\mathbf{k}_1,\nu_1)\psi(\mathbf{k}_2,\nu_2)\Big] \end{aligned} \tag{2.16}$$

with $(\mathbf{k}, \nu) = (\mathbf{k}_1 + \mathbf{k}_2, \nu_1 + \nu_2)$ here.

Finally, the source terms give rise to the following contribution

$$
\begin{aligned}
& J_{\mathrm{sc}}[\{\widetilde{\psi}_0\}, \{\psi_0\}, \{\widetilde{\mathrm{J}}_0\}, \{\mathrm{J}_0\}] \qquad (2.17) \\
&= \int \frac{d^d k_1}{(2\pi)^d} \frac{d\nu_1}{2\pi} \Big[\lambda_0 k_1^2 \, \widetilde{\psi}(-\mathbf{k}_1, -\nu_1) \, h(\mathbf{k}_1, \nu_1) \\
&+ \sum_{\alpha,\beta} D_0 k_1^2 \widetilde{\mathrm{J}}^\beta(-\mathbf{k}_1, -\nu_1) \, \mathcal{T}^{\alpha\beta}(\mathbf{k}_1) \, A^\beta(\mathbf{k}_1, \nu_1) \Big] \\
&+ i g_0 \int \frac{d^d k_1}{(2\pi)^d} \frac{d^d k_2}{(2\pi)^d} \frac{d\nu_1}{2\pi} \frac{d\nu_2}{2\pi} \sum_{\alpha,\beta} \\
&\Big[k_1^\alpha \, \mathcal{T}^{\alpha\beta}(\mathbf{k}_2) \, \widetilde{\psi}(-\mathbf{k}, -\nu) \psi(\mathbf{k}_1, \nu_1) \, A^\beta(\mathbf{k}_2, \nu_2) \\
&k_2^\alpha \, \mathcal{T}^{\alpha\beta}(\mathbf{k}) \widetilde{\mathrm{J}}^\beta(-\mathbf{k}, -\nu) \, h(\mathbf{k}_1, \nu_1) \psi(\mathbf{k}_2, \nu_2) \Big]
\end{aligned}
$$

with $(\mathbf{k}, \nu) = (\mathbf{k}_1 + \mathbf{k}_2, \nu_1 + \nu_2)$ in the second integral. The contribution of the sources is simply included for convenience. If one wishes to relate the dynamic susceptibilities to the Green's functions, one can simply differentiate the dynamic functional with respect to $h(\mathbf{x}, t)$, $\mathbf{A}(\mathbf{x}, t)$ and then set these sources equal to zero. The expression of the dynamic functional in Fourier space (2.14) through (2.16), although more cumbersome in notation, is directly applicable to the perturbation expansion.

As usual, the harmonic part (2.14) defines the propagators of the field theory, while the perturbation expansion is performed in terms of the non-linear vertices (2.15) and (2.16). Notice that the existence of the reversible forces (2.16) does not show up in dynamic mean-field theory (van Hove theory), which in field-theory language is based on the harmonic action (2.14) only. We shall see that while the choice (2.9), (2.10) for the functions $\widetilde{\lambda}_0(\mathbf{k})$ and $\widetilde{D}_0(\mathbf{k})$ yields a perfectly consistent field theory, the choice (2.11), (2.12) predicts a correction to the critical temperature which is *anisotropic*. One therefore needs to modify the theory accordingly in order to obtain a consistent description of the model. We shall address this issue in section 4. Here it suffices to say that once we have modified the Langevin equations describing the dynamics, we can also treat the resulting model by the means described above.

Once the dynamic functional is obtained, the perturbation expansion for all possible correlation functions of the dynamic and auxiliary fields, as well as for the associated vertex functions, is given by the one-particle irreducible Feynman diagrams. A straightforward scaling analysis yields that the upper critical dimension of the isotropic model H is $d_c = 4$ for both the relaxational *and* the mode-coupling vertices. For the anisotropic, effective two-temperature model H, however, the upper critical dimension is reduced to $d_c = 4 - d_\parallel$, as will be seen below. For $d \leq d_c$, the perturbation theory will be infrared-singular, and non-trivial critical exponents ensue, while for $d \geq d_c$ the perturbation theory contains ultraviolet divergences. In order to renormalize the field theory in the ultraviolet, it suffices to render all the non-vanishing two-, three-, and four-point functions finite by introducing multiplicative renormalization constants, following an additive renormalization corresponding to a fluctuation-induced shift r_{0c} of the critical temperature. This is achieved by demanding the renormalized vertex functions, or appropriate momentum and frequency derivatives thereof, to be finite when the fluctuation integrals are taken at a conveniently chosen normalization point, well outside the singularities of the infrared regime.

We shall employ the dimensional regularization scheme in order to compute the emerging momentum integrals, and choose the renormalized mass $\tau = 1$ as our normalization point, or, sufficient to one-loop order, $\tau_0 = r_0 - r_{0c} = \mu^2$. Notice that μ defines an intrinsic momentum scale of the renormalized theory. From the renormalization constants (Z factors) that render the field theory finite in the ultraviolet (UV), one may then derive the renormalization group (RG) flow functions which enter the Gell-Mann–Low equation. This partial differential equation describes how correlation functions change under scale transformations. In the vicinity of an RG fixed point, the theory becomes scale-invariant, and the information previously gained about the UV behavior can thus be employed to access the physically interesting power laws governing the infrared (IR) regime at the critical point ($\tau \propto T - T_c \to 0$) for long wavelengths (wave vector $\mathbf{q} \to 0$) and low frequencies ($\omega \to 0$).

3 Renormalization group analysis of the isotropic model H

3.1 Vertex and response function renormalization

The UV-divergent two-, three-, and four-point vertex functions or their derivatives that require multiplicative renormalization are $\partial_\omega \Gamma_{0\,\widetilde{\psi}\psi}(\mathbf{q}, \omega)$, $\partial_{q^2} \Gamma_{0\,\widetilde{\psi}\psi}(\mathbf{q}, \omega)$ together with $\partial_{q^4} \Gamma_{0\,\widetilde{\psi}\psi}(\mathbf{q}, \omega)$ and $\partial_{q^2} \Gamma_{0\,\widetilde{\psi}\widetilde{\psi}}(\mathbf{q}, \omega)$; the functions involving J, $\widetilde{\mathrm{J}}$, $\partial_\omega \Gamma_{0\widetilde{\mathrm{J}}\mathrm{J}}(\mathbf{q}, \omega)$, $\partial_{q^2} \Gamma_{0\widetilde{\mathrm{J}}\mathrm{J}}(\mathbf{q}, \omega)$ and $\partial_{q^2} \Gamma_{0\widetilde{\mathrm{J}}\widetilde{\mathrm{J}}}(\mathbf{q}, \omega)$; the three-point vertices $\partial_{q^\alpha} \Gamma_{0\,\widetilde{\psi}\psi\,\mathrm{J}^\alpha}(-\mathbf{q} - \mathbf{p}, -\omega; \mathbf{q}, \omega; \mathbf{p}, 0)$ and $\partial_{(q^\alpha p^2 + p^\alpha q^2)} \Gamma_{0\widetilde{\mathrm{J}}^\alpha\psi\psi}(-\mathbf{q} - \mathbf{p}, -\omega; \mathbf{q}, \omega; \mathbf{p}, 0)$; and finally the relaxation vertex $\partial_{q^2} \Gamma_{0\widetilde{\psi}\psi\psi\psi}(-\mathbf{q}, -\omega; \frac{\mathbf{q}}{3}, \frac{\omega}{3}; \frac{\mathbf{q}}{3}, \frac{\omega}{3}; \frac{\mathbf{q}}{3}, \frac{\omega}{3})$. On the other hand, we have four fluctuating fields ($\widetilde{\mathrm{J}}_0^\alpha$, J_0^α, $\widetilde{\psi}_0$, ψ_0) and the seven parameters $\widetilde{D}_0$, D_0, $\widetilde{\lambda}_0$, λ_0, τ_0, g_0 and u_0 available; this leaves us at liberty to choose one of the renormalization constants in a convenient manner. In addition to these parameters and since detailed balance does not hold for this non-equilibrium model, one also needs to consider the renormalization of the dynamic susceptibility of the order parameter, in order to determine the susceptibility exponent η.

Starting with the two-point functions $\Gamma_{0\,\widetilde{\psi}\psi}(\mathbf{q}, \omega)$ and $\Gamma_{0\widetilde{\mathrm{J}}\mathrm{J}}(\mathbf{q}, \omega)$ for the conserved order parameter and transverse current fluctuations, respectively, we immediately note that as a consequence of the momentum dependence of the mode-coupling vertices, one has for these two vertex functions

$$
\frac{\partial}{\partial (i\omega)} \Gamma_{0\,\widetilde{\psi}\psi\,(\widetilde{\mathrm{J}}\mathrm{J})}(\mathbf{q} = \mathbf{0}, \omega) \equiv 1 \qquad (3.1)
$$

to *all orders* in perturbation theory. Upon defining renormalized fields according to

$$\widetilde{\psi} = Z_{\widetilde{\psi}}^{1/2} \widetilde{\psi}_0 \ , \quad \psi = Z_{\psi}^{1/2} \psi_0 \ , \tag{3.2}$$

$$\widetilde{\jmath}^{\alpha} = Z_{\widetilde{\jmath}}^{1/2} \widetilde{\jmath}_0^{\alpha} \ , \quad \jmath^{\alpha} = Z_{\jmath}^{1/2} \jmath_0^{\alpha} \ , \tag{3.3}$$

which imply that $\Gamma_{\widetilde{\psi}\psi} = (Z_{\widetilde{\psi}} Z_{\psi})^{-1/2} \Gamma_{0\,\widetilde{\psi}\psi}$ and that $\Gamma_{\widetilde{\jmath}\jmath} = (Z_{\widetilde{\jmath}} Z_{\jmath})^{-1/2} \Gamma_{0\widetilde{\jmath}\jmath}$, we thus obtain the exact relations

$$Z_{\widetilde{\psi}} Z_{\psi} \equiv 1 \ , \tag{3.4}$$

$$Z_{\widetilde{\jmath}} Z_{\jmath} \equiv 1 \ . \tag{3.5}$$

At this point we utilize our freedom of choice [20] to set

$$Z_{\widetilde{\jmath}} \equiv Z_{\jmath} \equiv 1 \ . \tag{3.6}$$

The multiplicative renormalization factors for the coupling constants are defined through

$$\lambda = Z_{\lambda} \lambda_0 \ , \tag{3.7}$$

$$\widetilde{\lambda} = Z_{\widetilde{\lambda}} \widetilde{\lambda}_0 \ , \tag{3.8}$$

$$D = Z_D \, D_0 \, \mu^{-2} \ , \tag{3.9}$$

$$\widetilde{D} = Z_{\widetilde{D}} \widetilde{D}_0 \mu^{-2} \ , \tag{3.10}$$

$$\tau = Z_{\tau} \, \tau_0 \, \mu^{-2} \quad \text{with} \quad \tau_0 = r_0 - r_{0c} \ , \tag{3.11}$$

$$u = Z_u \, u_0 \, A_d \, \mu^{d-4} \ , \tag{3.12}$$

$$g = Z_g^{1/2} \, g_0 \, A_d^{1/2} \, \mu^{d/2-3} \ , \tag{3.13}$$

where the μ factors have been introduced to render the renormalized couplings dimensionless. The geometric numerical factor $A_d = \Gamma(3-d/2)/2^{d-1}\pi^{d/2}$ has been absorbed into u and g since it appears in the explicit integrals of the subsequent perturbational analysis. Finally, we still have at our disposal a symmetry of the theory which is valid even in a non-equilibrium situation, namely Galilean invariance (see Appendix A). This symmetry imposes the exact condition that

$$Z_g \, Z_{\jmath} \equiv 1 \ . \tag{3.14}$$

From this relation and Eq. (3.6) it follows that the dynamical vertex does not renormalize, i.e. $Z_g \equiv 1$.

