Switched Markov Jump Linear Systems: Analysis and Control Synthesis

Collin C. Lutz

Dissertation submitted to the Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in

Electrical Engineering

Daniel J. Stilwell, Chair

William T. Baumann

A. A. (Louis) Beex

Binoy Ravindran

Craig A. Woolsey

October 24, 2014

Blacksburg, Virginia

Keywords: Time-inhomogeneous Markov jump linear systems, switched Markov jump linear systems, stochastic systems, stochastic optimal control, networked control, control over communications, fault-tolerant control, energy-aware control

Switched Markov Jump Linear Systems: Analysis and Control Synthesis

Collin C. Lutz

ABSTRACT

Markov jump linear systems find application in many areas including economics, fault-tolerant control, and networked control. Despite significant attention paid to Markov jump linear systems in the literature, few authors have investigated Markov jump linear systems with time-inhomogeneous Markov chains (Markov chains with time-varying transition probabilities), and even fewer authors have considered time-inhomogeneous Markov chains with a priori unknown transition probabilities. This dissertation provides a formal stability and disturbance attenuation analysis for a Markov jump linear system where the underlying Markov chain is characterized by an a priori unknown sequence of transition probability matrices that assumes one of finitely-many values at each time instant. Necessary and sufficient conditions for uniform stochastic stability and uniform stochastic disturbance attenuation are reported. In both cases, conditions are expressed as a set of finite-dimensional linear matrix inequalities (LMIs) that can be solved efficiently. These finite-dimensional LMI analysis results lead to nonconservative LMI formulations for optimal controller synthesis with respect to disturbance attenuation. As a special case, the analysis also applies to a Markov jump linear system with *known* transition probabilities that vary in a finite set.

This research was made with Government support under and awarded by DoD, Air Force Office of Scientific Research, National Defense Science and Engineering Graduate (NDSEG) Fellowship, 32 CFR 168a.

To Jessie and Aidric

Acknowledgments

The author expresses thanks to his research advisor, Professor Daniel J. Stilwell, for his guidance and assistance during all stages of research. Professor Stilwell set an example to follow with his high standards and work ethic. Special thanks goes to the author's committee members for their dedication to teaching and advising. The author is indebted to many exceptional instructors spanning multiple departments at Virginia Tech. The author is thankful for the financial support of the National Defense Science and Engineering Graduate (NDSEG) Fellowship and the academic freedom granted therein. Finally, the author conveys his deepest gratitude to his wife, Jessie, whose love, support, and encouragement made this journey possible.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Motivation

A discrete-time Markov jump linear system is a stochastic discrete-time linear time-varying system where the time-variation of parameter matrices is determined by a realization of a Markov chain. The Markov chain may be time-homogeneous (characterized by constant transition probabilities) or time-inhomogeneous (characterized by time-varying transition probabilities), and terminology is slightly abused by referring to a time-(in)homogeneous Markov jump linear system. This work addresses a *switched Markov jump linear system*, which is simply a time-inhomogeneous Markov jump linear system where the underlying Markov chain is characterized by an a priori unknown sequence of transition probability matrices that assumes one of finitely-many values at each time instant; an a priori unknown *switching sequence* parameterizes the transition probability matrices at each time instant. As a special case, the analysis also applies to time-inhomogeneous Markov jump linear systems with *known* transition probabilities that vary in a finite set.

In the existing literature, stability (resp. disturbance attenuation) of a time-inhomogeneous Markov jump linear system is equivalent to an infinite-dimensional Lyapunov (resp. storage function) criterion that in general lacks a practical technique for solving. In this dissertation, it is shown that stochastically stable and contractive systems admit Lyapunov and storage functions with finite dependence on the future. This observation leads to necessary and sufficient conditions for uniform stochastic stability and uniform stochastic disturbance attenuation for a switched Markov jump linear system, expressed as a set of finite-dimensional linear matrix inequalities (LMIs), which can be solved efficiently using well-known techniques. These finite-dimensional LMI analysis results lead to nonconservative LMI formulations for optimal controller synthesis with respect to disturbance attenuation for a switched Markov jump linear system.

The Markov jump linear system abstraction finds application in many areas including economics [12], fault-tolerant control [9], and networked control [27, 61]. Despite the prevalence of Markov jump linear systems, little attention has been paid to the case when the Markov chain transition probabilities are time-varying, and almost no attention has been paid to the case when the Markov chain transition probabilities are time-varying and a priori unknown.

Time-varying Markov chain transition probabilities may arise in a variety of situations. Consider, for example, a control system where the plant and controller are connected via a wireless communications network subject to random network delays and/or packet loss (see, e.g., [68]). Network delays and packet loss probabilities are influenced by many factors, including ambient noise, distance between wireless nodes, obstacles between the transmitter and receiver, the presence of other wireless communication nodes on the same network, and sources of interference on the same frequency band [25, 50]. Thus, network delay and packet loss probabilities may vary with time due to, e.g., solar activity, mobile network nodes, evolving network topology, or adversarial disruption (jamming). In some of these scenarios, the time-varying Markov chain transition probabilities may be known in advance, while in other scenarios, the time-variation may be a priori unknown.

In some instances, the time-varying probabilistic structure may arise due to implementation details of the control law. Chapter 7 discusses an energy-saving scheme for an autonomous underwater vehicle (AUV) that can be cast as a switched Markov jump linear system.

1.2 Related Work

Time-homogeneous Markov jump linear systems have been studied quite extensively in the literature. Ji and Chizeck [33] study various second moment stability concepts and the almost sure asymptotic stability for the time-homogeneous case. Costa and Fragoso [11] provide coupled linear matrix inequality conditions equivalent to mean square stability, and Ji and Chizeck [32] characterize the jump linear quadratic Gaussian optimal control problem. More recently, Seiler and Sengupta [59] consider the $\mathcal{H}_\infty$ control problem and provide a stochastic bounded real lemma that can be used when the Markov chain is time-homogeneous. Geromel et al. [22] address the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ dynamic output feedback control design problems for time-homogeneous Markov jump linear systems. Lee and Dullerud provide necessary and sufficient conditions for a time-homogeneous Markov jump linear system to be almost surely uniformly exponentially stable [38] and almost surely uniformly strictly contractive [37]; LMI-based synthesis techniques are also reported for the almost sure uniform stabilization and disturbance attenuation of a time-homogeneous Markov jump linear system.

Time-inhomogeneous Markov jump linear systems have received less attention. For the case when the time-varying Markov chain transition probabilities are known, Krtolica et al. [36] provide a necessary and sufficient condition for mean square stability in the form of an infinite set of coupled matrix equations, and Aberkane [1] states a similar stability result in terms of an infinite set of LMIs. Fang and Loparo [17] reduce the infinite set of matrix equations in [36] to a finite set when the transition probabilities of the Markov chain are periodic. Aberkane [1] provides a necessary and sufficient condition for stochastic disturbance attenuation in the form of an infinite set of LMIs, which reduces to a finite set when the transition probabilities of the Markov chain are periodic.

For the case when the Markov chain transition probabilities are time-varying and a priori unknown, only sufficient conditions for uniform stochastic stability and uniform stochastic disturbance attenuation have been provided in the literature. Bolzern et al. [5] examine a continuous-time Markov jump linear system with time-varying a priori unknown Markov process transition rates (see, e.g., [39, Sec. 11.4.2]) and provide a sufficient condition for uniform stochastic stability subject to a dwell-time constraint. Lutz and Stilwell [43] examine a particular class of time-inhomogeneous Markov jump linear systems with a priori unknown transition probabilities, provide sufficient conditions for uniform mean square stability and uniform stochastic disturbance attenuation, and present a sufficient condition for uniform stochastic stability subject to an average dwell-time constraint.

1.3 Organization

This dissertation is organized as follows. Notation, concepts, and results needed throughout the dissertation are reviewed in Chapter 2. Markov jump linear systems and switched Markov jump linear systems are formally introduced in Chapter 3.

Chapter 4 is concerned with stability for the different types of systems defined in Chapter 3. Stability results for a Markov jump linear system are reviewed. To motivate the development, an example is given in Chapter 4 where the time-variation of the transition probabilities of the Markov chain causes instability. In Chapter 4, a necessary and sufficient LMI condition for uniform stability of a switched Markov jump linear system is developed. A computationally simpler condition for uniform stability that can be applied when the Markov chain is an independent sequence of random variables is also derived.

In Chapter 5, the disturbance attenuation properties of the systems defined in Chapter 3 are examined. Disturbance attenuation results for Markov jump linear systems are reviewed. A motivating example is presented in Chapter 5 where certain disturbance attenuation properties of a system are lost due to time-variation of the Markov chain transition probabilities. A necessary and sufficient condition for uniform stochastic disturbance attenuation of a switched Markov jump linear system is obtained in Chapter 5 and is expressed as a set of finite-dimensional LMIs. When the Markov chain is an independent sequence of random variables, a simpler condition for uniform disturbance attenuation of a switched Markov jump linear system is also developed in Chapter 5.

In Chapter 6, control synthesis for a switched Markov jump linear system is considered. The existence of a controller that ensures the closed-loop system is uniformly stable and contractive is shown to be equivalent to the feasibility of a set of finite-dimensional LMIs. This result leads to an iterative design procedure for constructing an optimal controller with respect to disturbance attenuation for a switched Markov jump linear system. A few examples are considered at the end of Chapter 6, and near-optimal (with respect to disturbance attenuation) controllers are constructed via an iterative procedure.

In Chapter 7, an energy-saving scheme is examined for a control system implemented on a microprocessor where the energy consumed by the microprocessor is nontrivial. The problem is cast as a switched Markov jump linear system, and the analysis results of Chapters 4 and 5 are applied.

Chapter 2

Preliminaries

2.1 Introduction

This chapter establishes a common notation and reviews background material from mathematical analysis, probability theory, matrix theory, and graph theory. Only the key ideas and tools needed for this work are presented. For a more in-depth review, references are provided at the end of the chapter.

2.2 Notation

The set of integers is denoted $\mathbb{Z}$. The positive and nonnegative integers are represented by $\mathbb{N}$ and $\mathbb{N}_0$, respectively. The negative and nonpositive integers are represented by $\mathbb{Z}^-$ and $\mathbb{Z}_0^-$, respectively. The standard Euclidean vector norm and corresponding induced matrix norm are both denoted by $\|\cdot\|$. The set of $n \times n$ symmetric matrices is denoted by $\mathbb{S}_n$, and $\mathbb{S}_n^+$ denotes the set of positive definite symmetric matrices. For an $n \times n$ symmetric matrix X, the notation $X > 0$ simply means $X \in \mathbb{S}_n^+$, and $X < 0$ means $-X > 0$. The set of $N \times N$ *stochastic matrices* is denoted $\mathbb{T}_N$ and consists of matrices with nonnegative elements where each row sums to one. The composition of two functions f, g is denoted by $f \circ g$, and the image of a function f is written $\operatorname{Im} f$. A *sequence* is a function f whose domain is a subset of the set of integers (usually $\mathbb{N}$ or $\mathbb{N}_0$) and may be equivalently viewed as an ordered list $(f(1), f(2), \dots)$. Given two sets $\mathcal{N}$ and $\mathcal{J}$, the Cartesian product is denoted by $\mathcal{N} \times \mathcal{J}$, and the Cartesian power of a set $\mathcal{J}$ is denoted by $\mathcal{J}^M$ where $M \in \mathbb{N}$. By convention, if $M = 0$ then $\mathcal{J}^M = \{\emptyset\}$, a singleton set (e.g., see [29, p. 57]), and $\mathcal{N} \times \mathcal{J}^M = \mathcal{N}$. If $\mathcal{J}$ is a set, the notation $\mathcal{J}^\infty$ is used to denote the set of all sequences of elements of $\mathcal{J}$. If $r = (r_1, r_2, \dots, r_M) \in \mathcal{J}^M$ where $M \in \mathbb{N}$, the shorthand $r_{m:n} = (r_m, r_{m+1}, \dots, r_n) \in \mathcal{J}^{n-m+1}$ is sometimes used where $1 \le m \le n \le M$.

2.3 Mathematical Analysis

There are many excellent texts on mathematical analysis, and so definitions and facts needed in the dissertation are only briefly stated. Consult the references at the end of the chapter for more details.

Definition 2.1. A *metric space* is a set M equipped with a *metric* $d : M \times M \to \mathbb{R}$ that satisfies

Nonnegativity: $d(x, y) \geq 0$ for all $x, y \in M$;

Identity of indiscernibles: $d(x, y) = 0$ if and only if $x = y$;

Symmetry: $d(x, y) = d(y, x)$ for all $x, y \in M$;

Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in M$.

Definition 2.2. A sequence $x : \mathbb{N} \to M$ taking values in a metric space M is *Cauchy* if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x(m), x(n)) < \epsilon$ for all $m, n > N$.

Definition 2.3. A metric space M is *complete* if every Cauchy sequence converges to an element of M.

Definition 2.4. A *vector space over a field* F is a nonempty set V of elements (called vectors) closed under the algebraic operations of *vector addition* and *scalar multiplication* satisfying the following conditions

Commutative: $u + v = v + u$ for all $u, v \in V$;

Associative: $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in F$;

Additive identity: there exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$;

Additive inverse: for every $v \in V$, there exists $w \in V$ such that $v + w = 0$;

Multiplicative identity: $1v = v$ for all $v \in V$ where 1 denotes the multiplicative identity in F;

Distributive: $a(u + v) = au + av$ and $(a + b)u = au + bu$ for all $a, b \in F$ and all $u, v \in V$.

Definition 2.5. Let V_1 and V_2 be vector spaces over the same field F. A *linear operator* T is a function $T : V_1 \to V_2$ such that $T(u + v) = Tu + Tv$ and $T(av) = aTv$ for all $a \in F$ and all $u, v \in V_1$. For linear operators, it is common to write Tv in lieu of $T(v)$.

Definition 2.6. A *norm* on a real or complex vector space V is a real-valued function on V whose value at $v \in V$ is denoted by $\|v\|$ and which has the properties

Nonnegativity: $\|v\| \geq 0$ for all $v \in V$;

Definiteness: $\|v\| = 0$ if and only if $v = 0$;

Absolute homogeneity: $\|av\| = |a| \, \|v\|$ for all scalars a and all $v \in V$;

Triangle inequality: $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$.

Definition 2.7. A *normed space* is a vector space V with a norm defined on it.

Definition 2.8. Let V_1 and V_2 be normed spaces. A linear operator $T : V_1 \to V_2$ is *bounded* if there exists $c \geq 0$ such that $\|Tv\| \leq c \, \|v\|$ for all $v \in V_1$. The norms on V_1 and V_2 are denoted by the same symbol, with little risk of confusion.

Definition 2.9. Let $T : V_1 \to V_2$ be a bounded linear operator with V_1 and V_2 being normed spaces. The *induced norm* or simply *norm* of T is defined $\|T\| = \sup_{v \neq 0} \|Tv\| / \|v\|$. The induced norm is sometimes denoted by $\|T\|_{V_1 \to V_2}$ to emphasize the role of the vector norms on V_1 and V_2.

Definition 2.10. An *inner product* on a vector space V is a function, denoted $\langle \cdot, \cdot \rangle$, from $V \times V$ into F with the following properties

Nonnegativity: $\langle v, v \rangle \geq 0$ for all $v \in V$;

Definiteness: $\langle v, v \rangle = 0$ if and only if $v = 0$;

Additivity in the first argument: $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$;

Homogeneity in the first argument: $\langle av, w \rangle = a \langle v, w \rangle$ for all $a \in F$ and all $v, w \in V$;

Conjugate symmetry: $\langle v, w \rangle = \overline{\langle w, v \rangle}$ for all $v, w \in V$.

Definition 2.11. An *inner product space* is a vector space V equipped with an inner product on V.

Definition 2.12. An inner product on V *induces* a norm on V given by $\|v\| = \sqrt{\langle v, v \rangle}$ and a metric on V given by $d(u, v) = \sqrt{\langle u - v, u - v \rangle}$.

Definition 2.13. A *Hilbert space* is a complete inner product space with respect to the metric induced by the inner product.

Definition 2.14. Let $T : H_1 \to H_2$ be a bounded linear operator where H_1 and H_2 are Hilbert spaces. The *Hilbert-adjoint operator* or *adjoint operator* T^* of T is the operator $T^* : H_2 \to H_1$ such that $\langle Tx, y \rangle = \langle x, T^* y \rangle$ for all $x \in H_1$ and all $y \in H_2$. The inner products on H_1 and H_2 are denoted by the same symbol, with little risk of confusion.

Lemma 2.15 (Thm. 3.9-2 of [35])**.** *The Hilbert-adjoint operator T^* of a bounded linear operator T exists, is unique, and is a bounded linear operator with norm $\|T^*\| = \|T\|$.*

2.4 Probability and Random Variables

A *probability space* is a triple $(\Omega, \mathfrak{F}, \mathbf{P})$ where Ω is the sample space, $\mathfrak{F}$ is a collection of events (subsets of Ω) closed under complements and countable unions, and $\mathbf{P}$ is a probability measure which assigns probabilities to events in $\mathfrak{F}$. A *random variable* is a measurable real-valued function defined on Ω. A *random process* (or *stochastic process*) is an indexed family of random variables. If the index set is countable, the random process is *discrete-time*; otherwise, the random process is *continuous-time*. Some important spaces of discrete-time random processes defined on $(\Omega, \mathfrak{F}, \mathbf{P})$ are the spaces of mean square summable stochastic processes defined

$$\begin{aligned}
\ell_e^2(\mathbb{Z}) &= \{w : \mathbb{Z} \times \Omega \to \mathbb{R}^m \text{ s.t. } \|w\|_{2,e} < \infty\} \text{ where } \|w\|_{2,e}^2 = \mathbf{E}\left[\sum_{k=-\infty}^{\infty} \|w(k)\|^2\right] \\
\ell_e^2(\mathbb{N}_0) &= \{w \in \ell_e^2(\mathbb{Z}) : w(k) = 0 \text{ for } k \notin \mathbb{N}_0\} \\
\ell_e^2(\mathbb{Z}_0^-) &= \{w \in \ell_e^2(\mathbb{Z}) : w(k) = 0 \text{ for } k \notin \mathbb{Z}_0^-\} \\
\ell_e^2[0, T] &= \{w \in \ell_e^2(\mathbb{Z}) : w(k) = 0 \text{ for } k \notin [0, T]\} \\
\ell_e^2[-T, 0] &= \{w \in \ell_e^2(\mathbb{Z}) : w(k) = 0 \text{ for } k \notin [-T, 0]\}.
\end{aligned}$$

The abbreviated notation ℓ_e^2 is often used for $\ell_e^2(\mathbb{N}_0)$. In fact, all of the spaces listed above are Hilbert spaces with inner product given by $\langle v, w\rangle_e = \mathbf{E}\left[\sum_{k=-\infty}^{\infty} w^\mathsf{T}(k)v(k)\right]$. Note that the dimension m of vector $w(k)$ is not explicitly reflected in the notation ℓ_e^2. The same symbol ℓ_e^2 is used for spaces with different vector dimensions, with little risk of confusion.

2.4.1 Results from probability theory

Definition 2.16. A collection of random variables $X_1, \ldots, X_n$ are *(mutually) independent* if

$$\mathbf{P}\{X_1 \in H_1, \ldots, X_n \in H_n\} = \mathbf{P}\{X_1 \in H_1\} \cdots \mathbf{P}\{X_n \in H_n\}.$$

for all measurable subsets $H_1, \ldots, H_n$ of $\mathbb{R}$. Equivalently,

$$\mathbf{P}\{X_1 \le x_1, \ldots, X_n \le x_n\} = \mathbf{P}\{X_1 \le x_1\} \cdots \mathbf{P}\{X_n \le x_n\}.$$

for all $x_1, \ldots, x_n \in \mathbb{R}$ (e.g., see [4][Sec. 20]).

Definition 2.17. A discrete-time random process $X : \mathbb{N}_0 \times \Omega \to \mathbb{R}$ is an *independent sequence of random variables* if every finite subset of $\{X(0), X(1), \ldots\}$ is a set of mutually independent random variables. If, in addition, $X(0), X(1), \ldots$ have the same probability distribution then X is an *independent and identically* *distributed (i.i.d.) sequence.*

Lemma 2.18. *Let X, Y be independent random variables defined on a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$, and let f and g be measurable real-valued functions defined on $\operatorname{Im} X$ and $\operatorname{Im} Y$, respectively. Then $f(X)$ and $g(Y)$ are independent random variables.*

Proof. Let A and B be measurable subsets of $\mathbb{R}$.

$$\begin{aligned}\mathbf{P}\{f(X) \in A, g(Y) \in B\} &= \mathbf{P}\left\{X \in f^{-1}[A], Y \in g^{-1}[B]\right\}\\ &= \mathbf{P}\left\{X \in f^{-1}[A]\right\}\mathbf{P}\left\{Y \in g^{-1}[B]\right\}\\ &= \mathbf{P}\{f(X) \in A\}\,\mathbf{P}\{g(Y) \in B\}\end{aligned}$$

where $f^{-1}[A] = \{x \in \operatorname{Im} X : f(x) \in A\}$ is the inverse image of A under f. □

The law of iterated expectations, also known as the law of total expectation, will prove useful in the sequel.

Lemma 2.19 (Law of iterated expectations)**.** *Let X, Y, Z be random variables defined on a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$. Then*

$$\mathbf{E}[X \mid Y] = \mathbf{E}[\mathbf{E}[X \mid Y, Z] \mid Y].$$

The Cauchy-Schwarz inequality can be verified on any probability space.

Lemma 2.20 (Cauchy-Schwarz inequality for random variables)**.** *Let X, Y be random variables defined on a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ such that $\mathbf{E}\left[X^2\right]$ and $\mathbf{E}\left[Y^2\right]$ are finite. Then*

$$|\mathbf{E}[XY]| \leq \sqrt{\mathbf{E}[X^2]\,\mathbf{E}[Y^2]}.$$

The Cauchy-Schwarz inequality may also be extended to random vectors.

Lemma 2.21 (Cauchy-Schwarz inequality for random vectors)**.** *Let X, Y be random vectors defined on a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ such that $\mathbf{E}\left[X^\intercal X\right]$ and $\mathbf{E}\left[Y^\intercal Y\right]$ are finite. Then*

$$\left|\mathbf{E}\left[X^\intercal Y\right]\right| \leq \sqrt{\mathbf{E}[X^\intercal X]\,\mathbf{E}[Y^\intercal Y]}.$$

A generalized version of the Cauchy-Schwarz inequality is sometimes useful.

Lemma 2.22 (Generalized Cauchy-Schwarz inequality). *Let X, Y be random vectors in $\mathbb{R}^n$, and let A be an $n \times n$ symmetric matrix whose entries are random variables. If $A > 0$ almost surely then*

$$\left|\mathbf{E}\left[X^\mathsf{T} A Y\right]\right| \leq \sqrt{\mathbf{E}\left[X^\mathsf{T} A X\right] \mathbf{E}\left[Y^\mathsf{T} A Y\right]}.$$

Proof. Let $U = A^{1/2}X$ and $V = A^{1/2}Y$. Note that $A^{1/2} > 0$ almost surely. Then

$$\begin{aligned}\left|\mathbf{E}\left[X^\mathsf{T} A Y\right]\right| &= \left|\mathbf{E}\left[U^\mathsf{T} V\right]\right| \\ &\leq \sqrt{\mathbf{E}\left[U^\mathsf{T} U\right] \mathbf{E}\left[V^\mathsf{T} V\right]} \qquad (2.1) \\ &= \sqrt{\mathbf{E}\left[X^\mathsf{T} A X\right] \mathbf{E}\left[Y^\mathsf{T} A Y\right]}\end{aligned}$$

where (2.1) follows from Lemma 2.21. □

2.4.2 Markov chains

A *finite Markov chain* $\theta : \mathbb{N}_0 \times \Omega \to \mathcal{N}$ is a discrete-time random process that takes values in a finite set $\mathcal{N} = \{1, \ldots, N\}$ and satisfies the *Markov property*

$$\mathbf{P}\left\{\theta(k_{n+1}) = i_{n+1} \mid \theta(k_n) = i_n, \ldots, \theta(k_1) = i_1\right\} = \mathbf{P}\left\{\theta(k_{n+1}) = i_{n+1} \mid \theta(k_n) = i_n\right\}$$

where $0 \leq k_1 < k_2 < \cdots < k_n < k_{n+1}$ are arbitrary points in time. Thus, given the present state of the Markov chain, the future is independent of the past. If $\theta(k) = i$, the Markov chain is in *mode* i at time k.

For $k \in \mathbb{N}$, define $p_{ij}(k) = \mathbf{P}\{\theta(k) = j \mid \theta(k-1) = i\}$ and let $P(k)$ be the $N \times N$ matrix with entries $p_{ij}(k)$. Let $p_i(k) = \mathbf{P}\{\theta(k) = i\}$ and define the row vector $p(k) = [p_1(k) \; p_2(k) \; \cdots \; p_N(k)]$. Note that the initial distribution $p(0)$ and the sequence $P : \mathbb{N} \to \mathbb{T}_N$ of transition probability matrices give a full probabilistic description of random process θ since

$$\mathbf{P}\left\{\theta(k) = i_k, \theta(k-1) = i_{k-1}, \ldots, \theta(0) = i_0\right\} = \mathbf{P}\left\{\theta(0) = i_0\right\} \prod_{j=1}^{k} \mathbf{P}\left\{\theta(j) = i_j \mid \theta(j-1) = i_{j-1}\right\}.$$

where $k \in \mathbb{N}$, $i_0, i_1, \ldots, i_k \in \mathcal{N}$, and $\mathbf{P}\left\{\theta(j) = i_j \mid \theta(j-1) = i_{j-1}\right\}$ is an element of the matrix $P(j)$.

Definition 2.23. The Markov chain θ is *time-homogeneous* if P is constant. Otherwise, the Markov chain is *time-inhomogeneous*.

Remark 2.24. An independent sequence of random variables is a special type of Markov chain. If θ is an independent sequence of random variables taking values in a finite set then θ is trivially a Markov chain. On the other hand, a Markov chain is an independent sequence of random variables exactly when $p_{ij}(k) = p_j(k)$ for all $i \in \mathcal{N}$ and $k \in \mathbb{N}$, i.e., each row of $P(k)$ is identical. For example, a series of coin flips is an independent and identically distributed sequence and is trivially a time-homogeneous Markov chain. On the other hand, consider three coins, each with a different probability of landing heads up. Suppose the first coin is flipped, followed by the second and third coins. If the coins continue to be flipped in this order indefinitely then the sequence of flips is a time-inhomogeneous independent sequence of random variables.

Lemma 2.25. *The distribution $p(k)$ may be calculated via vector and matrix multiplications as*

$$p(k) = p(0)P(1)P(2)\cdots P(k). \tag{2.2}$$

Proof. Let $\mathcal{N}_0 \cup \mathcal{N}_1$ be a partition of $\mathcal{N}$ such that $\mathcal{N}_0 = \{i \in \mathcal{N} : p_i(k-1) = 0\}$ and $\mathcal{N}_1 = \{i \in \mathcal{N} : p_i(k-1) \neq 0\}$. Then

$$\begin{aligned}
p_j(k) &= \sum_{i=1}^{N} \mathbf{P}\left\{\theta(k) = j, \theta(k-1) = i\right\} \\
&= \sum_{i\in\mathcal{N}_0} 0 + \sum_{i\in\mathcal{N}_1} p_{ij}(k)p_i(k-1) && (2.3)\\
&= \sum_{i\in\mathcal{N}_0} p_{ij}(k)p_i(k-1) + \sum_{i\in\mathcal{N}_1} p_{ij}(k)p_i(k-1) && (2.4)\\
&= \begin{bmatrix} p_1(k-1) & p_2(k-1) & \cdots & p_N(k-1) \end{bmatrix} \begin{bmatrix} p_{1j}(k) \\ p_{2j}(k) \\ \vdots \\ p_{Nj}(k) \end{bmatrix}. && (2.5)
\end{aligned}$$

Equations (2.3) and (2.4) indicate that while $\mathbf{P}\left\{\theta(k) = j \mid \theta(k-1) = i\right\}$ is technically undefined for $i \in \mathcal{N}_0$, the vector multiplication in (2.5) still gives the correct result regardless of the entries $p_{ij}(k)$ for $i \in \mathcal{N}_0$. Equation (2.5) is simply the multiplication of $p(k-1)$ and the j-th column of $P(k)$. Thus $p(k) = p(k-1)P(k)$. Equation (2.2) follows by iteration. □

Definition 2.26. Let $P(k)$ be a sequence of transition probability matrices, and suppose there exists a distribution $\mu = [\mu_1\ \mu_2\ \ldots\ \mu_N]$, $\mu_i \geq 0$, $\sum_{i=1}^{N} \mu_i = 1$ such that

$$\mu_j = \sum_{i=1}^{N} \mu_i p_{ij}(k) \tag{2.6}$$

for all $j \in \mathcal{N}$ and $k \in \mathbb{N}$. Then μ is *invariant for the sequence $P(k)$*.

Remark 2.27. Note that the conditions specified in (2.6) may be written in vector-matrix notation:

$$\mu = \mu P(k)$$

for all $k \in \mathbb{N}$. Thus, μ is a left eigenvector of each matrix $P(k)$ with eigenvalue 1.

Lemma 2.28. *Suppose there exists an invariant distribution μ for the sequence $P(k)$ and $p(0) = \mu$. Then $p(k) = \mu$ for all $k \in \mathbb{N}$.*

Proof. The result follows via Lemma 2.25 and Remark 2.27. $\square$

Definition 2.29. If μ exists as in Definition 2.26 then μ is also called an *equilibrium distribution for the Markov chain θ.*

Many time-inhomogeneous Markov chains do not admit an equilibrium distribution. In contrast, an irreducible time-homogeneous Markov chain admits an equilibrium distribution if and only if every mode of the Markov chain is positive recurrent (see [46, Thm. 1.7.7] for details). A large class of examples of time-inhomogeneous Markov chains that do admit an equilibrium distribution is provided by time-inhomogeneous random walks on groups (see [55]).

Definition 2.30. A Markov chain θ is *instantaneously reversible* if there exists $\mu = [\mu_1 \ \mu_2 \ \dots \ \mu_N]$ such that $\mu_i \geq 0$, $\sum_{i=1}^{N} \mu_i = 1$, and

$$\mu_i p_{ij}(k) = \mu_j p_{ji}(k) \tag{2.7}$$

for all $i, j \in \mathcal{N}$ and $k \in \mathbb{N}_0$.

Lemma 2.31. *If a Markov chain θ is instantaneously reversible and μ satisfies Definition 2.30, then μ is an equilibrium distribution of the Markov chain.*

Proof. Sum both sides of equation (2.7) over i to arrive at (2.6). $\square$

If a Markov chain is instantaneously reversible and $p(0) = \mu$, then

$$\mathbf{P}\left\{\theta(k-1) = i, \theta(k) = j\right\} = \mathbf{P}\left\{\theta(k) = i, \theta(k-1) = j\right\} \tag{2.8}$$

for all $k \in \mathbb{N}$. In words, relation (2.8) says that the probability of visiting (i, j) in forward-time is equal to the probability of visiting (i, j) in reverse-time. Reversibility for time-inhomogeneous Markov chains has been studied by Ge et al. [21]; see also [55].

2.5 Matrix Theory

A few standard results from matrix theory are needed in the sequel.

Lemma 2.32 (Inverse of partitioned matrix)**.** *Let*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

be a block matrix with A and D nonsingular. Then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}B(D - CA^{-1}B)^{-1} \\ -D^{-1}C(A - BD^{-1}C)^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}.$$

Proof. See Chapter 0 of [28]. □

Lemma 2.33 (Sherman-Morrison-Woodbury matrix inversion lemma)**.** *Suppose A, C, $A + BCD$, and $C^{-1} + DA^{-1}B$ are nonsingular. Then*

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}.$$

Proof. See Chapter 0 of [28]. □

Lemma 2.34 (Singular value decomposition (SVD))**.** *Let $A \in \mathbb{R}^{n \times m}$ be given, let $q = \min(m, n)$, and suppose that* $\operatorname{rank} A = r$.

a) There are unitary matrices $U \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{m \times m}$, and a square diagonal matrix

$$\Sigma_q = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_q \end{bmatrix}$$

such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 = \sigma_{r+1} = \cdots = \sigma_q$ and $A = U\Sigma V^\mathsf{T}$, in which

$$\begin{aligned} &\Sigma = \Sigma_q && \text{if } m = n \\ &\Sigma = \begin{bmatrix} \Sigma_q & 0 \end{bmatrix} \in \mathbb{R}^{n \times m} && \text{if } m > n \\ &\Sigma = \begin{bmatrix} \Sigma_q \\ 0 \end{bmatrix} \in \mathbb{R}^{n \times m} && \text{if } m < n \end{aligned}$$

b) *The parameters* $\sigma_1, \ldots, \sigma_r$ *are the positive square roots of the decreasingly ordered nonzero eigenvalues of* $AA^\top$*, which are the same as the decreasingly ordered nonzero eigenvalues of* $A^\top A$*.*

Proof. See, e.g., Theorem 2.6.3 of [28]. □

Lemma 2.35 (Fundamental subspaces via SVD)**.** *Let* $A \in \mathbb{R}^{n\times m}$ *and suppose* $r = \operatorname{rank} A < \min(m, n)$*. Let*

$$A = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^\top \\ V_2^\top \end{bmatrix} = U\Sigma V^\top$$

be an SVD of A *where* $\Sigma_r = \operatorname{diag}(\sigma_1, \ldots, \sigma_r) > 0$ *and*

$$\begin{aligned} U_1 &\in \mathbb{R}^{n\times r}, \quad U_2 \in \mathbb{R}^{n\times(n-r)} \\ V_1 &\in \mathbb{R}^{m\times r}, \quad V_2 \in \mathbb{R}^{m\times(m-r)}. \end{aligned}$$

Then

a) *The columns of* U_1 *form an orthogonal basis for* $\operatorname{Im} A$*.*

b) *The columns of* U_2 *form an orthogonal basis for* $\ker A^\top$*.*

c) *The columns of* V_1 *form an orthogonal basis for* $\operatorname{Im} A^\top$*.*

d) *The columns of* V_2 *form an orthogonal basis for* $\ker A$*.*

Proof. See, e.g., Theorem A.4.2 of [62]. □

Definition 2.36. The *Moore-Penrose pseudoinverse* of a matrix $A \in \mathbb{R}^{n\times m}$ with $r = \operatorname{rank} A$ is defined to be the $m \times n$ matrix

$$A^\dagger := V \begin{bmatrix} \Sigma_r^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^\top$$

where $\Sigma_r = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$ and U, V, and σ_i $i = 1, 2, \ldots, r$ are from Lemma 2.34.

Lemma 2.37. *The Moore-Penrose pseudoinverse is unique and satisfies the following properties:*

$$\begin{aligned} AA^\dagger A = A, &\qquad A^\dagger AA^\dagger = A^\dagger \\ (AA^\dagger)^\top = AA^\dagger, &\quad (A^\dagger A)^\top = A^\dagger A \end{aligned}$$

Proof. See, e.g., Theorem 2.2.1 of [62]. □

2.5.1 Linear matrix inequalities

A linear matrix inequality (LMI) is an inequality of the form

$$F(y) = F_0 + \sum_{i=1}^{m} y_i F_i > 0 \tag{2.9}$$

where $F_i \in \mathbb{S}_n$ are constant matrices for $i = 0, \ldots, m$, and $y \in \mathbb{R}^m$ is the variable. LMIs are commonly encountered in matrix form where a symmetric matrix $Y \in \mathbb{S}_p$ is the variable, and the inequality is of the form

$$F(Y) > 0 \in \mathbb{R}^{n\times n} \tag{2.10}$$

where $F(\cdot)$ is pointwise symmetric and linear as a mapping from $\mathbb{R}^{p\times p}$ to $\mathbb{R}^{n\times n}$. Equation (2.10) can be easily written in the form of (2.9). Let $V_1, \ldots, V_m$ be a basis for the vector space of symmetric matrices in $\mathbb{R}^{p\times p}$. Any matrix $Y \in \mathbb{S}_p$ may be written as $Y = \sum_{i=1}^{m} y_i V_i$ where $y_1, \ldots, y_m$ are scalars. By linearity of $F(\cdot)$,

$$F(Y) = \sum_{i=1}^{m} y_i F(V_i).$$

Defining $F_0 = 0$ and $F_i = F(V_i)$ for $i = 1, \ldots, m$ gives (2.9).

LMIs are valuable as a theoretical and computational tool. Efficient methods for solving LMIs have been developed [7, 8], and many excellent solvers are readily available (e.g., see [42, 63]).

Lemma 2.38 (Congruence transformation)**.** *Let $T \in \mathbb{R}^{n\times n}$ be nonsingular and $A \in \mathbb{S}_n$. Then $A > 0$ if and only if $T^\mathsf{T} A T > 0$.*

The Schur complement is often used to convert nonlinear, convex inequalities to LMI form.

Lemma 2.39 (Schur complement)**.** *For all $A \in \mathbb{S}_n$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{S}_m$, the following statements are equivalent:*

$$\begin{bmatrix} A & B \\ B^\mathsf{T} & C \end{bmatrix} > 0 \tag{2.11a}$$

$$C > 0,\ A - BC^{-1}B^\mathsf{T} > 0 \tag{2.11b}$$

$$A > 0,\ C - B^\mathsf{T} A^{-1} B > 0 \tag{2.11c}$$

Lemma 2.40 (Concavity of the Schur complement). *Let* $\mu_1, \dots, \mu_N \geq 0$ *be such that* $\sum_{j=1}^N \mu_j = 1$. *Suppose* $A_1, \dots, A_N \in \mathbb{S}_n^+$, $C_1, \dots, C_N \in \mathbb{S}_m^+$, *and* $B_1, \dots, B_N \in \mathbb{R}^{n \times m}$. *Then*

$$\sum_{j=1}^N \mu_j A_j - \left(\sum_{j=1}^N \mu_j B_j\right) \left(\sum_{j=1}^N \mu_j C_j\right)^{-1} \left(\sum_{j=1}^N \mu_j B_j^\mathsf{T}\right) \geq \sum_{j=1}^N \mu_j \left(A_j - B_j C_j^{-1} B_j^\mathsf{T}\right)$$

Proof. See Corollary 1.5.3 of [3]. □

Lemma 2.41 (Projection Lemma). *Given a symmetric matrix* $\mathcal{H} \in \mathbb{R}^{m \times m}$ *and two matrices* $\mathcal{P}$, $\mathcal{Q}$ *of column dimension* m, *consider the problem of finding some matrix* K *of compatible dimensions such that*

$$\mathcal{H} + \mathcal{P}^\mathsf{T} K^\mathsf{T} \mathcal{Q} + \mathcal{Q}^\mathsf{T} K \mathcal{P} < 0. \tag{2.12}$$

Denote by $W_\mathcal{P}$, $W_\mathcal{Q}$ *any matrices whose columns form bases of the null spaces of* $\mathcal{P}$ *and* $\mathcal{Q}$, *respectively. There exists* K *satisfying* (2.12) *if and only if the following two conditions hold*

$$W_\mathcal{P}^\mathsf{T} \mathcal{H} W_\mathcal{P} < 0 \quad \textit{or} \quad \ker \mathcal{P} = \{0\} \tag{2.13a}$$

$$W_\mathcal{Q}^\mathsf{T} \mathcal{H} W_\mathcal{Q} < 0 \quad \textit{or} \quad \ker \mathcal{Q} = \{0\}. \tag{2.13b}$$

Moreover, suppose (2.13a) *and* (2.13b) *hold. Let* $\mathcal{Q}^\mathsf{T} = FG$ *and* $\mathcal{P} = MN$ *be full rank factorizations of* $\mathcal{Q}^\mathsf{T}$ *and* $\mathcal{P}$ *(see Section 0.4.6 of [28]). Then all solutions to* (2.12) *are given by*

$$K = G^\dagger T M^\dagger + Z - G^\dagger G Z M M^\dagger$$

where Z *is an arbitrary matrix and*

$$T := -R^{-1} F^\mathsf{T} Y N^\mathsf{T} \left(N Y N^\mathsf{T}\right)^{-1} + S^{1/2} L \left(N Y N^\mathsf{T}\right)^{-1/2}$$
$$S := R^{-1} - R^{-1} F^\mathsf{T} \left(Y - Y N^\mathsf{T} \left(N Y N^\mathsf{T}\right)^{-1} N Y\right) F R^{-1}$$

where L *is an arbitrary matrix such that* $\|L\| < 1$ *and* R *is an arbitrary positive definite matrix such that*

$$Y := \left(F R^{-1} F^\mathsf{T} - \mathcal{H}\right)^{-1} > 0.$$

Proof. See Theorem 2.3.12 of [62] or Lemma 3.1 of [20]. □

The Projection Lemma eliminates K from the feasibility problem in (2.12) and projects the inequality onto two subspaces, $\ker \mathcal{P}$ and $\ker \mathcal{Q}$. Note that (2.13a) and (2.13b) may be equivalently stated $y^\mathsf{T}\mathcal{H}y < 0$ for all $y \in \ker \mathcal{P} \cup \ker \mathcal{Q}$, $y \neq 0$.

Lemma 2.42. *Let matrices $A \in \mathbb{R}^{n\times \ell}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{k\times \ell}$, $R \in \mathbb{S}_n^+$, and $Q \in \mathbb{S}_\ell^+$ be given. There exists a matrix X such that*

$$(A + BXC)^\mathsf{T} R(A + BXC) < Q \tag{2.14}$$

if and only if the following two conditions hold

$$W^\mathsf{T}\left[AQ^{-1}A^\mathsf{T} - R^{-1}\right]W < 0 \quad \text{or} \quad \ker B^\mathsf{T} = \{0\} \tag{2.15a}$$

$$V^\mathsf{T}\left[A^\mathsf{T}RA - Q\right]V < 0 \quad \text{or} \quad \ker C = \{0\} \tag{2.15b}$$

where W and V are any matrices whose columns form bases of $\ker B^\mathsf{T}$ *and* $\ker C$*, respectively.*

Proof. The result is a slight modification of Theorem 2.3.11 in Skelton et al. [62]. For completeness, the proof is provided; see [62] for more details. By the Schur complement, $R > 0$ and inequality (2.14) is equivalent to

$$-\begin{bmatrix} Q & (A+BXC)^\mathsf{T} \\ A+BXC & R^{-1}\end{bmatrix} = -\begin{bmatrix} Q & A^\mathsf{T} \\ A & R^{-1}\end{bmatrix} + \begin{bmatrix} C^\mathsf{T} \\ 0 \end{bmatrix}(-X)^\mathsf{T}\begin{bmatrix} 0 & B^\mathsf{T}\end{bmatrix} + \begin{bmatrix} 0 \\ B\end{bmatrix}(-X)\begin{bmatrix} C & 0\end{bmatrix} < 0. \tag{2.16}$$

Application of Lemma 2.41 to (2.16) and the Schur complement yield (2.15a) and (2.15b). □

Lemma 2.43. *Suppose $P \in \mathbb{S}_n^+$ and $G \in \mathbb{R}^{n\times n}$. Then*

$$G^\mathsf{T}P^{-1}G \geq G + G^\mathsf{T} - P. \tag{2.17}$$

Proof. $P > 0$ implies $(G - P)^\mathsf{T}P^{-1}(G - P) \geq 0$, which implies (2.17). □

Lemma 2.44. *Suppose $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, and $X, Y \in \mathbb{S}_n^+$. The follow LMIs are equivalent:*

$$\begin{bmatrix} A & B \\ C & D\end{bmatrix}^\mathsf{T}\begin{bmatrix} X & 0 \\ 0 & I\end{bmatrix}\begin{bmatrix} A & B \\ C & D\end{bmatrix} - \begin{bmatrix} Y & 0 \\ 0 & I\end{bmatrix} < 0 \tag{2.18}$$

$$\begin{bmatrix} -X^{-1} & A & B & 0 \\ A^\top & -Y & 0 & C^\top \\ B^\top & 0 & -I & D^\top \\ 0 & C & D & -I \end{bmatrix} < 0 \tag{2.19}$$

Proof. Rewrite (2.18) as

$$\begin{bmatrix} A^\top XA - Y & A^\top XB \\ B^\top XA & B^\top XB - I \end{bmatrix} - \begin{bmatrix} C^\top \\ D^\top \end{bmatrix} (-I) \begin{bmatrix} C & D \end{bmatrix} < 0 \tag{2.20}$$

By Lemma 2.39, inequality (2.20) is equivalent to

$$\begin{bmatrix} A^\top XA - Y & A^\top XB & C^\top \\ B^\top XA & B^\top XB - I & D^\top \\ C & D & -I \end{bmatrix} < 0 \tag{2.21}$$

Now rewrite (2.21) as

$$\begin{bmatrix} -Y & 0 & C^\top \\ 0 & -I & D^\top \\ C & D & -I \end{bmatrix} - \begin{bmatrix} A^\top \\ B^\top \\ 0 \end{bmatrix} (-X) \begin{bmatrix} A & B & 0 \end{bmatrix} < 0 \tag{2.22}$$

Finally, by Lemma 2.39 inequality (2.22) is equivalent to (2.19). □

Lemma 2.45. *Suppose $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$, $X \in \mathbb{S}_n^+$, $Y \in \mathbb{S}_n^+$, $W_1 \in \mathbb{R}^{n\times q}$, and $W_2 \in \mathbb{R}^{p\times q}$. The follow LMIs are equivalent:*

$$\begin{bmatrix} W_1 & 0 \\ W_2 & 0 \\ 0 & I \end{bmatrix}^\top \begin{bmatrix} AXA^\top - Y & AXC^\top & B \\ CXA^\top & -I + CXC^\top & D \\ B^\top & D^\top & -I \end{bmatrix} \begin{bmatrix} W_1 & 0 \\ W_2 & 0 \\ 0 & I \end{bmatrix} < 0 \tag{2.23}$$

$$\begin{bmatrix} W_1 \\ W_2 \end{bmatrix}^\top \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^\top - \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} < 0 \tag{2.24}$$

Proof. Define

$$W = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}, \quad F = \begin{bmatrix} AXA^\mathsf{T} - Y & AXC^\mathsf{T} \\ CXA^\mathsf{T} & -I + CXC^\mathsf{T} \end{bmatrix}, \quad G = \begin{bmatrix} B \\ D \end{bmatrix}$$

Then (2.23) is equivalently expressed

$$\begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix}^\mathsf{T} \begin{bmatrix} F & G \\ G^\mathsf{T} & -I \end{bmatrix} \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix} < 0 \tag{2.25}$$

Carry out the multiplication in (2.25) and apply the Schur complement to see that (2.25) is equivalent to

$$W^\mathsf{T} F W - W^\mathsf{T} G (-I)^{-1} G^\mathsf{T} W < 0 \tag{2.26}$$

The expression in (2.26) is easily shown to be equivalent to the expression in (2.24). □

Lemma 2.46. *Suppose $A \in \mathbb{S}_n$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{S}_m$, $W \in \mathbb{R}^{(n+m)\times p}$, and $D \in \mathbb{S}_n$. Then*

$$W^\mathsf{T} \begin{bmatrix} A & B \\ B^\mathsf{T} & C \end{bmatrix} W < 0, \quad D \le A \quad \Rightarrow \quad W^\mathsf{T} \begin{bmatrix} D & B \\ B^\mathsf{T} & C \end{bmatrix} W < 0.$$

Similarly if $W \in \mathbb{R}^{n\times p}$ then

$$W^\mathsf{T} A W < 0, \quad D \le A \quad \Rightarrow \quad W^\mathsf{T} D W < 0.$$

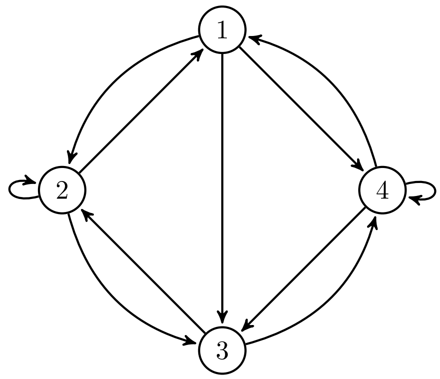
Proof. Note that

$$W^\mathsf{T} \begin{bmatrix} D - A & 0 \\ 0 & 0 \end{bmatrix} W \le 0$$

so

$$W^\mathsf{T} \begin{bmatrix} D & B \\ B^\mathsf{T} & C \end{bmatrix} W = W^\mathsf{T} \left(\begin{bmatrix} A & B \\ B^\mathsf{T} & C \end{bmatrix} + \begin{bmatrix} D - A & 0 \\ 0 & 0 \end{bmatrix} \right) W \le W^\mathsf{T} \begin{bmatrix} A & B \\ B^\mathsf{T} & C \end{bmatrix} W < 0.$$

The second statement follows analogously. □



$$\Lambda = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

Figure 2.1: A directed graph and its adjacency matrix.

