Optimal Dynamic Pricing for Two Perishable and Substitutable Products

Feng Li

Grado Department of Industrial and Systems Engineering

Virginia Polytechnic Institute and State University

Thesis Submitted to the Faculty of

Virginia Polytechnic Institute and State University

in fulfillment of the requirements for the degree of

Master of Science

In

Industrial and Systems Engineering

Kyle Y. Lin, Chair

Ebru K. Bish

Barbara M. P. Fraticelli

June 2003

Blacksburg, Virginia

Keywords: Revenue management, Dynamic pricing

Product variety, Quality differentiated.

Optimal Dynamic Pricing
for Two Perishable and Substitutable Products

Feng Li

ABSTRACT

This thesis presents a dynamic pricing model where a seller offers two types of a generic product to a random number of customers. Customers show up sequentially. When a customer arrives, he will —depending on the prices—either purchase one unit of type 1 product or one unit of type 2 product, or will leave empty-handed. The sale ends either when the entire stock is sold out, or when the customers are exhausted. The seller's task is to post the optimal prices for the two product types to each customer to maximize the expected total revenue. We use dynamic programming to formulate this problem, and derive the optimal policy for special cases. For general cases, we develop an algorithm to approximate the optimal policy and use numerical examples to demonstrate the efficiency of the algorithm.

Finally, we apply the results to a continuous-time model where customers arrive according to a Poisson process. We develop a heuristic policy and use numerical examples to show the heuristic policy is very effective.

Acknowledgments

First, I want to thank my advisor, Dr. Lin. As an advisor, he gave me much guidance, counsel, patience and support. I am especially grateful to him for his always being there whenever I need his help.

Second, I want to thank Dr. Bish and Dr. Fraticelli for their help and valuable advices to my thesis.

I am also thankful to all my friends at Virginia Tech and Association of Chinese Social Society (ACSS). They kept me in company when I am thousands of miles away from home.

Finally, I thank my mother and father for their great love to me, always.

Contents

List of Tables

List of Figures

Chapter 1

Introduction

Dynamic pricing, in general, is to dynamically adjust the product price, as the fixed inventory is depleted, to maximize the expected revenue stream over a finite planning horizon. Demand in the process is stochastic and price sensitive. Items left unsold at the end of horizon are disposed with little salvage value. Inventory is not replenishable, and unsatisfied demand is lost with no penalty cost. The objective is to find a dynamic pricing policy that maximizes expected revenue. Dynamic pricing has been introduced by Kincaid and Darling (1963). It is now widely practiced in capacity-constrained service industries such as airlines, hotels, rental cars, and seasonal fashion goods. Interested readers are referred to van Ryzin and McGill (1999) for a thorough discussion of the model's motivation and its role in revenue management.

Product variety is an important factor in dynamic pricing. Carrying a range of models for a generic product type makes the problem of forecasting and inventory management at the retailer level much more difficult. Since the models within a generic product type are substitutable products, each model will potentially compete with and hurt the sale of other models. Each consumer will weigh the improved performance offered by a higher quality model against the low price charged for a lower quality model, and will then make a choice as to which model to buy. Thus, the possibility exists that a customer who prefers a product with a high quality level will choose to purchase a product with low quality level because of its lower price. This phenomenon is known in the marketing literature as *cannibalization*. Overall, the prices of products interact with the quality levels of the products in a complex way to determine the consumer's purchasing behavior and hence the retailer's profit.

In this thesis, we propose a dynamic pricing model in this situation where a seller sells given inventories of two perishable products, to a random number of customers over a finite time horizon. We assume that the two products are quality-differentiated, but of the same generic type (for instance, two different computer processors or two air tickets of different departure times, but with the same departure place and destination). Customers show up sequentially, and will either purchase an item or leave empty-handed. The sale ends either when the entire stock is sold out, or when the customers are exhausted. The seller's objective is to maximize the total expected revenue by choosing different prices of the two stocks for each coming customer. In the end of this thesis, we will apply results from this model to continuous-time problems where customers arrive according to a general point process.

1.1 Research Objectives

There are three objectives in our research:

1. To find the structural properties of optimal pricing policies in the model.

2. To develop algorithms that compute the optimal policy for special cases, and to approximate it in general by tight bounds.

3. To apply results from this model to draw insights to continuous-time dynamic pricing problems.

1.2 Literature Review

There is an extensive literature on dynamic pricing at the operational, revenue management decision level. Miller (1968) studied a finite horizon, continuous-time Markov decision process where only finitely many actions (prices) are allowed. Littlewood (1972) posed a stochastic two-fare, single-leg problem in the airlines. Weatherford and Bodily (1992) gave a review of research on revenue management, where they adopt the term perishable asset revenue management to describe this class of problems. Gallego and van Ryzin (1994) proposed a model where customers arrive in accordance with a Poisson process, whose reservation prices are identical and independently distributed (i.i.d.). The retailer is allowed to change the price in real-time in order to maximize the expected total revenue when the sale concludes. In this model, the optimal policy can be shown in closed form only when the reservation price is exponentially distributed, but not in general. The authors studied the structural properties and proposed the fixed price heuristic policy, which is asymptotically optimal as the expected sales volume tends to infinity. However, this heuristic can not address adequately the situation when inventory is relatively small compared to the number of customers.

Several studies analyzed dynamic pricing policies in the case of product variety. Aydin and Ryan (1999) considered a retailer's production line selection and pricing problem under cannibalization and stochastic utility. They used Multinomial logit model (MNL) to describe customer discrete choice and showed that the profit margins of all models are equal at optimality under the assumption that initial inventories of the n given substitutable models are unlimited. Gallego and van Ryzin (1997) studied a multiproduct problem where demand for each product is a stochastic point process with an intensity that is a function of the vector of prices for products and the time at which these prices are offered. An upper bound on the optimal expected revenue is established by analyzing a deterministic version of the problem.

The pricing models are related to, but different from, another type of better known and extensively studied model, which we refer to as the multiclass yield management models. These models, which were initially constructed for the airline seat allocation problem, allow the same items to be sold at different prices at the same time due to market segmentation.

Customer choice behavior is an important phenomenon within revenue management systems (previously called yield management). Yet there are few models or methodologies available to exploit this phenomenon. The only theoretical models and methods that partially address choice behavior issues are dynamic pricing models, such as those studied by Bitran et al. (1998), Feng and Gallego (1995) and Gallego and van Ryzin (1994), (1997). While these models allow demand depend on the current price, they assume only one product is sold at one price at any point in time. Thus, customers face a binary choice; to buy or not to buy. In reality, airlines offer many fares simultaneously and customers choose among them based on whether or not they can meet various restrictions (e.g. Saturday night stay, minimum-stay and maximum-stay). The MNL is both a theoretically sound and empirically well-tested model of consumer choice behavior. It is the most widely used discrete choice model in practical applications. Jain, Vilcassim and Chintagunta (1994) cite three major reasons for the use of the MNL model. They are: (1) conceptual appeal, since the MNL model is grounded in economic theory; (2) analytical tractability and ease of econometric estimation; (3) excellent empirical performance as measured by model fit and other criteria.

Lin (2002) took a different approach modeling the arrival process of customers from those proposed in the literature. He assumed that the total number of customers is a non-negative integer-valued random variable instead of a Poisson arrival process. Customers arrive sequentially, while the seller knows the distribution of the total number of customers. Given an initial stock, the objective in his paper is to choose different prices for each arriving customer in order to maximize the total expected revenue. He formulated the optimality equation and derived the structural properties of the optimal policy and the optimal value function. The basic model has a wide range of applications to the continuous time dynamic pricing models where the arrival process is a general point process. The optimal policy in this model is well approximated in general.

This thesis is an extension to Lin's paper, where two given models are going to be considered in the basic model. We can foresee that the cannibalization problem exists, since these two models are substitutable. We use the MNL to handle the consumer choice behavior in this case in Chapter 2.

Chapter 2

The Mathematical Model

Consider a seller that has two types of products, say type 1 and type 2, to sell to a random number of customers. The customers show up sequentially, while the seller has to choose a price $\mathbf{P} = (p_1, p_2)$, possibly different for each arriving customer. Based on the prices, each arriving customer will immediately decide whether he buys one unit of type 1 product, one unit of type 2 product, or leaves empty-handed.

2.1 The Customer Choice Process

We use the multinomial logit model (MNL) to describe consumer discrete choice, and we assume the utility of product i for each customer is

$$U_i = \alpha_i - \beta p_i + Z_i, \qquad i = 1, 2,$$

while the utility of no purchase is

$$U_0 = Z_0.$$

The parameter α_i models product i's quality, brand image, etc.; p_i denotes the price of product i, while β is the price response coefficient. The random variables Z_i , i= 1, 2, model the idiosyncratic preference of each customer, and are independent and identically distributed Gumbel random variables with the following distribution function

$$F_{Z_i}(z) = \exp(-(e^{-\frac{z}{\mu}+\gamma})) \qquad i = 1, 2,$$

where μ is a scale parameter and γ is Euler's constant, which is approximately equal to 0.5772. Each customer will independently make the decision that yields the highest utility. Let (p_1, p_2) denote the price vector. Then it can be shown (see Mahajan, S. and van Ryzin, G. (1999)) that a customer will buy one unit of product i with probability

$$q_i(p_1, p_2) = P\{U_i = \max_{j=0,1,2} U_j\} = \frac{e^{\alpha_i - \beta p_i}}{\sum_{j=0}^{2} e^{\alpha_j - \beta p_j}} \qquad i = 1, 2, \tag{1}$$

where we have defined $\alpha_0 - \beta p_0 \equiv 0$ and interpreted product 0 as the no-purchase decision. In addition, we have let $\mu = 1$ without loss of generality. Given α_i and β, we can see that, once we choose a price vector (p_1, p_2), we could get the corresponding probability vector $\mathbf{Q}$=(q_1,q_2) from the MNL model. Without loss of generality, we assume the salvage value of any unsold item is zero.

Let $M(\mathbf{P}) \equiv \mathbf{P} \cdot \mathbf{Q} = (p_1, p_2)\binom{q_1}{q_2} = p_1 q_1 + p_2 q_2$ be the expected revenue collected from the arriving customer when the seller uses the policy (p_1, p_2). We interpret the seller's decision as choosing the price vector $\mathbf{P}$. It is noted that the policy $\mathbf{P}\rightarrow \infty$, i.e.,

$p_1 \to \infty$ and $p_2 \to \infty$, corresponds to the case when the seller decides not to sell to a particular customer. From Equation (1), $M(\infty, \infty) = 0$. Since $\lim_{\mathbf{P}\to\infty}$ M($\mathbf{P}$)= 0, the expected revenue tends to zero as the prices go to infinity. Also when $\mathbf{P} = (0, 0)$, we have $M(0, 0) = 0$. This also implies $M(\mathbf{P})$ is continuous and bounded in $[0, \infty)$.

Lemma 2.1 *There is a one-to-one correspondence between* $\mathbf{P}$ *and* $\mathbf{Q}$ *in the MNL model.*

Proof: From Equation (1), we see that $\mathbf{Q}$ is a function of $\mathbf{P}$, and it is sufficient to show that $\mathbf{P}$ is also a function of $\mathbf{Q}$. Suppose otherwise that there exist $\mathbf{P}$ and $\bar{\mathbf{P}}$ such that $q_i(\mathbf{P})= q_i(\bar{\mathbf{P}})$ for $i = 0, 1, 2$.

Let

$$K = \sum_{j=0}^{n} e^{\alpha_j - \beta p_j} \qquad j = 0, 1, 2,$$

and by Equation (1), we have

$$K = \frac{1}{1 - \sum_{j=0}^{n} q_j}.$$

We can see that K is the same for the two different price vectors. Hence, again by Equation 1, we have that terms of $e^{\alpha_i - \beta p_i}, i = 0, 1, 2$ for the two price vectors are all the same. Also, because function $e^{\alpha_i - \beta p_i}$ is strictly decreasing as p_i increases, we know with certainty that the two different price vectors are actually the same. That is,

$$e^{\alpha_i - \beta p_i} = e^{\alpha_i - \beta \bar{p}_i} \Rightarrow p_i = \bar{p}_i \qquad i = 1, 2.$$

The proof is completed. □

From Lemma 2.1, we can interpret the seller's decision as choosing the probability of successfully selling one item $\mathbf{Q}$, or as choosing the price vector $\mathbf{P}$. In the rest of the paper, we will alternately refer to them as the policy of the seller.

