Octonions and the Exceptional Lie Algebra $\mathfrak{g}_2$

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Abstract

The octonions are an eight dimensional algebra invented in the 1840s as an analog to the quaternions. In this paper, we present two constructions of the octonions and show the octonions' relationship to the exceptional Lie algebra $\mathfrak{g}_2$.

We first introduce the octonions as an eight dimensional vector space over a field of characteristic zero with a multiplication defined using a table. We also show that the multiplication rules for octonions can be derived from a special graph with seven vertices call the Fano Plane.

Next we explain the Cayley-Dickson construction, which exhibits the octonions as the set of ordered pairs of quaternions. This approach parallels the realization of the complex numbers as ordered pairs of real numbers.

The rest of the thesis is devoted to following a paper by N. Jacobson written in 1939 entitled "Cayley Numbers and Normal Simple Lie Algebras of Type G". We prove that the algebra of derivations on the octonions is a Lie algebra of type G_2. The proof proceeds by showing the set of derivations on the octonions is a Lie algebra, has dimension fourteen, and is semisimple. Next, we complexify the algebra of derivations on the octonions and show the complexification is simple. This suffices to show the complexification of the algebra of derivations is isomorphic to $\mathfrak{g}_2$ since $\mathfrak{g}_2$ is the only semisimple complex Lie algebra of dimension fourteen. Finally, we conclude the algebra of derivations on the octonions is a simple Lie algebra of type G_2.

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1 Introduction

Our story begins in the 1830's when a mathematician named William Hamilton became the first person to realize complex numbers as ordered pairs of real numbers. Fascinated with the idea of a complex plane, Hamilton set out to find a three dimensional analog. In particular, he wanted a three dimensional algebra having both addition and multiplication. Hamilton tried for many years to come up with a good system. In a letter to his son, Hamilton describes the end of his search [Ha]:

> ... your mother was walking with me, along the Royal Canal, to which she had perhaps driven; and although she talked with me now and then, yet an *under-current* of thought was going on in my mind, which gave at last a *result*, whereof it is not too much to say that I felt at once the importance. An *electric* circuit seemed to *close*; and a spark flashed forth, the herald (as I *foresaw, immediately*) of many long years to come of definitely directed thought and work, by *myself* if spared, and at all events on the part of *others*, if I should even be allowed to live long enough distinctly to communicate the discovery. Nor could I resist the impulse - unphilosophical as it may have been - to cut with a knife on a stone of the Brougham Bridge, as we passed it, the fundamental formula with symbols i, j, k; namely,
>
> $$ij = k \quad jk = i \quad ki = j \quad i^2 = j^2 = k^2 = -1$$
>
> which contains the *Solution* of the *Problem* ...

What you may ask, are these i, j, and k? They are the basis elements for the algebra Hamilton sought. Can this be the birth of the octonions? No, not quite yet. These are the quaternions. To Hamilton's suprise, the algebra he needed was not three dimensional, but four. Thus, the quaternions were born - a four dimensional normed division algebra over the reals having basis $\{1, i, j, k\}$.

So, where do these octonions come in? Well, one of Hamilton's colleagues, a man by the name of John Graves was intrigued by Hamilton's discovery and Graves set out to see whether the pattern might continue. That is, if we have a normed division algebra of dimension $1 = 2^0$ ($\mathbf{R}$), and one of dimension 2 (**C**), and now one of dimension $4 = 2^2$ (the quaternions, **H**) then might there be a normed division algebra of dimension 8?

As Graves discovered, the answer is yes. In an 1843 letter to Hamilton, Graves describes his discovery of an eight dimensional algebra having all the properties of a normed division algebra. However, while Hamilton's quaternions had the slightly eccentric property of being noncommutative (and Graves' octonions shared this), the octonions further had the encumberment of being nonassociative. Adding nail to coffin, Graves' contemporary, Arthur Cayley, beat him to print in a small note at the end of a 1845 paper. So, while the octonions enjoyed the attention of mathematicians for a time, Graves soon moved on to other interests and most mathematicians followed suit. Thus, we have it that while strolling the streets of 21st century earth, it's possible you may catch snippets of conversation about a now little known thing called quaternions but you will never hear of octonions.

However, in recent years, interest has begun to pique again about these strange little octonions. It turns out that the octonions have a way of popping up in the most unrelated branches of mathematics and tying together seemingly disparate ideas. For example, each of the five exceptional Lie algebras have an octonionic construction [B, p. 148]. Other applications appear in areas such as superstring theory, projective geometry, Moufang loops, topology, and Jordan algebras. An excellent paper by John Baez [B] goes into detail on a number of such applications.

This thesis presents the octonions, discusses their membership in the set of normed division algebras, and shows their relationship to the exceptional simple Lie algebra $\mathfrak{g}_2$. Namely, it proves that the algebra of derivations on the octonions is a Lie algebra of type G_2. We describe the contents of each section following this introduction below.

In section two, the octonions are introduced. We define normed division algebras and introduce several important concepts regarding them. As we introduce the concepts, we apply each one to our most familiar normed division algebras - the reals and the complexes. Then, we introduce quaternions and further apply the normed division algebra concepts to quaternions. Having completed our introduction to normed division algebras, we then define the octonions using their most standard construction as a vector space over the reals. We apply the normed division algebra concepts to the octonions. We also show how the Fano Plane can be used to remember the octonions' multiplication rules. The section is finished with an observation of octonion subalgebras.

In section three, we delve deeper into the octonions, presenting a second construction of the octonions which reflects their historical origin. The Cayley-Dickson construction delineates the natural relationship between the reals, the complexes, the quaternions, and the octonions. We will show that the complexes can be viewed as ordered pairs of reals, quaternions can be viewed as ordered pairs of complexes, and octonions can be viewed as ordered pairs of quaternions. In fact, multiplcation is defined in a uniform way throughout these sets of pairs.

Section four is where we show the octonion's relationship to the exceptional Lie algebra $\mathfrak{g}_2$. In 4.1, we introduce section four by discussing the notion of Lie algebra and briefly presenting the classification of complex Lie algebras. The rest of this thesis then follows a paper by N. Jacobson written in 1939 entitled "Cayley Numbers and Normal Simple Lie Algebras of Type G" [J1]. Among other things, the Jacobson paper proves that the Lie algebra of derivations on the octonions is a simple Lie algebra of type G_2. In this paper we present a revised version of Jacobson's proof that operates in the case when the underlying field of the octonions has characteristic zero. We furthermore prove the specific case that the complexification of the Lie algebra of derivations on the octonions is isomorphic to the complex exceptional Lie algebra $\mathfrak{g}_2$.

Section 4.2 officially begins our proof by showing that the set of derivations on the octononians constitutes a Lie algebra. We present the definition of derivation and prove a few necessary lemmas about derivations. We let $\mathcal{D}$ denote derivations on the octonions. Finally, we define the operations on $\mathcal{D}$ and show these satisfy the Lie algebra requirements.

In section 4.3 we prove $\mathcal{D}$ has dimension fourteen. Since $\mathfrak{g}_2$ is the only complex simple Lie algebra of dimension fourteen, we will later use this result to eliminate the possibility of $\mathcal{D}$ having any type other than G_2. In proving $\mathcal{D}$ has dimension fourteen (Thm 4.10), we prove that an octonion map is a derivation if and only if it has the standard form presented in Table 4. Thus, in addition to proving dimension, we have created for ourselves a very convenient way to construct derivations.

The next section (4.4) shows that $\mathcal{D}$ is semisimple. The proof of $\mathcal{D}$ semisimple proceeds by showing that $\mathcal{D}$ has no nontrivial abelian ideals (Thm 4.13). Next, a result from one of our reference sources indicates this condition is sufficient to make $\mathcal{D}$ semisimple (Prop 4.14). We then pass the argument to the complexification $\mathcal{D}_{\mathbf{C}}$ of $\mathcal{D}$ and show $\mathcal{D}_{\mathbf{C}}$ is semisimple (Cor 4.15). This complexification will later enable us to use our familiar classification system of Lie algebras to determine $\mathcal{D}$'s type.

In our final section, we complete the proof. First, we show that $\mathcal{D}_{\mathbf{C}}$ has at most two components when written as a direct sum of simple Lie algebras. A counting argument using the classification of complex simple Lie algebras (see section 4.1) then yields the main result: $\mathcal{D}_{\mathbf{C}}$ is isomorphic to the Lie algebra $\mathfrak{g}_2$ (Thm 4.21). Finally, using a result from Knapp, we conclude that $\mathcal{D}$ must be a Lie algebra of type G_2 (Cor 4.22).

For those fascinated by the octonions, Baez's paper[B] is a great entry point to the vast and varied world of octonians. The results in this paper represent only the tip of the iceberg is comparison to the body of literature already written on this subject and the various relevances of these most interesting mathematical objects.

2 An Introduction to Octonions

2.1 Preliminaries

The octonions belong to a special family of algebras called normed division algebras. A theorem of Hurwitz asserts there exist only four such algebras: the reals $\mathbf{R}$, the complexes $\mathbf{C}$, the quaternions $\mathbf{H}$, and the octonions $\mathbf{O}$ [B, p. 150]. Before defining the octonions, we introduce several basic concepts which apply to normed division algebras and we illustrate how they work for the complexes. Then we introduce the quaternions using our new concepts.

The complexes are a vector space over the real numbers having dimension 2 with basis $\{1, i\}$. We can write each complex number uniquely as a linear combination $a + bi$ where a and b are real numbers. Complex numbers can also be multiplied given by the rule $(a + bi)(c + di) = (ac - db) + (ad + cb)i$. This makes the complex numbers into an algebra over $\mathbf{R}$.

Definition 2.1 (Algebra) *A is an algebra over $\mathbf{R}$ if A is a real vector space having a distributive multiplication map with the properties that $\mathbf{R}$ is in the center of A and 1 is the multiplicative identity of A.*

Note that here we are not assuming algebras are associative. In fact, this paper addresses an example of nonassociative algebra - the octonions.

Recall that complexes have a conjugation function. We conjugate a complex number a+bi by $\overline{a + bi} = a - bi$. Note that

$$\begin{aligned}
\overline{(a + bi)(c + di)} &= \overline{(ac - db) + (ad + cb)i} \\
&= (ac - db) - (ad + cb)i \\
&= (ac - (-d)(-b)) + (a(-d) + c(-b))i \\
&= (c - di)(a - bi) \\
&= \overline{(c + di)}\ \overline{(a + bi)}.
\end{aligned}$$

We now introduce two maps from $\mathbf{C}$ to $\mathbf{R}$ which will be relevant to our discussion of octonions. The norm function on $\mathbf{C}$ is the map from $\mathbf{C}$ to $\mathbf{R}$ given by $N(a+bi) = (a+bi)\overline{(a + bi)} = (a+bi)(a-bi) = a^2+b^2$. Additionally, trace is a map from $\mathbf{C}$ to $\mathbf{R}$ given by $tr(a + bi) = (a + bi) + \overline{(a + bi)} = (a + bi) + (a - bi) = 2a$. If we strict norm and trace to the reals, then $N(x) = x^2$ and $tr(x) = 2x$ for all $x \in \mathbf{R}$.

There are two more definitions we need to consider:

Definition 2.1 (Division Algebra) *A finite dimensional algebra A is a division algebra if given a, b in A such that $ab = 0$, then either $a = 0$ or $b = 0$.*

Note that when we add the condition that A be associative, the usual definition follows. That is, A is a finite dimensional associative division algebra when every nonzero element has a multiplicative inverse.