In order to discuss the RG flow diagram, it is convenient to introduce the following rescaled coupling constants

$$\widetilde{u}_0 = \frac{\widetilde{\lambda}_0}{\lambda_0} \, u_0 \ , \tag{3.15}$$

$$\widetilde{f}_0 = \frac{\widetilde{\lambda}_0}{\lambda_0} \, \frac{g_0^2}{\lambda_0 D_0} \ , \tag{3.16}$$

and the ratio between the noise temperature of the order parameter and the noise temperature of the mass-energy current, respectively [21]

$$T_0 = \frac{\widetilde{\lambda}_0}{\lambda_0} \, \frac{D_0}{\widetilde{D}_0} \ , \tag{3.17}$$

with the renormalized versions of these coupling constants being defined as

$$\widetilde{u} = \frac{Z_{\widetilde{\lambda}} Z_u}{Z_{\lambda}} \, \widetilde{u}_0 \, A_d \, \mu^{d-4} \, , \tag{3.18}$$

$$\widetilde{f} = \frac{Z_{\widetilde{\lambda}}}{Z_{\lambda}^2 Z_D} \, \widetilde{f}_0 \, A_d \, \mu^{d-4} \, , \tag{3.19}$$

$$T = \frac{Z_{\widetilde{\lambda}} Z_D}{Z_{\lambda} Z_{\widetilde{D}}} \, T_0 \, , \tag{3.20}$$

whence the upper critical dimension of the dynamical vertex $\widetilde{f}$ comes out to be the same as that of the static vertex $\widetilde{u}$, i.e. $d_c = 4$. Also, in equilibrium, $Z_{\widetilde{\lambda}} = Z_{\lambda}$, $Z_{\widetilde{D}} = Z_D$, wherefrom we conclude that $T = T_0 = 1$.

The analysis is now carried through by considering all one-loop Feynman diagrams which contribute to the different vertex functions listed above. The renormalization factors are determined in such a way that the renormalized vertex functions depend on the renormalized couplings in the same way as the zero-loop vertex functions depend on the bare couplings at a normalization point, chosen here to be $(\mathbf{q}, \omega) = (\mathbf{0}, 0)$ and $\tau = 1$ (to one-loop order, $\tau_0 = \mu^2$), well outside the IR region. The introduction of this normalization point (NP) renders the renormalized couplings dependent on the momentum scale μ, as pointed out above.

From $\partial_{q^2} \Gamma_{\widetilde{\psi}\psi}(\mathbf{q}, \omega)|_{\rm NP}$ and $\partial_{q^4} \Gamma_{\widetilde{\psi}\psi}(\mathbf{q}, \omega)|_{\rm NP}$, one obtains the following results for the renormalization factors Z_{λ} and Z_{τ} [22],

$$Z_{\lambda} \, Z_{\tau} = 1 + \frac{2(d-1)}{(d-2)d} \, \widetilde{f}_0 \, \frac{A_d}{\epsilon} \mu^{-\epsilon} - \frac{\widetilde{u}_0}{d-2} \, \frac{A_d}{\epsilon} \mu^{-\epsilon} \, , \tag{3.21}$$

$$Z_{\lambda} = 1 + \frac{2(d-1)}{(d-2)d} \, \widetilde{f}_0 \, \frac{A_d}{\epsilon} \mu^{-\epsilon} \, , \tag{3.22}$$

where we have used the value of r_{0c} obtained from the condition of divergence of the static susceptibility (see below) and where $\epsilon = 4 - d$. From these equations one can obtain the value of Z_{τ}, namely

$$Z_{\tau} = 1 - \frac{\widetilde{u}_0}{d-2} \, \frac{A_d}{\epsilon} \mu^{-\epsilon} \, . \tag{3.23}$$

Applying the normalization condition to the derivative $\partial_{q^2} \Gamma_{\widetilde{\psi}\widetilde{\psi}}(\mathbf{q}, \omega)|_{\rm NP}$, one obtains the following expression for $Z_{\widetilde{\lambda}}/Z_{\psi}$

$$\frac{Z_{\widetilde{\lambda}}}{Z_{\psi}} = 1 + \frac{2(d-1)}{(d-2)d} \, \frac{\widetilde{f}_0}{T_0} \, \frac{A_d}{\epsilon} \, \mu^{-\epsilon} \, . \tag{3.24}$$

The normalization conditions for the derivatives of the vertex functions $\partial_{q^2} \Gamma_{\widetilde{\jmath}\jmath}(\mathbf{q}, \omega)|_{\rm NP}$ and $\partial_{q^2} \Gamma_{\widetilde{\jmath}\widetilde{\jmath}}(\mathbf{q}, \omega)|_{\rm NP}$ in turn yield the following expressions for Z_D and $Z_{\widetilde{D}}$,

$$Z_D = 1 + \frac{\widetilde{f}_0}{4(d+2)} \, \frac{A_d}{\epsilon} \, \mu^{-\epsilon} \, , \tag{3.25}$$

$$Z_{\widetilde{D}} = 1 + \frac{1}{4(d+2)} \, \widetilde{f}_0 \, T_0 \, \frac{A_d}{\epsilon} \, \mu^{-\epsilon} \, , \tag{3.26}$$

which become equal in equilibrium, $T_0 = 1$, as they should. From the normalization condition for the three point vertex $\partial_{q^\alpha} \Gamma_{\widetilde{\psi}\psi \mathrm{J}^\alpha}(-\mathbf{q}-\mathbf{p}, -\omega; \mathbf{q}, \omega; \mathbf{p}, 0)|_{\mathrm{NP}}$, one simply obtains, to one-loop order, that $Z_g = 1$, which confirms the result, valid to all orders, obtained from the condition of Galilean invariance.

On the other hand, from the normalization condition for $\partial_{(q^\alpha p^2 + p^\alpha q^2)} \Gamma_{\widetilde{\mathrm{J}}^\alpha \psi\psi}(-\mathbf{q}-\mathbf{p}, -\omega; \mathbf{q}, \omega; \mathbf{p}, 0)\,|_{NP}$, one obtains the following expression for Z_ψ

$$Z_\psi = 1 + \frac{4\widetilde{f}_0}{(d-2)d(d+2)} \left(1 - \frac{1}{T_0}\right) \frac{A_d}{\epsilon}\, \mu^{-\epsilon} . \quad (3.27)$$

Note that in equilibrium this expression is identical to 1, i.e., there are no corrections to $\eta = 0$ to one-loop order. Eq. (3.27) can be used in (3.24) to find $Z_{\widetilde{\lambda}}$,

$$Z_{\widetilde{\lambda}} = 1 + \frac{2(d-1)}{(d-2)d} \frac{\widetilde{f}_0}{T_0} \frac{A_d}{\epsilon}\, \mu^{-\epsilon} + \frac{4}{(d-2)d(d+2)} \widetilde{f}_0 \left(1 - \frac{1}{T_0}\right) \frac{A_d}{\epsilon}\, \mu^{-\epsilon} , \quad (3.28)$$

which again reduces to (3.22) in equilibrium.

Finally, the normalization condition for the relaxation vertex $\partial_{q^2} \Gamma_{\widetilde{\psi}\psi\psi\psi}(-\mathbf{q}, -\omega; \frac{\mathbf{q}}{3}, \frac{\omega}{3}; \frac{\mathbf{q}}{3}, \frac{\omega}{3}; \frac{\mathbf{q}}{3}, \frac{\omega}{3})|_{\mathrm{NP}}$ yields the following expression for the product $Z_\lambda\, Z_u\, Z_\psi$,

$$Z_\lambda\, Z_u\, Z_\psi = 1 - \frac{3}{2} \widetilde{u}_0 \frac{A_d}{\epsilon}\, \mu^{-\epsilon} + \frac{d-1}{d} \widetilde{f}_0 \frac{A_d}{\epsilon}\, \mu^{-\epsilon} , \quad (3.29)$$

from which one obtains, using (3.22) and (3.27), the following result for Z_u,

$$Z_u = 1 - \frac{3}{2} \widetilde{u}_0 \frac{A_d}{\epsilon}\, \mu^{-\epsilon} - \frac{4}{(d-2)d(d+2)} \widetilde{f}_0 \left(1 - \frac{1}{T_0}\right) \frac{A_d}{\epsilon}\, \mu^{-\epsilon} , \quad (3.30)$$

which is independent of the dynamic coupling $\widetilde{f}_0$ in equilibrium, as it should be.

The above results for the Z factors are sufficient to provide the beta functions which determine the fixed points of the theory. However, one still needs to consider the independent renormalizations needed for the susceptibilities, since one does not have the usual constraints imposed by detailed balance. To define the "static" limit of the intrinsically dynamic model under consideration, we compute the response functions for the order parameter and for the transverse current, and then take the limit $\omega \to 0$ there. In order to obtain these functions, one simply needs to take the derivative of the functional J_{sc}, equation (2.17), with respect to the external sources $h(\mathbf{x}, t)$ and $\mathbf{A}(\mathbf{x}, t)$ and then set these sources equal to zero. Some formal manipulations using the properties of the vertex functions yield

$$\chi_0(\mathbf{q}, \omega) = \Gamma_{0\,\widetilde{\psi}\,\psi}(-\mathbf{q}, -\omega)^{-1} \times \left[\lambda_0\, q^2 + g_0\, \Gamma_{0\,\widetilde{\psi}\,[\widetilde{\mathrm{J}}\nabla\psi]}(-\mathbf{q}, -\omega)\right] , \quad (3.31)$$

$$X_0(\mathbf{q}, \omega) = \Gamma_{0\widetilde{\mathrm{J}}\mathrm{J}}(-\mathbf{q}, -\omega)^{-1} \times \left[D_0 q^2 - g_0\, \Gamma_{0\widetilde{\mathrm{J}}\,[\widetilde{\psi}\nabla\psi]}(-\mathbf{q}, -\omega)\right] , \quad (3.32)$$

respectively [18,10,23]. Note that composite-operator vertex functions enter these expressions, which in general implies that new renormalization constants are required to remove the UV singularities of the response functions (equivalently, one may utilize the Z factors obtained from the multiplicative renormalization of the vertex functions plus appropriate additive renormalizations [18,10]). Yet one may show to *all orders* in perturbation theory [24] that

$$\Gamma_{0\widetilde{\mathrm{J}}\mathrm{J}}(\mathbf{q}, \omega) = i\omega + D_0 q^2 - g_0\, \Gamma_{0\widetilde{\mathrm{J}}\,[\widetilde{\psi}\nabla\psi]}(\mathbf{q}, \omega) , \quad (3.33)$$

and consequently

$$X_0(\mathbf{q}, \omega = 0) \equiv 1 , \quad (3.34)$$

which means that there is no additional renormalization required here. On the other hand, the static limit of the order parameter susceptibility is in fact singular, which leads us to define the corresponding renormalized response function via

$$\chi(\mathbf{q}, \omega) = Z\, \chi_0(\mathbf{q}, \omega) . \quad (3.35)$$

The new renormalization constant Z is determined by demanding that $\partial_{q^2} \chi^{-1}(\mathbf{q}, \omega)|_{\mathrm{NP}}$ be UV-finite. In this case, to one-loop order we obtain the simple result $Z = 1$, i.e., there are no corrections to this order of perturbation theory. Note that in equilibrium, one has $Z_\psi = Z$ to all orders as a consequence of detailed balance [25].