Suppose a seller starts with s_1 and s_2 items for type 1 and 2 respectively, and let X denote the number of customers. Knowing the distribution of X, we name its probability mass function as $p_X(i) = P(X = i)$, $i = 0, 1, 2, \ldots$. In the beginning of the sale, the seller simply waits for the first customer to show up. With probability $p_X(0)$, no customer shows up, so the sale ends and the total revenue is equal to zero. On the other hand with probability $1 - p_X(0)$, the first customer does show up, so the seller sets a product price and lets the customer decide if he/she wants to purchase an item. After the first customer leaves, the seller waits for the second customer. With probability $p_{X \mid X \geq 1}(1) = p_X(1)/(1 - p_X(0))$, there are no more customers and the sale ends, while with probability $1 - p_{X \mid X \geq 1}(1)$, the second customer does show up and the sale continues. Knowing p_X, we say a seller's stationary policy is $p(i, j, k)$ with the interpretation that he has i items and j items for type 1 and type 2, respectively, when (and if) the k^{th} customer shows up, where $k = 1, 2, \ldots, \infty$, $i = s_1, s_1 - 1, \ldots, 0$ and $j = s_2, s_2 - 1, \ldots, 0$. Note that k might not go to ∞, it depends on the distribution of X. Let Π denote the class of all stationary policies. For a given a policy $\pi \in \Pi$, denote the expected total revenue by $J_\pi(s_1, s_2, p_X)$. It is worth noting that the first two arguments s_1, s_2 in the function $J_\pi(s_1, s_2, p_X)$ are nonnegative integers, while the last argument X represents a probability distribution.

2.2 Formulation of the Optimal Policy

Since the expected revenue received from each arriving customer is bounded and we made the assumption that $E[X]$ is finite, it follows that the expected total revenue is bounded by $E[X] \times \max_{\mathbf{P} \in R^{+2}} M(\mathbf{P})$, and is therefore finite.

As it is well known (see Ross (1983)) that a stationary policy suffices to be optimal for this type of control problem, the seller's objective is to find a policy $\pi \in \Pi$ that maximizes the expected total revenue. Let

$$J(s_1, s_2, p_X) \equiv \max_{\pi \in \Pi} J_\pi(s_1, s_2, p_X)$$

denote the optimal expected total revenue that can be generated in state (s_1, s_2, p_X), i.e., if the seller starts with s_1, s_2 items and the probability mass function of the number of customers is p_X.

Since it can never hurt to have more items, or more customers, we summarize the monotonicity of the optimal value function $J(\cdot,\cdot,\cdot)$ in the following lemmas.

Lemma 2.2 *$J(s_1, s_2, p_X)$ increases in s_1 and in s_2.*

Proof: Suppose there are two sellers: seller 1 and seller 2. Seller 1 is in state (a, b, p_X) and seller 2 is in state (c, b, p_X), where $c > a$, and couple each customer's preference for the two sellers. Let seller 1 use the optimal policy. Upon arrival of a new customer, let seller 2 simply follow seller 1's policy. Then we know that seller 2's total revenue is equal to that of seller 1. Since, by definition, $J(c, b, p_X)$ is generated by using the optimal policy, we have that $J(c, b, p_X) \geq J(a, b, p_X)$.

The same approach can be used for other cases. □

Lemma 2.3 *$J(s_1, s_2, p_X)$ increases in X in the regular stochastic sense; that is, if $Y \geq_{\text{st}} X$, i.e., $P\{Y > i\} \geq P\{X > i\}$ for all $i \geq 0$, then $J(s_1, s_2, p_Y) \geq J(s_1, s_2, p_X)$.*

Proof: Suppose there are two sellers: seller 1 and seller 2. Seller 1 is in state (a, b, p_X) and seller 2 is in state (a, b, p_Y). Because $Y \geq_{\text{st}} X$, we can couple X and Y such that $X \sim \bar{X}$, $Y \sim \bar{Y}$, and $P\{\bar{Y} \geq \bar{X}\} = 1$. In addition, we couple each customer's preference for the two sellers. Let seller 1 use the optimal policy. Upon arrival of a new customer, let seller 2 follow seller 1's policy, until seller 1 sells out his inventory (denoted event E).

Consider two cases:

1. Event E occurs: When E occurs, seller 2 also sells out his inventory. The total revenue generated by seller 1 is equal to that generated by 2.
2. Event E never occurs: Then seller 1 must exhaust his customers, at which point the total revenue generated by seller 1 is equal to that generated by 2 and seller 2 still might have future customers. Therefore, the additional revenue generated by seller 2 is greater than or equal to 0.

We see that for each sample path, seller 2 does not actually use his optimal policy. But seller 2's total revenue is no less than that of seller 1. Hence, the lemma in turn is proved. $\square$

To formulate the optimality equation, first note that the seller does not have to make a decision in state (s_1, s_2, p_Z), since it may be the case that Z takes on the value 0. If on the other hand, a new customer shows up, the first thing the seller needs to do is update the distribution of the number of future customers, which is $X \sim (Z - 1 \mid Z \geq 1)$. Knowing $Z \geq 1$, the state becomes $(s_1, s_2, p_{(Z \mid Z \geq 1)})$, or equivalently, (s_1, s_2, p_{1+X}). Hence, after updating the distribution of the number of future customers to X, the seller needs to choose $\mathbf{P}$ for the arriving customer to maximize the expected total revenue collected from this customer and onward. The optimality equation is

$$\begin{aligned} J(s_1, s_2, p_{1+X}) &= \max_{p_1, p_2} \; q_1(p_1 + J(s_1 - 1, s_2, p_X)) \\ &\quad + q_2(p_2 + J(s_1, s_2 - 1, p_X)) + (1 - q_1 - q_2) J(s_1, s_2, p_X) \\ &= \max_{p_1, p_2} \Big\{ M(\mathbf{P}) - q_1(J(s_1, s_2, p_X) - J(s_1 - 1, s_2, p_X)) \\ &\quad - q_2(J(s_1, s_2, p_X) - J(s_1, s_2 - 1, p_X)) + J(s_1, s_2, p_X) \Big\}, \end{aligned} \tag{2}$$

where (see Equation (1))

$$q_1 = q_1(p_1, p_2) and \qquad q_2 = q_2(p_1, p_2),$$

for $s_1 \geq 1$ and $s_2 \geq 1$, with boundary conditions $J(0, 0, \cdot) = 0$.

Define

$$\begin{aligned} \mathbf{P}(s_1, s_2, p_X) = \arg\max_{p_1, p_2} \Big\{ & M(\mathbf{P}) - q_1(J(s_1, s_2, p_X) - J(s_1 - 1, s_2, p_X)) \\ & - q_2(J(s_1, s_2, p_X) - J(s_1, s_2 - 1, p_X)) \Big\}. \end{aligned}$$

Also, when s_2= 0, the optimality equation becomes

$$J(s_1, 0, p_{1+X}) = \max_{p_1} \Big\{ q_1(p_1, \infty)(p_1 + J(s_1 - 1, 0, p_X)) + (1 - q_1(p_1, \infty)) J(s_1, 0, p_X) \Big\},$$

and when s_1= 0, it becomes

$$J(0, s_2, p_{1+X}) = \max_{p_2} \Big\{ q_2(\infty, p_2)(p_2 + J(0, s_2 - 1, p_X)) + (1 - q_2(\infty, p_2)) J(0, s_2, p_X) \Big\},$$

where

$$q_1(p,\infty) = \frac{e^{\alpha_1-\beta p}}{1+e^{\alpha_1-\beta p}}$$

and

$$q_2(\infty,p) = \frac{e^{\alpha_2-\beta p}}{1+e^{\alpha_2-\beta p}}.$$

Define

$$M_1(p) = pq_1(p,\infty) = \frac{pe^{\alpha_1-\beta p}}{1+e^{\alpha_1-\beta p}}$$

and

$$M_2(p) = pq_2(\infty,p) = \frac{pe^{\alpha_2-\beta p}}{1+e^{\alpha_2-\beta p}}.$$

Hence,

$$J(s_1,0,p_{1+X}) = \max_{p_1}\left\{M_1(p_1) - q_1(p_1,\infty)(J(s_1,0,p_X) - J(s_1-1,0,p_X)) + J(s_1,0,p_X)\right\}$$

and

$$J(0,s_2,p_{1+X}) = \max_{p_2}\left\{M_2(p_2) - q_2(\infty,p_2)(J(0,s_2,p_X) - J(0,s_2-1,p_X)) + J(0,s_2,p_X)\right\}.$$

Correspondingly, we define

$$p_1(s_1,0,p_X) = \arg\max_{p_1}\left\{M_1(p_1) - q_1(p_1,\infty)(J(s_1,0,p_X) - J(s_1-1,0,p_X))\right\}$$

and

$$p_2(0, s_2, p_X) = \arg\max_{p_2} \Big\{ M_2(p_2) - q_2(\infty, p_2)(J(0, s_2, p_X) - J(0, s_2 - 1, p_X)) \Big\}.$$

Lemma 2.4 *$M_1(p)$ and $M_2(p)$ have only one global maximum on $[0, \infty)$ respectively. Define them as p_1^* and p_2^* correspondingly, such that $M_1(p_1^*) = \max_p M_1(p)$, $M_2(p_2^*) = \max_p M_2(p)$.*

Proof: We know

$$M_1(p) = pq_1(p, \infty) = \frac{pe^{\alpha_1 - \beta p}}{1 + e^{\alpha_1 - \beta p}}.$$

Taking the first derivative, we have

$$M_1'(p) = -e^{\alpha_1 - \beta p}(\beta p - 1 - e^{\alpha_1 - \beta p})/(1 + e^{\alpha_1 - \beta p})^2 = 0.$$

Since $e^{\alpha_1 - \beta p}$ is always greater than 0, the solution must satisfy

$$\beta p - 1 = e^{\alpha_1 - \beta p}. \tag{3}$$

We can see that $\beta p - 1$ is a strictly increasing function of p, while $e^{\alpha_1 - \beta p}$ is a strictly decreasing one. Also, when $p = 0$, $\beta p - 1$ = -1, while $e^{\alpha_1 - \beta p} = e^{\alpha_1} > 0$. Therefore, there is only one solution to $M_1'(p) = 0$, denoted as p_1^*.

When $p < p_1^*$, $\beta p - 1 - e^{\alpha_1 - \beta p} < 0$, so we have $M_1'(p) > 0$, which means $M_1(p)$ is strictly increasing. When $p > p_1^*$, $\beta p - 1 - e^{\alpha_1 - \beta p} > 0$, so we have $M_1'(p) < 0$, which means $M_1(p)$

is strictly decreasing. Overall, $M_1(p)$ has only one global maximum at p_1^*.

The proof is the same for $M_2(p)$. $\square$

Lemma 2.5 *For given α_1, α_2 and β, $M(\mathbf{P})$ has only one global maximum on ${R^+}^2$, i.e., $p_1 \in [0,\infty)$ and $p_2 \in [0,\infty)$, define it as $\mathbf{P}^*$, such that $M(\mathbf{P}^*) = \max_{p_1,p_2} M(\mathbf{P})$.*

Proof: We know

$$M(\mathbf{P}) = p_1q_1 + p_2q_2 = \frac{p_1e^{\alpha_1-\beta p_1} + p_2e^{\alpha_2-\beta p_2}}{1+e^{\alpha_1-\beta p_1}+e^{\alpha_2-\beta p_2}}.$$

Let $K = 1+e^{\alpha_1-\beta p_1}+e^{\alpha_2-\beta p_2}$ and take the first derivative on $M(\mathbf{P})$, we then have

$$\begin{aligned}\frac{\partial M}{\partial p_1} &= \frac{e^{\alpha_1-\beta p_1}}{K^2}\{e^{\alpha_1-\beta p_1} + (1+\beta p_2-\beta p_1)e^{\alpha_2-\beta p_2} + 1 - \beta p_1\},\\ \frac{\partial M}{\partial p_2} &= \frac{e^{\alpha_2-\beta p_2}}{K^2}\{e^{\alpha_2-\beta p_2} + (1+\beta p_1-\beta p_2)e^{\alpha_1-\beta p_1} + 1 - \beta p_2\}.\end{aligned}$$

Letting $\frac{\partial M}{\partial p_1} = \frac{\partial M}{\partial p_2} = 0$, and because $\frac{e^{\alpha_2-\beta p_2}}{K^2} > 0$, we have that

$$e^{\alpha_1-\beta p_1} + (1+\beta p_2-\beta p_1)e^{\alpha_2-\beta p_2} + 1 - \beta p_1 = 0, \tag{4}$$

and

$$e^{\alpha_2-\beta p_2} + (1+\beta p_1-\beta p_2)e^{\alpha_1-\beta p_1} + 1 - \beta p_2 = 0. \tag{5}$$

Therefore,

$$\beta(p_1 - p_2)(1 + e^{\alpha_1-\beta p_1} + e^{\alpha_2-\beta p_2}) = 0.$$

Since $\beta > 0$ and $1 + e^{\alpha_1-\beta p_1} + e^{\alpha_2-\beta p_2} > 0$, we have $p_1 = p_2$.