Definition 2.1 (Normed Division Algebra) *A finite dimensional algebra A is a normed division algebra if it is a normed vector space with norm N such that $N(ab) = N(a)N(b)$ holds for all a, b in A.*

Every normed division algebra is a division algebra since the division algebra condition follows from the norm condition. To see this, suppose $ab = 0$ for some a, b in R. Then $N(ab) = N(a)N(b) = 0$. Since fields are integral domains, this implies $N(a) = 0$ or $N(b) = 0$ which implies that $a = 0$ or $b = 0$.

It is easy to verify that $N((a + bi)(c + di)) = N(a + bi)N(c + di)$ making our familiar complexes and reals into normed division algebras.

As a final precursor to introducing the octonions, a primer is needed on one of the octonion's predecessors - the quaternions.

As stated in the introduction, the quaternions are a four dimensional vector space, $\mathbf{H}$, over the reals having basis $\{1, i, j, k\}$. We can write any element of the quaternions uniquely as four real numbers $a + bi + cj + dk$. Quaternions form an algebra since they have a multiplication governed by Hamilton's rules:

$$ij = k = -ji, \qquad jk = i = -kj, \qquad ki = j = -ik,$$

and

$$i^2 = j^2 = k^2 = -1.$$

Quaternion multiplication is illustrated in Figure 1. Quaternions can be conjugated as $\overline{a + bi + cj + dk} = a - bi - cj - dk$ in the expected fashion. Just like complexes, quaternions display the trait that

$$\overline{xy} = \overline{y}\,\overline{x}.$$

Norm and trace also have similarly predictable manifestations in the quaternions: $N(x) = x\overline{x}$ and $tr(x) = x + \overline{x}$ for all $x \in \mathbf{H}$. It is straightforward to verify that $N(ab) = N(a)N(b)$ for all $a, b \in \mathbf{H}$, so the quaternions are in fact a normed division algebra as we expected. Note that quaternions are noncommutative since $jk = -kj \neq kj$.

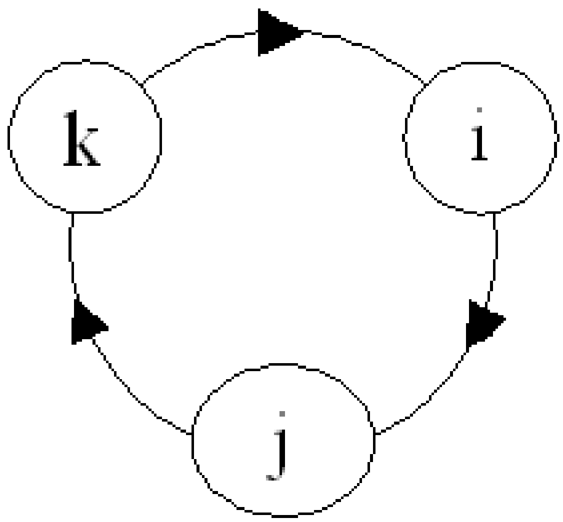

Figure 1: Quaternion Multiplication

2.2 An Eight Dimensional Vector Space over the Reals

What are the octonions? The simplest way to think of them is as an eight dimensional vector space over the reals having a special multiplication. We denote the octonions by **O**. There exist several other constructions of the octonions; this section explains the simplest of them. We name the basis: $\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and give the multiplication of these basis elements by the Table 1.

Table 1: Octonion Multiplication Table

	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
e_2	e_2	$-e_3$	-1	e_1	$-e_6$	e_7	e_4	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	e_6	$-e_5$	$-e_4$
e_4	e_4	$-e_5$	e_6	$-e_7$	-1	e_1	$-e_2$	e_3
e_5	e_5	e_4	$-e_7$	$-e_6$	$-e_1$	-1	e_3	e_2
e_6	e_6	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	-1	e_1
e_7	e_7	e_6	e_5	e_4	$-e_3$	$-e_2$	$-e_1$	-1

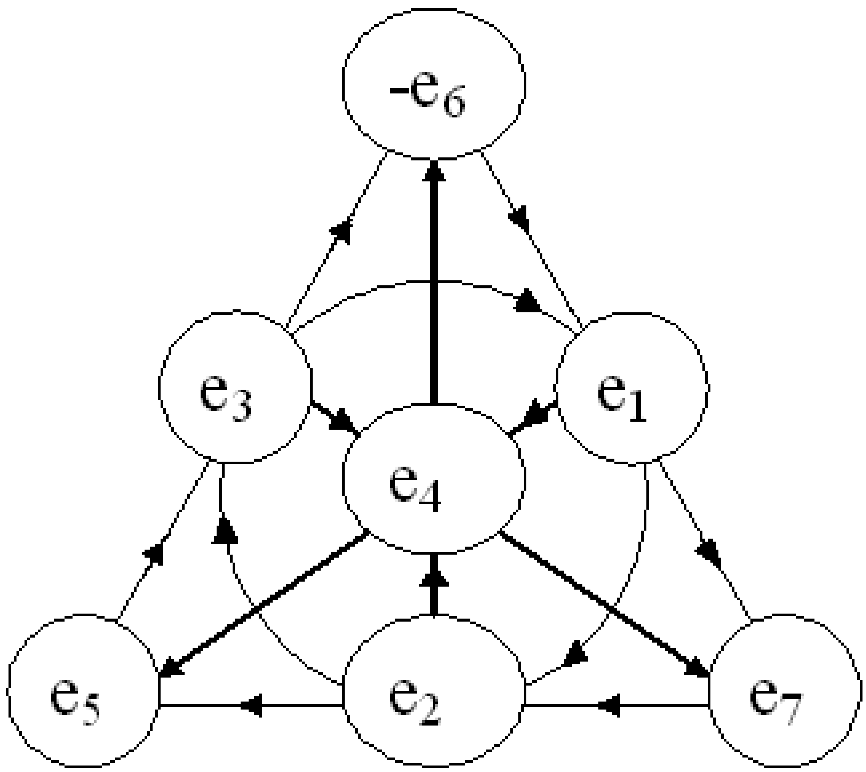

Figure 2: The Fano Plane

Every element in **O** is a linear combination of these basis elements where the scalars are in the field **R**. By the distributive laws, this completely defines octonion multiplication.

One of the most interesting properties of octonions is that they are nonassociative. To see this, take the example of multiplying three basis elements e_1, e_2, and e_4: $e_1(e_2e_4) = e_1(-e_6) = -(e_1e_6) = -e_7$. Whereas: $(e_1e_2)e_4 = (e_3)e_4 = e_7 \neq -e_7$.

While the octonions are not associative in general, they do satisfy a particular type of associativity called alternate associativity.

Definition 2.1 (Alternate Associative) *An algebra is alternate associative if any two elements generate an associative subalgebra.*

An equivalent and more commonly used definition is satisfaction of the following conditions:

$$a(ba) = (ab)a$$
$$a(ab) = (aa)b$$
$$a(bb) = (ab)b$$

for all elements a, b in the algebra. The equivalency of these definitions is proved by a theorem of Emil Artin [S, thm 3.1]. A theorem by Zorn shows that the octonions are indeed alternate associative [S, thm 3.17].

Let's consider a few properties of the octonions using the definitions in our preliminary section.

Let $x \in \mathbf{O}$. Then $x = \lambda_0 + \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 e_5 + \lambda_6 e_6 + \lambda_7 e_7$ for some $\lambda_i \in \mathbf{R}$.

The conjugate of x is defined to be,

$$\overline{x} = \lambda_0 - \lambda_1 e_1 - \lambda_2 e_2 - \lambda_3 e_3 - \lambda_4 e_4 - \lambda_5 e_5 - \lambda_6 e_6 - \lambda_7 e_7.$$

That is, the scalar λ_0 retains its sign and each other coefficient switches its sign. This definition is analogous to that for complexes because in complexes the scalar is unchanged and the coefficient of i changes sign. Like the complexes and quaternions,

$$\overline{xy} = \overline{y}\ \overline{x}$$

for all $x, y \in \mathbf{O}$.

Since the octonian algebra is a normed division algebra, it must have a norm. The norm is defined as $N(x) = x\overline{x}$. This agrees with the norm of a complex number because $N(a + bi) = (a + bi)\overline{(a + bi)} = (a + bi)(a - bi) = a^2 + b^2$. A routine calculation will verify that $N(x)$ is indeed in the base field.

The last standard mapping we need to we need to know about for octonions is trace. By definition, the trace of x is, $tr(x) = x + \overline{x} = 2\lambda_0$. Naturally, the trace of any octonion lives inside the base field.

Several interesting features of the multiplication table (Table 1) deserve to be pointed out. First of all, $e_i^2 = -1$ for $i = 1$ to 7. Also, $e_i e_j = -e_j e_i$ for all $i \neq j$. However, even with the help of these little rules, octonion multiplication would be difficult in the least without some other helpful mnemonic for recalling the multiplication table. Fortunately, such a mnemonic exists. The fano plane (Figure 2) captures the multiplicative relationship between every basis element except the identity basis element. To see how it works, observe the bottom line of the triangle in Figure 2. If we multiply from right to left as e_7e_2 then it yields e_5. Going "backwards," e_5e_2 yields $-e_7$. The line loops around the back end as well: e_5e_7 yields e_2.

It is worth noting that there are several interesting subalgebras in the octonions. Consider the relationships along the circle of the fano plane: $e_1e_2 = e_3$, $e_3e_1 = e_2$, and $e_2e_3 = e_1$. Adding or multiplying linear combinations of 1, e_1, e_2, and e_3 yield another such linear combination. Thus, $(1, e_1, e_2, e_3)$ is a subalgebra. As a matter of fact, this subalgebra is isomorphic to the quaternions for if we define a map $\Phi : \mathbf{O}|_{(1,e_1,e_2,e_3)} \to \mathbf{H}$ by $\Phi(1) = 1, \Phi(e_1) = i, \Phi(e_2) = j$, and $\Phi(e_3) = k$, then it is easy to see Φ forms an isomorphism.

From observing the fano plane, there are seven such sets of three basis elements: the upper right line, the upper left line, the base line, the circle, and the three medians. Thus, there are seven distinct subalgebras isomorphic to the quaternions within the octonions.

Concretely,

$$(1, e_1, e_2, e_3)$$
$$(1, e_1, e_4, e_5)$$
$$(1, e_4, e_2, e_6)$$
$$(1, e_3, e_4, e_7)$$
$$(1, e_1, e_6, e_7)$$
$$(1, e_2, e_5, e_7)$$
$$(1, e_3, e_5, e_6)$$

each form a subalgebra of the octonions isomorphic to the quaternions. This will prove useful in future sections.

As we shall see in the next section, the octonions can actually be viewed as a vector space over any of these quaternions.

3 The Cayley-Dickson Construction

The Cayley-Dickson construction is a second construction of the octonions which clarifies the relationship between the four normed division algebras: reals $\mathbf{R}$, complexes $\mathbf{C}$, quaternions $\mathbf{H}$, and octonions $\mathbf{O}$ [B].

Recall that $\mathbf{C}$ is a two dimensional vector space over $\mathbf{R}$. Thus we may view $\mathbf{C}$ as ordered pairs of real numbers. In a similar fashion, $\mathbf{H}$ can be viewed as ordered pairs of complex numbers and **O** can be realized as ordered pairs of quaternions.

Now suppose that A is one of the normed division algebras, **R**, **C**, **H**, or **O**. The Cayley-Dickson construction provides a uniform method for defining mulitplication and conjugation of $A \times A$ so that the resulting algebra is the "next" normed division algebra in the series.

We continue the assumption that A is a normed division algebra. Define conjugation on $A \times A$ as follows: $\overline{(a,b)} = (\overline{a}, -b)$. Define multiplication by $(a,b)(c,d) = (ac - \overline{d}b, da + b\overline{c})$.

