New Methods for Finding Non-Left-Orderable and Unique Product Groups

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Abstract

In this paper, we present techniques for proving a group to be non-left-orderable or a unique product group. These methods involve the existence of a mapping from the group to $\mathbb{R}$ which obeys a left-multiplication criterion. By determining the existence or non-existence of such a mapping, the desired information about the group can be concluded. As examples, we apply this technique to groups of transformations in hyperbolic 2- and 3- space, and Fibonacci groups.

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CHAPTER 1

Introduction

The immediate goals for this paper are to find new examples of unique product groups and non-left-orderable groups — namely, groups of transformations in hyperbolic space and Fibonacci groups. In addition, there are two larger questions which hope to be addressed using the techniques outlined here. The first is to determine whether there exists a unique product group which is not left-orderable; the second is to find new examples of torsion-free groups which are not unique product groups. Recall that G is a unique product group if, given nonempty subsets $X, Y \subseteq G$, there exists a $g \in G$ which can be uniquely expressed in the form xy with $x \in X$ and $y \in Y$. Also recall a group G is left-orderable if there exists a total ordering $\leq$ on G such that $a \leq b \iff ca \leq cb$ for all $a, b, c \in G$. It is known that left-orderable groups are unique product groups; however, the existence of a non-left-orderable unique product group is still an open problem. Examples of torsion-free groups which are not unique product groups are known — one such group is given by David Promislow [**7**] — but these examples are few. Besides the two goals listed above, another use for our methods is to prove that a group is in fact a unique product group. By Thomas Delzant's paper [**2**] we have the following result:

PROPOSITION 1.1. *Let G be a residually finite hyperbolic group. Then G has a subgroup H of finite index such that H is a unique product group.*

In the context of Delzant's paper, "hyperbolic" means word hyperbolic in the Gromov sense. This proposition provides motivation for finding hyperbolic groups which are also unique product groups. One method for doing so arises when the group acts on a metric space. In this case, the following criterion can be used:

PROPOSITION 1.2. *Let X be a metric space with metric d, and let G be a group acting on X. If, for all $p, q \in X$ and all non-identity $g \in G$,*

$$d(gp, q) > d(p, q) \text{ or } d(g^{-1}p, q) > d(p, q),$$

then G is a unique product group.

This proposition is implicit from [**2**] though it is not explicitly stated. It can be easily shown by choosing a point $p \in X$. Then there exists an element $g \in G$ such that $d(gp, p)$ is maximal; the unique product property must apply to g. Showing that the criterion of this proposition applies for a given group — for example, a torsion-free group of transformations in hyperbolic n-space — tells us that the unique product property also applies. One motivation for finding unique product groups lies in the area of group rings. By [**6**] and [**8**], we see that if k is an uncountable field and G is a unique product group, then the group algebra $k[G]$ is a domain.

Another problem addressed in this paper is whether a given group is non-left-orderable. A second criterion can be applied which allows us to determine such groups:

PROPOSITION 1.3. *Let H be a countable left-orderable group. Then there exists a mapping $\theta : H \to \mathbb{R}$ such that*

$$\theta(gh) > \theta(h) \text{ or } \theta(g^{-1}h) > \theta(h)$$

for all $g, h \in H$, where $g \neq 1$.

By [**5**, Lemma 2.4, Proposition 2.5] and [**9**, Lemma 2.2], we can assume a countable left-orderable group H is a subgroup of $\mathrm{Aut}(\mathbb{R})$ and that for all non-identity $g, h \in H$, $g(0) \neq 0$ and $g \neq h \Rightarrow g(0) \neq h(0)$. Then, defining θ by $\theta(g) = g(0)$ for all $g \in H$, the condition follows.

By proving that such a mapping θ cannot exist for certain torsion-free groups, we can show that these groups are not left-orderable. In other words, these groups cannot act on the real line (or the circle) by orientation-preserving homeomorphisms.

The above two conditions can be generalized by the statement: Let G be a group. G is left orderable implies there is a map θ: G$\to \mathbb{R}$ such that $\theta gx > \theta x$ or $\theta g^{-1}x > \theta x$ for all $g, x \in G$ with $g \neq 1$ implies G is a unique product group.

CHAPTER 2

Unique product property of Fuchsian groups

Using 1.2 we can determine that certain groups acting on hyperbolic n-space obey the unique product property. The first such group is a torsion-free Fuchsian group; i.e. a torsion-free discrete group of Möbius transformations acting on the hyperbolic 2-space $\mathbb{H}^2$. Recall that a group is *discrete* when its associated topology is discrete.