Setting $p_1 = p_2 = p^*$ in Equation (4) or (5), we obtain the only solution of $\frac{\partial M}{\partial P} = 0$ from the equation:

$$1 + e^{\alpha_1-\beta p^*} + e^{\alpha_2-\beta p^*} = \beta p^*. \tag{6}$$

Next, we take the second derivative to obtain the Hessian matrix $H(\mathbf{P})$.

$$\begin{aligned}
\frac{\partial^2 M}{\partial p_1^2} &= \frac{1}{(1 + e^{\alpha_1-\beta p_1} + e^{\alpha_2-\beta p_2})^3}\{\beta e^{\alpha_1-\beta p_1}[-2 - 4e^{\alpha_2-\beta p_2} + \beta p_1 - 2e^{\alpha_1-\beta p_1} \\
&\quad -\beta p_1 e^{\alpha_1-\beta p_1+\alpha_2-\beta p_2} + \beta e^{\alpha_1-\beta p_1+\alpha_2-\beta p_2} p_2 - \beta e^{\alpha_1-\beta p_1} p_1 \\
&\quad +\beta p_1 e^{2\alpha_2-2\beta p_2} - 2e^{\alpha_1-\beta p_1+\alpha_2-\beta p_2} - 2e^{2\alpha_2-2\beta p_2} + 2\beta p_1 e^{\alpha_2-\beta p_2} \\
&\quad -\beta e^{\alpha_2-\beta p_2} p_2 - \beta e^{2\alpha_2-2\beta p_2} p_2]\}, \\
\frac{\partial^2 M}{\partial p_2^2} &= \frac{1}{(1 + e^{\alpha_2-\beta p_2} + e^{\alpha_1-\beta p_1})^3}\{\beta e^{\alpha_2-\beta p_2}[-2 - 4e^{\alpha_1-\beta p_1} + \beta p_2 - 2e^{\alpha_2-\beta p_2} \\
&\quad -\beta p_2 e^{\alpha_2-\beta p_2+\alpha_1-\beta p_1} + \beta e^{\alpha_2-\beta p_2+\alpha_1-\beta p_1} p_1 - \beta e^{\alpha_2-\beta p_2} p_2 \\
&\quad +\beta p_2 e^{2\alpha_1-2\beta p_1} - 2e^{\alpha_2-\beta p_2+\alpha_1-\beta p_1} - 2e^{2\alpha_1-2\beta p_1} + 2\beta p_2 e^{\alpha_1-\beta p_1} \\
&\quad -\beta e^{\alpha_1-\beta p_1} p_1 - \beta e^{2\alpha_1-2\beta p_1} p_1]\}, \\
\frac{\partial^2 M}{\partial p_1 \partial p_2} &= \frac{-\beta e^{\alpha_1-\beta p_1} p_1 + e^{\alpha_1-\beta p_1}}{(1 + e^{\alpha_1-\beta p_1} + e^{\alpha_2-\beta p_2})^2}\beta e^{\alpha_2-\beta p_2}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{-\beta e^{\alpha_2-\beta p_2}p_2+e^{\alpha_2-\beta p_2}}{(1+e^{\alpha_1-\beta p_1}+e^{\alpha_2-\beta p_2})^2}\beta e^{\alpha_1-\beta p_1}\\
&+\frac{2[e^{\alpha_1-\beta p_1}p_1+e^{\alpha_2-\beta p_2}p_2]}{(1+e^{\alpha_1-\beta p_1}+e^{\alpha_2-\beta p_2})^3}\beta^2 e^{\alpha_1-\beta p_1}e^{\alpha_2-\beta p_2}.
\end{aligned}
$$

When $p_1 = p_2 = p^*$, we have

$$
\begin{aligned}
\frac{\partial^2 M}{\partial p_1^2} &= \frac{-e^{\alpha_1-\beta p^*}}{p^*},\\
\frac{\partial^2 M}{\partial p_2^2} &= \frac{-e^{\alpha_2-\beta p^*}}{p^*},\\
\frac{\partial^2 M}{\partial p_1 \partial p_2} &= 0.
\end{aligned}
$$

Hence, $H(p^*, p^*)$ is negative definite(ND), and we know that (p^*, p^*) is a local maximum.

Next, we first find a region where the maximum point of the outside of the region is less than some points within the region and also (p^*, p^*) is the unique maximum of the region. Therefore we in turn prove our argument.

In the following two equations

$$
\begin{aligned}
M(p^*, p^*) &= M_1(p_1^*) + \frac{p_2 e^{\alpha_2-\beta p_2}}{1+e^{\alpha_2-\beta p_2}},\\
M(p^*, p^*) &= M_2(p_2^*) + \frac{p_1 e^{\alpha_1-\beta p_1}}{1+e^{\alpha_1-\beta p_1}},
\end{aligned}
$$

p^*, p_1^* and p_2^* are known. (For instance, p_1^* is derived by Equation (3).) Hence, we can solve the two equations. Note that there are two solutions to each of them (see Lemma 2.4), and we choose the correspondingly larger ones as solutions and denote them as $\bar{p}_1$ and $\bar{p}_2$.

Now we prove that in the region $\{(p_1, p_2), p_1 \geq \bar{p}_1 \text{ or } p_2 \geq \bar{p}_2\}$, we have $M(\mathbf{P}) < M(p^*, p^*)$.

Case 1: When $\{(p_1, p_2), p_1 \geq \bar{p}_1\}$, we know

$$\begin{aligned} M(p_1,p_2) &= \frac{p_1 e^{\alpha_1-\beta p_1} + p_2 e^{\alpha_2-\beta p_2}}{1+e^{\alpha_1-\beta p_1}+e^{\alpha_2-\beta p_2}} < \frac{p_1 e^{\alpha_1-\beta p_1}}{1+e^{\alpha_1-\beta p_1}} + \frac{p_2 e^{\alpha_2-\beta p_2}}{1+e^{\alpha_2-\beta p_2}} \\ &\leq M_2(p_2^*) + \frac{\bar{p}_1 e^{\alpha_1-\beta \bar{p}_1}}{1+e^{\alpha_1-\beta \bar{p}_1}} = M(p^*, p^*). \end{aligned}$$

Case 2: When$\{(p_1, p_2), p_2 \geq \bar{p}_2\}$, we know

$$\begin{aligned} M(p_1,p_2) &= \frac{p_1 e^{\alpha_1-\beta p_1} + p_2 e^{\alpha_2-\beta p_2}}{1+e^{\alpha_1-\beta p_1}+e^{\alpha_2-\beta p_2}} < \frac{p_1 e^{\alpha_1-\beta p_1}}{1+e^{\alpha_1-\beta p_1}} + \frac{p_2 e^{\alpha_2-\beta p_2}}{1+e^{\alpha_2-\beta p_2}} \\ &\leq M_1(p_1^*) + \frac{\bar{p}_2 e^{\alpha_2-\beta \bar{p}_2}}{1+e^{\alpha_2-\beta \bar{p}_2}} = M(p^*, p^*). \end{aligned}$$

Also, we can see that in $\{(p_1, p_2), p_1 = 0$ and $p_2 \leq \bar{p}_2\}$ and $\{(p_1, p_2), p_2 = 0$ and $p_1 \leq \bar{p}_1\}$, then $M(\mathbf{P}) < M(p^*, p^*)$.

Now we only need to show that $M(p^*, p^*)$ is the global maximum in the region $P = \{(p_1, p_2), p_1 \leq \bar{p}_1$ and $p_2 \leq \bar{p}_2\}$. Knowing that this region is a compact set and $M(\mathbf{P})$ is continuous and differentiable on the set, there exists a maximum. Given again that (p^*, p^*) is the only solution of $\frac{\partial M}{\partial P} = 0$ and on the boundary, all $M(\mathbf{P}) < M(p^*, p^*)$, we know that if there is a point with a value larger than $M(p^*, p^*)$, then there must be at least a point that satisfies $\frac{\partial M}{\partial P} = 0$ besides (p^*, p^*). Hence $M(p^*, p^*)$ is the global maximum. The proof is completed. $\square$

Chapter 3

Optimal Policy for Special Cases

3.1 When the Number of Customers Is Deterministic

Consider the case when the seller knows the exact number of customers. We say it is in state (s_1, s_2, n) if there are s_1, s_2 items in stock, and n customers waiting in line. Let $\omega(s_1, s_2, n) = (\omega_1, \omega_2)$ denote the optimal policy, where ω_1 is the optimal price for product 1, ω_2 is the optimal price for product 2, and $R(s_1, s_2, n)$ be the maximized expected total revenue in state (s_1, s_2, n), respectively.

The optimality condition for $R(s_1, s_2, n)$ is

$$\begin{aligned} R(s_1, s_2, n) = \max_{p_1, p_2} \{ & M(\mathbf{P}) - q_1(p_1, p_2)(R(s_1, s_2, n-1) - R(s_1 - 1, s_2, n-1)) \\ & - q_2(p_1, p_2)(R(s_1, s_2, n-1) - R(s_1, s_2 - 1, n-1))\} + R(s_1, s_2, n-1), \end{aligned}$$

for $s_1 \geq 1$, $s_2 \geq 1$ and $n \geq 1$, with the following boundary conditions.

Condition 1:

$$R(0, s_2, 1+n) = q_2(\infty, p_2)(p_2 + R(0, s_2 - 1, n)) + (1 - q_2(\infty, p_2))R(0, s_2, n)$$

and

$$R(s_1, 0, 1+n) = q_1(p_1, \infty)(p_1 + R(s_1 - 1, 0, n)) + (1 - q_1(p_1, \infty))R(s_1, 0, n).$$

Condition 2:

$$R(\cdot, \cdot, 0) = R(0, 0, \cdot) = 0,$$

$$R(s_1, s_2, n) = R(n, s_2, n) \qquad \text{for } s_1 \geq n,$$

$$R(s_1, s_2, n) = R(s_1, n, n) \qquad \text{for } s_2 \geq n,$$

and

$$R(s_1, s_2, n) = R(n, n, n) = nM(\mathbf{P}^*) \qquad \text{for } s_1, s_2 \geq n.$$

Letting $n = 1$, with these boundary conditions, we have the following results.

1. When $s_1 \geq 1$ and $s_2 \geq 1$, then $R(s_1, s_2, 1) = R(1, 1, 1) = M(\mathbf{P}^*)$.

2. When $s_1 = 0$ and $s_2 \geq 1$, then $R(0, s_2, 1) = R(0, 1, 1) = M_1(p_1^*)$.

3. When $s_1 \geq 1$ and $s_2 = 0$, then $R(s_1, 0, 1) = R(1, 0, 1) = M_2(p_2^*)$.

4. When $s_1 = s_2 = 0$, then $R(0, 0, 1) = 0$.

Therefore, we know all the optimal revenues and policies for $n = 1$. Next, we can let $n = 2$, and we can determine all the optimal revenues and optimal policies by the optimality condition above. Then we can solve them for $n = 3, 4, \ldots$. So in this way, all $R(s_1, s_2, n)$ and $\omega(s_1, s_2, n)$ can be solved recursively (see Table 10 through 15 in the Appendix for examples) .

3.2 When the Number of Customers Is Geometric

Consider the case when X, the number of customers, has a geometric distribution with the following probability mass function.

$$P\{X = i\} = (1 - \lambda)^i \lambda, \qquad i = 0, 1, \ldots,$$

where $0 < \lambda < 1$. Since X is memoryless, i.e. $(X - 1 \mid X \geq 1) \sim X$, we can compute $J(s_1, s_2, p_X)$ by conditioning on whether $X = 0$ or $X \geq 1$. That is,

$$\begin{aligned} J(s_1, s_2, p_X) &= (1 - p_X(0)) J(s_1, s_2, p_{X \mid X \geq 1}) + p_X(0) \cdot 0 \\ &= (1 - \lambda)\ J(s_1, s_2, p_{1+X}). \end{aligned}$$

Note that we use $(X \mid X \geq 1)$ to denote a nonnegative integer-valued random variable that takes on the value i with probability $P\{X = i \mid X \geq 1\}$, $i = 0, 1, \ldots$. Together with the optimality equation (2), we have

$$\begin{aligned}\frac{\lambda}{1-\lambda} J(s_1, s_2, p_X) = \max_{p_1, p_2}\ & M(\mathbf{P}) - q_1(p_1, p_2)(J(s_1, s_2, p_X) - J(s_1 - 1, s_2, p_X)) \\ & - q_2(p_1, p_2)(J(s_1, s_2, p_X) - J(s_1, s_2 - 1, p_X)). \end{aligned} \tag{7}$$

For $s_1 = 0$, while $s_2 \geq 1$, with boundary conditions $J(0, 0, \cdot) = 0$, we also have

$$\frac{\lambda}{1-\lambda} J(0, s_2, p_X) = \max_{p_2}\ q_2(\infty, p_2)(p_2 + J(0, s_2 - 1, p_X)) - q_2(\infty, p_2) J(0, s_2, p_X)$$

and for $s_2 = 0$, while $s_1 \geq 1$,

$$\frac{\lambda}{1-\lambda} J(s_1, 0, p_X) = \max_{p_1}\ q_1(p_1, \infty)(p_1 + J(s_1 - 1, 0, p_X)) - q_1(p_1, \infty) J(s_1, 0, p_X).$$

Let $s_1 = s_2 = 1$, we have

$$\begin{aligned}\frac{\lambda}{1-\lambda} J(1, 1, p_X) = \max_{p_1, p_2}\ & M(\mathbf{P}) - q_1(p_1, p_2)(J(1, 1, p_X) - J(0, 1, p_X)) \\ & - q_2(p_1, p_2)(J(1, 1, p_X) - J(1, 0, p_X)), \end{aligned} \tag{8}$$

$$\begin{aligned}\frac{\lambda}{1-\lambda} J(0, 1, p_X) &= \max_{p_2}\ q_2(\infty, p_2)(p_2 + J(0, 0, p_X)) - q_2(\infty, p_2) J(0, 1, p_X) \\ &= \max_{p_2}\ q_2(\infty, p_2) p_2 - q_2(\infty, p_2) J(0, 1, p_X)\end{aligned}$$

and

$$\frac{\lambda}{1-\lambda}J(1,0,p_X) = \max_{p_1}\ q_1(p_1,\infty)(p_1 + J(0,0,p_X)) - q_1(p_1,\infty)J(1,0,p_X)$$
$$= \max_{p_1}\ q_1(p_1,\infty)p_1 - q_1(p_1,\infty)J(1,0,p_X).$$

Note that as $J(1,0,p_X)$ and $J(0,1,p_X)$ both increase from zero to infinity (as do the left hand sides of the preceding equations), the right hand sides of the preceding equations decrease from $M(\mathbf{P}^*)$ to zero. Therefore, we can solve $J(1,0,p_X)$, $J(0,1,p_X)$, $\mathbf{P}(1,0,p_X)$ and $\mathbf{P}(0,1,p_X)$. Based on these and Equation (8), next we can solve $J(1,1,p_X)$ and $\mathbf{P}(1,1,p_X)$. We then plug $s_1 = 2$ and $s_2 = 2$ into Equation (7) and in the same way, we can solve $J(2,1,p_X)$, $J(1,2,p_X)$, $\mathbf{P}(2,1,p_X)$ and $\mathbf{P}(1,2,p_X)$. Based on these, next we can solve $J(2,2,p_X)$ and $\mathbf{P}(2,2,p_X)$. As a result, $J(s_1,s_2,p_X)$ and $\mathbf{P}(s_1,s_2,p_X)$, $s = 1,2,\ldots$, can be solved recursively (see Table 1 for example).