Now, we assert that the complexes are isomorphic to $A \times A$ when $A = \mathbf{R}$. To see the isomorphism, define $\Phi((a,b)) = a + bi$. It is clear that $\mathbf{C} \cong \mathbf{R} \times \mathbf{R}$ as vector spaces. Thus we only need to check that Φ preserves multiplication. By definition of multiplication,

$$\Phi((a,b)(c,d)) = \Phi(ac - \overline{d}b, da + b\overline{c})$$

Since $\overline{a} = a$ for all $a \in \mathbf{R}$,

$$\Phi((a,b)(c,d)) = \Phi(ac - db, da + bc)$$

By definition of Φ,

$$\Phi((a,b)(c,d)) = (ac - db) + (da + bc)i$$

Complex multiplication yields,

$$(ac - db) + (da + bc)i = (a + bi)(c + di)$$

Hence,

$$\Phi((a,b)(c,d)) = \Phi(a,b)\Phi(c,d).$$

Next, the quaternions are constructed using $A \times A$ where $A = \mathbf{C}$. The isomorphism is defined as $\Phi((a_1, a_2), (b_1, b_2)) = a_1 + a_2 i + b_1 j + b_2 k$. A routine, straightforward calculation shows that this map preserves the multiplication operation.

Note, by this construction the quaternions can be viewed as a 2 dimensional vector space over the complexes. More explicitly, we take basis {1,j} and if $\alpha = a + bi + cj + ck$ then we can also write $\alpha = (a + bi) + (c + di)j$ where (a+bi) and (c+di) are in the complexes. This works since $(c + di)j = cj + dk$.

At the next level, the octonions are realized as $A \times A$ where $A = \mathbf{H}$. The isomorphism between this construction and the notation used by our usual octonion multiplication table is as follows:

$$\begin{aligned}\Phi(((a_1, a_2), (b_1, b_2)), ((c_1, c_2), (d_1, d_2))) &= a_1 + a_2 e_1 + b_1 e_2 + b_2 e_3 \\ &\quad + c_1 e_4 + c_2 e_5 - d_1 e_6 + d_2 e_7.\end{aligned}$$

Proving the isomorphism is tedious but straightforward. This construction lets us view the octonions as a two dimensional vector space over the quaternions. To satisfy any curiosity, the negative in front of e_6 comes from the somewhat backward multiplication between e_4 and e_2. Explicitly, if

$$x = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7,$$

then we can write

$$x = (a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3) + (a_4 + a_5 e_1 - a_6 e_2 + a_7 e_3) e_4$$

since

$$(a_4) e_4 = a_4 e_4,$$

$$(a_5 e_1) e_4 = a_5 e_5,$$

$$\begin{aligned}(-a_6 e_2) e_4 &= a_6(-e_2 e_4) \\ &= a_6(e_4 e_2) \\ &= a_6 e_6,\end{aligned}$$

and

$$(a_7 e_3) e_4 = e_7.$$

The fact that $(a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3)$ and $(a_4 + a_5 e_1 - a_6 e_2 + a_7 e_3)$ are in the quaternions yields our conclusion that x is an ordered pair of quaternions.

Note that by this construction, we have $\mathbf{H}$ and e_4 generating $\mathbf{O}$ as an algebra.

If we wished to, we could continue the process to obtain higher dimensional algebras. The dimensions increase by powers of 2 - reals 1, complexes 2, quaternions 4, octonions 8, sedonians 16, and so on. However, our algebras become more unwieldy at each iteration. With the step from reals to complexes, we loose the property that every element is its own conjugate. With the step from complexes to quaternions, we loose commutivity. Arriving at octonions, we loose associativity. Finally, at sedonians, we loose the division algebra property [B, p. 154].

Now that we've introduced octonions and examined their properties, it's time to introduce the proof of the main result of this thesis - the result that we can construct the exceptional Lie algebra $\mathfrak{g}_2$ from the octonions.

4 The Lie Algebra $\mathfrak{g}_2$

In this section we cover Jacobson's theorem that the set of derivations on the octonions is isomorphic to an exceptional Lie algebra of type G_2 [J1].

4.1 Lie Algebra Concepts

A few definitions concerning Lie algebras are necessary before we can begin the proof.

Definition 4.1 (Lie Algebra) *L is a Lie algebra over a field $\mathbf{F}$ if L is a vector space over $\mathbf{F}$ and it has an operation called bracket [,] satisfying:*

- *bilinearity: $[x + x', y] = [x, y] + [x', y]$, $[x, y + y'] = [x, y] + [x, y']$, and $[\alpha x, y] = [x, \alpha y] = \alpha[x, y]$ for all $x, x', y, y' \in L$ and $\alpha \in \mathbf{F}$.*
- *antisymmetry: $[x, y] = -[y, x]$ for all $x, y \in L$.*
- *the Jacobi identity: $[x, [y, z]]+[y, [z, x]]+[z, [x, y]] = 0$ for all $x, y, z \in L$.*

Note that if L is an associative algebra, then A becomes a Lie algebra using the commutator:

$$[a, b] = ab - ba.$$

Lie algebras offer a slightly different perspective on a few of our familiar definitions.

Definition 4.1 (Commute) *Given a Lie algebra L, two elements $x, y \in L$ are said to commute if $[x, y] = 0$.*

Definition 4.1 (Lie Ideal) *Given a Lie algebra L, $I \subseteq L$ is a Lie ideal if given some $x \in A$, $y \in I$, $[x, y] \in I$.*

Definition 4.1 (Simple Lie Algebra) *A Lie algebra L is a simple Lie algebra if $[L, L] \neq 0$ and if L has no ideals other than (0) and itself.*

Definition 4.1 (Semisimple Lie Algebra) *A Lie algebra L is semisimple if it is the direct sum of simple Lie algebras. Specifically, $L = \bigoplus_{i=1}^{r} L_i$. Where the $\bigoplus$ implies $[L_i, L_j] = 0$ for all $i \neq j$ and each L_i is simple.*

Simple Lie algebras over the complex numbers have been fully classified. There are two major divisions: the classical Lie algebras and the exceptional Lie algebras. Among the classical Lie algebras, there are four infinite families of complex simple Lie algebras: A_l, B_l, C_l, and D_l. These algebras are assumed to be over the complexes.

Unlike the classical Lie algebras, there are only a finite number of complex exceptional Lie algebras. Specifically, there are five. These are the algebras of type E_6, E_7, E_8, F_4, and G_2. Given that these algebras are being considered over the complexes, Table 2 summarizes the rules for finding the possible dimensions of each of these [K, p. 508-517].

Table 2: Dimensions of Simple Lie Algebras

Simple Lie Algebra Family	Dimension	Restriction
A_l	$l(l+2)$	$l \geq 1$
B_l	$l(2l+1)$	$l \geq 2$
C_l	$l(2l+1)$	$l \geq 3$
D_l	$l(2l-1)$	$l \geq 4$
E_6	78	
E_7	133	
E_8	248	
F_4	52	
G_2	14	

The algebra we are interested in is $\mathfrak{g}_2$, the simple Lie algebra of type G_2. In the next section we will begin to exhibit a concrete realization of an algebra of type G_2 as the Lie algebra of derivations on the octonions.

The proof proceeds as follows. First we present the set of derivations on the octonions, $\mathcal{D}$. We show $\mathcal{D}$ forms a Lie algebra. Next, we prove that $\mathcal{D}$ has dimension fourteen. Then we show $\mathcal{D}$ is semisimple by utilizing a result from one of our sources - Humphreys' "Introduction to Lie Algebras." We then use $\mathcal{D}$ semisimple to show the complexification of $\mathcal{D}$ is a simple Lie algebra. According to our classification, there exists only one complex simple Lie algebra of dimension fourteen and this is $\mathfrak{g}_2$. This makes the complexification of $\mathcal{D}$ isomorphic to $\mathfrak{g}_2$. Finally, our proof finishes with the observation that the complexification of $\mathcal{D}$ isomorphic to $\mathfrak{g}_2$ implies $\mathcal{D}$ is of type G_2.

4.2 The Algebra of Derivations on O

Suppose A is an algebra over a field F. Define a derivation over A as a function $D : A \to A$ such that:

$$D(x + y) = D(x) + D(y)$$

$$D(\alpha x) = \alpha D(x)$$

$$D(xy) = xD(y) + D(x)y$$

for all $\alpha \in F$, and for all $x, y \in A$. That is, D is linear and the derivation condition, $D(xy) = xD(y) + D(x)y$, holds.

We denote $\mathcal{D}(A)$ to be the set of derivations over A.

For practice, let's prove a few useful little results about derivations.

Proposition 4.1 *Let A be a division algebra over the real numbers. If $D \in \mathcal{D}(A)$, then $D(\alpha) = 0$ for all α in $\mathbf{R}$.*

Proof Choose nonzero $x \in A$ and $\alpha \in \mathbf{R}$. Then $D(\alpha x) = \alpha D(x) + D(\alpha)x$ by the definition of derivation. However, it is also true that $D(\alpha x) = \alpha D(x)$ by the definition of derivation. Substituting, $\alpha D(x) = \alpha D(x) + D(\alpha)x$ implies $D(\alpha)x = 0$. Since $x \neq 0$ and A is a division algebra, this yields our conclusion. $\square$

Now we will begin developing our understanding of the derivations on $\mathbf{O}$.

Proposition 4.2 *If $D \in \mathcal{D}(\mathbf{O})$, then $tr(D(x)) = 0$ for all $x \in \mathbf{O}$.*

Proof Suppose D is a derivation on the octonions and let $x = a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7$ be an arbitrary element in the octonions. Then,

$$\begin{aligned} D(x) &= D(a_0 + \textstyle\sum_{i=1}^{7} a_ie_i) \\ &= D(a_0) + \textstyle\sum_{i=1}^{7} D(a_ie_i) \\ &= 0 + \textstyle\sum_{i=1}^{7} a_iD(e_i) \end{aligned}$$

So,

$$tr(D(x)) = tr(\sum_{i=1}^{7} a_iD(e_i)).$$

Therefore, it is sufficient to show $tr(D(e_i)) = 0$ for all i.