We will characterize $\mathbb{H}^2$ using the upper half-plane model, in which every point is represented by a complex number $a+bi \in \mathbb{C}$; $b>0$. We will also refer to the real axis, the line $b=0$, as $\partial\mathbb{H}^2$. The hyperbolic metric d in this model is defined by

$$\cosh d(w,z) = 1 + \frac{|w-z|^2}{2\operatorname{Im}(w)\operatorname{Im}(z)}$$

For ease of calculation, we introduce a modified version of the metric, denoted $\|w-z\|$ and defined by

Definition 2.1.

$$\|w-z\| = \frac{|w-z|^2}{\operatorname{Im}(w)\operatorname{Im}(z)}.$$

Since the hyperbolic arccosine function is strictly increasing on its entire domain, it can be easily verified that $\|w-z\| > \|x-z\| \Rightarrow d(w,z) > d(x,z)$. For our purposes, transformations in $\mathbb{H}^2$ can be represented as matrices $g \in SL_2(\mathbb{R})$; that is, matrices of the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,b,c,d \in \mathbb{R}, ad-bc=1.$$

Viewing $z \in \mathbb{H}^2$ as a complex number, the transformation g acts on z as the fractional linear transformation

$$gz = \frac{az+b}{cz+d}.$$

Every transformation of $\mathbb{H}^2$ can be placed into one of three distinct classifications, depending on its fixed points in $\mathbb{H}^2$, or alternately, the square of the trace of its matrix representation:

Definition 2.2. *Let $g \in SL_2(\mathbb{R})$ act on $\mathbb{H}^2$. Then*

(1) *g is* elliptic $\iff \operatorname{tr}^2(g) < 4 \iff$ *g fixes a single point in $\mathbb{H}^2$.*
(2) *g is* hyperbolic $\iff \operatorname{tr}^2(g) > 4 \iff$ *g fixes two distinct points on $\partial\mathbb{H}^2$.*
(3) *g is* parabolic $\iff \operatorname{tr}^2(g) = 4 \iff$ *g fixes a single point on $\partial\mathbb{H}^2$.*

Elliptic, parabolic, and hyperbolic transformations are analogous to Euclidean rotations, translations, and horolations, respectively. Along with fixing certain points, these elements also keep certain geodesics in $\mathbb{H}^2$ invariant. The figures below demonstrate the fixed points and an invariant geodesic for representative transformations.