3.3 When the Number of Customers Is Bounded

Consider the case where X is bounded by $x_{\max}$, i.e. $P\{x_{\max} \geq X\} = 1$. Define the failure rate function of X as follows

$$r_X(k) \equiv P\{X = k \mid X \geq k\} \qquad k = 0,1,\ldots,x_{\max}.$$

That is, when the k^{th} customer arrives, the probability that he is the last one is $r_X(k)$, and in particular, $r_X(x_{\max}) = 1$. Since X is bounded, the boundary conditions can be set up properly. As a result, the optimal policy can be computed using backward dynamic programming.

We illustrate the procedures as follows. Consider a seller with s_1, s_2 items for product 1, 2 and the number of customers is distributed as X. Suppose that immediately after the seller deals with the k^{th} customer, the inventory is down to i and j for products 1 and 2 respectively. Therefore, the seller is in state $(i, j, (p_{X-k\,|\,X\geq k}))$. The optimality equation can be formulated upon conditioning on whether $X = k$ or not. For $i, j \geq 1$ and $k \geq 0$,

$$
\begin{aligned}
J(i, j, (p_{X-k\,|\,X\geq k})) &= P\Big\{X = k\,|\,X \geq k\Big\} \times 0 + P\Big\{X \geq k+1\,|\,X \geq k\Big\} \\
&\quad \times \max_{p_1, p_2}\Big\{q_1\Big(p_1 + J(i-1, j, (p_{X-k-1\,|\,X\geq k+1}))\Big) \\
&\quad + q_2\Big(p_2 + J(i, j-1, (p_{X-k-1\,|\,X\geq k+1}))\Big) \\
&\quad + (1 - q_1 - q_2)J(i, j, (p_{X-k-1\,|\,X\geq k+1}))\Big\} \\
&= \Big(1 - r_X(k)\Big)\max_{p_1, p_2}\Big\{q_1\Big(p_1 + J(i-1, j, (p_{X-k-1\,|\,X\geq k+1}))\Big) \\
&\quad + q_2\Big(p_2 + J(i, j-1, (p_{X-k-1\,|\,X\geq k+1}))\Big) \\
&\quad + (1 - q_1 - q_2)J(i, j, (p_{X-k-1\,|\,X\geq k+1}))\Big\}.
\end{aligned}
$$

The boundary conditions are

$$
J(0, 0, \cdot) = J(\cdot, \cdot, (p_{X-x_{\max}\,|\,X\geq x_{\max}})) = 0.
$$

Consequently, the optimal policy can be solved recursively (see Tables 2 and 3 for example).

Table 1: Optimal revenue and policy when the number of customers follows a geometric distribution ($\alpha_1 = 1$, $\alpha_2 = 2$, $\beta = 1$).

λ	$J(5, 10, p_X)$	q_1	q_2	p_1	p_2
0.05	17.5026	0.13627	0.34322	2.34018	2.41644
0.10	9.8712	0.14336	0.37972	2.20198	2.22791
0.15	6.4814	0.14428	0.38926	2.17346	2.18094
0.20	4.6283	0.14443	0.39198	2.16624	2.16779
0.25	3.4828	0.14448	0.39275	2.16405	2.16404
0.30	2.7115	0.14452	0.39296	2.16324	2.16298
0.35	2.1588	0.14455	0.39301	2.16290	2.16269
0.40	1.7438	0.14457	0.39302	2.16274	2.16262
0.45	1.4209	0.14457	0.39302	2.16266	2.16260
0.50	1.1626	0.14458	0.39301	2.16263	2.16260
0.55	0.9512	0.14458	0.39301	2.16261	2.16260
0.60	0.7751	0.14458	0.39301	2.16261	2.16260
0.65	0.6260	0.14458	0.39301	2.16260	2.16260
0.70	0.4983	0.14458	0.39301	2.16260	2.16260
0.75	0.3875	0.14458	0.39301	2.16260	2.16260
0.80	0.2907	0.14458	0.39301	2.16260	2.16260
0.85	0.2052	0.14458	0.39301	2.16260	2.16260
0.90	0.1292	0.14458	0.39301	2.16260	2.16260
0.95	0.0612	0.14458	0.39301	2.16260	2.16260

Table 2: Optimal revenue and policy when the number of customers follows a binomial distribution ($\alpha_1 = 1$, $\alpha_2 = 4$, $\beta = 1$, $x_{\max} = 20$, $p = 0.60$).

k	$J(2, 5, p_{X-k\|X>=k})$	q_1	q_2	p_1	p_2
20	0	0.03279	0.65870	3.24146	3.24146
19	0.1564	0.03279	0.65870	3.24146	3.24146
18	0.3479	0.03279	0.65870	3.24147	3.24146
17	0.5859	0.03279	0.65870	3.24150	3.24146
16	0.8857	0.03279	0.65871	3.24156	3.24146
15	1.2686	0.03279	0.65870	3.24161	3.24148
14	1.7633	0.03280	0.65865	3.24138	3.24169
13	2.4072	0.03287	0.65842	3.23998	3.24258
12	3.2445	0.03308	0.65768	3.23510	3.24540
11	4.3197	0.03367	0.65573	3.22202	3.25277
10	5.6612	0.03499	0.65135	3.19314	3.26926
9	7.2588	0.03760	0.64286	3.14001	3.30099
8	9.0478	0.04203	0.62858	3.05891	3.35379
7	10.9212	0.04857	0.60769	2.95696	3.43023
6	12.7679	0.05688	0.58083	2.85142	3.52796
5	14.5094	0.06600	0.54995	2.76112	3.64093
4	16.1089	0.07468	0.51740	2.69786	3.76228
3	17.5596	0.08196	0.48509	2.66444	3.88631
2	18.8692	0.08737	0.45426	2.65756	4.00903
1	20.0508	0.09089	0.42552	2.67154	4.12793
0	21.1187	—	—	—	—

Table 3: Optimal revenue and policy when the number of customers follows a binomial distribution ($\alpha_1 = 1$, $\alpha_2 = 4$, $\beta = 1$, $x_{\max} = 20$, $p = 0.60$).

$J(2, 5, p_{1+X})$	q_1	q_2	p_1	p_2
22.0867	0.09277	0.39909	2.70061	4.24156

Chapter 4

Structural Properties

By Lemma 2.3, $J(s_1, s_2, p_X)$ increases as X becomes stochastically larger. However, the optimal policy need not have such a monotonic property in X. In this section, we use examples with a deterministic number of customers to demonstrate how the shape of the optimal policy depends on the following two factors:

1. The difference between α_1 and α_2.

2. The difference between s_1 and s_2.

Example 1: $\alpha_1 = \alpha_2$**,** $s_1 = s_2$

We can see from Figure 1 that without factors 1 and 2, the optimal policies are monotonic. When $n \geq 5$ (see section 3.1 for details), q_1 and q_2 both decrease and p_1 and p_2 both increase as n (the number of customers) increases. (Please see Table 10 in Appendix for original data.)

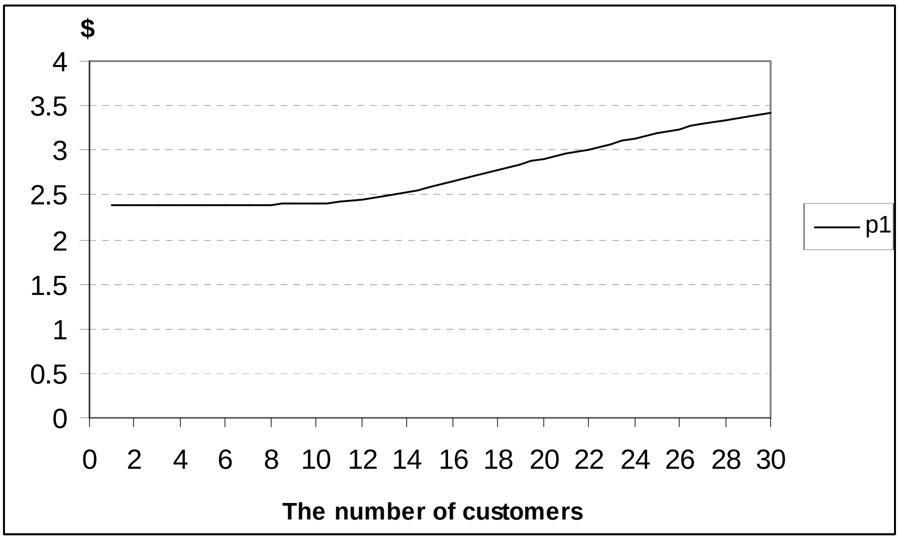

Figure 1: Optimal policy is monotonic with $\alpha_1 = \alpha_2 = 2$, $s_1 = s_2 = 5$.

Example 2: $\alpha_1 \neq \alpha_2$**,** $s_1 = s_2$

Figure 2 shows that when $n \geq 5$, as n increases, q_2 decreases and p_2 increases all the way, while q_1 increases first and then decreases, and p_1 decreases first and then increases. Our explanation to this phenomenon is that before the two type products begin to compete for customers (in this case, it is when the number of customers is no greater than the inventory), the products are sold at the same price. That implies product 2 (the product with a higher quality) will have a larger chance to be sold. If we still sell them at same price when competition for customers happens, then we know for sure that more likely customers will compete for product 2, because they have a better opportunity to buy it. Hence, in order to make full use of this competition so that the seller can obtain more revenue, the seller should increase the price for product 2. If the difference of the two quality parameters is large enough, the price of product 1 should also decrease in order to

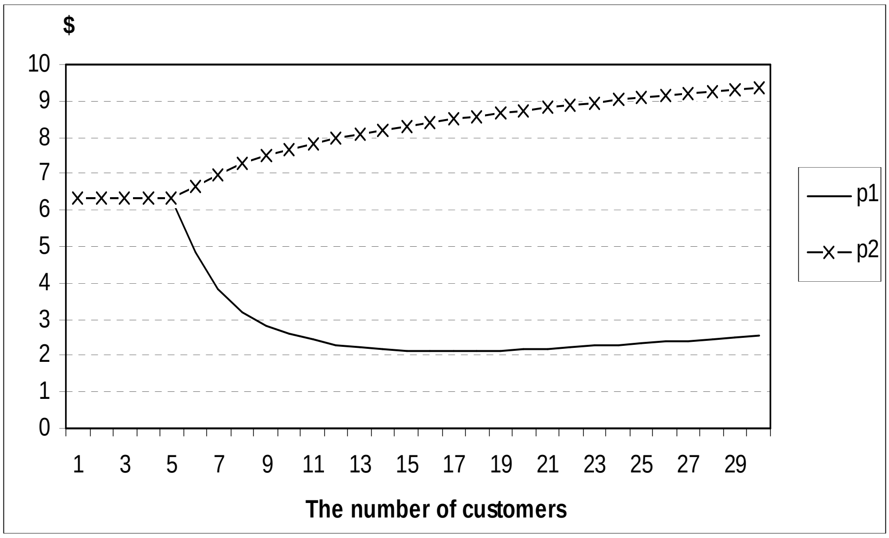

Figure 2: Optimal prices with $\alpha_1 = 1$, $\alpha_2 = 8$ and $s_1 = s_2 = 5$.

decrease the probability of no sale. We can conjecture that if the two quality parameters are slightly different, their prices might both increase when competition for customers starts (a similar example is case 1). (Please see Table 11 in Appendix for original data.)