Now, fix an i and define $D(e_i) = a_{i0} + \sum_{j=1}^{7} a_{ij}e_j$ for some $a_{ij} \in \mathbf{R}$. Then,

$$\begin{aligned} D(e_i^2) &= e_i D(e_i) + D(e_i)e_i \\ &= e_i(a_{i0} + \textstyle\sum_{j=1}^{7} a_{ij}e_j) + (a_{i0} + \sum_{j=1}^{7} a_{ij}e_j)e_i \\ &= 2a_{i0}e_i + \textstyle\sum_{j=1}^{7} a_{ij}e_ie_j + \sum_{j=1}^{7} a_{ij}e_je_i \\ &= 2a_{i0}e_i - a_{ii}e_i^2 + \textstyle\sum_{j=1,j\neq i}^{7} a_{ij}e_ie_j - a_{ii}e_i^2 + \sum_{j=1,j\neq i}^{7} a_{ij}e_ie_j \end{aligned}$$

since $e_n^2 = -1$ for all n. Now, since $e_ie_j = -e_je_i$ for all $i \neq j$,

$$\begin{aligned} D(e_i^2) &= 2a_{i0}e_i - 2a_{ii}e_i^2 + \textstyle\sum_{j=1,j\neq i}^{7} a_{ij}e_ie_j - \sum_{j=1,j\neq i}^{7} a_{ij}e_ie_j \\ &= 2a_{i0}e_i - 2a_{ii} \end{aligned}$$

However, since $e_i^2 = -1$,

$$D(e_i^2) = D(-1) = 0.$$

Therefore,

$$2a_{i0}e_i - 2a_{ii} = D(e_i^2) = 0.$$

Equating basis coefficients indicates $a_{i0} = 0$. Therefore

$$D(e_i) = a_{i0} + \sum_{j=1}^{7} a_{ij}e_j = \sum_{j=1}^{7} a_{ij}e_j.$$

Therefore $tr(D(e_i)) = 0$. □

Proposition 4.3 *$D(\overline{x}) = \overline{D(x)}$ for all $x \in \mathbf{O}$.*

Proof Choose arbitrary $x \in \mathbf{O}$. By our last proposition, $D(x) \in \mathbf{O}'$. Then

$$tr(D(x)) = 0 = D(x) + \overline{D(x)}$$

Therefore,

$$\overline{D(x)} = -D(x)$$

Now, let $x = a_0 + \sum_{i=1}^{7} a_ie_i$.

$$\begin{aligned} D(\overline{x}) &= D(\overline{a_0 + \textstyle\sum_{i=1}^{7} a_ie_i}) \\ &= D(a_0 - \textstyle\sum_{i=1}^{7} a_ie_i) \\ &= a_0D(1) - \textstyle\sum_{i=1}^{7} a_iD(e_i) \\ &= 0 - \textstyle\sum_{i=1}^{7} a_iD(e_i) \\ &= -D(x) \\ &= \overline{D(x)}. \quad \square \end{aligned}$$

Now we'll consider the interplay between derivations on quaternions and derivations on octonions. Take $\mathbf{H} = (1, e_1, e_2, e_3)$ so that we use e_1, e_2, and e_3 as a basis for the quaternions in this discussion. Then $\mathcal{D}(\mathbf{H})$ denotes the set of derivations on the quaternions and $\mathcal{D}(\mathbf{O})$ denotes the set of derivations on the octonions.

Let $D \in \mathcal{D}(\mathbf{H})$. Then, since D is linear, D is fully defined by its behavior on the basis $1, e_1, e_2, e_3$. We can extend D by defining $D(e_4) = ce_4$ for some $c \in \mathbf{H}$. Specifically, from our work on the Cayley-Dickson construction, we know that every element of the octonions can be writtenly uniquely as an ordered pair $a+be_r$ where $a, b \in \mathbf{H}$. Then $D(a+be_4) = D(a)+bD(e_4)+D(b)e_4$ and our extension is well defined. This extension of D puts the range of D in $\mathbf{O}$ since for any quaternion $c = c_0 + c_1e_1 + c_2e_2 + c_3e_3$,

$$\begin{aligned} ce_4 &= (c_0 + c_1e_1 + c_2e_2 + c_3e_3)e_4 \\ &= c_0e_4 + c_1e_5 - c_2e_6 + c_3e_7 \\ &\in \mathbf{O} \end{aligned}$$

The next proposition confirms what you may already suspect; such an extended D retains the properties of a derivation.

Proposition 4.4 *Let $D \in \mathcal{D}(\mathbf{H})$. If we define $D(e_4) = ce_4$ for some $c \in \mathbf{H}$, then $D \in \mathcal{D}(\mathbf{O})$.*

(It is worth noting that if we used any of the other quaternion subalgebras for $\mathbf{H}$ this result would also hold simply by placing some other fourth basis element not in $\mathbf{H}$ in the role of e_4.)

Proof Linearity of D on $\mathbf{O}$ follows naturally from our construction. We need only prove the derivation condition,

$$D(xy) = xD(y) + D(x)y$$

holds for all $x, y \in \mathbf{O}$. However, since this condition already holds for $\mathbf{H}$ and since D is linear on $\mathbf{O}$, we need only prove

$$D(e_ia) = e_iD(a) + D(e_i)a$$

holds for $i = 4, \ldots, 7$ and for arbitrary a in $\mathbf{H}$. We will only show the calculation for $i = 4$ since the proof is nearly identical for the others. Let $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7$.

$$
\begin{aligned}
D(e_4 a) &= a_0 D(e_4) + a_1 D(e_4 e_1) + a_2 D(e_4 e_2) + a_3 D(e_4 e_3) \\
&\quad + a_4 D(e_4 e_4) + a_5 D(e_4 e_5) + a_6 D(e_4 e_6) + a_7 D(e_4 e_7)
\end{aligned}
$$

$$
\begin{aligned}
D(e_4 a) &= a_0 D(e_4) + a_1 e_4 D(e_1) + a_1 D(e_4) e_1 + a_2 e_4 D(e_2) + a_2 D(e_4) e_2 \\
&\quad + a_3 e_4 D(e_3) + a_3 D(e_4) e_3 + a_4 e_4 D(e_4) + a_4 D(e_4) e_4 + a_5 e_4 D(e_5) \\
&\quad + a_5 D(e_4) e_5 + a_6 e_4 D(e_6) + a_6 D(e_4) e_6 + a_7 e_4 D(e_7) + a_7 D(e_4) e_7
\end{aligned}
$$

$$
D(e_4 a) = \textstyle\sum_{i=1}^{7} e_4(a_i D(e_i)) + D(e_4) a_0 + \sum_{i=1}^{7} D(e_4)(a_i e_i)
$$

$$
D(e_4 a) = e_4 \textstyle\sum_{i=1}^{7} a_i D(e_i) + D(e_4)(a_0 + \sum_{i=1}^{7} a_i e_i)
$$

$$
D(e_4 a) = e_4 D(a) + D(e_4) a. \quad \square
$$

From now on, let $\mathcal{D}$ denote the set of derivations over the octonions. That is, $\mathcal{D} = \mathcal{D}(\mathbf{O})$. We finish this section by showing that $\mathcal{D}$ is a Lie algebra.

Note that any linear combination of linear maps is a linear map. We can also show that a linear combination of derivations retains the derivation condition:

$$
\begin{aligned}
(\alpha D + \beta E)(ab) &= \alpha D(ab) + \beta E(ab) \\
&= a\alpha D(b) + \alpha D(a) b + a\beta E(b) + \beta E(a) b \\
&= a(\alpha D(b) + \beta E(b)) + (\alpha D(a) + \beta E(a)) b \\
&= a(\alpha D + \beta E)(b) + (\alpha D + \beta E)(a) b.
\end{aligned}
$$

Therefore $\mathcal{D}$ has closure under linear combinations and is thus a vector space.

Now, given any two derivations $E, D \in \mathcal{D}$, we can form the composition of E and D as linear maps on $\mathbf{O}$. This is a type of multiplication for elements of $\mathcal{D}$. Note however, that ED is not necessarily a derivation, so $\mathcal{D}$ does not have multiplicative closure. We define the bracket on $\mathcal{D}$ as follows: if $E, D \in \mathcal{D}$, then

$$
[D, E] = DE - ED
$$

The next result is standard, we include it for the sake of completeness.

Lemma 4.5 *$[D, E] \in \mathcal{D}$ for all $D, E \in \mathcal{D}$.*

Proof Choose arbitrary $D, E \in \mathcal{D}$. Let $\alpha \in \mathbf{R}$. Then,

$$\begin{aligned} [D,E](\alpha a) &= (DE - ED)(\alpha a) \\ &= D(E(\alpha a)) - E(D(\alpha a)) \\ &= D(\alpha(E(a)) - E(\alpha D(a)) \\ &= \alpha D(E(a)) - \alpha E(D(a)) \\ &= \alpha[D,E](a) \end{aligned}$$

So that this result follows from D, E linear. Then,

$$\begin{aligned} [D,E](a+b) &= (DE - ED)(a+b) \\ &= D(E(a+b)) - E(D(a+b)) \\ &= D(E(a) + E(b)) - E(D(a) + D(b)) \\ &= D(E(a)) + D(E(b)) - E(D(a)) - E(D(b)) \\ &= D(E(a)) - E(D(a)) + D(E(b)) - E(D(b)) \\ &= [D,E](a) + [D,E](b) \end{aligned}$$

So that $[D, E]$'s linearity follows from D's and E's linearity. Finally,

$$\begin{aligned} [D,E](ab) &= (DE - ED)(ab) \\ &= D(E(ab)) - E(D(ab)) \\ &= D(E(a)b + aE(b)) - E(D(a)b + aD(b)) \\ &= D(E(a)b) + D(aE(b)) - E(D(a)b) - E(aD(b)) \\ &= D(E(a))b + E(a)D(b) + D(a)E(b) + aD(E(b)) \\ &\quad -E(D(a))b - D(a)E(b) - E(a)D(b) - aE(D(b)) \\ &= (D(E(a)) - E(D(a)))b + a(D(E(b)) - E(D(b))) \\ &= [D,E](a)b + a[D,E](b). \ \square \end{aligned}$$

Since our bracket was defined using the commutator, it is well known that [,] satisifies the Jacobi identity. Therefore,

Theorem 4.6 *$\mathcal{D}$ is a Lie algebra.*

4.3 Dimension of $\mathcal{D}$

In this section we will show that $\mathcal{D}$ has dimension equal to the dimension of $\mathfrak{g}_2$, namely, dimension 14. From there we will only need to show the complexification of $\mathcal{D}$ is simple since $\mathfrak{g}_2$ is the only complex simple Lie algebra of dimension 14. To prove the dimension, we will consider the form of a typical element D in $\mathcal{D}$ and, using facts from the last section, show D must take a certain form. This form will have precisely 14 variables.

Let's begin. Choose a typical element $D \in \mathcal{D}$. Recall that D's range is a subset of the elements of trace zero. Thus, a typical element in the range of D would have form $\sum_{i=1}^{7} \lambda_i e_i$ (i.e. "λ_0" $= 0$). Because D is a linear map, D is determined by its behavior on the basis. Furthermore, D is completely determined by its effect on e_1, e_2, and e_4 for if we define

$$D(e_1) = \sum_{i=1}^{7} e_i \lambda_i,$$

$$D(e_2) = \sum_{i=1}^{7} e_i \mu_i,$$

and

$$D(e_4) = \sum_{i=1}^{7} e_i \nu_i,$$

then the derivation rules

$$\begin{aligned} D(e_3) &= D(e_1)e_2 + e_1 D(e_2) \\ D(e_5) &= D(e_1)e_4 + e_1 D(e_4) \\ D(e_6) &= D(e_4)e_2 + e_4 D(e_2) \\ D(e_7) &= D(e_3)e_4 + e_3 D(e_4) \\ &= D(e_1 e_2)e_4 + e_3 D(e_4) \\ &= [D(e_1)e_2 + e_1 D(e_2)]e_4 + e_3 D(e_4) \end{aligned}$$

yield D's behavior on the rest of the basis. Taking these arbitrary definitions for $D(e_1)$, $D(e_2)$, and $D(e_4)$ and writing out what D is on the other basis elements, we get the results of Table 3.

Note that this table has 21 variables: $\lambda_1, \ldots, \lambda_7, \mu_1, \ldots, \mu_1, \nu_1, \ldots, \nu_7$. We will prove $\mathcal{D}$ has dimension 14 by showing that only 14 of these variables are independent.