2.6 Graph Theory

Only a few concepts from graph theory are needed. A *directed graph* is a pair (V, E) where V is a set of *vertices*, and E is a set of ordered pairs $(s, r) \in V \times V$ where (s, r) is a *directed edge* from vertex s to vertex r (e.g., see [52][Sec. 7.3]). Only graphs with a finite number of vertices are considered, so without loss of generality $V = \{1, \dots, J\}$ for some finite J. The set of directed edges E may be represented by a $J \times J$ *adjacency matrix* Λ where Λ_{sr}, the sr-th element of Λ, is equal to one if $(s, r) \in E$, and $\Lambda_{sr} = 0$ if $(s, r) \notin E$ (see Fig. 2.1). A *directed path* is an M-tuple ($M \geq 2$) of elements $(r_1, r_2, \dots, r_M)$ in V composed of one or more directed edges in E: $(r_1, r_2) \in E$, $(r_2, r_3) \in E$, etc. (e.g., see [52][Sec. 7.4]). The path $(r_1, r_2, \dots, r_M)$ requires $M - 1$ edge traversals so is a path of length $M - 1$. If $r_1 = r_M$, the directed path $(r_1, \dots, r_M)$ is called a *cycle*. If Ψ is the set of all infinite-length paths on a directed graph (V, E), then Ψ is *generated* by the directed graph (V, E).

Definition 2.47. A directed graph is *strongly connected* or *irreducible* if, given any pair of vertices (s, r), there exists a directed path from s to r.

Definition 2.48. A square matrix Λ is *irreducible* if for any index (s, r) there exists some $l \in \mathbb{N}$ such that $\left[\Lambda^l\right]_{sr} > 0$ where $\left[\Lambda^l\right]_{sr}$ denotes the sr-th entry of the matrix Λ^l, the l-th power of Λ.

Let Λ be the adjacency matrix of a directed graph and $l \in \mathbb{N}$. The sr-th entry of Λ^l is the number of directed paths of length l from vertex s to vertex r (e.g., see [6, Ex. 10.1.8] or [52][Sec. 8.4]). Thus, a directed graph is irreducible if and only if its adjacency matrix Λ is irreducible.

A *subgraph* of a directed graph (V, E) is a directed graph (W, F) where $W \subset V$ and $F \subset E$ (e.g., see [52][Sec. 8.2]). Any directed graph (V, E) may be decomposed into a partition of strongly connected subgraphs such that each subgraph is not contained in a larger strongly connected subgraph (see Fig 2.2 and [52][Sec. 8.4]). If each of these *strongly connected components* is reduced to a single vertex, the resulting

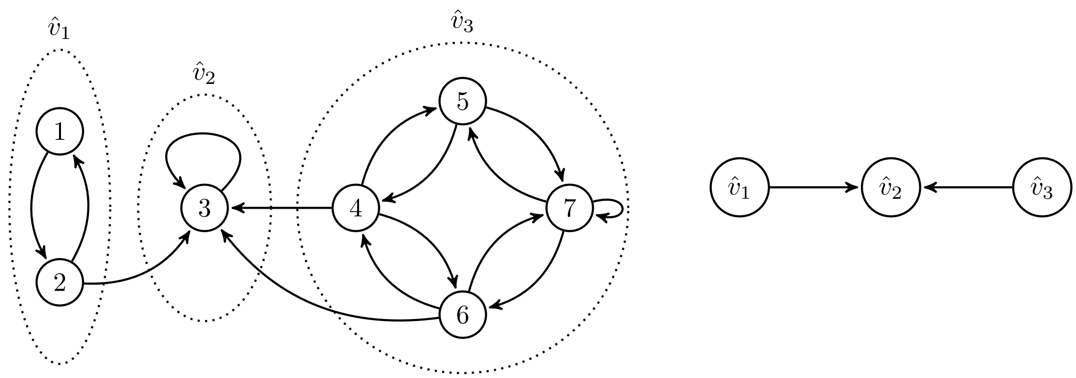


Figure 2.2: Decomposition of a directed graph into strongly connected components and the resulting condensation graph.

directed graph is called the *condensation* of (V, E) and contains no directed cycles (e.g., see [6][Ex. 10.1.9]).

2.7 Notes and References

For facts concerning elementary real analysis, see Rudin [53]. Axler [2] provides an excellent reference for linear algebra. Readers interested in Hilbert spaces and functional analysis should consult Kreyszig [35] or Reed and Simon [51]. Consult Garcia [39] for an introductory reference on probability theory; see Billingsley [4] or Folland [19] for a more advanced approach. Time-homogeneous Markov chains have been studied quite extensively, and many good references are available (see, e.g., [46]). In contrast, the theory concerning time-inhomogeneous Markov chains is much less developed and remains an area of active research [13,55–57]. For an introduction to LMIs with application to control theory, see Boyd et al. [7]. Horn and Johnson [28] is a standard reference for matrix analysis. See Bondy and Murty [6] or Wilson [67] for an introduction to graph theory; the graph theory terminology used in this work is generally consistent with Rosen [52].

Chapter 3

Randomly Jumping Systems

3.1 Introduction

A randomly jumping system is a linear system with randomly jumping parameters. If the random jumps can be modeled by the transitions of a Markov chain, the randomly jumping system is called a Markov jump linear system. The transition probabilities of the Markov chain may be time-varying or time-invariant. This work is mainly concerned with Markov chains that have time-varying transition probabilities. The switched Markov jump linear system abstraction of Section 3.3 represents a Markov jump linear system with time-varying Markov chain transition probabilities that are a priori unknown.

3.2 Markov Jump Linear Systems

In this section, a Markov jump linear system is defined. The Markov chain associated with a Markov jump linear system is characterized by known transition probabilities. Fix a probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ and let $\theta : \mathbb{N}_0 \times \Omega \to \mathcal{N}$ be a finite Markov chain defined on the probability space with initial distribution $p(0)$ and sequence $P : \mathbb{N} \to \mathbb{T}_N$ of transition probability matrices. Let

$$\mathcal{G} = \{(A(1), B(1), C(1), D(1)), \ldots, (A(N), B(N), C(N), D(N))\}$$

be a finite set of matrices where $A(i) \in \mathbb{R}^{n\times n}$, $B(i) \in \mathbb{R}^{n\times m}$, $C(i) \in \mathbb{R}^{p\times n}$, and $D(i) \in \mathbb{R}^{p\times m}$. The discrete-time *Markov jump linear system*, denoted $(\mathcal{G}, P, p(0))$, is defined by the difference equation

$$\begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} A(\theta(k)) & B(\theta(k)) \\ C(\theta(k)) & D(\theta(k)) \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \tag{3.1}$$

and initial condition $x(0)$, where $x(k) \in \mathbb{R}^n$ is the state vector, $w(k) \in \mathbb{R}^m$ is a disturbance vector, and $z(k) \in \mathbb{R}^p$ is an error vector.

Definition 3.1. If θ is time-homogeneous, $(\mathcal{G}, P, p(0))$ is a *time-homogeneous Markov jump linear system.* Otherwise, $(\mathcal{G}, P, p(0))$ is a *time-inhomogeneous Markov jump linear system.*

The (random) state transition matrix

$$\Phi(k,j) = \begin{cases} A(\theta(k-1))A(\theta(k-2))\cdots A(\theta(j)) & : \ k > j \\ I & : \ k = j \end{cases} \tag{3.2}$$

is defined only for $k \geq j$. The solution of the state in (3.1) can be established via forward iteration (see, e.g., [54])

$$x(k) = \Phi(k,0)x(0) + \sum_{j=0}^{k-1} \Phi(k,j+1)B(\theta(j))w(j) \tag{3.3}$$

where $k \geq 1$. The solution formula in (3.3) appears as the sum of a zero-input ($w \equiv 0$) response and a zero-state response ($x(0) = 0$).

Definition 3.2 (Def. 3 of [59])**.** The Markov jump linear system $(\mathcal{G}, P, p(0))$ is *weakly controllable* if for every initial (x_0, θ_0) and any final (x_f, θ_f), there exists a finite time k_f and an input w_c such that $\mathbf{P}\{x(k_f) = x_f, \theta(k_f) = \theta_f \mid x(0) = x_0, \theta(0) = \theta_0\} > 0$.

3.2.1 Independent jump linear systems

An *independent jump linear system* is simply a Markov jump linear system where θ is an independent sequence of random variables. As discussed in Remark 2.24, if θ is an independent sequence of random variables, then each row of $P(k)$ is identical. If, in addition, θ is time-homogeneous, then $\theta(1), \theta(2), \ldots$ is an independent and identically distributed sequence.

3.3 Switched Markov Jump Linear Systems

In this section, a time-inhomogeneous Markov jump linear system with a priori unknown time-varying transition probabilities is examined. It is assumed that the sequence P of transition probability matrices is not known in advance but takes values in some finite set of matrices $\{\Pi(1), \ldots, \Pi(J)\}$ where $\Pi(s) \in \mathbb{T}_N$ for $s \in \mathcal{J} = \{1, \ldots, J\}$. Thus, $P(k) = \Pi(\psi(k))$ for some a priori unknown sequence $\psi : \mathbb{N} \to \mathcal{J}$. The notation $\pi_{ij}(\psi(k))$ denotes the ij-th element of matrix $\Pi(\psi(k))$.

A *switched Markov jump linear system*, denoted $(\mathcal{G}, \Pi, \Psi, p(0))$, is defined to be the family of Markov jump linear systems

$$\{(\mathcal{G}, \Pi \circ \psi, p(0)) : \psi \in \Psi\} \tag{3.4}$$

where Ψ is the application-specific set of all sequences that may occur. Depending on the application, Ψ could be the set of *all* sequences in $\mathcal{J}$. Alternatively, some applications may disallow certain sequences from occurring due to problem-specific information available. Each member of the family in (3.4) is driven by a different time-inhomogeneous Markov chain with transition probabilities given by $\Pi(\psi(k))$, $k \in \mathbb{N}$ for some sequence $\psi \in \Psi$. The *switched* modifier here is used to draw analogy to deterministic switched systems (see, e.g., [40]), and ψ is often referred to as a switching sequence.

For $M \in \mathbb{N}$ and $k \in \mathbb{N}_0$, the notation

$$\psi_M(k) = (\psi(k+1), \psi(k+2), \ldots, \psi(k+M))$$

is used to denote the next M values of the switching sequence. If $M = 0$, the convention that $\psi_M(k) = \emptyset$, the empty sequence, is used. Additionally, the set of all sequences of length M that occur in Ψ is denoted

$$\Psi_M = \{\psi_M(t) : \psi \in \Psi, t \in \mathbb{N}_0\}$$

and is a subset of $\mathcal{J}^M$. If $M = 0$, then $\Psi_M = \{\emptyset\}$, and $\mathcal{N} \times \Psi_M = \mathcal{N}$ (see Section 2.2).

3.3.1 Switched independent jump linear systems

A *switched independent jump linear system* is simply a switched Markov jump linear system where θ is an independent sequence of random variables for all $\psi \in \Psi$. If θ is an independent sequence of random variables for all $\psi \in \Psi$, then each row of $\Pi(s)$ must be identical for any $s \in \mathcal{J}$ that occurs in some sequence of Ψ. Since $\pi_{ij}(s)$ is constant for $i = 1, \ldots, N$, the simplified notation $\pi_j(s)$ denotes the ij-th element of matrix

$\Pi(s)$ where i is irrelevant.

3.4 Notes and References

Markov jump linear systems have been thoroughly examined in the literature, with the time-homogeneous case receiving the greatest scrutiny [10, 11, 22, 32, 33, 37, 38, 45, 59]. Markov jump linear systems with time-inhomogeneous structure have also been examined, albeit less extensively [1, 17, 36]. Switched Markov jump linear systems have received very little scrutiny [5].

Chapter 4

Stability

4.1 Introduction

This chapter begins by reviewing stability concepts and results for Markov jump linear systems and independent jump linear systems. To motivate the need for formal stability results for switched Markov jump linear systems, an example is given in Remark 4.9 of Section 4.4 where switching of the Markov chain transition probabilities causes instability. The uniform stability (uniform over all possible sequences of Markov chain transition probabilities) of a switched Markov jump linear system is examined in Section 4.4, and an LMI condition equivalent to uniform stability is obtained. When the Markov chain is an independent sequence of random variables, a computationally simpler LMI condition is shown to be equivalent to uniform stability in Section 4.5.

4.2 Markov Jump Linear Systems

Among the various ways to address stochastic stability for Markov jump linear systems, mean square stability is most appropriate for the approach taken in this dissertation.

Definition 4.1. The Markov jump linear system $(\mathcal{G}, P, p(0))$ is *exponentially mean square stable* if there exist $c \geq 1$ and $0 \leq \lambda < 1$ such that

$$\mathbf{E}\left[\Phi^{\mathsf{T}}(k,j)\Phi(k,j) \mid \theta(j) = i\right] \leq c\lambda^{k-j} I \tag{4.1}$$

for all $i \in \mathcal{N}$ and for all $k, j \in \mathbb{N}_0$ such that $k \geq j \geq 0$.

It is well-known that exponential mean square stability implies almost sure (nonuniform) asymptotic stability where $\mathbf{P}\{\lim_{k\to\infty} \|x(k)\| = 0\} = 1$ (see [33]).

4.2.1 Time-inhomogeneous Markov chain

Exponential mean square stability of the time-inhomogeneous Markov jump linear system $(\mathcal{G}, P, p(0))$ with known transition probabilities may be characterized by a stochastic Lyapunov criterion.

Proposition 4.2 (Thm. 2 of [36], Prop. 1 of [1])**.** *The time-inhomogeneous Markov jump linear system* $(\mathcal{G}, P, p(0))$ *is exponentially mean square stable if and only if there exist* $\eta, \rho, \nu > 0$ *and a function* $X : \mathcal{N} \times \mathbb{N}_0 \to \mathbb{S}_n^+$ *such that*

$$\eta I \leq X(i,k) \leq \rho I \tag{4.2a}$$

$$A^{\mathsf{T}}(i)\tilde{X}(i,k+1)A(i) - X(i,k) \leq -\nu I \tag{4.2b}$$

for all $i \in \mathcal{N}$ *and all* $k \in \mathbb{N}_0$ *where* $\tilde{X}(i,k+1) = \sum_{j=1}^{N} p_{ij}(k+1)X(j,k+1)$. *Moreover, if* (4.2) *holds, one may take* $c = \rho/\eta$ *and* $\lambda = 1 - \nu/\rho$ *in Definition 4.1.*

The matrix inequalities in (4.2) are a stochastic version of the familiar Lyapunov stability criterion (e.g., [54, Thm. 23.3]) for discrete-time linear time-varying systems. If X satisfies (4.2) then $V(i,k,y) = y^{\mathsf{T}}X(i,k)y$ where $i \in \mathcal{N}$, $k \in \mathbb{N}_0$, $y \in \mathbb{R}^n$ is a stochastic Lyapunov function for $(\mathcal{G}, P, p(0))$ and satisfies

$$\eta\|y\|^2 \leq V(i,k,y) \leq \rho\|y\|^2$$

$$\mathbf{E}\left[V(\theta(k+1),k+1,x(k+1)) - V(\theta(k),k,x(k)) \mid x(k) = y, \theta(k) = i\right] \leq -\nu\|y\|^2$$

for all $i \in \mathcal{N}$, $k \in \mathbb{N}_0$, and $y \in \mathbb{R}^n$. Of course, the utility of Proposition 4.2 for assessing stability of a given system is limited due to the infinite number of matrices being prohibitively difficult to compute in practice.

4.2.2 Time-homogeneous Markov chain

If the Markov jump linear system $(\mathcal{G}, P, p(0))$ is time-homogeneous, the stochastic Lyapunov criterion in (4.2) may be simplified.

Proposition 4.3 (Thm. 2.1 of [32], Thm. 2 of [11])**.** *The time-homogeneous Markov jump linear system* $(\mathcal{G}, P, p(0))$ *is exponentially mean square stable if and only if there exist a function* $X : \mathcal{N} \to \mathbb{S}_n^+$ *such that*

$$A^{\mathsf{T}}(i)\tilde{X}(i)A(i) - X(i) < 0 \tag{4.3}$$

for all $i \in \mathcal{N}$ where $\tilde{X}(i) = \sum_{j=1}^{N} p_{ij} X(j)$. (The time dependence of $p_{ij}(k)$ has been omitted since P is constant.)

4.3 Independent Jump Linear Systems

If θ is an independent sequence of random variables, the stability definition in (4.1) is equivalent to a seemingly less stringent requirement.

Lemma 4.4. *Suppose that θ is an independent sequence of random variables. The following are equivalent.*

a) There exist $c_1 > 0$ and $0 \leq \lambda_1 < 1$ such that

$$\mathbf{E}\left[\Phi^{\mathsf{T}}(k,j)\Phi(k,j) \mid \theta(j) = i\right] \leq c_1 \lambda_1^{k-j} I \tag{4.4}$$

for all $i \in \mathcal{N}$ and for all $k, j \in \mathbb{N}_0$ such that $k \geq j \geq 0$.

b) There exist $c_2 > 0$ and $0 \leq \lambda_2 < 1$ such that

$$\mathbf{E}\left[\Phi^{\mathsf{T}}(k,j)\Phi(k,j)\right] \leq c_2 \lambda_2^{k-j} I \tag{4.5}$$

for all $k, j \in \mathbb{N}_0$ such that $k \geq j \geq 0$.

Proof. Note that (4.4) implies (4.5) by applying the expectation operator with respect to i to both sides of (4.4). Conversely, suppose (4.5) holds so that

$$\begin{aligned}
\mathbf{E}\left[\Phi^{\mathsf{T}}(k,j)\Phi(k,j) \mid \theta(j) = i\right] &= A^{\mathsf{T}}(i)\mathbf{E}\left[\Phi^{\mathsf{T}}(k,j+1)\Phi(k,j+1) \mid \theta(j) = i\right] A(i) \\
&= A^{\mathsf{T}}(i)\mathbf{E}\left[\Phi^{\mathsf{T}}(k,j+1)\Phi(k,j+1)\right] A(i) \qquad (4.6) \\
&\leq c_2 \lambda_2^{k-j-1} A(i)^{\mathsf{T}} A(i) \\
&\leq \left(c_2 \lambda_2^{-1} \max_{i \in \mathcal{N}} \lambda_{\max}(A^{\mathsf{T}}(i)A(i))\right) \lambda_2^{k-j} I \qquad (4.7)
\end{aligned}$$

where (4.6) follows by the independence hypothesis, and $\lambda_{\max}(\cdot)$ denotes the maximum eigenvalue. Thus let $\lambda_1 = \lambda_2$ and let c_1 be defined by the term in parentheses in (4.7). □

4.3.1 Time-inhomogeneous sequence

If θ is an independent sequence of random variables, exponential mean square stability may be characterized by a simpler stochastic Lyapunov criterion than Proposition 4.2.

Proposition 4.5. *The time-inhomogeneous independent jump linear system* $(\mathcal{G}, P, p(0))$ *is exponentially mean square stable if and only if there exist* $\eta, \rho, \nu > 0$ *and a function* $X : \mathbb{N}_0 \to \mathbb{S}_n^+$ *such that*

$$\eta I \leq X(k) \leq \rho I \tag{4.8a}$$

$$\sum_{j=1}^{N} p_j(k) A^{\mathsf{T}}(j) X(k+1) A(j) - X(k) \leq -\nu I. \tag{4.8b}$$

for all $k \in \mathbb{N}_0$.

Proof. The proof is provided in Appendix A. □

If X satisfies (4.8) then $V(k, y) = y^{\mathsf{T}} X(k) y$ where $k \in \mathbb{N}_0$ and $y \in \mathbb{R}^n$ is a stochastic Lyapunov function for the independent jump linear system $(\mathcal{G}, P, p(0))$ and satisfies

$$\eta \|y\|^2 \leq V(k, y) \leq \rho \|y\|^2$$

$$\mathbf{E}\left[V(k+1, x(k+1)) - V(k, x(k)) \mid x(k) = y\right] \leq -\nu \|y\|^2$$

for all $k \in \mathbb{N}_0$ and $y \in \mathbb{R}^n$.

4.3.2 Time-homogeneous sequence

If θ is an independent sequence of random variables and time-homogeneous, then the stochastic Lyapunov criterion in (4.8) reduces to solving a single LMI with a single matrix variable.

Proposition 4.6 (Cor. 1 of [11])**.** *The time-homogeneous independent jump linear system* $(\mathcal{G}, P, p(0))$ *is exponentially mean square stable if and only if there exists* $X \in \mathbb{S}_n^+$ *such that*

$$\sum_{j=1}^{N} p_j A^{\mathsf{T}}(j) X A(j) - X < 0.$$

(The time dependence of $p_j(k)$ *has been omitted since* P *is constant.)*

Remark 4.7. Proposition 4.6 requires solving a linear matrix inequality with $n(n+1)/2$ unknowns (since X is symmetric). Proposition 4.3, on the other hand, requires solving a set of linear matrix inequalities with $Nn(n+1)/2$ unknowns.

4.4 Switched Markov Jump Linear Systems

When addressing time-inhomogeneous Markov jump linear systems where the sequence of transition probability matrices is not known a priori, the definition of stability must be modified so that it applies uniformly over all possible sequences of transition probability matrices.

Definition 4.8. The switched Markov jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ is *uniformly exponentially mean square stable* if there exist $c \geq 1$ and $0 \leq \lambda < 1$ such that

$$\mathbf{E}\left[\Phi^{\mathsf{T}}(k,j)\Phi(k,j) \mid \theta(j) = i\right] \leq c\lambda^{k-j} I \tag{4.9}$$

for all $i \in \mathcal{N}$, all $k, j \in \mathbb{N}_0$ such that $k \geq j \geq 0$, and all $\psi \in \Psi$.

Uniformity in Definition 4.8 refers to the uniform decay rate for all $\psi \in \Psi$. Thus, uniform exponential mean square stability ensures that each individual Markov jump linear system in the family $(\mathcal{G}, \Pi, \Psi, p(0))$ is exponentially mean square stable, and all members share a common uniform decay rate.

Remark 4.9. Consider the time-homogeneous Markov jump linear systems $(\mathcal{G}, \Pi \circ \psi_s, p(0))$ where $\psi_s \equiv s$, $s \in \mathcal{J}$ are constant sequences. Stability of each time-homogeneous subsystem is not sufficient for uniform stability of the switched Markov jump linear system. Consider the following example inspired by [40, p. 19].

$$A(1) = \begin{bmatrix} 0.999 & -0.04 \\ 0 & 0.999 \end{bmatrix}, \; A(2) = \begin{bmatrix} 0.999 & 0 \\ 0.005 & 0.999 \end{bmatrix}, \; A(3) = \begin{bmatrix} 0.999 & 0 \\ 0.04 & 0.999 \end{bmatrix}, \; \Psi = \mathcal{J}^{\infty},$$
$$\Pi(1) = \begin{bmatrix} 1/4 & 2/7 & 13/28 \\ 1/4 & 2/7 & 13/28 \\ 1/4 & 2/7 & 13/28 \end{bmatrix}, \; \Pi(2) = \begin{bmatrix} 1/2 & 2/7 & 3/14 \\ 1/2 & 2/7 & 3/14 \\ 1/2 & 2/7 & 3/14 \end{bmatrix}, \; p(0) = \begin{bmatrix} 1/4 & 2/7 & 13/28 \end{bmatrix}.$$

Since each row of $\Pi(s)$, $s = 1, 2$ is identical, each $\psi \in \Psi$ gives rise to an independent sequence θ of random variables. Thus

$$\begin{aligned} \mathbf{E}\left[x(k) \mid x(0) = x_0\right] &= \mathbf{E}\left[A(\theta(k-1))A(\theta(k-2)) \cdots A(\theta(0))\right] x_0 \\ &= \mathbf{E}\left[A(\theta(k-1))\right] \mathbf{E}\left[A(\theta(k-2))\right] \cdots \mathbf{E}\left[A(\theta(0))\right] x_0 \\ &= \mathbf{E}\left[A(\theta(k-1))\right] \mathbf{E}\left[x(k-1) \mid x(0) = x_0\right] \end{aligned}$$

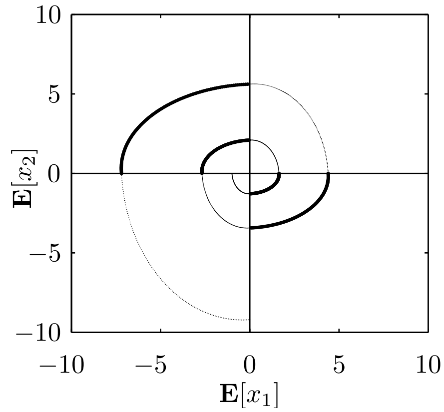


Figure 4.1: The unstable trajectory of $\mathbf{E}[x(k) \mid x(0)]$ with $x(0) = [-1\ 0]^{\mathsf{T}}$ and a particular switching sequence ψ. The smaller points indicate that $\psi(k) = 1$, while the larger points indicate that $\psi(k) = 2$.

where

$$\mathbf{E}\left[A(\theta(k))\right] = \begin{cases} \begin{bmatrix} 0.999 & -0.01 \\ 0.02 & 0.999 \end{bmatrix} & : \ \psi(k) = 1 \text{ or } k = 0 \\ \begin{bmatrix} 0.999 & -0.02 \\ 0.01 & 0.999 \end{bmatrix} & : \ \psi(k) = 2 \end{cases}.$$

Note that each time-homogeneous subsystem $(\mathcal{G}, \Pi \circ \psi_s, p(0))$, $s = 1, 2$ is exponentially mean square stable by the condition in Proposition 4.6. Furthermore, $A(i)$ is Schur for $i = 1, 2, 3$. Even so, Fig. 4.1 shows that a particular switching sequence can generate an unstable trajectory so that the switched Markov jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ is *not* uniformly stable. The switching sequence ψ in Fig. 4.1 is such that $\psi(k) = 1$ when $\mathbf{E}\left[x(k)\right]$ is in the first or third quadrant, and $\psi(k) = 2$ when $\mathbf{E}\left[x(k)\right]$ is in the second or fourth quadrant. Note that $\|E[x(k)]\| \leq E[\|x(k)\|] \leq \sqrt{E[\|x(k)\|^2]}$ so that $\|E[x(k)]\| \to \infty$ in Fig. 4.1 implies $E[\|x(k)\|^2] \to \infty$ so that the Markov jump linear system shown in Fig. 4.1 is *not* exponentially mean square stable.

The goal in this section is to establish a necessary and sufficient condition for uniform stability that is more tractable than an infinite set of LMIs. It is well-known that any stable discrete-time linear time-varying system admits a time-varying quadratic Lyapunov function; it is less well-known that the usual construction (e.g., see [54, Thm. 23.3]) can be modified so that at each time instant, the Lyapunov function depends on only a finite number of the past system parameter matrices [38, Lem. 4]. Inspired by this fact, the following lemma constructs a stochastic Lyapunov function for a stable time-inhomogeneous Markov jump linear system that depends on only a finite number of the future transition probability matrices.

Lemma 4.10. *Suppose system* $(\mathcal{G}, \Pi, \Psi, p(0))$ *is uniformly exponentially mean square stable with stability constant* c *and decay rate* λ *in Definition 4.8. Let* $M = \max(\left\lceil \frac{\log(1/c)}{\log(\lambda)} - 2 \right\rceil, 0)$ *so that* $c\lambda^{M+2} < 1$*. Then for each* $\psi \in \Psi$

$$Y_\psi(i,k) := \sum_{j=k}^{k+M+1} \mathbf{E}\left[\Phi^\mathsf{T}(j,k)\Phi(j,k) \mid \theta(k) = i\right] \tag{4.10}$$

satisfies

$$\eta I \leq Y_\psi(i,k) \leq \rho I \tag{4.11a}$$

$$A^\mathsf{T}(i)\tilde{Y}_\psi(i,k+1)A(i) - Y_\psi(i,k) \leq -\nu I. \tag{4.11b}$$

for all $i \in \mathcal{N}$ *and all* $k \in \mathbb{N}_0$ *where* $\tilde{Y}_\psi(i,k+1) = \sum_{j=1}^N \pi_{ij}(\psi(k+1))Y_\psi(j,k+1)$*,* $\eta = 1$*,* $\rho = c/(1-\lambda)$*, and* $\nu = 1 - c\lambda^{M+2}$*.*

Proof. By (4.9) and (4.10)

$$I \leq Y_\psi(i,k) \leq c\sum_{j=k}^{\infty} \lambda^{j-k} I = \frac{c}{1-\lambda} I. \tag{4.12}$$

so $\eta = 1$ and $\rho = c/(1-\lambda)$. For convenience, define $\Gamma(j,k) = \Phi^\mathsf{T}(j,k)\Phi(j,k)$. Then

$$\begin{aligned}
A^\mathsf{T}(i)\tilde{Y}_\psi(i,k+1)A(i) &= A^\mathsf{T}(i)\sum_{l=1}^{N} \pi_{il}(\psi(k+1)) \sum_{j=k+1}^{k+M+2} \mathbf{E}\left[\Gamma(j,k+1) \mid \theta(k+1) = l\right] A(i) \\
&= \sum_{j=k+1}^{k+M+2} \mathbf{E}\left[A^\mathsf{T}(\theta(k))\Gamma(j,k+1)A(\theta(k)) \mid \theta(k) = i\right] && (4.13) \\
&= \sum_{j=k+1}^{k+M+2} \mathbf{E}\left[\Gamma(j,k) \mid \theta(k) = i\right] \\
&= Y_\psi(i,k) - I + \mathbf{E}\left[\Gamma(k+M+2,k) \mid \theta(k) = i\right] && (4.14)
\end{aligned}$$

where (4.13) follows by interchanging the order of summation and recognizing an iterated expectation. Equations (4.9) and (4.14) show (4.11b) with $\nu = 1 - c\lambda^{M+2} > 0$ by the hypothesis on M. □

Remark 4.11. Note that $\sum_{j=k}^{k+M+1} \Phi^\mathsf{T}(j,k)\Phi(j,k)$ from (4.10) is a function of the random variables $(\theta(k), \theta(k+1), \ldots, \theta(k+M))$. The joint probability distribution

$$\mathbf{P}\left\{\theta(k+1) = i_1, \ldots, \theta(k+M) = i_M \mid \theta(k) = i_0\right\} = \prod_{l=1}^{M} \mathbf{P}\left\{\theta(k+l) = i_l \mid \theta(k+l-1) = i_{l-1}\right\}$$

$$= \prod_{l=1}^{M} \pi_{i_{l-1} i_l}(\psi(k+l))$$

is required to compute the expectation in the definition of $Y_\psi(i,k)$ in Lemma 4.10. Since the joint probability distribution is determined by the conditional value of $\theta(k)$ and the value of $\psi_M(k) = (\psi(k+1), \ldots, \psi(k+M))$, $Y_\psi(i,k)$ may be computed with knowledge of only i and $\psi_M(k)$. Since $\mathcal{N}$ and $\mathcal{J}$ are finite sets, $\cup_{\psi \in \Psi} \operatorname{Im} Y_\psi$ is a finite set of matrices with no more than NJ^M elements. Fix $\psi \in \Psi$ arbitrarily. For $i \in \mathcal{N}$, $k \in \mathbb{N}_0$, and $y \in \mathbb{R}^n$, define

$$V_\psi(i,k,y) := y^\mathsf{T} Y_\psi(i,k) y. \tag{4.15}$$

By (4.11), V_ψ is a quadratic stochastic Lyapunov function for system $(\mathcal{G}, \Pi \circ \psi, p(0))$. Thus, uniform stability of the family $(\mathcal{G}, \Pi, \Psi, p(0))$ guarantees the existence of a finite set of matrices that may be used to construct a time-varying quadratic stochastic Lyapunov function for any member of the family.

The next theorem, inspired by [38, Thm. 9], provides a necessary and sufficient condition, expressed as a set of finite-dimensional LMIs, for uniform exponential mean square stability of a switched Markov jump linear system.

Theorem 4.12. *The switched Markov jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable if and only if there exist $M \in \mathbb{N}_0$ and a function $X : \mathcal{N} \times \Psi_M \to \mathbb{S}_n^+$ such that*

$$A^\mathsf{T}(i) \sum_{j=1}^{N} \pi_{ij}(r_1) X(j, r_2, \ldots, r_{M+1}) A(i) - X(i, r_1, \ldots, r_M) < 0 \tag{4.16}$$

for any $(r_1, \ldots, r_{M+1}) \in \Psi_{M+1}$ and $i \in \mathcal{N}$.

Proof. Suppose there exist M and X such that (4.16) holds. Let $\psi \in \Psi$ be arbitrary. Define $Y_\psi(i,k) := X(i, \psi_M(k))$. Since $\mathcal{N} \times \Psi_{M+1} \subset \mathcal{N} \times \mathcal{J}^{M+1}$ is a finite set, inequality (4.16) holds uniformly, so there exist $\eta, \rho, \nu > 0$ such that

$$\eta I \leq Y_\psi(i,k) \leq \rho I$$
$$A^\mathsf{T}(i) \tilde{Y}_\psi(i,k+1) A(i) - Y_\psi(i,k) \leq -\nu I$$

for all $i \in \mathcal{N}$, $k \in \mathbb{N}_0$, and $\psi \in \Psi$. Thus, $y^\mathsf{T} Y_\psi(i,k) y$ is a stochastic Lyapunov function for the single system $(\mathcal{G}, \Pi \circ \psi, p(0))$ and guarantees exponential mean square stability by Proposition 4.2 with $c = \rho/\eta$ and $\lambda = 1 - \nu/\rho$. Since ψ was arbitrary and the same c and λ work for any $\psi \in \Psi$, $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable.

Conversely, assume that $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable with stability constant c and decay rate λ. Fix $M \in \mathbb{N}_0$ such that $c\lambda^{M+2} < 1$. Let $(i, r_1, \dots, r_{M+1}) \in \mathcal{N} \times \Psi_{M+1}$ be arbitrary. By definition of Ψ_{M+1}, there exist $\psi \in \Psi$ and $t \in \mathbb{N}_0$ such that $\psi_{M+1}(t) = (r_1, \dots, r_{M+1})$. Construct Y_ψ as in Lemma 4.10 and recall from Remark 4.11 that $Y_\psi(i,t)$ depends only on $(i, \psi_M(t))$. Thus, define $X(i, r_1, \dots, r_M) := Y_\psi(i,t)$ and define $X(i, r_2, \dots, r_{M+1}) := Y_\psi(i, t+1)$. One recovers every inequality in (4.16) from (4.11). $\square$

Remark 4.13. For any $M \in \mathbb{N}_0$, $\operatorname{Im} X$ in Theorem 4.12 is finite. Thus, for each $M \in \mathbb{N}_0$, the number of LMIs specified in (4.16) is finite. The stability of a switched Markov jump linear system may be investigated using an iterative algorithm. First, set $M = 0$ and check if the LMIs in (4.16) are feasible. If not, increment M and repeat. If the switched Markov jump linear system is stable, Theorem 4.12 says that this algorithm will stop in a finite amount of time with some finite value of M. A conservative estimate for M is based on the uniform decay rate of the switched Markov jump linear system (see Lemma 4.10).

Remark 4.14. Theorem 4.12 provides a practical approach for investigating the stability of a *single* time-inhomogeneous Markov jump linear system with *known* transition probability matrices that vary in a finite set (let Ψ be the set containing a single sequence).

Remark 4.15. Consider the case when $J = 1$ and $\Psi = \{(1, 1, \dots)\}$. The switched Markov jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ reduces to a single time-homogeneous Markov jump linear system $(\mathcal{G}, \Pi \circ \psi_1, p(0))$ where $\psi_1 \equiv 1$. For any M, the set Ψ_M contains only a single element $(1, \dots, 1)$, and the set $\mathcal{N} \times \Psi_M$ contains only N elements. For $i \in \mathcal{N}$, define $Z(i) := X(i, 1, \dots, 1)$ where X is as in Theorem 4.12. Then (4.16) reduces to

$$A^\mathsf{T}(i) \sum_{j=1}^{N} \pi_{ij}(1) Z(j) A(i) - Z(i) < 0$$

which is the well-known stability criterion (see [32, Thm. 2.1] or [11, Thm. 2]) for time-homogeneous Markov jump linear systems; this well-known result is a corollary of Theorem 4.12.

Remark 4.16. If for some $M \in \mathbb{N}_0$ and some function X, (4.16) is satisfied, then for any $K > M$ there exists $Z : \mathcal{N} \times \Psi_K \to \mathbb{S}_n^+$ such that (4.16) is satisfied with Z in place of X and K in place of M. For each $(i, r_1, \dots, r_K) \in \mathcal{N} \times \Psi_K$, define $Z(i, r_1, \dots, r_K) := X(i, r_1, \dots, r_M)$.

If Ψ is generated via a directed graph (see Section 2.6), then—for purposes of stability—one need only check that (4.16) holds on the strongly connected components of the directed graph. Suppose that Ψ is generated via a directed graph (V, E), and decompose the directed graph (V, E) into n disjoint strongly connected components $(V_1, E_1), \dots, (V_n, E_n)$ (see Section 2.6). For each strongly connected component

Figure 4.2: The directed graph on the left contains two strongly connected components, shown on the right.

(V_l, E_l), define $\Psi(V_l, E_l)$ to be the set of sequences generated by (V_l, E_l) (see Section 2.6). Note that $\Psi(V_1, E_1) \cup \cdots \cup \Psi(V_n, E_n) \subset \Psi$. For example, suppose Ψ is generated by the directed graph with two strongly connected components shown in Fig. 4.2. In this case, $\Psi(V_1, E_1) = \{(1, 1, \dots)\}$, $\Psi(V_2, E_2) = \{(2, 2, \dots)\}$, and $\Psi(V_1, E_1) \cup \Psi(V_2, E_2) \subsetneq \Psi$ since, for example, $(1, 1, 1, 2, 2, \dots) \in \Psi$.

Theorem 4.17. *Suppose that Ψ is generated by the directed graph (V, E) with disjoint strongly connected components $(V_1, E_1), \dots, (V_n, E_n)$, and for $l = 1, 2, \dots, n$, let $\Psi(V_l, E_l)$ be the set of sequences generated by (V_l, E_l). Let $\hat{\Psi} = \Psi(V_1, E_1) \cup \cdots \cup \Psi(V_n, E_n)$. The switched Markov jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable if and only if there exist $M \in \mathbb{N}_0$ and a function $X : \mathcal{N} \times \hat{\Psi}_M \to \mathbb{S}_n^+$ such that*

$$A^{\mathsf{T}}(i) \sum_{j=1}^{N} \pi_{ij}(r_1) X(j, r_2, \dots, r_{M+1}) A(i) - X(i, r_1, \dots, r_M) < 0 \tag{4.17}$$

for any $(r_1, \dots, r_{M+1}) \in \hat{\Psi}_{M+1}$ and $i \in \mathcal{N}$.

Proof. Since $\hat{\Psi} \subset \Psi$, necessity of (4.17) follows from Theorem 4.12. The key to proving sufficiency of (4.17) stems from the fact that the condensation of (V, E) (see Section 2.6) contains no directed cycles, so the tail of any sequence in Ψ must lie in one of the strongly connected components. Indeed, suppose that there exist M and X such that (4.17) holds. Then proceeding as in the proof of Theorem 4.12, there exists some $c \geq 1$ and $0 \leq \lambda < 1$ such that $(\mathcal{G}, \Pi \circ \psi, p(0))$ with $\psi \in \hat{\Psi}$ is exponentially mean square stable where c and λ hold uniformly for $\psi \in \hat{\Psi}$. For simplicity of the argument, $n = 2$ is considered, but the general result is analogous. Let $\psi \in \Psi \setminus \hat{\Psi}$ be arbitrary. Suppose that

$$\begin{cases} \psi(k) \in V_1 \ : \ 0 \leq k \leq k_0 \\ \psi(k) \in V_2 \ : \ k_0 + 1 \leq k < \infty \end{cases}.$$

Let $j, k \in \mathbb{N}_0$ such that $0 \leq j \leq k_0 \leq k$ (the cases where $k_0 < j$ or $k_0 > k$ are trivial). Then

$$\begin{aligned} \mathbf{E}\left[\Phi^{\mathsf{T}}(k, j)\Phi(k, j) \mid \theta(j)\right] &= \mathbf{E}\left[\mathbf{E}\left[\Phi^{\mathsf{T}}(k, j)\Phi(k, j) \mid \theta(j), \dots, \theta(k_0)\right] \mid \theta(j)\right] \\ &= \mathbf{E}\left[\Phi^{\mathsf{T}}(k_0, j)\mathbf{E}\left[\Phi^{\mathsf{T}}(k, k_0)\Phi(k, k_0) \mid \theta(k_0)\right] \Phi(k_0, j) \mid \theta(j)\right] \\ &\leq c\lambda^{k-k_0}\mathbf{E}\left[\Phi^{\mathsf{T}}(k_0, j)\Phi(k_0, j) \mid \theta(j)\right] \end{aligned}$$

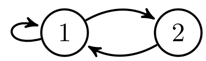


Figure 4.3: Directed graph that determines Ψ in the example of Section 4.4.1.

$$\begin{aligned} &\leq c\lambda^{k-k_0} c\lambda^{k_0-j} I \\ &= c^2 \lambda^{k-j} I \end{aligned}$$

Thus, $(\mathcal{G}, \Pi \circ \psi, p(0))$ is exponentially mean square stable with stability constant c^2 and decay rate λ. In general, the switched Markov jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable with stability constant c^n and decay rate λ. □

4.4.1 Example

Consider the switched Markov jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ with

$$A(1) = \begin{bmatrix} 0.08 & 0.15 & 0.30 \\ 0.20 & 0.60 & 0.10 \\ 0.50 & 0.20 & 0.40 \end{bmatrix}, \quad A(2) = \begin{bmatrix} 0.30 & 0.60 & 0.50 \\ 0.30 & 0.70 & 0.20 \\ 0.90 & 0.70 & 0.10 \end{bmatrix},$$

$$\Pi(1) = \begin{bmatrix} 0.90 & 0.10 \\ 0.80 & 0.20 \end{bmatrix}, \quad \Pi(2) = \begin{bmatrix} 0.60 & 0.40 \\ 0.75 & 0.25 \end{bmatrix}, \quad p(0) = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}.$$

Let Ψ be the set of sequences that could arise from traversing the directed graph in Fig. 4.3. Note that matrix $A(2)$ is not Schur and that $\Pi(2)$ places a larger conditional probability on $\theta(k) = 2$ than $\Pi(1)$. Solving the convex feasibility problem posed in Proposition 4.3, the time-homogeneous Markov jump linear system $(\mathcal{G}, \Pi \circ \psi_2, p(0))$ where $\psi_2 \equiv 2$ is *not* exponentially mean square stable, while the time-homogeneous Markov jump linear system $(\mathcal{G}, \Pi \circ \psi_1, p(0))$ where $\psi_1 \equiv 1$ is exponentially mean square stable. Note from Fig. 4.3 that $\psi_1 \in \Psi$ while $\psi_2 \notin \Psi$. The LMIs in Theorem 4.12 are feasible with $M = 1$ so $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable. If $\psi_2 \in \Psi$ then $(\mathcal{G}, \Pi, \Psi, p(0))$ cannot be uniformly exponentially mean square stable (see the discussion after Definition 4.8). Thus, uniform stability may sometimes be gained by imposing constraints on the set Ψ.

4.5 Switched Independent Jump Linear Systems

If θ is an independent sequence of random variables for all $\psi \in \Psi$, a result similar to Theorem 4.12 can be obtained that is less computationally complex. The derivation closely mirrors that of Section 4.4, so the discourse in this section is brief. Under the independence hypothesis, a simpler characterization of uniform exponential mean square stability is useful.

Lemma 4.18. *The switched independent jump linear system* $(\mathcal{G}, \Pi, \Psi, p(0))$ *is uniformly exponentially mean square stable if and only if there exist* $c \geq 1$ *and* $0 \leq \lambda < 1$ *such that*

$$\mathbf{E}\left[\Phi^{\mathsf{T}}(k,j)\Phi(k,j)\right] \leq c\lambda^{k-j} I \tag{4.18}$$

for all $k, j \in \mathbb{N}_0$ *such that* $k \geq j \geq 0$*, and all* $\psi \in \Psi$*.*

Proof. See Lemma 4.4. □

Lemma 4.19. *Suppose the switched independent jump linear system* $(\mathcal{G}, \Pi, \Psi, p(0))$ *is uniformly exponentially mean square stable and* $M \in \mathbb{N}_0$ *is such that* $c\lambda^{M+1} < 1$ *where* c, λ *are as in* (4.18)*. Then for each* $\psi \in \Psi$

$$Y_\psi(k) := \sum_{j=k}^{k+M} \mathbf{E}\left[\Phi^{\mathsf{T}}(j,k)\Phi(j,k)\right] \tag{4.19}$$

satisfies

$$\eta I \leq Y_\psi(k) \leq \rho I \tag{4.20a}$$

$$\sum_{j=1}^{N} \pi_j(\psi(k)) A^{\mathsf{T}}(j) Y_\psi(k+1) A(j) - Y_\psi(k) \leq -\nu I. \tag{4.20b}$$

for all $k \in \mathbb{N}$ *where* $\eta = 1$*,* $\rho = c/(1-\lambda)$*, and* $\nu = 1 - c\lambda^{M+1}$*.*

Proof. The inequalities in (4.20a) follow directly from (4.19). Proceeding as in the proof of Proposition 4.5,

$$\begin{aligned}\sum_{j=1}^{N} \pi_j(\psi(k)) A^{\mathsf{T}}(j) Y_\psi(k+1) A(j) - Y_\psi(k) &= -I + \mathbf{E}\left[\Phi^{\mathsf{T}}(k+M+1,k)\Phi(k+M+1,k)\right] \\ &\leq -(1 - c\lambda^{M+1}) I\end{aligned}$$

By hypothesis, $\nu = 1 - c\lambda^{M+1} > 0$. □

Remark 4.20. Note that $\sum_{j=k}^{k+M} \Phi^{\mathsf{T}}(j,k)\Phi(j,k)$ from (4.10) is a function of $(\theta(k), \ldots, \theta(k+M-1))$. One may easily compute $Y_\psi(k)$ in (4.10) given the joint probability distribution

$$\mathbf{P}\left\{\theta(k), \ldots, \theta(k+M-1)\right\} = \mathbf{P}\left\{\theta(k)\right\} \cdots \mathbf{P}\left\{\theta(k+M-1)\right\}$$

For $k \geq 1$, $\mathbf{P}\{\theta(k)\}$ is an element of the matrix $\Pi(\psi(k))$ so $Y_\psi(k)$ may be computed with knowledge of only $(\psi(k), \ldots, \psi(k+M-1))$. Since $\mathcal{J}$ is a finite set, $\cup_{\psi \in \Psi} \operatorname{Im} Y_\psi$ is a finite set of matrices with no more than J^M elements. Fix $\psi \in \Psi$ arbitrarily. For $k \in \mathbb{N}_0$ and $y \in \mathbb{R}^n$, define

$$V_\psi(k, y) := y^\mathsf{T} Y_\psi(k) y.$$

By (4.20), V_ψ is a quadratic stochastic Lyapunov function for system $(\mathcal{G}, \Pi \circ \psi, p(0))$. Thus, uniform stability of the family $(\mathcal{G}, \Pi, \Psi, p(0))$ guarantees the existence of a finite set of matrices that may be used to construct a time-varying quadratic stochastic Lyapunov function for any member of the family.

Theorem 4.21. *The switched independent jump linear system* $(\mathcal{G}, \Pi, \Psi, p(0))$ *is uniformly exponentially mean square stable if and only if there exist* $M \in \mathbb{N}_0$ *and a function* $X : \Psi_M \to \mathbb{S}_n^+$ *such that*

$$\sum_{j=1}^{N} \pi_j(r_0) A^\mathsf{T}(j) X(r_1, \ldots, r_M) A(j) - X(r_0, \ldots, r_{M-1}) < 0 \tag{4.21}$$

for any $(r_0, \ldots, r_M) \in \Psi_{M+1}$.

Proof. Suppose there exist M and X such that (4.21) holds. Let $\psi \in \Psi$ be arbitrary. Define $Y_\psi(k) := X(\psi(k), \psi(k+1), \ldots, \psi(k+M-1))$. Since $\Psi_{M+1} \subset \mathcal{J}^{M+1}$ is a finite set, inequality (4.21) holds uniformly, so there exist $\eta, \rho, \nu > 0$ such that

$$\eta I \leq Y_\psi(k) \leq \rho I$$
$$\sum_{j=1}^{N} \pi_j(\psi(k)) A^\mathsf{T}(j) Y_\psi(k+1) A(j) - Y_\psi(k) \leq -\nu I$$

for all $k \in \mathbb{N}_0$ and $\psi \in \Psi$. Thus, $y^\mathsf{T} Y_\psi(k) y$ is a stochastic Lyapunov function for the single system $(\mathcal{G}, \Pi \circ \psi, p(0))$ but is only defined for $k \in \mathbb{N}$. The first half of the proof of Proposition 4.5 shows that

$$\mathbf{E}\left[\Phi^\mathsf{T}(k, j)\Phi(k, j)\right] \leq (\rho/\eta)\lambda^{k-j} I \tag{4.22}$$

for $k \geq j \geq 1$ where $\lambda = 1 - \nu/\rho$. When $j = 0$,

$$\begin{aligned}
\mathbf{E}\left[\Phi^\mathsf{T}(k, 0)\Phi(k, 0)\right] &= \mathbf{E}\left[A^\mathsf{T}(\theta(0))\mathbf{E}\left[\Phi^\mathsf{T}(k, 1)\Phi(k, 1)\right] A(\theta(0))\right] && (4.23)\\
&\leq (\rho/\eta)\lambda^{k-1}\mathbf{E}\left[A^\mathsf{T}(\theta(0))A(\theta(0))\right] && (4.24)\\
&\leq \left(\rho\lambda^{-1} \max_{i \in \mathcal{N}} \lambda_{\max}(A^\mathsf{T}(i)A(i))/\eta\right) \lambda^k I && (4.25)
\end{aligned}$$

where (4.23) follows from the independence hypothesis on θ, and (4.24) follows from (4.22). Let c be the maximum of ρ/η and the expression in parentheses in (4.25) so that the single system $(\mathcal{G}, \Pi \circ \psi, p(0))$ is exponentially mean square stable with this c and λ. Since ψ was arbitrary and the same c and λ work for any $\psi \in \Psi$, $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable.