Example 3: $\alpha_1 = \alpha_2$, $s_1 \neq s_2$

From Figure 3, we can see that when $n \geq 3$, as n increases, q_1 decreases and p_1 increases all the way, while q_2 increases first and then decreases, and p_2 decreases first and then increases. Our explanation of this phenomenon is that because the quality parameters are the same, the two products are the same products. When competition for customers happens, the price of product 1 (the one with a lower initial inventory) will increase because of competition, while the price of the other one should correspondingly decrease because of competition. Even though they are the same products, this is how MNL model denotes

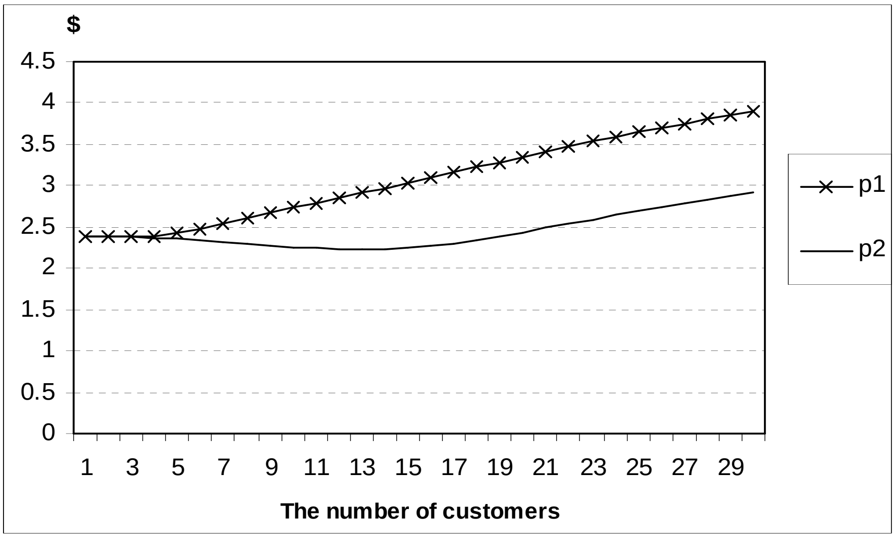

Figure 3: Optimal prices with $\alpha_1 = \alpha_2 = 2$, $s_1 = 3$ and $s_2 = 8$.

competition. (Please see Table 12 in Appendix for original data.)

Example 4: $\alpha_1 \neq \alpha_2$, $s_1 \neq s_2$

Figure 4 shows clearly the effect of both of the two factors. We see that when $n \geq 4$, as n increases, q_1 decreases first, then increases and finally decreases again; q_2 increases first and then decreases. We can explain this phenomenon with similar arguments from previous examples.

Figure 5 shows that when we make the difference of quality parameters large enough, factor 1 becomes more significant and finally it dominates factor 2. We see that although there is a difference of initial inventories, there is no decreasing behavior of q_1 at the beginning, which does happen in Figure 4.

Figure 6 shows that when the difference of initial inventories becomes large enough,

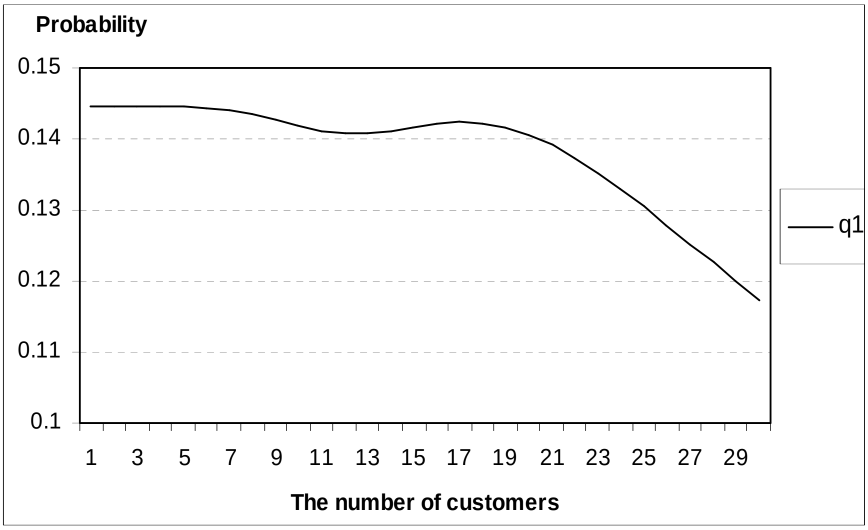

Figure 4: Optimal policy with $\alpha_1 = 1$, $\alpha_2 = 2$, $s_1 = 4$ and $s_2 = 8$.

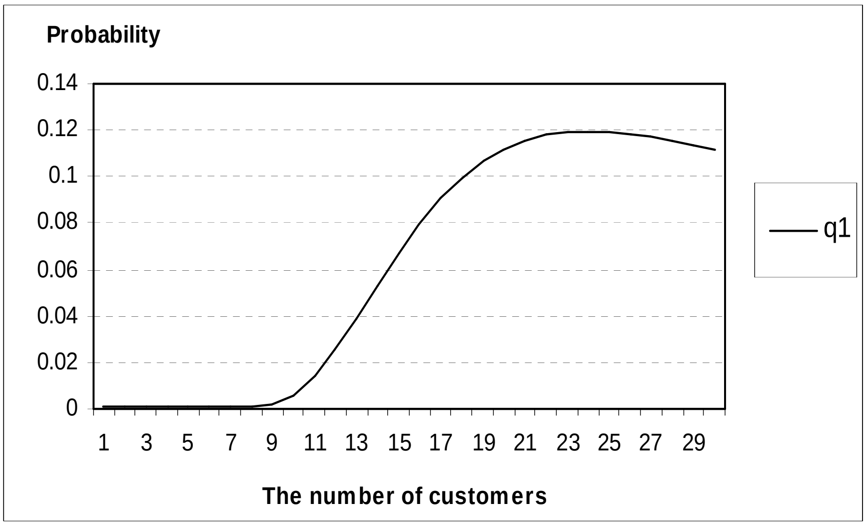

Figure 5: Optimal policy with $\alpha_1 = 1$, $\alpha_2 = 8$, $s_1 = 4$ and $s_2 = 8$.

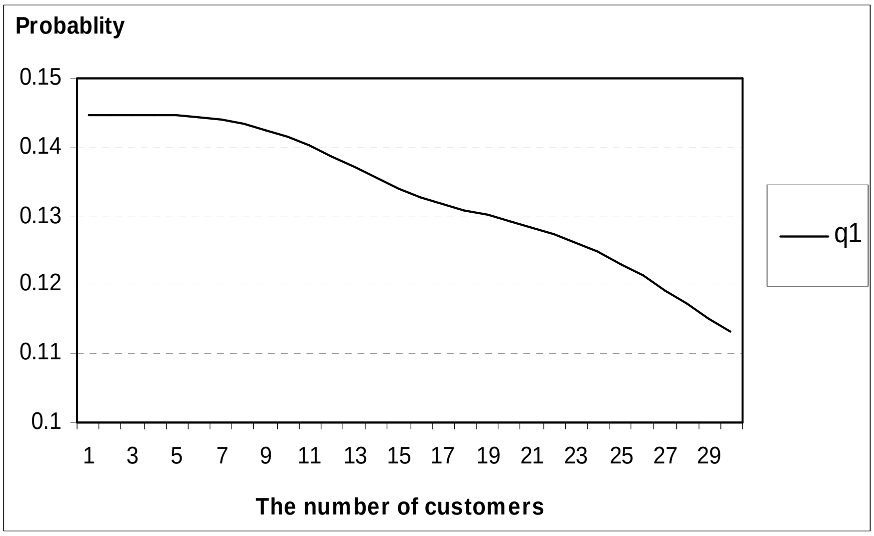

Figure 6: Optimal policy with $\alpha_1 = 1$, $\alpha_2 = 2$, $s_1 = 4$ and $s_2 = 10$.

factor 2 becomes more significant and finally it dominates factor 1. We see that although there is a difference in the quality parameters, there is no increasing behavior of q_1 all the way, which does happen in Figure 4. (Please see Tables 13, 14 and 15 in Appendix for original data.)

Finally, we make the following remark on the structural property of the optimal policies. First, the optimal policy need not to be monotonic in the total number of customers, as we can see from the counterexamples shown. Second, factors 1 and 2 will only be in effect at certain areas. In the long run, the increasing behavior of p_X will finally dominate, which means that q_1 and q_2 will both have a decreasing tail as X keeps stochastically increasing.

Chapter 5

Optimal Policy for General Cases

In this section we study the case when X does not fall into the special cases in Section 3. Specifically, we assume X is unbounded and does not have a geometric tail. Lin (2002) proposes, in his paper, an efficient way to approximate the optimal policy in general cases using the results derived from previous special cases. We bound the optimal policy by following his approach. In most cases, the bounds can be made tighter by exerting computational effort, and the optimal policy can be approximated numerically.

Suppose the seller has s_1 and s_2 items for product 1 and 2 for sale, and the number of customers is distributed as X, with a given probability mass function $P\{X = j\} = p_j$, $j = 0, 1, \ldots$. Choose a positive integer $x_{\max}$ and define X_L by truncating X at $x_{\max}$ with the following probability mass functions,

$$P\{X_L = j\} = \begin{cases} p_j, & j = 0, \ldots, x_{\max} - 1 \\ \sum_{i=x_{\max}}^{\infty} p_i, & j = x_{\max} \\ 0, & j \geq x_{\max} + 1. \end{cases}$$

Since X_L is bounded, we can solve $J(s_1, s_2, p_{X_L})$ and $p(s_1, s_2, p_{X_L})$ using the results in Section 3.3. In addition since $X \geq_{\text{st}} X_L$, Lemma 2.3 implies that

$$J(s_1, s_2, p_X) \geq J(s_1, s_2, p_{X_L}).$$

To get the bound from the other side, we consider the case where X has the property that there exists a positive integer k and $\epsilon > 0$ such that

$$\inf_{j \geq k} r_X(j) \geq \epsilon. \tag{9}$$

That is, we assume the failure rate of X is bounded below when X is large. Note that this is a weaker condition than that X has an increasing failure rate when j is large, and can accommodate most practical problems (for instance, poisson, binomial and negative binomial). Choose $x_{\max} \geq k$ and define

$$r_{\min} \equiv \inf_{j \geq x_{\max}} r_X(j) > 0.$$

Define X_U with the following failure rate function,

$$r_{X_U}(j) = \begin{cases} r_X(j) & j = 0, \ldots, x_{\max} - 1 \\ r_{\min} & j \geq x_{\max}. \end{cases}$$

Since $X_U \geq_{\text{st}} X$, therefore Lemma 2.3 implies that

$$J(s_1, s_2, p_{X_U}) \geq J(s_1, s_2, p_X).$$

Furthermore, $J(s_1, s_2, p_{X_U})$ and $p(s_1, s_2, p_{X_U})$ can be computed since $X_U - x_{\max}|X_U \geq x_{\max}$ has a geometric distribution with parameter (or failure rate) $r_{\min} > 0$.

An example of X_L and X_U is as shown in Table 4.

Table 4: The probability mass function of X_L and the failure rate function of X_U, when $x_{\max} = 5$ and X follows a Poisson distribution with $\lambda = 3$.

i	0	1	2	3	4	5	6	7	$\cdots$
$p_X(i)$	0.0498	0.1494	0.2239	0.2241	0.1681	0.1008	0.0504	0.0216	$\cdots$
$p_{X_L}(i)$	0.0498	0.1494	0.2239	0.2241	0.1681	0.1847			
$r_X(i)$	0.0490	0.1572	0.2789	0.3884	0.4763	0.5457	0.6007	0.6447	$\cdots$
$r_{X_U}(i)$	0.0490	0.1572	0.2789	0.3884	0.4763	0.5457	0.5457	0.5457	$\cdots$

Theorem 5.1 *Assuming Equation (9) holds, then as $x_{\max} \to \infty$, we have that*

1. $J(s_1, s_2, p_{X_U}) - J(s_1, s_2, p_{X_L}) \to 0$.

2. $p_1(s_1, s_2, p_{X_U}) - p_1(s_1, s_2, p_{X_L}) \to 0$.

3. $p_2(s_1, s_2, p_{X_U}) - p_2(s_1, s_2, p_{X_L}) \to 0$.