Proposition 4.7 *Suppose $D \in \mathcal{D}$ and we let*

$$D(e_1) = \sum_{i=1}^{7} \lambda_i e_i$$

Table 3: General Form of Derivations on the Octonions

	$D(e_1)$	$D(e_2)$	$D(e_3)$	$D(e_4)$	$D(e_5)$	$D(e_6)$	$D(e_7)$
1	0	0	$-\lambda_2 - \mu_1$	0	$-\lambda_4 - \nu_1$	$-\mu_4 - \nu_2$	$\lambda_6 + \mu_5 - \nu_3$
e_1	λ_1	μ_1	$-\lambda_3$	ν_1	$-\lambda_5$	$\mu_5 - \nu_3$	$-\lambda_7 - \mu_4 - \nu_2$
e_2	λ_2	μ_2	$-\mu_3$	ν_2	$\lambda_6 - \nu_3$	$-\mu_6$	$\lambda_4 - \mu_7 + \nu_1$
e_3	λ_3	μ_3	$\lambda_1 + \mu_2$	ν_3	$-\lambda_7 + \nu_2$	$-\mu_7 + \nu_1$	$\lambda_5 - \mu_6$
e_4	λ_4	μ_4	$-\lambda_6 - \mu_5$	ν_4	$-\nu_5$	$-\nu_6$	$-\lambda_2 - \mu_1 - \nu_7$
e_5	λ_5	μ_5	$\lambda_7 + \mu_4$	ν_5	$\lambda_1 + \nu_4$	$-\mu_1 + \nu_7$	$-\lambda_3 - \nu_6$
e_6	λ_6	μ_6	$\lambda_4 - \mu_7$	ν_6	$-\lambda_2 - \nu_7$	$\mu_2 + \nu_4$	$\mu_3 + \nu_5$
e_7	λ_7	μ_7	$-\lambda_5 + \mu_6$	ν_7	$\lambda_3 + \nu_6$	$-\mu_3 - \nu_5$	$\lambda_1 + \mu_2 + \nu_4$

$$D(e_2) = \sum_{i=1}^{7} \mu_i e_i$$

$$D(e_4) = \sum_{i=1}^{7} \nu_i e_i$$

fully defining D. Then,

$$\lambda_1 = \mu_2 = \nu_4 = \mu_1 + \lambda_2 = \nu_1 + \lambda_4 = \nu_2 + \mu_4 = \lambda_6 + \mu_5 - \nu_3 = 0.$$

Proof First we show $\lambda_1 = \mu_2 = \nu_4 = 0$.

Note, $D(e_1 e_1) = D(-1) = 0$. Thus,

$$e_1 D(e_1) + D(e_1) e_1 = 0$$

$$e_1 (\sum_1^7 \lambda_i e_i) + (\sum_1^7 \lambda_i e_i) e_1 = 0$$

$$-\lambda_1 + \sum_2^7 \lambda_i e_1 e_i - \lambda_1 + \sum_2^7 \lambda_i e_i e_1 = 0.$$

Since basis elements anticommute,

$$-\lambda_1 + \sum_2^7 \lambda_i e_1 e_i - \lambda_1 - \sum_2^7 \lambda_i e_1 e_i = 0$$

$$-\lambda_1 - \lambda_1 = 0$$

$$\lambda_1 = 0.$$

The proofs for $\mu_2 = 0$ and $\nu_4 = 0$ are very similar. The proof for $\mu_2 = 0$ begins with the fact that $D(e_2e_2) = 0$ and the proof for $\nu_4 = 0$ begins with the fact that $D(e_4e_4) = 0$.

Now we show $\mu_1 + \lambda_2 = \nu_1 + \lambda_4 = \nu_2 + \mu_4 = 0$. Recall that trace maps **O** to **R**. Since derivations map every real number to zero, $D(tr(x)) = 0$ for all $x \in \mathbf{O}$.

Consider,

$$D(tr(e_1e_2)) = 0$$

implies

$$D(e_1e_2 + \overline{e_1e_2}) = 0$$

Since $\overline{ab} = \overline{b}\overline{a}$ for all $a, b \in \mathbf{O}$ by our study of conjugation in relation to the Cayley-Dickson construction,

$$D(e_1e_2) + D((\overline{e_2})(\overline{e_1})) = 0$$

Then by our derivation condition,

$$D(e_1)e_2 + e_1D(e_2) + (D(\overline{e_2})\overline{e_1} + \overline{e_2}D(\overline{e_1})) = 0$$

By lemma 4.3,

$$D(e_1)e_2 + e_1D(e_2) + (\overline{D(e_2)}\overline{e_1} + \overline{e_2}\overline{D(e_1)}) = 0$$

Recall that $\overline{e_i} = -e_i$ and that $\overline{D(x)} = -D(x)$. Therefore,

$$D(e_1)e_2 + e_1D(e_2) + (D(e_2)e_1 + e_2D(e_1)) = 0$$

Substituting,

$$(\sum_{i=1}^{7} \lambda_i e_i)e_2 + e_1(\sum_{i=1}^{7} \mu_i e_i) + (\sum_{i=1}^{7} \mu_i e_i)e_1 + e_2(\sum_{i=1}^{7} \lambda_i e_i) = 0$$

Then, since $e_i^2 = -1$ for all i,

$$-\lambda_2 + \sum_{i=1, i\neq 2}^{7} \lambda_i e_i e_2 - \mu_1 + \sum_{i=1, i\neq 1}^{7} \mu_i e_1 e_i - \mu_1 + \sum_{i=1, i\neq 1}^{7} \mu_i e_i e_1 - \lambda_2 + \sum_{i=1 i\neq 2}^{7} \lambda_i e_2 e_i = 0$$

Using $e_ie_j = -e_je_i$ for all i, j,

$$-\lambda_2 + \sum_{i=1, i\neq 2}^{7} \lambda_i e_i e_2 - \mu_1 - \sum_{i=1, i\neq 1}^{7} \mu_i e_i e_2 - \mu_1 + \sum_{i=1, i\neq 1}^{7} \mu_i e_i e_1 - \lambda_2 - \sum_{i=1, i\neq 2}^{7} \lambda_i e_i e_2 = 0$$

And now we can cancel:

$$-\lambda_2 - \mu_1 - \mu_1 - \lambda_2 = 0$$

$$-2\lambda_2 - 2\mu_1 = 0$$

$$\lambda_2 + \mu_1 = 0$$

The proof for $\nu_1 + \lambda_4$ is similar enough to omit it. Suffice it to say, the proof begins with the information that $D(tr(e_1e_4)) = 0$. Similarly, we omit the proof for $\nu_2 + \mu_4 = 0$. That proof begins with the information that $D(tr(e_2e_4)) = 0$.

To show $\lambda_6 + \mu_5 - \nu_3 = 0$, we again begin with the same idea:

$$D(tr(e_3e_4)) = 0$$

$$D(e_3e_4 + \overline{e_3e_4}) = 0$$

$$D(e_3e_4) + D(\overline{e_4e_3}) = 0$$

$$D(e_3e_4) + D(e_4e_3) = 0$$

$$D(e_1e_2)e_4 + e_3D(e_4) + D(e_4)e_3 + e_4D(e_1e_2) = 0$$

$$[D(e_1)e_2 + e_1D(e_2)]e_4 + e_3D(e_4) + D(e_4)e_3 + e_4[D(e_1)e_2 + e_1D(e_2)] = 0$$

By the same type of reasoning, we arrive at,

$$\lambda_6 + \mu_5 - \nu_3 = 0. \ \square$$

Table 4: Linearly Independent Form of Derivations on the Octonions

	$D(e_1)$	$D(e_2)$	$D(e_3)$	$D(e_4)$	$D(e_5)$	$D(e_6)$	$D(e_7)$
1	0	0	0	0	0	0	0
e_1	0	$-\lambda_2$	$-\lambda_3$	$-\lambda_4$	$-\lambda_5$	$-\lambda_6$	$-\lambda_7$
e_2	λ_2	0	$-\mu_3$	$-\mu_4$	$-\mu_5$	$-\mu_6$	$-\mu_7$
e_3	λ_3	μ_3	0	$\lambda_6+\mu_5$	$-\lambda_7-\mu_4$	$-\lambda_4+\mu_7$	$\lambda_5-\mu_6$
e_4	λ_4	μ_4	$-\lambda_6-\mu_5$	0	$-\nu_5$	$-\nu_6$	$-\nu_7$
e_5	λ_5	μ_5	$\lambda_7+\mu_4$	ν_5	0	$\nu_7+\lambda_2$	$-\lambda_3-\nu_6$
e_6	λ_6	μ_6	$\lambda_4-\mu_7$	ν_6	$-\lambda_2-\nu_7$	0	$\mu_3+\nu_5$
e_7	λ_7	μ_7	$-\lambda_5+\mu_6$	ν_7	$\lambda_3+\nu_6$	$-\mu_3-\nu_5$	0

Note that with these results we have determined a great deal about what derivations on the octonions look like. Combining this new information with Table 3 gives us a $\mathcal{D}$ whose behavior is summarized in Table 4.

Observe that the top row is all zeros. This reflects the fact the $\mathcal{D}$ sends all scalars to zero. We expected this result from the first lemma we proved about derivations. Observe also there are now only 14 variables in the table. This implies that the dimension of $\mathcal{D}$ can be no more than 14. In the next theorem, we will see that satisfaction of the form in Table 4 for a map over the octonions is sufficient to prove the map is a derivation. This will imply that $\mathcal{D}$ does indeed have dimension 14.

Theorem 4.8 *Every octonion map of the form in Table 4 is a derivation.*

Proof We begin by choosing an arbitrary mapping D of the form in Table 4. The proof proceeds by exhibiting three derivations on the octonions, E, F, and G, such that

$$E + F + G = D$$

Since the $\mathcal{D}$ is an algebra, it has additive closure. This places D in $\mathcal{D}$.

Let E be a linear map satisfying

$$E(e_1) = \lambda_2 e_2 + \lambda_3 e_3,$$

$$E(e_2) = \mu_3 e_3,$$

$$E(e_3) = e_1 E(e_2) + E(e_1) e_2,$$

and

$$E(e_4) = \nu_5 e_5 + \nu_6 e_6 + \nu_7 e_7.$$

and $E(\alpha) = 0$ for all $\alpha \in \mathbf{R}$.

We show E is in $\mathcal{D}(\mathbf{O})$.

By proposition 3.4, it is sufficient to show E is a derivation on the quaternion $(1, e_1, e_2, e_3)$ for then E will extend to the octonions by the map $E(e_4) = ce_4$ where $c = \nu_6 e_1 - \nu_6 e_2 + \nu_7 e_3$.

Because E is already linear, we only need to verify that $E(ab) = aE(b) + E(a)b$ for all $a, b \in \mathbf{H}$. Furthermore, since E is linear, it will be sufficient to show $E(e_i b) = e_i E(b) + E(e_i)b$ for i=1, 2, 3 and for arbitrary $b \in \mathbf{H}$.