Conversely, assume that $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable. Let $M \in \mathbb{N}_0$ be such that $c\lambda^{M+1} < 1$ where c and λ are as in (4.18). Let $(r_0, r_1, \ldots, r_M) \in \Psi_{M+1}$ be arbitrary. By definition of Ψ_{M+1}, there exist $\psi \in \Psi$ and $t \in \mathbb{N}_0$ such that $(\psi(t), \psi(t+1), \ldots, \psi(t+M)) = (r_0, r_1, \ldots, r_M)$. Construct Y_ψ as in (4.19) and recall from Remark 4.20 that $Y_\psi(t)$ depends only on $(\psi(t), \ldots, \psi(t+M-1))$. Thus, define $X(r_0, \ldots, r_{M-1}) := Y_\psi(t)$ and define $X(r_1, \ldots, r_M) := Y_\psi(t+1)$. One recovers every inequality in (4.21) from (4.20). $\square$

Remark 4.22. For each M, Theorem 4.21 requires solving a set of LMIs with no more than $J^M n(n+1)/2$ unknowns (some M-length sequences of elements in $\mathcal{J}$ may not be in Ψ_M so there are at most J^M unknown symmetric matrices). In contrast, for each M, Theorem 4.12 requires solving a set of LMIs with no more than $NJ^M n(n+1)/2$ unknowns.

Chapter 5

Disturbance Attenuation

5.1 Introduction

This chapter begins by reviewing disturbance attenuation concepts and results for Markov jump linear systems and independent jump linear systems. To motivate the need for formal disturbance attenuation results for switched Markov jump linear systems, an example is given in Remark 5.12 of Section 5.4 where a single switch of the Markov chain transition probabilities causes a system to lose the contractiveness property. The uniform contractiveness (uniform over all possible sequences of Markov chain transition probabilities) of a switched Markov jump linear system is examined in Section 5.4, and an LMI condition equivalent to uniform contractiveness is obtained. When the Markov chain is an independent sequence of random variables, a computationally simpler LMI condition is shown to be equivalent to uniform contractiveness in Section 5.5.

5.2 Markov Jump Linear Systems

At least two notions of disturbance attenuation for Markov jump linear systems have been examined in the literature [37, 59]. Mean square attenuation best suits the approach taken in this dissertation.

Definition 5.1. The Markov jump linear system $(\mathcal{G}, P, p(0))$ is *mean square strictly contractive* if there exists $\gamma \in (0,1)$ such that whenever $x(0) = 0$

$$\|z\|_{2,e} \leq \gamma \|w\|_{2,e} \tag{5.1}$$

for all $w \in \ell_e^2$.

Remark 5.2. If $(\mathcal{G}, P, p(0))$ is not mean square strictly contractive, there may still exist some strict bound $0 < \gamma_0 < \infty$ such that $\|z\|_{2,e} < \gamma_0 \|w\|_{2,e}$ uniformly for all $w \in \ell_e^2$. Stated another way, there exist $0 < \gamma_0 < \infty$ and $\gamma \in (0,1)$ such that $\|z\|_{2,e} \leq \gamma\gamma_0 \|w\|_{2,e}$ for all $w \in \ell_e^2$, which is equivalent to $\|(1/\gamma_0)z\|_{2,e} \leq \gamma \|w\|_{2,e}$ for all $w \in \ell_e^2$. Hence, it suffices to consider the mean square strict contractiveness of $(\mathcal{G}_0, P, p(0))$ where $\mathcal{G}_0 = \{(A(i), B(i), (1/\gamma_0)C(i), (1/\gamma_0)D(i)) : i \in \mathcal{N}\}$. Taking the infimum of the set $\{\gamma_0 : \|z\|_{2,e} < \gamma_0 \|w\|_{2,e}, \forall w \in \ell_e^2\}$ produces the ℓ_e^2-induced norm of system $(\mathcal{G}, P, p(0))$ viewed as a linear operator on ℓ_e^2 (see Definition 2.9). The ℓ_e^2-induced norm of system $(\mathcal{G}, P, p(0))$ is denoted by $\|(\mathcal{G}, P, p(0))\|_\infty$.

5.2.1 Time-inhomogeneous Markov chain

Mean square strict contractiveness of the time-inhomogeneous Markov jump linear system $(\mathcal{G}, P, p(0))$ with known transition probabilities may be characterized by a stochastic Kalman-Yakubovich-Popov (KYP) criterion.

Proposition 5.3 (Thm. 1 of [1]). *Assume $(\mathcal{G}, P, p(0))$ is weakly controllable and $p_i(k) > 0$ for all $i \in \mathcal{N}$ and $k \in \mathbb{N}_0$. The time-inhomogeneous Markov jump linear system $(\mathcal{G}, P, p(0))$ is exponentially mean square stable and mean square strictly contractive if and only if there exist $\eta, \rho, \nu > 0$ and a function $X : \mathcal{N} \times \mathbb{N}_0 \to \mathbb{S}_n^+$ such that*

$$\eta I \leq X(i,k) \leq \rho I \tag{5.2a}$$

$$\begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix}^\mathsf{T} \begin{bmatrix} \tilde{X}(i,k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix} - \begin{bmatrix} X(i,k) & 0 \\ 0 & I \end{bmatrix} \leq -\nu I \tag{5.2b}$$

for all $i \in \mathcal{N}$ and all $k \in \mathbb{N}_0$ where $\tilde{X}(i,k+1) = \sum_{j=1}^{N} p_{ij}(k+1) X(j,k+1)$.

The matrix inequalities in (5.2) are a stochastic version of the KYP lemma (e.g., see [14, Cor. 12]) for discrete-time linear time-varying systems. When $(\mathcal{G}, P, p(0))$ is contractive, the solutions to the inequalities in (5.2) are obtained by solving associated Riccati difference equations [1]. Conversely, if X satisfies (5.2) then $V(i,k,y) = y^\mathsf{T} X(i,k) y$ where $i \in \mathcal{N}$, $k \in \mathbb{N}_0$, $y \in \mathbb{R}^n$ is a *stochastic storage function* (see [65]) for $(\mathcal{G}, P, p(0))$ and satisfies

$$\eta \|y\|^2 \leq V(i,k,y) \leq \rho \|y\|^2$$
$$\mathbf{E}\left[V(\theta(k+1), k+1, x(k+1)) - V(\theta(k), k, x(k)) + \|z(k)\|^2\right] \leq \gamma^2 \mathbf{E}\left[\|w(k)\|^2\right] \tag{5.3}$$

for all $i \in \mathcal{N}$, $k \in \mathbb{N}_0$, $y \in \mathbb{R}^n$, and $w \in \ell_e^2$. That (5.3) follows from (5.2b) can be seen by multiplying inequality (5.2b) on the left by $[x^\mathsf{T}(k)\ w^\mathsf{T}(k)]$ and on the right by $[x^\mathsf{T}(k)\ w^\mathsf{T}(k)]^\mathsf{T}$ and applying the expectation operator as in [1]. The stochastic storage function V may be thought of as a generalized measure of the internally stored energy in the system. Inequality (5.3) is sometimes called the *dissipation inequality* (see [65]) since the expected increase in stored energy is not greater than the expected energy supply rate $\mathbf{E}\left[\gamma^2\|w(k)\|^2 - \|z(k)\|^2\right]$ [66, Sec. 11, Rmk. 9].

Note that a hypothesis in the statement of Proposition 5.3 can be expressed as a requirement on the sequence P of stochastic matrices and the initial distribution $p(0)$.

Proposition 5.4. *For all $i \in \mathcal{N}$ and all $k \in \mathbb{N}_0$, $p_i(k) > 0$ if and only if for all $k \in \mathbb{N}$, each column of $P(k)$ is nonzero, and $p_i(0) > 0$ for all $i \in \mathcal{N}$.*

Proof. The result can be shown using induction and the identity

$$\mathbf{P}\{\theta(k) = i\} = \sum_{l=1}^{N} p_{li}(k)\mathbf{P}\{\theta(k-1) = l\}. \qquad \square$$

5.2.2 Time-homogeneous Markov chain

If the Markov jump linear system $(\mathcal{G}, P, p(0))$ is time-homogeneous, the stochastic KYP criterion in (5.2) may be simplified.

Proposition 5.5 (Thm. 2 of [59])**.** *Assume $(\mathcal{G}, P, p(0))$ is weakly controllable. The time-homogeneous Markov jump linear system $(\mathcal{G}, P, p(0))$ is exponentially mean square stable and mean square strictly contractive if and only if there exist a function $X : \mathcal{N} \to \mathbb{S}_n^+$ such that*

$$\begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix}^\mathsf{T} \begin{bmatrix} \tilde{X}(i) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix} - \begin{bmatrix} X(i) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{5.4}$$

for all $i \in \mathcal{N}$ where $\tilde{X}(i) = \sum_{j=1}^N p_{ij} X(j)$. (The time dependence of $p_{ij}(k)$ has been omitted since P is constant.)

Remark 5.6. The weak controllability hypothesis of Proposition 5.5, as stated in [59], may be replaced by a weaker hypothesis on the initial distribution $p(0)$ and the transition probability matrix P (see Remark 5.32 of Section 5.4).

5.3 Independent Jump Linear Systems

If θ is an independent sequence of random variables, Propositions 5.3 and 5.5 may still be used to investigate contractiveness of $(\mathcal{G}, P, p(0))$ since θ is trivially a Markov chain. However, different results that are more computationally efficient are also available.

5.3.1 Time-inhomogeneous sequence

If θ is an independent sequence and time-inhomogeneous, mean square contractiveness of $(\mathcal{G}, P, p(0))$ may be characterized by a simpler stochastic KYP criterion than Proposition 5.3.

Proposition 5.7. *The time-inhomogeneous independent jump linear system $(\mathcal{G}, P, p(0))$ is exponentially mean square stable and mean square strictly contractive if and only if there exist $\eta, \rho, \nu > 0$ and a function $X : \mathbb{N}_0 \mapsto \mathbb{S}_n^+$ such that*

$$\eta I \le X(k) \le \rho I \tag{5.5a}$$

$$\sum_{j=1}^{N} p_j(k) \begin{bmatrix} A(j) & B(j) \\ C(j) & D(j) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} X(k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(j) & B(j) \\ C(j) & D(j) \end{bmatrix} - \begin{bmatrix} X(k) & 0 \\ 0 & I \end{bmatrix} \le -\nu I \tag{5.5b}$$

for all $k \in \mathbb{N}_0$.

Proof. The proof Proposition 5.7 may be found in Appendix A. □

If X satisfies (5.5) then $V(k, y) = y^{\mathsf{T}} X(k) y$ where $k \in \mathbb{N}_0$ and $y \in \mathbb{R}^n$ is a stochastic storage function for $(\mathcal{G}, P, p(0))$ and satisfies

$$\eta \|y\|^2 \le V(k, y) \le \rho \|y\|^2$$
$$\mathbf{E}\left[V(k+1, x(k+1)) - V(k, x(k)) + \|z(k)\|^2\right] \le \gamma^2 \mathbf{E}\left[\|w(k)\|^2\right]$$

for all $k \in \mathbb{N}_0$, $y \in \mathbb{R}^n$, and $w \in \ell_e^2$.

5.3.2 Time-homogeneous sequence

If θ is an independent sequence of random variables and time-homogeneous, then the stochastic KYP criterion in (5.5) reduces to solving a single LMI with a single matrix variable.

Proposition 5.8 (Thm. 4 of [61])**.** *Suppose that the time-homogeneous independent jump linear system $(\mathcal{G}, P, p(0))$ is weakly controllable. The time-homogeneous independent jump linear system $(\mathcal{G}, P, p(0))$ is* *exponentially mean square stable and mean square strictly contractive if and only if there exists $X \in \mathbb{S}_n^+$ such that*

$$\sum_{j=1}^{N} p_j \begin{bmatrix} A(j) & B(j) \\ C(j) & D(j) \end{bmatrix}^\mathsf{T} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(j) & B(j) \\ C(j) & D(j) \end{bmatrix} - \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} < 0$$

(The time-dependence of $p_j(k)$ has been omitted since P is constant.)

Remark 5.9. The weak controllability hypothesis of Proposition 5.8, as stated in [61], is shown to be unnecessary in Remark 5.49 of Section 5.5.

5.4 Switched Markov Jump Linear Systems

Disturbance attenuation for a time-inhomogeneous Markov jump linear system where the sequence of transition probability matrices is not known a priori is now addressed. Accordingly, the definition of disturbance attenuation must be modified so that it applies uniformly over all possible sequences of transition probability matrices.

Definition 5.10. The switched Markov jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ is *uniformly mean square strictly contractive* if there exists $\gamma \in (0, 1)$ such that whenever $x(0) = 0$

$$\|z\|_{2,e} \leq \gamma \|w\|_{2,e} \tag{5.6}$$

for all $w \in \ell_e^2$ and all $\psi \in \Psi$.

Uniformity in Definition 5.10 refers to the uniform bound, γ, holding for all $\psi \in \Psi$. Thus, uniform mean square strict contractiveness ensures that each individual Markov jump linear system in the family $(\mathcal{G}, \Pi, \Psi, p(0))$ is mean square strictly contractive, and all members share a common bound, γ.

Remark 5.11. Define $\|(\mathcal{G}, \Pi, \Psi, p(0))\|_\infty = \inf\{\gamma_0 \in (0, \infty) : \|z\|_{2,e} < \gamma_0 \|w\|_{2,e}, \forall w \in \ell_e^2, \forall \psi \in \Psi\}$. As in Remark 5.2, if $(\mathcal{G}, \Pi, \Psi, p(0))$ is not uniformly contractive, the quantity $\|(\mathcal{G}, \Pi, \Psi, p(0))\|_\infty$ may still be finite. Note that $\|(\mathcal{G}, \Pi, \Psi, p(0))\|_\infty$ is uniform over all $\psi \in \Psi$.

Remark 5.12. Consider the time-homogeneous systems $(\mathcal{G}, \Pi \circ \psi_s, p(0))$ where $\psi_s \equiv s$, $s \in \mathcal{J}$ are constant sequences. Contractiveness of each time-homogeneous subsystem is not sufficient for uniform contractiveness of the switched Markov jump linear system. Consider the scalar switched Markov jump linear system

$$A(1) = 0.3, \quad B(1) = 1.5, \quad C(1) = 0.2, \quad D(1) = 0,$$

$$A(2) = 0.4, \quad B(2) = 0.5, \quad C(2) = 0.85, \quad D(2) = 0,$$

$$\Pi(1) = \begin{bmatrix} 0.8 & 0.2 \\ 0.8 & 0.2 \end{bmatrix}, \quad \Pi(2) = \begin{bmatrix} 0.3 & 0.7 \\ 0.3 & 0.7 \end{bmatrix}, \quad p(0) = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}, \quad \Psi = \mathcal{J}^\infty$$

Note that each time-homogeneous subsystem $(\mathcal{G}, \Pi \circ \psi_s, p(0))$, $s = 1, 2$ is exponentially mean square stable and mean square strictly contractive by the condition in Proposition 5.8. Furthermore, the linear time-invariant discrete-time system with parameter matrices $(A(i), B(i), C(i), D(i))$ is contractive for $i = 1, 2$. Even so, a particular switching sequence shows that the switched Markov jump linear system is not uniformly contractive. Each row of $\Pi(s)$, $s = 1, 2$ is identical so each $\psi \in \Psi$ gives rise to an independent sequence θ of random variables. Fix $\tau \in \mathbb{N}$ arbitrarily and consider the input and switching sequences

$$w(k) = \begin{cases} 1 \ : \ k = \tau \\ 0 \ : \ k \neq \tau \end{cases}, \qquad \psi(k) = \begin{cases} 1 \ : \ 1 \leq k \leq \tau \\ 2 \ : \ k \geq \tau + 1 \end{cases}.$$

Let $x(0) = 0$. Then

$$z(k) = \begin{cases} 0 & : \ 0 \leq k \leq \tau \\ C(\theta(k))\Phi(k, \tau+1)B(\theta(\tau)) & : \ k \geq \tau + 1 \end{cases}$$

and

$$\begin{aligned} &\sum_{k=0}^{\infty} \mathbf{E}\left[z^{\mathsf{T}}(k)z(k)\right] \\ &\quad = \sum_{k=\tau+1}^{\infty} \mathbf{E}\left[B^2(\theta(\tau))C^2(\theta(k))\Phi^2(k, \tau+1)\right] \\ &\quad = \mathbf{E}\left[B^2(\theta(\tau))\right] \left(\mathbf{E}\left[C^2(\theta(\tau+1))\right] + \sum_{k=\tau+2}^{\infty} \mathbf{E}\left[C^2(\theta(k))\right] \mathbf{E}\left[A^2(\theta(k-1))\right] \cdots \mathbf{E}\left[A^2(\theta(\tau+1))\right] \right) \end{aligned} \tag{5.7}$$

where (5.7) follows since $\theta(\tau), \theta(\tau+1), \ldots, \theta(k)$ are independent. Now

$$\mathbf{E}\left[A^2(\theta(k))\right] = \begin{cases} (0.8)(0.3)^2 + (0.2)(0.4)^2 \ : \ 0 \leq k \leq \tau \\ (0.3)(0.3)^2 + (0.7)(0.4)^2 \ : \ k \geq \tau + 1 \end{cases} = \begin{cases} 0.104 \ : \ 0 \leq k \leq \tau \\ 0.139 \ : \ k \geq \tau + 1 \end{cases} \tag{5.8a}$$

$$\mathbf{E}\left[B^2(\theta(k))\right] = \begin{cases} (0.8)(1.5)^2 + (0.2)(0.5)^2 \ : \ 0 \le k \le \tau \\ (0.3)(1.5)^2 + (0.7)(0.5)^2 \ : \ k \ge \tau + 1 \end{cases} = \begin{cases} 1.85 \ : \ 0 \le k \le \tau \\ 0.85 \ : \ k \ge \tau + 1 \end{cases} \tag{5.8b}$$

$$\mathbf{E}\left[C^2(\theta(k))\right] = \begin{cases} (0.8)(0.2)^2 + (0.2)(0.85)^2 \ : \ 0 \le k \le \tau \\ (0.3)(0.2)^2 + (0.7)(0.85)^2 \ : \ k \ge \tau + 1 \end{cases} = \begin{cases} 0.1765 \quad : \ 0 \le k \le \tau \\ 0.51775 \ : \ k \ge \tau + 1 \end{cases} \tag{5.8c}$$

By (5.7) and (5.8)

$$\begin{aligned} \sum_{k=0}^{\infty} \mathbf{E}\left[z^{\mathsf{T}}(k) z(k)\right] &= (1.85)(0.51775) \sum_{l=0}^{\infty} (0.139)^l \\ &= \frac{(1.85)(0.51775)}{1 - (0.139)} > 1 \end{aligned} \tag{5.9}$$

Since $\|w\|_{2,e} = 1$, inequality (5.9) shows that the switched Markov jump linear system is not uniformly contractive. Remarkably, ψ switches only once in this example.

Remark 5.13. Let $\psi_s \equiv s$, $s \in \mathcal{J}$ be the constant switching sequences in Ψ. The example in Remark 5.12 shows that, in general, there can be separation between $\|(\mathcal{G}, \Pi, \Psi, p(0))\|_\infty$ and $\max_{s \in \mathcal{J}} \|(\mathcal{G}, \Pi \circ \psi_s, p(0))\|_\infty$.

The goal of this section is to establish a KYP-like result for switched Markov jump linear systems in terms of finite-dimensional LMIs similar to Theorem 4.12. The main result can be found in Theorem 5.29. The necessity of the LMIs is the difficult part of the proof and hinges on the existence of the matrix-valued functions in Lemma 5.14. Like Lemma 4.10, at any time instant each matrix-valued function depends only on a finite number of the future transition probability matrices.

Lemma 5.14. *Assume $p_i(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$. If $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable and uniformly mean square strictly contractive then there exist $\eta, \rho, \nu > 0$ and $M \in \mathbb{N}_0$ such that for each $\psi \in \Psi$, there exists $Y_\psi : \mathcal{N} \times \mathbb{N}_0 \to \mathbb{S}_n^+$ such that $Y_\psi(i,k)$ depends only on i and $\psi_M(k)$ and satisfies*

$$\eta I \le Y_\psi(i,k) \le \rho I \tag{5.10a}$$

$$\begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \tilde{Y}_\psi(i,k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix} - \begin{bmatrix} Y_\psi(i,k) & 0 \\ 0 & I \end{bmatrix} \le -\nu I \tag{5.10b}$$

for all $i \in \mathcal{N}$ and all $k \in \mathbb{N}_0$ where $\tilde{Y}_\psi(i,k+1) = \sum_{j=1}^{N} \pi_{ij}(\psi(k+1)) Y_\psi(j,k+1)$.

The construction of the functions Y_ψ, $\psi \in \Psi$ requires the intermediate results contained in this section up to and including Lemma 5.26, and the proof of Lemma 5.14 follows directly from Lemma 5.27. For the moment, suppose that functions Y_ψ, $\psi \in \Psi$ have been found that satisfy Lemma 5.14 and define

$$V_\psi(i,k,y) = y^\mathsf{T} Y_\psi(i,k) y \tag{5.11}$$

for $i \in \mathcal{N}$, $k \in \mathbb{N}_0$, and $y \in \mathbb{R}^n$. Then V is a quadratic stochastic storage function for system $(\mathcal{G}, \Pi \circ \psi, p(0))$ that at each time instant depends only on i and $\psi_M(k)$. Since $\mathcal{N}$ and $\mathcal{J}$ are finite sets, $\cup_{\psi\in\Psi} \operatorname{Im} Y_\psi$ is a finite set of matrices with no more than NJ^M elements. Thus, uniform stability and contractiveness of the switched Markov jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ guarantees the existence of a finite set of matrices that may be used to construct a time-varying quadratic stochastic storage function for any individual Markov jump linear system in the family.

Riccati difference equations defined in terms of the following operators play a key role in the construction of the functions Y_ψ, $\psi \in \Psi$ in Lemma 5.14.

Definition 5.15. Let $\mathcal{G}$ be given. For $i \in \mathcal{N}$ and $X \in \mathbb{S}_n$, define

$$\begin{aligned}
\mathcal{L}(i,X) &= A^\mathsf{T}(i) X A(i) + C^\mathsf{T}(i) C(i) \\
\mathcal{R}(i,X) &= B^\mathsf{T}(i) X A(i) + D^\mathsf{T}(i) C(i) \\
\mathcal{W}(i,X) &= I - B^\mathsf{T}(i) X B(i) - D^\mathsf{T}(i) D(i) \\
\mathcal{M}(i,X) &= \begin{bmatrix} \mathcal{L}(i,X) & \mathcal{R}^\mathsf{T}(i,X) \\ \mathcal{R}(i,X) & -\mathcal{W}(i,X) \end{bmatrix}
\end{aligned}$$

For $i \in \mathcal{N}$ let $\mathbb{X}_i = \{X \in \mathbb{S}_n : \mathcal{W}(i,X) \text{ invertible}\}$. For $i \in \mathcal{N}$ and $X \in \mathbb{X}_i$, define

$$\mathcal{S}(i,X) = \mathcal{L}(i,X) + \mathcal{R}^\mathsf{T}(i,X) \mathcal{W}^{-1}(i,X) \mathcal{R}(i,X).$$

Given a modified set of matrices $\{(A(i), B(i), C_\epsilon(i), D_\epsilon(i)) : i \in \mathcal{N}\}$, let $\mathcal{L}_\epsilon(i,X)$, $\mathcal{R}_\epsilon(i,X)$, $\mathcal{W}_\epsilon(i,X)$, and $\mathcal{S}_\epsilon(i,X)$ be defined as above but with $C_\epsilon(i)$ in place of $C(i)$ and $D_\epsilon(i)$ in place of $D(i)$.

Lemma 5.16. *Let $i \in \mathcal{N}$ and $X, Y \in \mathbb{S}_n$. The following algebraic identity holds*

$$\begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix}^\mathsf{T} \begin{bmatrix} X & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix} - \begin{bmatrix} Y & 0 \\ 0 & I \end{bmatrix} = \mathcal{M}(i,X) - \begin{bmatrix} Y & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.12}$$

Proof. The result can be shown by expanding the left side of (5.12) □

Remark 5.17. By Lemma 5.16, inequality (5.10b) may be rewritten in terms of the operators in Definition 5.15 as

$$\begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \tilde{Y}_\psi(i,k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix} - \begin{bmatrix} Y_\psi(i,k) & 0 \\ 0 & I \end{bmatrix}$$
$$= \mathcal{M}(i, \tilde{Y}_\psi(i,k+1)) - \begin{bmatrix} Y_\psi(i,k) & 0 \\ 0 & 0 \end{bmatrix}. \tag{5.13}$$

By the Schur complement, (5.13) is negative definite if and only if

$$\mathcal{W}(i, \tilde{Y}_\psi(i,k+1)) > 0 \text{ and } Y_\psi(i,k) > \mathcal{S}(i, \tilde{Y}_\psi(i,k+1)). \tag{5.14}$$

Using the inequalities in (5.14) as a guide, consider the finite-horizon Riccati difference equations defined by the recursive relation and final condition

$$X_\psi(i,k,T) = \mathcal{S}(i, \tilde{X}_\psi(i,k+1,T)) \tag{5.15a}$$
$$X_\psi(i,T+1,T) = 0 \tag{5.15b}$$

where $i \in \mathcal{N}$, $T \in \mathbb{N}_0$ (the horizon), $0 \leq k \leq T$, $\psi \in \Psi$, and $\tilde{X}_\psi(i,k+1,T) = \sum_{j=1}^N \pi_{ij}(\psi(k+1))X_\psi(j,k+1,T)$. For a fixed $\psi \in \Psi$ and $T \in \mathbb{N}_0$, the solution $X_\psi(\cdot,\cdot,T)$ to (5.15) may be computed iteratively backwards-in-time starting with the final condition. However, one must first verify that the inverse specified in (5.15a) is well-defined. The algebraic identity and special input in the next lemma aid in this task.

Lemma 5.18. *Let $k \in \mathbb{N}_0$, $\theta(k) \in \mathcal{N}$, $X \in \mathbb{S}_n$, and $x(k)$, $z(k)$, $w(k)$ be as in* (3.1). *Then*

$$z^{\mathsf{T}}(k)z(k) - w^{\mathsf{T}}(k)w(k) + x^{\mathsf{T}}(k+1)Xx(k+1) = \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^{\mathsf{T}} \mathcal{M}(\theta(k), X) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}. \tag{5.16}$$

If $X \in \mathbb{X}_{\theta(k)}$ and $w(k) = \mathcal{W}^{-1}(\theta(k), X)\mathcal{R}(\theta(k), X)x(k)$ then

$$\begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^{\mathsf{T}} \mathcal{M}(\theta(k), X) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} = x^{\mathsf{T}}(k)\mathcal{S}(\theta(k), X)x(k). \tag{5.17}$$

Proof. The proof of Lemma 5.18 follows from simple matrix algebra. □

Remark 5.19. If $\mathcal{W}(\theta(k), X) > 0$, the input specified in Lemma 5.18 arises from maximizing the quadratic form in (5.16) over $w(k)$ for a fixed $x(k)$ [41, Lem. 2.1].

The following lemma establishes that the Riccati recursive relation in (5.15a) is well-defined when $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly mean square strictly contractive.

Lemma 5.20. *Assume $p_i(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$. If $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly mean square strictly contractive then there exists $\nu > 0$ such that*

$$\mathcal{W}(i, \tilde{X}_\psi(i, k+1, T)) \geq \nu I$$

for all $\psi \in \Psi$, $i \in \mathcal{N}$, $T \in \mathbb{N}_0$, and $0 \leq k \leq T$ where X_ψ is defined by the recursive relation and final condition in (5.15).

Proof. Let $x(0) = 0$. Arbitrarily fix $\psi \in \Psi$ and $T \in \mathbb{N}_0$. Let $i \in \mathcal{N}$ and consider w of the form

$$w(k) = \begin{cases} 0 & : \ k \neq T \\ \chi\{\theta(T) = i\} y & : \ k = T \end{cases} \tag{5.18}$$

where y is an arbitrary vector, and $\chi\{\theta(T) = i\}$ is the indicator function of the set $\{\theta(T) = i\} \subset \Omega$. Note that $\mathbf{E}\left[\chi\{\theta(T) = i\}\right] = \mathbf{P}\left\{\theta(T) = i\right\}$. With w defined in (5.18), $x(k) = 0$ for $k \leq T$ and $z(k) = 0$ for $k \leq T - 1$. Definition 5.10 gives

$$\|z\|_{2,e}^2 - \|w\|_{2,e}^2 \leq -\nu \|w\|_{2,e}^2 \tag{5.19}$$

for any $\psi \in \Psi$ and $w \in \ell_e^2$ where $\nu = 1 - \gamma^2$. Then

$$\begin{aligned}
-\nu \|w\|_{2,e}^2 &= -\nu \mathbf{P}\left\{\theta(T) = i\right\} y^\mathsf{T} y && (5.20)\\
&\geq \sum_{k=0}^{T} \mathbf{E}\left[\|z(k)\|^2 - \|w(k)\|^2\right] && (5.21)\\
&= \mathbf{E}\left[z^\mathsf{T}(T) z(T) - w^\mathsf{T}(T) w(T) + x^\mathsf{T}(T+1) \tilde{X}_\psi(\theta(T), T+1, T) x(T+1)\right] && (5.22)\\
&= \mathbf{E}\left[\begin{bmatrix} x(T) \\ w(T) \end{bmatrix}^\mathsf{T} \mathcal{M}(\theta(T), \tilde{X}_\psi(\theta(T), T+1, T)) \begin{bmatrix} x(T) \\ w(T) \end{bmatrix}\right] && (5.23)\\
&= \mathbf{E}\left[-w^\mathsf{T}(T) \mathcal{W}(\theta(T), \tilde{X}_\psi(\theta(T), T+1, T)) w(T)\right] && (5.24)\\
&= -\mathbf{P}\left\{\theta(T) = i\right\} y^\mathsf{T} \mathcal{W}(i, \tilde{X}_\psi(i, T+1, T)) y
\end{aligned}$$

where (5.20) follows from (5.18); (5.21) follows from (5.19); (5.22) follows since $X_\psi(i, T+1, T) = 0$; (5.23) follows from (5.16); and, (5.24) follows since $x(T) = 0$. Since y was an arbitrary vector, $\mathcal{W}(i, \tilde{X}_\psi(i, T+$

$1, T)) \geq \nu I$.

Now fix $0 \leq t \leq T$ and assume $\mathcal{W}(i, \tilde{X}_\psi(i, k+1, T)) \geq \nu I$ for $t \leq k \leq T$ and $i \in \mathcal{N}$. Consider w of the form

$$w(k) = \begin{cases} 0 & : \ k \leq t-2 \\ \chi\{\theta(k) = i\} y & : \ k = t-1 \\ \mathcal{W}^{-1}(\theta(k), \tilde{X}_\psi(\theta(k), k+1, T)) \mathcal{R}(\theta(k), \tilde{X}_\psi(\theta(k), k+1, T)) x(k) & : \ t \leq k \leq T \\ 0 & : \ k \geq T+1 \end{cases}. \tag{5.25}$$

Then $x(k) = 0$ for $k \leq t-1$ and $z(k) = 0$ for $k \leq t-2$. Define $\mathcal{V}(k) = x^\mathsf{T}(k) X_\psi(\theta(k), k, T) x(k)$ and $\tilde{\mathcal{V}}(k) = x^\mathsf{T}(k) \tilde{X}_\psi(\theta(k-1), k, T) x(k)$. Then

$$\sum_{k=0}^{T} \mathbf{E}\left[\|z(k)\|^2 - \|w(k)\|^2\right] \tag{5.26}$$

$$= \sum_{k=t-1}^{T} \mathbf{E}\left[\|z(k)\|^2 - \|w(k)\|^2 + \mathcal{V}(k+1) - \mathcal{V}(k) + \mathcal{V}(k) - \mathcal{V}(k+1)\right]$$

$$= \sum_{k=t-1}^{T} \mathbf{E}\left[\|z(k)\|^2 - \|w(k)\|^2 + \tilde{\mathcal{V}}(k+1) - \mathcal{V}(k)\right] \tag{5.27}$$

$$= \sum_{k=t}^{T} \mathbf{E}\left[\begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^\mathsf{T} \mathcal{M}(\theta(k), \tilde{X}_\psi(\theta(k), k+1, T)) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} - \mathcal{V}(k)\right]$$

$$+ \mathbf{E}\left[-w^\mathsf{T}(t-1) \mathcal{W}(\theta(t-1), \tilde{X}_\psi(\theta(t-1), t, T)) w(t-1)\right] \tag{5.28}$$

$$= \mathbf{E}\left[-w^\mathsf{T}(t-1) \mathcal{W}(\theta(t-1), \tilde{X}_\psi(\theta(t-1), t, T)) w(t-1)\right] \tag{5.29}$$

$$= -\mathbf{P}\{\theta(t-1) = i\} y^\mathsf{T} \mathcal{W}(i, \tilde{X}_\psi(i, t, T)) y$$

$$\leq -\nu \mathbf{E}\left[w^\mathsf{T}(t-1) w(t-1)\right] \tag{5.30}$$

$$= -\nu \mathbf{P}\{\theta(t-1) = i\} y^\mathsf{T} y$$

where (5.27) follows after applying an iterated expectation, recognizing a telescoping sum, and realizing $\mathcal{V}(t-1) = \mathcal{V}(T+1) = 0$; (5.28) follows from $x(t-1) = 0$; (5.29) follows from (5.17), (5.15a), and definition of $\mathcal{V}(k)$; and, (5.30) follows from (5.19). Since y was an arbitrary vector, $\mathcal{W}(i, \tilde{X}_\psi(i, t, T)) \geq \nu I$. The result follows by induction. □

Remark 5.21. The input specified in (5.25) is similar to disturbance inputs constructed in [59] and [1]. The techniques used in Lemma 5.20 show that if

$$w(k) = \begin{cases} \mathcal{W}^{-1}(\theta(k), \tilde{X}_\psi(\theta(k), k+1, T))\mathcal{R}(\theta(k), \tilde{X}_\psi(\theta(k), k+1, T))x(k) : 0 \leq k \leq T \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad : k \geq T+1 \end{cases} \tag{5.31}$$

then $\sum_{k=0}^{T} \mathbf{E}\left[z^\mathsf{T}(k)z(k) - w^\mathsf{T}(k)w(k)\right] = \mathbf{E}\left[x^\mathsf{T}(0)X_\psi(\theta(0), 0, T)x(0)\right]$. The disturbance input (5.31) maximizes the quantity in (5.26) (see [41, Lem. 2.1]).

The following property is key for finding a uniform upper bound on solutions to (5.15).

Lemma 5.22. *Fix $\psi \in \Psi$ and $t \in \mathbb{N}_0$. Define ψ_t to be a shifted version of ψ so that $\psi_t(k) = \psi(t+k)$ for $k \in \mathbb{N}$, and define $p_t(0) = p(t)$. If $(\mathcal{G}, \Pi \circ \psi, p(0))$ is exponentially mean square stable and mean square strictly contractive, then $(\mathcal{G}, \Pi \circ \psi_t, p_t(0))$ is exponentially mean square stable and mean square strictly contractive. Furthermore,*

$$X_\psi(i, t, T) = X_{\psi_t}(i, 0, T-t) \tag{5.32}$$

for $i \in \mathcal{N}$ and $0 \leq t \leq T$ where X_ψ and X_{ψ_t} are defined by (5.15).

Proof. Consider the Markov jump linear system modulated by a shifted random process

$$\begin{bmatrix} x_t(k+1) \\ z_t(k) \end{bmatrix} = \begin{bmatrix} A(\theta_t(k)) & B(\theta_t(k)) \\ C(\theta_t(k)) & D(\theta_t(k)) \end{bmatrix} \begin{bmatrix} x_t(k) \\ w_t(k) \end{bmatrix} \tag{5.33}$$

where $\theta_t(k) = \theta(t+k)$ for $k \in \mathbb{N}_0$ and $w_t \in \ell_e^2$. Note that this system may be denoted by $(\mathcal{G}, \Pi \circ \psi_t, p_t(0))$. Now $(\mathcal{G}, \Pi \circ \psi_t, p_t(0))$ is exponentially mean square stable since

$$\begin{aligned} \mathbf{E}\left[\Phi_t^\mathsf{T}(k,j)\Phi_t(k,j) \mid \theta_t(j)\right] &= \mathbf{E}\left[\Phi^\mathsf{T}(t+k, t+j)\Phi(t+k, t+j) \mid \theta(t+j)\right] \\ &\leq c\lambda^{t+k-(t+j)} I \end{aligned}$$

where Φ_t is the random state transition matrix for the system in (5.33). Now let $x_t(0) = x(0) = 0$, and let $w_t \in \ell_e^2$ be arbitrary. Define w such that $w(k) = 0$ when $k < t$, and $w(k) = w_t(k-t)$ when $k \geq t$. Note that $w_t \in \ell_e^2$ implies $w \in \ell_e^2$, and $\|w\|_{2,e} = \|w_t\|_{2,e}$. Furthermore, $z(k) = 0$ for $0 \leq k \leq t-1$, and $x(k) = 0$ for $0 \leq k \leq t$. It is easily shown that $z_t(k) = z(t+k)$ for $k \in \mathbb{N}_0$ and $\|z_t\|_{2,e} = \|z\|_{2,e}$. Since $(\mathcal{G}, \Pi \circ \psi, p(0))$ is mean square strictly contractive, $\|z_t\|_{2,e} = \|z\|_{2,e} \leq \gamma \|w\|_{2,e} = \gamma \|w_t\|_{2,e}$. Since $w_t \in \ell_e^2$ was arbitrary, $(\mathcal{G}, P_t, p_t(0))$ is mean square strictly contractive.

Now to prove (5.32), note that the base case $X_\psi(i, T, T) = X_{\psi_t}(i, T-t, T-t) = \mathcal{S}(i, 0)$ holds for all $i \in \mathcal{N}$.

For the inductive hypothesis, assume for some $0 \le k \le T-t-1$ that $X_\psi(i, T-k, T) = X_{\psi_t}(i, T-t-k, T-t)$ for all $i \in \mathcal{N}$. Then

$$\begin{aligned} X_\psi(i, T-(k+1), T) &= \mathcal{S}\left(i, \sum_{j=1}^N \pi_{ij}(\psi(T-k)) X_\psi(j, T-k, T)\right) \\ &= \mathcal{S}\left(i, \sum_{j=1}^N \pi_{ij}(\psi_t(T-t-k)) X_{\psi_t}(j, T-t-k, T-t)\right) \qquad (5.34) \\ &= X_{\psi_t}(i, T-t-(k+1), T-t) \end{aligned}$$

where (5.34) follows from the inductive hypothesis and the fact that $\psi(T-k) = \psi_t(T-t-k)$. Equation (5.32) follows by induction. □

A uniform upper bound on solutions to (5.15) is established in the following lemma using Lemma 5.22 and a technique similar to [26, Sec. B.2.3].

Lemma 5.23. *Assume $p_i(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$. If $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable and uniformly mean square strictly contractive then there exists $\rho > 0$ such that*

$$0 \le X_\psi(i, k, T) \le \rho I \qquad (5.35)$$

for all $i \in \mathcal{N}$, any $T \in \mathbb{N}_0$, all $\psi \in \Psi$, and all $0 \le k \le T+1$ where X_ψ is defined in (5.15).

Proof. Arbitrarily fix $\psi \in \Psi$ and $T \in \mathbb{N}_0$. Define w as in (5.31). Then

$$\begin{aligned} \mathbf{E}\left[x^\mathsf{T}(0) X_\psi(\theta(0), 0, T) x(0)\right] &= \sum_{k=0}^T \mathbf{E}\left[z^\mathsf{T}(k) z(k) - w^\mathsf{T}(k) w(k)\right] \qquad (5.36) \\ &\le \|z\|_{2,e}^2 - \|w\|_{2,e}^2 \qquad (5.37) \end{aligned}$$

where (5.36) follows from Remark 5.21. By linearity, $z = z_{x_0} + z_w$ where z_{x_0} is the zero-input response and z_w is the zero-state response (e.g., see [54, Ch. 20]). By the Cauchy-Schwarz inequality

$$\|z\|_{2,e}^2 \le \|z_w\|_{2,e}^2 + \|z_{x_0}\|_{2,e}^2 + 2\|z_w\|_{2,e}\|z_{x_0}\|_{2,e}. \qquad (5.38)$$

Since $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable,

$$\|z_{x_0}\|_{2,e}^2 = \sum_{k=0}^\infty \mathbf{E}\left[x^\mathsf{T}(0) \Phi^\mathsf{T}(k, 0) C^\mathsf{T}(\theta(k)) C(\theta(k)) \Phi(k, 0) x(0)\right]$$

$$\leq \max_{i\in\mathcal{N}} \left(\lambda_{\max}(C^{\mathsf{T}}(i)C(i))\right) \sum_{k=0}^{\infty} c\lambda^k \mathbf{E}\left[\|x(0)\|^2\right]$$
$$= \delta\mathbf{E}\left[\|x(0)\|^2\right] \tag{5.39}$$

for all $\psi \in \Psi$ where $\delta = \max_{i\in\mathcal{N}} \left(\lambda_{\max}(C^{\mathsf{T}}(i)C(i))\right) c/(1-\lambda)$. By Definition 5.10,

$$\|z_w\|_{2,e}^2 - \|w\|_{2,e}^2 \leq -\nu \|w\|_{2,e}^2 \tag{5.40}$$
$$\|z_w\|_{2,e} < \|w\|_{2,e} \tag{5.41}$$

for all $\psi \in \Psi$ and $w \in \ell_e^2$ where $\nu = 1-\gamma^2$. Then

$$\|z\|_{2,e}^2 - \|w\|_{2,e}^2 \leq -\nu \|w\|_{2,e}^2 + \delta\mathbf{E}[\,\|x(0)\|^2\,] + 2\sqrt{\delta\mathbf{E}[\,\|x(0)\|^2\,]}\,\|w\|_{2,e} \tag{5.42}$$
$$= \left(\delta + \frac{\delta}{\nu}\right)\mathbf{E}[\,\|x(0)\|^2\,] - \nu\left(\|w\|_{2,e} - \frac{\sqrt{\delta}}{\nu}\sqrt{\mathbf{E}[\,\|x(0)\|^2\,]}\right)^2 \tag{5.43}$$
$$\leq \rho\mathbf{E}[\,\|x(0)\|^2\,] \tag{5.44}$$

for all $\psi \in \Psi$ and all $w \in \ell_e^2$ where $\rho = \delta + \delta/\nu$; (5.42) follows from (5.38), (5.39), (5.40), and (5.41); and, (5.43) follows by completing the square. Choose any $i \in \mathcal{N}$ and let $x(0) = \chi\{\theta(0) = i\}y$ where y is an arbitrary vector. Then (5.37) and (5.44) yield

$$\mathbf{P}\{\theta(0) = i\}\, y^{\mathsf{T}} X_\psi(i,0,T) y \leq \rho\mathbf{P}\{\theta(0) = i\}\, y^{\mathsf{T}} y.$$

Since y was arbitrary

$$X_\psi(i,0,T) \leq \rho I. \tag{5.45}$$

Since ψ, i, and T were arbitrary, equation (5.35) follows for $k = 0$. The general case follows from Lemma 5.22. That $0 \leq X_\psi(i,k,T)$ for all $i \in \mathcal{N}$, $T \in \mathbb{N}_0$, $\psi \in \Psi$, and $0 \leq k \leq T+1$ can be seen clearly from (5.15). □

Consider now the perturbed finite-horizon Riccati difference equations defined by the recursive relation and final condition

$$X_\psi(i,k,T,\epsilon) = \mathcal{S}(i, \tilde{X}_\psi(i,k+1,T,\epsilon)) + \epsilon I \tag{5.46a}$$
$$X_\psi(i,T+1,T,\epsilon) = 0 \tag{5.46b}$$

where $i \in \mathcal{N}$, $T \in \mathbb{N}_0$, $0 \le k \le T$, $\psi \in \Psi$, $\epsilon \ge 0$, and $\tilde{X}_\psi(i,k+1,T,\epsilon) = \sum_{j=1}^N \pi_{ij}(\psi(k+1))X_\psi(j,k+1,T,\epsilon)$. For fixed $\psi \in \Psi$, $T \in \mathbb{N}_0$, and ϵ, the solution $X_\psi(\cdot,\cdot,T,\epsilon)$ to (5.46) may be computed iteratively backwards-in-time starting with the final condition. An augmented and perturbed system utilized in the following theorem shows that solutions to (5.46) are uniformly positive definite as well as uniformly bounded.

Theorem 5.24. *Assume $p_i(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$. If $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable and uniformly mean square strictly contractive then there exist $\eta, \rho, \nu > 0$ such that for all $\epsilon \in [0, \eta]$*

$$\begin{aligned} \nu I &\le \mathcal{W}(i, \tilde{X}_\psi(i,k+1,T,\epsilon)) \\ \epsilon I &\le X_\psi(i,k,T,\epsilon) \le \rho I \end{aligned} \tag{5.47}$$

for all $i \in \mathcal{N}$, $T \in \mathbb{N}_0$, $0 \le k \le T$, and $\psi \in \Psi$ where X_ψ is defined by the recursive relation and final condition in (5.46).

Proof. Consider the augmented switched Markov jump linear system $(\mathcal{G}_\epsilon, \Pi, \Psi, p(0))$ where

$$\mathcal{G}_\epsilon = \{(A(1), B(1), C_\epsilon(1), D_\epsilon(1)), \ldots, (A(N), B(N), C_\epsilon(N), D_\epsilon(N))\}$$
$$C_\epsilon(i) = \begin{bmatrix} C(i) \\ \sqrt{\epsilon} I \end{bmatrix}, \quad D_\epsilon(i) = \begin{bmatrix} D(i) \\ 0 \end{bmatrix}.$$

First note $(\mathcal{G}_\epsilon, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable for any ϵ since $\mathcal{G}$ and $\mathcal{G}_\epsilon$ share the same matrices $A(i)$, $i \in \mathcal{N}$. Since $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly mean square strictly contractive, there exists $\eta > 0$ small enough so that for all $\epsilon \in [0, \eta]$ the augmented system $(\mathcal{G}_\epsilon, \Pi, \Psi, p(0))$ is uniformly mean square strictly contractive. By Lemma 5.20, there exists $\nu > 0$ such that for all $\epsilon \in [0, \eta]$

$$\nu I \le \mathcal{W}_\epsilon(i, \tilde{X}_\psi(i,k+1,T,\epsilon)) = \mathcal{W}(i, \tilde{X}_\psi(i,k+1,T,\epsilon))$$

for all $i \in \mathcal{N}$, $T \in \mathbb{N}_0$, $\psi \in \Psi$, $0 \le k \le T$ where

$$\begin{aligned} X_\psi(i,k,T,\epsilon) &= \mathcal{S}_\epsilon(i, \tilde{X}_\psi(i,k+1,T,\epsilon)) = \mathcal{S}(i, \tilde{X}_\psi(i,k+1,T,\epsilon)) + \epsilon I \\ X_\psi(i,T+1,T,\epsilon) &= 0. \end{aligned} \tag{5.48}$$

By Lemma 5.23, there exists $\rho > 0$ such that for all $\epsilon \in [0, \eta]$

$$0 \le X_\psi(i,k,T,\epsilon) \le \rho I$$

for all $i \in \mathcal{N}$, $T \in \mathbb{N}_0$, $\psi \in \Psi$, and $0 \le k \le T+1$. That $X_\psi(i,k,T,\epsilon) \ge \epsilon I$ for $i \in \mathcal{N}$ and $0 \le k \le T$ follows clearly from (5.48). □

The following lemma allows comparison of the solutions of two Riccati difference equations in (5.46) with different values for ϵ.