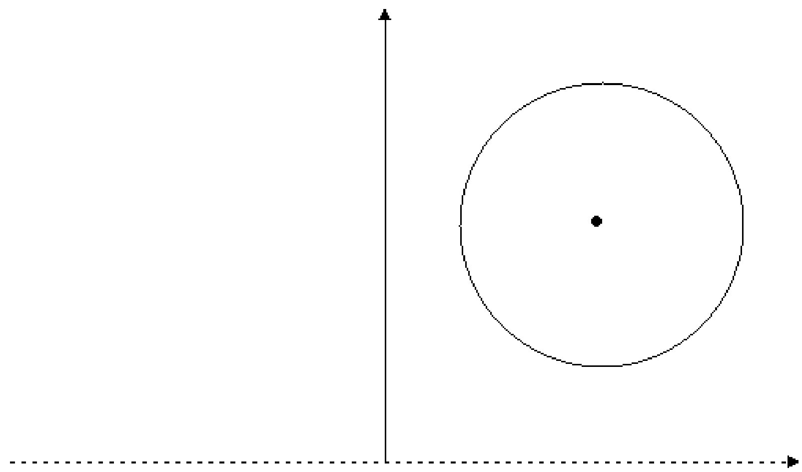

FIGURE 1. Elliptic Transformation

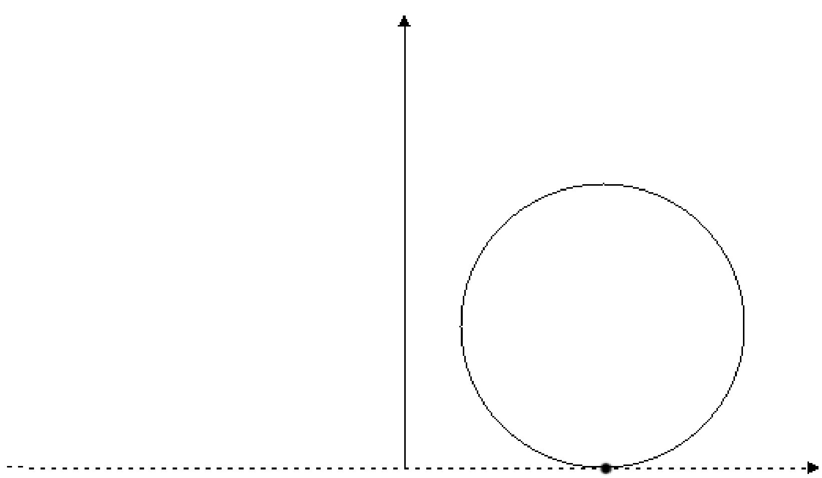

FIGURE 2. Parabolic Transformation

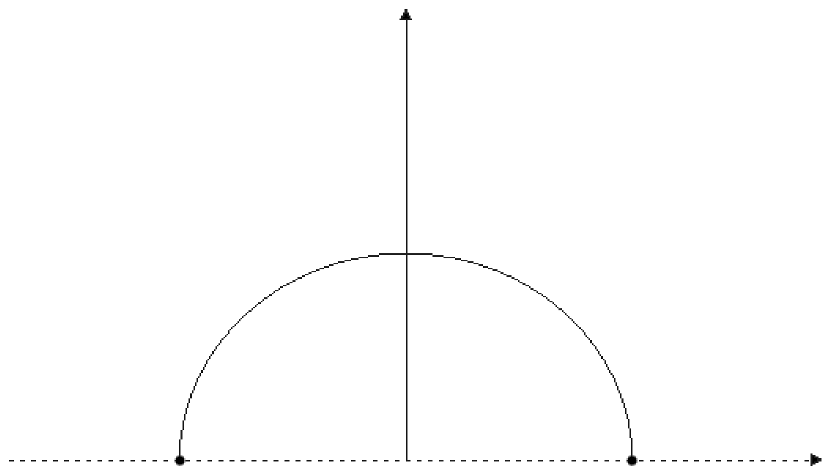

FIGURE 3. Hyperbolic Transformation

We will require the following result:

LEMMA 2.3. *A torsion-free group of Möbius transformations contains no elliptic elements.*

PROOF. Let g be an elliptic transformation and consider the group $\langle g \rangle$. From [**3**, Proposition 2.1], a group of Möbius tranformations fixes a point in $\mathbb{H}^2$ if and only if the group is finite. Since $\langle g \rangle$ fixes a point, namely, the point fixed by g, $|\langle g \rangle|$ is finite. Thus g has finite order and so a group containing g cannot be torsion-free. □

We can now prove the following theorem:

THEOREM 2.4. *A torsion-free Fuchsian group is a unique product group.*

PROOF. Let $p, q \in \mathbb{H}^2$ be distinct and let G be a torsion-free Fuchsian group. There exists an isometry α of $\mathbb{H}^2$ such that $\alpha(p) = i$ and $\alpha(q) = \frac{i}{n}$, where $d(p,q) = d(i, \frac{i}{n})$. ($\alpha$ is essentially a shift which moves p and q to the vertical axis). Also, since $d(i, \frac{i}{n}) = d(i, ni)$, we can further assume that $0 < n < 1$. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ be a nonidentity element. The isometry α acts on g as $\alpha g \alpha^{-1}$. Thus the classification of g as elliptic, hyperbolic, or parabolic remains invariant under α, since $\mathrm{tr}(\alpha g \alpha^{-1}) = \mathrm{tr}(\alpha \alpha^{-1} g) = \mathrm{tr}(g)$. To show that G is a unique product group, it is now sufficient to show that $\|gi - \frac{i}{n}\|$ or $\|g^{-1}i - \frac{i}{n}\|$ is greater than $\|i - \frac{i}{n}\|$:

$$\frac{1}{n}\|i - \frac{i}{n}\| = \left(\frac{n-1}{n}\right)^2$$

and

$$\frac{1}{n}\|gi-\frac{i}{n}\| = \frac{|\frac{ai+b}{ci+d}-\frac{i}{n}|^2}{\mathrm{Im}\left(\frac{ai+b}{ci+d}\right)}$$
$$= \frac{|\frac{ai+b}{ci+d}\frac{d-ci}{d-ci}-\frac{i}{n}|^2}{\mathrm{Im}\left(\frac{ai+b}{ci+d}\frac{d-ci}{d-ci}\right)} = \frac{|\frac{bd+ac+(ad-bc)i}{c^2+d^2}-\frac{i}{n}|^2}{\mathrm{Im}\left(\frac{bd+ac+(ad-bc)i}{c^2+d^2}\right)}$$
$$= \frac{\left(\frac{bd+ac}{c^2+d^2}\right)^2+\left(\frac{1}{c^2+d^2}-\frac{1}{n}\right)^2}{\frac{1}{c^2+d^2}}$$
$$= \frac{(bd+ac)^2}{c^2+d^2}+\left(\frac{n-\left(c^2+d^2\right)}{n}\right)^2\frac{1}{c^2+d^2}$$
$$\geq \frac{\left(n-\left(c^2+d^2\right)\right)^2}{n^2\left(c^2+d^2\right)}.$$