Let $b = b_0 + b_1 e_1 + b_2 e_2 + b_3 e_3$ an arbitrary element in $\mathbf{H}$. Starting from the right hand side,

$$\begin{aligned}
E(e_1 b) &= E(b_0 e_1 + b_1 e_1 e_1 + b_2 e_1 e_2 + b_3 e_1 e_3) \\
&= E(b_0 e_1) + E(b_1 e_1 e_1) + E(b_2 e_1 e_2) + E(b_3 e_1 e_3) \\
&= E(b_0 e_1) + E(-b_1) + E(b_2 e_3) + E(-b_3 e_2) \\
&= b_0 E(e_1) - b_1 E(1) + b_2 E(e_3) - b_3 E(e_2) \\
&= b_0(\lambda_2 e_2 + \lambda_3 e_3) - b_1 0 + b_2(e_1 E(e_2) + E(e_1)e_2) - b_3(\mu_3 e_3) \\
&= b_0(\lambda_2 e_2 + \lambda_3 e_3) + b_2(e_1(\mu_3 e_3) + (\lambda_2 e_2 + \lambda_3 e_3)e_2) - b_3(\mu_3 e_3) \\
&= b_0 \lambda_2 e_2 + b_0 \lambda_3 e_3 - b_2 \mu_3 e_2 - b_2 \lambda_2 - b_2 \lambda_3 e_1 - b_3 \mu_3 e_3
\end{aligned}$$

Now considering the left hand side,

$$\begin{aligned}
E(e_1)b + e_1 E(b) &= (\lambda_2 e_2 + \lambda_3 e_3)(b_0 + b_1 e_1 + b_2 e_2 + b_3 e_3) \\
&\quad + e_1(b_0 E(1) + b_1 E(e_1) + b_2 E(e_2) + b_3 E(e_3)) \\
&= b_0 \lambda_2 e_2 + b_0 \lambda_3 e_3 - b_1 \lambda_2 e_3 + b_1 \lambda_3 e_2 - b_2 \lambda_2 - b_2 \lambda_3 e_3 \\
&\quad + b_3 \lambda_2 e_1 - b_3 \lambda_3 + b_0 E(1) e_1 + b_1 e_1(\lambda_2 e_2 + \lambda_3 e_3) \\
&\quad + b_2 e_1(\mu_3 e_3) + b_3 e_1(e_1(\mu_3 e_3) + (\lambda_2 e_2 + \lambda_3 e_3)e2) \\
&= b_0 \lambda_2 e_2 + b_0 \lambda_3 e_3 - b_2 \mu_3 e_2 - b_2 \lambda_2 - b_2 \lambda_3 e_1 - b_3 \mu_3 e_3 \\
&\quad + (b_1 \lambda_2 e_3 - b_1 \lambda_2 e_3) + (b_1 \lambda_3 e_2 - b_1 \lambda_3 e_2) \\
&\quad + (b_3 \lambda_2 e_1 - b_3 \lambda_2 e_1) + (b_3 \lambda_3 - b_3 \lambda_3) + b_0 E(1) e_1 \\
&= b_0 \lambda_2 e_2 + b_0 \lambda_3 e_3 - b_2 \mu_3 e_2 - b_2 \lambda_2 - b_2 \lambda_3 e_1 - b_3 \mu_3 e_3 \\
&= E(e_1 b)
\end{aligned}$$

Next, we define an automorphism Φ of the quaternions such that $e_1 \mapsto e_2$, $e_2 \mapsto e_3$, and $e_3 \mapsto e_1$. Applying Φ to the above calculation yields $E(e_2 b) = e_2 E(b) + E(e_2)b$. Applying Φ a second time yields $E(e_3 b) = e_3 E(b) +$

$E(e_3)b$. Therefore E is a derivation over the quaternion $(1, e_1, e_2, e_3)$ and thus a derivation over the octonions by the extension $E(e_4) = (\nu_1 e_1 - \nu_6 e_2 + \nu_7 e_3)e_4$ (using Proposition 3.4).

E is the first of our three derivations whose sum will equal D. We now create the second, F.

Let F be a linear map satisfying

$$F(e_1) = \lambda_6 e_6 + \lambda_7 e_7,$$

$$F(e_2) = 0,$$

$$F(e_4) = 0,$$

$$F(e_6) = e_4 F(e_2) + F(e_4)e_2,$$

$$F(e_7) = e_3 F(e_4) + F(e_3)e_4.$$

and $F(\alpha) = 0$ for all $\alpha \in \mathbf{R}$.

We need to show F is in $\mathcal{D}(\mathbf{O})$.

The proof for F is quite similar to the proof for E. So, we do not show the calculation here. To prove F is a derivation over the octonions, we show it is a derivation over the quaternions $(1, e_1, e_6, e_7)$ and then extend it to $\mathbf{O}$ using Proposition 3.4 again by $F(e_4) = ce_4$ where $c = 0$. The proof proceeds as before by showing $F(ab) = aF(b) + F(a)b$ for all $a, b \in \mathbf{H}$ where this time the quaternions are $(1, e_1, e_6, e_7)$.

We now have E and F derivations over the octonions. Only one more derivation is needed to complete our proof.

Let G be a linear map satisfying

$$G(e_1) = \lambda_4 e_4 + \lambda_5 e_5,$$

$$G(e_2) = \mu_6 e_6 + \mu_7 e_7,$$

$$G(e_4) = 0,$$

$$G(e_6) = e_4 F(e_2) + F(e_4)e_2,$$

$$G(e_7) = e_3 F(e_4) + F(e_3)e_4.$$

and $G(\alpha) = 0$ for all $\alpha \in \mathbf{R}$.

We need to show G is in $\mathcal{D}(\mathbf{O})$.

Again, the proof for G is so similar to the proof for E that omit it. The initial construction of G is over the quaternions $(1, e_1, e_4, e_5)$. The extension to $\mathcal{D}(\mathbf{O})$ goes by $F(e_2) = ce_2$ where $c = \mu_6 e_4 - \mu_7 e_5$.

Now, E, F, G are elements of the algebra of derivations over the octonions. Thus their sum is also a derivations over the octonions.

Notice carefully that that $D = E+F+G$ because D agrees with $E+F+G$ on e_1, e_2, and e_4. Therefore D is a derivation. □

In summary,

Theorem 4.9 *A linear mapping on the octonions D has the form in Table 4 if and only if D is a derivation over the octonions.*

Notice that Table 4 leaves precisely 14 independent variables for the way a derivation can be defined. Namely, the independent variables are $\lambda_2 \dots \lambda_7$, $\mu_3 \dots \mu_7$, and $\nu_5 \dots \nu_7$. Therefore,

Theorem 4.10 *$\mathcal{D}$, the algebra of derivations over the octonions, has dimension 14.*

4.4 $\mathcal{D}$ is Semisimple

In this section we will show $\mathcal{D}$ is semisimple. Recall, a Lie algebra is semisimple when it is a direct sum of simple Lie algebras. Our approach will be to use a result from Humphreys [Hu]. On page 22 we find that a Lie algebra is semisimple if and only if it has no nonzero abelian ideals. Thus, our approach will be to show $\mathcal{D}$ has no nonzero abelian ideals.

Our proof of $\mathcal{D}$ having no nonzero abelian ideals will proceed as follows. First we will show that if such an ideal I exists, then any element $D \in I$ will have the condition $D(e_1) = 0$. We will then further use this information to show if I exists then $D(e_i) = 0$ for all i and for all $D \in I$. This in turn implies that I is zero, a contradiction of our selection of I.

But first, let's define the basis for $\mathcal{D}$. Most of our work is complete by Table 4. We just need to construct the elements.

Let D_1 be the derivation defined by Table 4 in which $\lambda_2 = 1$ and all other λ_i, μ_i, and ν_i are zero. Similarly, let D_2 be the derivation with $\lambda_3 = 1$ and all other variables 0. Following this pattern, we define D_i to be the derivation of the form in Table 4 where the i^{th} variable is one and all others are zero where the "i^{th} variable" means the i^{th} member of $[\lambda_2, \dots, \lambda_7, \mu_2, \dots, \mu_7, \nu_4, \dots, \mu_7]$.

Clearly, $\{D_1, \dots, D_{14}\}$ is a basis for $\mathcal{D}$.

Define D' to mean the restriction of $D \in \mathcal{D}$ to $\mathbf{R} + \mathbf{R}e_1$. Then define $\mathcal{D}'$ to be the set of all such D'.

Now, it's quite clear that $\{D'_1 \ldots, D'_6\}$ is a basis for $\mathcal{D}'$.

Working out the bracket relationships among these basis elements yields the results in Table 5.

Table 5: Brackets of Derivations on the Octonions

$[Y, X]$	D'_1	D'_2	D'_3	D'_4	D'_5	D'_6
D'_1	0	0	0	D'_5	D'_4	0
D'_2	0	0	D'_5	$2D'_6$	D'_3	$-2D'_4$
D'_3	0	D'_5	0	0	$-2D'_2$	D'_1
D'_4	$-D'_5$	$-2D'_6$	0	0	0	D'_2
D'_5	$-D'_4$	$-D'_3$	$2D'_2$	0	0	0
D'_6	0	$2D'_4$	$-D'_1$	$-D'_2$	0	0

Suppose $I' \subset \mathcal{D}'$ is an abelian ideal. We will first show an element in I of a special form has to be zero. Then we will handle the general case.

Lemma 4.11 *If there is a $D' \in I'$ such that $D' = aD'_4 + bD'_5$ for some $a, b \in \mathbf{R}$, then D' is zero.*

Proof Since $D' \in I'$, $[D'_3, D'] \in I'$. Note that we can use Table 4 to define a D and a D_3 such that $\mu_2, \ldots, \mu_7, \nu_4, \ldots, \nu_7$ are zero. Then,

$$\begin{aligned}
[D_3, D](e_1) &= D_3 D(e_1) - D D_3(e_1) \\
&= D_3(ae_5 + be_6) - D(e_4) \\
&= aD_3(e_5) + bD(e_6) - D(e_4) \\
&= 0 - be_3 - be_3 \\
&= -2be_3.
\end{aligned}$$

Here we are using Table 4 to determine D_3 and D. Therefore, $[D'_3, D'] = -2be_3 = -2bD'_2 \in I'$. I abelian implies that $-2bD'_2$ commutes with D'. Thus,

$$\begin{aligned}
0 &= [-2bD'_2, D'] \\
&= [-2bD'_2, aD'_4 + bD'_5] \\
&= [-2bD'_2, aD'_4] + [-2bD'_2, bD'_5] \\
&= -4abD'_6 + -2b^2 D'_3
\end{aligned}$$

since $[D'_2, D'_4] = 2D'_6$ and $[D'_2, D'_5] = D'_3$ by Table 5. Therefore, $b = 0$.

Also, if we point out that $[D', aD'_4] = aD'_5 \in I'$, then similar reasoning will yield the result that $a = 0$.

In particular, $a = b = 0$ implies $D' = 0$. $\Box$

And now the general case of our first step.

Lemma 4.12 *If $D \in I$ where I is a nonzero abelian ideal in $\mathcal{D}$, then $D(e_1) = 0$.*

Proof Suppose we have an arbitrary $D' = a_1D'_1 + a_2D'_2 + a_3D'_3 + a_4D'_4 + a_5D'_5 + a_6D'_6 \in I'$ where $I' = \mathcal{D}' \cap I$. It will be sufficient to show $D' = 0$.

Then,

$$\begin{aligned}
[D'_1, D'] &= [D'_1, a_1D'_1 + a_2D'_2 + a_3D'_3 + a_4D'_4 + a_5D'_5 + a_6D'_6] \\
&= [D'_1, a_1D'_1] + [D'_1, a_2D'_2] + [D'_1, a_3D'_3] + [D'_1, a_4D'_4] \\
&\quad +[D'_1, a_5D'_5] + [D'_1, a_6D'_6] \\
&= a_4D'_5 + a_5D'_4 \\
&\quad \in I'
\end{aligned}$$

using Table 5. The last lemma implies $a_4D'_5 + a_5D_4 = 0$, so $a_4 = a_5 = 0$.

Similarly,

$$\begin{aligned}
[D'_2, D'] &= [D'_2, a_1D'_1] + [D'_2, a_2D'_2] + [D'_2, a_3D'_3] + [D'_2, a_4D'_4] \\
&\quad +[D'_2, a_5D'_5] + [D'_2, a_6D'_6] \\
&= a_3D'_5 + 2a_4D'_6 + a_5D'_3 - 2a_6D'_4 \\
&= a_3D'_5 + 0 + 0 - 2a_6D'_4 \\
&= \in I'
\end{aligned}$$

using Table 5. Again, the lemma implies $a_3D'_5 - 2a_6D'_4 = 0$, so $a_3 = a_6 = 0$.

Also,

$$\begin{aligned}
[D'_6, D'] &= 2a_2D'_4 - a_3D_1 - a_4D'_2 \\
&= 2a_2D'_4 + 0 + 0 + 0D'_5 \\
&= \in I'
\end{aligned}$$

using Table 5. Again, the lemma implies $2a_2D'_4 + 0D'_5 = 0$, so $a_2 = 0$.