The fluctuation-induced T_c shift is determined from the criticality condition $\chi_0^{-1}(\mathbf{q} = \mathbf{0}, \omega = 0) = 0$ at $r_0 = r_{0c}$ ($\tau_0 = 0$) with the result

$$r_{0c} = -\frac{1}{2} \widetilde{u}_0 \int_k \frac{1}{r_{0c} + k^2} + \frac{d-1}{d} \frac{D_0}{\lambda_0} \widetilde{f}_0 \left(1 - \frac{1}{T_0}\right) \int_k \frac{1}{k^2 (r_{0c} + D_0/\lambda_0 + k^2)} , \quad (3.36)$$

which determines r_{0c} implicitly. The momentum integrals in (3.36) should be evaluated with a *finite* upper cutoff, which underlines the non-universality of the T_c shift, i.e., its dependence on short-distance properties. This expression is however sufficient, in its present form, to provide for the additive renormalization (mass renormalization) necessary to make $\Gamma_{\widetilde{\psi}\psi}(\mathbf{q}, \omega)$ finite and we have implicitly used it to obtain the results (3.21) and (3.22).

3.2 Discussion of the RG flow equations

3.2.1 RG equations for the vertex and response functions

By means of the above renormalization constants, we can now write down the RG (Gell-Mann–Low) flow equations for the vertex functions and the dynamic susceptibilities. The latter connect the asymptotic theory, where the IR singularities become manifest, with a region in parameter space where the loop integrals are finite and ordinary 'naive' perturbation expansion is applicable, and follow from the simple observation that the 'bare' vertex functions do not depend on the renormalization scale μ,

$$\mu \frac{d}{d\mu}\bigg|_0 \Gamma_{0\widetilde{J}^k\widetilde{\psi}^r J^l \psi^s}(\{\mathbf{q},\omega\};\{a_0\}) = 0 \ , \tag{3.37}$$

with $\{a_0\} = \lambda_0$, $\widetilde{\lambda}_0$, D_0, $\widetilde{D}_0$, τ_0, u_0 and g_0. Replacing the bare parameters and fields in Eq. (3.37) with the renormalized ones, we thus find the following partial differential equations for the renormalized vertex functions,

$$\left[\mu\frac{\partial}{\partial\mu} + \sum_{\{a\}} \zeta_a \, a \frac{\partial}{\partial a} + \frac{r}{2}\zeta_{\widetilde{\psi}} + \frac{s}{2}\zeta_\psi\right] \times \Gamma_{\widetilde{J}^k\widetilde{\psi}^r J^l \psi^s}(\mu,\{\mathbf{q},\omega\};\{a\}) = 0 \ . \tag{3.38}$$

Here, we have introduced Wilson's flow functions

$$\zeta_\psi = \mu\frac{\partial}{\partial\mu}\bigg|_0 \ln Z_\psi \ , \tag{3.39}$$

$$\zeta_{\widetilde{\psi}} = \mu\frac{\partial}{\partial\mu}\bigg|_0 \ln Z_{\widetilde{\psi}} \ , \tag{3.40}$$

and

$$\zeta_a = \mu\frac{\partial}{\partial\mu}\bigg|_0 \ln \frac{a}{a_0} \tag{3.41}$$

(the index '0' indicates that the renormalized fields and parameters are to be expressed in terms of their bare counterparts prior to performing the derivatives with respect to the momentum scale μ). Note that $\zeta_{\widetilde{J}} = \zeta_J \equiv 0$, $\zeta_g \equiv d/2 - 3$ as a consequence of Eqs. (3.6) and (3.14).

The Gell-Mann–Low equation (3.38) is readily solved with the method of characteristics $\mu \to \mu\ell$; this defines running couplings as the solutions to the first-order differential RG flow equations

$$\ell \frac{da(\ell)}{d\ell} = \zeta_a(\ell)\, a(\ell) \ , \ a(1) = a \ . \tag{3.42}$$

The solutions of the partial differential equations (3.38) then read

$$\begin{aligned}\Gamma_{\widetilde{J}^k\widetilde{\psi}^r J^l \psi^s}(\mu,\{\mathbf{q},\omega\};\{a\}) &= \\ &= \exp\left\{\frac{1}{2}\int_1^\ell \left[r\,\zeta_{\widetilde{\psi}}(\ell') + s\,\zeta_\psi(\ell')\right]\frac{d\ell'}{\ell'}\right\} \\ &\quad \times \Gamma_{\widetilde{J}^k\widetilde{\psi}^r J^l \psi^s}(\mu\ell,\{\mathbf{q},\omega\};\{a(\ell)\}) \ .\end{aligned} \tag{3.43}$$

In the same manner, one can solve the RG equations for the dynamic susceptibilities, with the results

$$\begin{aligned}X(\mu,\{\mathbf{q},\omega\};\{a\}) &= \\ &= X(\mu\ell,\{\mathbf{q},\omega\};\{a(\ell)\}) \ ,\end{aligned} \tag{3.44}$$

and

$$\begin{aligned}\chi(\mu,\{\mathbf{q},\omega\};\{a\}) &= \\ &= \exp\left\{-\int_1^\ell \zeta(\ell')\frac{d\ell'}{\ell'}\right\} \\ &\quad \times \chi(\mu\ell,\{\mathbf{q},\omega\};\{a(\ell)\}) \ ,\end{aligned} \tag{3.45}$$

where, in analogy with Eq. (3.39),

$$\zeta = \mu\frac{\partial}{\partial\mu}\bigg|_0 \ln Z \ . \tag{3.46}$$

In terms of the renormalized couplings $\widetilde{u}$, $\widetilde{f}$ and T, as defined by Eqs. (3.18)–(3.20), we find, using the results of subsection 3.1 for the Z factors, the following explicit results for the zeta functions to one-loop order:

$$\zeta_\psi = \frac{4}{(d-2)d(d+2)}\,\widetilde{f}\,\frac{1-T}{T} \ , \tag{3.47}$$

$$\zeta_{\widetilde{\psi}} = -\frac{4}{(d-2)d(d+2)}\,\widetilde{f}\,\frac{1-T}{T} \ , \tag{3.48}$$

$$\zeta = 0 + O(\widetilde{u}^2,\widetilde{f}^2) \ , \tag{3.49}$$

$$\zeta_\lambda = -\frac{2(d-1)}{(d-2)d}\,\widetilde{f} \ , \tag{3.50}$$

$$\zeta_{\widetilde{\lambda}} = -\frac{2(d-1)}{(d-2)d}\,\frac{\widetilde{f}}{T} + \frac{4}{(d-2)d(d+2)}\,\widetilde{f}\,\frac{1-T}{T} \ , \tag{3.51}$$

$$\zeta_D = -\frac{\widetilde{f}}{4(d+2)} \ , \tag{3.52}$$

$$\zeta_{\widetilde{D}} = -\frac{1}{4(d+2)}\,\widetilde{f}\,T \ , \tag{3.53}$$

$$\zeta_\tau = -2 + \frac{\widetilde{u}}{d-2} \ , \tag{3.54}$$

$$\zeta_u = -\epsilon + \frac{3}{2}\,\widetilde{u} - \frac{4}{(d-2)d(d+2)}\,\widetilde{f}\,\frac{1-T}{T} \ , \tag{3.55}$$

with $\zeta_g \equiv d/2 - 3$, identically. These results enable us now to study the scaling behavior of the non-equilibrium model H with dynamical noise in the vicinity of the different RG fixed points, which are given by the zeros of the appropriate RG beta functions ($\{v\} = \widetilde{u}$, $\widetilde{f}$ and T)

$$\beta_v = \mu\frac{\partial}{\partial\mu}\bigg|_0 v \ . \tag{3.56}$$

According to

$$\ell\frac{dv(\ell)}{d\ell} = \beta_v(\{v(\ell)\}) \ , \tag{3.57}$$

these govern the flow of the effective couplings $\widetilde{u}$, $\widetilde{f}$, and T under scale transformations $\mu \to \mu\ell$, and the fixed points $\{v^*\}$ where all $\beta_v(\{v^*\}) = 0$ thus describe scale-invariant regimes.

The RG analysis of the theory then requires the study of the behavior of three independent coupling constants under the RG flow, the static coupling $\widetilde{u}$ defined in (3.18), the mode-coupling vertex $\widetilde{f}$ defined in (3.19), and the parameter T defined in (3.20), which denotes the ratio of the noise temperature of the order parameter to the noise temperature of the transverse mass current, with the corresponding beta functions being $\beta_{\widetilde{u}}$, $\beta_{\widetilde{f}}$ and β_T.

Evaluating the Gell-Mann–Low flow equations near an IR-stable RG fixed point, we may derive the following scaling laws for the two-point correlation functions of the order parameter and conserved currents, respectively,

$$C_\psi(\tau, \mathbf{q}, \omega) = q^{-2-\tilde{z}_\psi+\eta_\psi}\, \hat{C}_\psi\left(\frac{\tau}{q^{1/\nu}}, \frac{\omega}{q^{z_\psi}}\right) , \quad (3.58)$$

$$C_{\mathrm{J}}(\tau, \mathbf{q}, \omega) = q^{-2-\tilde{z}_{\mathrm{J}}}\, \hat{C}_{\mathrm{J}}\left(\frac{\tau}{q^{1/\nu}}, \frac{\omega}{q^{z_{\mathrm{J}}}}\right) , \quad (3.59)$$

and for the order parameter susceptibility

$$\chi(\tau, \mathbf{q}, \omega) = q^{-2+\eta}\, \hat{\chi}\left(\frac{\tau}{q^{1/\nu}}, \frac{\omega}{q^{z_\psi}}\right) . \quad (3.60)$$

Here, the different critical exponents are given in terms of the following fixed-point values (indicated by a '*') of the zeta functions

$$\eta = -\zeta^* , \quad (3.61)$$

$$\eta_\psi = -\zeta_\psi^* , \quad (3.62)$$

$$\nu^{-1} = -\zeta_\tau^* , \quad (3.63)$$

$$z_\psi = 4 + \zeta_\lambda^* , \quad (3.64)$$

$$\widetilde{z}_\psi = 4 + \zeta_{\widetilde{\lambda}}^* , \quad (3.65)$$

$$z_{\mathrm{J}} = 2 + \zeta_D^* , \quad (3.66)$$

$$\widetilde{z}_{\mathrm{J}} = 2 + \zeta_{\widetilde{D}}^* . \quad (3.67)$$

The first three exponents correspond in equilibrium to the static critical exponents (with $\eta = \eta_\psi$), while the last four yield the dynamical critical exponents (again, in equilibrium $z_\psi = \widetilde{z}_\psi$ and $z_{\mathrm{J}} = \widetilde{z}_{\mathrm{J}}$).

3.2.2 RG fixed points and critical exponents

The relevant RG beta functions, namely $\beta_{\widetilde{u}}$, $\beta_{\widetilde{f}}$ and β_T, are given, to one-loop order, by the following expressions:

$$\begin{aligned}\beta_{\widetilde{u}} &= -\widetilde{u}\left(\zeta_\lambda - \zeta_{\widetilde{\lambda}} - \zeta_u\right)\\ &= -\widetilde{u}\left(\epsilon - \frac{3}{2}\,\widetilde{u} + \frac{2(d-1)}{(d-2)d}\,\frac{1-T}{T}\,\widetilde{f}\right) , \quad (3.68)\end{aligned}$$

$$\begin{aligned}\beta_{\widetilde{f}} &= -\widetilde{f}\left(\epsilon + 2\zeta_\lambda + \zeta_D - \zeta_{\widetilde{\lambda}}\right)\\ &= -\widetilde{f}\left(\epsilon - \frac{17d^2+14d-48}{4(d-2)d(d+2)}\,\widetilde{f}\right.\\ &\qquad \left. + \frac{2(d^2+d-4)}{(d-2)d(d+2)}\,\frac{\widetilde{f}}{T}\right) , \quad (3.69)\end{aligned}$$

and

$$\begin{aligned}\beta_T &= T\left(\zeta_{\widetilde{\lambda}} + \zeta_D - \zeta_\lambda - \zeta_{\widetilde{D}}\right)\\ &= -\left[\frac{2(d^2+d-4)}{(d-2)d(d+2)}\right.\\ &\qquad \left. + \frac{T}{4(d+2)}\right](1-T)\,\widetilde{f} , \quad (3.70)\end{aligned}$$

where we have used Eqs. (3.50)–(3.55) for the one-loop Wilson zeta functions. From Eq. (3.70) one sees that there exists a fixed point either if $T = 1$, which is the ordinary equilibrium fixed point at which the noise temperatures become equal, or when the mode coupling vertex $\widetilde{f} = 0$, which can yield a non-equilibrium fixed point (see below).