Lemma 5.25 (Lem. 2.6 of [37]). *For $i \in \mathcal{N}$ and $X \in \mathbb{X}_i$, define*

$$\mathcal{F}(i,X) = A(i) + B(i)\mathcal{W}^{-1}(i,X)\mathcal{R}(i,X). \tag{5.49}$$

Let $Y \in \mathbb{X}_i$ and $\Delta = X - Y$. Then the following algebraic identities hold.

$$\mathcal{S}(i,X) - \mathcal{S}(i,Y) = \mathcal{F}^{\mathsf{T}}(i,Y)\Delta\mathcal{F}(i,Y) + \mathcal{F}^{\mathsf{T}}(i,Y)\Delta B(i)\mathcal{W}^{-1}(i,X)B^{\mathsf{T}}(i)\Delta\mathcal{F}(i,Y) \tag{5.50}$$

$$= \mathcal{F}^{\mathsf{T}}(i,X)\Delta\mathcal{F}(i,Y) \tag{5.51}$$

Before proceeding, the following technical lemma is needed which is similar in nature to [37, Thm. 2.7(a)]. The following lemma examines the random state transition matrix defined by

$$\phi(k,j,T) = \mathcal{F}(\theta(k-1), \tilde{X}_\psi(\theta(k-1),k,T,\epsilon)) \cdots \mathcal{F}(\theta(j), \tilde{X}_\psi(\theta(j),j+1,T,\epsilon)) \tag{5.52}$$

when k and j are such that $0 \le j < k \le T+1$, and $\phi(k,j,T) = I$ when $k = j$. Here, $\mathcal{F}(i,X)$ is defined as in (5.49), and X_ψ is defined in (5.46) for a stable and contractive system $(\mathcal{G},\Pi,\Psi,p(0))$. Note that ϕ is only defined for $0 \le j \le k \le T+1$ and that dependence of ϕ on ψ and ϵ is suppressed. The state transition matrix in (5.52) arises from the recurrence $x(k+1) = \mathcal{F}(\theta(k), \tilde{X}_\psi(\theta(k),k+1,T,\epsilon))x(k)$, which is only defined for $0 \le k \le T$.

Lemma 5.26. *Assume $p_i(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$. If $(\mathcal{G},\Pi,\Psi,p(0))$ is uniformly exponentially mean square stable and uniformly mean square strictly contractive then there exists $\eta > 0$ such that for any $\epsilon \in (0,\eta)$ there exist $0 \le \lambda_\epsilon < 1$ and $c_\epsilon > 0$ such that*

$$\mathbf{E}\left[\phi^{\mathsf{T}}(k,j,T)\phi(k,j,T) \mid \theta(j) = i\right] \le c_\epsilon \lambda_\epsilon^{k-j} I \tag{5.53}$$

for all $i \in \mathcal{N}$, $T \in \mathbb{N}_0$, $\psi \in \Psi$, and all $0 \le j \le k \le T+1$ where ϕ is defined in (5.52).

Proof. Let $\eta > 0$ be as in Theorem 5.24 and fix $\epsilon \in (0,\eta)$. Fix $\epsilon_0 \in (\epsilon,\eta)$ and let $\bar{\epsilon} = \epsilon_0 - \epsilon$. Define

$$Z_\psi(i,k,T,\bar{\epsilon}) = X_\psi(i,k,T,\epsilon_0) - X_\psi(i,k,T,\epsilon)$$

$$\tilde{Z}_{\psi}(i,k,T,\bar{\epsilon}) = \sum_{j=1}^{N} \pi_{ij}(\psi(k)) Z_{\psi}(j,k,T,\bar{\epsilon})$$

for $i \in \mathcal{N}$, $T \in \mathbb{N}_0$, $0 \leq k \leq T+1$, $\psi \in \Psi$ where X_ψ are solutions to (5.46). Let $T \in \mathbb{N}_0$ and $\psi \in \Psi$ be arbitrary, and, for notational convenience, define

$$F(i,k) = \mathcal{F}(i, \tilde{X}_{\psi}(i,k+1,T,\epsilon))$$

where $i \in \mathcal{N}$ and $0 \leq k \leq T$. Then

$$\begin{aligned}
Z_{\psi}(i,k,T,\bar{\epsilon}) &= \mathcal{S}(i, \tilde{X}_{\psi}(i,k+1,T,\epsilon_0)) + \epsilon_0 I - \mathcal{S}(i, \tilde{X}_{\psi}(i,k+1,T,\epsilon)) - \epsilon I && (5.54)\\
&\geq F^{\mathsf{T}}(i,k)(\tilde{X}_{\psi}(i,k+1,T,\epsilon_0) - \tilde{X}_{\psi}(i,k+1,T,\epsilon))F(i,k) + \bar{\epsilon} I && (5.55)\\
&= F^{\mathsf{T}}(i,k)\tilde{Z}_{\psi}(i,k+1,T,\bar{\epsilon})F(i,k) + \bar{\epsilon} I && (5.56)
\end{aligned}$$

where (5.54) follows from (5.46a), and (5.55) follows from (5.50). Additionally, from (5.56) and (5.47)

$$\bar{\epsilon} I \leq Z_{\psi}(i,k,T,\bar{\epsilon}) \leq \bar{\rho} I \tag{5.57}$$

for all $\psi \in \Psi$, $T \in \mathbb{N}_0$, $i \in \mathcal{N}$, and all $0 \leq k \leq T$ where $\bar{\rho} = \rho - \epsilon$. Inequality (5.57) yields

$$-\bar{\epsilon} I \leq -\bar{\epsilon}/\bar{\rho}\, Z_{\psi}(i,k,T,\bar{\epsilon}). \tag{5.58}$$

Then

$$\begin{aligned}
\mathbf{E}\big[\, F^{\mathsf{T}}(\theta(k),k) Z_{\psi}(\theta(k+1),k+1,T,\bar{\epsilon}) F(\theta(k),k) \mid \theta(k) = i \,\big] &\leq Z_{\psi}(i,k,T,\bar{\epsilon}) - \bar{\epsilon} I && (5.59)\\
&\leq \lambda_{\epsilon} Z_{\psi}(i,k,T,\bar{\epsilon}) && (5.60)
\end{aligned}$$

where $\lambda_\epsilon = 1 - \bar{\epsilon}/\bar{\rho}$; (5.59) follows from (5.56); and, (5.60) follows from (5.58). Inequality (5.57) easily shows $0 \leq \lambda_\epsilon < 1$. Now

$$\begin{aligned}
&\mathbf{E}\big[\, \phi^{\mathsf{T}}(k+1,k-1,T) Z_{\psi}(\theta(k+1),k+1,T,\bar{\epsilon}) \phi(k+1,k-1,T) \mid \theta(k-1) = i \,\big]\\
&\quad = \mathbf{E}\big[\, F^{\mathsf{T}}(\theta(k-1),k-1)\, \mathbf{E}\big[\, F^{\mathsf{T}}(\theta(k),k) Z_{\psi}(\theta(k+1),k+1,T,\bar{\epsilon}) F(\theta(k),k) \mid \theta(k) \,\big]\\
&\qquad \times F(\theta(k-1),k-1) \mid \theta(k-1) = i \,\big]\\
&\quad \leq \lambda_{\epsilon} \mathbf{E}\left[F^{\mathsf{T}}(\theta(k-1),k-1) Z_{\psi}(\theta(k),k,T,\bar{\epsilon}) F(\theta(k-1),k-1) \mid \theta(k-1) = i \right] && (5.61)
\end{aligned}$$

$$\leq \lambda_\epsilon^2 Z_\psi(i, k-1, T, \bar{\epsilon}). \tag{5.62}$$

where (5.61) follows from (5.60), and (5.62) follows by another application of (5.60). Thus

$$\begin{aligned}
&\mathbf{E}\left[\phi^{\mathsf{T}}(k,j,T)\phi(k,j,T) \mid \theta(j) = i\right] \\
&\quad \leq (1/\bar{\epsilon})\mathbf{E}\big[\, \phi^{\mathsf{T}}(k,j,T) Z_\psi(\theta(k), k, T, \bar{\epsilon})\phi(k,j,T) \mid \theta(j) = i \,\big] && (5.63)\\
&\quad \leq (1/\bar{\epsilon})\lambda_\epsilon^{k-j} Z_\psi(i,j,T,\bar{\epsilon}) && (5.64)\\
&\quad \leq (\bar{\rho}/\bar{\epsilon})\lambda_\epsilon^{k-j} I. && (5.65)
\end{aligned}$$

where (5.63) follows from (5.57); (5.64) follows by iterating as in (5.61) and (5.62); and, (5.65) follows from (5.57). Hence, the result follows with $c_\epsilon = \bar{\rho}/\bar{\epsilon}$. □

An explicit construction for Y_ψ in Lemma 5.14 is now provided. The construction in the following lemma ensures that for each $k \in \mathbb{N}_0$, $Y_\psi(i,k)$ depends only on i and $\psi_M(k)$.

Lemma 5.27. *Assume $p_i(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$. If $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable and uniformly mean square strictly contractive then there exist $\eta, \rho > 0$ such that for all $\epsilon \in (0,\eta)$, there exist $M \in \mathbb{N}_0$ and $\nu > 0$ such that*

$$Y_\psi(i,k) := X_\psi(i,k,k+M,\epsilon) \tag{5.66}$$

satisfies

$$\epsilon I \leq Y_\psi(i,k) \leq \rho I \tag{5.67a}$$

$$\begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \tilde{Y}_\psi(i,k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix} - \begin{bmatrix} Y_\psi(i,k) & 0 \\ 0 & I \end{bmatrix} \leq -\nu I \tag{5.67b}$$

for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$ where $\tilde{Y}_\psi(i,k+1) = \sum_{j=1}^{N} \pi_{ij}(\psi(k+1)) Y_\psi(j,k+1)$ and X_ψ is defined in (5.46)*.*

Proof. Let η, ρ be as in Theorem 5.24 and choose $\epsilon \in (0,\eta)$ so that (5.67a) is verified automatically. Let λ_ϵ and c_ϵ be as in Lemma 5.26. Choose $M \in \mathbb{N}_0$ such that $c_\epsilon \lambda_\epsilon^{M+1} < \epsilon/\rho$. Then

$$\begin{aligned}
&\mathcal{S}(i, \tilde{Y}_\psi(i,k+1)) - Y_\psi(i,k) + \epsilon I \\
&\quad = \mathcal{S}(i, \tilde{X}_\psi(i,k+1,k+M+1,\epsilon)) - \mathcal{S}(i, \tilde{X}_\psi(i,k+1,k+M,\epsilon))
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{F}^{\mathsf{T}}(i, \tilde{X}_{\psi}(i, k+1, k+M+1, \epsilon))(\tilde{X}_{\psi}(i, k+1, k+M+1, \epsilon) - \tilde{X}_{\psi}(i, k+1, k+M, \epsilon)) \\
&\quad \times \mathcal{F}(i, \tilde{X}_{\psi}(i, k+1, k+M, \epsilon)). \qquad (5.68)
\end{aligned}
$$

where (5.68) follows from (5.51). But the middle term in (5.68) can be written

$$
\begin{aligned}
&\tilde{X}_{\psi}(i, k+1, k+M+1, \epsilon) - \tilde{X}_{\psi}(i, k+1, k+M, \epsilon) \\
&= \sum_{j=1}^{N} \pi_{ij}(\psi(k+1)) \Big(\mathcal{S}(j, \tilde{X}_{\psi}(j, k+2, k+M+1, \epsilon)) - \mathcal{S}(j, \tilde{X}_{\psi}(j, k+2, k+M, \epsilon)) \Big) \qquad (5.69) \\
&= \mathbf{E}\Big[\mathcal{F}^{\mathsf{T}}(\theta(k+1), \tilde{X}_{\psi}(\theta(k+1), k+2, k+M+1, \epsilon)) \\
&\quad \times \Big(\tilde{X}_{\psi}(\theta(k+1), k+2, k+M+1, \epsilon) - \tilde{X}_{\psi}(\theta(k+1), k+2, k+M, \epsilon) \Big) \\
&\quad \times \mathcal{F}(\theta(k+1), \tilde{X}_{\psi}(\theta(k+1), k+2, k+M, \epsilon)) \mid \theta(k) = i \Big] \qquad (5.70)
\end{aligned}
$$

where (5.69) follows from (5.46a), and (5.70) results after applying (5.68) to (5.69). Proceeding in an iterative fashion,

$$
\begin{aligned}
&\mathcal{S}(i, \tilde{Y}_{\psi}(i, k+1)) - Y_{\psi}(i, k) + \epsilon I \\
&\quad = \mathbf{E}\Big[\phi^{\mathsf{T}}(k+M+1, k, k+M+1) \\
&\quad\quad \times \Big(\tilde{X}_{\psi}(\theta(k+M), k+M+1, k+M+1, \epsilon) - \tilde{X}_{\psi}(\theta(k+M), k+M+1, k+M, \epsilon) \Big) \\
&\quad\quad \times \phi(k+M+1, k, k+M) \mid \theta(k) = i \Big]. \qquad (5.71)
\end{aligned}
$$

Note that the middle term in (5.71) satisfies

$$
\epsilon I \leq \tilde{X}_{\psi}(\theta(k+M), k+M+1, k+M+1, \epsilon) - 0 \leq \rho I \qquad (5.72)
$$

for all values of $\theta(k+M) \in \mathcal{N}$. Let $y \in \mathbb{R}^n$ be arbitrary, and for convenience define $\phi_1 = \phi(k+M+1, k, k+M+1)$, $\phi_2 = \phi(k+M+1, k, k+M)$, and $X = \tilde{X}_{\psi}(\theta(k+M), k+M+1, k+M+1, \epsilon)$. Then

$$
\begin{aligned}
&y^{\mathsf{T}} \Big(\mathcal{S}(i, \tilde{Y}_{\psi}(i, k+1)) - Y_{\psi}(i, k) + \epsilon I \Big) y \\
&= y^{\mathsf{T}} \mathbf{E}\left[\phi_1^{\mathsf{T}} X \phi_2 \mid \theta(k) = i \right] y \\
&= \mathbf{E}\left[v_1^{\mathsf{T}} X v_2 \mid \theta(k) = i \right] \\
&\leq \sqrt{ \mathbf{E}\left[v_1^{\mathsf{T}} X v_1 \mid \theta(k) = i \right] \mathbf{E}\left[v_2^{\mathsf{T}} X v_2 \mid \theta(k) = i \right] } \qquad (5.73) \\
&\leq \sqrt{ \rho^2 \mathbf{E}\left[v_1^{\mathsf{T}} v_1 \mid \theta(k) = i \right] \mathbf{E}\left[v_2^{\mathsf{T}} v_2 \mid \theta(k) = i \right] } \qquad (5.74)
\end{aligned}
$$

$$
\begin{aligned}
&= \rho \sqrt{\left(y^{\mathsf{T}} \mathbf{E}\left[\phi_1^{\mathsf{T}} \phi_1 \mid \theta(k) = i\right] y\right) \left(y^{\mathsf{T}} \mathbf{E}\left[\phi_2^{\mathsf{T}} \phi_2 \mid \theta(k) = i\right] y\right)} \\
&\leq \rho \sqrt{\left(c_\epsilon \lambda_\epsilon^{M+1} y^{\mathsf{T}} y\right) \left(c_\epsilon \lambda_\epsilon^{M+1} y^{\mathsf{T}} y\right)} && (5.75) \\
&= \rho c_\epsilon \lambda_\epsilon^{M+1} y^{\mathsf{T}} y \\
&< \epsilon y^{\mathsf{T}} y && (5.76)
\end{aligned}
$$

where $v_1 = \phi_1 y$ and $v_2 = \phi_2 y$; (5.73) follows from Lemma 2.22; (5.74) follows from (5.72); (5.75) follows from Lemma 5.26; and, (5.76) follows by choice of M. Let $\epsilon_0 \in (0, \epsilon)$ be such that

$$
\mathcal{S}(i, \tilde{Y}_\psi(i, k+1)) - Y_\psi(i, k) + \epsilon I \leq \epsilon_0 I < \epsilon I. \tag{5.77}
$$

By (5.77)

$$
\mathcal{S}(i, \tilde{Y}_\psi(i, k+1)) - Y_\psi(i, k) \leq -\nu I. \tag{5.78}
$$

where $\nu = \epsilon - \epsilon_0$. Inequality (5.78) and the Schur complement formula yield (5.67b). □

Lemma 5.27 uses techniques similar to those found in [37, Thm. 2.7(b)] where it is shown that the time-varying version of the KYP inequality associated with a uniformly stable and contractive linear time-varying system admits a solution with finite memory of past parameters.

Remark 5.28. The construction in Lemma 5.27 ensures that $Y_\psi(i, k)$ may be computed with knowledge of only i and $\psi_M(k)$. Indeed, if $t \neq k$ but $\psi_M(k) = \psi_M(t)$ then $Y_\psi(i, k) = Y_\psi(i, t)$. This claim can be easily established using the recursive relation (5.46a) and base case $X_\psi(i, k + M + 1, k + M, \epsilon) = X_\psi(i, t + M + 1, t + M, \epsilon) = 0$.

The following theorem, inspired by [37, Thm. 3.3], provides a necessary and sufficient condition, expressed as a set of finite-dimensional LMIs, for uniform exponential mean square stability and uniform mean square strict contractiveness of a switched Markov jump linear system.

Theorem 5.29. *Assume $p_i(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$. The switched Markov jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable and uniformly mean square strictly contractive if and only if there exist $M \in \mathbb{N}_0$ and a function $X : \mathcal{N} \times \Psi_M \to \mathbb{S}_n^+$ such that*

$$
\begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \sum_{j=1}^N \pi_{ij}(r_1) X(j, r_2, \ldots, r_{M+1}) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix} - \begin{bmatrix} X(i, r_1, \ldots, r_M) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{5.79}
$$

for any $(r_1, \ldots, r_{M+1}) \in \Psi_{M+1}$ and $i \in \mathcal{N}$.

Proof. Suppose there exist M and X such that (5.79) holds. Note that the upper left block of (5.79) implies (4.16) so uniform exponential mean square stability of $(\mathcal{G}, \Pi, \Psi, p(0))$ follows from Theorem 4.12. Since $\mathcal{N} \times \Psi_{M+1} \subset \mathcal{N} \times \mathcal{J}^{M+1}$ is a finite set, inequality (5.79) holds uniformly, so there exists $0 < \nu < 1$ such that

$$\begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \sum_{j=1}^{N} \pi_{ij}(r_1) X(j, r_2, \ldots, r_{M+1}) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix} - \begin{bmatrix} X(i, r_1, \ldots, r_M) & 0 \\ 0 & I \end{bmatrix} \leq -\nu I \quad (5.80)$$

for any $(i, r_1, \ldots, r_{M+1}) \in \mathcal{N} \times \Psi_{M+1}$. Let $\psi \in \Psi$ be arbitrary. Define $Y_\psi(i, k) := X(i, \psi(k+1), \ldots, \psi(k + M))$. Using (5.13) and (5.16) to rewrite (5.80), it follows that

$$\begin{aligned} &\mathbf{E}[\, \|z(k)\|^2 + x^{\mathsf{T}}(k+1) Y_\psi(\theta(k+1), k+1) x(k+1) - x^{\mathsf{T}}(k) Y_\psi(\theta(k), k) x(k)\,] \\ &\quad \leq (1-\nu) \mathbf{E}[\, \|w(k)\|^2\,]. \end{aligned} \quad (5.81)$$

Inequality (5.81), positive definiteness of $Y_\psi(i, k)$, and $x(0) = 0$ yield

$$\sum_{k=0}^{l} \mathbf{E}\left[\|z(k)\|^2\right] \leq (1-\nu) \sum_{k=0}^{l} \mathbf{E}\left[\|w(k)\|^2\right]$$

for all $l \in \mathbb{N}_0$. Since $\psi \in \Psi$ was arbitrary, Definition 5.10 is satisfied with $\gamma = \sqrt{1-\nu}$ so $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly mean square strictly contractive.

Conversely, assume that $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable and uniformly mean square strictly contractive. Let η, ρ be as in Lemma 5.27, fix $\epsilon \in (0, \eta)$, and let M and ν be defined as in Lemma 5.27. Let $(i, r_1, \ldots, r_{M+1}) \in \mathcal{N} \times \Psi_{M+1}$ be arbitrary. By definition of Ψ_{M+1}, there exist $\psi \in \Psi$ and $t \in \mathbb{N}_0$ such that $\psi_{M+1}(t) = (r_1, \ldots, r_{M+1})$. Construct Y_ψ as in Lemma 5.27 and recall from Remark 5.28 that $Y_\psi(i, t)$ depends only on $(i, \psi_M(t))$. Thus, define $X(i, r_1, \ldots, r_M) := Y_\psi(i, t)$ and define $X(i, r_2, \ldots, r_{M+1}) := Y_\psi(i, t+1)$. One recovers every inequality in (5.79) from (5.67). □

Remark 5.30. Theorem 5.29 may be used to approximate $\|(\mathcal{G}, \Pi, \Psi, p(0))\|_\infty$. First, set γ_U equal to some upper bound on $\|(\mathcal{G}, \Pi, \Psi, p(0))\|_\infty$, and set γ_L equal to some lower bound on $\|(\mathcal{G}, \Pi, \Psi, p(0))\|_\infty$. Use a bisection search to approximate the smallest $\gamma_0 \in [\gamma_L, \gamma_U]$ such that the LMIs in Theorem 5.29 with $M = 0$ are feasible for the system $(\mathcal{G}_0, \Pi, \Psi, p(0))$ where $\mathcal{G}_0 = \{(A(i), B(i), (1/\gamma_0)C(i), (1/\gamma_0)D(i)) : i \in \mathcal{N}\}$. Next, use a bisection search to approximate the smallest $\gamma_1 \in [\gamma_L, \gamma_0]$ such that the LMIs in Theorem 5.29 with $M = 1$ are feasible for the system $(\mathcal{G}_1, \Pi, \Psi, p(0))$ where $\mathcal{G}_1 = \{(A(i), B(i), (1/\gamma_1)C(i), (1/\gamma_1)D(i)) : i \in \mathcal{N}\}$. If $\|(\mathcal{G}, \Pi, \Psi, p(0))\|_\infty = \gamma^*$, Theorem 5.29 says that $\gamma_M \to \gamma^*$ as $M \to \infty$.

Remark 5.31. Theorem 5.29 provides a practical method for investigating the contractiveness of a *single* time-inhomogeneous Markov jump linear system with *known* transition probability matrices that vary in a finite set (let Ψ be the set containing a single sequence).

Remark 5.32. Consider the case when $J = 1$ and $\Psi = \{(1, 1, \dots)\}$. The the switched Markov jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ reduces to a single time-homogeneous Markov jump linear system $(\mathcal{G}, \Pi \circ \psi_1, p(0))$ where $\psi_1 \equiv 1$. For any M, the set Ψ_M contains only a single element $(1, \dots, 1)$, and the set $\mathcal{N} \times \Psi_M$ contains only N elements. For $i \in \mathcal{N}$, define $Z(i) := X(i, 1, \dots, 1)$ where X is as in Theorem 5.29. Then (5.79) reduces to

$$\begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix}^\mathsf{T} \begin{bmatrix} \tilde{Z}(i) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(i) & B(i) \\ C(i) & D(i) \end{bmatrix} - \begin{bmatrix} Z(i) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{5.82}$$

where $\tilde{Z}(i) = \sum_{j=1}^N \pi_{ij}(1) Z(j)$. Inequality (5.82) is the same inequality found in Proposition 5.5. Theorem 5.29, however, does not require $(\mathcal{G}, \Pi \circ \psi_1, p(0))$ to be weakly controllable as in Proposition 5.5. Thus, the weak controllability hypothesis of Proposition 5.5 can be replaced by the weaker (see Proposition 5.33) hypothesis that $p_i(k) > 0$ for all $i \in \mathcal{N}$ and all $k \in \mathbb{N}_0$. Recalling Proposition 5.4, this hypothesis is equivalent to $p_i(0) > 0$ for all $i \in \mathcal{N}$, and each column of $\Pi(1)$ is nonzero.

Proposition 5.33. *Let $\Pi(1)$ be a stochastic matrix and $\psi_1 \equiv 1$. If the time-homogeneous Markov jump linear system $(\mathcal{G}, \Pi \circ \psi_1, p(0))$ is weakly controllable and $p_i(0) > 0$ for all $i \in \mathcal{N}$, then $p_i(k) > 0$ for all $i \in \mathcal{N}$ and $k \in \mathbb{N}_0$.*

Proof. The contrapositive is proved. Suppose the conclusion of the conditional statement is false. Then by Proposition 5.4, $\Pi(1)$ has a zero column and/or $p_i(0) = 0$ for some $i \in \mathcal{N}$. If the j-th column of $\Pi(1)$ is zero, then $\pi_{ij}(1) = 0$ for all $i \in \mathcal{N}$ and $\mathbf{P}\{x(k) = x_f, \theta(k) = j\} \leq \mathbf{P}\{\theta(k) = j\} = \sum_{i=1}^N \pi_{ij}(1) \mathbf{P}\{\theta(k-1) = i\} = 0$ for all $k \geq 1$. Thus, the final state (x_f, j) has zero probability for all $k \geq 1$ and any input $w_c \in \ell_e^2$ so $(\mathcal{G}, \Pi \circ \psi_1, p(0))$ is not weakly controllable. □

5.4.1 Example

Consider the switched Markov jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ where

$$A(1) = \begin{bmatrix} 0.08 & 0.15 & 0.30 \\ 0.20 & 0.60 & 0.10 \\ 0.50 & 0.20 & 0.40 \end{bmatrix}, \qquad B(1) = \begin{bmatrix} 0.10 & 0.70 \\ 0.50 & 0.80 \\ 0.20 & 0.40 \end{bmatrix},$$

$$C(1) = \begin{bmatrix} 0.18 & 0.03 & 0.01 \\ 0.01 & 0.07 & 0.06 \\ 0.02 & 0.03 & 0.15 \end{bmatrix}, \qquad D(1) = \begin{bmatrix} 0.01 & 0 \\ 0.08 & 0.05 \\ 0 & 0.01 \end{bmatrix},$$

$$A(2) = \begin{bmatrix} -0.06 & 0.40 & 0.70 \\ 0.35 & -0.07 & 0.10 \\ 0.23 & -0.04 & 0.51 \end{bmatrix}, B(2) = \begin{bmatrix} 0.41 & -0.75 \\ 0.90 & 0.47 \\ 0.54 & 0.28 \end{bmatrix},$$

$$C(2) = \begin{bmatrix} 0.03 & -0.02 & 0.03 \\ 0.07 & 0.09 & 0.10 \\ 0.07 & 0.02 & 0.08 \end{bmatrix}, \quad D(2) = \begin{bmatrix} 0 & 0.03 \\ 0.01 & -0.11 \\ 0 & 0.05 \end{bmatrix},$$

and

$$\Pi(1) = \begin{bmatrix} 0.46 & 0.54 \\ 0.40 & 0.60 \end{bmatrix}, \Pi(2) = \begin{bmatrix} 0.01 & 0.99 \\ 0.05 & 0.95 \end{bmatrix},$$

$$\Psi = \mathcal{J}^\infty, \qquad p(0) = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}.$$

Note that the linear time-invariant system with fixed matrices $(A(1), B(1), C(1), D(1))$ is exponentially stable but not contractive. On the other hand, the linear time-invariant system with fixed matrices $(A(2), B(2), C(2), D(2))$ is exponentially stable and contractive. The LMIs in Theorem 5.29 are not feasible with $M = 0$, but are feasible with $M = 1$. Thus, the switched Markov jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable and uniformly mean square strictly contractive.

5.5 Switched Independent Jump Linear Systems

If each $\psi \in \Psi$ gives rise to an independent sequence θ of random variables, then a more computationally efficient result (see Theorem 5.46) may be used to check contractiveness. Although the result does not follow immediately from Theorem 5.29, a similar mathematical machinery may be employed. The derivation closely mirrors that of Section 5.4, so the explanatory discussion in this section is brief. Recall from Section 3.3.1 that the notation $\pi_j(s)$ where $j \in \mathcal{N}$ and $s \in \mathcal{J}$ denotes the ij-th element of matrix $\Pi(s)$ where i is irrelevant since each row of $\Pi(s)$ is identical (θ is an independent sequence).

Definition 5.34. Given $\mathcal{G}$, $s \in \mathcal{J}$, and $X \in \mathbb{S}_n$, define

$$L(s, X) = \sum_{j=1}^{N} \pi_j(s) \mathcal{L}(j, X)$$

$$R(s,X) = \sum_{j=1}^{N} \pi_j(s)\mathcal{R}(j,X)$$
$$W(s,X) = \sum_{j=1}^{N} \pi_j(s)\mathcal{W}(j,X)$$
$$M(s,X) = \sum_{j=1}^{N} \pi_j(s)\mathcal{M}(j,X)$$

where $\mathcal{L}(i,X)$, $\mathcal{R}(i,X)$, $\mathcal{W}(i,X)$, and $\mathcal{M}(i,X)$, $i \in \mathcal{N}$, are defined in Definition 5.15. For $s \in \mathcal{J}$ let $\mathbb{Y}_s = \{X \in \mathbb{S}_n : W(s,X) \text{ invertible}\}$. For $X \in \mathbb{Y}_s$ define

$$S(s,X) = L(s,X) + R^{\mathsf{T}}(s,X)W^{-1}(s,X)R(s,X)$$

Given a modified set of matrices $\{(A(i), B(i), C_\epsilon(i), D_\epsilon(i)) : i \in \mathcal{N}\}$, let $L_\epsilon(s,X)$, $R_\epsilon(s,X)$, $W_\epsilon(s,X)$, $S_\epsilon(s,X)$ be defined as above but with $C_\epsilon(i)$ in place of $C(i)$ and $D_\epsilon(i)$ in place of $D(i)$.

When θ is an independent sequence of random variables, the following finite-horizon Riccati difference equations are used to establish an LMI condition equivalent to contractiveness.

$$X_\psi(k,T) = S(\psi(k), X_\psi(k+1,T)) \tag{5.83a}$$
$$X_\psi(T+1,T) = 0 \tag{5.83b}$$

where $T \in \mathbb{N}_0$ (the horizon), $0 \le k \le T$, and $\psi \in \Psi$. For a fixed $\psi \in \Psi$ and $T \in \mathbb{N}_0$, the solution $X_\psi(\cdot,T)$ to (5.83) may be computed iteratively backwards-in-time starting with the final condition. Note that the Riccati difference equations in (5.83) differ from the Riccati difference equations specified in (5.15). The following lemma is used to establish that the inverse specified in (5.83a) is well-defined when the switched independent jump linear system is contractive.

Lemma 5.35. *Fix $\psi \in \Psi$. If $X \in \mathbb{Y}_{\psi(k)}$ and $w(k) = W^{-1}(\psi(k),X)R(\psi(k),X)x(k)$ then*

$$\mathbf{E}\left[\begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^{\mathsf{T}} \mathcal{M}(\theta(k),X) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \mid x(k)\right] = x^{\mathsf{T}}(k)S(\psi(k),X)x(k). \tag{5.84}$$

Proof. The proof of Lemma 5.35 follows from straightforward algebra. □

The next lemma verifies that the inverse specified in (5.83a) is indeed well-defined when the switched independent jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly mean square strictly contractive.

Lemma 5.36. *If the switched independent jump linear system* $(\mathcal{G}, \Pi, \Psi, p(0))$ *is uniformly mean square strictly contractive then there exists* $\nu > 0$ *such that*

$$W(\psi(k), X_\psi(k+1, T)) \geq \nu I$$

for all $\psi \in \Psi$*,* $T \in \mathbb{N}_0$*, and* $0 \leq k \leq T$ *where* X_ψ *is defined by the recursive relation and final condition in* (5.83)*.*

Proof. The proof is very similar to the proof of Lemma 5.20 so only a sketch is provided. Let $y \in \mathbb{R}^m$ be arbitrary and use the input

$$w(k) = \begin{cases} 0 \;:\; k \neq T \\ y \;:\; k = T \end{cases} \tag{5.85}$$

to show that $W(\psi(T), X_\psi(T+1, T)) \geq \nu I$. Now fix $0 \leq t \leq T$, assume $W(\psi(k), X_\psi(k+1, T)) \geq \nu I$ for $t \leq k \leq T$, and use the identity (5.84) and the input

$$w(k) = \begin{cases} 0 & : \; k \leq t-2 \\ y & : \; k = t-1 \\ W^{-1}(\psi(k), X_\psi(k+1, T)) R(\psi(k), X_\psi(k+1, T)) x(k) & : \; t \leq k \leq T \\ 0 & : \; k \geq T+1 \end{cases} \tag{5.86}$$

to show that $W(\psi(t-1), X_\psi(t, T)) \geq \nu I$. The result then follows by induction. □

Remark 5.37. In contrast to Lemma 5.20, the special inputs, (5.85) and (5.86), in the proof of Lemma 5.36 do not use an indicator function. Thus, the hypothesis $p_i(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$ from Lemma 5.20 is not needed here.

Remark 5.38. The techniques used in Lemmas 5.20 and 5.36 show that if

$$w(k) = \begin{cases} W^{-1}(\psi(k), X_\psi(k+1, T)) R(\psi(k), X_\psi(k+1, T)) x(k) : 0 \leq k \leq T \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad : k \geq T+1 \end{cases} \tag{5.87}$$

then $\sum_{k=0}^{T} \mathbf{E}\left[z^\top(k) z(k) - w^\top(k) w(k)\right] = \mathbf{E}\left[x^\top(0) X_\psi(0, T) x(0)\right]$.

Lemma 5.39. *Fix* $\psi \in \Psi$ *and* $t \in \mathbb{N}_0$*. Define* ψ_t *to be a shifted version of* ψ *so that* $\psi_t(k) = \psi(t+k)$ *for* $k \in \mathbb{N}$*, and define* $p_t(0) = p(t)$*. If* $(\mathcal{G}, \Pi \circ \psi, p(0))$ *is exponentially mean square stable and mean square strictly* *contractive, then $(\mathcal{G}, \Pi \circ \psi_t, p_t(0))$ is exponentially mean square stable and mean square strictly contractive. Furthermore,*

$$X_\psi(t, T) = X_{\psi_t}(0, T - t) \tag{5.88}$$

for $0 \le t \le T$ where X_ψ and X_{ψ_t} are defined by (5.83).

Proof. The proof is analogous to the proof of Lemma 5.22. □

Lemma 5.40. *If the switched independent jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable and uniformly mean square strictly contractive then there exists $\rho > 0$ such that*

$$0 \le X_\psi(k, T) \le \rho I \tag{5.89}$$

for all $T \in \mathbb{N}_0$, all $\psi \in \Psi$, and all $0 \le k \le T + 1$ where X_ψ is defined in (5.83).

Proof. The proof is very similar to the proof of Lemma 5.23, so only a sketch is provided. Define w as in (5.87) and proceed as in the proof of Lemma 5.23 to arrive at the inequality

$$\mathbf{E}\left[x^\mathsf{T}(0) X_\psi(0, T) x(0)\right] \le \rho \mathbf{E}\left[\|x(0)\|^2\right]$$

where ρ is defined as in Lemma 5.23. Now let $x(0) = y$, a constant random variable, where $y \in \mathbb{R}^n$ is arbitrary to get $X_\psi(0, T) \le \rho I$. Apply Lemma 5.39 to get inequality (5.89). □

Consider the perturbed finite-horizon Riccati difference equations defined by the recursive relation and final condition

$$X_\psi(k, T, \epsilon) = S(\psi(k), X_\psi(k + 1, T, \epsilon)) + \epsilon I \tag{5.90a}$$

$$X_\psi(T + 1, T, \epsilon) = 0 \tag{5.90b}$$

where $T \in \mathbb{N}_0$, $0 \le k \le T$, $\psi \in \Psi$, and $\epsilon \ge 0$. For fixed $\psi \in \Psi$, $T \in \mathbb{N}_0$, and ϵ, the solution $X_\psi(\cdot, T, \epsilon)$ to (5.90) may be computed iteratively backwards-in-time starting with the final condition. An augmented and perturbed system utilized in the following theorem shows that solutions to (5.90) are uniformly positive definite as well as uniformly bounded.

Theorem 5.41. *If the switched independent jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable and uniformly mean square strictly contractive then there exist $\eta, \rho, \nu > 0$ such that for* *all* $\epsilon \in [0, \eta]$

$$\begin{aligned} \nu I &\leq W(\psi(k), X_\psi(k+1, T, \epsilon)) \\ \epsilon I &\leq X_\psi(k, T, \epsilon) \leq \rho I \end{aligned} \tag{5.91}$$

for all $T \in \mathbb{N}_0$*,* $0 \leq k \leq T$*, and* $\psi \in \Psi$ *where* X_ψ *is defined by the recursive relation and final condition in* (5.90)*.*

Proof. Given the augmented and perturbed system $(\mathcal{G}_\epsilon, \Pi, \Psi, p(0))$ from the proof of Theorem 5.24, note that

$$\begin{aligned} W_\epsilon(s, X) &= W(s, X) \\ S_\epsilon(s, X) &= S(s, X) + \epsilon I. \end{aligned}$$

The rest of the proof is almost identical to the proof of Theorem 5.24. □

The following lemma allows comparison of the solutions of two Riccati difference equations in (5.90) with different values for ϵ.

Lemma 5.42. *For* $i \in \mathcal{N}$*,* $s \in \mathcal{J}$*, and* $X \in \mathbb{Y}_s$*, define*

$$\mathcal{A}(i, s, X) = A(i) + B(i)W^{-1}(s, X)R(s, X). \tag{5.92}$$

Let $Y \in \mathbb{Y}_s$*, and let* $\Delta = X - Y$*. Then the following algebraic identities hold.*

$$\begin{aligned} &S(s, X) - S(s, Y) \\ &\quad = \sum_{j=1}^{N} \pi_j(s)\mathcal{A}^\mathsf{T}(j, s, Y)\Delta\mathcal{A}(j, s, Y) \\ &\qquad + \left[\sum_{j=1}^{N} \pi_j(s)\mathcal{A}^\mathsf{T}(j, s, Y)\Delta B(j)\right] W^{-1}(s, X) \left[\sum_{j=1}^{N} \pi_j(s)B^\mathsf{T}(j)\Delta\mathcal{A}(j, s, Y)\right] && (5.93) \\ &\quad = \sum_{j=1}^{N} \pi_j(s)\mathcal{A}^\mathsf{T}(j, s, X)\Delta\mathcal{A}(j, s, Y). && (5.94) \end{aligned}$$

Proof. Equation (5.93) follows from the proof of [58, Lemma C.1]. To show (5.94), define

$$\tilde{W}(s, \Delta) = W(s, X) - W(s, Y) = -\sum_{j=1}^{N} \pi_j(s)B^\mathsf{T}(j)\Delta B(j). \tag{5.95}$$

Equations (5.92) and (5.95) yield

$$\sum_{j=1}^{N} \pi_j(s) \left[(\mathcal{A}(j,s,Y) - A(j))^{\mathsf{T}} \Delta B(j) W^{-1}(s,X) \right] = -R^{\mathsf{T}}(s,Y) W^{-1}(s,Y) \tilde{W}(s,\Delta) W^{-1}(s,X). \tag{5.96}$$

The following identities are straightforward to verify.

$$W^{-1}(s,Y)\tilde{W}(s,\Delta)W^{-1}(s,X) = W^{-1}(s,Y) - W^{-1}(s,X) \tag{5.97}$$

$$\sum_{j=1}^{N} \pi_j(s) A(j)^{\mathsf{T}} \Delta B(j) + R^{\mathsf{T}}(s,Y) = R^{\mathsf{T}}(s,X) \tag{5.98}$$

Equations (5.96), (5.97), and (5.98) yield

$$\sum_{j=1}^{N} \pi_j(s) \mathcal{A}^{\mathsf{T}}(j,s,Y) \Delta B(j) W^{-1}(s,X) = R^{\mathsf{T}}(s,X) W^{-1}(s,X) - R^{\mathsf{T}}(s,Y) W^{-1}(s,Y). \tag{5.99}$$

By equation (5.92),

$$\sum_{j=1}^{N} \pi_j(s) \left[(\mathcal{A}(j,s,X) - \mathcal{A}(j,s,Y))^{\mathsf{T}} \Delta \mathcal{A}(j,s,Y) \right] = \left(R^{\mathsf{T}}(s,X) W^{-1}(s,X) - R^{\mathsf{T}}(s,Y) W^{-1}(s,Y) \right) \sum_{j=1}^{N} \pi_j(s) B^{\mathsf{T}}(j) \Delta \mathcal{A}(j,s,Y).$$

Thus, (5.94) follows from (5.99). □

The following lemma examines the random state transition matrix defined by

$$\phi(k,j,T) = \mathcal{A}(\theta(k-1), \psi(k-1), X_\psi(k,T,\epsilon)) \cdots \mathcal{A}(\theta(j), \psi(j), X_\psi(j+1,T,\epsilon)) \tag{5.100}$$

when k and j are such that $0 \leq j < k \leq T+1$, and $\phi(k,j,T) = I$ when $k = j$. Here, $\mathcal{A}(i,s,X)$ is defined as in (5.92), and X_ψ is defined in (5.90) for a stable and contractive system $(\mathcal{G}, \Pi, \Psi, p(0))$. Note that ϕ is only defined for $0 \leq j \leq k \leq T+1$ and that dependence of ϕ on ψ and ϵ is suppressed. The state transition matrix in (5.100) arises from the recurrence $x(k+1) = \mathcal{A}(\theta(k), \psi(k), X_\psi(k+1,T,\epsilon))x(k)$, which is only defined for $0 \leq k \leq T$.

Lemma 5.43. *If the switched independent jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable and uniformly mean square strictly contractive then there exists $\eta > 0$ such that for any*

$\epsilon \in (0, \eta)$ *there exist* $0 \leq \lambda_\epsilon < 1$ *and* $c_\epsilon > 0$ *such that*

$$\mathbf{E}\left[\phi^\mathsf{T}(k, j, T)\phi(k, j, T)\right] \leq c_\epsilon \lambda_\epsilon^{k-j} I \tag{5.101}$$

for all $T \in \mathbb{N}_0$, $\psi \in \Psi$, *and all* $0 \leq j \leq k \leq T + 1$ *where* ϕ *is defined in* (5.100).

Proof. An outline of the proof is provided. For more details, refer to the proof of Lemma 5.26. Let $\eta > 0$ be as in Theorem 5.41 and fix $\epsilon \in (0, \eta)$. Fix $\epsilon_0 \in (\epsilon, \eta)$ and let $\bar{\epsilon} = \epsilon_0 - \epsilon$. Define

$$Z_\psi(k, T, \bar{\epsilon}) = X_\psi(k, T, \epsilon_0) - X_\psi(k, T, \epsilon).$$

Use (5.90a) and (5.93), and proceed as in the proof of Lemma 5.26 to get

$$\begin{aligned} \mathbf{E}\left[\mathcal{A}^\mathsf{T}(\theta(k), \psi(k), X_\psi(k+1, T, \epsilon))Z_\psi(k+1, T, \bar{\epsilon})\mathcal{A}(\theta(k), \psi(k), X_\psi(k+1, T, \epsilon))\right] &\leq Z_\psi(k, T, \bar{\epsilon}) - \bar{\epsilon} I \\ &\leq \lambda_\epsilon Z_\psi(k, T, \bar{\epsilon}) \end{aligned} \tag{5.102}$$

where $\lambda_\epsilon = 1 - \bar{\epsilon}/\bar{\rho}$ and $\bar{\rho} = \rho - \epsilon$. Apply the independence hypothesis on θ, iterate (5.102), and use the inequality $\bar{\epsilon} I \leq Z_\psi(k, T, \bar{\epsilon}) \leq \bar{\rho} I$ to get (5.101) with $c_\epsilon = \bar{\rho}/\bar{\epsilon}$. □

Lemma 5.44. *If the switched independent jump linear system* $(\mathcal{G}, \Pi, \Psi, p(0))$ *is uniformly exponentially mean square stable and uniformly mean square strictly contractive then there exist* $\eta, \rho > 0$ *such that for all* $\epsilon \in (0, \eta)$, *there exist* $M \in \mathbb{N}$ *and* $\nu > 0$ *such that*

$$Y_\psi(k) := X_\psi(k, k + M - 1, \epsilon) \tag{5.103}$$

satisfies

$$\epsilon I \leq Y_\psi(k) \leq \rho I \tag{5.104a}$$

$$\sum_{j=1}^{N} \pi_j(\psi(k)) \begin{bmatrix} A(j) & B(j) \\ C(j) & D(j) \end{bmatrix}^\mathsf{T} \begin{bmatrix} Y_\psi(k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(j) & B(j) \\ C(j) & D(j) \end{bmatrix} - \begin{bmatrix} Y_\psi(k) & 0 \\ 0 & I \end{bmatrix} \leq -\nu I \tag{5.104b}$$

for all $\psi \in \Psi$ *and* $k \in \mathbb{N}_0$ *where* X_ψ *is defined in* (5.90).

Proof. Let η, ρ be as in Theorem 5.41 and choose $\epsilon \in (0, \eta)$ so (5.104a) is verified automatically. Let λ_ϵ and

c_ϵ be as in Lemma 5.43. Choose $M \in \mathbb{N}$ such that $c_\epsilon \lambda_\epsilon^M < \epsilon/\rho$. Then

$$
\begin{aligned}
&S(\psi(k), Y_\psi(k+1)) - Y_\psi(k) + \epsilon I \\
&\quad = S(\psi(k), X_\psi(k+1, k+M, \epsilon)) - S(\psi(k), X_\psi(k+1, k+M-1, \epsilon)) && (5.105)\\
&\quad = \sum_{j=1}^{N} \pi_j(\psi(k)) \Big[\mathcal{A}^\mathsf{T} \left(j, \psi(k), X_\psi(k+1, k+M, \epsilon)\right) \left(X_\psi(k+1, k+M, \epsilon) - X_\psi(k+1, k+M-1, \epsilon)\right) \\
&\qquad\quad \times \mathcal{A}\left(j, \psi(k), X_\psi(k+1, k+M-1, \epsilon)\right) \Big] && (5.106)
\end{aligned}
$$

where (5.105) follows from (5.90a) and (5.106) follows from (5.94). Using (5.90a) and (5.94), the middle term in (5.106) may be written

$$
\begin{aligned}
&X_\psi(k+1, k+M, \epsilon) - X_\psi(k+1, k+M-1, \epsilon) \\
&\quad = \sum_{j=1}^{N} \pi_j(\psi(k+1)) \Big[\mathcal{A}^\mathsf{T} \left(j, \psi(k+1), X_\psi(k+2, k+M, \epsilon)\right) \left(X_\psi(k+2, k+M, \epsilon) - X_\psi(k+2, k+M-1, \epsilon)\right) \\
&\qquad\quad \times \mathcal{A}\left(j, \psi(k+1), X_\psi(k+2, k+M-1, \epsilon)\right) \Big]. && (5.107)
\end{aligned}
$$

Then

$$
\begin{aligned}
&S(\psi(k), Y_\psi(k+1)) - Y_\psi(k) + \epsilon I \\
&\quad = \mathbf{E}\big[\phi^\mathsf{T}(k+M, k, k+M) \left(X_\psi(k+M, k+M, \epsilon) - X_\psi(k+M, k+M-1, \epsilon)\right) \\
&\qquad\quad \times \phi(k+M, k, k+M-1)\big] && (5.108)\\
&\quad = \mathbf{E}\big[\phi^\mathsf{T}(k+M, k, k+M) X_\psi(k+M, k+M, \epsilon) \phi(k+M, k, k+M-1)\big] \\
&\quad \le \rho c_\epsilon \lambda_\epsilon^M y^\mathsf{T} y && (5.109)\\
&\quad < \epsilon y^\mathsf{T} y && (5.110)
\end{aligned}
$$

where ϕ is defined in (5.100); equality (5.108) follows by proceeding iteratively as in (5.107) and using the independence hypothesis on θ; inequality (5.109) follows from arguments similar to those found in the proof of Lemma 5.27; and, inequality (5.110) follows by choice of M. Let $\epsilon_0 \in (0, \epsilon)$ be such that

$$S(\psi(k), Y_\psi(k+1)) - Y_\psi(k) + \epsilon I \le \epsilon_0 I < \epsilon I$$

so that

$$S(\psi(k), Y_\psi(k+1)) - Y_\psi(k) \le -\nu I \tag{5.111}$$

where $\nu = \epsilon - \epsilon_0$. Expanding the left side of (5.104b) yields

$$\sum_{j=1}^{N} \pi_j(\psi(k)) \begin{bmatrix} A(j) & B(j) \\ C(j) & D(j) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} Y_\psi(k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(j) & B(j) \\ C(j) & D(j) \end{bmatrix} - \begin{bmatrix} Y_\psi(k) & 0 \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} L(\psi(k), Y_\psi(k+1)) - Y_\psi(k) & R^{\mathsf{T}}(\psi(k), Y_\psi(k+1)) \\ R(\psi(k), Y_\psi(k+1)) & -W(\psi(k), Y_\psi(k+1)) \end{bmatrix}$$

The Schur complement formula combined with inequality (5.111) yields (5.104b). □

Remark 5.45. The construction in Lemma 5.44 ensures that $Y_\psi(k)$ may be computed with knowledge of only $(\psi(k), \ldots, \psi(k+M-1))$. Indeed, if $t \neq k$ but $(\psi(k), \ldots, \psi(k+M-1)) = (\psi(t), \ldots, \psi(t+M-1))$ then $Y_\psi(k) = Y_\psi(t)$. This claim can be easily established using the recursive relation (5.90a) and base case $X_\psi(k+M, k+M-1, \epsilon) = X_\psi(t+M, t+M-1, \epsilon) = 0$.