Proof : Suppose a seller is in state (s_1, s_2, p_{X_L}) and uses the policy as if he were in state (s_1, s_2, p_{X_U}). The difference between $J(s_1, s_2, p_{X_U})$ and the expected total revenue this seller can generate is no more than

$$P\{X \geq x_{\max}\}M(P^*)(\frac{1}{r_{\min}} - 1),$$

because $p_{X_L}(i) = p_X(i) = p_{X_U}(i)$ for $i < x_{\max}$, and conditional on $X_U \geq x_{\max}$, the random variable $X_U - x_{\max}$ follows a geometric distribution with parameter $r_{\min}$. On the other hand, the expected total revenue generated by this seller is a lower bound on $J(s_1, s_2, p_{X_L})$, because by definition $J(s_1, s_2, p_{X_L})$ is the optimal value function in state (s_1, s_2, p_{X_L}). Therefore,

$$J(s_1, s_2, p_{X_U}) - J(s_1, s_2, p_{X_L}) \leq P\{X \geq x_{\max}\}M(P^*)(\frac{1}{r_{\min}} - 1).$$

Because

$$P\{X \geq x_{\max}\}M(P^*)(\frac{1}{r_{\min}} - 1) \to 0,$$

as $x_{\max} \to \infty$, part (a) follows. From the definition of optimal policy, we have

$$\begin{aligned}\mathbf{P}(s_1, s_2, p_{X_L}) = \arg\max_{p_1, p_2} \Big\{ & M(\mathbf{P}) - q_1(J(s_1, s_2, p_{X_L}) - J(s_1 - 1, s_2, p_{X_L})) \\ & -q_2(J(s_1, s_2, p_{X_L}) - J(s_1, s_2 - 1, p_{X_L}))\Big\}\end{aligned}$$

and

$$\begin{aligned}\mathbf{P}(s_1, s_2, p_{X_U}) = \arg\max_{p_1, p_2} \Big\{ & M(\mathbf{P}) - q_1(J(s_1, s_2, p_{X_U}) - J(s_1 - 1, s_2, p_{X_U})) \\ & -q_2(J(s_1, s_2, p_{X_U}) - J(s_1, s_2 - 1, p_{X_U}))\Big\}.\end{aligned}$$

Parts (b) and (c) then follow because from part (a), $J(s_1, s_2, p_{X_U}) - J(s_1, s_2, p_{X_L}) \to 0$, $J(s_1 - 1, s_2, p_{X_U}) - J(s_1 - 1, s_2, p_{X_L}) \to 0$ and $J(s_1, s_2 - 1, p_{X_U}) - J(s_1, s_2 - 1, p_{X_U}) \to 0$, as $x_{\max} \to \infty$.

The proof is completed. (See Tables 5 and 6 for example.) $\square$

Theorem 5.2 *As $x_{\max} \to \infty$, we have that*

1. $J(s_1, s_2, p_X) - J(s_1, s_2, p_{X_L}) \to 0.$

2. $J(s_1, s_2, p_{X_U}) - J(s_1, s_2, p_X) \to 0.$

3. $p_i(s_1, s_2, p_{X_U}) - p_i(s_1, s_2, p_X) \to 0 \qquad i = 1, 2.$

4. $p_i(s_1, s_2, p_X) - p_i(s_1, s_2, p_{X_L}) \to 0 \qquad i = 1, 2.$

Proof : Because of

$$J(s_1, s_2, p_{X_L}) \leq J(s_1, s_2, p_X) \leq J(s_1, s_2, p_{X_U}),$$

and Theorem 5.1, parts (a) and (b) hold.

From the definition of an optimal policy, we have

$$\begin{aligned}\mathbf{P}(s_1, s_2, p_X) = \arg\max_{p_1, p_2} \Big\{ & M(\mathbf{P}) - q_1(J(s_1, s_2, p_X) - J(s_1 - 1, s_2, p_X)) \\ & - q_2(J(s_1, s_2, p_X) - J(s_1, s_2 - 1, p_X)) \Big\}.\end{aligned}$$

Parts (c) and (d) then follow because from parts (a) and (b), $J(s_1, s_2, p_{X_U}) - J(s_1, s_2, p_X) \to 0$, $J(s_1, s_2, p_X) - J(s_1, s_2, p_{X_L}) \to 0$, $J(s_1 - 1, s_2, p_{X_U}) - J(s_1 - 1, s_2, p_X) \to 0$, $J(s_1 - 1, s_2, p_X) - J(s_1 - 1, s_2, p_{X_L}) \to 0$, $J(s_1, s_2 - 1, p_{X_U}) - J(s_1, s_2 - 1, p_X) \to 0$ and $J(s_1, s_2 - 1, p_X) - J(s_1, s_2 - 1, p_{X_U}) \to 0$, as $x_{\max} \to \infty$.

The proof is completed. $\square$

Table 5: Optimal Revenue and Policy when the number of customers follows a Poisson distribution ($\alpha_1 = 1$, $\alpha_2 = 2$, $\beta = 1$, $\lambda = 20$).

$x_{\max}$	$J(5, 10, p_{X_L})$	$J(5, 10, p_{X_U})$	$q_1(5, 10, p_{X_L})$	$q_1(5, 10, p_{X_U})$	$q_2(5, 10, p_{X_L})$	$q_2(5, 10, p_{X_U})$
1	2.32520302	205.38433525	0.14458	0.00000	0.39301	0.00001
2	3.48780448	173.30223982	0.14458	0.00002	0.39301	0.00005
3	4.65040546	147.19317325	0.14458	0.00013	0.39301	0.00027
4	5.81300325	125.34502738	0.14458	0.00056	0.39301	0.00116
5	6.97557552	106.81450622	0.14457	0.00187	0.39302	0.00395
6	8.13804444	91.00383875	0.14454	0.00521	0.39304	0.01108
7	9.30020256	77.50739928	0.14447	0.01218	0.39310	0.02615
8	10.46156385	66.04380099	0.14432	0.02417	0.39321	0.05240
9	11.62109384	56.41343541	0.14407	0.04111	0.39338	0.09014
10	12.77674307	48.45893315	0.14372	0.06095	0.39363	0.13532
11	13.92476709	42.03012004	0.14327	0.08068	0.39389	0.18164
12	15.05895550	36.96316910	0.14280	0.09796	0.39403	0.22380
13	16.17002511	33.07599709	0.14237	0.11169	0.39388	0.25910
14	17.24547592	30.17517694	0.14209	0.12186	0.39324	0.28702
15	18.27012095	28.06857293	0.14203	0.12903	0.39198	0.30828
16	19.22732738	26.57834952	0.14220	0.13389	0.39003	0.32404
17	20.10081993	25.55036004	0.14256	0.13712	0.38743	0.33549
18	20.87673767	24.85836327	0.14304	0.13923	0.38432	0.34366
19	21.54553262	24.40373809	0.14352	0.14062	0.38088	0.34939
20	22.10329735	24.11238239	0.14394	0.14154	0.37732	0.35332
21	22.55222999	23.93042270	0.14424	0.14216	0.37387	0.35595
22	22.90015792	23.81983211	0.14439	0.14258	0.37069	0.35767
23	23.15926790	23.75452036	0.14440	0.14286	0.36793	0.35876
24	23.34435643	23.71709762	0.14431	0.14305	0.36566	0.35943
25	23.47097944	23.69632214	0.14415	0.14318	0.36389	0.35983
26	23.55384496	23.68516027	0.14398	0.14326	0.36260	0.36006
27	23.60568155	23.67936177	0.14381	0.14331	0.36170	0.36018
28	23.63666825	23.67645078	0.14367	0.14333	0.36111	0.36025
29	23.65437203	23.67503891	0.14356	0.14335	0.36075	0.36028
30	23.66404509	23.67437734	0.14348	0.14336	0.36054	0.36029
31	23.66910412	23.67407777	0.14343	0.14336	0.36042	0.36030
32	23.67163972	23.67394664	0.14340	0.14337	0.36036	0.36031
33	23.67285919	23.67389111	0.14339	0.14337	0.36033	0.36031
34	23.67342276	23.67386834	0.14338	0.14337	0.36032	0.36031
35	23.67367339	23.67385931	0.14337	0.14337	0.36031	0.36031
36	23.67378079	23.67385583	0.14337	0.14337	0.36031	0.36031
37	23.67382520	23.67385453	0.14337	0.14337	0.36031	0.36031
38	23.67384294	23.67385405	0.14337	0.14337	0.36031	0.36031
39	23.67384980	23.67385389	0.14337	0.14337	0.36031	0.36031
40	23.67385237	23.67385383	0.14337	0.14337	0.36031	0.36031

Table 6: Optimal Revenue and Policy when the number of customers follows a Poisson distribution ($\alpha_1 = 1$, $\alpha_2 = 2$, $\beta = 1$, $\lambda = 20$) (continued).

$p_1(5, 10, p_{X_L})$	$p_1(5, 10, p_{X_U})$	$p_2(5, 10, p_{X_L})$	$p_2(5, 10, p_{X_U})$
2.16260	13.79803	2.16260	14.07498
2.16260	11.66867	2.16260	11.94062
2.16260	9.93863	2.16260	10.20507
2.16260	8.49440	2.16260	8.75468
2.16261	7.27425	2.16259	7.52756
2.16289	6.24033	2.16255	6.48574
2.16347	5.36848	2.16245	5.60494
2.16459	4.64277	2.16227	4.86910
2.16643	4.05075	2.16196	4.26571
2.16912	3.57929	2.16156	3.78161
2.17262	3.21295	2.16131	3.40147
2.17668	2.93496	2.16167	3.10873
2.18090	2.72875	2.16330	2.88726
2.18482	2.57911	2.16689	2.72248
2.18812	2.47271	2.17296	2.60175
2.19074	2.39835	2.18177	2.51453
2.19297	2.34709	2.19320	2.45234
2.19525	2.31209	2.20686	2.40858
2.19806	2.28831	2.22208	2.37821
2.20172	2.27224	2.23804	2.35748
2.20626	2.26143	2.25383	2.34358
2.21150	2.25422	2.26863	2.33449
2.21706	2.24947	2.28176	2.32868
2.22252	2.24641	2.29278	2.32509
2.22749	2.24449	2.30152	2.32294
2.23171	2.24332	2.30807	2.32170
2.23504	2.24263	2.31271	2.32101
2.23750	2.24224	2.31581	2.32064
2.23921	2.24203	2.31776	2.32045
2.24032	2.24191	2.31892	2.32035
2.24099	2.24186	2.31958	2.32031
2.24138	2.24183	2.31993	2.32029
2.24160	2.24182	2.32011	2.32028
2.24171	2.24181	2.32020	2.32027
2.24176	2.24181	2.32024	2.32027
2.24179	2.24181	2.32026	2.32027
2.24180	2.24181	2.32026	2.32027
2.24181	2.24181	2.32027	2.32027
2.24181	2.24181	2.32027	2.32027
2.24181	2.24181	2.32027	2.32027

There are some cases where the assumption in Equation (9) is not satisfied. For such special cases, we can often define X_U by considering the characteristics of X, so that the upper-bound of $J(s_1, s_2, p_X)$ and the upper-bound of $p(s_1, s_2, p_X)$ can be computed accordingly. Note that as $x_{\max}$ increases, X_L increases and X_U decreases, both in the regular stochastic sense. Therefore, the bounds can be made tighter by choosing a larger value of $x_{\max}$, and the optimal policy can be approximated numerically. In general, the result is satisfactory if $x_{\max} \gg s$ and $P\{X > x_{\max}\} \approx 0$.

Chapter 6

Applications to Continuous-Time Models

The generic model proposed has many applications in continuous-time dynamic pricing models where the customers arrive according to a general point process. In this section, we consider a continuous-time model where customers arrive according to a Poisson process for the ease of illustration. However, the methods proposed in this section are applicable to many other demand arrival processes including nonhomogeneous Poisson process, and the stochastic processes that do not have independent increments.

We first show a heuristic policy that is easy to compute and implement for the continuous-time model, and then an upper-bound for the optimal expected revenue in the continuous-time model. Finally, we give a numerical example.

6.1 Heuristic and Upper Bound

Consider a seller that sells s_1, s_2 items in the time interval $[0, T]$. The customers arrive in accordance with a Poisson process with rate λ, while the interaction between the seller and customers is the same as in the generic model.

Note that at time 0, the total number of customers in [0, T] follows a Poisson distribution with mean λT. Let N_m denote a Poisson random variable with mean m, then a reasonable policy is to use $\mathbf{P}(i, j, N_{\lambda T} - k | N_{\lambda T} \geq k)$ if the inventory is (i, j) when the k^{th} customer shows up. This policy yields the expected total revenue equal to $J(s_1, s_2, N_{\lambda T})$, and is called the *arrival-order-based* policy (AOB) (see Lin (2002)), because it sets the product price based on the order each customer arrives without considering the arrival time of each customer.

Consider that at any time $t \in [0, T]$, the additional number of customers that will show up in the time interval $(t, T]$ has a Poisson distribution with mean $\lambda(T - t)$. Therefore, when a customer shows up at time t, with (s_1, s_2) in inventory, we use the policy $\mathbf{P}(s_1, s_2, N_{\lambda(T-t)})$, and it is called the *future-distribution-based* heuristic (FDB) (see Lin (2002)).

For a problem with a large value of s_1, or s_2, or λT, real-time implementation of the FDB policy may raise computational issues. In this case, we could construct a three-dimensional table of $\mathbf{P}(s_1, s_2, N_m)$ for different s_1, s_2 and m. The FDB policies can then be quoted from this table and be computed using linear interpolation when necessary. (See Table 8 as a part of the three-dimensional table of $\mathbf{P}(s_1, s_2, N_m)$.)