As a prelude to the study of the behavior of the model at the non-equilibrium fixed point and in order to render the analysis more transparent, we start by reviewing the results for the equilibrium fixed point. Firstly, we notice that in equation (3.70) one has $\beta_T > 0$ for $T > 1$ and $\beta_T < 0$ for $T < 1$. This implies that in the IR regime $\ell \to 0$, T increases if $T < 1$ and it decreases if $T > 1$. The equilibrium fixed point is thus *stable* with respect to perturbations in the noise temperature. Secondly, at this fixed point one obtains the following values for the coupling constants $\widetilde{u}^*$ and $\widetilde{f}^*$,

$$\widetilde{u}^* = \frac{2}{3}\,\epsilon + O(\epsilon^2) , \quad (3.71)$$

$$\widetilde{f}^* = \frac{24}{19}\,\epsilon + O(\epsilon^2) , \quad (3.72)$$

where we have performed an expansion around $d = 4$ in expressions (3.68) and (3.69) and where we have used the fact that $\beta_{\widetilde{u}}$ is independent of $\widetilde{f}$ when $T = 1$. When these values are substituted in the expressions for the zeta functions (3.47) to (3.55), one obtains for the critical exponents, Eqs. (3.61) to (3.67), the results

$$\eta = \eta_\psi = O(\epsilon^2) , \quad (3.73)$$

$$\nu^{-1} = 2 - \frac{1}{3}\,\epsilon + O(\epsilon^2) , \quad (3.74)$$

$$z_\psi = \widetilde{z}_\psi = 4 - \frac{18}{19}\,\epsilon + O(\epsilon^2) , \quad (3.75)$$

$$z_{\mathrm{J}} = \widetilde{z}_{\mathrm{J}} = 2 - \frac{\epsilon}{19} + O(\epsilon^2) . \quad (3.76)$$

The values for the static critical exponents are (due to the existence of detailed balance) the ones one obtains if one performs an RG analysis of the static model described by the free energy (2.1). As for the dynamical exponents, whereas z_{J} shows a small negative correction due to the dynamical coupling between the two modes, the dynamical exponent z_ψ, which describes the behavior of the characteristic frequency of the order parameter $\omega_\psi(q) \propto q^{z_\psi}$, displays a strong suppression already in $d = 3$ ($\epsilon = 1$). This can be understood from the fact that in the absence of dynamical coupling and at the critical point, the dynamics of the conserved current is purely diffusive and therefore much faster than the dynamics of the also *conserved* order parameter which shows the characteristic critical slowing down [26]. The existence of dynamical coupling between the two modes will then provide the slower order parameter field with an additional and faster decay channel, so that $z_\psi < 4$. Notice furthermore that Eq. (3.69) implies for *any* fixed point with non-trivial and finite mode-coupling $0 < \widetilde{f}^* < \infty$, in equilibrium (where $\zeta_\lambda = \zeta_{\widetilde{\lambda}}$) the relation $\zeta_\lambda^* + \zeta_D^* = -\epsilon = d - 4$, i.e., the exponent identity

$$z_\psi + z_{\rm J} \equiv 6 - \epsilon = d + 2 \ , \tag{3.77}$$

which is of course satisfied by the explicit one-loop results (3.75) and (3.76).

If one now takes $\widetilde{f} = 0$ and T finite in Eqs. (3.68) to (3.70), one obtains a line of fixed points (i.e., one fixed point for each value of T), provided that $\widetilde{u} = \frac{2}{3}\,\epsilon + O(\epsilon^2)$. This line of fixed points corresponds to a system where the two modes are completely decoupled, as can be seen from the computation of the critical exponents. One has

$$\eta = \eta_\psi = O(\epsilon^2) \, , \tag{3.78}$$

$$\nu^{-1} = 2 - \frac{1}{3}\,\epsilon + O(\epsilon^2) \, , \tag{3.79}$$

$$z_\psi = \widetilde{z}_\psi = 4 + O(\epsilon^2) \, , \tag{3.80}$$

$$z_{\rm J} = \widetilde{z}_{\rm J} \equiv 2 \, . \tag{3.81}$$

with the last equality holding identically, given that $\widetilde{f} = 0$. The static critical exponents and the dynamical exponents z_ψ, $\widetilde{z}_\psi$ are simply the exponents one would obtain for the equilibrium purely relaxational critical dynamics model B with conserved order parameter, with the $O(\epsilon^2)$ term representing two-loop contributions coming from the static vertex. For the dynamic critical exponent, one knows that in fact

$$z_\psi \equiv 4 - \eta \tag{3.82}$$

exactly [1,18]. However, from Eq. (3.69) we see that the model B fixed point is IR-*unstable* against perturbations in $\widetilde{f}$ since $\beta_{\widetilde{f}} < 0$ for small $\widetilde{f}$.

The cases $\widetilde{f} = 0$ with $T = 0$ or $T = \infty$ require extra care, as one needs to consider the flow of $\bar{f}' = \widetilde{f}/T$ in the first case, and the flow of $\bar{f} = \widetilde{f}\,T$ in the second case, since these two quantities might, respectively, have a finite value at the hypothetical fixed points. If $\widetilde{f} = 0$ and $T = 0$, then $\beta_{\bar{f}'} = -\epsilon \bar{f}'$ with $\bar{f}' = \widetilde{f}/T$, which shows that a fixed point cannot exist for any finite value of $\bar{f}'$. Physically, the absence of a fixed point is again due to the fact that the order parameter displays critical slowing down compared to its diffusive relaxation away from the critical point. The conserved current is, on the other hand, always governed by faster, diffusive dynamics. If the conserved current were to be placed at infinite temperature, it would be slaving the order parameter, which is impossible. Thus no fixed point corresponding to such a situation exists, in contrast with the isotropic non-equilibrium SSS model [10]. There, the order parameter is not conserved, thus permitting a $T = 0$ fixed point (albeit an unstable one) where the purely diffusive (as there is no feedback from the order parameter) conserved variable actually becomes the slower dynamical mode, with $z_\psi = 2 - \epsilon/2 + O(\epsilon^2)$ for the order parameter [10,11].

If $\widetilde{f} = 0$ and $T = \infty$, one finds, on the other hand, for the RG beta function corresponding to the new coupling $\bar{f} = \widetilde{f}\,T$:

$$\beta_{\bar{f}} = -\left(\epsilon - \frac{1}{4(d+2)}\,\bar{f}\right)\,\bar{f} \, , \tag{3.83}$$

which has a *non-equilibrium* fixed point for $\bar{f}^* = 4(d+2)\epsilon$. It is also easy to see that, since $\beta_{\bar{f}} < 0$ for $\bar{f} < \bar{f}^*$, $\beta_{\bar{f}} > 0$ for $\bar{f} > \bar{f}^*$, this genuine non-equilibrium fixed point is IR-*stable* with respect to perturbations on the value of $\bar{f}$. However, it is *unstable* against perturbations on the value of T, since $\beta_T > 0$ for large T. The values of the critical exponents at this unstable fixed point are given by

$$\eta = \eta_\psi = O(\epsilon^2) \, , \tag{3.84}$$

$$\nu^{-1} = 2 - \frac{1}{3}\,\epsilon + O(\epsilon^2) \, , \tag{3.85}$$

$$z_\psi = \widetilde{z}_\psi = 4 + O(\epsilon^2) \, , \tag{3.86}$$

$$z_{\rm J} = 2 \, , \tag{3.87}$$

$$\widetilde{z}_{\rm J} = 2 - \epsilon + O(\epsilon^2) = d - 2 + O(\epsilon^2) \, . \tag{3.88}$$

Eq. (3.87) holds to all orders given the structure of the vertices at this fixed point. The $O(\epsilon^2)$ corrections to the static critical exponents and to $z_\psi = \widetilde{z}_\psi$ stem from the two-loop contributions induced by the *static* vertex only. We thus expect the exact relation (3.82) to hold here as well. Moreover, as we have $\zeta_\lambda = \zeta_{\widetilde{\lambda}}$ at $T = \infty$, the very existence of *any* non-trivial, finite fixed point $0 < \bar{f} < \infty$ demands that $\zeta_\lambda^* + \zeta_{\widetilde{D}}^* = -\epsilon$, i.e., $z_\psi + \widetilde{z}_{\rm J} = 6 - \epsilon = d + 2$, which generalizes the corresponding equilibrium scaling relation (3.77). With $\zeta_\lambda = -\eta_\psi$ we therefore arrive at

$$\widetilde{z}_{\rm J} = d - 2 + \eta_\psi \ . \tag{3.89}$$

One sees that this fixed point is characterized by model B exponents for the order parameter and by anomalous noise correlations for the conserved current, giving rise to a value of $\widetilde{z}_{\rm J} \neq 2$ for $d < d_c = 4$. Again, the model B exponents obtained for the order parameter can be understood from the fact that the conserved current is effectively at zero temperature, and therefore its dynamics does not influence the dynamics of the order parameter, being however affected by it through the residual one-way coupling between these two dynamic variables. Such a fixed point is also present in the non-equilibrium SSS model [10], with similar model A-type behavior for the order parameter, and anomalous noise properties for the dynamically coupled conserved fields. In the present context its existence is due to the fact that here the slower variable is at higher temperature, slaving the faster conserved modes, and suffering no feedback from the latter. The characteristics of this novel, unstable non-equilibrium fixed point should be contrasted with the pure model B fixed point discussed above where there is no coupling at all: $\widetilde{f} = 0$, and where the temperature ratio T, albeit finite, does not matter, for the two Langevin equations are fully decoupled. The RG fixed point structure, and their stability is summarized in Fig. 3.1.

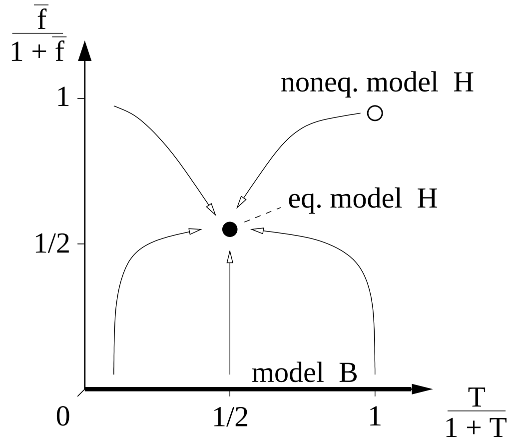

Fig. 3.1. Equilibrium and non-equilibrium fixed points for the isotropic non-equilibrium model H, plotted for $d = 3$ ($\epsilon = 1$). The equilibrium fixed point in the center of the flow diagram is infrared-stable. The model B fixed line is unstable with respect to the mode-coupling, the non-equilibrium fixed point at $T = \infty$ is unstable in the T direction.