We may further assume that $c^2+d^2>1$: since g must be hyperbolic or parabolic, $(a+d)^2 \geq 4$, which implies $a^2+c^2 \geq 1$ or $c^2+d^2 \geq 1$. If $a^2+c^2 \geq 1$, we achieve the result by substituting g^{-1} for g in the proof. The final case, $a^2=d^2=1$ and $c^2=0$, will be covered later in the proof. So $n^2 < c^2+d^2$, which implies:

$$n^2\left(1-\left(c^2+d^2\right)\right) > \left(c^2+d^2\right)\left(1-\left(c^2+d^2\right)\right)$$
$$\Rightarrow n^2+\left(c^2+d^2\right)^2 > \left(c^2+d^2\right)+n^2\left(c^2+d^2\right)$$
$$\Rightarrow \frac{n^2-2n\left(c^2+d^2\right)+\left(c^2+d^2\right)^2}{n^2} > \frac{n^2\left(c^2+d^2\right)-2n\left(c^2+d^2\right)+\left(c^2+d^2\right)}{n^2}$$
$$\Rightarrow \frac{\left(n-\left(c^2+d^2\right)\right)^2}{n^2} > \frac{\left(c^2+d^2\right)(n-1)^2}{n^2}$$
$$\Rightarrow \frac{\left(n-\left(c^2+d^2\right)\right)^2}{\left(c^2+d^2\right)n^2} > \left(\frac{n-1}{n}\right)^2$$
$$\Rightarrow \frac{1}{n}\|gi-\frac{i}{n}\| > \frac{1}{n}\|i-\frac{i}{n}\|$$
$$\therefore \|gi-\frac{i}{n}\| > \|i-\frac{i}{n}\|.$$

Also, if $a^2=d^2=1$ and $c^2=0$, then $b^2 \neq 0$ since g is a nonidentity element. Then

$$\frac{(bd+ac)^2}{c^2+d^2}+\left(\frac{n-\left(c^2+d^2\right)}{n}\right)^2\frac{1}{c^2+d^2} = b^2+\left(\frac{n-1}{n}\right)^2 > \left(\frac{n-1}{n}\right)^2$$

Therefore $d(gi,\frac{i}{n}) > d(i,\frac{i}{n})$, so $d(gp,q) > d(p,q)$ and the result follows from 1.2. □

CHAPTER 3

Unique product property of Kleinian groups

The next step logically is to develop an analogue of 2.4 for torsion-free *Kleinian* groups; that is, torsion-free discrete groups of Möbius transformations in the hyperbolic 3-space $\mathbb{H}^3$. While it may be possible to refine the results below to include further torsion-free Kleinian groups, it can at least be shown that some such groups, most notably torsion-free subgroups of Picard's group and Bianchi groups, are unique product groups.
The models and calculations used here are similar to those used in the previous section. We will represent $\mathbb{H}^3$ using the upper half-space model, in which every point can be represented by a quaternion $a+bi+cj \in \mathbb{H}, c>0$. The complex plane c $=0$ will be called $\partial\mathbb{H}^3$. The metric d on $\mathbb{H}^3$, for $w=a+bi+cj, z=q+ri+sj$, is defined by

$$\cosh d(w,z) = 1 + \frac{|w-z|^2}{2cs}.$$

As in the previous section, we will use a simplified version of this metric, $\|w-z\|$, defined by:

Definition 3.1. $\|w-z\| = \frac{|w-z|^2}{cs}$.

It can again be easily verified that $\|w-z\| > \|x-z\| \Rightarrow d(w,z) > d(x,z)$ in $\mathbb{H}^3$. For our purposes, we may view transformations in $\mathbb{H}^3$ as matrices in $SL_2(\mathbb{C})$, i.e. matrices of the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; a,b,c,d \in \mathbb{C}, ad-bc=1.$$

For a transformation g represented as above and for $z = s+tj \in \mathbb{H}^3, s \in \mathbb{C}$, g acts on z as the fractional linear transformation defined in the previous section. This action may also be written in an alternate form, called the *Poincaré extension:*

Definition 3.2 (Poincaré extension). $gz = \frac{(as+b)\left(\overline{cs+d}\right)+a\bar{c}t^2+|ad-bc|tj}{|cs+d|^2+|c|^2t^2}$.