An upper-bound on the optimal expected total revenue in the continuous-time model can be derived by supposing that the seller, at time 0, has the ability to know the exact number of future customers by time T. Certainly the seller has nothing to lose with this ability, and therefore the optimal expected total revenue with this ability is an upper-bound of that in the original problem. By conditioning on the number of customers and using

Table 7: Optimal policy of $\mathbf{P}(s_1, s_2, N_{40})$ where N_{40} represents a Poisson random number of customers with mean = 40.

s_1	s_2	$J(s_1, s_2, N_{40})$	q_1	q_2	p_1	p_2
0	0	0.00000	0.00000	0.00000	∞	∞
0	1	4.55349	0.00000	0.02783	∞	5.55349
0	2	8.36651	0.00000	0.05511	∞	4.84164
0	3	11.72562	0.00000	0.08185	∞	4.41744
0	4	14.74751	0.00000	0.10803	∞	4.11104
0	5	17.49564	0.00000	0.13363	∞	3.86924
0	6	20.00978	0.00000	0.15864	∞	3.66838
0	7	22.31711	0.00000	0.18304	∞	3.49589
0	8	24.43736	0.00000	0.20681	∞	3.34429
1	0	3.60588	0.02645	0.00000	4.60588	∞
1	1	8.11005	0.02624	0.02769	4.58518	5.53133
1	2	11.87244	0.02603	0.05482	4.56426	4.81939
1	3	15.17959	0.02582	0.08139	4.54312	4.39511
1	4	18.14818	0.02562	0.10738	4.52178	4.08867
1	5	20.84166	0.02541	0.13278	4.50026	3.84688
1	6	23.29982	0.02521	0.15757	4.47859	3.64607
1	7	25.54985	0.02501	0.18172	4.45679	3.47369
1	8	27.61152	0.02482	0.20522	4.43489	3.32227
2	0	6.47774	0.05220	0.00000	3.89902	∞
2	1	10.93164	0.05176	0.02755	3.87859	5.50898
2	2	14.64245	0.05131	0.05454	3.85797	4.79695
2	3	17.89669	0.05087	0.08095	3.83718	4.37262
2	4	20.81103	0.05043	0.10676	3.81624	4.06617
2	5	23.44892	0.04999	0.13196	3.79518	3.82440
2	6	25.85019	0.04955	0.15654	3.77403	3.62368
2	7	28.04205	0.04911	0.18046	3.75281	3.45145
2	8	30.04432	0.04868	0.20370	3.73156	3.30025
3	0	8.90296	0.07725	0.00000	3.48030	∞
3	1	13.30571	0.07654	0.02743	3.46021	5.48647
3	2	16.96405	0.07582	0.05427	3.43999	4.77437
3	3	20.16452	0.07511	0.08053	3.41966	4.35002
3	4	23.02377	0.07439	0.10617	3.39925	4.04357

Table 8: Optimal policy of $\mathbf{P}(s_1, s_2, N_{40})$ where N_{40} represents a Poisson random number of customers with mean = 40 (continued).

s_1	s_2	$J(s_1, s_2, N_{40})$	q_1	q_2	p_1	p_2
3	5	25.60528	0.07368	0.13119	3.37878	3.80186
3	6	27.94890	0.07297	0.15556	3.35829	3.60127
3	7	30.08190	0.07226	0.17926	3.33782	3.42922
3	8	32.02416	0.07155	0.20225	3.31742	3.27830
4	0	10.99917	0.10157	0.00000	3.17993	∞
4	1	15.34997	0.10055	0.02731	3.16029	5.46385
4	2	18.95508	0.09953	0.05402	3.14058	4.75171
4	3	22.10104	0.09851	0.08013	3.12084	4.32736
4	4	24.90451	0.09748	0.10561	3.10109	4.02095
4	5	27.42900	0.09645	0.13046	3.08137	3.77935
4	6	29.71441	0.09542	0.15464	3.06173	3.57891
4	7	31.78809	0.09438	0.17813	3.04222	3.40711
4	8	33.67001	0.09335	0.20089	3.02288	3.25651
5	0	12.83092	0.12512	0.00000	2.94481	∞
5	1	17.12910	0.12376	0.02719	2.92573	5.44119
5	2	20.68035	0.12240	0.05378	2.90667	4.72904
5	3	23.77121	0.12102	0.07975	2.88766	4.30472
5	4	26.51839	0.11964	0.10509	2.86874	3.99840
5	5	28.98546	0.11824	0.12977	2.84997	3.75694
5	6	31.21237	0.11684	0.15377	2.83139	3.55672
5	7	33.22657	0.11543	0.17707	2.81307	3.38522
5	8	35.04816	0.11402	0.19962	2.79507	3.23501
6	0	14.43928	0.14787	0.00000	2.75137	∞
6	1	18.68430	0.14614	0.02708	2.73299	5.41856
6	2	22.18122	0.14438	0.05356	2.71473	4.70644
6	3	25.21662	0.14261	0.07940	2.69663	4.28219
6	4	27.90727	0.14081	0.10460	2.67874	3.97601
6	5	30.31678	0.13900	0.12913	2.66114	3.73475
6	6	32.48522	0.13717	0.15297	2.64387	3.53481
6	7	34.44017	0.13532	0.17609	2.62702	3.36367
6	8	36.20187	0.13346	0.19844	2.61066	3.21394

$R(s_1, s_2, n)$ defined in Section 3.1, this upper-bound is equal to

$$\sum_{k=0}^{\infty} R(s_1, s_2, n)\; e^{\lambda T}\; \frac{(\lambda T)^k}{k!}. \tag{10}$$

6.2 A Numerical Example

This section presents a numerical example where $\alpha_1 = 1$, $\alpha_2 = 2$, $\beta = 1$ and $\lambda = 1$ person/min. We compare the optimal revenues in Table 9 with different values of s_1, s_2 and T, as well as the upper bound in Equation (10). As shown in Table 9, the expected revenue is about 98% of the upper bound, and it is in turn very close to the optimal expected revenue. Also we can see that, in the case where the number of customers is not relatively large compared with the inventory, the myopic policy is almost as good as the optimal policy, but it works poorly when the number of customers is much larger than the inventory. While the FDB heuristic works well enough all the time.

Table 9: Expected Revenues for Poisson Arrival Processes.

T	s_1	s_2	Myopic[a]/U.B.	AOB/U.B.	FDB/U.B.	FDB[b]	Upper Bound
40	6	8	0.7629	0.9763	0.9833	35.9589	36.5707
40	8	6	0.7517	0.9778	0.9791	34.0357	34.7600
40	3	4	0.5985	0.9743	0.9843	23.0393	23.4057
40	4	3	0.5934	0.9749	0.9856	22.1238	22.4475
20	6	8	0.9732	0.9824	0.9836	21.9671	22.3239
20	8	6	0.9341	0.9869	0.9884	21.1107	21.3584
20	3	4	0.7861	0.9717	0.9844	16.9378	17.2068
20	4	3	0.7759	0.9731	0.9885	16.1654	16.3538

[a] In this case, it is that q_1=0.1446, q_2=0.3930.

[b] Simulation results with 99.75% confidence interval ± 0.1.

Chapter 7

Conclusions

In this thesis, we propose a dynamic pricing model where a seller offers two models and the customers arrive sequentially. The seller adjusts the price for each customer in order to maximize the expected total revenue. We find some structural properties of the optimal policy, and develop algorithms to compute it when the distribution of the number of customers is deterministic, geometric, or bounded. In the general case, we can approximate the optimal policy. This model has many applications in continuous-time dynamic pricing problems where the customers arrive according to a general point process. Upon the arrival of each customer, the seller can first identify the distribution of the number of future customers and then use the *future-distribution-based* heuristic, which considers only the distribution of the number of future customers, but not the functional forms of their arrival times. The model also gives an upper bound for various dynamic pricing problems.

Finally, we present some insights into the case of three or more substitutable products. The phenomenon of *cannibalization* in this case will become more complicated for each pair of the different products will compete with each other.

Without loss of generality, we assume that the seller has n different models of a generic product type and any of them is substitutable for the others. Let $J(\cdot, \cdot, \cdots, \cdot)$, $p(\cdot, \cdot, \cdots, \cdot)$ denote the optimal expected total revenue and the optimal policy respectively. Since it can never hurt to have more items, or more customers, we summarize the monotonicity of the optimal value function $J(\cdot, \cdot, \cdots, \cdot)$ in the following lemma.

Lemma 7.1 *$J(s_1, s_2, \cdots, s_n, p_X)$ increases in any of $s_1, s_2, \cdots, s_n$, and X in the regular stochastic sense, that is, if $Y \geq_{\text{st}} X$, i.e., $P\{Y > i\} \geq P\{X > i\}$ for all $i \geq 0$, then $J(s_1, s_2, \cdots, s_n, p_Y) \geq J(s_1, s_2, \cdots, s_n, p_X)$.*

Proof: Same approach as in Lemma 2.2 and 2.3. $\square$

For $n \geq 2$, there are cases that optimal policies have no monotonicity property when X stochastically increases.

We give our explanation as follows. For the multiproduct case, MNL model factors in not only the difference of product quality parameters, but also the difference of the initial inventories, which affects optimal policies jointly as X stochastically increases. As a result, optimal policies show significantly different behaviors in distinct cases. Furthermore, we can expect that the effect of the difference of product quality and initial inventories will disappear when X becomes stochastically large enough. In the long run, the probability of selling any given model will decrease because there are relatively enough future customers compared with the total inventory of products . When initial inventories are fixed, optimal policies have monotonicity tails as X becomes stochastically large enough. When X is stochastically large enough compared with the total initial inventories, it makes sense that the optimal prices for each product will all keep increasing when we still keep stochastically increasing X.

References

Aydin, G. and Ryan, J.K.. Product line selection and pricing with stochastic utility. *Manuscript*, 1999.

Bitran, G. R. and Mondschein, S. V.. Periodic pricing of seasonal products in retailing. *Management Science*, **43**(1), 64-79, 1997.

Brumelle, S. L. and McGill, J.I.. Airline seat allocation with multiple nested fare classes. *Operations Research*, **41**(1), 127-137, 1993.

Dobson, G. and Kalish, S.. Positioning and pricing a product line. *Marketing Science*, **7**, 107-125, 1988.

Feng, Y. and Gallego, G.. Optimal stopping times for end of season sales and optimal stopping times for promotional fares. *Management Science*, **41**, 1371-1391, 1995.

Feng, Y. and Xiao, B.. A continuous-time yield management model with multiple prices and reversible price changes. *Management Science*, **46**(5), 644-657, 2000.

Gallego, G. and van Ryzin, G.. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Science*, **40**(8), 999-1020, 1994.

Gallego, G. and van Ryzin, G.. A multiproduct dynamic pricing problem and its applications to network yield management. *Operations Research*, **45**(1), 24-41, 1997.

Kincaid, W.M. and D. Darling. An inventory pricing problem. *J. Math. Anal. Appl.*, **7**, 183-208, 1963.

Lin, K. Y.. A sequential dynamic pricing model and its applications. *Manuscript*, 2002.

Littlewood, K.. Forecasting and control of passengers. 12*th AGIFORS Sympos. Proc.*, 103-105, 1972.

Mahajan, S. and van Ryzin, G.. *Quantitative models for supply chain management*, chapter 17, 505-506, 2000.

McGill, J. I. and van Ryzin, G. J.. Revenue management: research overview and prospects. *Transportation Science*, **33**(2), 233-256, 1999.

Ross, Sheldon M.. *Introduction to stochastic dynamic programming*, *Academic Press*, 1983.

Talluri, K and van Ryzin, G.. A discrete choice model of yield management. *Manuscript*, 2001.

Weatherford, L. R. and Bodily, S.E.. A taxonomy and research overview of perishable-asset revenue management: yield management, overbooking and pricing. *Operations Research*, **40**, 831-844, 1992.

Zhao, W. and Zheng, Y.-S.. Optimal dynamic pricing for perishable assets with nonhomogeneous demand. *Management Science*, **46**(3), 375-388, 2000.