Theorem 5.46. *The switched independent jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable and uniformly mean square strictly contractive if and only if there exist $M \in \mathbb{N}$ and a function $X : \Psi_M \to \mathbb{S}_n^+$ such that*

$$\sum_{j=1}^{N} \pi_j(r_0) \begin{bmatrix} A(j) & B(j) \\ C(j) & D(j) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} X(r_1, \ldots, r_M) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(j) & B(j) \\ C(j) & D(j) \end{bmatrix} - \begin{bmatrix} X(r_0, \ldots, r_{M-1}) & 0 \\ 0 & I \end{bmatrix} < 0 \qquad (5.112)$$

for any $(r_0, \ldots, r_M) \in \Psi_{M+1}$.

Proof. Suppose there exist M and X such that (5.112) holds. Note that the upper left block of (5.112) implies (4.21) so uniform exponential mean square stability of $(\mathcal{G}, \Pi, \Psi, p(0))$ follows from Theorem 4.21. Since Ψ_{M+1} is a finite set, inequality (5.112) holds uniformly, so there exists $0 < \nu < 1$ such that

$$\sum_{j=1}^{N} \pi_j(r_0) \begin{bmatrix} A(j) & B(j) \\ C(j) & D(j) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} X(r_1, \ldots, r_M) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(j) & B(j) \\ C(j) & D(j) \end{bmatrix} - \begin{bmatrix} X(r_0, \ldots, r_{M-1}) & 0 \\ 0 & I \end{bmatrix} \leq -\nu I. \qquad (5.113)$$

for any $(r_0, \ldots, r_M) \in \Psi_{M+1}$. Let $\psi \in \Psi$ be arbitrary. Define $Y_\psi(k) := X(\psi(k), \ldots, \psi(k+M-1))$. Then

$$\mathbf{E}[\, \|z(k)\|^2 - \|w(k)\|^2 + x^{\mathsf{T}}(k+1) Y_\psi(k+1) x(k+1) - x^{\mathsf{T}}(k) Y_\psi(k) x(k) \,]$$

$$= \mathbf{E}\left[\begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^{\mathsf{T}} \left(\begin{bmatrix} A(\theta(k)) & B(\theta(k)) \\ C(\theta(k)) & D(\theta(k)) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} Y_{\psi}(k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(\theta(k)) & B(\theta(k)) \\ C(\theta(k)) & D(\theta(k)) \end{bmatrix} - \begin{bmatrix} Y_{\psi}(k) & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}\right] \tag{5.114}$$

$$= \mathbf{E}\left[\begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^{\mathsf{T}} \sum_{j=1}^{N} \pi_j(\psi(k)) \left(\begin{bmatrix} A(j) & B(j) \\ C(j) & D(j) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} Y_{\psi}(k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(j) & B(j) \\ C(j) & D(j) \end{bmatrix} - \begin{bmatrix} Y_{\psi}(k) & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}\right] \tag{5.115}$$

$$\leq -\nu \mathbf{E}\left[\|w(k)\|^2\right] \tag{5.116}$$

where (5.114) follows from Lemmas 5.16 and 5.18; (5.115) follows by an iterated expectation and the independence hypothesis on θ; and, (5.116) follows from (5.113). Thus

$$\mathbf{E}[\, \|z(k)\|^2 + x^{\mathsf{T}}(k+1) Y_{\psi}(k+1) x(k+1) - x^{\mathsf{T}}(k) Y_{\psi}(k) x(k)\,] \leq (1-\nu)\mathbf{E}[\, \|w(k)\|^2\,]. \tag{5.117}$$

Inequality (5.117), positive definiteness of $Y_{\psi}(k)$, and $x(0) = 0$ yield

$$\sum_{k=0}^{l} \mathbf{E}\left[\|z(k)\|^2\right] \leq (1-\nu) \sum_{k=0}^{l} \mathbf{E}\left[\|w(k)\|^2\right]$$

for all $l \in \mathbb{N}_0$. Since $\psi \in \Psi$ was arbitrary, Definition 5.10 is satisfied with $\gamma = \sqrt{1-\nu}$ so $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly mean square strictly contractive.

Conversely, assume that $(\mathcal{G}, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable and uniformly mean square strictly contractive. Let η, ρ be as in Lemma 5.44, fix $\epsilon \in (0, \eta)$, and let $M \in \mathbb{N}$ and ν be defined as in Lemma 5.44. Let $(r_0, \ldots, r_M) \in \Psi_{M+1}$ be arbitrary. By definition of Ψ_{M+1}, there exists $\psi \in \Psi$ and $t \in \mathbb{N}_0$ such that $(\psi(t), \ldots, \psi(t+M)) = (r_0, \ldots, r_M)$. Construct Y_{ψ} as in Lemma 5.44 and recall from Remark 5.45 that $Y_{\psi}(t)$ depends only on $(\psi(t), \ldots, \psi(t+M-1))$. Thus define $X(r_0, \ldots, r_{M-1}) := Y_{\psi}(t)$ and define $X(r_1, \ldots, r_M) := Y_{\psi}(t+1)$. One recovers every inequality in (5.112) from (5.104). $\square$

Remark 5.47. Note that Theorem 5.46 does not require the hypothesis $p_i(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$ as in Theorem 5.29.

Remark 5.48. For each M, Theorem 5.46 requires solving a set of LMIs with no more than $J^M n(n+1)/2$ unknowns (there are at most J^M unknown symmetric positive definite matrices that must satisfy (5.112)).

Thus, Theorem 5.46 is more computationally efficient than Theorem 5.29, which requires solving a set of LMIs with no more than $NJ^M n(n+1)/2$ unknowns.

Remark 5.49. Consider the case when $J = 1$ and $\Psi = \{(1, 1, \dots)\}$. The switched independent jump linear system $(\mathcal{G}, \Pi, \Psi, p(0))$ reduces to a single time-homogeneous independent jump linear system $(\mathcal{G}, \Pi \circ \psi_1, p(0))$ where $\psi_1 \equiv 1$. For any M, the set Ψ_M contains only a single element, the M-tuple $(1, \dots, 1)$. Define $Z := X(1, \dots, 1)$ where X is as in Theorem 5.46. Then (5.112) reduces to

$$\sum_{j=1}^{N} \pi_j(1) \begin{bmatrix} A(j) & B(j) \\ C(j) & D(j) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} Z & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(j) & B(j) \\ C(j) & D(j) \end{bmatrix} - \begin{bmatrix} Z & 0 \\ 0 & I \end{bmatrix} < 0 \tag{5.118}$$

Inequality (5.118) is the same inequality found in Proposition 5.8. Theorem 5.46, however, does not require $(\mathcal{G}, \Pi \circ \psi_1, p(0))$ to be weakly controllable as in Proposition 5.8. Thus, the weak controllability hypothesis of Proposition 5.8 is unnecessary.

Chapter 6

Control Synthesis

6.1 Introduction

The aim of this chapter is to construct feedback controllers that ensure the design objectives of stability and a specified level of disturbance attenuation are met. The open-loop plant to be controlled is expressed

$$x(k+1) = A(\theta(k))x(k) + \begin{bmatrix} B_1(\theta(k)) & B_2(\theta(k)) \end{bmatrix} \begin{bmatrix} w(k) \\ u(k) \end{bmatrix} \tag{6.1a}$$

$$\begin{bmatrix} z(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} C_1(\theta(k)) \\ C_2(\theta(k)) \end{bmatrix} x(k) + \begin{bmatrix} D_{11}(\theta(k)) & D_{12}(\theta(k)) \\ D_{21}(\theta(k)) & D_{22}(\theta(k)) \end{bmatrix} \begin{bmatrix} w(k) \\ u(k) \end{bmatrix} \tag{6.1b}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $w(k) \in \mathbb{R}^{m_1}$ is a disturbance vector, $z(k) \in \mathbb{R}^{p_1}$ is an error vector, $y(k) \in \mathbb{R}^{p_2}$ is the output vector available to the controller, $u(k) \in \mathbb{R}^{m_2}$ is the controlled input vector, and $\theta : \mathbb{N}_0 \times \Omega \to \mathcal{N}$ is a finite Markov chain characterized by the a priori unknown sequence $\Pi \circ \psi$ of transition probabilities and initial distribution $p(0)$, where $\psi \in \Psi$ is an a priori unknown switching sequence. Without loss of generality (see [24]), it is assumed that $D_{22}(i) = 0$ for all $i \in \mathcal{N}$.

The results developed in this chapter address the question of existence of a controller for the plant in (6.1) that ensures the closed-loop system is uniformly exponentially mean square stable and uniformly mean square strictly contractive.

To motivate the types of controllers examined in this chapter, consider the control system in Fig. 6.1 where the sensor and controller are connected via a wireless communications network subject to Markovian packet loss. The closed-loop system may be modeled as a Markov jump linear system where $\theta(k)$ indicates

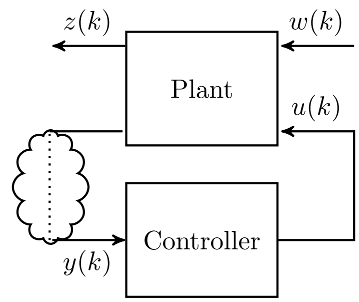


Figure 6.1: Control system where the controller and actuator are physically connected, while the sensor and controller are connected via a network.

whether or not the most recent packet was dropped (see, e.g., [60, 61, 68]). If network communications are time-stamped, then $\theta(k)$ is measurable at time k. Suppose that the transition probabilities governing packet loss are time-varying and a priori unknown. Many factors affect packet loss probabilities, but it might be reasonable to assume that the current packet loss probabilities are a function of the signal-to-noise ratio (SNR) (e.g., see the discussion in [50]) so that measurement of the SNR at time k yields the current transition probabilities $\Pi(\psi(k))$ of the Markov chain. In this scenario, both $\theta(k)$ and $\psi(k)$ are measurable at time k. Accordingly, in Section 6.4, synthesis of a feedback controller for the plant in (6.1) is examined that at time k has access to the plant output $y(k)$, the current mode $\theta(k)$ of the Markov chain, and the current value $\psi(k)$ of the switching sequence.

Before the control synthesis results of Section 6.4 can be stated, some preliminary results need to be developed. A few technical results that are needed for control synthesis are presented in Section 6.2. In Section 6.3, a controller is constructed that at each time k has access to the plant output $y(k)$, the current mode $\theta(k)$ of the Markov chain, and knowledge of some finite number of future values $\psi(k+1), \ldots, \psi(k+L)$ of the switching sequence. Depending upon the application, the controller construction in Section 6.3 may be useful in its own right. The results of Section 6.3, however, are applied in Section 6.4 under an additional assumption to construct a controller dependent only on the present and the past.

6.2 Technical Prerequisites

The synthesis results of the chapter require the technical results reported in this section. Stability and disturbance attenuation results for a Markov jump linear system where the parameter matrices depend explicitly on time, as well as on the current mode of the Markov chain are stated in Section 6.2.1. The adjoint state equation associated with a time-varying Markov jump linear system is examined in Section 6.2.2, and Section 6.2.3 contains uniform stability and uniform disturbance attenuation results for a time-varying Markov jump linear system where the sequence of transition probability matrices of the Markov chain is a priori unknown.

6.2.1 Time-varying Markov jump linear systems

Let $\mathcal{G}_k = \{(A(i,k), B(i,k), C(i,k), D(i,k)) : i \in \mathcal{N}, k \in \mathbb{N}_0\}$ where $A(i,k) \in \mathbb{R}^{n\times n}$, $B(i,k) \in \mathbb{R}^{n\times m}$, $C(i,k) \in \mathbb{R}^{p\times n}$, and $D(i,k) \in \mathbb{R}^{p\times m}$ are bounded for $i \in \mathcal{N}$ and $k \in \mathbb{N}_0$. For technical reasons, consider the time-varying, time-inhomogeneous Markov jump linear system, denoted $(\mathcal{G}_k, P, p(0))$ and described by the difference equation

$$\begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} A(\theta(k),k) & B(\theta(k),k) \\ C(\theta(k),k) & D(\theta(k),k) \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \tag{6.2}$$

and initial condition $x(0)$, where $x(k) \in \mathbb{R}^n$ is the state vector, $w(k) \in \mathbb{R}^m$ is a disturbance vector, $z(k) \in \mathbb{R}^p$ is an error vector, and $\theta : \mathbb{N}_0 \times \Omega \to \mathcal{N}$ is a finite Markov chain characterized by the sequence $P : \mathbb{N} \to \mathbb{T}_N$ of transition probability matrices and initial distribution $p(0)$.

The (random) state transition matrix is expressed

$$\Phi(k,j) = \begin{cases} A(\theta(k-1),k-1)A(\theta(k-2),k-2)\cdots A(\theta(j),j) & : \ k > j \\ I & : \ k = j \end{cases} \tag{6.3}$$

Exponential mean square stability of the time-varying, time-inhomogeneous Markov jump linear system $(\mathcal{G}_k, P, p(0))$ in (6.2) may be characterized by a stochastic Lyapunov criterion similar to Proposition 4.2.

Proposition 6.1. *The time-varying, time-inhomogeneous Markov jump linear system $(\mathcal{G}_k, P, p(0))$ in (6.2) is exponentially mean square stable if and only if there exist $\eta, \rho, \nu > 0$ and a function $X : \mathcal{N} \times \mathbb{N}_0 \to \mathbb{S}_n^+$ such that*

$$\eta I \le X(i,k) \le \rho I \tag{6.4a}$$

$$A^{\mathsf{T}}(i,k)\tilde{X}(i,k+1)A(i,k) - X(i,k) \le -\nu I \tag{6.4b}$$

for all $i \in \mathcal{N}$ and all $k \in \mathbb{N}_0$ where $\tilde{X}(i,k+1) = \sum_{j=1}^N p_{ij}(k+1)X(j,k+1)$. Moreover, if (6.4) holds, one may take $c = \rho/\eta$ and $\lambda = 1 - \nu/\rho$ in Definition 4.1.

Proof. The proof follows the same format as the proof of Proposition 4.2 (found in [36]). Namely, $V(i,k,y) = y^{\mathsf{T}}X(i,k)y$ is a stochastic Lyapunov function if (6.4) holds, and $X(i,k) = \mathbf{E}\left[\sum_{j=k}^{\infty} \Phi^{\mathsf{T}}(j,k)\Phi(j,k) \mid \theta(k) = i\right]$ satisfies (6.4) if the system is exponentially mean square stable. □

Mean square strict contractiveness of the time-varying, time-inhomogeneous Markov jump linear system $(\mathcal{G}_k, P, p(0))$ in (6.2) may be characterized by a stochastic Kalman-Yakubovich-Popov (KYP) criterion similar to Proposition 5.3.

Proposition 6.2 (Thm. 2 of [1]). *Assume the time-varying, time-inhomogeneous Markov jump linear system $(\mathcal{G}_k, P, p(0))$ in (6.2) is weakly controllable and $p_i(k) > 0$ for all $i \in \mathcal{N}$ and $k \in \mathbb{N}_0$. The system $(\mathcal{G}_k, P, p(0))$ is exponentially mean square stable and mean square strictly contractive if and only if there exist $\eta, \rho, \nu > 0$ and a function $X : \mathcal{N} \times \mathbb{N}_0 \to \mathbb{S}_n^+$ such that*

$$\eta I \leq X(i,k) \leq \rho I \tag{6.5a}$$

$$\begin{bmatrix} A(i,k) & B(i,k) \\ C(i,k) & D(i,k) \end{bmatrix}^\mathsf{T} \begin{bmatrix} \tilde{X}(i,k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(i,k) & B(i,k) \\ C(i,k) & D(i,k) \end{bmatrix} - \begin{bmatrix} X(i,k) & 0 \\ 0 & I \end{bmatrix} \leq -\nu I \tag{6.5b}$$

for all $i \in \mathcal{N}$ and all $k \in \mathbb{N}_0$ where $\tilde{X}(i,k+1) = \sum_{j=1}^{N} p_{ij}(k+1) X(j,k+1)$.

6.2.2 Adjoint system

As discussed in Section 2.3, every bounded linear operator defined on a Hilbert space has an associated Hilbert-adjoint operator, and the induced norm of the bounded linear operator and its adjoint are equal. Thus, it is sometimes fruitful to work with the Hilbert-adjoint operator when trying to establish an upper bound on the induced norm of the original linear operator. The Hilbert-adjoint of a time-varying Markov jump linear system, viewed as an input-output operator, is derived in this section.

Lemma 6.3. *Let $x(0) = 0$ and consider the bounded linear operator $G : \ell_e^2[0,T] \to \ell_e^2[0,T]$ defined by the system in (6.2), i.e., $w \overset{G}{\mapsto} z$. The Hilbert-adjoint, $G^* : \ell_e^2[0,T] \to \ell_e^2[0,T]$, is the linear map $v \overset{G^*}{\mapsto} f$ defined by the difference equation*

$$\begin{bmatrix} s(k-1) \\ f(k) \end{bmatrix} = \begin{bmatrix} A^\mathsf{T}(\theta(k),k) & C^\mathsf{T}(\theta(k),k) \\ B^\mathsf{T}(\theta(k),k) & D^\mathsf{T}(\theta(k),k) \end{bmatrix} \begin{bmatrix} s(k) \\ v(k) \end{bmatrix} \tag{6.6}$$

and final condition $s(T) = 0$ where $0 \leq k \leq T$, $s(k) \in \mathbb{R}^n$ is the adjoint state vector, $v(k) \in \mathbb{R}^p$ is the adjoint disturbance vector, and $f(k) \in \mathbb{R}^m$ is the adjoint error vector.

Proof. By Definition 2.14, it suffices to show that G^* is the operator that satisfies $\langle Gw, v \rangle_e = \langle w, G^* v \rangle_e$ for all $v, w \in \ell_e^2[0,T]$.

$$\langle Gw, v \rangle_e = \langle z, v \rangle_e$$

$$
\begin{aligned}
&= \mathbf{E}\left[\sum_{k=0}^{T} v^{\mathsf{T}}(k)z(k) + s^{\mathsf{T}}(k)x(k+1) - s^{\mathsf{T}}(k-1)x(k)\right] \qquad (6.7)\\
&= \mathbf{E}\left[\sum_{k=0}^{T} \begin{bmatrix} s(k) \\ v(k) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} - s^{\mathsf{T}}(k-1)x(k)\right]\\
&= \mathbf{E}\left[\sum_{k=0}^{T} \begin{bmatrix} s(k) \\ v(k) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} A(\theta(k),k) & B(\theta(k),k) \\ C(\theta(k),k) & D(\theta(k),k) \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} - s^{\mathsf{T}}(k-1)x(k)\right]\\
&= \mathbf{E}\left[\sum_{k=0}^{T} \begin{bmatrix} s(k-1) \\ f(k) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} - s^{\mathsf{T}}(k-1)x(k)\right]\\
&= \mathbf{E}\left[\sum_{k=0}^{T} f^{\mathsf{T}}(k)w(k)\right]\\
&= \langle w, G^{*}v\rangle_{e}
\end{aligned}
$$

where (6.7) follows since $s(T) = x(0) = 0$. □

The solution to (6.6) with arbitrary final condition (i.e., $s(T)$ is not necessarily zero) is expressed

$$s(k) = \Phi^{\mathsf{T}}(T+1,k+1)s(T) + \sum_{j=k+1}^{T} \Phi^{\mathsf{T}}(j,k+1)C^{\mathsf{T}}(\theta(j),j)v(j)$$

for $-1 \le k \le T$ and is the sum of the zero-input ($v \equiv 0$) response and zero-state ($s(T) = 0$) response, respectively. Since $\lambda_{\max}\left(\Phi^{\mathsf{T}}(k,j)\Phi(k,j)\right) = \lambda_{\max}\left(\Phi(k,j)\Phi^{\mathsf{T}}(k,j)\right)$ the adjoint state equation is stable if and only if the original state equation is stable. By Lemma 2.15, $\|G\|_{\ell_e^2[0,T]\to\ell_e^2[0,T]} = \|G^{*}\|_{\ell_e^2[0,T]\to\ell_e^2[0,T]}$. Thus, the adjoint state equation in (6.6) is mean square strictly contractive if and only if the original state equation in (6.2) is mean square strictly contractive.

If the time-varying Markov jump linear system in (6.2) is exponentially mean square stable, let $T \to \infty$ so that the adjoint system is generated by the difference equation

$$\begin{bmatrix} s(k-1) \\ f(k) \end{bmatrix} = \begin{bmatrix} A^{\mathsf{T}}(\theta(k),k) & C^{\mathsf{T}}(\theta(k),k) \\ B^{\mathsf{T}}(\theta(k),k) & D^{\mathsf{T}}(\theta(k),k) \end{bmatrix} \begin{bmatrix} s(k) \\ v(k) \end{bmatrix} \qquad (6.8)$$

with final condition $\lim_{T\to\infty} s(T) = 0$ where $k \in \mathbb{N}_0$. Now make the following change of variables

$$\acute{s}(t) := s(-t), \quad \acute{f}(t) := f(-t), \quad \acute{v}(t) := v(-t), \quad \acute{\theta}(t) := \theta(-t),$$

$$\begin{bmatrix} \acute{A}^{\mathsf{T}}(i,t) & \acute{C}^{\mathsf{T}}(i,t) \\ \acute{B}^{\mathsf{T}}(i,t) & \acute{D}^{\mathsf{T}}(i,t) \end{bmatrix} := \begin{bmatrix} A^{\mathsf{T}}(i,-t) & C^{\mathsf{T}}(i,-t) \\ B^{\mathsf{T}}(i,-t) & D^{\mathsf{T}}(i,-t) \end{bmatrix}$$

where $t \in \mathbb{Z}_0^-$ and $i \in \mathcal{N}$. Then by (6.8),

$$\begin{bmatrix} \acute{s}(t+1) \\ \acute{f}(t) \end{bmatrix} = \begin{bmatrix} \acute{A}^{\mathsf{T}}(\acute{\theta}(t),t) & \acute{C}^{\mathsf{T}}(\acute{\theta}(t),t) \\ \acute{B}^{\mathsf{T}}(\acute{\theta}(t),t) & \acute{D}^{\mathsf{T}}(\acute{\theta}(t),t) \end{bmatrix} \begin{bmatrix} \acute{s}(t) \\ \acute{v}(t) \end{bmatrix} \tag{6.9}$$

where $t \in \mathbb{Z}_0^-$. The Markov jump linear system (6.9) progresses forward in time from an initial condition in the infinite past.

6.2.3 Time-varying switched Markov jump linear systems

In this section, the time-varying, time-inhomogeneous Markov jump linear system in (6.2) is examined when the sequence P of transition probability matrices which describes θ is a priori unknown but varies in the finite set $\{\Pi(1), \ldots, \Pi(J)\}$. Thus, $P(k) = \Pi(\psi(k))$ where $\psi \in \Psi$ is some a priori unknown switching sequence in the set Ψ of all possible switching sequences. Define the switched Markov jump linear system

$$(\mathcal{G}_k, \Pi, \Psi, p(0)) = \{(\mathcal{G}_k, \Pi \circ \psi, p(0)) : \psi \in \Psi\}. \tag{6.10}$$

Many of the results derived in Sections 4.4 and 5.4 can be extended to apply to the switched Markov jump linear system in (6.10). The relevant results are stated here with minimal discussion, and the proofs largely mirror those in Sections 4.4 and 5.4.

Lemma 6.4. *Suppose system $(\mathcal{G}_k, \Pi, \Psi, p(0))$ is uniformly exponentially mean square stable and $M \in \mathbb{N}_0$ is such that $c\lambda^{M+2} < 1$ where c, λ are as in (4.9). Then for each $\psi \in \Psi$*

$$Y_\psi(i,k) := \sum_{j=k}^{k+M+1} \mathbf{E}\left[\Phi^{\mathsf{T}}(j,k)\Phi(j,k) \mid \theta(k) = i\right] \tag{6.11}$$

satisfies

$$\eta I \leq Y_\psi(i,k) \leq \rho I \tag{6.12a}$$

$$A^{\mathsf{T}}(i,k)\tilde{Y}_\psi(i,k+1)A(i,k) - Y_\psi(i,k) \leq -\nu I. \tag{6.12b}$$

for all $i \in \mathcal{N}$ and all $k \in \mathbb{N}_0$ where $\tilde{Y}_\psi(i,k+1) = \sum_{j=1}^N \pi_{ij}(\psi(k+1))Y_\psi(j,k+1)$, $\eta = 1$, $\rho = c/(1-\lambda)$, and $\nu = 1 - c\lambda^{M+2}$.

Proof. Replace $A(i)$ with $A(i,k)$ in the proof of Lemma 6.4 and proceed in the same manner. $\square$

Lemma 6.5. *The switched Markov jump linear system* $(\mathcal{G}_k, \Pi, \Psi, p(0))$ *is uniformly exponentially mean square stable if and only if there exist* $\eta, \rho, \nu > 0$ *and functions* $Y_\psi : \mathcal{N} \times \mathbb{N}_0 \to \mathbb{S}_n^+$, $\psi \in \Psi$ *such that*

$$\eta I \leq Y_\psi(i,k) \leq \rho I \tag{6.13a}$$

$$A^\mathsf{T}(i,k)\tilde{Y}_\psi(i,k+1)A(i,k) - Y_\psi(i,k) \leq -\nu I. \tag{6.13b}$$

for all $\psi \in \Psi$, $i \in \mathcal{N}$, *and* $k \in \mathbb{N}_0$ *where* $\tilde{Y}_\psi(i,k+1) = \sum_{j=1}^N \pi_{ij}(\psi(k+1))Y_\psi(j,k+1)$.

Proof. Necessity of (6.13) follows from Lemma 6.4. For sufficiency, (6.13) implies that Y_ψ is a Lyapunov function for the individual system $(\mathcal{G}_k, \Pi \circ \psi, p(0))$ and satisfies (6.4) in Proposition 6.1. Thus the individual system $(\mathcal{G}_k, \Pi \circ \psi, p(0))$ is exponentially mean square stable with decay rate $\lambda = 1 - \nu/\rho$ and $c = \rho/\eta$. Since η, ρ, ν are independent of ψ in (6.13), λ and c apply uniformly to all systems in the family $(\mathcal{G}_k, \Pi, \Psi, p(0))$. $\square$

Let $\mathcal{L}(i,k,X)$, $\mathcal{R}(i,k,X)$, $\mathcal{W}(i,k,X)$, $\mathcal{M}(i,k,X)$, and $\mathcal{S}(i,k,X)$ be defined analogously to Definition 5.15 by replacing $A(i)$, $B(i)$, $C(i)$, $D(i)$ with $A(i,k)$, $B(i,k)$, $C(i,k)$, $D(i,k)$, respectively. Consider the Riccati difference equations

$$X_\psi(i,k,T,\epsilon) = \mathcal{S}(i,k,\tilde{X}_\psi(i,k+1,T,\epsilon)) + \epsilon I \tag{6.14a}$$

$$X_\psi(i,T+1,T,\epsilon) = 0 \tag{6.14b}$$

where $i \in \mathcal{N}$, $T \in \mathbb{N}_0$, $0 \leq k \leq T$, $\psi \in \Psi$, $\epsilon > 0$, and $\tilde{X}_\psi(i,k+1,T,\epsilon) = \sum_{j=1}^N \pi_{ij}(\psi(k+1))X_\psi(j,k+1,T,\epsilon)$. Following the same steps in Section 5.4, one may show that solutions to the Riccati difference equations (6.14) are well-defined, uniformly positive definite, and uniformly bounded when $(\mathcal{G}_k, \Pi, \Psi, p(0))$ is uniformly stable and contractive. Proceeding as in Section 5.4, one can prove the following result.

Lemma 6.6. *Assume* $p_i(k) > 0$ *for all* $\psi \in \Psi$, $i \in \mathcal{N}$, *and* $k \in \mathbb{N}_0$. *If* $(\mathcal{G}_k, \Pi, \Psi, p(0))$ *is uniformly exponentially mean square stable and uniformly mean square strictly contractive then there exist* $\eta, \rho > 0$ *such that for all* $\epsilon \in (0, \eta)$, *there exist* $M \in \mathbb{N}_0$ *and* $\nu > 0$ *such that*

$$Y_\psi(i,k) := X_\psi(i,k,k+M,\epsilon) \tag{6.15}$$

satisfies

$$\epsilon I \leq Y_\psi(i,k) \leq \rho I \tag{6.16a}$$

$$\begin{bmatrix} A(i,k) & B(i,k) \\ C(i,k) & D(i,k) \end{bmatrix}^\mathsf{T} \begin{bmatrix} \tilde{Y}_\psi(i,k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(i,k) & B(i,k) \\ C(i,k) & D(i,k) \end{bmatrix} - \begin{bmatrix} Y_\psi(i,k) & 0 \\ 0 & I \end{bmatrix} \leq -\nu I \tag{6.16b}$$

for all $\psi \in \Psi$, $i \in \mathcal{N}$, *and* $k \in \mathbb{N}_0$ *where* $\tilde{Y}_\psi(i,k+1) = \sum_{j=1}^{N} \pi_{ij}(\psi(k+1))Y_\psi(j,k+1)$ *and* X_ψ *is defined in* (6.14).

Proof. Time-varying counterparts to Lemma 5.20, Lemmas 5.22–5.23, Theorem 5.24, and Lemmas 5.25–5.27 involving the Riccati difference equations in (6.14) and the matrices $A(i,k)$, $B(i,k)$, $C(i,k)$, $D(i,k)$ can be proved with only slight modifications to the original proofs. Lemma 6.6 is the time-varying counterpart to Lemma 5.27. □

Lemma 6.7. *Assume* $p_i(k) > 0$ *for all* $\psi \in \Psi$, $i \in \mathcal{N}$, *and* $k \in \mathbb{N}_0$. *The switched Markov jump linear system* $(\mathcal{G}_k, \Pi, \Psi, p(0))$ *is uniformly exponentially mean square stable and uniformly mean square strictly contractive if and only if there exist* $\eta, \rho, \nu > 0$ *and functions* $Y_\psi : \mathcal{N} \times \mathbb{N}_0 \to \mathbb{S}_n^+$, $\psi \in \Psi$ *such that*

$$\eta I \leq Y_\psi(i,k) \leq \rho I \tag{6.17a}$$

$$\begin{bmatrix} A(i,k) & B(i,k) \\ C(i,k) & D(i,k) \end{bmatrix}^\mathsf{T} \begin{bmatrix} \tilde{Y}_\psi(i,k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(i,k) & B(i,k) \\ C(i,k) & D(i,k) \end{bmatrix} - \begin{bmatrix} Y_\psi(i,k) & 0 \\ 0 & I \end{bmatrix} \leq -\nu I \tag{6.17b}$$

for all $\psi \in \Psi$, $i \in \mathcal{N}$, *and* $k \in \mathbb{N}_0$ *where* $\tilde{Y}_\psi(i,k+1) = \sum_{j=1}^{N} \pi_{ij}(\psi(k+1))Y_\psi(j,k+1)$.

Proof. Necessity of (6.17) follows from Lemma 6.6. For sufficiency,

$$\mathbf{E}\left[\|z(k)\|^2 - \|w(k)\|^2 + x^\mathsf{T}(k+1)Y_\psi(\theta(k+1),k+1)x(k+1) - x^\mathsf{T}(k)Y_\psi(\theta(k),k)x(k)\right] \tag{6.18}$$

$$= \mathbf{E}\left[\begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^\mathsf{T} \left(\begin{bmatrix} A(\theta(k),k) & B(\theta(k),k) \\ C(\theta(k),k) & D(\theta(k),k) \end{bmatrix}^\mathsf{T} \begin{bmatrix} \tilde{Y}_\psi(\theta(k),k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(\theta(k),k) & B(\theta(k),k) \\ C(\theta(k),k) & D(\theta(k),k) \end{bmatrix} - \begin{bmatrix} Y_\psi(\theta(k),k) & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}\right] \tag{6.19}$$

$$\leq -\nu \mathbf{E}[\,\|w(k)\|^2\,]. \tag{6.20}$$

where the appearance of $\tilde{Y}_\psi(\theta(k),k+1)$ in (6.19) follows from an iterated expectation applied to the third term of the sum in (6.18). Note that $Y_\psi(i,k)$ is positive definite and $x(0) = 0$. Sum both sides of inequality (6.20) to get

$$\sum_{k=0}^{l} \mathbf{E}\left[\|z(k)\|^2\right] \leq (1-\nu)\sum_{k=0}^{l} \mathbf{E}\left[\|w(k)\|^2\right]$$

for all $l \in \mathbb{N}_0$. Since $\psi \in \Psi$ was arbitrary, Definition 5.10 is satisfied with $\gamma = \sqrt{1-\nu}$ so $(\mathcal{G}_k, \Pi, \Psi, p(0))$ is uniformly mean square strictly contractive. □

6.3 Control Synthesis with Finite Knowledge of the Future

In this section, the goal is to synthesize a feedback controller for the plant in (6.1) under the assumption that at time k, the controller has access to the plant output $y(k)$, a perfect measurement of the current mode $\theta(k)$ of the Markov chain, and knowledge of some finite number of future values $\psi(k+1), \ldots, \psi(k+L)$ of the switching sequence.

In some applications, knowledge of $\psi(k+1), \ldots, \psi(k+L)$ may be available at time k. For example, if the sequence of transition probability matrices of the Markov chain is in fact *known* (i.e., $\Psi = \{\psi\}$), then the results of Section 6.3 provide synthesis results for the class of time-inhomogeneous Markov jump linear systems where the sequence of transition probability matrices is known and varies in a finite set. To the author's knowledge, the nonconservative controller synthesis reported in Section 6.3 for this class of Markov jump linear systems is the first of its kind. In other applications, the sequence of transition probability matrices may not be fully known a priori, but a receding horizon view of the sequence is available.

In any case, the results of Section 6.3 are important as theoretical tools and are applied in Section 6.4 to construct controllers with access to only the plant output $y(k)$, the current mode $\theta(k)$ of the Markov chain, and the current value $\psi(k)$ of the switching sequence. As discussed in the introduction of the chapter, this type of controller may be more realistic for some applications.

6.3.1 Dynamic output feedback controller

Suppose that the output $y(k)$ of the plant in (6.1) is available for feedback, and suppose that $\theta(k)$ and the next L values, $\psi_L(k)$, of the switching sequence are available to the controller at time k where $L \in \mathbb{N}_0$ is some finite horizon and $\psi_L(k) = (\psi(k+1), \psi(k+2), \ldots, \psi(k+L))$ as defined in Section 3.3. Consider a *finite-future-dependent output feedback controller* of the form

$$\hat{x}(k+1) = \hat{A}(\theta(k), \psi_L(k))\hat{x}(k) + \hat{B}(\theta(k), \psi_L(k))y(k) \tag{6.21a}$$

$$u(k) = \hat{C}(\theta(k), \psi_L(k))\hat{x}(k) + \hat{D}(\theta(k), \psi_L(k))y(k) \tag{6.21b}$$

where $\hat{x}(k) \in \mathbb{R}^n$ is the controller state; $\hat{A} : \mathcal{N} \times \Psi_L \to \mathbb{R}^{n\times n}$, $\hat{B} : \mathcal{N} \times \Psi_L \to \mathbb{R}^{n\times p_2}$, $\hat{C} : \mathcal{N} \times \Psi_L \to \mathbb{R}^{m_2\times n}$, and $\hat{D} : \mathcal{N} \times \Psi_L \to \mathbb{R}^{m_2\times p_2}$ are the controller parameter matrices; and, Ψ_L is the set of sequences of length L that occur in Ψ, as defined in Section 3.3. Synthesis methods for the controller matrices $\{(\hat{A}(i,r), \hat{B}(i,r), \hat{C}(i,r), \hat{D}(i,r)) : i \in \mathcal{N}, r \in \Psi_L\}$ that ensure design objectives of stability and/or disturbance attenuation are investigated in Section 6.3.5.

Define $\bar{x}(k) = [x^{\mathsf{T}}(k)\ \hat{x}^{\mathsf{T}}(k)]^{\mathsf{T}}$. The closed-loop system is expressed

$$\begin{bmatrix} \bar{x}(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} \bar{A}(\theta(k), \psi_L(k)) & \bar{B}(\theta(k), \psi_L(k)) \\ \bar{C}(\theta(k), \psi_L(k)) & \bar{D}(\theta(k), \psi_L(k)) \end{bmatrix} \begin{bmatrix} \bar{x}(k) \\ w(k) \end{bmatrix} \tag{6.22}$$

where

$$\bar{A}(i,r) = \begin{bmatrix} A(i) + B_2(i)\hat{D}(i,r)C_2(i) & B_2(i)\hat{C}(i,r) \\ \hat{B}(i,r)C_2(i) & \hat{A}(i,r) \end{bmatrix}, \qquad \bar{B}(i,r) = \begin{bmatrix} B_1(i) + B_2(i)\hat{D}(i,r)D_{21}(i) \\ \hat{B}(i,r)D_{21}(i) \end{bmatrix},$$
$$\bar{C}(i,r) = \begin{bmatrix} C_1(i) + D_{12}(i)\hat{D}(i,r)C_2(i) & D_{12}(i)\hat{C}(i,r) \end{bmatrix}, \quad \bar{D}(i,r) = D_{11}(i) + D_{12}(i)\hat{D}(i,r)D_{21}(i).$$

for $i \in \mathcal{N}$ and $r \in \Psi_L$. The closed-loop matrices can be rewritten

$$\bar{A}(i,r) = \mathcal{A}(i) + \mathcal{B}_2(i)K(i,r)\mathcal{C}_2(i), \qquad \bar{B}(i,r) = \mathcal{B}_1(i) + \mathcal{B}_2(i)K(i,r)\mathcal{D}_{21}(i), \tag{6.23a}$$

$$\bar{C}(i,r) = \mathcal{C}_1(i) + \mathcal{D}_{12}(i)K(i,r)\mathcal{C}_2(i), \quad \bar{D}(i,r) = D_{11}(i) + \mathcal{D}_{12}(i)K(i,r)\mathcal{D}_{21}(i), \tag{6.23b}$$

where

$$\mathcal{A}(i) = \begin{bmatrix} A(i) & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B}_1(i) = \begin{bmatrix} B_1(i) \\ 0 \end{bmatrix}, \quad \mathcal{B}_2(i) = \begin{bmatrix} 0 & B_2(i) \\ I & 0 \end{bmatrix}, \quad \mathcal{C}_1(i) = \begin{bmatrix} C_1(i) & 0 \end{bmatrix}, \tag{6.24a}$$

$$\mathcal{C}_2(i) = \begin{bmatrix} 0 & I \\ C_2(i) & 0 \end{bmatrix}, \ \mathcal{D}_{12}(i) = \begin{bmatrix} 0 & D_{12}(i) \end{bmatrix}, \ \mathcal{D}_{21}(i) = \begin{bmatrix} 0 \\ D_{21}(i) \end{bmatrix}, \quad K(i,r) = \begin{bmatrix} \hat{A}(i,r) & \hat{B}(i,r) \\ \hat{C}(i,r) & \hat{D}(i,r) \end{bmatrix}. \tag{6.24b}$$

Note that the closed-loop matrices in (6.23) depend affinely on the controller matrices.

6.3.2 Full state feedback controller

If $y(k) = x(k)$, consider a *finite-future-dependent state feedback controller* of the form

$$u(k) = F(\theta(k), \psi_L(k))x(k) \tag{6.25}$$

where $F : \mathcal{N} \times \Psi_L \to \mathbb{R}^{m_2 \times n}$. Synthesis methods for the controller matrices $\{F(i,r) : i \in \mathcal{N}, r \in \Psi_L\}$ that ensure design objectives of stability and/or disturbance attenuation are examined in Section 6.3.5.

The closed-loop system with the plant in (6.1) and the controller in (6.25) is expressed

$$\begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} \bar{F}(\theta(k), \psi_L(k)) & B_1(\theta(k)) \\ \bar{H}(\theta(k), \psi_L(k)) & D_{11}(\theta(k)) \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \tag{6.26}$$

with

$$\bar{F}(i,r) = A(i) + B_2(i)F(i,r), \quad \bar{H}(i,r) = C_1(i) + D_{12}(i)F(i,r) \tag{6.27}$$

where $i \in \mathcal{N}$ and $r \in \Psi_L$.

6.3.3 Lyapunov and KYP criteria for the closed-loop system

Lyapunov and KYP criteria, similar to Theorems 4.12 and 5.29, for the closed-loop switched Markov jump linear systems of Sections 6.3.1 and 6.3.2 are reported in this section. These criteria are used in the control synthesis results of Sections 6.3.4 and 6.3.5.

Output feedback

Let $T \in \mathbb{N}_0$ and $r = (r_1, r_2, \ldots, r_{T+1}) \in \Psi_{T+1}$. For the remainder of this chapter, the notation established in Section 2.2 is used, where $r_{m:n} = (r_m, r_{m+1}, \ldots, r_n)$ for m and n such that $1 \le m \le n \le T+1$.

Lemma 6.8. *The closed-loop switched Markov jump linear system in* (6.22) *with the output feedback controller in* (6.21) *is uniformly exponentially mean square stable if and only if there exist $M \in \mathbb{N}_0$ and a function $X : \mathcal{N} \times \Psi_{M+L} \to \mathbb{S}^+_{2n}$ such that*

$$\bar{A}^\mathsf{T}(i, r_{1:L}) \sum_{j=1}^{N} \pi_{ij}(r_1) X(j, r_{2:M+L+1}) \bar{A}(i, r_{1:L}) - X(i, r_{1:M+L}) < 0 \tag{6.28}$$

for any $i \in \mathcal{N}$ and any $r = (r_1, r_2, \ldots, r_{M+L+1}) \in \Psi_{M+L+1}$.

Proof. For sufficiency of (6.28), define $Y_\psi(i,k) := X(i, \psi_{M+L}(k))$ and $\bar{A}(i,k) := \bar{A}(i, \psi_L(k))$. Since $\mathcal{N} \times \Psi_{M+L+1}$ is a finite set, there exist $\eta, \rho, \nu > 0$ such that (6.13) holds with $\bar{A}$ in place of A. Now appeal to Lemma 6.5.

For necessity of (6.28), define $\bar{A}(i,k) := \bar{A}(i, \psi_L(k))$. Let $(i,r) \in \mathcal{N} \times \Psi_{M+L+1}$ be arbitrary. By definition of Ψ_{M+L+1}, there exist $\psi \in \Psi$ and $t \in \mathbb{N}_0$ such that $\psi_{M+L+1}(t) = r$. Construct Y_ψ for the closed-loop system using Lemma 6.4. By construction, $Y_\psi(i,t)$ depends only on i and $\psi_{M+L}(t)$. Indeed, if $t \neq k$ but $\psi_{M+L}(t) = \psi_{M+L}(k)$ then $Y_\psi(i,t) = Y_\psi(i,k)$. To see why, note that

$$\bar{A}(i, t+\tau) = \bar{A}(i, \psi_L(t+\tau)) = \bar{A}(i, \psi_L(k+\tau) = \bar{A}(i, k+\tau)$$

for $\tau = 0, 1, \ldots, M$ and

$$\begin{aligned}\mathbf{P}\left\{\theta(t+M) = i_M, \ldots, \theta(t+1) = i_1 \mid \theta(t) = i_0\right\} &= \pi_{i_0 i_1}(\psi(t+1))\pi_{i_1 i_2}(\psi(t+2)) \cdots \pi_{i_{M-1} i_M}(\psi(t+M)) \\ &= \pi_{i_0 i_1}(\psi(k+1))\pi_{i_1 i_2}(\psi(k+2)) \cdots \pi_{i_{M-1} i_M}(\psi(k+M)) \\ &= \mathbf{P}\left\{\theta(k+M) = i_M, \ldots, \theta(k+1) = i_1 \mid \theta(k) = i_0\right\}.\end{aligned}$$

Therefore

$$Y_\psi(i,t) = \mathbf{E}\left[\sum_{j=t}^{t+M+1} \Phi^\mathsf{T}(j,t)\Phi(j,t) \mid \theta(t) = i\right] = \mathbf{E}\left[\sum_{j=k}^{k+M+1} \Phi^\mathsf{T}(j,k)\Phi(j,k) \mid \theta(k) = i\right] = Y_\psi(i,k)$$

where Φ is the closed-loop state transition matrix. Thus, define $X(i, r_{1:M+L}) := Y_\psi(i,t)$ and define $X(i, r_{2:M+L+1}) := Y_\psi(i, t+1)$. One recovers every inequality in (6.28) via Lemma 6.4. □

Lemma 6.9. *Assume $p_i(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$. The closed-loop switched Markov jump linear system in* (6.22) *with the output feedback controller in* (6.21) *is uniformly exponentially mean square stable and uniformly mean square strictly contractive if and only if there exist $M \in \mathbb{N}_0$ and a function $X : \mathcal{N} \times \Psi_{M+L} \to \mathbb{S}_{2n}^+$ such that*

$$\begin{bmatrix} \bar{A}(i, r_{1:L}) & \bar{B}(i, r_{1:L}) \\ \bar{C}(i, r_{1:L}) & \bar{D}(i, r_{1:L}) \end{bmatrix}^\mathsf{T} \begin{bmatrix} \sum_{j=1}^N \pi_{ij}(r_1) X(j, r_{2:T+1}) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A}(i, r_{1:L}) & \bar{B}(i, r_{1:L}) \\ \bar{C}(i, r_{1:L}) & \bar{D}(i, r_{1:L}) \end{bmatrix} - \begin{bmatrix} X(i, r_{1:T}) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{6.29}$$

for any $i \in \mathcal{N}$ and any $r = (r_1, r_2, \ldots, r_{T+1}) \in \Psi_{T+1}$ where $T := M + L$.

Proof. For sufficiency of (6.29), define $Y_\psi(i,k) := X(i, \psi_T(k))$, and define $\bar{A}(i,k) := \bar{A}(i, \psi_L(k))$, $\bar{B}(i,k) := \bar{B}(i, \psi_L(k))$, $\bar{C}(i,k) := \bar{C}(i, \psi_L(k))$, and $\bar{D}(i,k) := \bar{D}(i, \psi_L(k))$. Since $\mathcal{N} \times \Psi_{T+1}$ is a finite set, there exist $\eta, \rho, \nu > 0$ such that (6.17) holds with $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ in place of A, B, C, D. Now appeal to Lemma 6.7.

For necessity of (6.29), again define $\bar{A}(i,k) := \bar{A}(i, \psi_L(k))$, $\bar{B}(i,k) := \bar{B}(i, \psi_L(k))$, $\bar{C}(i,k) := \bar{C}(i, \psi_L(k))$, and $\bar{D}(i,k) := \bar{D}(i, \psi_L(k))$. Let $(i,r) \in \mathcal{N} \times \Psi_{T+1}$ be arbitrary. By definition of Ψ_{T+1}, there exist $\psi \in \Psi$ and $t \in \mathbb{N}_0$ such that $\psi_{T+1}(t) = r$. Construct Y_ψ for the closed-loop system using Lemma 6.6. Since the time-variation of the closed-loop matrices is parameterized in terms of ψ, $Y_\psi(i,t)$ may be computed with knowledge of only i and $\psi_T(t)$. Indeed, if $t \neq k$ but $\psi_T(t) = \psi_T(k)$ then $Y_\psi(i,t) = Y_\psi(i,k)$. To establish this claim, note that the base case holds from (6.14b):

$$X_\psi(i, t+M+1, t+M, \epsilon) = X_\psi(i, k+M+1, k+M, \epsilon) = 0. \tag{6.30}$$

Now

$$\begin{aligned} X_\psi(i, t+M, t+M, \epsilon) &= \mathcal{S}(i, t+M, \tilde{X}_\psi(i, t+M+1, t+M, \epsilon)) + \epsilon I &(6.31)\\ &= \mathcal{S}(i, k+M, \tilde{X}_\psi(i, k+M+1, k+M, \epsilon)) + \epsilon I &(6.32)\\ &= X_\psi(i, k+M, k+M, \epsilon) \end{aligned}$$

where (6.31) follows from the recursive relation (6.14a), and (6.32) follows from (6.30) and the fact that

$$\begin{aligned} \begin{bmatrix} \bar{A}(i,t+M) & \bar{B}(i,t+M) \\ \bar{C}(i,t+M) & \bar{D}(i,t+M) \end{bmatrix} &= \begin{bmatrix} \bar{A}(i,\psi_L(t+M)) & \bar{B}(i,\psi_L(t+M)) \\ \bar{C}(i,\psi_L(t+M)) & \bar{D}(i,\psi_L(t+M)) \end{bmatrix} &(6.33)\\ &= \begin{bmatrix} \bar{A}(i,\psi_L(k+M)) & \bar{B}(i,\psi_L(k+M)) \\ \bar{C}(i,\psi_L(k+M)) & \bar{D}(i,\psi_L(k+M)) \end{bmatrix} &(6.34)\\ &= \begin{bmatrix} \bar{A}(i,k+M) & \bar{B}(i,k+M) \\ \bar{C}(i,k+M) & \bar{D}(i,k+M) \end{bmatrix} &(6.35) \end{aligned}$$

where (6.33) and (6.35) follow by definition, and (6.34) follows since $\psi_L(t+M) = \psi_L(k+M)$ by assumption. Induction yields $Y_\psi(i,t) = Y_\psi(i,k)$. Thus, define $X(i, r_{1:T}) := Y_\psi(i,t)$ and define $X(i, r_{2:T+1}) := Y_\psi(i, t+1)$. One recovers every inequality in (6.29) from (6.16). □

State feedback

Lemma 6.10. *The closed-loop switched Markov jump linear system in* (6.26) *with the state feedback controller in* (6.25) *is uniformly exponentially mean square stable if and only if there exist* $M \in \mathbb{N}_0$ *and a function* $X : \mathcal{N} \times \Psi_{M+L} \to \mathbb{S}_n^+$ *such that*

$$\bar{F}^\mathsf{T}(i, r_{1:L}) \sum_{j=1}^{N} \pi_{ij}(r_1) X(j, r_{2:M+L+1}) \bar{F}(i, r_{1:L}) - X(i, r_{1:M+L}) < 0 \tag{6.36}$$

for any $i \in \mathcal{N}$ *and any* $r = (r_1, r_2, \ldots, r_{M+L+1}) \in \Psi_{M+L+1}$.