This is the preferred formulation for our calculations in $\mathbb{H}^3$. A transformation in hyperbolic 3-space can again be classified as elliptic, hyperbolic, and parabolic according to the square of the trace of its matrix representation or, alternately, the points it fixes; replacing all instances of $\mathbb{H}^2$ with $\mathbb{H}^3$ (and $\partial\mathbb{H}^2$ with $\partial\mathbb{H}^3$) in Definition 2.2 gives the analogous definitions. In addition, the fact that our matrices have complex entries brings rise to a fourth classification:

Definition 3.3. *Let $g \in SL_2(\mathbb{C})$ act on $\mathbb{H}^3$. Then g is* strictly loxodromic $\iff \operatorname{tr}^2(g) \notin [0,+\infty)$.

We will now prove the following theorem, a 3-dimensional analogue to 2.4:

THEOREM 3.4. *A torsion-free Kleinian group G is a unique product group provided*
$|a|, |c|$, *or* $|d| \geq 1$
for all strictly loxodromic $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.

PROOF. Let $p, q \in \mathbb{H}^3$ be distinct and let G be a torsion-free Kleinian group. The following lemma, from [**1**], implies that there again can be no elliptic elements in G:

LEMMA 3.5. *A subgroup G of $\mathcal{M}$, the group of Möbius transformations, such that G acts on $\mathbb{H}^3$, is discrete $\iff$ for every compact $K \subseteq \mathbb{H}^3$, $g(K) \cap K = \emptyset$ for all but finitely many $g \in G$.*

If there exists an elliptic $g \in G$ with fixed point $x \in \mathbb{H}^3$, $g^k(\{x\}) = \{x\}$ for all k. Since G is torsion-free, there are infinitely many such g^k; thus G cannot be a discrete group and so is not a Kleinian group.
Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ be a nonidentity element. As in the proof of 2.4, we assert the existence of an isometry α of $\mathbb{H}^3$ such that $\alpha(p) = j$ and $\alpha(q) = \frac{j}{n}$, where $d(p,q) = d(j, \frac{j}{n})$ and $0 < n < 1$. It is again sufficient to show that $\|gj - \frac{j}{n}\| > \|j - \frac{j}{n}\|$ or $\|g^{-1}j - \frac{j}{n}\| > \|j - \frac{j}{n}\|$. We know

$$\frac{1}{n}\|j - \frac{j}{n}\| = \left(\frac{n-1}{n}\right)^2$$

And, using the Poincaré extension for gj:

$$\frac{1}{n}\|gj - \frac{j}{n}\| = \frac{\left|\frac{b\bar{d}+a\bar{c}}{|c|^2+|d|^2} + \frac{j}{|c|^2+|d|^2} - \frac{j}{n}\right|^2}{\frac{1}{|c|^2+|d|^2}}$$

$$= \left(|c|^2 + |d|^2\right)\left(\left|\frac{b\bar{d}+a\bar{c}}{|c|^2+|d|^2}\right|^2 + \left[\frac{n - \left(|c|^2+|d|^2\right)}{n\left(|c|^2+|d|^2\right)}\right]^2\right)$$

$$\geq \frac{\left(n - \left(|c|^2+|d|^2\right)\right)^2}{n^2\left(|c|^2+|d|^2\right)}.$$

For g hyperbolic or parabolic, the proof that $\|gj - \frac{j}{n}\| > \|j - \frac{j}{n}\|$ or $\|g^{-1}j - \frac{j}{n}\| > \|j - \frac{j}{n}\|$ is now identical to the one used for 2.4. For g strictly loxodromic,

$$|c| \text{ or } |d| \geq 1 \Rightarrow \frac{1}{n}\|gj - \frac{j}{n}\| \geq \frac{\left(n - \left(|c|^2+|d|^2\right)\right)^2}{n^2\left(|c|^2+|d|^2\right)} > \left(\frac{n-1}{n}\right)^2$$

and

$$|a| \text{ or } |c| \geq 1 \Rightarrow \frac{1}{n}\|g^{-1}j - \frac{j}{n}\| \geq \frac{\left(n - \left(|c|^2+|a|^2\right)\right)^2}{n^2\left(|c|^2+|a|^2\right)} > \left(\frac{n-1}{n}\right)^2.$$