Finally,

$$\begin{aligned}
[D'_5, D'] &= -a_1D'_4 - a_2D'_3 + 2a_3D'_2 \\
&= -a_1D'_4 + 0 + 0 + 0D'_5 \\
&= \in I'
\end{aligned}$$

using Table 5. Again, the lemma implies $-a_1D'_4 + 0D'_5 = 0$, so $a_1 = 0$.

Therefore, $D' = 0$. $\Box$

We are now ready to prove the following result:

Proposition 4.13 *$\mathcal{D}$ has no nonzero abelian Lie ideals.*

Proof Suppose not. That is, suppose there exists a nonzero abelian Lie ideal I. Let $D \in I$. Then we can express D as $D = \lambda_1 D_1 + \ldots + \nu_7 D_{14}$. By our previous work, $D(e_1) = 0$, so $\lambda_1, \ldots, \lambda_6$ must be zero for D. Thus $D = \mu_3 D_y + \ldots + \nu_7 D_{14}$. We now show $\mu_3 = \mu_4 = \ldots = \nu_7 = 0$.

Take note that $[D_1, D] \in I$ since I is an ideal. Therefore, $[D_1, D](e_1) = 0$. Well,

$$\begin{aligned}
[D_1, D](e_1) &= (D_1 D - D D_1)(e_1) \\
&= D_1(D(e_1)) - D(D_1(e_1)) \\
&= D_1(0) - D(e_2) \\
&= 0 - D(e_2) \\
&= -D(e_2) \\
&= -(\mu_3 e_3 + \mu_4 e_4 + \mu_5 e_5 + \mu_6 e_6 + \mu_7 e_7) \\
&= 0
\end{aligned}$$

Therefore, $\mu_3 = \mu_4 = \ldots = \mu_7 = 0$.

Next note $[D_3, D] \in I$. Thus, $[D_3, D](e_1) = 0$. Hence,

$$\begin{aligned}
[D_3, D](e_1) &= (D_3 D - D D_3)(e_1) \\
&= D_3(D(e_1)) - D(D_3(e_1)) \\
&= D_3(0) - D(e_4) \\
&= 0 - D(e_4) \\
&= -D(e_4) \\
&= -(\nu_5 e_5 + \nu_6 e_6 + \nu_7 e_7) \\
&= 0
\end{aligned}$$

Therefore, $\nu_5 = \nu_6 = \nu_7 = 0$. Therefore $D = 0$.

Since D was an arbitrary element in I, $I = (0)$. However, this contradicts our choice of I. Therefore, no such I exists. □

This conclusion, together with our discussion at the beginning of this subsection proves the following:

Proposition 4.14 *$\mathcal{D}$ is semisimple.*

Now, our next argument uses $\mathcal{D}$ semisimple to prove $\mathcal{D}$ simple. However, some parts of our argument use the more standard notion of having a Lie algebra over the complexes as opposed to the particular case we have which is a Lie algebra over the reals. Thus, we will tensor with $\mathbf{C}$ to complexify $\mathcal{D}$.

From here on we view $\mathbf{C}$ as a field. We do not view $\mathbf{C}$ as being isomorphic to any subalgebra of $\mathbf{O}$ as we did in the Cayley-Dickson construction.

Let $\mathbf{O_C}$ denote $\mathbf{O} \otimes_{\mathbf{R}} \mathbf{C}$. Then $\mathbf{O_C}$ is viewed as the octonions having base field complexes instead of reals. That is,

$$\mathbf{O} = \mathbf{R} + \mathbf{R}e_1 \ldots + \mathbf{R}e_7$$

and

$$\mathbf{O_C} = \mathbf{C} + \mathbf{C}e_1 \ldots + \mathbf{C}e_7.$$

From this viewpoint, $dim_{\mathbf{C}}(\mathbf{O_C}) = dim_{\mathbf{R}}(\mathbf{O}) = 8$.

Furthermore, let $\mathcal{D}_{\mathbf{C}}$ denote $\mathcal{D} \otimes_{\mathbf{R}} \mathbf{C}$. Then

$$\mathcal{D} = \mathbf{R}D_1 + \ldots + \mathbf{R}D_{14}$$

whereas

$$\mathcal{D}_{\mathbf{C}} = \mathbf{C}D_1 \ldots + \mathbf{C}D_{14}.$$

Therefore, $dim_{\mathbf{C}}(\mathcal{D}_{\mathbf{C}}) = dim_{\mathbf{R}}(\mathcal{D}) = 14$.

Then, using a result from one of our sources, [K, p. 348], $\mathcal{D}$ semisimple implies,

Corollary 4.15 *$\mathcal{D}_{\mathbf{C}}$ is semisimple.*

4.5 Simplicity of $\mathcal{D}$

In this section, we show that $\mathcal{D}$ is simple. To do this, it will be sufficient to show $\mathcal{D}_{\mathbf{C}}$ is simple. We've already established that $\mathcal{D}_{\mathbf{C}}$ is semisimple. This means $\mathcal{D}_{\mathbf{C}}$ is the direct sum of simple Lie algebras. That is, $\mathcal{D}_{\mathbf{C}} = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \ldots \oplus \mathcal{D}_r$ for some $r \in \mathbf{N}$ and each $\mathcal{D}_i$ simple. Thus, if $D \in \mathcal{D}_{\mathbf{C}}$ then $D = \sum D_i$ with $D_i \in \mathcal{D}_i$. We will prove $\mathcal{D}_{\mathbf{C}}$ is simple by showing $r = 1$. However, before we do this, we need to prove a few propositions about $\mathcal{D}_{\mathbf{C}}$.

Proposition 4.16 *Two derivations are commutative if and only if their components are commutative.*

Proof Let $E \in \mathcal{D}_{\mathbf{C}}$ with $E = \sum E_i$ such that $E_i \in \mathcal{D}_i$ for each i and let $D \in \mathcal{D}_{\mathbf{C}}$ with $D = \sum D_i$ such that $D_i \in \mathcal{D}_i$ for each i. Note that $[E_i, D_j] = 0$ for all $i \neq j$ by the definiton of the direct sum of Lie algebras. Then, D commutes with E if and only if

$$[E, D] = 0$$

if and only if

$$[\sum_1^r E_i, \sum_1^r D_i] = 0$$

if and only if

$$[E_i, D_i] = 0$$

for each i since $[E_i, D_j] = 0$ necessarily for all $i \neq j$ by definition of lie algebra direct sum. □

For some $D \in \mathcal{D}_{\mathbf{C}}$, we define the centralizer of D to be

$$Cent(D) = \{E \in \mathcal{D}_{\mathbf{C}} \,|\, [E, D] = 0\}.$$

That is, $Cent(D)$ is the set of derivations that commute with D.

Proposition 4.17 *Fix a nonzero $D \in \mathcal{D}_{\mathbf{C}}$. Then $Cent(D)$ is a nontrivial Lie subalgebra of $\mathcal{D}_{\mathbf{C}}$. Furthermore, $Cent(D)$ is the direct sum of r subalgebras:*

$$Cent(D) = \bigoplus_{i=1}^r (Cent(D) \cap \mathcal{D}_i)$$

such that

$$Cent(D) \cap \mathcal{D}_i \neq \{0\}$$

for each i.

Proof To see that $Cent(D)$ is a Lie subalgebra, we need to show it is a vector subspace and it has closure under the bracket. Choose arbitrary $E, F \in Cent(D)$ and $\alpha, \beta \in \mathbf{C}$. Then $[E, D] = 0$ and $[F, D] = 0$. Well,

$$\begin{aligned} [\alpha E + \beta F, D] &= [\alpha F, D] + [\beta E, D] \\ &= \alpha[F, D] + \beta[E, D] \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Therefore, $\alpha E + \beta F \in Cent(D)$. Now consider $[[F, E], D]$. By the Jacobi identity,

$$[[F, E], D] = [F, [E, D]] + [[F, D], E].$$

Since $[E, D] = [F, D] = 0$, it now follows that

$$[F, [E, D]\,] = [F, 0] = 0$$

and

$$[\,[F, D], E] = [0, E] = 0.$$

Hence

$$[\,[F, E], D] = 0$$

and $[F, E] \in Cent(D)$.

$Cent(D)$ is nontrivial since $D \in Cent(D)$ by [D,D]=0.

To see that $Cent(D)$ is a direct sum of r subalgebras, consider an element $E \in Cent(D)$. We can write $E = \sum_{i=1}^{r} E_i$ with $E_i \in \mathcal{D}_i$. By the previous proposition, $E \in Cent(D)$ implies $[E_i, D_j] = 0$ for each i, j. Hence $E_i \in Cent(D)$ for all i. It follows that $Cent(D) = \bigoplus_{i=1}^{r} Cent(D) \cap \mathcal{D}_i$.

Finally, we will show $Cent(D) \cap \mathcal{D}_i$ is nonzero for each i. Fix an arbitrary i. We know $D = D_1 + \ldots + D_r$. If $D_i \neq 0$, we have $D_i \in Cent(D) \cap \mathcal{D}_i$ by the last proposition. If, on the other hand, $D_i = 0$, then the whole subalgebra $\mathcal{D}_i \subseteq Cent(D) \cap \mathcal{D}_i$ for $[0, F_i] = 0$ for all $F_i \in \mathcal{D}_i$. $\square$

These results are the equipment we need to execute our proof of $\mathcal{D}_{\mathbf{C}}$ simple. We now proceed by fixing a derivation D and looking at $Cent(D)$. By examining the restriction of commutativity with D, we will find that $Cent(D)$ is the direct sum of at most two nonzero ideals. By our work above, this will imply $r \leq 2$. To show $r \neq 2$, we will demonstrate that no two simple Lie algebras exist whose dimensions adds up to 14 - the dimension of $\mathcal{D}_{\mathbf{C}}$. Thus r must equal 1 making $\mathcal{D}_{\mathbf{C}}$ simple.

Note that Theorem 4.9 (the result substantiating Table 4) still stands in $\mathcal{D}_{\mathbf{C}}$ since our calculations were independent of base field as long as the field had characteristic zero. Now, fix a certain derivation $D \in \mathcal{D}_{\mathbf{C}}$ such that $D(e_1) = -e_2$, $D(e_2) = e_1$, and $D(e_4) = 0$. Then the rest of D is defined as

$$(D(e_1), \ldots, D(e_7)) = (-e_2, e_1, 0, 0, e_6, -e_5, 0)$$

since derivations in $\mathcal{D}_{\mathbf{C}}$ are still determined by the form given in Table 4.

Lemma 4.18 *Suppose* $E \in Cent(D)$ *and* E *uses the notation of Table 3 (specifically,* $E(e_1) = \sum_{i=1}^{7} \lambda_i e_i$*,* $E(e_2) = \sum_{i=1}^{7} \mu_i e_i$*, and* $E(e_4) = \sum_{i=1}^{7} \nu_i e_i$*). Then,*

$$E(e_1) = e_2\lambda_2 + e_5\lambda_5 + e_6\lambda_6$$

$$E(e_2) = e_1\mu_1 + e_5\mu_5 + e_6\mu_6$$

$$E(e_4) = e_3\nu_3 + e_7\nu_7$$

such that $-\mu_1 - \lambda_2 = -\mu_5 + \lambda_6 = \mu_6 + \lambda_5 = -\nu_3 + 2\lambda_6 = 0$.