Proof. The proof mirrors that of Lemma 6.8. □

Lemma 6.11. *Assume $p_i(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$. The closed-loop switched Markov jump linear system in (6.26) with the state feedback controller in (6.25) is uniformly exponentially mean square stable and uniformly mean square strictly contractive if and only if there exist $M \in \mathbb{N}_0$ and a function $X : \mathcal{N} \times \Psi_{M+L} \to \mathbb{S}_n^+$ such that*

$$\begin{bmatrix} \bar{F}(i, r_{1:L}) & B_1(i) \\ \bar{H}(i, r_{1:L}) & D_{11}(i) \end{bmatrix}^\mathsf{T} \begin{bmatrix} \sum_{j=1}^N \pi_{ij}(r_1) X(j, r_{2:T+1}) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{F}(i, r_{1:L}) & B_1(i) \\ \bar{H}(i, r_{1:L}) & D_{11}(i) \end{bmatrix} - \begin{bmatrix} X(i, r_{1:T}) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{6.37}$$

for any $i \in \mathcal{N}$ and any $r = (r_1, r_2, \ldots, r_{T+1}) \in \Psi_{T+1}$ where $T := M + L$.

Proof. The proof mirrors that of Lemma 6.9. □

6.3.4 Stabilizability and detectability

An important question is the existence of a controller of the form (6.25) that ensures the closed-loop system is uniformly exponentially mean square stable. A related question is the existence of an observer that ensures the error between the system state and the observer state decays exponentially in mean square.

Definition 6.12. The open-loop plant in (6.1) is *uniformly mean square stabilizable by a finite-future-dependent state feedback controller* if there exists $L \in \mathbb{N}_0$ and a function $F : \mathcal{N} \times \Psi_L \to \mathbb{R}^{m_2 \times n}$ such that

$$\begin{aligned} x(k+1) &= A(\theta(k))x(k) + B_2(\theta(k))u(k) \\ u(k) &= F(\theta(k), \psi_L(k))x(k) \end{aligned}$$

is uniformly exponentially mean square stable.

Definition 6.13. The open-loop plant in (6.1) is *uniformly mean square detectable by a finite-future-dependent observer* if there exists $L \in \mathbb{N}_0$ and a function $O : \mathcal{N} \times \Psi_L \to \mathbb{R}^{m_2 \times n}$ such that the error

$$e(k+1) = [A(\theta(k)) + O(\theta(k), \psi_L(k))C_2(\theta(k))]\, e(k), \quad e(0) = e_0$$

is uniformly exponentially mean square stable where

$$e(k) := x(k) - \hat{x}(k)$$

and

$$\begin{aligned}
x(k+1) &= A(\theta(k))x(k) + B_2(\theta(k))u(k), \quad x(0) = x_0 \\
y(k) &= C_2(\theta(k))x(k) \\
\hat{x}(k+1) &= A(\theta(k))\hat{x}(k) + B_2(\theta(k))u(k) - O(\theta(k), \psi_L(k))\left(y(k) - \hat{y}(k)\right), \quad \hat{x}(0) = \hat{x}_0 \\
\hat{y}(k) &= C_2(\theta(k))\hat{x}(k).
\end{aligned}$$

In Section 6.3.5, it is shown that a mean square stabilizing output feedback controller of the form (6.21) exists if and only if the open-loop plant in (6.1) is uniformly mean square stabilizable by a finite-future-dependent state feedback controller and uniformly mean square detectable by a finite-future-dependent observer.

LMI conditions equivalent to the concepts of stabilizability and detectability in Definitions 6.12 and 6.13 are now derived.

Lemma 6.14. *The open-loop plant in* (6.1) *is uniformly mean square stabilizable by a finite-future-dependent state feedback controller if and only if there exists* $T \in \mathbb{N}_0$ *and* $U : \mathcal{N} \times \Psi_T \to \mathbb{S}_n^+$ *such that*

$$W^{\mathsf{T}}(i)\left[A(i)U(i, r_{1:T})A^{\mathsf{T}}(i) - \left(\sum_{j=1}^{N} \pi_{ij}(r_1)U^{-1}(j, r_{2:T+1})\right)^{-1}\right]W(i) < 0 \tag{6.38}$$

for all $i \in \mathcal{N}$ *and any* $r = (r_1, r_2, \ldots, r_{T+1}) \in \Psi_{T+1}$*, where* $W(i)$ *is any basis of* $\ker B_2^{\mathsf{T}}(i)$.

Proof. Suppose the open-loop plant in (6.1) is uniformly mean square stabilizable by a finite-future-dependent controller. Let L and F be as in Definition 6.12, and let $\bar{F}$ be the closed-loop system matrix in (6.27). By Lemma 6.10, there exist M and X such that (6.36) holds. Two applications of the Schur complement show that inequality (6.36) is equivalent

$$\left(A(i) + B_2(i)F(i, r_{1:L})\right)U(i, r_{1:T})\left(A(i) + B_2(i)F(i, r_{1:L})\right)^{\mathsf{T}} - \left(\sum_{j=1}^{N} \pi_{ij}(r_1)U^{-1}(j, r_{2:T+1})\right)^{-1} < 0 \tag{6.39}$$

where $T := M + L$ and $U := X^{-1}$. Pre- and post-multiply (6.39) by $W^{\mathsf{T}}(i)$ and $W(i)$, respectively, to arrive at (6.38).

Conversely, suppose there exist T and U such that (6.38) holds. Application of Lemma 2.42 implies that there exists $F : \mathcal{N} \times \Psi_{T+1} \to \mathbb{R}^{m_2 \times n}$ such that

$$(A(i) + B_2(i)F(i,r))^{\mathsf{T}} \sum_{j=1}^{N} \pi_{ij}(r_1) X(j, r_{2:T+1}) \left(A(i) + B_2(i)F(i,r)\right) - X(i, r_{1:T}) < 0 \tag{6.40}$$

holds where $X := U^{-1}$. Define $\bar{F}(i,k) := A(i) + B_2(i)F(i, \psi_{T+1}(k))$ and $Y_\psi(i,k) := X(i, \psi_T(k))$. Inequality (6.40) and Lemma 6.5 ensure that the closed-loop system is uniformly exponentially mean square stable. □

The matrices $W(i)$, $i \in \mathcal{N}$ in Lemma 6.14 may be readily computed using singular value decomposition (see Lemma 2.35). The inequalities in (6.38) are in general *not* LMIs. However, Lemma 6.14 can be stated in an equivalent LMI form using a trick from [22].

Lemma 6.15. *The open-loop plant in* (6.1) *is uniformly mean square stabilizable by a finite-future-dependent state feedback controller if and only if there exists* $T \in \mathbb{N}_0$, $U : \mathcal{N} \times \Psi_T \to \mathbb{S}_n^+$, $G : \mathcal{N} \times \Psi_{T+1} \to \mathbb{R}^{n \times n}$, $Z : \mathcal{N} \times \mathcal{N} \times \Psi_{T+1} \to \mathbb{S}_n^+$ *such that*

$$W^{\mathsf{T}}(i) \left[A(i)U(i, r_{1:T})A^{\mathsf{T}}(i) + \mathcal{Z}(i,r)\right] W(i) < 0 \tag{6.41a}$$

$$\begin{bmatrix} Z(i,j,r) & G^{\mathsf{T}}(i,r) \\ G(i,r) & U(j, r_{2:T+1}) \end{bmatrix} > 0 \tag{6.41b}$$

for all $i, j \in \mathcal{N}$ *and any* $r = (r_1, r_2, \ldots, r_{T+1}) \in \Psi_{T+1}$, *where* $W(i)$ *is any basis of* $\ker B_2^{\mathsf{T}}(i)$, *and*

$$\mathcal{Z}(i,r) := \sum_{j=1}^{N} \pi_{ij}(r_1) Z(i,j,r) - G(i,r) - G^{\mathsf{T}}(i,r).$$

Proof. First, suppose that the open-loop plant in (6.1) is uniformly mean square stabilizable by a finite-future-dependent state feedback controller. Thus, there exist T and U from Lemma 6.14 such that the inequalities in (6.38) hold. The nonlinear inequalities in (6.38) can be converted to LMIs using a trick from [22]. Let $\epsilon > 0$ be small enough so that

$$W^{\mathsf{T}}(i) \left[A(i)U(i, r_{1:T})A^{\mathsf{T}}(i) - \left(\sum_{j=1}^{N} \pi_{ij}(r_1) U^{-1}(j, r_{2:T+1})\right)^{-1} + \epsilon I\right] W(i) < 0 \tag{6.42}$$

for all $i \in \mathcal{N}$ and $r \in \Psi_{T+1}$. Define

$$G(i,r) := \left(\sum_{j=1}^{N} \pi_{ij}(r_1) U^{-1}(j, r_{2:T+1})\right)^{-1}, \quad Z(i,j,r) := G(i,r)U^{-1}(j, r_{2:T+1})G(i,r) + \epsilon I. \tag{6.43}$$

Note that

$$\begin{aligned}\sum_{j=1}^{N} \pi_{ij}(r_1) Z(i,j,r) &= G(i,r) \sum_{j=1}^{N} \pi_{ij}(r_1) U^{-1}(j, r_{2:T+1}) G(i,r) + \epsilon I \\ &= G(i,r) + \epsilon I. \end{aligned} \tag{6.44}$$

Noting that $G(i,r) = G^{\mathsf{T}}(i,r)$ by the definition in (6.43) and using the equality in (6.44)

$$\begin{aligned}\mathcal{Z}(i,r) &= -G(i,r) - G^{\mathsf{T}}(i,r) + \sum_{j=1}^{N} \pi_{ij}(r_1) Z(i,j,r) \\ &= -G(i,r) + \epsilon I \\ &= -\left(\sum_{j=1}^{N} \pi_{ij}(r_1) U^{-1}(j, r_{2:T+1})\right)^{-1} + \epsilon I. \end{aligned} \tag{6.45}$$

Inequality (6.41a) follows from (6.45) and inequality (6.42). To show inequality (6.41b), note by definition of Z and symmetry of G

$$Z(i,j,r) - G^{\mathsf{T}}(i,r) U^{-1}(j, r_{2:T+1}) G(i,r) = \epsilon I > 0. \tag{6.46}$$

Applying the Schur complement to (6.46) yields (6.41b).

Conversely, suppose that there exist T, U, G, and Z such that (6.41) holds. By the Schur complement, inequality (6.41b) implies

$$Z(i,j,r) > G^{\mathsf{T}}(i,r) U^{-1}(j, r_{2:T+1}) G(i,r). \tag{6.47}$$

Thus

$$\sum_{j=1}^{N} \pi_{ij}(r_1) Z(i,j,r) > G^{\mathsf{T}}(i,r) \sum_{j=1}^{N} \pi_{ij}(r_1) U^{-1}(j, r_{2:T+1}) G(i,r) \tag{6.48}$$

$$\geq G^{\mathsf{T}}(i,r) + G(i,r) - \left(\sum_{j=1}^{N} \pi_{ij}(r_1) U^{-1}(j, r_{2:T+1})\right)^{-1} \tag{6.49}$$

where (6.48) follows by summing both sides of (6.47), and (6.49) follows from Lemma 2.43. Therefore

$$-\left(\sum_{j=1}^{N} \pi_{ij}(r_1) U^{-1}(j, r_{2:T+1})\right)^{-1} \leq \mathcal{Z}(i,r) \tag{6.50}$$

so that inequality (6.41a) implies (6.38) via (6.50) and Lemma 2.46. $\square$

In addition to providing a numerical test for stabilizability, Lemma 6.15 implicitly provides a stabilizing state feedback controller. Indeed, suppose the LMIs in Lemma 6.15 have been solved, and T, U, G, and Z are given. As the proof of Lemma 6.15 shows, the inequalities in (6.41) ensure that the inequalities in (6.38) from Lemma 6.14 hold with the same T and U. As the proof of Lemma 6.14 shows, a stabilizing state feedback controller F that satisfies (6.40) is guaranteed to exist by Lemma 2.42. By the Schur complement, inequality (6.40) is equivalent to

$$\begin{bmatrix} -\left(\sum_{j=1}^{N} \pi_{ij}(r_1)X(j, r_{2:T+1})\right)^{-1} & A(i) + B_2(i)F(i, r) \\ (A(i) + B_2(i)F(i, r))^{\mathsf{T}} & -X(i, r_{1:T}) \end{bmatrix} < 0 \tag{6.51}$$

With U (and hence X) fixed, inequality (6.51) is an LMI in variable F and can be solved using standard numerical techniques. A solution to this LMI is guaranteed to exist. Alternatively, the proof of Lemma 2.42 and the formulas given in Lemma 2.41 can be used to algebraically construct a solution F.

Lemma 6.16. *The open-loop plant in* (6.1) *is uniformly mean square detectable by a finite-future-dependent observer if and only if there exists $T \in \mathbb{N}_0$ and $S : \mathcal{N} \times \Psi_T \to \mathbb{S}_n^+$ such that*

$$V^{\mathsf{T}}(i) \left[A^{\mathsf{T}}(i) \sum_{j=1}^{N} \pi_{ij}(r_1) S(j, r_{2:T+1}) A(i) - S(i, r_{1:T}) \right] V(i) < 0 \tag{6.52}$$

for all $i \in \mathcal{N}$ and any $r = (r_1, r_2, \ldots, r_{T+1}) \in \Psi_{T+1}$, where $V(i)$ is any basis of $\ker C_2(i)$.

Proof. Suppose the open-loop plant in (6.1) is uniformly mean square detectable by a finite-future-dependent observer. Let L and O be as in Definition 6.13. By Definition 6.13 and Lemma 6.10, there exist M and S such that

$$[A(i) + O(i, r_{1:L})C_2(i)]^{\mathsf{T}} \sum_{j=1}^{N} \pi_{ij}(r_1) S(j, r_{2:M+L+1}) [A(i) + O(i, r_{1:L})C_2(i)] - S(i, r_{1:M+L}) < 0 \tag{6.53}$$

Pre- and post-multiply (6.53) by $V^{\mathsf{T}}(i)$ and $V(i)$, respectively, to arrive at (6.52) with $T = M + L$.

Conversely, suppose there exist T and S such that (6.52) holds. Lemma 2.42 implies that there exists $O : \mathcal{N} \times \Psi_{T+1} \to \mathbb{R}^{n \times p_2}$ such that

$$[A(i) + O(i, r)C_2(i)]^{\mathsf{T}} \sum_{j=1}^{N} \pi_{ij}(r_1) S(j, r_{2:T+1}) [A(i) + O(i, r)C_2(i)] - S(i, r_{1:T}) < 0 \tag{6.54}$$

where $i \in \mathcal{N}$ and $r \in \Psi_{T+1}$. Define $\bar{O}(i, k) := A(i) + O(i, \psi_{T+1}(k))C_2(i)$ and $Y_\psi(i, k) := S(i, \psi_T(k))$. Inequality (6.54) and Lemma 6.5 ensure that the error $e(k)$ in Definition 6.13 is uniformly exponentially mean square stable. □

For a fixed T, the inequalities in (6.52) of Lemma 6.16 are LMIs in variable S. In addition to providing a numerical test for detectability, Lemma 6.16 implicitly provides an observer that ensures the error in Definition 6.13 decays exponentially in mean square. Indeed, if T and S satisfy the LMIs in (6.52), the proof of Lemma 6.16 ensures that a solution O exists to the inequality in (6.54). For fixed S, the Schur complement can be used to convert (6.54) to an LMI in variable O, which can be solved using standard numerical techniques.

6.3.5 Controller construction via LMIs

Output and state feedback controllers of the forms (6.21) and (6.25), respectively, are now constructed that ensure design objectives of stability and/or disturbance attention are met.

Output feedback

Theorem 6.17 shows that existence of an output feedback controller of the form (6.21) which ensures that the closed-loop system is stable and contractive is equivalent to the feasibility of a number of matrix inequalities. These matrix inequalities are *not* LMIs, but are converted to LMIs in Theorem 6.21. Theorem 6.17 is analogous to the LMI characterization of $\mathcal{H}_\infty$ controllers for deterministic linear time-invariant systems found in Gahinet and Apkarian [20].

Theorem 6.17. *Assume $p_i(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$. There exists a finite-future-dependent output feedback controller as in (6.21) such that the closed-loop system in (6.22) is uniformly exponentially mean square stable and uniformly mean square strictly contractive if and only if there exist $T \in \mathbb{N}_0$ and functions $S, U : \mathcal{N} \times \Psi_T \to \mathbb{S}_n^+$ such that*

$$\begin{bmatrix} W_1(i) & 0 \\ W_2(i) & 0 \\ 0 & I \end{bmatrix}^\mathsf{T} \begin{bmatrix} A(i)U(i,r_{1:T})A^\mathsf{T}(i) - \mathcal{U}(i,r) & A(i)U(i,r_{1:T})C_1^\mathsf{T}(i) & B_1(i) \\ C_1(i)U(i,r_{1:T})A^\mathsf{T}(i) & -I + C_1(i)U(i,r_{1:T})C_1^\mathsf{T}(i) & D_{11}(i) \\ B_1^\mathsf{T}(i) & D_{11}^\mathsf{T}(i) & -I \end{bmatrix} \begin{bmatrix} W_1(i) & 0 \\ W_2(i) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{6.55a}$$

$$\begin{bmatrix} V_1(i) & 0 \\ V_2(i) & 0 \\ 0 & I \end{bmatrix}^\mathsf{T} \begin{bmatrix} A^\mathsf{T}(i)\tilde{S}(i,r)A(i) - S(i,r_{1:T}) & A^\mathsf{T}(i)\tilde{S}(i,r)B_1(i) & C_1^\mathsf{T}(i) \\ B_1^\mathsf{T}(i)\tilde{S}(i,r)A(i) & -I + B_1^\mathsf{T}(i)\tilde{S}(i,r)B_1(i) & D_{11}^\mathsf{T}(i) \\ C_1(i) & D_{11}(i) & -I \end{bmatrix} \begin{bmatrix} V_1(i) & 0 \\ V_2(i) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{6.55b}$$

$$\begin{bmatrix} S(i, r_{1:T}) & I \\ I & U(i, r_{1:T}) \end{bmatrix} > 0 \tag{6.55c}$$

for all $i \in \mathcal{N}$ *and any* $r = (r_1, r_2, \ldots, r_{T+1}) \in \Psi_{T+1}$*, where* $\left[W_1^\mathsf{T}(i)\, W_2^\mathsf{T}(i)\right]^\mathsf{T}$ *is any basis of* $\ker\left[B_2^\mathsf{T}(i)\, D_{12}^\mathsf{T}(i)\right]$*;* $\left[V_1^\mathsf{T}(i)\, V_2^\mathsf{T}(i)\right]^\mathsf{T}$ *is any basis of* $\ker\left[C_2(i)\, D_{21}(i)\right]$*; and,*

$$\tilde{S}(i, r) := \sum_{j=1}^{N} \pi_{ij}(r_1) S(j, r_{2:T+1}), \quad \mathcal{U}(i, r) := \left(\sum_{j=1}^{N} \pi_{ij}(r_1) U^{-1}(j, r_{2:T+1})\right)^{-1}.$$

Proof. Suppose there exists a finite-future-dependent controller such that the closed-loop system is uniformly exponentially mean square stable and uniformly mean square strictly contractive, and let L, $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ be as in (6.21). By Lemma 6.9 there exist M and X such that (6.29) holds. Let $T = M + L$ and extend the controller matrices $\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$ to be defined on $\mathcal{N} \times \Psi_{T+1}$ so that for $r \in \Psi_{T+1}$

$$\begin{bmatrix} \hat{A}(i, r) & \hat{B}(i, r) \\ \hat{C}(i, r) & \hat{D}(i, r) \end{bmatrix} := \begin{bmatrix} \hat{A}(i, r_{1:L}) & \hat{B}(i, r_{1:L}) \\ \hat{C}(i, r_{1:L}) & \hat{D}(i, r_{1:L}) \end{bmatrix} \tag{6.56}$$

for all $r_{L+1:T+1} \in \Psi_{M+1}$. The closed-loop system with the controller defined on $\mathcal{N} \times \Psi_{T+1}$ in (6.56) thus satisfies

$$\begin{bmatrix} \bar{A}(i, r) & \bar{B}(i, r) \\ \bar{C}(i, r) & \bar{D}(i, r) \end{bmatrix}^\mathsf{T} \begin{bmatrix} \sum_{j=1}^{N} \pi_{ij}(r_1) X(j, r_{2:T+1}) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A}(i, r) & \bar{B}(i, r) \\ \bar{C}(i, r) & \bar{D}(i, r) \end{bmatrix} - \begin{bmatrix} X(i, r_{1:T}) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{6.57}$$

for any $r \in \Psi_{T+1}$ and $i \in \mathcal{N}$. By Lemma 2.44, inequality (6.57) is equivalent to

$$\begin{bmatrix} -\left(\sum_{j=1}^{N} \pi_{ij}(r_1) X(j, r_{2:T+1})\right)^{-1} & \bar{A}(i, r) & \bar{B}(i, r) & 0 \\ \bar{A}^\mathsf{T}(i, r) & -X(i, r_{1:T}) & 0 & \bar{C}^\mathsf{T}(i, r) \\ \bar{B}^\mathsf{T}(i, r) & 0 & -I & \bar{D}^\mathsf{T}(i, r) \\ 0 & \bar{C}(i, r) & \bar{D}(i, r) & -I \end{bmatrix} < 0 \tag{6.58}$$

Using the decompositions in (6.23), inequality (6.58) can be expressed

$$\mathcal{H}(i, r) + \mathcal{Q}^\mathsf{T}(i) K^\mathsf{T}(i, r) \mathcal{P}(i) + \mathcal{P}^\mathsf{T}(i) K(i, r) \mathcal{Q}(i) < 0 \tag{6.59}$$

where $r \in \Psi_{T+1}$, $i \in \mathcal{N}$, and

$$\mathcal{H}(i,r) = \begin{bmatrix} -\left(\sum_{j=1}^{N} \pi_{ij}(r_1) X(j, r_{2:T+1})\right)^{-1} & \mathcal{A}(i) & \mathcal{B}_1(i) & 0 \\ \mathcal{A}^\mathsf{T}(i) & -X(i, r_{1:T}) & 0 & \mathcal{C}_1^\mathsf{T}(i) \\ \mathcal{B}_1^\mathsf{T}(i) & 0 & -I & D_{11}^\mathsf{T}(i) \\ 0 & \mathcal{C}_1(i) & D_{11}(i) & -I \end{bmatrix}$$

$$\mathcal{P}(i) = \begin{bmatrix} \mathcal{B}_2^\mathsf{T}(i) & 0 & 0 & \mathcal{D}_{12}^\mathsf{T}(i) \end{bmatrix}$$

$$\mathcal{Q}(i) = \begin{bmatrix} 0 & \mathcal{C}_2(i) & \mathcal{D}_{21}(i) & 0 \end{bmatrix}$$

For a fixed $(i,r) \in \mathcal{N} \times \Psi_{T+1}$, Lemma 2.41 shows that (6.59) is feasible in $K(i,r)$ if and only if

$$W_{\mathcal{P}(i)}^\mathsf{T} \mathcal{H}(i,r) W_{\mathcal{P}(i)} < 0 \tag{6.60a}$$

$$W_{\mathcal{Q}(i)}^\mathsf{T} \mathcal{H}(i,r) W_{\mathcal{Q}(i)} < 0 \tag{6.60b}$$

where $W_{\mathcal{P}(i)}$ and $W_{\mathcal{Q}(i)}$ are any matrices whose columns form bases of $\ker \mathcal{P}(i)$ and $\ker \mathcal{Q}(i)$, respectively. Proceeding as in Section 5 of [20], inequalities (6.60a) and (6.60b) are respectively equivalent to

$$\begin{bmatrix} W_1(i) & 0 \\ W_2(i) & 0 \\ 0 & I \end{bmatrix}^\mathsf{T} \begin{bmatrix} A(i)U(i,r_{1:T})A^\mathsf{T}(i) - R(i,r) & A(i)U(i,r_{1:T})C_1^\mathsf{T}(i) & B_1(i) \\ C_1(i)U(i,r_{1:T})A^\mathsf{T}(i) & -I + C_1(i)U(i,r_{1:T})C_1^\mathsf{T}(i) & D_{11}(i) \\ B_1^\mathsf{T}(i) & D_{11}^\mathsf{T}(i) & -I \end{bmatrix} \begin{bmatrix} W_1(i) & 0 \\ W_2(i) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{6.61a}$$

$$\begin{bmatrix} V_1(i) & 0 \\ V_2(i) & 0 \\ 0 & I \end{bmatrix}^\mathsf{T} \begin{bmatrix} A^\mathsf{T}(i)\tilde{S}(i,r)A(i) - S(i,r_{1:T}) & A^\mathsf{T}(i)\tilde{S}(i,r)B_1(i) & C_1^\mathsf{T}(i) \\ B_1^\mathsf{T}(i)\tilde{S}(i,r)A(i) & -I + B_1^\mathsf{T}(i)\tilde{S}(i,r)B_1(i) & D_{11}^\mathsf{T}(i) \\ C_1(i) & D_{11}(i) & -I \end{bmatrix} \begin{bmatrix} V_1(i) & 0 \\ V_2(i) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{6.61b}$$

where $\left[W_1^\mathsf{T}(i)\, W_2^\mathsf{T}(i)\right]^\mathsf{T}$ is any basis of $\ker \left[B_2^\mathsf{T}(i)\, D_{12}^\mathsf{T}(i)\right]$; $\left[V_1^\mathsf{T}(i)\, V_2^\mathsf{T}(i)\right]^\mathsf{T}$ is any basis of $\ker [C_2(i)\, D_{21}(i)]$; $\tilde{S}(i,r) = \sum_{j=1}^{N} \pi_{ij}(r_1) S(j, r_{2:T+1})$; and, R, U, and S are defined by the partitions

$$X(i,r_{1:T}) = \begin{bmatrix} S(i,r_{1:T}) & N(i,r_{1:T}) \\ N^\mathsf{T}(i,r_{1:T}) & F(i,r_{1:T}) \end{bmatrix} \quad X^{-1}(i,r_{1:T}) = \begin{bmatrix} U(i,r_{1:T}) & L(i,r_{1:T}) \\ L^\mathsf{T}(i,r_{1:T}) & * \end{bmatrix}$$

$$\left(\sum_{j=1}^{N} \pi_{ij}(r_1) X(j, r_{2:T+1})\right)^{-1} = \begin{bmatrix} R(i,r) & M(i,r) \\ M^\mathsf{T}(i,r) & * \end{bmatrix};$$

where the entries denoted by $*$ are of no importance. Now note that

$$
\begin{aligned}
R^{-1}(i,r) &= \sum_{j=1}^{N} \pi_{ij}(r_1) S(j, r_{2:T+1}) \\
&\quad - \sum_{j=1}^{N} \pi_{ij}(r_1) N(j, r_{2:T+1}) \left(\sum_{j=1}^{N} \pi_{ij}(r_1) F(j, r_{2:T+1}) \right)^{-1} \sum_{j=1}^{N} \pi_{ij}(r_1) N^{\mathsf{T}}(j, r_{2:T+1}) \qquad (6.62) \\
&\geq \sum_{j=1}^{N} \pi_{ij}(r_1) \left(S(j, r_{2:T+1}) - N(j, r_{2:T+1}) F^{-1}(j, r_{2:T+1}) N^{\mathsf{T}}(j, r_{2:T+1}) \right) \qquad (6.63) \\
&= \sum_{j=1}^{N} \pi_{ij}(r_1) U^{-1}(j, r_{2:T+1}) \qquad (6.64)
\end{aligned}
$$

where (6.62) follow from Lemma 2.32, inequality (6.63) follows from Lemma 2.40, and (6.64) follows from Lemma 2.32. By Lemma 2.46, $R(i,r)$ in (6.61a) may be replaced with $\left(\sum_{j=1}^{N} \pi_{ij}(r_1) U^{-1}(j, r_{2:T+1}) \right)^{-1}$, which yields (6.55a). Note also by Lemma 2.32

$$
U^{-1}(i, r_{1:T}) = S(i, r_{1:T}) - N(i, r_{1:T}) F^{-1}(i, r_{1:T}) N^{\mathsf{T}}(i, r_{1:T}) \qquad (6.65)
$$

Without loss of generality, $N(i, r_{1:T})$ may be assumed invertible. If N is not invertible, perturb X so that N is invertible and (6.57) still holds. Thus, by (6.65) and Lemma 2.38, $U^{-1}(i, r_{1:T}) < S(i, r_{1:T})$. The Schur complement yields (6.55c).

Conversely, suppose there exist $T \in \mathbb{N}_0$ and functions S, U that satisfy (6.55). Define

$$
X(i,r) = \begin{bmatrix} S(i,r) & U^{-1}(i,r) - S(i,r) \\ U^{-1}(i,r) - S(i,r) & S(i,r) - U^{-1}(i,r) \end{bmatrix} \qquad (6.66)
$$

where $i \in \mathcal{N}$ and $r \in \Psi_T$. Inequality (6.55c) implies $S(i,r) - U^{-1}(i,r) > 0$; coupled with the Schur complement, this ensures $X(i,r) > 0$. By Lemma 2.32,

$$
X^{-1}(i,r) = \begin{bmatrix} U(i,r) & U(i,r) \\ U(i,r) & U(i,r) \left(U(i,r) - S^{-1}(i,r) \right)^{-1} U(i,r) \end{bmatrix}
$$

$$
\left(\sum_{j=1}^{N} \pi_{ij}(r_1) X(j, r_{2:T+1}) \right)^{-1} = \begin{bmatrix} \mathcal{U}(i,r) & \mathcal{U}(i,r) \\ \mathcal{U}(i,r) & \mathcal{U}(i,r) \left(\mathcal{U}(i,r) - \left(\tilde{S}(i,r) \right)^{-1} \right)^{-1} \mathcal{U}(i,r) \end{bmatrix}
$$

Thus, for the choice of X in (6.66), inequalities (6.61) hold with $R(i,r) = \mathcal{U}(i,r)$. Starting with the inequalities in (6.61), run the previous arguments in reverse to show that X in (6.66) satisfies (6.57) for some $K(i,r)$, $i \in \mathcal{N}$, $r \in \Psi_{T+1}$ which can be found via Lemma 2.41. With (6.57) in hand, define $\bar{A}(i,k) := \bar{A}(i,\psi_{T+1}(k))$, $\bar{B}(i,k) := \bar{B}(i,\psi_{T+1}(k))$, $\bar{C}(i,k) := \bar{C}(i,\psi_{T+1}(k))$, and $\bar{D}(i,k) := \bar{D}(i,\psi_{T+1}(k))$. Define $Y_\psi(i,k) := X(i,\psi_T(k))$. With these definitions, inequality (6.57) and Lemma 6.7 ensure that the closed-loop system is uniformly exponentially mean square stable and uniformly mean square strictly contractive. □

Remark 6.18. Consider the case when $\mathcal{N} = \{1\}$, $J = 1$, $\Pi(1) = 1$, and $\Psi = \{(1,1,\dots)\}$ so that (6.1) reduces to a deterministic linear time-invariant system. Let $U := U(1,1,\dots,1)$ and $S := S(1,1,\dots,1)$ in Theorem 6.17, and note that $U = \mathcal{U}$ and $S = \tilde{S}$. Then (6.55a)–(6.55c) reduce to equations (50)-(52) from Gahinet and Apkarian [20], which is the well-known LMI solution for deterministic linear time-invariant $\mathcal{H}_\infty$ control.

Existence of an output feedback controller of the form (6.21) that merely ensures closed-loop stability is equivalent to the simpler matrix inequalities stated in Corollary 6.19.

Corollary 6.19. *There exists a finite-future-dependent output feedback controller as in* (6.21) *such that the closed-loop system in* (6.22) *is uniformly exponentially mean square stable if and only if there exist* $T \in \mathbb{N}_0$ *and functions* $S, U : \mathcal{N} \times \Psi_T \to \mathbb{S}_n^+$ *such that*

$$W^\mathsf{T}(i)\left[A(i)U(i,r_{1:T})A^\mathsf{T}(i) - \left(\sum_{j=1}^N \pi_{ij}(r_1)U^{-1}(j,r_{2:T+1})\right)^{-1}\right]W(i) < 0 \tag{6.67a}$$

$$V^\mathsf{T}(i)\left[A^\mathsf{T}(i)\sum_{j=1}^N \pi_{ij}(r_1)S(j,r_{2:T+1})A(i) - S(i,r_{1:T})\right]V(i) < 0 \tag{6.67b}$$

$$\begin{bmatrix} S(i,r_{1:T}) & I \\ I & U(i,r_{1:T}) \end{bmatrix} > 0 \tag{6.67c}$$

for all $i \in \mathcal{N}$ *and any* $r = (r_1, r_2, \dots, r_{T+1}) \in \Psi_{T+1}$*, where* $W(i)$ *is any basis of* $\ker B_2^\mathsf{T}(i)$*, and* $V(i)$ *is any basis of* $\ker C_2(i)$*.*

Proof. Apply Theorem 6.17 to the plant in (6.1) with $B_1(i) = 0$, $C_1(i) = 0$, $D_{11}(i) = 0$, $D_{12}(i) = 0$, $D_{21}(i) = 0$, $D_{22}(i) = 0$. □

In fact, the third matrix inequality stated in Corollary 6.19 is unnecessary.

Corollary 6.20. *There exists a finite-future-dependent output feedback controller as in* (6.21) *such that the closed-loop system in* (6.22) *is uniformly exponentially mean square stable if and only if the open-loop plant in* (6.1) *is uniformly mean square stabilizable by a finite-future-dependent state feedback controller and uniformly mean square detectable by a finite-future-dependent observer.*

Proof. From Corollary 6.19 and Lemmas 6.14 and 6.16, stabilizability and detectability is clearly necessary. Conversely, if S and U satisfy (6.67a) and (6.67b), S may be replaced with αS and U with αU for any positive α without affecting (6.67a) or (6.67b). For α large enough, inequality (6.67c) holds with S replaced by αS and U replaced by αU. □

Note that the inequalities in (6.55) from Theorem 6.17 are in general *not* LMIs due to the presence of $\mathcal{U}(i,r)$. However, they can be converted to LMI form.

Theorem 6.21. *Assume $p_i(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$. There exists a finite-future-dependent output feedback controller as in (6.21) such that the closed-loop system in (6.22) is uniformly exponentially mean square stable and uniformly mean square strictly contractive if and only if there exists $T \in \mathbb{N}_0$ and functions $S, U : \mathcal{N} \times \Psi_T \to \mathbb{S}_n^+$, $G : \mathcal{N} \times \Psi_{T+1} \to \mathbb{R}^{n\times n}$, $Z : \mathcal{N} \times \mathcal{N} \times \Psi_{T+1} \to \mathbb{S}_n^+$ such that*

$$\begin{bmatrix} W_1(i) & 0 \\ W_2(i) & 0 \\ 0 & I \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} A(i)U(i,r_{1:T})A^{\mathsf{T}}(i) + \mathcal{Z}(i,r) & A(i)U(i,r_{1:T})C_1^{\mathsf{T}}(i) & B_1(i) \\ C_1(i)U(i,r_{1:T})A^{\mathsf{T}}(i) & -I + C_1(i)U(i,r_{1:T})C_1^{\mathsf{T}}(i) & D_{11}(i) \\ B_1^{\mathsf{T}}(i) & D_{11}^{\mathsf{T}}(i) & -I \end{bmatrix} \begin{bmatrix} W_1(i) & 0 \\ W_2(i) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{6.68a}$$

$$\begin{bmatrix} V_1(i) & 0 \\ V_2(i) & 0 \\ 0 & I \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} A^{\mathsf{T}}(i)\tilde{S}(i,r)A(i) - S(i,r_{1:T}) & A^{\mathsf{T}}(i)\tilde{S}(i,r)B_1(i) & C_1^{\mathsf{T}}(i) \\ B_1^{\mathsf{T}}(i)\tilde{S}(i,r)A(i) & -I + B_1^{\mathsf{T}}(i)\tilde{S}(i,r)B_1(i) & D_{11}^{\mathsf{T}}(i) \\ C_1(i) & D_{11}(i) & -I \end{bmatrix} \begin{bmatrix} V_1(i) & 0 \\ V_2(i) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{6.68b}$$

$$\begin{bmatrix} S(i,r_{1:T}) & I \\ I & U(i,r_{1:T}) \end{bmatrix} > 0 \tag{6.68c}$$

$$\begin{bmatrix} Z(i,j,r) & G^{\mathsf{T}}(i,r) \\ G(i,r) & U(j,r_{2:T+1}) \end{bmatrix} > 0 \tag{6.68d}$$

for all $i, j \in \mathcal{N}$ and $r = (r_1, r_2, \ldots, r_{T+1}) \in \Psi_{T+1}$, where $\left[W_1^{\mathsf{T}}(i)\, W_2^{\mathsf{T}}(i)\right]^{\mathsf{T}}$ is any basis of $\ker\left[B_2^{\mathsf{T}}(i)\, D_{12}^{\mathsf{T}}(i)\right]$; $\left[V_1^{\mathsf{T}}(i)\, V_2^{\mathsf{T}}(i)\right]^{\mathsf{T}}$ is any basis of $\ker\left[C_2(i)\, D_{21}(i)\right]$; and,

$$\tilde{S}(i,r) := \sum_{j=1}^{N} \pi_{ij}(r_1) S(j, r_{2:T+1}), \quad \mathcal{Z}(i,r) := \sum_{j=1}^{N} \pi_{ij}(r_1) Z(i,j,r) - G(i,r) - G^{\mathsf{T}}(i,r).$$

Proof. Inequality (6.55a) can be shown to be equivalent to inequalities (6.68a) and (6.68d) via the same technique used in the proof of Lemma 6.15. □

Suppose the LMIs in Theorem 6.21 have been solved and T, S, U, G, Z are given. As the proof of Theorem 6.21 shows (via Lemma 6.15), the inequalities in (6.68) ensure that the inequalities in (6.55) hold with the same T, S, and U. As the proof of Theorem 6.17 shows, one may construct X from S and U using (6.66) so that X satisfies (6.60). For each $(i, r) \in \mathcal{N} \times \Psi_{T+1}$, the formulas in Lemma 2.41 can be used to find some $K(i, r)$ that satisfies (6.59), which is equivalent to (6.57). Alternatively, K can be constructed by solving a second LMI problem. Namely, plug the solution X constructed in terms of S and U into the inequality (6.59) and solve for K numerically; with X fixed, inequality (6.59) is clearly an LMI in variable K and may be solved via standard numerical techniques. A solution K is guaranteed to exist by Lemma 2.41.

State feedback

Note that the stabilization problem with state feedback was considered in Section 6.3.4. See Lemmas 6.14 and 6.15. Theorems 6.22 and 6.23 below consider synthesis of state feedback controllers that, in addition to stability, ensure some bound on closed-loop disturbance attenuation. The matrix inequalities in Theorem 6.22 are *not* LMIs, but are converted to LMI form in Theorem 6.23.

Theorem 6.22. *Assume $p_i(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$. There exists a finite-future-dependent state feedback controller as in* (6.25) *such that the closed-loop system in* (6.26) *is uniformly exponentially mean square stable and uniformly mean square strictly contractive if and only if there exist $T \in \mathbb{N}_0$ and function $U : \mathcal{N} \times \Psi_T \to \mathbb{S}_n^+$ such that*

$$\begin{bmatrix} W_1(i) & 0 \\ W_2(i) & 0 \\ 0 & I \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} A(i)U(i,r_{1:T})A^{\mathsf{T}}(i) - \mathcal{U}(i,r) & A(i)U(i,r_{1:T})C_1^{\mathsf{T}}(i) & B_1(i) \\ C_1(i)U(i,r_{1:T})A^{\mathsf{T}}(i) & -I + C_1(i)U(i,r_{1:T})C_1^{\mathsf{T}}(i) & D_{11}(i) \\ B_1^{\mathsf{T}}(i) & D_{11}^{\mathsf{T}}(i) & -I \end{bmatrix} \begin{bmatrix} W_1(i) & 0 \\ W_2(i) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{6.69a}$$

$$\begin{bmatrix} -\mathcal{U}(i,r) & B_1(i) \\ B_1^{\mathsf{T}}(i) & -I + D_{11}^{\mathsf{T}}(i)D_{11}(i) \end{bmatrix} < 0 \tag{6.69b}$$

for all $i \in \mathcal{N}$ and $r = (r_1, r_2, \ldots, r_{T+1}) \in \Psi_{T+1}$, where $\left[W_1^{\mathsf{T}}(i)\, W_2^{\mathsf{T}}(i)\right]^{\mathsf{T}}$ is any basis of $\ker\left[B_2^{\mathsf{T}}(i)\, D_{12}^{\mathsf{T}}(i)\right]$ *and*

$$\mathcal{U}(i,r) := \left(\sum_{j=1}^{N} \pi_{ij}(r_1) U^{-1}(j, r_{2:T+1})\right)^{-1}.$$

Proof. Suppose there exists a finite-future-dependent state feedback controller such that the closed-loop system in uniformly exponentially mean square stable and uniformly mean square strictly contractive, and let L and F be as in (6.25). By Lemma 6.11 there exist M and X such that (6.37) holds. Multiply out the expression in (6.37) and note that the lower-right entry of the resulting partitioned matrix must be negative definite, i.e.,

$$-I + B_1^{\mathsf{T}}(i) \sum_{j=1}^{N} \pi_{ij}(r_1) X(j, r_{2:T+1}) B_1(i) + D_{11}^{\mathsf{T}}(i) D_{11}(i) < 0. \tag{6.70}$$

Application of the Schur complement to (6.70) yields

$$\begin{bmatrix} -\left(\sum_{j=1}^{N} \pi_{ij}(r_1) X(j, r_{2:T+1})\right)^{-1} & B_1(i) \\ B_1^{\mathsf{T}}(i) & -I + D_{11}^{\mathsf{T}}(i) D_{11}(i) \end{bmatrix} < 0$$

which is equivalent to (6.69b) with $U := X^{-1}$. Turning our attention to establishing (6.69a), note that two applications of the Schur complement to (6.37) yield

$$\begin{bmatrix} \bar{F}(i, r_{1:L}) & B_1(i) \\ \bar{H}(i, r_{1:L}) & D_{11}(i) \end{bmatrix} \begin{bmatrix} X^{-1}(i, r_{1:T}) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{F}(i, r_{1:L}) & B_1(i) \\ \bar{H}(i, r_{1:L}) & D_{11}(i) \end{bmatrix}^{\mathsf{T}} - \begin{bmatrix} \left(\sum_{j=1}^{N} \pi_{ij}(r_1) X(j, r_{2:T+1})\right)^{-1} & 0 \\ 0 & I \end{bmatrix} < 0 \tag{6.71}$$

Pre- and post-multiply inequality (6.71) by $\left[W_1^{\mathsf{T}}(i)\, W_2^{\mathsf{T}}(i)\right]$ and $\left[W_1^{\mathsf{T}}(i)\, W_2^{\mathsf{T}}(i)\right]^{\mathsf{T}}$, respectively, to arrive at

$$\begin{bmatrix} W_1(i) \\ W_2(i) \end{bmatrix}^{\mathsf{T}} \left(\begin{bmatrix} A(i) & B_1(i) \\ C_1(i) & D_{11}(i) \end{bmatrix} \begin{bmatrix} U(i, r_{1:T}) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(i) & B_1(i) \\ C_1(i) & D_{11}(i) \end{bmatrix}^{\mathsf{T}} - \begin{bmatrix} \mathcal{U}(i, r) & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} W_1(i) \\ W_2(i) \end{bmatrix} < 0 \tag{6.72}$$

where $U = X^{-1}$. Now Lemma 2.45 shows that (6.72) is equivalent to (6.69a).

Conversely, suppose that T and U have been found such that (6.69) holds. Define $X := U^{-1}$. Note that

$$\begin{aligned} &\begin{bmatrix} 0 \\ I \end{bmatrix}^{\mathsf{T}} \left(\begin{bmatrix} A(i) & B_1(i) \\ C_1(i) & D_{11}(i) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \sum_{j=1}^{N} \pi_{ij}(r_1) X(j, r_{2:T+1}) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(i) & B_1(i) \\ C_1(i) & D_{11}(i) \end{bmatrix} - \begin{bmatrix} X(i, r_{1:T}) & 0 \\ 0 & I \end{bmatrix} \right) \begin{bmatrix} 0 \\ I \end{bmatrix} \\ &= -I + B_1^{\mathsf{T}}(i) \sum_{j=1}^{N} \pi_{ij}(r_1) X(j, r_{2:T+1}) B_1(i) + D_{11}^{\mathsf{T}}(i) D_{11}(i) \\ &< 0 \end{aligned} \tag{6.73}$$

where inequality (6.73) follows from the Schur complement and (6.69b). Inequalities (6.72) and (6.73), combined with Lemma 2.42, show that for each $(i,r) \in \mathcal{N} \times \Psi_{T+1}$, there exists $F(i,r)$ such that

$$(\mathcal{M}(i) + \mathcal{B}(i)F(i,r)\mathcal{C})^{\mathsf{T}} \mathcal{R}(i,r) (\mathcal{M}(i) + \mathcal{B}(i)F(i,r)\mathcal{C}) < \mathcal{Q}(i, r_{1:T}) \tag{6.74}$$

where

$$\mathcal{M}(i) = \begin{bmatrix} A(i) & B_1(i) \\ C_1(i) & D_{11}(i) \end{bmatrix}, \quad \mathcal{B}(i) = \begin{bmatrix} B_2(i) \\ D_{12}(i) \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} I & 0 \end{bmatrix},$$

$$\mathcal{R}(i,r) = \begin{bmatrix} \sum_{j=1}^{N} \pi_{ij}(r_1) X(j, r_{2:T+1}) & 0 \\ 0 & I \end{bmatrix}, \quad \mathcal{Q}(i, r_{1:T}) = \begin{bmatrix} X(i, r_{1:T}) & 0 \\ 0 & I \end{bmatrix}.$$

Inequality (6.74) is equivalent to

$$\begin{bmatrix} \bar{F}(i,r) & B_1(i) \\ \bar{H}(i,r) & D_{11}(i) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} \sum_{j=1}^{N} \pi_{ij}(r_1) X(j, r_{2:T+1}) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{F}(i, r_{1:L}) & B_1(i) \\ \bar{H}(i, r_{1:L}) & D_{11}(i) \end{bmatrix} - \begin{bmatrix} X(i, r_{1:T}) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{6.75}$$

Define $\bar{F}(i,k) := \bar{F}(i, \psi_{T+1}(k))$, $\bar{H}(i,k) := \bar{H}(i, \psi_{T+1}(k))$, and $Y_\psi(i,k) := X(i, \psi_T(k))$. Inequality (6.75), finiteness of the set $\mathcal{N} \times \Psi_{T+1}$, and Lemma 6.7 ensure that the plant in (6.1) with the state feedback controller $u(k) = F(\theta(k), \psi_{T+1}(k))x(k)$ is uniformly exponentially mean square stable and uniformly mean square strictly contractive. □

Theorem 6.23. *Assume $p_i(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$. There exists a finite-future-dependent state feedback controller as in* (6.25) *such that the closed-loop system in* (6.26) *is uniformly exponentially mean square stable and uniformly mean square strictly contractive if and only if there exist $T \in \mathbb{N}_0$ and functions $U : \mathcal{N} \times \Psi_T \to \mathbb{S}_n^+$, $G : \mathcal{N} \times \Psi_{T+1} \to \mathbb{R}^{n \times n}$, $Z : \mathcal{N} \times \mathcal{N} \times \Psi_{T+1} \to \mathbb{S}_n^+$ such that*

$$\begin{bmatrix} W_1(i) & 0 \\ W_2(i) & 0 \\ 0 & I \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} A(i)U(i,r_{1:T})A^{\mathsf{T}}(i) + \mathcal{Z}(i,r) & A(i)U(i,r_{1:T})C_1^{\mathsf{T}}(i) & B_1(i) \\ C_1(i)U(i,r_{1:T})A^{\mathsf{T}}(i) & -I + C_1(i)U(i,r_{1:T})C_1^{\mathsf{T}}(i) & D_{11}(i) \\ B_1^{\mathsf{T}}(i) & D_{11}^{\mathsf{T}}(i) & -I \end{bmatrix} \begin{bmatrix} W_1(i) & 0 \\ W_2(i) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{6.76a}$$

$$\begin{bmatrix} \mathcal{Z}(i,r) & B_1(i) \\ B_1^{\mathsf{T}}(i) & -I + D_{11}^{\mathsf{T}}(i)D_{11}(i) \end{bmatrix} < 0 \tag{6.76b}$$

$$\begin{bmatrix} Z(i,j,r) & G^{\mathsf{T}}(i,r) \\ G(i,r) & U(j,r_{2:T+1}) \end{bmatrix} > 0 \tag{6.76c}$$

for all $i, j \in \mathcal{N}$ *and* $r = (r_1, r_2, \ldots, r_{T+1}) \in \Psi_{T+1}$*, where* $\left[W_1^{\mathsf{T}}(i)\, W_2^{\mathsf{T}}(i)\right]^{\mathsf{T}}$ *is any basis of* $\ker\left[B_2^{\mathsf{T}}(i)\, D_{12}^{\mathsf{T}}(i)\right]$ *and*

$$\mathcal{Z}(i,r) := \sum_{j=1}^{N} \pi_{ij}(r_1) Z(i,j,r) - G(i,r) - G^{\mathsf{T}}(i,r).$$

Proof. The result follows from Theorem 6.22 and the technique used in the proof of Lemma 6.15. □

6.3.6 Control synthesis algorithm

Theorems 6.21 and 6.23 can be used to construct output feedback and state feedback controllers, respectively, that approximately minimize the worst-case gain from $\|w\|_{2,e}$ to $\|z\|_{2,e}$ of the closed-loop system. For concreteness, an algorithm to design an output feedback controller of the form (6.21) that approximately minimizes the worst-case gain from $\|w\|_{2,e}$ to $\|z\|_{2,e}$ of the closed-loop system in (6.22) is examined. Construction of a state feedback controller of the form (6.25) that approximately minimizes the worst-case gain from $\|w\|_{2,e}$ to $\|z\|_{2,e}$ of the closed-loop system in (6.26) follows analogously.