$\square$

Groups which satisfy the condition of the above theorem include torsion-free subgroups of *Picard's group*, the group of matrices whose entries are Gaussian integers, and Bianchi groups in general. A *Bianchi group* is a discrete group of matrices whose entries lie in the ring of integers $\mathcal{O}_{\mathbb{Q}\left(\sqrt{-d}\right)}$ of the extension $\mathbb{Q}\left(\sqrt{-d}\right)$, where d is a squarefree positive integer. It is known that $\mathcal{O}_{\mathbb{Q}\left(\sqrt{-d}\right)} = \mathbb{Z}\left[\omega\right]$, where

$$\omega = \begin{cases} \sqrt{-d}, & \text{if } -d \equiv 2,3 \mod 4 \\ \frac{1+\sqrt{-d}}{2}, & \text{if } -d \equiv 1 \mod 4. \end{cases}$$

For $-d \equiv 2,3 \mod 4$, each matrix entry is of the form $a + b\sqrt{-d}; a, b \in \mathbb{Z}$ (assume $b \neq 0$). Since $|a + b\sqrt{-d}| = \sqrt{a^2 + b^2|d|} \geq \sqrt{1} \geq 1$, these Bianchi groups meet the unique product criterion. For $-d \equiv 1 \mod 4$, we may assume $|d| > 3$. Each matrix entry is of the form $a + b\left(\frac{1+\sqrt{-d}}{2}\right); a, b \in \mathbb{Z}$ (again, assume $b \neq 0$). Then $|a + b\left(\frac{1+\sqrt{-d}}{2}\right)|^2 = \left(a + \frac{b}{2}\right)^2 + \frac{b^2|d|}{4}$. For $b \neq 2a$, $\left(a + \frac{b}{2}\right)^2 \geq \frac{1}{4}$; since $\frac{b^2|d|}{4} \geq \frac{3}{4}$, it follows that $|a + b\left(\frac{1+\sqrt{-d}}{2}\right)|^2 \geq 1 \Rightarrow |a + b\left(\frac{1+\sqrt{-d}}{2}\right)| \geq 1$. If $b = -2a$, $a + b\left(\frac{1+\sqrt{-d}}{2}\right) = a\sqrt{-d}$; obviously $|a\sqrt{-d}| \geq 1$. Note that we may ignore the cases in which the matrix entries are equal to 0; since the determinant must equal 1, at least one of the lower triangular entries is nonzero.

CHAPTER 4

$F(n-1,n)$ is not left-orderable

In the next two sections, we will prove that certain groups are non-left-orderable by demonstrating that a function as described in 1.3 cannot act on them. The first such group is the Fibonacci group $F(n-1,n)$. This group is presented by

$$F(n-1,n) = \langle x_1, x_2, \ldots, x_n \mid x_1x_2\cdots x_{n-1} = x_n, \ldots, x_nx_1\cdots x_{n-2} = x_{n-1}\rangle,$$

where the product of $n-1$ consecutive generators gives the next one modulo n. In [**4**], it is shown that this group is not left-orderable by other means. However, [**4**] also provides motivation for finding other non-left-orderable 3-manifold groups, which can be accomplished using this method. We state the following:

THEOREM 4.1. *There exists no mapping $\theta : F(n-1,n) \to \mathbb{R}$ which satisfies*

$$\theta(gh) > \theta(h) \text{ or } \theta(g^{-1}h) > \theta(h)$$

for all $g, h \in F(n-1,n)$. Therefore $F(n-1,n)$ is not left-orderable.

PROOF. Suppose there exists a $\theta : F(n-1,n) \to \mathbb{R}$ which satisfies this condition. Then, by defining $x_0 = x_n$ and $x_{n+1} = x_1$, the following lemmas apply to $F(n-1,n)$:

LEMMA 4.2. *Let $1 \leq i \leq n$. Then $\theta(x_{i+1}) > \theta(x_i)$ or $\theta(x_i^{-1}) > \theta(x_i)$.*

PROOF. Because of the left-multiplication condition satisfied by θ, one of the following cases apply:

Case 1. $\theta(x_{i+1}) = \theta(x_{i+2}x_{i+3}\cdots x_i) > \theta(x_{i+3}\cdots x_i) > \cdots > \theta(x_i)$.