4 The anisotropic non-equilibrium model H

In this section, we study the critical behavior of our non-equilibrium version for model H with dynamical noise, as defined through Eqs. (2.4) and (2.6) and by the *anisotropic* noise correlators (2.11) and (2.12). We start by computing the T_c shift from the static susceptibility. As a consequence of the spatially anisotropic conserved noise with $T_0^\perp < T_0^\parallel$, it turns out that the transverse momentum space sector with *lower* noise temperature softens first. Thus, at the critical point, the longitudinal sector remains uncritical ('stiff'), similar to equilibrium anisotropic elastic phase transitions [27] and at Lifshitz points [28]. In Sec. 4.2, we turn to the perturbational renormalization of the two-temperature non-equilibrium model J to one-loop order, and finally discuss the resulting RG flow equations.

4.1 Dynamic field theory and the anisotropic $\mathbf{T_c}$ shift

The dynamic field theory which describes the anisotropic model H has already been presented in Sec. 2, the dynamic functional J being given by Eqs. (2.14) to (2.16), with the choice (2.11) and (2.12) for the noise correlators. However, the computation of the shift in the critical temperature shows that, as it stands, this theory is not fully consistent. The shift in the critical temperature is determined, as in the isotropic theory, by the condition $\chi_0^{-1}(\mathbf{q}, \omega) = 0$ in the limit $\mathbf{q} \to \mathbf{0}$, $\omega \to 0$, where $\chi_0(\mathbf{q}, \omega)$ is the order parameter dynamic susceptibility, which is given in terms of the relevant vertex functions by Eq. (3.31).

From the ensuing expression (to one-loop order), we may determine the fluctuation-induced shift of the critical temperature. Because of the dynamic anisotropy appearing in the noise correlators (2.11) and (2.12), however, the result depends on how the limit $\mathbf{q} \to \mathbf{0}$ is taken; upon defining $q_\parallel = q\cos\Theta$ and $q_\perp = q\sin\Theta$ and with $T_0^\parallel = (\widetilde{\lambda}_0^\parallel/\lambda_0)\,(D_0/\widetilde{D}_0^\parallel)$, $T_0^\perp = (\widetilde{\lambda}_0^\perp/\lambda_0)\,(D_0/\widetilde{D}_0^\perp)$ denoting the temperature ratios in the different sectors, we find

$$
\begin{aligned}
r_{0c}(\Theta) = {} & \frac{d_\parallel}{d}\frac{\widetilde{D}_0^\parallel}{D_0}\left(-\frac{1}{2}\,u_0 T_0^\parallel \int_k \frac{1}{r_{0c}+k^2}\right. \\
& \left. + \frac{g_0^2(d+1)}{\lambda_0^2(d+2)}(T_0^\parallel - 1)\int_k \frac{1}{k^2(r_{0c}+D_0/\lambda_0+k^2)}\right) \\
& + \frac{d_\perp}{d}\frac{\widetilde{D}_0^\perp}{D_0}\left(-\frac{1}{2}\,u_0 T_0^\perp \int_k \frac{1}{r_{0c}+k^2}\right. \\
& \left. + \frac{g_0^2(d+1)}{\lambda_0^2(d+2)}(T_0^\perp - 1)\int_k \frac{1}{k^2(r_{0c}+D_0/\lambda_0+k^2)}\right) \\
& - \frac{2g_0^2}{\lambda_0^2 d(d+2)}\int_k \frac{1}{k^2(r_{0c}+D_0/\lambda_0+k^2)} \\
& \times \left[\left(\frac{\widetilde{\lambda}_0^\parallel}{\lambda_0} - \frac{\widetilde{D}_0^\parallel}{D_0}\right)\cos^2\Theta + \left(\frac{\widetilde{\lambda}_0^\perp}{\lambda_0} - \frac{\widetilde{D}_0^\perp}{D_0}\right)\sin^2\Theta\right] ,
\end{aligned}
\tag{4.1}
$$

in contrast with Eq. (3.36) for the isotropic model H. Here, $d_\parallel$ and $d_\perp$ are, respectively, the dimensions of the parallel and transverse subspaces, with $d = d_\parallel + d_\perp$. As $T_c = T_c^0 + r_{0c}$, the phase transition will occur at the maximum of the function $r_{0c}(\Theta)$, which for $\left(\frac{\widetilde{\lambda}_0^\perp}{\lambda_0} - \frac{\widetilde{D}_0^\perp}{D_0}\right) < \left(\frac{\widetilde{\lambda}_0^\parallel}{\lambda_0} - \frac{\widetilde{D}_0^\parallel}{D_0}\right)$, or, equivalently, $T_0^\perp - 1 < (T_0^\parallel - 1)\frac{\widetilde{D}_0^\parallel}{\widetilde{D}_0^\perp}$, occurs at $\Theta = \pi/2$. The $d_\perp$-dimensional transverse sector in momentum space thus softens first, and the true T_c shift is given by

$$
\begin{aligned}
r_{0c}\left(\frac{\pi}{2}\right) = {} & \frac{d_\parallel}{d}\frac{\widetilde{D}_0^\parallel}{D_0}\left(-\frac{1}{2}\,u_0 T_0^\parallel \int_k \frac{1}{r_{0c}+k^2}\right. \\
& \left. + \frac{g_0^2(d+1)}{\lambda_0^2(d+2)}(T_0^\parallel - 1)\int_k \frac{1}{k^2(r_{0c}+D_0/\lambda_0+k^2)}\right) \\
& + \frac{d_\perp}{d}\frac{\widetilde{D}_0^\perp}{D_0}\left(-\frac{1}{2}\,u_0 T_0^\perp \int_k \frac{1}{r_{0c}+k^2} + \frac{g_0^2}{\lambda_0^2(d+2)}(T_0^\perp - 1)\right. \\
& \left. \times \left(d+1-\frac{2}{d_\perp}\right)\int_k \frac{1}{k^2(r_{0c}+D_0/\lambda_0+k^2)}\right) ,
\end{aligned}
\tag{4.2}
$$

where again this non-universal quantity must be computed with a finite UV cutoff. For $T_0^\parallel = T_0^\perp = T_0$, we recover the result (3.36) for the isotropic model. Notice that dynamical anisotropy ($T_0^\parallel \neq T_0^\perp$), *combined* with the reversible mode-coupling terms, has a very drastic effect here: It renders the system soft only in the momentum subspace with *lower* effective noise temperature. This effect has a simple physical interpretation: The T_c shift is due to thermal fluctuations, which are reduced in the transverse sector ($T_0^\perp < T_0^\parallel$), and therefore lead to a comparatively stronger downwards shift in the longitudinal sector. This result is completely analogous to our earlier findings for the non-equilibrium model J (describing isotropic ferromagnets) with anisotropic noise [11].

In order to characterize the critical properties of our model, we may neglect terms $\propto q_\parallel^4$ in the stiff momentum space sector, because $\tau_0^\parallel = r_0 - r_{0c}(\Theta = 0)$ remains positive at the phase transition where $\tau_0^\perp = r_0 - r_{0c}(\Theta = \pi/2)$ vanishes. In analogy with the situation at anisotropic elastic structural phase transitions [27], or with Lifshitz points in magnetic systems with competing interactions [28], as well as driven diffusive systems [5], we thus have to scale the soft and stiff wave vector components differently, $[q_\perp] = \mu$, whereas $[q_\parallel] = [q_\perp]^2 = \mu^2$. Consequently, while $[\widetilde{\lambda}_0^\perp] = \mu^0$ and $[\widetilde{D}_0^\perp] = \mu^2$, if we choose $[\omega] = \mu^4$, we find for the longitudinal noise strengths the scaling dimensions $[\widetilde{\lambda}_0^\parallel] = \mu^{-2}$ and $[\widetilde{D}_0^\parallel] = \mu^0$, which implies that they become *irrelevant* under scale transformations. Allowing for distinct couplings in the different sectors, one finds in the same manner that the ratios $[\lambda_0^\parallel/\lambda_0^\perp] = [D_0^\parallel/D_0^\perp] = [\lambda_0^\parallel u_0^\parallel/\lambda_0^\perp u_0^\perp] = \mu^{-2}$ and $[g_0^\parallel/g_0^\perp] = \mu^{-1}$ all have negative scaling dimension. Thus, for an investigation of the asymptotic critical behavior, the longitudinal parameters may be neglected as compared to their transverse counterparts, and can all be set to zero in the *effective* dynamic functional.

Upon rescaling the fields according to $\psi_0 \to (\widetilde{\lambda}_0^\perp/\lambda_0^\perp)^{1/2} \psi_0$, $\widetilde{\psi}_0 \to (\lambda_0^\perp/\widetilde{\lambda}_0^\perp)^{1/2}\widetilde{\psi}_0$, $\mathrm{J}_0^\alpha \to (\widetilde{D}_0^\perp/D_0^\perp)^{1/2}\mathrm{J}_0^\alpha$ and $\widetilde{\mathrm{J}}_0^\alpha \to (D_0^\perp/\widetilde{D}_0^\perp)^{1/2}\widetilde{\mathrm{J}}_0^\alpha$ and defining

$$c_0 = \frac{\lambda_0^\parallel}{\lambda_0^\perp}\,\tau_0^\parallel\ , \quad \widetilde{u}_0 = \frac{\widetilde{\lambda}_0^\perp}{\lambda_0^\perp}\,u_0^\perp\ ,$$
$$g_0 = \sqrt{\frac{\widetilde{D}_0^\perp}{D_0^\perp}}\,g_0^\perp\ , \quad \widetilde{g}_0 = \frac{\widetilde{\lambda}_0^\perp}{\lambda_0^\perp}\sqrt{\frac{D_0^\perp}{\widetilde{D}_0^\perp}}\,g_0^\perp\ , \tag{4.3}$$

and omitting the labels '$\perp$' again for λ_0 and r_0, the ensuing *effective* Langevin equations of motion become

$$\frac{\partial\psi_0}{\partial t} = \lambda_0\left[c_0\nabla_\parallel^2 + \nabla_\perp^2(r_0 - \nabla_\perp^2)\right]\psi_0 + \tag{4.4}$$
$$+\lambda_0\frac{\widetilde{u}_0}{6}\nabla_\perp^2\psi_0^3 - g_0\nabla_\perp\psi_0\cdot\mathbf{J}_0 + \eta\ ,$$

and

$$\frac{\partial\mathbf{J}_0}{\partial t} = \mathcal{T}\left[D_0\nabla_\perp^2\mathbf{J}_0 + \widetilde{g}_0\nabla_\perp\psi_0(r_0 - \nabla_\perp^2)\psi_0 + \zeta\right]\ , \tag{4.5}$$

where the transverse projector in the soft ($\perp$) subspace $\mathcal{T}$ is given, in Fourier space, by the expression $\mathcal{T}^{\alpha\beta}(\mathbf{k}) = \delta^{\alpha\beta} - k_\perp^\alpha k_\perp^\beta/k_\perp^2$. The noise correlators in turn read

$$\langle\eta(\mathbf{k},\nu)\eta(\mathbf{k}',\nu')\rangle = 2\widetilde{\lambda}_0\,k_\perp^2\,\delta(\mathbf{k}+\mathbf{k}')\,\delta(\nu+\nu') \tag{4.6}$$

and

$$\langle\zeta^\alpha(\mathbf{k},\nu)\zeta^\beta(\mathbf{k}',\nu')\rangle = 2\widetilde{D}_0\,k_\perp^2\,\delta(\mathbf{k}+\mathbf{k}')\,\delta(\nu+\nu') \times\left(\delta^{\alpha\beta} - \frac{k_\perp^\alpha k_\perp^\beta}{k_\perp^2}\right)\ , \tag{4.7}$$

where again, for convenience, we have used the Fourier space representation. These equations define the *two-temperature non-equilibrium model H*. In order to perform the RG analysis, one represents these equations in the form of a dynamic functional, precisely as in Sec. 2 above.