Appendix

Table 10: Optimal Revenue and Policy when the number of customers is deterministic ($\alpha_1 = 2$, $\alpha_2 = 2$, $\beta = 1$, $s_1 = 5$, $s_2 = 5$).

n	$R(s_1, s_2, n)$	q_1	q_2	p_1	p_2
1	1.3748	0.2895	0.2895	2.3748	2.3748
2	2.7496	0.2895	0.2895	2.3748	2.3748
3	4.1245	0.2895	0.2895	2.3748	2.3748
4	5.4993	0.2895	0.2895	2.3748	2.3748
5	6.8741	0.2895	0.2895	2.3748	2.3748
6	8.2476	0.2893	0.2893	2.3758	2.3758
7	9.6168	0.2890	0.2890	2.3789	2.3789
8	10.9775	0.2882	0.2882	2.3852	2.3852
9	12.3247	0.2870	0.2870	2.3951	2.3951
10	13.6531	0.2853	0.2853	2.4091	2.4091
11	14.9560	0.2829	0.2829	2.4286	2.4286
12	16.2242	0.2796	0.2796	2.4555	2.4555
13	17.4479	0.2751	0.2751	2.4913	2.4913
14	18.6183	0.2696	0.2696	2.5358	2.5358
15	19.7295	0.2632	0.2632	2.5877	2.5877
16	20.7789	0.2560	0.2560	2.6449	2.6449
17	21.7665	0.2484	0.2484	2.7057	2.7057
18	22.6942	0.2406	0.2406	2.7681	2.7681
19	23.5654	0.2328	0.2328	2.8311	2.8311
20	24.3838	0.2250	0.2250	2.8936	2.8936
21	25.1535	0.2175	0.2175	2.9549	2.9549
22	25.8785	0.2102	0.2102	3.0147	3.0147
23	26.5627	0.2031	0.2031	3.0727	3.0727
24	27.2096	0.1964	0.1964	3.1287	3.1287
25	27.8224	0.1900	0.1900	3.1828	3.1828
26	28.4041	0.1839	0.1839	3.2350	3.2350
27	28.9573	0.1781	0.1781	3.2852	3.2852
28	29.4843	0.1726	0.1726	3.3336	3.3336
29	29.9874	0.1673	0.1673	3.3802	3.3802
30	30.4684	0.1624	0.1624	3.4251	3.4251

Table 11: Optimal Revenue and Policy when the number of customers is deterministic ($\alpha_1 = 1$, $\alpha_2 = 8$, $\beta = 1$, $s_1 = 5$, $s_2 = 5$).

n	$R(s_1, s_2, n)$	q_1	q_2	p_1	p_2
1	5.3279	0.0008	0.8412	6.3279	6.3279
2	10.6559	0.0008	0.8412	6.3279	6.3279
3	15.9838	0.0008	0.8412	6.3279	6.3279
4	21.3118	0.0008	0.8412	6.3279	6.3279
5	26.6397	0.0008	0.8412	6.3279	6.3279
6	30.4888	0.0044	0.7894	4.8491	6.6577
7	33.3012	0.0158	0.7219	3.8124	6.9876
8	35.5059	0.0344	0.6536	3.2047	7.2607
9	37.3408	0.0563	0.5910	2.8349	7.4840
10	38.9349	0.0783	0.5362	2.5941	7.6700
11	40.3617	0.0988	0.4891	2.4280	7.8286
12	41.6653	0.1172	0.4487	2.3098	7.9669
13	42.8725	0.1330	0.4140	2.2259	8.0902
14	43.9999	0.1460	0.3839	2.1689	8.2024
15	45.0579	0.1564	0.3577	2.1339	8.3062
16	46.0532	0.1640	0.3348	2.1169	8.4034
17	46.9908	0.1693	0.3146	2.1147	8.4950
18	47.8747	0.1725	0.2967	2.1239	8.5818
19	48.7086	0.1740	0.2807	2.1421	8.6642
20	49.4958	0.1742	0.2663	2.1670	8.7425
21	50.2398	0.1733	0.2533	2.1967	8.8169
22	50.9437	0.1716	0.2415	2.2298	8.8878
23	51.6106	0.1693	0.2308	2.2652	8.9552
24	52.2433	0.1666	0.2210	2.3021	9.0195
25	52.8446	0.1636	0.2119	2.3398	9.0807
26	53.4168	0.1604	0.2036	2.3778	9.1392
27	53.9622	0.1571	0.1959	2.4159	9.1950
28	54.4829	0.1537	0.1887	2.4537	9.2483
29	54.9808	0.1503	0.1821	2.4910	9.2994
30	55.4575	0.1470	0.1758	2.5278	9.3484

Table 12: Optimal Revenue and Policy when the number of customers is deterministic ($\alpha_1 = 2$, $\alpha_2 = 2$, $\beta = 1$, $s_1 = 3$, $s_2 = 8$).

n	$R(s_1, s_2, n)$	q_1	q_2	p_1	p_2
1	1.3748	0.2895	0.2895	2.3748	2.3748
2	2.7496	0.2895	0.2895	2.3748	2.3748
3	4.1245	0.2895	0.2895	2.3748	2.3748
4	5.4915	0.2848	0.2927	2.3942	2.3670
5	6.8436	0.2759	0.2989	2.4323	2.3522
6	8.1769	0.2643	0.3071	2.4834	2.3333
7	9.4897	0.2514	0.3162	2.5423	2.3128
8	10.7822	0.2382	0.3256	2.6051	2.2925
9	12.0552	0.2253	0.3348	2.6692	2.2732
10	13.3102	0.2131	0.3435	2.7330	2.2555
11	14.5481	0.2017	0.3514	2.7953	2.2403
12	15.7684	0.1914	0.3582	2.8558	2.2289
13	16.9684	0.1821	0.3634	2.9150	2.2239
14	18.1423	0.1737	0.3663	2.9740	2.2277
15	19.2832	0.1661	0.3668	3.0340	2.2417
16	20.3841	0.1592	0.3649	3.0954	2.2658
17	21.4399	0.1528	0.3608	3.1581	2.2989
18	22.4470	0.1469	0.3549	3.2216	2.3392
19	23.4043	0.1413	0.3478	3.2852	2.3846
20	24.3120	0.1361	0.3397	3.3485	2.4336
21	25.1719	0.1312	0.3312	3.4108	2.4847
22	25.9861	0.1265	0.3223	3.4717	2.5367
23	26.7573	0.1221	0.3133	3.5310	2.5889
24	27.4885	0.1180	0.3044	3.5885	2.6407
25	28.1825	0.1141	0.2956	3.6440	2.6916
26	28.8420	0.1103	0.2871	3.6977	2.7415
27	29.4697	0.1068	0.2788	3.7494	2.7901
28	30.0680	0.1035	0.2708	3.7993	2.8373
29	30.6391	0.1003	0.2632	3.8473	2.8832
30	31.1851	0.0974	0.2558	3.8936	2.9276

Table 13: Optimal Revenue and Policy when the number of customers is deterministic ($\alpha_1 = 1$, $\alpha_2 = 2$, $\beta = 1$, $s_1 = 4$, $s_2 = 8$).

n	$R(s_1, s_2, n)$	q_1	q_2	p_1	p_2
1	1.1626	0.1446	0.3930	2.1626	2.1626
2	2.3252	0.1446	0.3930	2.1626	2.1626
3	3.4878	0.1446	0.3930	2.1626	2.1626
4	4.6504	0.1446	0.3930	2.1626	2.1626
5	5.8129	0.1445	0.3931	2.1630	2.1625
6	6.9753	0.1444	0.3932	2.1643	2.1623
7	8.1371	0.1440	0.3934	2.1669	2.1619
8	9.2983	0.1434	0.3938	2.1712	2.1611
9	10.4581	0.1427	0.3943	2.1772	2.1605
10	11.6153	0.1419	0.3946	2.1841	2.1611
11	12.7677	0.1412	0.3942	2.1912	2.1643
12	13.9119	0.1408	0.3928	2.1975	2.1716
13	15.0441	0.1408	0.3902	2.2030	2.1840
14	16.1597	0.1412	0.3862	2.2084	2.2022
15	17.2539	0.1417	0.3808	2.2150	2.2263
16	18.3219	0.1421	0.3743	2.2245	2.2562
17	19.3590	0.1423	0.3668	2.2380	2.2915
18	20.3617	0.1422	0.3585	2.2561	2.3313
19	21.3274	0.1416	0.3497	2.2789	2.3749
20	22.2544	0.1406	0.3405	2.3061	2.4213
21	23.1423	0.1391	0.3312	2.3370	2.4696
22	23.9913	0.1373	0.3218	2.3708	2.5190
23	24.8024	0.1352	0.3126	2.4068	2.5689
24	25.5769	0.1329	0.3035	2.4444	2.6187
25	26.3165	0.1305	0.2947	2.4830	2.6681
26	27.0231	0.1279	0.2861	2.5221	2.7167
27	27.6986	0.1252	0.2779	2.5614	2.7644
28	28.3447	0.1226	0.2700	2.6005	2.8110
29	28.9635	0.1199	0.2623	2.6394	2.8565
30	29.5566	0.1173	0.2550	2.6777	2.9007

Table 14: Optimal Revenue and Policy when the number of customers is deterministic ($\alpha_1 = 1$, $\alpha_2 = 8$, $\beta = 1$, $s_1 = 4$, $s_2 = 8$).

n	$R(s_1, s_2, n)$	q_1	q_2	p_1	p_2
1	5.3279	0.0008	0.8412	6.3279	6.3279
2	10.6559	0.0008	0.8412	6.3279	6.3279
3	15.9838	0.0008	0.8412	6.3279	6.3279
4	21.3118	0.0008	0.8412	6.3279	6.3279
5	26.6397	0.0008	0.8412	6.3279	6.3279
6	31.9677	0.0008	0.8412	6.3279	6.3279
7	37.2956	0.0008	0.8412	6.3279	6.3279
8	42.6236	0.0008	0.8412	6.3279	6.3279
9	47.1179	0.0020	0.8160	5.4943	6.4997
10	50.7098	0.0060	0.7762	4.5918	6.7290
11	53.6246	0.0138	0.7307	3.9149	6.9489
12	56.0686	0.0252	0.6844	3.4440	7.1425
13	58.1824	0.0387	0.6401	3.1152	7.3103
14	60.0563	0.0530	0.5990	2.8814	7.4568
15	61.7481	0.0669	0.5616	2.7138	7.5868
16	63.2948	0.0797	0.5277	2.5953	7.7044
17	64.7210	0.0906	0.4972	2.5145	7.8125
18	66.0433	0.0997	0.4697	2.4632	7.9131
19	67.2740	0.1067	0.4450	2.4351	8.0075
20	68.4225	0.1120	0.4226	2.4247	8.0965
21	69.4969	0.1156	0.4023	2.4279	8.1809
22	70.5040	0.1179	0.3839	2.4412	8.2608
23	71.4499	0.1191	0.3670	2.4620	8.3366
24	72.3403	0.1195	0.3515	2.4881	8.4087
25	73.1800	0.1191	0.3373	2.5179	8.4771
26	73.9736	0.1183	0.3242	2.5504	8.5422
27	74.7252	0.1171	0.3120	2.5845	8.6042
28	75.4383	0.1156	0.3007	2.6196	8.6633
29	76.1163	0.1139	0.2902	2.6553	8.7197
30	76.7620	0.1120	0.2804	2.6910	8.7735

Table 15: Optimal Revenue and Policy when the number of customers is deterministic ($\alpha_1 = 1$, $\alpha_2 = 2$, $\beta = 1$, $s_1 = 4$, $s_2 = 10$).

n	$R(s_1, s_2, n)$	q_1	q_2	p_1	p_2
1	1.1626	0.1446	0.3930	2.1626	2.1626
2	2.3252	0.1446	0.3930	2.1626	2.1626
3	3.4878	0.1446	0.3930	2.1626	2.1626
4	4.6504	0.1446	0.3930	2.1626	2.1626
5	5.8129	0.1445	0.3931	2.1630	2.1625
6	6.9753	0.1444	0.3932	2.1643	2.1623
7	8.1371	0.1440	0.3934	2.1669	2.1619
8	9.2983	0.1434	0.3938	2.1712	2.1611
9	10.4584	0.1426	0.3944	2.1775	2.1601
10	11.6171	0.1415	0.3952	2.1857	2.1588
11	12.7742	0.1402	0.3962	2.1958	2.1571
12	13.9290	0.1387	0.3972	2.2076	2.1556
13	15.0809	0.1371	0.3982	2.2206	2.1546
14	16.2284	0.1355	0.3988	2.2342	2.1549
15	17.3697	0.1341	0.3989	2.2478	2.1576
16	18.5022	0.1328	0.3982	2.2613	2.1635
17	19.6224	0.1318	0.3965	2.2749	2.1734
18	20.7264	0.1309	0.3938	2.2892	2.1881
19	21.8100	0.1301	0.3899	2.3050	2.2077
20	22.8691	0.1293	0.3850	2.3231	2.2323
21	23.8999	0.1284	0.3792	2.3439	2.2613
22	24.8996	0.1274	0.3726	2.3677	2.2944
23	25.8660	0.1261	0.3653	2.3944	2.3307
24	26.7979	0.1247	0.3577	2.4236	2.3695
25	27.6949	0.1230	0.3498	2.4550	2.4101
26	28.5571	0.1213	0.3417	2.4881	2.4519
27	29.3852	0.1193	0.3336	2.5225	2.4944
28	30.1802	0.1173	0.3256	2.5578	2.5371
29	30.9435	0.1152	0.3176	2.5936	2.5797
30	31.6764	0.1131	0.3098	2.6296	2.6219

Vita

Feng Li was born on March 5, 1977 in Handan, Hebei, China. He graduated from Dalian University of Technology in July 1999, with his B.S. degree in Chemical Engineering and Machinery. Two years later he started to pursue a master degree in Operations Research in Grado Department of the Industrial and Systems Engineering at Virginia Polytechnic Institute and State University. This thesis completes his M.S. degree in OR from Virginia Tech.