Case 2. For some $k \leq n$,

$$\theta(x_k^{-1}x_{k+1}\cdots x_i) = \theta(x_{k-1}^{-1}x_{k-2}^{-1}\cdots x_{i+1}^{-1}) > \theta(x_{k+1}\cdots x_i) > \cdots > \theta(x_i).$$

Then

$$\theta(x_k^{-1}x_{k-1}^{-1}\cdots x_{i+1}^{-1}) > \theta(x_{k-1}^{-1}x_{k-2}^{-1}\cdots x_{i+1}^{-1})$$

or

$$\theta(x_kx_{k-1}^{-1}\cdots x_{i+1}^{-1}) = \theta(x_{k+1}x_{k+2}\cdots x_i) > \theta(x_{k+1}x_{k+2}\cdots x_i),$$

a contradiction. Therefore

$$\theta(x_k^{-1}x_{k-1}^{-1}\cdots x_{i+1}^{-1}) > \theta(x_{k-1}^{-1}x_{k-2}^{-1}\cdots x_{i+1}^{-1}).$$

Applying this argument inductively for $k+1, k+2, \ldots, i$ gives

$$\theta(x_i^{-1}) = \theta(x_{i-1}^{-1}x_{i-2}^{-1}\cdots x_{i+1}^{-1}) > \cdots > \theta(x_{k-1}^{-1}x_{k-2}^{-1}\cdots x_{i+1}^{-1}) > \cdots > \theta(x_i),$$

the desired result.

□

An analogue of this lemma exists for $\theta(x_i^{-1})$:

LEMMA 4.3. *Let $1 \leq i \leq n$. Then $\theta(x_{i-1}^{-1}) > \theta(x_i^{-1})$ or $\theta(x_i) > \theta(x_i^{-1})$.*

PROOF. Again, two cases apply here:

Case 1. $\theta(x_{i-1}^{-1}) = \theta(x_{i-1}^{-1}x_{i-2}^{-2}\cdots x_i^{-1}) > \theta(x_{i-2}^{-2}\cdots x_i^{-1}) > \cdots > \theta(x_i^{-1})$.

Case 2. For some $k \leq n$,

$$\theta(x_k x_{k-1}^{-1}\cdots x_i^{-1}) = \theta(x_{k+1}x_{k+2}\cdots x_{i-1}) > \theta(x_{k-1}^{-1}\cdots x_i^{-1}) > \cdots > \theta(x_i^{-1}).$$

Then

$$\theta(x_k x_{k+1}\cdots x_{i-1}) > \theta(x_{k+1}x_{k+2}\cdots x_{i-1})$$

or

$$\theta(x_k^{-1}x_{k+1}\cdots x_{i-1}) = \theta(x_{k-1}^{-1}x_{k-2}^{-1}\cdots x_i^{-1}) > \theta(x_{k-1}^{-1}\cdots x_i^{-1}),$$

a contradiction. Thus

$$\theta(x_k x_{k+1}\cdots x_{i-1}) > \theta(x_{k+1}x_{k+2}\cdots x_{i-1}).$$

Applying this argument inductively for $k+1, k+2, \ldots, i$ gives

$$\theta(x_i) = \theta(x_{i+1}x_{i+2}\cdots x_{i-1}) > \cdots > \theta(x_{k+1}x_{k+2}\cdots x_{i-1}) > \theta(x_i^{-1}),$$

the desired result.

□

$2n+1$ applications of the results of 4.2 and 4.3, starting with $\theta(x_1)$, gives a chain of strict inequalities:

$$\theta(x_1) > \theta(x_{i_1}^{\epsilon_1}) > \cdots > \theta(x_{i_{2n+1}}^{\epsilon_{2n+1}}),$$

where $i_j \leq n$ and $\epsilon_j = \pm 1$. Since no more than $2n$ such $x_i^{\epsilon_i}$ in this chain can be distinct, the pigeonhole principle indicates $\theta(x_j^{\epsilon j}) > \theta(x_j^{\epsilon j})$ for some j in the chain. The resulting contradiction completes the proof. □

CHAPTER 5

$F(2, 2n)$ is not left-orderable

A somewhat more interesting application of 1.3 is proving the non-left-orderability of the Fibonacci group $F(2, 2n)$, presented by

$$F(2, 2n) = \langle x_1, x_2, \ldots, x_{2n} \mid x_1x_2 = x_3, \ldots, x_{2n}x_1 = x_2\rangle,$$

where the product of two consecutive generators gives the next generator modulo $2n$.

THEOREM 5.1. *There exists no mapping $\theta : F(2, 2n) \to \mathbb{R}$ which satisfies*

$$\theta(gh) > \theta(h) \text{ or } \theta(g^{-1}h) > \theta(h)$$

for all $g, h \in F(2, 2n)$. Therefore $F(2, 2n)$ is not left-orderable.

PROOF. Again, assume such a θ exists. We first prove the following lemma:

LEMMA 5.2. *For all $1 \leq i \leq 2n$, the following hold:*
(1) $\theta(x_{i+2}) > \theta(x_i)$ *or* $\theta(x_{i-1}^{-1}) > \theta(x_i)$
(2) $\theta(x_{i-1}) > \theta(x_i^{-1})$ *or* $\theta(x_{i+2}^{-1}) > \theta(x_i^{-1})$

Where we define $x_{2n+1} = x_1$, $x_{2n+2} = x_2$, and $x_0 = x_n$.