Suppose that the plant in (6.1) is uniformly mean square stabilizable by a finite-future-dependent state feedback controller and uniformly mean square detectable by a finite-future-dependent observer. Let γ be the minimum worst-case gain from $\|w\|_{2,e}$ to $\|z\|_{2,e}$ of the closed-loop system in (6.22) that is achievable by an output feedback controller of the form (6.21). Consider the following algorithm:

1. Set $\gamma_{-1} = -\epsilon$.

2. Set $T = 0$.

3. Use a bisection search to approximate the smallest $\gamma_T > 0$ such that the LMIs in (6.68) from Theorem 6.21 with C_1 replaced by $(1/\gamma_T)C_1$, D_{11} replaced by $(1/\gamma_T)D_{11}$, D_{12} replaced by $(1/\gamma_T)D_{12}$, and the current value of T are feasible.

4. If $(\gamma_T - \gamma_{T-1}) < \epsilon$ then stop. Otherwise, set $T = T + 1$ and go to step 3.

Here, ϵ is some small threshold that prompts the algorithm to exit when the sequence γ_T stops changing by a significant amount. Note that scaling the matrices C_1, D_{11} and D_{12} converts the general bound problem, $\|z\|_{2,e} < \gamma_T \|w\|_{2,e}$, to a contractiveness problem, $\|(1/\gamma_T)z\|_{2,e} < \|w\|_{2,e}$ (see Remark 5.2). Also note that

γ_T is monotonically nonincreasing and $\gamma_T \to \gamma$ as $T \to \infty$. To see why γ_T is monotonically nonincreasing, note that any controller of the form (6.21) that is dependent on the next T values of the sequence ψ can be extended as in (6.56) to be dependent on the next $T+1$ values of the sequence ψ. The closed-loop system with this extended controller will have the same ℓ_e^2-induced norm as the closed-loop system with the original controller.

When the algorithm terminates with some T, S, U, G, and Z, a controller $K(i,r)$, $(i,r) \in \mathcal{N} \times \Psi_{T+1}$ can be constructed by either solving a second LMI problem or by using the algebraic formulas in Lemma 2.41 (see the discussion immediately following Theorem 6.21).

6.4 Control Synthesis with Finite Knowledge of the Past

The aim of Section 6.4 is to construct a controller that at time k has access to the output $y(k)$ of the plant in (6.1), the current mode $\theta(k)$ of the Markov chain, and the current value $\psi(k)$ of the switching sequence. The controller is also assumed to have a finite memory of past observed values $\psi(k-1), \ldots, \psi(k-L)$ of the switching sequence.

Unfortunately, nonconservative synthesis for a controller with measurements $y(k)$, $\theta(k)$, and $\psi(k)$ is only possible in two special cases. If either of the following two assumptions hold, then controllers can be synthesized that depend only on the present and past and satisfy the design objectives of stability and a certain level of disturbance attenuation.

Assumption 6.24 (Common Invariant Distribution)**.** There exists a distribution $\mu = [\mu_1 \ \mu_2 \ \ldots \ \mu_N]$, $\mu_i \geq 0$, $\sum_{i=1}^N \mu_i = 1$ that is invariant for the collection of matrices $\{\Pi(1), \ldots, \Pi(J)\}$:

$$\mu = \mu\Pi(s)$$

for all $s \in \mathcal{J}$. Additionally, $p(0) = \mu$.

Assumption 6.25 (Switched Independent Jump Linear System)**.** The Markov chain θ is an independent sequence of random variables for all switching sequences $\psi \in \Psi$.

For $k \in \mathbb{N}_0$, define the reverse transition probabilities $q_{ij}(k) = \mathbf{P}\{\theta(k) = j \mid \theta(k+1) = i\}$, and let $Q(k)$ be the $N \times N$ reverse transition probability matrix with entries $q_{ij}(k)$.

If Assumption 6.24 holds then μ is invariant for the sequence $\Pi(\psi(k))$ for any $\psi \in \Psi$ (see Definition 2.26).

Thus, $p(k) = \mu$ for all $k \in \mathbb{N}_0$ (see Lemma 2.28). It follows that

$$
\begin{aligned}
q_{ij}(k) &= \mathbf{P}\{\theta(k) = j \mid \theta(k+1) = i\} \\
&= \frac{\mathbf{P}\{\theta(k+1) = i \mid \theta(k) = j\}\,\mathbf{P}\{\theta(k) = j\}}{\mathbf{P}\{\theta(k+1) = i\}} \\
&= \frac{\mathbf{P}\{\theta(k) = j\}}{\mathbf{P}\{\theta(k+1) = i\}}\pi_{ji}(\psi(k+1)) \\
&= \frac{\mu_j}{\mu_i}\pi_{ji}(\psi(k+1)).
\end{aligned}
$$

For $s \in \mathcal{J}$ and $i, j \in \mathcal{N}$, define

$$
\xi_{ij}(s) = \frac{\mu_j}{\mu_i}\pi_{ji}(s). \tag{6.77}
$$

For $s \in \mathcal{J}$, let $\Xi(s)$ be the $N \times N$ matrix with entries $\xi_{ij}(s)$. Thus, Assumption 6.24 ensures that the reverse transition probability matrix $Q(k)$ varies in the finite set $\{\Xi(1), \ldots, \Xi(J)\}$.

Alternatively, if Assumption 6.25 holds, then

$$
\begin{aligned}
q_{ij}(k) &= \mathbf{P}\{\theta(k) = j \mid \theta(k+1) = i\} \\
&= \mathbf{P}\{\theta(k) = j \mid \theta(k-1) = i\} \\
&= \pi_{ij}(\psi(k))
\end{aligned}
$$

Thus, Assumption 6.25 ensures that the reverse transition probability matrix $Q(k)$ varies in the finite set $\{\Pi(1), \ldots, \Pi(J)\}$.

Both assumptions ensure that the Markov chain running in reverse time is described by a finite number of reverse transition probability matrices. This property of the Markov chain is crucial for the results that follow.

Remark 6.26. Note that (6.77) implies that the diagonal elements of $\Xi(s)$ and $\Pi(s)$ are the same. If $N = 2$, one can show that $\Xi(s) = \Pi(s)$ since the off-diagonal elements of each matrix are determined by the diagonal elements (i.e., each row must sum to one). In this case, (6.77) shows that $\mu_i\pi_{ij}(s) = \mu_j\pi_{ji}(s)$, which implies that when $N = 2$, the Markov chain with the sequence $\Pi \circ \psi$ of forward transition probability matrices is instantaneously reversible (see Definition 2.30).

6.4.1 Dynamic output feedback controllers

Dynamic output feedback controllers of two different forms are considered, depending on whether Assumption 6.24 or 6.25 holds.

Common invariant distribution

If Assumption 6.24 holds, consider a *finite-past-dependent output feedback controller* of the form

$$\begin{bmatrix} \hat{x}(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} (\theta(k), \psi(k), \psi(k-1), \dots, \psi(k-L+1)) \begin{bmatrix} \hat{x}(k) \\ y(k) \end{bmatrix} \tag{6.78}$$

where $\hat{x}(k) \in \mathbb{R}^n$ is the controller state; $\hat{A} : \mathcal{N} \times \Psi_L^* \to \mathbb{R}^{n \times n}$, $\hat{B} : \mathcal{N} \times \Psi_L^* \to \mathbb{R}^{n \times p_2}$, $\hat{C} : \mathcal{N} \times \Psi_L^* \to \mathbb{R}^{m_2 \times n}$, and $\hat{D} : \mathcal{N} \times \Psi_L^* \to \mathbb{R}^{m_2 \times p_2}$ are the controller parameter matrices; and,

$$\Psi_L^* = \{(r_L, r_{L-1}, \dots, r_1) : r = (r_1, \dots, r_{L-1}, r_L) \in \Psi_L\}$$

is the set of reversed sequences in Ψ_L.

Proceeding as in Section 6.3.1, the closed-loop system with the plant in (6.1) and controller in (6.78) is expressed

$$\begin{bmatrix} \bar{x}(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} (\theta(k), \psi(k), \dots, \psi(k-L+1)) \begin{bmatrix} \bar{x}(k) \\ w(k) \end{bmatrix} \tag{6.79}$$

where $\bar{x}(k) = [x^{\mathsf{T}}(k)\ \hat{x}^{\mathsf{T}}(k)]^{\mathsf{T}}$, and the closed-loop matrices are defined in (6.23) with $(i, r) \in \mathcal{N} \times \Psi_L^*$.

For $k \in \mathbb{N}_0$, define $\sigma(k) = \psi(k+1)$. Since $\psi(k)$ is available to the controller at time k, $\sigma(k-1)$ is available to the controller at time k. The controller in (6.78) can be expressed in terms of σ as follows

$$\begin{bmatrix} \hat{x}(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} (\theta(k), \sigma(k-1), \dots, \sigma(k-L)) \begin{bmatrix} \hat{x}(k) \\ y(k) \end{bmatrix} \tag{6.80}$$

The closed-loop system in (6.79) is equivalently expressed

$$\begin{bmatrix} \bar{x}(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} (\theta(k), \sigma(k-1), \dots, \sigma(k-L)) \begin{bmatrix} \bar{x}(k) \\ w(k) \end{bmatrix} \tag{6.81}$$

For $t \in \mathbb{Z}_0^-$, define $\acute{\theta}(t) := \theta(-t)$, and $\acute{\sigma}(t) := \sigma(-t)$. Note that $\mathbf{P}\{\acute{\theta}(t+1) = j \mid \acute{\theta}(t) = i\} = \xi_{ij}(\acute{\sigma}(t+1))$. Consider the switched Markov jump linear system defined by the difference equation

$$\begin{bmatrix} \acute{s}(t+1) \\ \acute{f}(t) \end{bmatrix} = \begin{bmatrix} \bar{A}^\intercal & \bar{C}^\intercal \\ \bar{B}^\intercal & \bar{D}^\intercal \end{bmatrix} (\acute{\theta}(t), \acute{\sigma}(t+1), \dots, \acute{\sigma}(t+L)) \begin{bmatrix} \acute{s}(t) \\ \acute{v}(t) \end{bmatrix} \tag{6.82}$$

where $t \in \mathbb{Z}_0^-$, $\acute{s}(t) \in \mathbb{R}^{2n}$ is the state vector, $\acute{v}(t) \in \mathbb{R}^{p_1}$ is a disturbance vector, $\acute{f}(t) \in \mathbb{R}^{m_1}$ is an error vector, the initial condition is specified by $\lim_{T\to-\infty} \acute{s}(T) = 0$, and $\acute{\theta} : \mathbb{Z}_0^- \times \Omega \to \mathcal{N}$ is a finite Markov chain characterized by the a priori unknown sequence $\Xi \circ \acute{\sigma}$ of transition probability matrices for some a priori unknown switching sequence $\acute{\sigma} \in \Psi_0^*$ where

$$\Psi_0^* = \{\sigma : \mathbb{Z}_0^- \to \mathcal{J} \text{ s.t. } \sigma(t) = \psi(-t+1), \exists \psi \in \Psi, \forall t \in \mathbb{Z}_0^-\}$$

is the set of reversed and shifted sequences from Ψ.

Appealing to Section 6.2.2, the closed-loop system (6.81) is uniformly exponentially mean square stable and uniformly mean square strictly contractive if and only if the system (6.82) is uniformly exponentially mean square stable and uniformly mean square strictly contractive.

Switched independent jump linear system

Alternatively, if Assumption 6.25 holds, then it suffices for the controller to have access to a delayed measurement $\psi(k-1)$ at time k. Under Assumption 6.25, consider a *finite-past-dependent output feedback controller* of the form

$$\begin{bmatrix} \hat{x}(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} (\theta(k), \psi(k-1), \psi(k-2), \dots, \psi(k-L)) \begin{bmatrix} \hat{x}(k) \\ y(k) \end{bmatrix} \tag{6.83}$$

where $\hat{x}(k) \in \mathbb{R}^n$ is the controller state, and $\hat{A} : \mathcal{N} \times \Psi_L^* \to \mathbb{R}^{n\times n}$, $\hat{B} : \mathcal{N} \times \Psi_L^* \to \mathbb{R}^{n\times p_2}$, $\hat{C} : \mathcal{N} \times \Psi_L^* \to \mathbb{R}^{m_2\times n}$, and $\hat{D} : \mathcal{N} \times \Psi_L^* \to \mathbb{R}^{m_2\times p_2}$ are the controller parameter matrices.

Proceeding as in Section 6.3.1, the closed-loop system with the plant in (6.1) and controller in (6.83) is expressed

$$\begin{bmatrix} \bar{x}(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} (\theta(k), \psi(k-1), \dots, \psi(k-L)) \begin{bmatrix} \bar{x}(k) \\ w(k) \end{bmatrix} \tag{6.84}$$

where $\bar{x}(k) = [x^{\mathsf{T}}(k)\ \hat{x}^{\mathsf{T}}(k)]^{\mathsf{T}}$, and the closed-loop matrices are defined in (6.23) with $(i, r) \in \mathcal{N} \times \Psi_L^*$.

For $t \in \mathbb{Z}_0^-$, define $\acute{\theta}(t) := \theta(-t)$; and, for $t \in \mathbb{Z}^-$, define $\acute{\psi}(t) := \psi(-t)$. Note that $\mathbf{P}\{\acute{\theta}(t+1) = j \mid \acute{\theta}(t) = i\} = \pi_{ij}(\acute{\psi}(t+1))$. Consider the switched Markov jump linear system defined by the difference equation

$$\begin{bmatrix} \acute{s}(t+1) \\ \acute{f}(t) \end{bmatrix} = \begin{bmatrix} \bar{A}^{\mathsf{T}} & \bar{C}^{\mathsf{T}} \\ \bar{B}^{\mathsf{T}} & \bar{D}^{\mathsf{T}} \end{bmatrix} (\acute{\theta}(t), \acute{\psi}(t+1), \ldots, \acute{\psi}(t+L)) \begin{bmatrix} \acute{s}(t) \\ \acute{v}(t) \end{bmatrix} \tag{6.85}$$

where $t \in \mathbb{Z}_0^-$, $\acute{s}(t) \in \mathbb{R}^{2n}$ is the state vector, $\acute{v}(t) \in \mathbb{R}^{p_1}$ is a disturbance vector, $\acute{f}(t) \in \mathbb{R}^{m_1}$ is an error vector, the initial condition is specified by $\lim_{T \to -\infty} \acute{s}(T) = 0$, and $\acute{\theta} : \mathbb{Z}_0^- \times \Omega \to \mathcal{N}$ is a finite Markov chain characterized by the a priori unknown sequence $\Pi \circ \acute{\psi}$ of transition probability matrices for some a priori unknown switching sequence $\acute{\psi} \in \Psi^*$ where

$$\Psi^* = \{\sigma : \mathbb{Z}^- \to \mathcal{J} \text{ s.t. } \sigma(t) = \psi(-t), \exists \psi \in \Psi, \forall t \in \mathbb{Z}^-\}$$

is the set of reversed sequences from Ψ.

Appealing to Section 6.2.2, the closed-loop system (6.84) is uniformly exponentially mean square stable and uniformly mean square strictly contractive if and only if the system (6.85) is uniformly exponentially mean square stable and uniformly mean square strictly contractive.

6.4.2 Controller construction via LMIs

Common invariant distribution

If a common invariant distribution exists as in Assumption 6.24 for the set of transition probability matrices $\{\Pi(1), \ldots, \Pi(J)\}$, then the LMI condition in Theorem 6.27 below may be used to construct an output feedback controller of the form (6.78) that is dependent only on the present and past and ensures that the closed-loop system in (6.79) is uniformly exponentially mean square stable and uniformly mean square strictly contractive.

Theorem 6.27. *Assume $p_i(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$. Under Assumption 6.24, there exists a finite-past-dependent output feedback controller as in* (6.78) *such that the closed-loop system in* (6.79) *is uniformly exponentially mean square stable and uniformly mean square strictly contractive if and only if there exists $T \in \mathbb{N}_0$ and functions $S, U : \mathcal{N} \times \Psi_T^* \to \mathbb{S}_n^+$, $G : \mathcal{N} \times \Psi_{T+1}^* \to \mathbb{R}^{n \times n}$, $Z : \mathcal{N} \times \mathcal{N} \times \Psi_{T+1}^* \to \mathbb{S}_n^+$*

such that

$$\begin{bmatrix} W_1(i) & 0 \\ W_2(i) & 0 \\ 0 & I \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} A^{\mathsf{T}}(i)U(i,r_{1:T})A(i) + \mathcal{Z}(i,r) & A^{\mathsf{T}}(i)U(i,r_{1:T})B_1(i) & C_1^{\mathsf{T}}(i) \\ B_1^{\mathsf{T}}(i)U(i,r_{1:T})A(i) & -I + B_1^{\mathsf{T}}(i)U(i,r_{1:T})B_1(i) & D_{11}^{\mathsf{T}}(i) \\ C_1(i) & D_{11}(i) & -I \end{bmatrix} \begin{bmatrix} W_1(i) & 0 \\ W_2(i) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{6.86a}$$

$$\begin{bmatrix} V_1(i) & 0 \\ V_2(i) & 0 \\ 0 & I \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} A(i)\tilde{S}(i,r)A^{\mathsf{T}}(i) - S(i,r_{1:T}) & A(i)\tilde{S}(i,r)C_1^{\mathsf{T}}(i) & B_1(i) \\ C_1(i)\tilde{S}(i,r)A^{\mathsf{T}}(i) & -I + C_1(i)\tilde{S}(i,r)C_1^{\mathsf{T}}(i) & D_{11}(i) \\ B_1^{\mathsf{T}}(i) & D_{11}^{\mathsf{T}}(i) & -I \end{bmatrix} \begin{bmatrix} V_1(i) & 0 \\ V_2(i) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{6.86b}$$

$$\begin{bmatrix} S(i,r_{1:T}) & I \\ I & U(i,r_{1:T}) \end{bmatrix} > 0 \tag{6.86c}$$

$$\begin{bmatrix} Z(i,j,r) & G^{\mathsf{T}}(i,r) \\ G(i,r) & U(j,r_{2:T+1}) \end{bmatrix} > 0 \tag{6.86d}$$

for all $i, j \in \mathcal{N}$ *and* $r = (r_1, r_2, \ldots, r_{T+1}) \in \Psi^*_{T+1}$*, where* $\left[W_1^{\mathsf{T}}(i)\, W_2^{\mathsf{T}}(i)\right]^{\mathsf{T}}$ *is any basis of* $\ker\left[C_2(i)\, D_{21}(i)\right]$*;* $\left[V_1^{\mathsf{T}}(i)\, V_2^{\mathsf{T}}(i)\right]^{\mathsf{T}}$ *is any basis of* $\ker\left[B_2^{\mathsf{T}}(i)\, D_{12}^{\mathsf{T}}(i)\right]$*; and,*

$$\tilde{S}(i,r) := \sum_{j=1}^{N} \xi_{ij}(r_1)S(j,r_{2:T+1}), \quad \mathcal{Z}(i,r) := \sum_{j=1}^{N} \xi_{ij}(r_1)Z(i,j,r) - G(i,r) - G^{\mathsf{T}}(i,r).$$

Proof. Apply Theorem 6.21 to the closed-loop system in (6.82). □

Note that Theorem 6.27 is a restatement of Theorem 6.21 with the following replacements:

$$\begin{bmatrix} A(i) & B_1(i) & B_2(i) \\ C_1(i) & D_{11}(i) & D_{12}(i) \\ C_2(i) & D_{21}(i) & D_{22}(i) \end{bmatrix} \mapsto \begin{bmatrix} A^{\mathsf{T}}(i) & C_1^{\mathsf{T}}(i) & C_2^{\mathsf{T}}(i) \\ B_1^{\mathsf{T}}(i) & D_{11}^{\mathsf{T}}(i) & D_{21}^{\mathsf{T}}(i) \\ B_2^{\mathsf{T}}(i) & D_{12}^{\mathsf{T}}(i) & D_{22}^{\mathsf{T}}(i) \end{bmatrix}, \quad \pi \mapsto \xi, \quad \Psi \mapsto \Psi^*.$$

Now to construct a controller using Theorem 6.27, suppose the LMIs in Theorem 6.27 have been solved and T, S, U, G, Z satisfying (6.86) are given. Define X in terms of S and U as in (6.66), and define

$$\acute{\mathcal{P}}(i) = \begin{bmatrix} \mathcal{C}_2(i) & 0 & 0 & \mathcal{D}_{21}(i) \end{bmatrix}, \quad \acute{\mathcal{Q}}(i) = \begin{bmatrix} 0 & \mathcal{B}_2^{\mathsf{T}}(i) & \mathcal{D}_{12}^{\mathsf{T}}(i) & 0 \end{bmatrix}$$

where $i \in \mathcal{N}$. As in the proof of Theorem 6.17, it follows that

$$W^\mathsf{T}(i)\acute{\mathcal{H}}(i,r)W(i) < 0$$
$$V^\mathsf{T}(i)\acute{\mathcal{H}}(i,r)V(i) < 0$$

where $W(i)$ and $V(i)$ are any matrices whose columns form bases of $\ker \acute{\mathcal{P}}(i)$ and $\ker \acute{\mathcal{Q}}(i)$, respectively, and

$$\acute{\mathcal{H}}(i,r) = \begin{bmatrix} -\left(\sum_{j=1}^{N} \xi_{ij}(r_1) X(j, r_{2:T+1})\right)^{-1} & \mathcal{A}^\mathsf{T}(i) & \mathcal{C}_1^\mathsf{T}(i) & 0 \\ \mathcal{A}(i) & -X(i, r_{1:T}) & 0 & \mathcal{B}_1(i) \\ \mathcal{C}_1(i) & 0 & -I & D_{11}(i) \\ 0 & \mathcal{B}_1^\mathsf{T}(i) & D_{11}^\mathsf{T}(i) & -I \end{bmatrix}$$

with $(i,r) \in \mathcal{N} \times \Psi_{T+1}^*$. Now Lemma 2.41 guarantees the existence of $K(i,r)$, $(i,r) \in \mathcal{N} \times \Psi_{T+1}^*$ such that

$$\acute{\mathcal{H}}(i,r) + \acute{\mathcal{P}}^\mathsf{T}(i)K^\mathsf{T}(i,r)\acute{\mathcal{Q}}(i) + \acute{\mathcal{Q}}^\mathsf{T}(i)K(i,r)\acute{\mathcal{P}}(i) < 0. \tag{6.87}$$

Note that (6.87) is an LMI in variable K since X is fixed. Thus, K that satisfies (6.87) can be found either by using the formulas in Lemma 2.41 or by solving the LMI (6.87) for K. Note the intentional difference between the placement of K and K^T in the expressions (6.87) and (6.59). Using the partition in (6.24b) for K and the expressions in (6.23), inequality (6.87) is equivalently expressed

$$\begin{bmatrix} -\left(\sum_{j=1}^{N} \xi_{ij}(r_1) X(j, r_{2:T+1})\right)^{-1} & \bar{A}^\mathsf{T}(i,r) & \bar{C}^\mathsf{T}(i,r) & 0 \\ \bar{A}(i,r) & -X(i, r_{1:T}) & 0 & \bar{B}(i,r) \\ \bar{C}(i,r) & 0 & -I & \bar{D}(i,r) \\ 0 & \bar{B}^\mathsf{T}(i,r) & \bar{D}^\mathsf{T}(i,r) & -I \end{bmatrix} < 0 \tag{6.88}$$

where $i \in \mathcal{N}$ and $r \in \Psi_{T+1}^*$. As in the proof of Theorem 6.17, the inequalities in (6.88) ensure that the system in (6.82) is uniformly exponentially mean square stable and uniformly mean square strictly contractive, which ensures that the system in (6.79) is uniformly exponentially mean square stable and uniformly mean square strictly contractive.

Switched independent jump linear system

Theorem 6.28 may be applied to any switched independent jump linear system for the purpose of nonconservative controller synthesis.

Theorem 6.28. *Assume $p_i(k) > 0$ for all $\psi \in \Psi$, $i \in \mathcal{N}$, and $k \in \mathbb{N}_0$. Under Assumption 6.25, there exists a finite-past-dependent output feedback controller as in (6.83) such that the closed-loop system is uniformly exponentially mean square stable and uniformly mean square strictly contractive if and only if there exists $T \in \mathbb{N}_0$ and functions $S, U : \mathcal{N} \times \Psi_T^* \to \mathbb{S}_n^+$, $G : \mathcal{N} \times \Psi_{T+1}^* \to \mathbb{R}^{n\times n}$, $Z : \mathcal{N} \times \mathcal{N} \times \Psi_{T+1}^* \to \mathbb{S}_n^+$ such that*

$$\begin{bmatrix} W_1(i) & 0 \\ W_2(i) & 0 \\ 0 & I \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} A^{\mathsf{T}}(i)U(i,r_{1:T})A(i) + \mathcal{Z}(i,r) & A^{\mathsf{T}}(i)U(i,r_{1:T})B_1(i) & C_1^{\mathsf{T}}(i) \\ B_1^{\mathsf{T}}(i)U(i,r_{1:T})A(i) & -I + B_1^{\mathsf{T}}(i)U(i,r_{1:T})B_1(i) & D_{11}^{\mathsf{T}}(i) \\ C_1(i) & D_{11}(i) & -I \end{bmatrix} \begin{bmatrix} W_1(i) & 0 \\ W_2(i) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{6.89a}$$

$$\begin{bmatrix} V_1(i) & 0 \\ V_2(i) & 0 \\ 0 & I \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} A(i)\tilde{S}(i,r)A^{\mathsf{T}}(i) - S(i,r_{1:T}) & A(i)\tilde{S}(i,r)C_1^{\mathsf{T}}(i) & B_1(i) \\ C_1(i)\tilde{S}(i,r)A^{\mathsf{T}}(i) & -I + C_1(i)\tilde{S}(i,r)C_1^{\mathsf{T}}(i) & D_{11}(i) \\ B_1^{\mathsf{T}}(i) & D_{11}^{\mathsf{T}}(i) & -I \end{bmatrix} \begin{bmatrix} V_1(i) & 0 \\ V_2(i) & 0 \\ 0 & I \end{bmatrix} < 0 \tag{6.89b}$$

$$\begin{bmatrix} S(i,r_{1:T}) & I \\ I & U(i,r_{1:T}) \end{bmatrix} > 0 \tag{6.89c}$$

$$\begin{bmatrix} Z(i,j,r) & G^{\mathsf{T}}(i,r) \\ G(i,r) & U(j,r_{2:T+1}) \end{bmatrix} > 0 \tag{6.89d}$$

for all $i, j \in \mathcal{N}$ and $r = (r_1, r_2, \ldots, r_{T+1}) \in \Psi_{T+1}^*$, where $\left[W_1^{\mathsf{T}}(i)\, W_2^{\mathsf{T}}(i)\right]^{\mathsf{T}}$ is any basis of $\ker\,[C_2(i)\, D_{21}(i)]$; $\left[V_1^{\mathsf{T}}(i)\, V_2^{\mathsf{T}}(i)\right]^{\mathsf{T}}$ is any basis of $\ker\left[B_2^{\mathsf{T}}(i)\, D_{12}^{\mathsf{T}}(i)\right]$; and,

$$\tilde{S}(i,r) := \sum_{j=1}^{N} \pi_{ij}(r_1)S(j,r_{2:T+1}), \quad \mathcal{Z}(i,r) := \sum_{j=1}^{N} \pi_{ij}(r_1)Z(i,j,r) - G(i,r) - G^{\mathsf{T}}(i,r).$$

Proof. Apply Theorem 6.21 to the closed-loop system in (6.85). □

Note the only difference between the condition in Theorem 6.27 and that of Theorem 6.28 is the presence of ξ in the former and π in the latter. If the LMIs in (6.89) from Theorem 6.28 are feasible, a controller can be constructed by following the steps outlined immediately after the proof of Theorem 6.27.

6.4.3 Control synthesis algorithm

Depending on whether Assumption 6.24 or 6.25 holds, Theorem 6.27 or Theorem 6.28 may be used to construct output feedback controllers that approximately minimize the worst-case gain from $\|w\|_{2,e}$ to $\|z\|_{2,e}$

of the closed-loop system. The algorithm is analogous to the one proposed in Section 6.3.6. For concreteness, suppose that Assumption 6.24 holds and consider the following iterative algorithm using Theorem 6.27:

1. Set $\gamma_{-1} = -\epsilon$.
2. Set $T = 0$.
3. Use a bisection search to approximate the smallest $\gamma_T > 0$ such that the LMIs in (6.86) from Theorem 6.27 with B_1 replaced by $(1/\gamma_T)B_1$, D_{11} replaced by $(1/\gamma_T)D_{11}$, D_{21} replaced by $(1/\gamma_T)D_{21}$, and the current value of T are feasible.
4. If $(\gamma_T - \gamma_{T-1}) < \epsilon$ then stop. Otherwise, set $T = T + 1$ and go to step 3.

Here, ϵ is some small threshold that prompts the algorithm to exit when the sequence γ_T stops changing by a significant amount. In contrast to the algorithm in Section 6.3.6, note that the matrices B_1, D_{11} and D_{21} are scaled by γ_T.

6.4.4 Remarks on Assumptions 6.24 and 6.25

The assumptions required in Section 6.4 are somewhat restrictive. Whether or not these assumptions can be weakened or disposed of remains an open question. The important consequence of both Assumptions 6.24 and 6.25 is that the Markov chain running in reverse time is described by a finite number of reverse transition probability matrices; this fact allows us to apply the results of Section 6.3 to the adjoint system, where the Markov chain is reversed. A first step toward weakening Assumption 6.24 might be to remove the restriction that $p(0) = \mu$. To that end, ergodicity results for time-inhomogeneous Markov chains may prove fruitful. Roughly speaking, a time-inhomogeneous Markov chain is strongly ergodic if there exists some probability distribution q such that $\lim_{k\to\infty} p(m)P(m+1)P(m+2)\cdots P(k) = q$ independently of the starting time m and the initial distribution $p(m)$ (see, e.g., [30, Chap. V]).

6.5 Examples

6.5.1 Scalar example from Remark 5.12

Consider the plant in (6.1) with parameter matrices from the example in Remark 5.12:

$$A(1) = 0.3, \quad \begin{bmatrix} B_1(1) & B_2(1) \end{bmatrix} = \begin{bmatrix} 1.5 & 1 \end{bmatrix}, \quad \begin{bmatrix} C_1(1) \\ C_2(1) \end{bmatrix} = \begin{bmatrix} 0.2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} D_{11}(1) & D_{12}(1) \\ D_{21}(1) & D_{22}(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{6.90a}$$

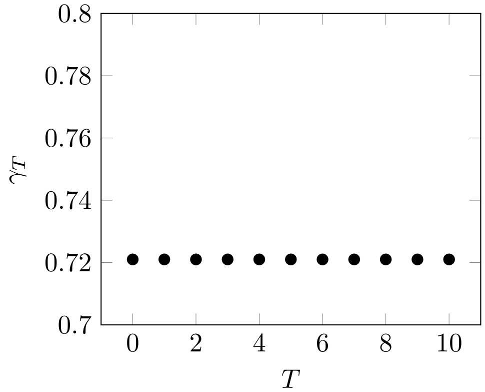


Figure 6.2: Plot generated by applying Theorem 6.28 and the algorithm in Section 6.4.3 to the plant with parameter matrices specified in (6.90).

$$A(2) = 0.4, \quad \begin{bmatrix} B_1(2) & B_2(2) \end{bmatrix} = \begin{bmatrix} 0.5 & 1 \end{bmatrix}, \quad \begin{bmatrix} C_1(2) \\ C_2(2) \end{bmatrix} = \begin{bmatrix} 0.85 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} D_{11}(2) & D_{12}(2) \\ D_{21}(2) & D_{22}(2) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{6.90b}$$

$$\Pi(1) = \begin{bmatrix} 0.8 & 0.2 \\ 0.8 & 0.2 \end{bmatrix}, \quad \Pi(2) = \begin{bmatrix} 0.3 & 0.7 \\ 0.3 & 0.7 \end{bmatrix}. \tag{6.90c}$$

Notice that $C_2(2) = 0$, $D_{21}(2) = 0$, and $D_{22}(2) = 0$, so the plant in (6.1) with the matrices in (6.90) provides no information about the state to the controller when $\theta(k) = 2$. Alternatively, when $\theta(k) = 1$, $y(k) = x(k) + w(k)$. In Remark 5.12, it was established that the open-loop system is *not* contractive. By using Theorem 5.46 and scaling the problem as in Remark 5.2, the worst-case gain from $\|w\|_{2,e}$ to $\|z\|_{2,e}$ of the open-loop system is approximately 1.2.

Fig. 6.2 shows the result of applying Theorem 6.28 and the algorithm in Section 6.4.3 to the plant with parameter matrices specified in (6.90). From Fig. 6.2, the best disturbance attenuation of the closed-loop system that is achievable by a controller of the form (6.83) is approximately 0.721. In addition, Fig. 6.2 indicates that increasing T does not yield better performance. Solving a second LMI problem yields a controller which guarantees that the worst-case gain from $\|w\|_{2,e}$ to $\|z\|_{2,e}$ of the closed-loop system is no greater than 0.721:

$$\begin{bmatrix} \hat{x}(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} \hat{A}(\theta(k), \psi(k-1)) & \hat{B}(\theta(k), \psi(k-1)) \\ \hat{C}(\theta(k), \psi(k-1)) & \hat{D}(\theta(k), \psi(k-1)) \end{bmatrix} \begin{bmatrix} \hat{x}(k) \\ y(k) \end{bmatrix}$$

where

$$\begin{bmatrix} \hat{A}(1,1) & \hat{B}(1,1) \\ \hat{C}(1,1) & \hat{D}(1,1) \end{bmatrix} = \begin{bmatrix} -0.0283 & -0.0015 \\ -0.7969 & -1.1002 \end{bmatrix}, \quad \begin{bmatrix} \hat{A}(1,2) & \hat{B}(1,2) \\ \hat{C}(1,2) & \hat{D}(1,2) \end{bmatrix} = \begin{bmatrix} 0.0363 & 0.0035 \\ -0.8720 & -1.1059 \end{bmatrix}, \tag{6.91a}$$

$$\begin{bmatrix} \hat{A}(2,1) & \hat{B}(2,1) \\ \hat{C}(2,1) & \hat{D}(2,1) \end{bmatrix} = \begin{bmatrix} 0.0314 & 0 \\ 1.1423 & 0 \end{bmatrix}, \quad \begin{bmatrix} \hat{A}(2,2) & \hat{B}(2,2) \\ \hat{C}(2,2) & \hat{D}(2,2) \end{bmatrix} = \begin{bmatrix} -0.0178 & 0 \\ 1.2052 & 0 \end{bmatrix}. \tag{6.91b}$$

Notice that when $\theta(k) = 2$, $\hat{B}$ and $\hat{D}$ are zero; this reflects the fact that $y(k) = 0$ when $\theta(k) = 2$.

The closed-loop system with the plant matrices in (6.90) and the controller matrices in (6.91) was simulated 1×10^5 times, with each simulation being $1,000$ time-units long. The disturbance $w(k)$ was sampled uniformly from the interval $(-1, 1)$, the switching sequence ψ was generated randomly, and $\theta(k)$ was sampled from the appropriate distribution $\Pi(\psi(k))$. Table 6.1 shows the average (over 1×10^5 realizations) and maximum gains encountered from the disturbance input to the error output of the closed-loop system with the plant matrices in (6.90) and the controller matrices in (6.91).

6.5.2 Mass-spring-damper example

Consider the mass-spring-damper system shown in Fig. 6.3 that consists of two frictionless carts with masses m_1 and m_2 coupled together with a spring and damper. The first cart is attached to a fixed end point through a spring. The goal is to control the position and velocity of the second cart by applying a force F to it. A disturbance force F_d is applied to the first cart. Only a noisy measurement of the position and velocity of the first cart is available to the controller, and the measurement is sent over a wireless network at each sampling time that is subject to packet loss (see Fig. 6.1). The controller is able to detect a lost or corrupted measurement, but retransmission does not occur. Packet losses in the wireless channel are assumed to be

Table 6.1: Gains encountered from disturbance input to error output of the closed-loop system with the controller matrices given in (6.91).

Quantity	Value
$\dfrac{\left(\text{mean}\left(\sum_{k=0}^{1000} z^{\mathsf{T}}(k)z(k)\right)\right)^{1/2}}{\left(\text{mean}\left(\sum_{k=0}^{1000} w^{\mathsf{T}}(k)w(k)\right)\right)^{1/2}}$	0.350
$\max \dfrac{\left(\sum_{k=0}^{1000} z^{\mathsf{T}}(k)z(k)\right)^{1/2}}{\left(\sum_{k=0}^{1000} w^{\mathsf{T}}(k)w(k)\right)^{1/2}}$	0.401

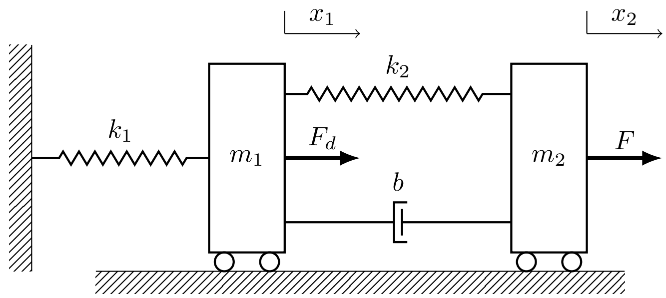


Figure 6.3: Mass-spring-damper system.

Markovian. The transition probabilities governing packet loss are a priori unknown, but vary in a finite set. As discussed in the chapter introduction, it is assumed that measurement of the signal-to-noise ratio at the receiver yields the current transition probabilities of the Markov chain. For this example, it is assumed that network delays are negligible, but delays may also be modeled within the Markov jump linear system framework (see, e.g., [68]).

Let x_i be the position of the i-th cart as indicated in Fig. 6.3. The continuous-time, open-loop dynamics of the system in Fig. 6.3 are expressed

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (-k_2 - k_1)/m_1 & k_2/m_1 & -b/m_1 & b/m_1 \\ k_2/m_2 & -k_2/m_2 & b/m_2 & -b/m_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix} F_d(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} F(t) \quad (6.92)$$

The open-loop system in (6.92) was discretized using a zero-order hold on the inputs and a sampling time of 0.02 s. The discrete-time error output is expressed

$$z(k) = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

where $x(k)$ is the discretized state from (6.92); $\alpha > 0$ and $\beta > 0$ are relative weights on the position and velocity, respectively, of the second cart; and, $u(k)$ is the discretized signal $F(t)$. The controlled input is included in $z(k)$ to ensure that large control effort is penalized.

Let $\theta(k) = 1$ indicate that the measurement at time k was lost or corrupted, and let $\theta(k) = 2$ indicate a successful transmission of the measurement at time k. Let $w_1(k)$ and $w_2(k)$ be additive measurement noise for the position and velocity, respectively, of the first cart, and let $w_3(k)$ be the discretized signal $F_d(t)$. The discrete-time output available to the controller is expressed

$$y(k) = \begin{bmatrix} \theta(k)-1 & 0 & 0 & 0 \\ 0 & 0 & \theta(k)-1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} (\theta(k)-1)\delta & 0 & 0 \\ 0 & (\theta(k)-1)\rho & 0 \end{bmatrix} w(k)$$

where $w(k) = [\, w_1(k)\, w_2(k)\, w_3(k)\,]^{\mathsf{T}}$, and $\delta, \rho > 0$ are weights that determine the relative size of the measurement noise.

It is assumed that at each time instant, the Markov chain θ may be described by one of the following transition probability matrices

$$\Pi(1) = \begin{bmatrix} 0.85 & 0.15 \\ 0.1 & 0.9 \end{bmatrix}, \qquad \Pi(2) = \begin{bmatrix} 0.55 & 0.45 \\ 0.3 & 0.7 \end{bmatrix}, \tag{6.93a}$$

$$\Pi(3) = \begin{bmatrix} 0.925 & 0.075 \\ 0.05 & 0.95 \end{bmatrix}, \quad \Pi(4) = \begin{bmatrix} 0.5 & 0.5 \\ 1/3 & 2/3 \end{bmatrix}. \tag{6.93b}$$

Note that for the matrices in (6.93), the probability of a packet loss following a packet loss is large ($\pi_{11}(s) \geq \pi_{12}(s)$), and the probability of a successful packet following a successful packet is also large ($\pi_{22}(s) \geq \pi_{21}(s)$). This property of the transition probability matrices is characteristic of bursty errors encountered in wireless networks [16, 23, 25]. Also, it is easily verified that

$$\begin{bmatrix} 0.4 & 0.6 \end{bmatrix} \Pi(s) = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}$$

for all $s \in \mathcal{J} = \{1, 2, 3, 4\}$ so that Assumption 6.24 is satisfied with $\mu = [\, 0.4 \;\; 0.6\,]$. For this example, the switching set is not restricted, so $\Psi = \mathcal{J}^{\infty}$.

Using Theorem 6.27 and the algorithm in Section 6.4.3, a finite-past-dependent output feedback controller of the form (6.78) is constructed for the mass-spring-damper system of Fig. 6.3 with the parameter values in Table 6.2. Fig. 6.4 shows the results of applying the algorithm in Section 6.4.3.

Each point on the graph in Fig. 6.4 corresponds to a controller

$$\begin{bmatrix} \hat{x}(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} (\theta(k), \psi(k), \psi(k-1), \ldots, \psi(k-T)) \begin{bmatrix} \hat{x}(k) \\ y(k) \end{bmatrix}$$

with a memory of the past T transition probability matrices. As Fig. 6.4 shows, diminishing improvements in disturbance attenuation are obtained as the controller's memory of past transition probability matrices is increased. Clearly, there is a trade-off between controller complexity and disturbance attenuation. For exam-

Table 6.2: Parameter values for the mass-spring-damper system in Section 6.5.2.

Parameter	Value	Units
m_1	0.5	kg
m_2	1.0	kg
k_1	12.0	N/m
k_2	7.0	N/m
b	0.2	Ns/m
α	10	-
β	1	-
δ	0.08	-
ρ	0.12	-

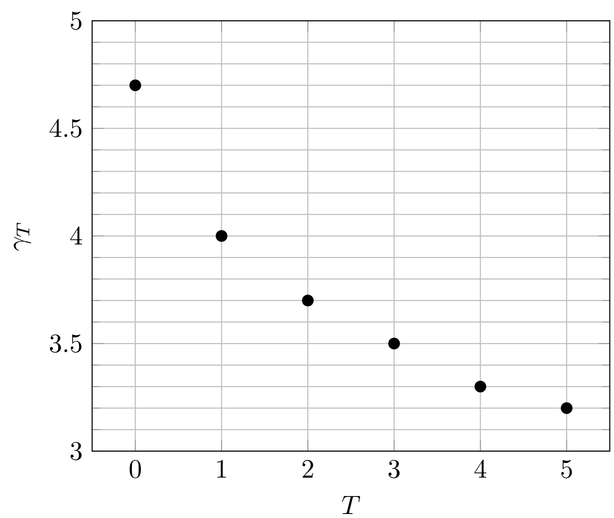


Figure 6.4: Plot generated by applying Theorem 6.27 and the algorithm in Section 6.4.3 to the mass-spring-damper system in Section 6.5.2.

ple, with $T = 5$ and $\Psi = \mathcal{J}^{\infty}$, one must compute $2 \times 4^6 = 8,192$ different controller matrices corresponding to $K(i,r)$ with $(i,r) \in \mathcal{N} \times \mathcal{J}^{T+1}$ (see the discussion immediately following Theorem 6.27). Alternatively, with $T = 0$, only $2 \times 4^1 = 8$ different controller matrices must be computed. Computing the controller matrices for $T = 0$ takes approximately 4 seconds on a standard desktop computer with 8 GB of system memory and an Intel Core i7 processor clocked at 3.40 GHz. On the other hand, for $T = 5$, computing the controller matrices takes approximately 60 minutes.

The closed-loop system is simulated with the two different controllers:

$$\begin{bmatrix} \hat{x}(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} (\theta(k), \psi(k), \psi(k-1), \ldots, \psi(k-5)) \begin{bmatrix} \hat{x}(k) \\ y(k) \end{bmatrix} \tag{6.94a}$$

$$\begin{bmatrix} \hat{x}(k+1) \\ u(k) \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} (\theta(k), \psi(k)) \begin{bmatrix} \hat{x}(k) \\ y(k) \end{bmatrix} \tag{6.94b}$$

where the controller matrices in (6.94a) correspond to $T = 5$ in Fig. 6.4, and the controller matrices in (6.94b) correspond to $T = 0$ in Fig. 6.4. A simulation of the closed-loop system was performed 5×10^3 times, each $1,000$ time-units long, where

- $\bar{x}(0) = 0$;
- $w_i(k)$, $i = 1, 2, 3$ were sampled uniformly from the interval $(-1, 1)$;
- ψ was chosen as follows:
  - for simulation iterations 1–100, $\psi \equiv 1$,
  - for simulation iterations 101–200, $\psi \equiv 2$,
  - for simulation iterations 201–300, $\psi \equiv 3$,
  - for simulation iterations 301–400, $\psi \equiv 4$,
  - and, for simulation iterations 401–5000, $\psi(k)$ is sampled uniformly from $\{1, 2, 3, 4\}$;
- $\theta(0)$ was sampled from μ;
- and, $\theta(k)$ for $k \geq 1$ was sampled from the appropriate distribution $\Pi(\psi(k))$.

For each iteration of the simulation, the same realizations for w, ψ, and θ were applied to both closed-loop systems with controllers (6.94a) and (6.94b), respectively. Table 6.3 shows the average (over 5×10^3 realizations) and maximum gains encountered from the disturbance input to the error output using the controllers in (6.94a) and (6.94b). Table 6.3 shows that in terms of disturbance attenuation, the controller in (6.94a) outperforms the controller in (6.94b), which is expected from Fig. 6.4. In all but 3 out of 5×10^3 simulation iterations, the gain from w to z of a realization, which is expressed

$$\frac{\left(\sum_{k=0}^{1000} z^{\mathsf{T}}(k) z(k)\right)^{1/2}}{\left(\sum_{k=0}^{1000} w^{\mathsf{T}}(k) w(k)\right)^{1/2}},$$

was smaller with the $T = 5$ controller when compared to the gain with the $T = 0$ controller.

Both of the controllers in (6.94) guarantee that the closed-loop system is uniformly exponentially mean square stable. Fig. 6.5 shows representative realizations of the quantities $x^{\mathsf{T}}(k)x(k)$ and $\hat{x}^{\mathsf{T}}(k)\hat{x}(k)$ for the closed-loop systems when $w \equiv 0$, $x(0) = [\,1 \;\; -1 \;\; 0 \;\; 0\,]^{\mathsf{T}}$, and $\hat{x}(0) = 0$.