We emphasize the fact that the anisotropy of the T_c shift in Eq. (4.1) only occurs in the contribution $\propto g_0^2$, i.e., the anisotropy in the T_c shift is due to the purely dynamical mode-coupling terms. In the non-equilibrium model B with dynamical anisotropy [9], the criticality condition for the response function remains isotropic, at least to one-loop order. Thus, if one does not assume different critical temperatures in the purely diffusive non-linear Langevin equation to begin with, these are *not* generated, and one is not immediately led to the two-temperature model B as the correct effective theory for the phase transition. In the presence of reversible mode-couplings, however, anisotropic noise correlations, specifically for a conserved order parameter field, have a much more drastic effect: For both models J [11] and model H such violations of the detailed-balance conditions render the system inherently anisotropic at criticality, and certainly prevent any restoration of the equilibrium critical properties.

4.2 Renormalization of the two-temperature model H

We start by noticing that, as with the isotropic non-equilibrium model H, the two-temperature model H, being a genuinely non-equilibrium model as well, does not allow us to invoke a fluctuation-dissipation theorem in order to relate vertex and response function renormalizations, and we have to compute almost all the Z factors independently. These consist of the wave function renormalization factors Z_ψ, $Z_{\widetilde{\psi}}$, Z_{J}, $Z_{\widetilde{\mathrm{J}}}$, as given by (3.2) and (3.3), and the coupling constant renormalizations, which we define here through

$$\lambda = Z_\lambda\,\lambda_0\,, \tag{4.8}$$
$$c = Z_c\,c_0\,, \tag{4.9}$$
$$D = Z_D\,D_0\,\mu^{-2}\,, \tag{4.10}$$
$$\tau = Z_\tau\,\tau_0\,\mu^{-2}\ \text{ with }\ \tau_0 = r_0 - r_{0c}\,, \tag{4.11}$$
$$\widetilde{u} = Z_{\widetilde{u}}\,\widetilde{u}_0\,A(d_\parallel,d_\perp)\,\mu^{d+d_\parallel-4}\,, \tag{4.12}$$
$$g = Z_g^{1/2}\,g_0\,A(d_\parallel,d_\perp)^{1/2}\,\mu^{(d+d_\parallel)/2-3}\,, \tag{4.13}$$
$$\widetilde{g} = Z_{\widetilde{g}}^{1/2}\,\widetilde{g}_0\,A(d_\parallel,d_\perp)^{1/2}\,\mu^{(d+d_\parallel)/2-3}\,, \tag{4.14}$$

where $A(d_\parallel,d_\perp)$ is given by

$$A(d_\parallel,d_\perp) = \frac{\Gamma(3-d/2-d_\parallel/2)\,\Gamma(d/2)}{2^{d-1}\pi^{d/2}\,\Gamma(d_\perp/2)}\ . \tag{4.15}$$

Again, this factor is included because it appears in subsequent formulas.

As before, Eq. (3.1) implies that the relations (3.4) and (3.5) for the wave function renormalizations hold. Furthermore, the vertex structure of the model leads to

$$\Gamma_{0\,\widetilde{\psi}\psi}(\mathbf{q}_\parallel,\mathbf{q}_\perp=\mathbf{0},\omega=0) = \lambda_0\,c_0\,q_\parallel^2\,, \tag{4.16}$$

which must hold to all orders. This entails that

$$Z_\lambda\,Z_c \equiv 1, \tag{4.17}$$

which leaves us with one Z factor less to determine. One has thus a total of seven independent renormalization factors. As the vertex functions or their derivatives which we must render finite are the same as for the isotropic model H, given at the beginning of Sec. 3.1, with the exception that one has to substitute the derivatives ∂_q with respect to $\mathbf{q}$ by derivatives $\partial_{q_\perp}$ with respect to $\mathbf{q}_\perp$, one sees that the renormalization conditions on these vertex functions determine all the renormalization factors, i.e., there is no freedom left, as opposed to the isotropic model H, to arbitrarily fix one of the renormalizations. To these renormalization factors, one adds, as above, the renormalization factor Z, defined in Eq. (3.35), necessary to render the order parameter susceptibility finite. It is determined by the condition that $\partial_{q_\perp^2} \chi^{-1}(\mathbf{q},\omega)|_{\mathrm{NP}}$ be UV-finite [29]. Furthermore, Galilean invariance still holds in the two-temperature model (see Appendix A), which entails that

$$Z_g\, Z_{\mathrm{J}} \equiv 1, \tag{4.18}$$

to all orders, although, as pointed above, one can now no longer suppose that $Z_{\mathrm{J}} = 1$.

Next we introduce the coupling constants

$$v_0 = \frac{\widetilde{u}_0}{c_0^{d_\parallel/2}} \,, \tag{4.19}$$

$$f_0 = \frac{g_0^2}{\lambda_0 D_0 c_0^{d_\parallel/2}} \,, \tag{4.20}$$

$$\bar{f}_0 = \frac{g_0 \widetilde{g}_0}{\lambda_0 D_0 c_0^{d_\parallel/2}} \,, \tag{4.21}$$

$$\widetilde{f}_0 = \frac{\widetilde{g}_0^2}{\lambda_0 D_0 c_0^{d_\parallel/2}} \,, \tag{4.22}$$

and their renormalized counterparts

$$v = Z_{\widetilde{u}}\, Z_\lambda^{d_\parallel/2}\, v_0\, A(d_\parallel, d_\perp)\, \mu^{-\epsilon} \,, \tag{4.23}$$

$$f = \frac{Z_g}{Z_\lambda^{1-d_\parallel/2} Z_D}\, f_0\, A(d_\parallel, d_\perp)\, \mu^{-\epsilon} \,, \tag{4.24}$$

$$\bar{f} = \frac{(Z_g Z_{\widetilde{g}})^{1/2}}{Z_\lambda^{1-d_\parallel/2} Z_D}\, \bar{f}_0\, A(d_\parallel, d_\perp)\, \mu^{-\epsilon} \,, \tag{4.25}$$

$$\widetilde{f} = \frac{Z_{\widetilde{g}}}{Z_\lambda^{1-d_\parallel/2} Z_D}\, \widetilde{f}_0\, A(d_\parallel, d_\perp)\, \mu^{-\epsilon} \,, \tag{4.26}$$

where $\epsilon = 4 - d - d_\parallel$. Notice that all these renormalized non-linear couplings become scale-invariant at the *reduced* upper critical dimension

$$d_c(d_\parallel) = 4 - d_\parallel \,. \tag{4.27}$$

Such a lowering of the critical dimension is typical of models with anisotropic scaling [27,28,5], for the fluctuations are critical merely in the transverse sector. Employing the renormalization conditions, one then obtains to one-loop order the following results for the Z factors

$$Z_\lambda = 1 + \frac{6(d_\perp - 2)(d_\perp - 1)}{(d-2)d_\perp(d_\perp + 2)}\, \bar{f}_0\, \frac{A(d_\parallel, d_\perp)}{\epsilon}\, \mu^{-\epsilon} \,, \tag{4.28}$$

$$Z_\tau = 1 - \frac{2}{d + d_\parallel - 2} \left(\frac{v_0}{2} - \frac{d_\perp - 1}{d_\perp}\, \bar{f}_0 \right) \frac{A(d_\parallel, d_\perp)}{\epsilon}\, \mu^{-\epsilon} - \frac{6(d_\perp - 2)(d_\perp - 1)}{(d-2)d_\perp(d_\perp + 2)}\, \bar{f}_0\, \frac{A(d_\parallel, d_\perp)}{\epsilon}\, \mu^{-\epsilon} \,, \tag{4.29}$$

$$Z_D = 1 + \frac{1}{4(d_\perp + 2)}\, \bar{f}_0\, \frac{A(d_\parallel, d_\perp)}{\epsilon}\, \mu^{-\epsilon} \,, \tag{4.30}$$

$$Z_\psi = 1 - \frac{2(d_\perp - 1)}{(d-2)d_\perp}\, f_0\, \frac{A(d_\parallel, d_\perp)}{\epsilon}\, \mu^{-\epsilon} + \frac{6(d_\perp - 2)(d_\perp - 1)}{(d-2)d_\perp(d_\perp + 2)}\, \bar{f}_0\, \frac{A(d_\parallel, d_\perp)}{\epsilon}\, \mu^{-\epsilon} \,, \tag{4.31}$$

$$Z_{\mathrm{J}} = 1 + \frac{1}{4(d_\perp + 2)}\, \bar{f}_0\, \frac{A(d_\parallel, d_\perp)}{\epsilon}\, \mu^{-\epsilon} - \frac{d}{4d_\perp(d_\perp + 2)}\, \tilde{f}_0\, \frac{A(d_\parallel, d_\perp)}{\epsilon}\, \mu^{-\epsilon} \,, \tag{4.32}$$

$$Z_{\widetilde{u}} = 1 - \frac{3}{2}\, v_0\, \frac{A(d_\parallel, d_\perp)}{\epsilon}\, \mu^{-\epsilon} + \frac{2(d_\perp - 1)}{(d-2)d_\perp}\, f_0\, \frac{A(d_\parallel, d_\perp)}{\epsilon}\, \mu^{-\epsilon} + \frac{d_\perp - 1}{d_\perp} \left(1 - \frac{12(d_\perp - 2)}{(d-2)(d_\perp + 2)} \right) \tilde{f}_0\, \frac{A(d_\parallel, d_\perp)}{\epsilon}\, \mu^{-\epsilon} \,, \tag{4.33}$$

$$\begin{aligned} Z_{\widetilde{g}} = 1 &+ \frac{4}{(d-2)d_\perp} \left(d_\perp - 1 - \frac{2}{d_\perp + 2} \right) f_0\, \frac{A(d_\parallel, d_\perp)}{\epsilon}\, \mu^{-\epsilon} \\ &- \frac{4}{d_\perp(d_\perp + 2)} \left(3(d_\perp - 1)\frac{d_\perp - 2}{d - 2} - \frac{d_\perp + 16}{16} \right) \\ &\times \bar{f}_0\, \frac{A(d_\parallel, d_\perp)}{\epsilon}\, \mu^{-\epsilon} \\ &- \frac{d}{4d_\perp(d_\perp + 2)}\, \tilde{f}_0\, \frac{A(d_\parallel, d_\perp)}{\epsilon}\, \mu^{-\epsilon} \,, \end{aligned} \tag{4.34}$$

with $Z_g = Z_{\mathrm{J}}^{-1}$, a result which can be confirmed explicitly to one-loop order from the renormalization condition of $\partial_{q_\perp^\alpha} \Gamma_{\widetilde{\psi}\psi \mathrm{J}^\alpha}(-\mathbf{q} - \mathbf{p}, -\omega; \mathbf{q}, \omega; \mathbf{p}, 0)\, |_{NP}$. Subsequently, rendering $\partial_{q_\perp^2} \chi^{-1}(\mathbf{q}, \omega = 0)|_{\mathrm{NP}}$ UV-finite yields the additional Z factor for the response function

$$Z = 1 + \frac{4(d_\perp - 4)(d_\perp - 1)}{(d-2)d_\perp(d_\perp + 2)}\, \bar{f}_0\, \frac{A(d_\parallel, d_\perp)}{\epsilon}\, \mu^{-\epsilon} \,. \tag{4.35}$$

These Z factors can now be used to compute the relevant beta functions and Wilson zeta functions of the theory.