PROOF.
(1) From the properties of θ, we know $\theta(x_ix_{i-1}x_i) = \theta(x_{i+2}) > \theta(x_i)$ or $\theta((x_ix_{i-1})^{-1}x_i) = \theta(x_{i-1}^{-1}) > \theta(x_i)$.
(2) Similarly, $\theta(x_{i+1}x_i^{-1}) = \theta(x_{i-1}) > \theta(x_i^{-1})$ or $\theta(x_{i+1}^{-1}x_i^{-1}) = \theta(x_{i+2}^{-1}) > \theta(x_i^{-1})$.

□

$4n + 1$ applications of the result of 5.2, starting with $\theta(x_1)$, gives a chain of inequalities:

$$\theta(x_1) > \theta(x_{i_1}^{\epsilon_1}) > \cdots > \theta(x_{i_{4n+1}}^{\epsilon_{4n+1}}),$$

where $i_j \leq 2n$ and $\epsilon_j = \pm 1$. Since the number of such distinct $x_i^{\epsilon_i}$ is no more than $4n$, the pigeonhole principle indicates $\theta(x_j^{\epsilon j}) > \theta(x_j^{\epsilon j})$ for some j in this chain. The resulting contradiction completes the proof. □

The argument above can also be illustrated by using a directed graph, like the one below for $F(2, 8)$. An edge directed from a vertex A to vertex B means $\theta(B) > \theta(A)$. Thus any cycle containing A indicates the contradiction $\theta(A) > \theta(A)$. Since the properties of θ require us to always trace an edge outward from a vertex if one exists, a quick inspection shows us that a cycle will be encountered along any path. For $2n \geq 8$, $F(2, 2n)$ is an infinite torsion-free group; thus our result provides new examples of non-left-orderable infinite torsion-free groups.

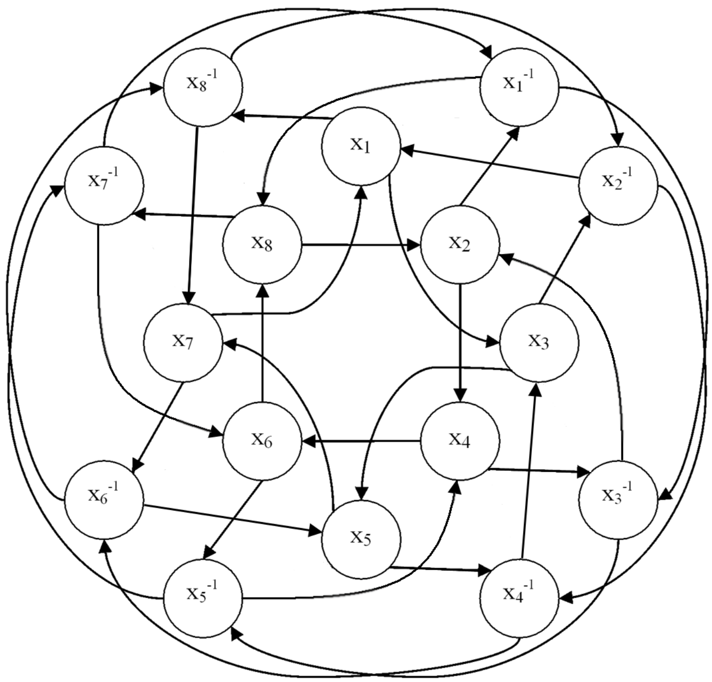

FIGURE 4. The generators of F(2,8) and their inverses.

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Vita

Steven M. Hair, the son of Randy and Patricia Hair, was born on December 11, 1979 in Shermans Dale, Pennsylvania. He graduated from West Perry High School in June 1998. He began attending Virginia Tech in August 1998, and graduated from this institution in December 2001 with a B.S. in Mathematics. During his time as an undergraduate, Steven was a National Merit Scholar, member of the Phi Beta Kappa honor society, the Pi Mu Epsilon mathematics honor society and the Sigma Pi Sigma physics honor society. In his senior year, he was also awarded the Layman prize for outstanding undergraduate research in mathematics.

Upon completing his undergraduate work, Steven enrolled in the M.S. program at Virginia Tech. As a graduate student, he worked as a Graduate Teaching Assistant, teaching freshman calculus classes, as well as pursuing research. This thesis completes the requirements for his M.S. degree in Mathematics.

Following graduation, Steven plans to pursue a Ph.D. in Mathematics at another institution.