Proof Assume $E \in Cent(D)$. Hence $[E, D] = 0$. Now,

$$[E, D](e_1) = 0$$

which is equivalent to

$$E(D(e_1)) - D(E(e_1)) = 0$$

Since $D(e_1) = -e_2$, we have

$$E(-e_2) = D(\sum_1^7 \lambda_i e_i)$$

Hence,

$$-\sum_1^7 \mu_i e_i = \sum_1^7 \lambda_i D(e_i)$$

which implies,

$$-\mu_1 e_1 - \mu_2 e_2 - \mu_3 e_3 - \mu_4 e_4 - \mu_5 e_5 - \mu_6 e_6 - \mu_7 e_7 = -\lambda_1 e_2 + \lambda_2 e_1 + \lambda_5 e_6 - \lambda_6 e_5.$$

Equating basis coefficients yields $-\mu_1 - \lambda_2 = 0$, $-\mu_5 + \lambda_6 = 0$, and $\mu_6 + \lambda_5 = 0$ which are some of the conclusions in our lemma. We also know that $\lambda_6 + \mu_5 - \nu_3 = 0$ since E is a derivation. Since we just found $\mu_5 = \lambda_6$, substituting yields $2\lambda_6 - \nu_3 = 0$.

The rest of the rules for E's results on e_1, e_2, and e_4 follow similarly from the fact that E is a derivation (thus having the form in our table) and the fact that $[D, E](e_2) = [D, E](e_4) = 0$.

Now, writing out our results for E explicitly,

$$
\begin{array}{rcl}
E(e_1) & = & \lambda_2 e_2 + \lambda_5 e_5 + \lambda_6 e_6 \\
E(e_2) & = & \mu_1 e_1 + \mu_5 e_5 + \mu_6 e_6 \\
E(e_3) & = & (-\lambda_6 - \mu_5)e_4 + (-\lambda_5 + \mu_6)e_7 \\
E(e_4) & = & \nu_3 e_3 + \nu_7 e_7 \\
E(e_5) & = & -\lambda_5 e_1 + (\lambda_6 - \nu_3)e_2 + (-\lambda_2 - \nu_7)e_6 \\
E(e_6) & = & (\mu_5 - \nu_3)e_1 - \mu_6 e_2 + (-\mu_1 + \nu_7)e_5 \\
E(e_7) & = & (\lambda_5 - \mu_6)e_3 + (-\lambda_2 - \mu_1 - \nu_7)e_4. \ \square
\end{array}
$$

Corollary 4.19 *$Cent(D)$ has the basis D, E_1, E_2, E_3, where:*
$(E_1(e_1), E_1(e_2), \ldots, E_1(e_7)) = (e_5, -e_6, -2e_7, 0, -e_1, e_2, 2e_3)$.
$(E_2(e_1), E_2(e_2), \ldots, E_2(e_7)) = (e_6, e_5, -2e_4, 2e_3, -e_2, -e_1, 0)$.
$(E_3(e_1), E_3(e_2), \ldots, E_3(e_7)) = (-e_2, e_1, 0, 2e_7, -e_6, e_5, -2e_4)$.

Proof Since the E of our last lemma in an arbitrary element in $Cent(D)$, we can read off a basis for $Cent(D)$ by separating E into it's components. Taking the four relationships from the last lemma and noting $\mu_1 = -\lambda_2$, $\mu_5 = \lambda_6$, $\mu_6 = -\lambda_5$, and $\nu_3 = 2\lambda_6$ by Table 4 yields the following further restrictions on E,

$$
\begin{array}{rcl}
E(e_1) & = & \lambda_2 e_2 + \lambda_5 e_5 + \lambda_6 e_6 \\
E(e_2) & = & -\lambda_2 e_1 + \lambda_6 e_5 - \lambda_5 e_6 \\
E(e_3) & = & -2\lambda_6 e_4 - 2\lambda_5 e_7 \\
E(e_4) & = & 2\lambda_6 e_3 + \nu_7 e_7 \\
E(e_5) & = & -\lambda_5 e_1 - \lambda_6 e_2 + (-\lambda_2 - \nu_7)e_6 \\
E(e_6) & = & -\lambda_6 e_1 + \lambda_5 e_2 + (\lambda_1 + \nu_7)e_5 \\
E(e_7) & = & 2\lambda_5 e_3 - \nu_7 e_4
\end{array}
$$

Notice that E is then defined by only four variables - λ_2, λ_5, λ_6, and ν_7. Notice also that D is simply equal to an E where $\lambda_2 = 1$ and the other variables are zero. To split E into the rest of its pieces, we let E_1 be the derivation with $\lambda_5 = 1$ and all other variables 0. We let E_2 be the derivation with $\lambda_6 = 1$ and all other variables 0. We let E_3 be the deivation wth $\lambda_2 = -1$, $\nu_7 = 2$, and all other variables 0.

The rules for each basis elements behavior on $\{1, e_1, \ldots, e_7\}$ is then easily derived from Table 4. $\square$

Lemma 4.20 *The basis elements E_1, E_2, and E_3 are in the same Lie ideal of $Cent(D)$.*

Proof We prove $[E_1, E_2] = -2E_3$, $[E_2, E_3] = -2E_1$, and $[E_3, E_1] = -2E_2$ for then $E_3 \in (E_2)$, $E_1 \in (E_3)$, and $E_2 \in (E_1)$. Therefore $(E_3) \subseteq (E_2) \subseteq (E_1) \subseteq (E_3) = (E_1, E_2, E_3)$. We proceed by examining the E_i's behavior on the basis elements e_1, e_2, and e_4 since rules for all else will follow consistently.

For e_1:

$$\begin{aligned}
[E_1, E_2](e_1) &= E_1(E_2(e_1)) - E_2(E_1(e_1)) \\
&= E_1(e_6) - E_2(e_5) \\
&= e_2 - (-e_2) \\
&= 2e_2 \\
&= -2(-e_2) \\
&= -2E_3(e_1)
\end{aligned}$$

For e_2:

$$\begin{aligned}
[E_1, E_2](e_2) &= E_1(E_2(e_2)) - E_2(E_1(e_2)) \\
&= E_1(e_5) - E_2(-e_6) \\
&= -e_1 + (-e_1) \\
&= -2e_1 \\
&= -2E_3(e_2)
\end{aligned}$$

For e_4:

$$\begin{aligned}
[E_1, E_2](e_4) &= E_1(E_2(e_4)) - E_2(E_1(e_4)) \\
&= E_1(2e_3) - E_2(0) \\
&= 2(-2e_7) \\
&= -2(2e_7) \\
&= -2E_3(e_4)
\end{aligned}$$

The calculations to show $[E_2, E_3] = -2E_1$, and $[E_3, E_1] = -2E_2$ are the same.

In fact, it can be shown that E_1, E_2, E_3 span a copy of *sl2*, a simple Lie algebra.

The immediate consequence of this is that E_1, E_2, and E_3 must all lie in the same component $Cent(D) \cap \mathcal{D}_i$ of $Cent(D)$. $\square$

Since E_1, E_2, E_3, and D form a basis for $Cent(D)$, this implies that either $(D) = Cent(D) \cap \mathcal{D}_1$, and $(E_1, E_2, E_3) = Cent(D) \cap \mathcal{D}_2$ up to possible reordering of the $\mathcal{D}_i$, or $Cent(D) = Cent(D) \cap \mathcal{D}_1$. Recalling that $Cent(D) \cap \mathcal{D}_i \neq \{0\}$ for each i, we see that this implies $r \leq 2$.

Now we need only show $r \neq 2$.

To summarize our results so far, we have $\mathcal{D}_{\mathbf{C}}$ a direct sum of either one or two simple Lie algebras. Furthermore, the dimension of $\mathcal{D}$ is 14 so the dimension of $\mathcal{D}_{\mathbf{C}}$ is 14. So either $\mathcal{D}_{\mathbf{C}}$ is a simple Lie algebra of dimension 14 or $\mathcal{D}_{\mathbf{C}}$ is a direct sum of two simple Lie algebras whose dimensions add up to 14. We will show the latter is not a possibility.

Recall from our introduction to this proof that there are 4 infinite families of classical simple Lie algebras: A_l, B_l, C_l, and D_l considered over base field $\mathbf{C}$. Plus there are 5 exceptional classes: E_6, E_7, E_8, F_4, and G_2 considered over base field $\mathbf{C}$. Table 2 summarizes the rules for finding the possible dimensions of each of these [K, p. 508-517].

Table 6 lists the only simple complex Lie algebras having dimension less than or equal to 14. Note carefully, that no two algebras in Table 6 have dimensions adding up to 14.

Table 6: Simple Lie Algebras of Dimension Less Than or Equal to Fourteen

Simple Lie Algebra	Dimension
A_1	3
A_2	8
B_2	10
G_2	14

Therefore $r \neq 2$. This proves,

Theorem 4.21 *$\mathcal{D}_{\mathbf{C}}$ is a simple Lie algebra. Futhermore, $\mathcal{D}_{\mathbf{C}}$ is isomorphic to the exceptional Lie algebra $\mathfrak{g}_2$.*

Using a result from Knapp, we find that $\mathcal{D}_{\mathbf{C}}$ simple implies the real Lie algebra $\mathcal{D}$ is also simple [K, p. 348]. Therefore,

Theorem 4.22 *$\mathcal{D}$ is a real Lie algebra of type G_2.*

A References

References

[B] J. Baez, The Octonians, *Bull. Amer. Math. Soc.*, **145**, Vol. 39, No. 2 (2001), 145-205.

[C] J. H. Conway, "On Quaternions and Octonians: Their Geometry, Arithmetic, and Symmetry," A K Peters Ltd, Massachusetts, 2003.

[F] J.R. Faulkner and J.C. Ferrar, Exceptional Lie Algebras and Related Algebraic and Geometric Structures, *Bull. London Math. Soc.* 9 (1977), No. 1, 1-35.

[Ha] W.R. Hamilton, Letter to Rev. A.H. Hamilton, 15 August 1865, "History of Mathematics," 31 March 2004, <http://www.maths.tcd.ie/pub/HistMath/People/Hamilton /Letters/BroomeBridge.html>.

[Hu] J.E. Humphreys, "Introduction to Lie Algebras and Representation Theory," Springer Verlag, 3rd printing, New York, Heidelberg, and Berlin, 1997.

[J1] N. Jacobson, Cayley Numbers and Normal Simple Lie Algebras of Type G, *Duke Math. Jour.*, **775**, Vol. 5 (1939), 775-783.

[J2] N. Jacobson, "Lie Algebras," Dover Publications, New York, 1962.

[K] A.W. Knapp, "Lie Groups Beyond an Introduction," Birkh*ä*user, Boston, Basel, and Berlin, 1996.

[S] R.D. Schafer, "An Introduction to Nonassociative Algebras," Academic Press, New York and London, 1966.

B Vita

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The author was born on April 16, 1981 in Norfolk, Virginia to parents Peter and Margaret McLewin. After completing all twelve years of grade school as a homeschooler, she came to Virginia Tech to pursue a degree in Mathematics.

Her undergraduate achievements include a semester abroad in London, two appointments to co-president positions at the local chapter of the Association for Women in Computing, and several scholarships. Kelly's industrial accomplishments include the attainment of four excellent internships in the field of computer science and the inclusion of her code in premier product lines. Upon completion of her undergraduate studies, she received a Commonwealth Scholar diploma, *cum laude*. She was accepted into the five year Bachelors/Masters program for the Department of Mathematics in fall of 2002 and enjoys a sense of satisfaction as she now nears the end of this labor.

In addition to having a passion for logical reasoning, Kelly loves dance, theatre, travel, and sushi. She is also an accomplished martial artist.

In June she plans to move to Baltimore to work for Northrop Grumman as a systems engineer. She looks forward to marrying her dear fiancé, Daniel Stevens, sometime in the not-too-distant future.