4.3 Discussion of the RG flow equations

Being in possession of the expressions for the Z factors to one-loop order, one can, in an analogous manner to what was done in sections 3.2.1 and 3.2.2, compute the relevant Wilson zeta functions, also to one-loop order. These are given by

$$\zeta_\psi = \frac{2(d_\perp - 1)}{(d-2)d_\perp}\, f \; - \; 6\, \frac{(d_\perp - 2)(d_\perp - 1)}{(d-2)d_\perp(d_\perp + 2)}\, \bar{f} \,, \tag{4.36}$$

$$\zeta_{\mathrm{J}} = -\frac{1}{4(d_\perp + 2)}\, \bar{f} \; + \; \frac{d}{4d_\perp(d_\perp + 2)}\, \widetilde{f} \,, \tag{4.37}$$

$$\zeta = -\frac{4(d_\perp - 4)(d_\perp - 1)}{(d-2)d_\perp(d_\perp + 2)}\,\bar{f}\,, \tag{4.38}$$

$$\zeta_\lambda = -\frac{6(d_\perp - 2)(d_\perp - 1)}{(d-2)d_\perp(d_\perp + 2)}\,\bar{f}\,, \tag{4.39}$$

$$\zeta_D = -\frac{1}{4(d_\perp + 2)}\,\bar{f}\,, \tag{4.40}$$

$$\zeta_\tau = -2 + \frac{1}{d + d_\parallel - 2}\,v - \frac{2(d_\perp - 1)}{d_\perp} \times \left(\frac{1}{d + d_\parallel - 2} - \frac{3(d_\perp - 2)}{(d-2)(d_\perp + 2)}\right)\bar{f}\,, \tag{4.41}$$

$$\zeta_{\widetilde{u}} = -\epsilon + \frac{3}{2}\,v - \frac{2(d_\perp - 1)}{(d-2)d_\perp}\,f - \frac{d_\perp - 1}{d_\perp}\left(1 - \frac{12(d_\perp - 2)}{(d-2)(d_\perp + 2)}\right)\bar{f}\,, \tag{4.42}$$

$$2\zeta_{\widetilde{g}} = d + d_\parallel - 6 - \frac{4}{(d-2)d_\perp}\left(d_\perp - 1 - \frac{2}{d_\perp + 2}\right) f + \frac{4}{d_\perp(d_\perp + 2)}\left(\frac{3(d_\perp - 2)(d_\perp - 1)}{d-2} - \frac{d_\perp + 16}{16}\right)\bar{f} + \frac{d}{4d_\perp(d_\perp + 2)}\,\widetilde{f}\,, \tag{4.43}$$

with $\zeta_c = -\zeta_\lambda$ from Eq. (4.17) and $2\zeta_g = d + d_\parallel - 6 - \zeta_{\mathrm{J}}$ from (4.18). These equations reduce to the equilibrium ones in the limit $d_\parallel = 0$, i.e., $d_\perp = d$.

In the anisotropic two-temperature model H, the scaling laws (3.58)–(3.60) generalize to

$$C_\psi(\tau, \mathbf{q}_\parallel, \mathbf{q}_\perp, \omega) = q_\perp^{-2-\widetilde{z}_\psi + \eta_\psi}\,\hat{C}_\psi\left(\frac{\tau}{q_\perp^{1/\nu}}, \frac{q_\parallel}{q_\perp^{1+\Delta}}, \frac{\omega}{q_\perp^{z_\psi}}\right), \tag{4.44}$$

$$C_{\mathrm{J}}(\tau, \mathbf{q}_\parallel, \mathbf{q}_\perp, \omega) = q_\perp^{-2-\widetilde{z}_{\mathrm{J}}}\,\hat{C}_{\mathrm{J}}\left(\frac{\tau}{q_\perp^{1/\nu}}, \frac{q_\parallel}{q_\perp^{1+\Delta}}, \frac{\omega}{q_\perp^{z_{\mathrm{J}}}}\right), \tag{4.45}$$

$$\chi(\tau, \mathbf{q}_\parallel, \mathbf{q}_\perp, \omega) = q_\perp^{-2+\eta}\,\hat{\chi}\left(\frac{\tau}{q_\perp^{1/\nu}}, \frac{q_\parallel}{q_\perp^{1+\Delta}}, \frac{\omega}{q_\perp^{z_\psi}}\right), \tag{4.46}$$

and the critical exponents are defined via

$$\eta = -\zeta^*\,, \tag{4.47}$$

$$\eta_\psi = -\zeta_\psi^*\,, \tag{4.48}$$

$$\nu^{-1} = -\zeta_\tau^*\,, \tag{4.49}$$

$$\Delta = 1 - \frac{\zeta_c^*}{2} = 1 + \frac{\zeta_\lambda^*}{2}\,, \tag{4.50}$$

$$z_\psi = 4 + \zeta_\lambda^*\,, \tag{4.51}$$

$$z_{\mathrm{J}} = 2 + \zeta_D^*\,, \tag{4.52}$$

where the exponent Δ originates from the intrinsic anisotropy of the system [5,9,11].

From the zeta functions (4.39)–(4.43), one can compute the beta functions for the coupling constants v, f and $\widetilde{f}$, which determine the fixed points, with $\bar{f}$ being given by $\bar{f} = \sqrt{f\widetilde{f}}$. To one-loop order, these beta functions read

$$\beta_v = \left(\zeta_{\widetilde{u}} + \frac{d_\parallel}{2}\,\zeta_\lambda\right) v = \left[-\epsilon + \frac{3}{2}\,v - \frac{2(d_\perp - 1)}{(d-2)d_\perp}\,f - \frac{d_\perp - 1}{d_\perp}\left(1 + \frac{3(d_\parallel - 4)(d_\perp - 2)}{(d-2)(d_\perp + 2)}\right)\bar{f}\right] v\,, \tag{4.53}$$

$$\beta_f = \left[2(\zeta_g + 1) - \zeta_D + \left(\frac{d_\parallel}{2} - 1\right)\zeta_\lambda\right] f = -\epsilon\, f + \left[\left(\frac{1}{2(d_\perp + 2)} - \frac{3(d_\parallel - 2)(d_\perp - 2)(d_\perp - 1)}{(d-2)d_\perp(d_\perp + 2)}\right)\bar{f} - \frac{d}{4d_\perp(d_\perp + 2)}\,\widetilde{f}\right] f\,, \tag{4.54}$$

$$\beta_{\widetilde{f}} = \left[2(\zeta_{\widetilde{g}} + 1) - \zeta_D + \left(\frac{d_\parallel}{2} - 1\right)\right]\widetilde{f} = \left[-\epsilon - \frac{4}{(d-2)d_\perp}\left(d_\perp - 1 - \frac{2}{d_\perp + 2}\right) f - \frac{1}{d_\perp(d_\perp + 2)}\left(4 + \frac{3(d_\parallel - 6)(d_\perp - 2)(d_\perp - 1)}{d-2}\right)\bar{f} + \frac{d}{4d_\perp(d_\perp + 2)}\widetilde{f}\right]\widetilde{f}\,, \tag{4.55}$$

and it follows from $\bar{f} = \sqrt{f\widetilde{f}}$ that

$$\beta_{\bar{f}} = \frac{1}{2}\left(\frac{\beta_f}{f} + \frac{\beta_{\widetilde{f}}}{\widetilde{f}}\right)\bar{f}\,, \tag{4.56}$$

which is used, together with (4.54) and (4.55), to compute $\beta_{\bar{f}}$.

Since the beta functions β_f and $\beta_{\widetilde{f}}$ do not depend on the static coupling v, the determination of the fixed points reduces to the solution of the system of quadratic equations given by $\beta_f = \beta_{\widetilde{f}} = 0$, the equation for $\beta_{\bar{f}}$ being automatically satisfied. Introducing the following condensed notation

$$a = \frac{2\,(d-2)\,d_\perp\,(d_\perp + 2)}{(d-2)d_\perp - 6(d_\parallel - 2)(d_\perp - 2)(d_\perp - 1)}\,, \tag{4.57}$$

$$\alpha = \frac{4\,(d-2)\,d_\perp\,(d_\perp + 2)}{(d-2)(d_\perp - 8) - 12(d_\parallel - 4)(d_\perp - 2)(d_\perp - 1)}\,, \tag{4.58}$$

$$c = \frac{(d-2)\,d}{2[(d-2)d_\perp - 6(d_\parallel - 2)(d_\perp - 2)(d_\perp - 1)]}\,, \tag{4.59}$$

$$\gamma = \frac{8\,[(d_\perp - 1)(d_\perp + 2) - 2]}{(d-2)(d_\perp - 8) - 12(d_\parallel - 4)(d_\perp - 2)(d_\perp - 1)}\,, \tag{4.60}$$

one can write the solutions of the system of quadratic equations as

$$f_\pm^* = \frac{\alpha - a - 2\alpha\gamma c \pm \sqrt{(a-\alpha)^2 + 4ac\alpha\gamma}}{2\,\gamma\,(\gamma c - 1)}\,\epsilon\,, \tag{4.61}$$

$$\widetilde{f}_\pm^* = \frac{a - \alpha - 2ac\gamma \pm \sqrt{(a-\alpha)^2 + 4ac\alpha\gamma}}{2\,c\,(\gamma c - 1)}\,\epsilon\,. \tag{4.62}$$

The existence of fixed points depends on the existence of at least one positive root for each of these two equations, which gives a series of conditions on the coefficients a, c, α and γ. However, it is immediate to see from Eqs. (4.61) and (4.62) that, even when those solutions exist, they diverge if $\gamma c = 1$. Taking into account that $d = d_\perp + d_\parallel$, this condition defines a sixth-order equation determining the relation between $d_\perp$ and $d_\parallel$. With the minimal subtraction prescription that $d = 4 - d_\parallel$, i.e., setting $\epsilon = 0$ in this equation, one obtains the numerical solution $d_\parallel = 0.838454$; already for $0 \leq d_\parallel = 1$, the RG flow takes the mode coupling to infinity. This result is completely analogous to the result we have previously obtained in our study of the two-temperature non-equilibrium model J [11].

Formally, and following our study of model J, we may expand about the equilibrium model H, and thus obtain critical exponents in the limit $d_\parallel \ll 1$. To first order in $d_\parallel \epsilon$, we find

$$f^* = \frac{24}{19}\,\epsilon + \frac{1442}{6137}\,d_\parallel \epsilon\,, \tag{4.63}$$

$$\widetilde{f}^* = \frac{24}{19}\,\epsilon + \frac{11246}{6137}\,d_\parallel\,\epsilon\,, \tag{4.64}$$

$$v^* = \frac{2}{3}\,\epsilon + \frac{143}{323}\,d_\parallel \epsilon\,, \tag{4.65}$$

leading to the critical exponents

$$\eta = -\frac{12}{19}\,d_\parallel \epsilon\,, \quad \eta_\psi = -\frac{21}{646}\,d_\parallel \epsilon\,, \tag{4.66}$$

$$\nu^{-1} = 2 - \frac{1}{3}\,\epsilon - \frac{41}{646}\,d_\parallel \epsilon\,, \tag{4.67}$$

$$z = 4 - \frac{18}{19}\,\epsilon - \frac{2820}{6137}\,d_\parallel \epsilon\,, \tag{4.68}$$

$$z_{\mathrm{J}} = 2 - \frac{1}{19}\,\epsilon - \frac{372}{6137}\,d_\parallel \epsilon\,, \tag{4.69}$$

$$\Delta = 1 - \frac{9}{19}\,\epsilon - \frac{1410}{6137}\,d_\parallel \epsilon\,. \tag{4.70}$$

Notice that this procedure amounts to an expansion with respect to *two* dimensional parameters, namely $\epsilon = 4 - d - d_\parallel$, and $d_\parallel \epsilon$. Moreover, the divergence of the non-expanded fixed point $\widetilde{f}^*$ at $d_\parallel \approx 0.838454$ indicates that an extrapolation of the formal results (4.66) to (4.70) to any physical dimension $d_\parallel \geq 1$ is unlikely to work. On the other hand, we cannot exclude that, also for model H, this divergence merely represents an one-loop artifact, and is cured if one calculates the RG beta functions to higher loop orders. Yet another possibility might well be that the divergence of f^*, $\widetilde{f}^*$ and v^* indicates the absence of a *simple* non-equilibrium stationary state of the two-temperature model H in the vicinity of its critical point. For example, in a *uniformly* rather than randomly driven non-equilibrium version of model J, a similar divergence has been found recently [30]. In that case, computer simulations have revealed that the system enters a regime of spatio-temporal chaos at long times; perhaps the absence of a finite RG fixed point in the randomly driven two-temperature models J and H might indicate similar behavior. A somewhat less drastic implication may be that merely perturbation theory breaks down, and non-perturbative approaches could possibly characterize the scaling behavior at the transition of the two-temperature model H successfully.