Table 6.3: Gains encountered from disturbance input to error output with the $T = 5$ controller in (6.94a) and the $T = 0$ controller in (6.94b).

Quantity	$T = 5$	$T = 0$
$\dfrac{\left(\text{mean}\left(\sum_{k=0}^{1000} z^{\mathsf{T}}(k)z(k)\right)\right)^{1/2}}{\left(\text{mean}\left(\sum_{k=0}^{1000} w^{\mathsf{T}}(k)w(k)\right)\right)^{1/2}}$	0.315	0.343
$\max \dfrac{\left(\sum_{k=0}^{1000} z^{\mathsf{T}}(k)z(k)\right)^{1/2}}{\left(\sum_{k=0}^{1000} w^{\mathsf{T}}(k)w(k)\right)^{1/2}}$	0.378	0.418

6.6 Notes and References

The problem of LMI-based synthesis of $\mathcal{H}_\infty$ controllers for deterministic linear time-invariant systems was solved independently by Gahinet and Apkarian [20] and Iwasaki and Skelton [31]. The book by Skelton et al. [62] demonstrates the versatility of LMI-based methods in control theory. The Projection Lemma, Lemma 2.41, and its counterpart, Lemma 2.42, are important tools for control synthesis, and are related to Finsler's Lemma (the original paper is in German [18]; see also [64]).

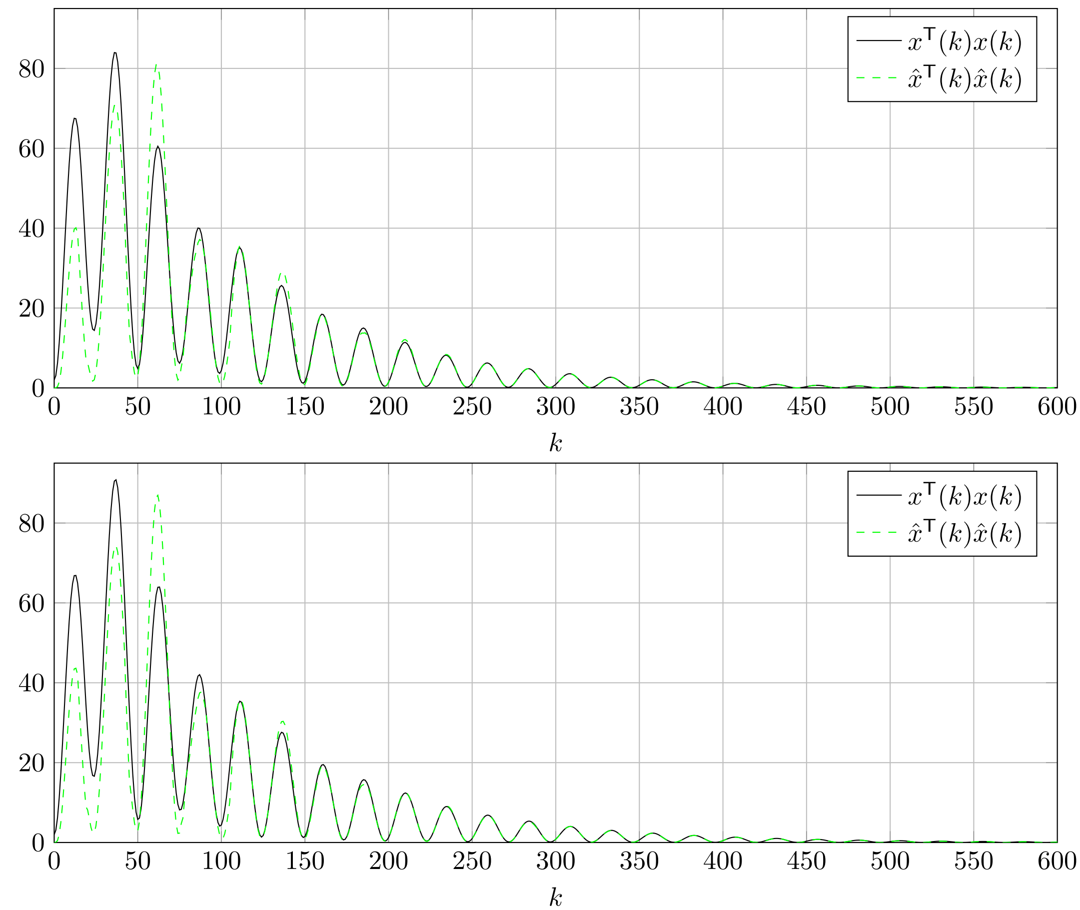


Figure 6.5: Zero-input response ($w \equiv 0$) of the closed-loop system with the controller in (6.94a) (top plot) and with the controller in (6.94b) (bottom plot).

Chapter 7

Energy-Aware Control

7.1 Introduction

Long duration autonomous underwater vehicles (AUVs) motivate much of the development in this chapter. For long duration AUVs operating at slow speeds, the energy consumed by the electronics on-board the vehicle may greatly exceed the energy consumed by the propulsion system. Thus, reducing energy consumed by the embedded microprocessor and the other electronics on-board the vehicle is important to extend the duration of the vehicle's mission.

A variety of techniques may be used to reduce the energy consumed by the processor. For example, system designers may choose a low-power processor appropriate for the computing needs of the system. In addition, power-management techniques such as dynamic voltage and frequency scaling (DVFS) or idle and sleep states of the processor may provide energy-saving benefits. In these cases, system designers may need worst-case estimates of the total processing power required by the system and the time required to execute certain tasks within the system. These estimates can be conservative due to unpredictability introduced by the system architecture (e.g., pipelined execution units, cache memory hierarchy, multiple processing cores [34, 47]) or unpredictability introduced by a task itself (e.g., image processing, optimization routines).

A few interesting questions emerge when considering how to reduce the conservatism associated with a particular energy-saving technique:

1. What if system designers intentionally choose a processor that is undersized in relation to the worst-case processing power required of the system with the knowledge that task deadlines will occasionally be violated? In particular, what if the control algorithm occasionally fails to generate an updated control signal before the next sampling time?

2. If DVFS is used, what if the processor voltage and frequency are occasionally reduced so much that the control algorithm fails to generate an updated control signal before the next sampling time?

3. If low-power idle and sleep states of the processor are used, what if the control signal is not updated before the next sampling time due to the time required to wake the processor from a sleep or idle state?

The remainder of this chapter is organized as follows. In Section 7.2, the power consumption of a slowly moving AUV is examined, and the power consumed by the electronics on-board the vehicle is shown to be large compared to the propulsive load. Section 7.3 shows how a control system subject to probabilistic control signal failures can be cast as a switched Markov jump linear system. Finally, in Section 7.4, the formal results of Chapters 4 and 5 are applied to an AUV that may occasionally fail to update the control signal before the next sampling time.

7.2 Power Analysis for an AUV

The power consumption of an AUV may be divided into three categories: the propulsive load, the hotel load, and the instrument load. The propulsive load, P_p, is defined to be the electrical power required to overcome the force of drag and maintain a constant speed. The drag force, F_d, and power required to overcome drag, P_d, are modeled by

$$\begin{aligned} F_d &= \frac{1}{2}\rho v^2 C_d A \\ P_d &= F_d v \end{aligned}$$

where ρ is the fluid density ($\rho = 1000$ kg/m^3 for fresh water), v is the speed of the object relative to the fluid, C_d is the drag coefficient, and A is the vehicle's cross-sectional area. The propulsive load is calculated by

$$P_p = \frac{P_d}{\eta_e}$$

where the overall propulsive efficiency η_e represents the available mechanical power generated by one unit of electrical power supplied to the propeller's motor.

The hotel load, P_h, is defined to be the power consumed by all essential electronics on-board the vehicle, and the instrument load is defined to be the power consumed by any payload instruments. For the remainder of the chapter, the instrument load is neglected, and emphasis is placed on the hotel and propulsive loads. The total available electrical energy, E, is determined by the type and number of batteries used to power

Table 7.1: Parameter values for an AUV.

Parameter	Value
C_d	0.19
A	0.0241 m^2
η_e	0.40
P_h	7 W
E	1.9 kW-h

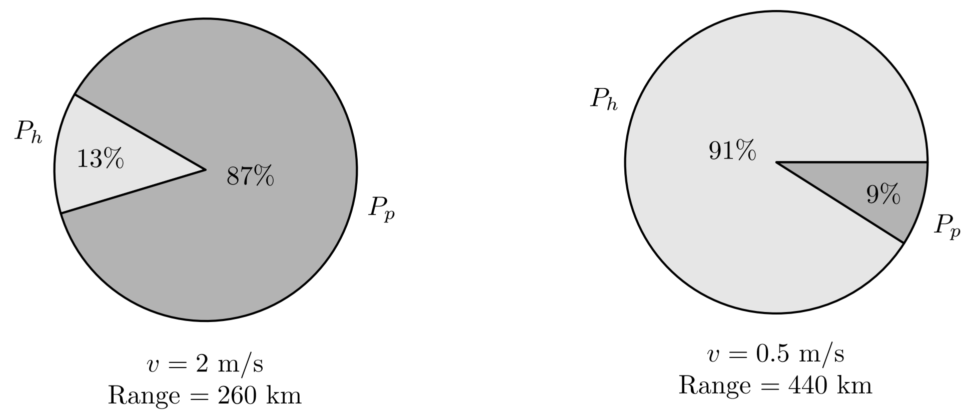


Figure 7.1: Power consumption due to hotel load and propulsive load for an AUV at different operating speeds.

the AUV. Table 7.1 provides parameter values for an AUV developed at Virginia Tech. The AUV has a maximum range of approximately 260 km when operating at 2 m/s and a maximum range of about 440 km when operating at 0.5 m/s. Since the power required to overcome drag depends on the cube of operating speed, a small change in the vehicle's operating speed can have a large effect on the propulsive load. As shown in Figure 7.1, the hotel load accounts for 13% of the overall power consumption at 2 m/s and 91% of overall power consumption at 0.5 m/s. Thus, for long duration missions operating at slow speeds, the hotel load becomes the critical component of power consumption.

7.3 Modeling Control Signal Failures

A control system subject to Markovian control signal failures can be cast as a randomly jumping system. The control system experiences a *missed control update* if the control algorithm fails to generate an updated control signal before the next sampling time. Suppose the open-loop plant is expressed

$$x_p(k+1) = A_p x_p(k) + \begin{bmatrix} B_{p1} & B_{p2} \end{bmatrix} \begin{bmatrix} w(k) \\ u(k) \end{bmatrix} \tag{7.1a}$$

$$\begin{bmatrix} z(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} C_{p1} \\ C_{p2} \end{bmatrix} x_p(k) + \begin{bmatrix} D_{p11} & D_{p12} \\ D_{p21} & D_{p22} \end{bmatrix} \begin{bmatrix} w(k) \\ u(k) \end{bmatrix} \tag{7.1b}$$

where $x_p(k)$ is the plant state, $u(k)$ is the control input to the plant, $w(k)$ is a disturbance vector, $z(k)$ is an error output to be kept small, and $y(k)$ is the output measurement provided to the controller. The failure-prone dynamic output feedback controller can be expressed

$$x_c(k+1) = \begin{cases} A_c x_c(k) + B_c y(k) & : \theta(k) = 2 \\ x_c(k) & : \theta(k) = 1 \end{cases} \tag{7.2a}$$

$$u(k) = \begin{cases} C_c x_c(k) + D_c y(k) & : \theta(k) = 2 \\ u(k-1) & : \theta(k) = 1 \end{cases} \tag{7.2b}$$

where $x_c(k)$ is the controller state, $\theta(k) = 1$ indicates a missed control update, and $\theta(k) = 2$ indicates a successful control update. Define $x(k) = [x_p^{\mathsf{T}}(k)\ x_c^{\mathsf{T}}(k)\ u^{\mathsf{T}}(k-1)]^{\mathsf{T}}$. If $\theta(k) = 1$ then

$$\begin{aligned} x(k+1) &= A(1)x(k) + B(1)w(k) \\ z(k) &= C(1)x(k) + D(1)w(k) \end{aligned}$$

where

$$A(1) = \begin{bmatrix} A_p & 0 & B_{p2} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad B(1) = \begin{bmatrix} B_{p1} \\ 0 \\ 0 \end{bmatrix}, \quad C(1) = \begin{bmatrix} C_{p1} & 0 & D_{p12} \end{bmatrix}, \quad D(1) = \begin{bmatrix} D_{p11} \end{bmatrix}.$$

If $\theta(k) = 2$, it is assumed that the feedback interconnection is well-posed (e.g., see [15, Ch. 5]) so that the matrix $I - D_{p22}D_c$ is invertible. By Lemmas 2.32 and 2.33, if $\theta(k) = 2$ then

$$\begin{bmatrix} u(k) \\ y(k) \end{bmatrix} = \begin{bmatrix} D_c H C_{p2} & (I + D_c H D_{p22})C_c \\ H C_{p2} & D_{p22}(I + D_c H D_{p22})C_c \end{bmatrix} \begin{bmatrix} x_p(k) \\ x_c(k) \end{bmatrix} + \begin{bmatrix} D_c H D_{p21} \\ H D_{p21} \end{bmatrix} w(k) \tag{7.3}$$

where $H = (I - D_{p22}D_c)^{-1}$. If $\theta(k) = 2$, the identity in (7.3) may be used to show (e.g., see [26, Lem. 4.1.2])

$$\begin{aligned} x(k+1) &= A(2)x(k) + B(2)w(k) \\ z(k) &= C(2)x(k) + D(2)w(k) \end{aligned}$$

where

$$A(2) = \begin{bmatrix} A_p + B_{p2}D_cHC_{p2} & B_{p2}(I + D_cHD_{p22})C_c & 0 \\ B_cHC_{p2} & A_c + B_cHD_{p22}C_c & 0 \\ D_cHC_{p2} & (I + D_cHD_{p22})C_c & 0 \end{bmatrix}, \quad B(2) = \begin{bmatrix} B_{p1} + B_{p2}D_cHD_{p21} \\ B_cHD_{p21} \\ D_cHD_{p21} \end{bmatrix}$$

$$C(2) = \begin{bmatrix} C_{p1} + D_{p12}D_cHC_{p2} & D_{p12}(I + D_cHD_{p22})C_c & 0 \end{bmatrix}, \; D(2) = \begin{bmatrix} D_{p11} + D_{p12}D_cHD_{p21} \end{bmatrix}.$$

Thus, the closed-loop system subject to probabilistic control signal failures can be expressed

$$\begin{bmatrix} x(k+1) \\ z(k) \end{bmatrix} = \begin{bmatrix} A(\theta(k)) & B(\theta(k)) \\ C(\theta(k)) & D(\theta(k)) \end{bmatrix} \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \tag{7.4}$$

where $\theta(k) = 1$ indicates a missed control update, and $\theta(k) = 2$ indicates a successful control update.

7.4 Energy-Aware Implementation for an AUV

An energy-aware control implementation for the Virginia Tech Self-Mooring AUV [44] is now analyzed. The results of Chapters 4 and 5 are used to obtain formal guarantees of stability and disturbance attenuation. The dynamic model for the AUV, found in McCarter et al. [44], was developed using the system identification process described in [48]. An inner-loop $\mathcal{H}_\infty$ attitude controller was designed in [44] using the same weighting filters for control design via loop shaping as in [49]. The plant and dynamic output feedback controller are discretized at a sampling rate of 10 Hz. In this example, it is assumed that the sequence of control signal updates may be modeled by an independent sequence of random variables. As in Section 7.3, the closed-loop system subject to probabilistic control signal failures can be described by (7.4) where $A(i) \in \mathbb{R}^{58\times 58}$, $B(i) \in \mathbb{R}^{58\times 14}$, $C(i) \in \mathbb{R}^{14\times 58}$, and $D(i) \in \mathbb{R}^{14\times 14}$ for $i \in \mathcal{N} = \{1, 2\}$.

7.4.1 Fixed probability of a missed control update

If the probability of a missed control update, $\mathbf{P}\{\theta(k) = 1\}$, is constant (and hence P is constant), a bisection search and Proposition 5.8 can be used to approximate $\|(\mathcal{G}, P, p(0))\|_\infty$ with arbitrary precision. Fig. 7.2 plots the ℓ_e^2-induced norm of the time-homogeneous independent jump linear system $(\mathcal{G}, P, p(0))$ versus the probability of a missed control update. For $\mathbf{P}\{\theta(k) = 1\} > 0.45$, $(\mathcal{G}, P, p(0))$ is *not* exponentially mean square stable (the LMI in Proposition 4.6 becomes infeasible), and the ℓ_e^2-induced norm is unbounded.

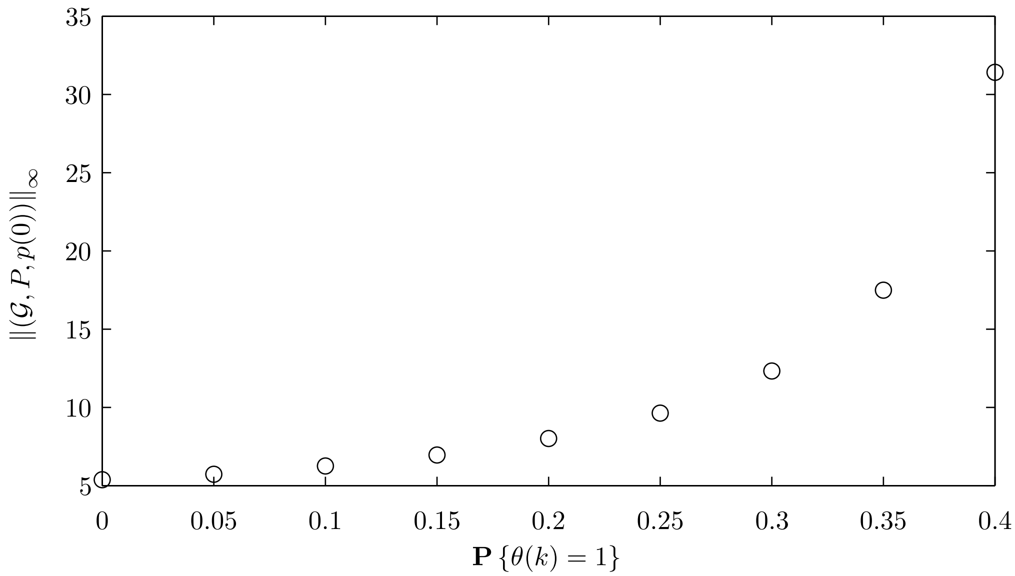


Figure 7.2: The approximate ℓ_e^2-induced norm of the time-homogeneous independent jump linear system versus the probability of a missed control update.

Table 7.2: Results of applying Theorem 5.46.

M	γ_M
0	8.9
1	8.6
2	8.4

7.4.2 A priori unknown time-varying probability of a missed control update

Suppose that the probability of a missed control update varies with time in a finite set, $\mathbf{P}\{\theta(k) = 1\} \in \{0.01, 0.15, 0.20\}$. Thus, define

$$\Pi(1) = \begin{bmatrix} 0.01 & 0.99 \\ 0.01 & 0.99 \end{bmatrix}, \quad \Pi(2) = \begin{bmatrix} 0.15 & 0.85 \\ 0.15 & 0.85 \end{bmatrix}, \quad \Pi(3) = \begin{bmatrix} 0.20 & 0.80 \\ 0.20 & 0.80 \end{bmatrix}$$

and let $\Psi = \mathcal{J}^\infty$. Let the initial distribution $p(0)$ be arbitrary. The uniform induced norm $\|(\mathcal{G}, \Pi, \Psi, p(0))\|_\infty$ is approximated using Theorem 5.46 and Remark 5.30. For each M, Table 7.2 shows the smallest γ_M such that the LMIs in Theorem 5.29 are feasible for the system $(\mathcal{G}_M, \Pi, \Psi, p(0))$ where

$$\mathcal{G}_M = \{(A(1), B(1), (1/\gamma_M)C(1), (1/\gamma_M)D(1)), (A(2), B(2), (1/\gamma_M)C(2), (1/\gamma_M)D(2))\}.$$

As discussed in Remark 5.30, $\gamma_M \to \|(\mathcal{G}, \Pi, \Psi, p(0))\|_\infty$ as $M \to \infty$. For $M \geq 3$, the semidefinite programming problem associated with the LMIs in Theorem 5.29 becomes very large due to the large state dimension

($n = 58$) in this example. Let $\psi_s \equiv s$, $s = 1, 2, 3$, be the constant switching sequences in Ψ. From Fig. 7.2,

$$\|(\mathcal{G}, \Pi \circ \psi_1, p(0))\|_\infty = 5.5, \quad \|(\mathcal{G}, \Pi \circ \psi_2, p(0))\|_\infty = 7.0, \quad \|(\mathcal{G}, \Pi \circ \psi_3, p(0))\|_\infty = 8.0.$$

Since $\psi_s \in \Psi$ for $s = 1, 2, 3$, $\|(\mathcal{G}, \Pi, \Psi, p(0))\|_\infty \geq 8.0$. On the other hand, Table 7.2 indicates that $\|(\mathcal{G}, \Pi, \Psi, p(0))\|_\infty \leq 8.4$. Thus, Theorem 5.46 ensures that the closed-loop system remains stable regardless of the time-variation of the probability of a missed control update and provides an upper bound on the worst-case performance that could be experienced due to switching of the probability of a missed control update. These formal guarantees are important since switching of the transition probability matrices can cause a system to become unstable (see Remark 4.9), or can cause a system to lose the contractiveness property (see Remark 5.12).

Chapter 8

Conclusions

Despite finding application in many areas, Markov jump linear systems with time-varying, possibly unknown, transition probabilities lack tools that are of practical use for assessing stability and disturbance attenuation, and for synthesizing optimal controllers. Nonconservative analysis and control synthesis results that are of practical use have been developed in this dissertation for switched Markov jump linear systems. As a special case, these results may also be applied to Markov jump linear systems with known transition probabilities that vary in a finite set.

In proving the analysis results of Chapters 4 and 5, a key observation was that a stochastically stable and contractive system admits Lyapunov and storage functions with finite dependence on the future transition probabilities of the Markov chain. This observation led to analysis criteria for stability and disturbance attenuation in terms of a set of finite-dimensional linear matrix inequalities. If the switching set Ψ is generated via a directed graph, Theorem 4.17 shows that—for purposes of stability—one need only consider switching sequences generated by the strongly connected components of the directed graph. When the Markov chain is an independent sequence of random variables, the computationally simpler LMI criteria found in Sections 4.5 and 5.5 for stability and disturbance attenuation, respectively, may be used.

A few academic examples were considered in Chapters 4 and 5. The example found in Remark 4.9 illustrates that time-variation of the transition probabilities can cause instability in the mean square. The example in Section 4.4.1 shows that stability may sometimes be gained by restricting the switching set Ψ. The example found in Remark 5.12 of Section 5.4 shows that a single switch of the Markov chain transition probabilities may cause a system to lose the contractiveness property. Finally, the example found in Section 5.4.1 shows that the constituent systems of a switched Markov jump linear system, considered as linear time-invariant systems, need not be contractive in order for the switched Markov jump linear system to be contractive.

Chapter 6 considers control synthesis for a switched Markov jump linear system. Stability and disturbance attenuation for the closed-loop system may be characterized by Lyapunov and storage function criteria and stated in terms of a set of finite-dimensional matrix inequalities. These matrix inequalities are *not* LMIs due to the presence of the unknown controller parameter matrices in addition to the standard unknown matrix-valued function corresponding to the Lyapunov or storage function. The controller parameter matrices can be eliminated from the feasibility problem by using the Projection Lemma (Lemma 2.41). The resulting matrix inequalities may be equivalently represented as LMIs after a few manipulations and substitutions. After solving the resulting LMIs, a controller can be constructed by solving a second LMI problem, or by using the formulas of Lemma 2.41. If the transition probabilities of the Markov chain are known and vary in a finite set, the results of Chapter 6 provide construction of a controller that depends on the future transition probabilities. Alternatively, a controller that depends only on the present and past transition probabilities may be constructed in two special cases. In both of these cases, the critical property required is that the Markov chain running in reverse-time is also characterized by finitely-many transition probability matrices.

A controller for the system in the example of Remark 5.12 is constructed in Section 6.5.1 such that the closed-loop system is stable and contractive. A mass-spring-damper system where the sensor is connected to the controller via a network is considered in Section 6.5.2. Two different controllers are constructed, and the trade-off between controller complexity and disturbance attenuation is discussed.

In Chapter 7, an energy-saving strategy for an autonomous underwater vehicle is cast as a switched Markov jump linear system, and the analysis results of Chapters 4 and 5 are applied. In this case, the stochastic nature of the system is due to implementation details of the control law, as opposed to wireless communications.

Future research may focus on weakening or removing the assumptions required for finite-past-dependent output feedback controllers in Section 6.4. Another future direction may consider the case when the sequence P of transition probability matrices varies in a convex polytope with finitely-many vertices, as opposed to a finite set.

Appendix A

Proofs

Proof of Proposition 4.5. Suppose there exists a function X that satisfies (4.8). Define the stochastic Lyapunov function $V(k,y) = y^{\mathsf{T}} X(k) y$ and let $w \equiv 0$.

$$\begin{aligned}
&\mathbf{E}\left[V(k,x(k)) - V(k-1,x(k-1)) \mid x(k-1) = y\right] \\
&\quad = y^{\mathsf{T}} \left(\mathbf{E}\left[A^{\mathsf{T}}(\theta(k-1)) X(k) A(\theta(k-1))\right] - X(k-1)\right) y \\
&\quad \leq -\nu y^{\mathsf{T}} y \qquad\qquad (A.1)
\end{aligned}$$

where (A.1) follows from (4.8b).

$$\mathbf{E}\left[V(k,x(k)) \mid x(k-1) = y\right] \leq (1 - \nu/\rho) V(k-1,y) \tag{A.2}$$

where (A.2) follows from (A.1) and (4.8a). Let $\lambda = 1 - \nu/\rho$. Then $0 \leq \lambda < 1$ by (4.8).

$$\begin{aligned}
\mathbf{E}\left[V(k,x(k)) \mid x(k-2) = y\right] &= \mathbf{E}\left[\mathbf{E}\left[V(k,x(k)) \mid x(k-1)\right] \mid x(k-2) = y\right] && (A.3) \\
&\leq \lambda \mathbf{E}\left[V(k-1,x(k-1)) \mid x(k-2) = y\right] && (A.4) \\
&\leq \lambda^2 V(k-2,y) && (A.5)
\end{aligned}$$

where (A.3) follows from the law of iterated expectations, (A.4) results after application of (A.2), and (A.5) follows from a second application of (A.2). In general,

$$\begin{aligned}
y^{\mathsf{T}} \mathbf{E}\left[\Phi^{\mathsf{T}}(k,j)\Phi(k,j)\right] y &\leq (1/\eta) y^{\mathsf{T}} \mathbf{E}\left[\Phi^{\mathsf{T}}(k,j) X(k) \Phi(k,j)\right] y && (A.6) \\
&= (1/\eta) \mathbf{E}\left[V(k,x(k)) \mid x(j) = y\right] && (A.7)
\end{aligned}$$

$$\leq (1/\eta)\lambda^{k-j} V(j, y) \tag{A.8}$$
$$\leq (\rho/\eta)\lambda^{k-j} y^{\mathsf{T}} y$$

where (A.6) follows from (4.8a), (A.7) follows from the independence hypothesis, and (A.8) follows by iteration as in (A.4) and (A.5). Since y was arbitrary, $(\mathcal{G}, P, p(0))$ is exponentially mean square stable.

Conversely, suppose $(\mathcal{G}, P, p(0))$ is exponentially mean square stable. Let c and λ be as in (4.5) and define

$$X(k) = \sum_{j=k}^{\infty} \mathbf{E}\left[\Phi^{\mathsf{T}}(j,k)\Phi(j,k)\right].$$

Note $I \leq X(k) \leq c/(1-\lambda)I$.

$$\begin{aligned}
\sum_{j=1}^{N} p_j(k) A^{\mathsf{T}}(j) X(k+1) A(j) &= \sum_{j=1}^{N} p_j(k) A^{\mathsf{T}}(j) \sum_{l=k+1}^{\infty} \mathbf{E}\left[\Phi^{\mathsf{T}}(l,k+1)\Phi(l,k+1)\right] A(j) \\
&= \sum_{l=k+1}^{\infty} \mathbf{E}\left[A^{\mathsf{T}}(\theta(k))\Phi^{\mathsf{T}}(l,k+1)\Phi(l,k+1)A(\theta(k))\right] \qquad \text{(A.9)} \\
&= X(k) - I \qquad \text{(A.10)}
\end{aligned}$$

where (A.9) follows by interchanging the order of summation and applying the independence hypothesis. □

Proof of Proposition 5.7. Suppose there exist $\eta, \rho, \nu > 0$ and a function $X : \mathbb{N}_0 \mapsto \mathbb{S}_n^+$ such that (5.5) holds. Without loss of generality, suppose $0 < \nu < 1$. Note that the upper left block of (5.5) implies (4.8b) so exponential mean square stability of $(\mathcal{G}, P, p(0))$ follows from Proposition 4.5. Now

$$\begin{aligned}
&\mathbf{E}\left[z^{\mathsf{T}}(k)z(k) - w^{\mathsf{T}}(k)w(k) + x^{\mathsf{T}}(k+1)X(k+1)x(k+1) - x^{\mathsf{T}}(k)X(k)x(k)\right] \\
&\quad = \mathbf{E}\left[\begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^{\mathsf{T}} \left(\mathcal{M}(\theta(k), X(k+1)) - \begin{bmatrix} X(k) & 0 \\ 0 & 0 \end{bmatrix}\right) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}\right] \qquad \text{(A.11)} \\
&\quad = \mathbf{E}\left[\begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^{\mathsf{T}} \left(\begin{bmatrix} A(\theta(k)) & B(\theta(k)) \\ C(\theta(k)) & D(\theta(k)) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} X(k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(\theta(k)) & B(\theta(k)) \\ C(\theta(k)) & D(\theta(k)) \end{bmatrix}\right.\right. \\
&\qquad \left.\left. - \begin{bmatrix} X(k) & 0 \\ 0 & I \end{bmatrix}\right) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}\right] \qquad \text{(A.12)}
\end{aligned}$$

$$
\begin{aligned}
&= \mathbf{E}\left[\mathbf{E}\left[\begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^{\mathsf{T}} \left(\begin{bmatrix} A(\theta(k)) & B(\theta(k)) \\ C(\theta(k)) & D(\theta(k)) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} X(k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(\theta(k)) & B(\theta(k)) \\ C(\theta(k)) & D(\theta(k)) \end{bmatrix}\right.\right.\right. \\
&\qquad \left.\left.\left. - \begin{bmatrix} X(k) & 0 \\ 0 & I \end{bmatrix}\right) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix} \mid x(k), w(k) \right]\right] \\
&= \mathbf{E}\left[\begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^{\mathsf{T}} \sum_{j=1}^{N} p_j(k) \left(\begin{bmatrix} A(j) & B(j) \\ C(j) & D(j) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} X(k+1) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A(j) & B(j) \\ C(j) & D(j) \end{bmatrix}\right.\right. \\
&\qquad \left.\left. - \begin{bmatrix} X(k) & 0 \\ 0 & I \end{bmatrix}\right) \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}\right] \qquad \text{(A.13)} \\
&\leq -\nu \mathbf{E}\left[x^{\mathsf{T}}(k)x(k) + w^{\mathsf{T}}(k)w(k)\right] \qquad \text{(A.14)} \\
&\leq -\nu \mathbf{E}\left[w^{\mathsf{T}}(k)w(k)\right]
\end{aligned}
$$

where equality (A.11) follows from Lemma 5.18; equality (A.12) follows from Lemma 5.16; equality (A.13) follows from the independence hypothesis on θ; and, inequality (A.14) follows from (5.5). Thus

$$
\begin{aligned}
&\mathbf{E}\left[z^{\mathsf{T}}(k)z(k) + x^{\mathsf{T}}(k+1)X(k+1)x(k+1) - x^{\mathsf{T}}(k)X(k)x(k)\right] \\
&\quad \leq (1-\nu)\mathbf{E}\left[w^{\mathsf{T}}(k)w(k)\right]. \qquad \text{(A.15)}
\end{aligned}
$$

Let $x(0) = 0$, sum both sides of inequality (A.15), and use the positive-definiteness of $X(k)$ to get

$$\|z\|_{2,e}^2 \leq \gamma^2 \|w\|_{2,e}^2$$

where $\gamma = \sqrt{1-\nu}$.

Conversely, suppose $(\mathcal{G}, P, p(0))$ is exponentially mean square stable and mean square strictly contractive. Fix $t \in \mathbb{N}_0$ and consider the independent jump linear system defined by the difference equation

$$\begin{bmatrix} x_t(k+1) \\ z_t(k) \end{bmatrix} = \begin{bmatrix} A(\theta_t(k)) & B(\theta_t(k)) \\ C(\theta_t(k)) & D(\theta_t(k)) \end{bmatrix} \begin{bmatrix} x_t(k) \\ w_t(k) \end{bmatrix}$$

where $\theta_t(k) = \theta(t+k)$ for $k \in \mathbb{N}_0$ and $w_t \in \ell_e^2$. Note that this system may be denoted by $(\mathcal{G}, P_t, p_t(0))$ where

$P_t(k) = P(t+k)$ for $k \in \mathbb{N}$ and $p_t(0)$ is the distribution $\mathbf{P}\{\theta(t)\}$. Define the random state transition matrix

$$\Phi_t(k,j) = \begin{cases} A(\theta_t(k-1))A(\theta_t(k-2))\cdots A(\theta_t(j)) & : \; k > j \\ I & : \; k = j \end{cases}$$

Note that $(\mathcal{G}, P_t, p_t(0))$ is exponentially mean square stable since

$$\begin{aligned} \mathbf{E}\left[\Phi_t^{\mathsf{T}}(k,j)\Phi_t(k,j)\right] &= \mathbf{E}\left[\Phi^{\mathsf{T}}(t+k,t+j)\Phi(t+k,t+j)\right] \\ &\le c\lambda^{t+k-(t+j)} I. \end{aligned}$$

Now let $x_t(0) = 0$, $x(0) = 0$, and let $w_t \in \ell_e^2$ be arbitrary. Define

$$w(k) = \begin{cases} 0 & : \; k < t \\ w_t(k-t) & : \; k \ge t \end{cases}. \tag{A.16}$$

Note that $w_t \in \ell_e^2$ implies $w \in \ell_e^2$ and $\|w\|_{2,e} = \|w_t\|_{2,e}$. Now

$$\begin{aligned} z_t(k) &= C(\theta_t(k)) \sum_{j=0}^{k-1} \Phi_t(k,j+1)B(\theta_t(j))w_t(j) + D(\theta_t(k))w_t(k) && \text{(A.17)} \\ &= C(\theta(t+k)) \sum_{j=0}^{k-1} \Phi(t+k,t+j+1)B(\theta(t+j))w(t+j) + D(\theta(t+k))w(t+k) \\ &= C(\theta(t+k)) \sum_{l=t}^{t+k-1} \Phi(t+k,l+1)B(\theta(l))w(l) + D(\theta(t+k))w(t+k) \\ &= C(\theta(t+k)) \sum_{l=0}^{t+k-1} \Phi(t+k,l+1)B(\theta(l))w(l) + D(\theta(t+k))w(t+k) \\ &= z(t+k) && \text{(A.18)} \end{aligned}$$

where (A.17) follows from (3.3). Note that z in (A.18) is the solution to (3.1) for the particular w in (A.16) and $x(0) = 0$. Also note that $z(k) = 0$ for $0 \le k \le t-1$. Now

$$\begin{aligned} \|z_t\|_{2,e}^2 &= \sum_{k=0}^{\infty} \mathbf{E}\left[\|z_t(k)\|^2\right] \\ &= \sum_{k=0}^{\infty} \mathbf{E}\left[\|z(t+k)\|^2\right] \\ &= \sum_{l=0}^{t-1} \mathbf{E}\left[\|z(l)\|^2\right] + \sum_{l=t}^{\infty} \mathbf{E}\left[\|z(l)\|^2\right] \\ &= \|z\|_{2,e}^2 \end{aligned}$$

$$\leq \gamma^2 \|w\|_{2,e}^2 \tag{A.19}$$
$$= \gamma^2 \|w_t\|_{2,e}^2$$

where (A.19) follows from contractiveness of $(\mathcal{G}, P, p(0))$. Since $w_t \in \ell_e^2$ was arbitrary, $(\mathcal{G}, P_t, p_t(0))$ is mean square strictly contractive. With this fact, one may proceed as in Section 5.5 to construct a solution to (5.5) using the finite-horizon Riccati difference equations

$$\begin{aligned} X(k,T,\epsilon) &= \sum_{j=1}^{N} p_j(k)\mathcal{L}(j, X(k+1,T,\epsilon)) + \sum_{j=1}^{N} p_j(k)\mathcal{R}^{\mathsf{T}}(j, X(k+1,T,\epsilon)) \\ &\quad \times \left(\sum_{j=1}^{N} p_j(k)\mathcal{W}(j, X(k+1,T,\epsilon))\right)^{-1} \sum_{j=1}^{N} p_j(k)\mathcal{R}(j, X(k+1,T,\epsilon)) + \epsilon I \\ X(T+1,T,\epsilon) &= 0. \end{aligned}$$

□

Bibliography

[1] S. Aberkane, "Bounded real lemma for nonhomogeneous Markovian jump linear systems," *IEEE Trans. Autom. Control*, vol. 58, no. 3, pp. 797–801, Mar. 2013.

[2] S. Axler, *Linear Algebra Done Right*, 2nd ed., ser. Undergraduate Texts in Mathematics. Springer, 1997.

[3] R. Bhatia, *Positive Definite Matrices.* Princeton University Press, 2007.

[4] P. Billingsley, *Probability and Measure*, 3rd ed., ser. Wiley Series in Prob. Math. Stat. John Wiley & Sons, 1995.

[5] P. Bolzern, P. Colaneri, and G. De Nicolao, "Markov jump linear systems with switching transition rates: mean square stability with dwell-time," *Automatica*, vol. 46, no. 6, pp. 1081–1088, 2010.

[6] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications.* Elsevier Science Publishing Co., 1976.

[7] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, ser. SIAM Stud. Appl. Math. SIAM, 1994, vol. 15.

[8] S. Boyd and L. Vandenberghe, *Convex Optimization.* Cambridge University Press, 2004.

[9] H. J. Chizeck, "Fault tolerant optimal control," Sc.D. dissertation, Massachusetts Inst. Technol., Cambridge, MA, 1982.

[10] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, *Discrete-Time Markov Jump Linear Systems.* Springer, 2005.

[11] O. L. Costa and M. D. Fragoso, "Stability results for discrete-time linear systems with Markovian jumping parameters," *J. Math. Anal. Appl.*, vol. 179, no. 1, pp. 154–178, 1993.

[12] J. B. do Val and T. Başar, "Receding horizon control of jump linear systems and a macroeconomic policy problem," *J. Econ. Dyn. Control*, vol. 23, no. 8, pp. 1099–1131, 1999.

[13] R. Douc, E. Moulines, and J. S. Rosenthal, "Quantitative bounds on convergence of time-inhomogeneous Markov chains," *Ann. Appl. Probab.*, vol. 14, no. 4, pp. 1643–1665, 2004.

[14] G. E. Dullerud and S. Lall, "A new approach for analysis and synthesis of time-varying systems," *IEEE Trans. Autom. Control*, vol. 44, no. 8, pp. 1486–1497, Aug. 1999.

[15] G. E. Dullerud and F. Paganini, *A course in robust control theory*, ser. Texts Appl. Math. Springer New York, 2000, vol. 36.

[16] E. O. Elliott, "Estimates of error rates for codes on burst-noise channels," *Bell Syst. Tech. J.*, vol. 42, no. 5, pp. 1977–1997, 1963.

[17] Y. Fang and K. A. Loparo, "Stochastic stability of jump linear systems," *IEEE Trans. Autom. Control*, vol. 47, no. 7, pp. 1204–1208, 2002.

[18] V. P. Finsler, "Über das vorkommen definiter und semidefiniter formen in scharen quadratischer formen," *Commentarii Mathematici Helvetici*, vol. 9, no. 1, pp. 188–192, 1936.

[19] G. B. Folland, *Real Analysis: Modern Techniques and Their Applications*, 2nd ed., ser. Pure Appl. Math. John Wiley & Sons, 1999.

[20] P. Gahinet and P. Apkarian, "A linear matrix inequality approach to H_∞ control," *Int. J. Robust Nonlin.*, vol. 4, no. 4, pp. 421–448, 1994.

[21] H. Ge, D.-Q. Jiang, and M. Qian, "Reversibility and entropy production of inhomogeneous Markov chains," *J. Appl. Prob.*, vol. 43, no. 4, pp. 1028–1043, 2006.

[22] J. C. Geromel, A. P. C. Gonçalves, and A. R. Fioravanti, "Dynamic output feedback control of discrete-time Markov jump linear systems through linear matrix inequalities," *SIAM J. Control Optim.*, vol. 48, no. 2, pp. 573–593, 2009.

[23] E. N. Gilbert, "Capacity of a burst-noise channel," *Bell Syst. Tech. J.*, vol. 39, no. 5, pp. 1253–1265, 1960.

[24] K. Glover and J. C. Doyle, "State-space formulae for all stabilizing controllers that satisfy an h_∞-norm bound and relations to risk sensitivity," *Syst. Control Lett.*, vol. 11, no. 3, pp. 167–172, 1988.

[25] A. Goldsmith, *Wireless Communications.* Cambridge University Press, 2005.

[26] M. Green and D. J. Limebeer, *Linear robust control.* Prentice-Hall, 1995.

[27] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, “A survey of recent results in networked control systems,” *Proc. IEEE*, vol. 95, no. 1, pp. 138–162, 2007.

[28] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd ed. Cambridge University Press, 2013.

[29] K. Hrbacek and T. Jech, *Introduction to Set Theory*, 3rd ed. Marcel Dekker, 1999.

[30] D. L. Isaacson and R. W. Madsen, *Markov Chains, Theory and Applications.* John Wiley & Sons, 1976.

[31] T. Iwasaki and R. E. Skelton, “All controllers for the general $\mathcal{H}_\infty$ control problem: LMI existence conditions and state space formulas,” *Automatica*, vol. 30, no. 8, pp. 1307–1317, 1994.

[32] Y. Ji and H. J. Chizeck, “Jump linear quadratic Gaussian control: steady-state solution and testable conditions,” *Control Theory Adv. Technol.*, vol. 6, no. 3, pp. 289–319, Sep. 1990.

[33] Y. Ji, H. J. Chizeck, X. Feng, and K. A. Loparo, “Stability and control of discrete-time jump linear systems,” *Control Theory Adv. Technol.*, vol. 7, no. 2, pp. 247–270, Jun. 1991.

[34] H. Kopetz, *Real-Time Systems: Design Principles for Distributed Embedded Applications*, 2nd ed. Springer, 2011.

[35] E. Kreyszig, *Introductory Functional Analysis with Applications.* John Wiley & Sons, 1978.

[36] R. Krtolica, Ü. Özgüner, H. Chan, H. Göktas, J. Winkelman, and M. Liubakka, “Stability of linear feedback systems with random communication delays,” *Int. J. Control*, vol. 59, no. 4, pp. 925–953, 1994.

[37] J.-W. Lee and G. E. Dullerud, “Optimal disturbance attenuation for discrete-time switched and Markovian jump linear systems,” *SIAM J. Control Optim.*, vol. 45, no. 4, pp. 1329–1358, 2006.

[38] ——, “Uniform stabilization of discrete-time switched and Markovian jump linear systems,” *Automatica*, vol. 42, no. 2, pp. 205–218, 2006.

[39] A. Leon-Garcia, *Probability, Statistics, and Random Processes for Electrical Engineering*, 3rd ed. Pearson Education, 2008.

[40] D. Liberzon, *Switching in systems and control.* Birkhäuser Boston, 2003.

[41] D. Limebeer, M. Green, and D. Walker, "Discrete time H^∞ control," in *Proc. 28th IEEE Conf. Decision and Control.* Tampa, FL, USA: IEEE, 1989, pp. 392–396.

[42] J. Löfberg, "YALMIP : A toolbox for modeling and optimization in MATLAB," in *Proc. IEEE Int. Symp. Computer-Aided Control Syst. Des.* Taipei, Taiwan: IEEE, 2004, pp. 284–289. [Online]. Available: http://users.isy.liu.se/johanl/yalmip

[43] C. C. Lutz and D. J. Stilwell, "Energy-aware control: ℓ_2 gain for closed-loop systems implemented with stochastic schedulers," in *Proc. Amer. Control Conf.* Washington, DC, USA: IEEE, 2013, pp. 5313–5319.

[44] B. McCarter, R. Briggs, S. Portner, D. Stilwell, W. Neu, R. Coe, R. Duelley, D. Malley, and J. Mims, "Design and testing of a self-mooring AUV," in *Proc. MTS/IEEE Oceans*, Hampton Roads, VA, USA, Oct. 2012, pp. 1–8.

[45] T. Morozan, "Stabilization of some stochastic discrete-time control systems," *Stoch. Anal. Appl.*, vol. 1, no. 1, pp. 89–116, 1983.

[46] J. R. Norris, *Markov Chains.* Cambridge University Press, 1997.

[47] M. Paolieri, E. Quiñones, F. J. Cazorla, G. Bernat, and M. Valero, "Hardware support for WCET analysis of hard real-time multicore systems," in *Proc. Int. Symp. Comput. Archit.* Austin, TX, USA: ACM, 2009, pp. 57–68.

[48] J. Petrich and D. J. Stilwell, "Model simplification for AUV pitch-axis control design," *Ocean Eng.*, vol. 37, no. 7, pp. 638–651, 2010.

[49] ——, "Robust control for an autonomous underwater vehicle that suppresses pitch and yaw coupling," *Ocean Eng.*, vol. 38, no. 1, pp. 197–204, 2011.

[50] N. J. Ploplys, P. A. Kawka, and A. G. Alleyne, "Closed-loop control over wireless networks," *IEEE Control Syst. Mag.*, vol. 24, no. 3, pp. 58–71, 2004.

[51] M. Reed and B. Simon, *Methods of Modern Mathematical Physics: Vol. I Functional Analysis.* Academic Press, 1980.

[52] K. H. Rosen, *Discrete mathematics and its applications*, 5th ed. McGraw-Hill, 2003.

[53] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed. McGraw-Hill, 1976.

[54] W. J. Rugh, *Linear system theory.* Prentice-Hall, 1996.

[55] L. Saloff-Coste and J. Zúñiga, "Convergence of some time inhomogeneous Markov chains via spectral techniques," *Stoch. Proc. Appl.*, vol. 117, no. 8, pp. 961–979, 2007.

[56] ——, "Merging for time inhomogeneous finite Markov chains, part I: singular values and stability," *Electron. J. Probab.*, vol. 14, pp. 1456–1494, 2009.

[57] ——, "Merging for inhomogeneous finite Markov chains, part II: Nash and log-Sobolev inequalities," *Ann. Prob.*, vol. 39, no. 3, pp. 1161–1203, 2011.

[58] P. Seiler, "Coordinated control of unmanned aerial vehicles," Ph.D. dissertation, Univ. California, Berkeley, CA, 2001.

[59] P. Seiler and R. Sengupta, "A bounded real lemma for jump systems," *IEEE Trans. Autom. Control*, vol. 48, no. 9, pp. 1651–1654, Sep. 2003.

[60] ——, "Analysis of communication losses in vehicle control problems," in *Proc. Amer. Control Conf.* Arlington, VA, USA: IEEE, 2001, pp. 1491–1496.

[61] ——, "An H_∞ approach to networked control," *IEEE Trans. Autom. Control*, vol. 50, no. 3, pp. 356–364, 2005.

[62] R. E. Skelton, T. Iwasaki, and K. M. Grigoriadis, *A unified algebraic approach to linear control design.* Taylor & Francis, 1998.

[63] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimization Meth. & Soft.*, vol. 11, no. 1-4, pp. 625–653, 1999.

[64] F. Uhlig, "A recurring theorem about pairs of quadratic forms and extensions: a survey," *Linear Algebra Appl.*, vol. 25, pp. 219–237, 1979.

[65] J. C. Willems, "Dissipative dynamical systems part I: General theory," *Arch. Rational Mech. Anal.*, vol. 45, no. 5, pp. 321–351, 1972.

[66] ——, "Dissipative dynamical systems part II: Linear systems with quadratic supply rates," *Arch. Rational Mech. Anal.*, vol. 45, no. 5, pp. 352–393, 1972.

[67] R. J. Wilson, *Introduction to Graph Theory*, 3rd ed. Longman New York, 1985.

[68] L. Xiao, A. Hassibi, and J. P. How, "Control with random communication delays via a discrete-time jump system approach," in *Proc. Amer. Control Conf.* Chicago, IL, USA: IEEE, 2000, pp. 2199–2204.