5 Summary and final remarks

We have studied two non-equilibrium generalizations of the dynamical model H with both conserved scalar order parameter and dynamically coupled conserved transverse currents that describes second-order liquid-gas or binary-fluid phase transitions. Specifically, we were interested in the effect of detailed balance violations on the asymptotic critical behavior. We have investigated both (a) isotropic violations of the equilibrium conditions, which can be formulated in terms of different effective noise temperatures for the order parameter and conserved currents, respectively, and (b) spatially anisotropic detailed balance violations, i.e., dynamical noise which is governed by different strenghts in longitudinal and transverse momentum space sectors.

In principle, there are several possible scenarios: (1) In the vicinity of the critical point, detailed balance may effectively become restored as a consequence of the diverging correlation length that essentially averages over the different local noise temperatures; (2) a novel, stable renormalization group fixed point may emerge that describes a new universality class with genuine non-equilibrium scaling behavior; (3) there might be no stable RG fixed point at all, indicating perhaps complex spatio-temporal chaotic behavior rather than a simple stationary non-equilibrium state.

We find that situation (1) applies to the isotropic non-equilibrium model H, while scenario (3) appears to describe the effective two-temperature model that emerges upon allowing for spatially anisotropic noise correlations. Remarkably, case (2) is never realized in any dynamical model with reversible mode-couplings. In fact, a surprisingly simple overall picture emerges (we have already presented a brief overview in Ref. [12]). Namely, quite generally, the equilibrium dynamical models as listed in Ref. [1] with *non-conserved* order parameter turn out to be quite robust against detailed-balance violations. The purely relaxational models A and C do not even have a genuine non-equilibrium fixed point at all. For the SSS model, generalizing models E (for planar ferromagnets, $n = 2$) and model G (for isotropic antiferromagnets, $n = 3$) to arbitrary order parameter space dimension n, two non-equilibrium fixed points do exist, corresponding to ratios $T = 0$ and $T = \infty$ for the noise temperatures of the order parameter and dynamically coupled conserved fields, but neither of these is stable. Thus, near the critical point, detailed balance becomes eventually restored, and the asymptotic critical exponents are those of the equilibrium model [10]. Of course, such systems might remain in the crossover region for quite a while, masking the asymptotic regime. Essentially, this scenario (1) also applies when the conserved noise for the coupled non-critical fields is rendered anisotropic as well. Additional fixed points emerge, but the isotropic equilibrium one remains stable [11].

When detailed balance is violated *isotropically* in the models B, D (purely relaxational), J (isotropic ferromagnets, $n = 3$) and H with *conserved* order parameter, basically the same statements apply, and scenario (1) is realized again [10,12]. However, once one allows for spatially *anisotropic* or dynamical noise, separating a soft transverse and stiff longitudinal momentum space sector, which enforces anisotropic scaling, the behavior changes dramatically. In the relaxational two-temperature models B and D, the ensuing asymptotic theory however turns out to be equivalent to an equilibrium model with *long-range* correlations of the uniaxial dipolar or elastic type [9,12]. This corresponds to case (2) above. In stark contrast, in the anisotropic non-equilibrium versions of models J [11] and H which are characterized by relevant reversible mode-couplings to additional conserved variables, no stable renormalization group fixed point can be found (at least to one-loop order), which represents scenario (3). We do at this point not really know what the absence of an RG fixed point means physically in this situation; perhaps, as in the uniformly driven non-equilibrium model J [30], the long-time behavior is governed by spatio-temporal chaos. It would certainly be worthwhile to explore this issue further, e.g., through computer simulations.

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A Galilean invariance in the dynamic model H

In this appendix we derive the basic Ward identity which was used in the analysis of the non-equilibrium isotropic and two-temperature model H, namely Eq. (3.14), which follows from the Galilean invariance of the Langevin equations which describe the model [13]. In order to prove such an identity, we write the source-free equations (2.4) and (2.6) in a slightly different form:

$$\frac{\partial \psi_0}{\partial t} + g_0 \,\mathrm{j}_0 \cdot \nabla \psi_0 = \lambda_0 \nabla^2 (r_0 - \nabla^2) \psi_0 + \frac{\lambda_0 u_0}{6} \nabla^2 \psi_0^3 + \eta \,, \qquad \text{(A.1)}$$

and

$$\frac{\partial \mathrm{j}_0}{\partial t} + g_0 \,\mathrm{j}_0 \cdot \nabla \mathrm{j}_0 = = \mathcal{T} \left[D_0 \nabla^2 \mathrm{j}_0 + g_0 \nabla \psi_0 (r_0 - \nabla^2) \psi_0 + \zeta \right] \,, \quad \text{(A.2)}$$

where we have added the convective term $g_0 \,\mathrm{j}_0 \cdot \nabla \mathrm{j}_0$ to the left-hand side of (A.2). This term is normally not included explicitly in the analysis, as it generates diagrammatic contributions proportional to g_0^2/D_0^2. Since the scaling dimension of this effective coupling is μ^{2-d} ($\mu^{2-d-d_\parallel}$ for the two temperature model H), as can be seen from the dimensional analysis of sections 3.1 and 4.2, this coupling is irrelevant in the renormalization group sense and therefore normally neglected.

Under a Galilean transformation, to a reference frame moving with respect to the laboratory frame with velocity $\mathbf{v}$, the coordinates and fields change according to

$$\mathbf{r}' = \mathbf{r} - g_0 \,\mathbf{v}\, t \,, \qquad \text{(A.3)}$$

$$t' = t \,, \qquad \text{(A.4)}$$

$$\psi_0'(\mathbf{r}', t') = \psi_0'(\mathbf{r} - g_0 \,\mathbf{v}\, t, t) = \psi_0(\mathbf{r}, t) \,, \qquad \text{(A.5)}$$

$$\mathrm{j}_0'(\mathbf{r}', t') = \mathrm{j}_0'(\mathbf{r} - g_0 \,\mathbf{v}\, t, t) = \mathrm{j}_0(\mathbf{r}, t) - \mathbf{v} \,, \qquad \text{(A.6)}$$

where the prime ($'$) refers to parameters and dynamic variables measured in the moving frame.

Using these transformation laws, it is easy to show that the time and space derivatives of ψ_0 and j_0 are transformed according to

$$\nabla_{\mathbf{r}'} \mid_{t'} = \nabla_{\mathbf{r}} \mid_t \,, \qquad \text{(A.7)}$$

$$\frac{\partial}{\partial t'} \mid_{\mathbf{r}'} = \frac{\partial}{\partial t} \mid_{\mathbf{r}} + g_0 \,\mathbf{v} \cdot \nabla_{\mathbf{r}} \mid_t \,, \qquad \text{(A.8)}$$

where $|_{\mathbf{r}}$ etc. simply indicates which variable is being held constant when the derivative is taken. With these relations, it is easy to show that the material derivative d/dt which appears on the left-hand side of (A.1) and (A.2), i.e.

$$\frac{d}{dt} = \frac{\partial}{\partial t} \mid_{\mathbf{r}} + g_0 \,\mathrm{j}_0 \cdot \nabla_{\mathbf{r}} \mid_t = \frac{\partial}{\partial t'} \mid_{\mathbf{r}'} + g_0 \,\mathrm{j}_0' \cdot \nabla_{\mathbf{r}'} \mid_{t'} \,, \qquad \text{(A.9)}$$

is invariant under a Galilean transformation, i.e. it preserves its form on going from one reference frame to another, as indicated in Eq. (A.9). The right-hand side of Eqs. (A.1) and (A.2) can also be seen from Eqs. (A.5) to (A.7) to be trivially invariant, given the fact that $\mathbf{v}$ is a constant vector. It is thus shown that the Langevin equations describing model H are invariant under a Galilean transformation (the distribution of the noise being the same in both reference frames).

This invariance must be preserved under renormalization, i.e., when we substitute $\psi_0, \mathrm{j}_0, g_0, \ldots$, by their renormalized counterparts ψ, j, g, etc. For this to happen, the renormalization factors Z_g and Z_{J} have to compensate each other in Eq. (A.9), i.e., one must have $Z_g Z_{\mathrm{J}} \equiv 1$, which is Eq. (3.14). Notice that the same reasoning also applies to the two-temperature model H, once g_0 is substituted by $g_0 = \sqrt{\widetilde{D}_0^\perp / D_0^\perp}\, g_0^\perp$.

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20. In the equilibrium theory, this is a very natural choice, as the transverse currents are non-critical modes. Thus their static response is only weakly temperature-dependent and essentially constant near the critical point.
21. Note that in Ref. [10] we have defined, for convenience, T_0 as the inverse of Eq. (3.17) here. The subsequent RG analysis is of course independent of this choice.
22. In some of the integrals to be evaluated one obtains contributions of the form $(\tau_0 + D_0/\lambda_0)^{(d-4)/2}$. In the IR limit $\tau_0 \to 0$, these contributions are finite below four dimensions and may be neglected, following the minimal subtraction prescription. A more expedite way to deal with these terms simply consists in taking the asymptotic limit $\lambda_0/D_0 \to 0$, keeping T_0 finite.
23. The vertex functions and susceptibilities of the variables $\widetilde{\mathbf{J}}$ and $\mathbf{J}$ are, of course, tensor functions. But given that these functions are purely transverse, one can write them as, e.g., $\Gamma^{\alpha\beta}_{0\widetilde{\mathrm{J}}\mathrm{J}}(\mathbf{q},\omega) = \mathcal{T}^{\alpha\beta}(\mathbf{q})\,\Gamma_{0\widetilde{\mathrm{J}}\mathrm{J}}(\mathbf{q},\omega)$, where $\mathcal{T}^{\alpha\beta}(\mathbf{q})$ denotes the transverse projector defined above. We have therefore only explicitly given the scalar part $\Gamma_{0\widetilde{\mathrm{J}}\mathrm{J}}(\mathbf{q},\omega)$ in the main text.
24. This equality follows from the relation $\left\langle \frac{\delta J}{\delta \mathrm{J}_0^\alpha(\mathbf{x},t)} \right\rangle + H_0^\alpha(\mathbf{x},t) = 0$, where $H_0^\alpha(\mathbf{x},t)$ is a source function which couples linearly to J_0^α and which was added to the functional J. Taking the functional derivative of this equation with respect to $\widetilde{\mathrm{J}}_0^\beta$ and performing a Fourier transformation, one obtains the desired result. This equality can be traced to the fact that in the functional J (and in the Langevin equations) J_0^α appears in the combination $\mathrm{J}_0^\alpha - A^\alpha$ in all terms except the term containing the time derivative $\partial_t \mathrm{J}_0^\alpha$. This is a trivial consequence of the fact that the free energy H depends at most quadratically on $\mathbf{J}_0$.
25. Note that although $Z_\psi = Z$ to all orders of perturbation theory in equilibrium, the equality $Z_\psi = Z = 1$ holds only to one-loop order, i.e., the susceptibility (and consequently the exponent η) acquires higher order corrections.
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