ANALYSIS AND FINITE ELEMENT APPROXIMATION OF AN OPTIMAL SHAPE CONTROL PROBLEM FOR THE STEADY–STATE NAVIER–STOKES EQUATIONS

by

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(Abstract)

An optimal shape control problem for the steady–state Navier–Stokes equations is considered from an analytical point of view. We examine a rather specific model problem dealing with 2–dimensional channel flow of incompressible viscous fluid: we wish to determine the shape of a bump on a part of the boundary in order to minimize the energy dissipation.

To formulate the problem in a comprehensive manner, we study some properties of the Navier–Stokes equations. The penalty method is applied to relax the difficulty of dealing with incompressibility in conjunction with domain perturbations and regularity requirements for the solutions. The existence of optimal solutions for the penalized problem is presented.

The computation of the shape gradient and its treatment play a central role in the shape sensitivity analysis. To describe the domain perturbation and to derive the shape gradient, we study the material derivative method and related shape calculus. The shape sensitivity analysis using the material derivative method and Lagrange multiplier technique is presented. The use of Lagrange multiplier techniques,

from which an optimality system is derived, is justified by applying a method from functional analysis.

Finite element discretizations for the domain and discretized description of the problem are given. We study finite element approximations for the weak penalized optimality system. To deal with inhomogeneous essential boundary condition, the framework of a Lagrange multiplier technique is applied. The split formulation decoupling the traction force from the velocity is proposed in conjunction with the penalized optimality system and optimal error estimates are derived.

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CHAPTER I

INTRODUCTION

The optimal shape design problem is to find the shape of an object which is optimal with respect to some specified mechanical or physical criteria. Here, *shape* is the description of a geometrical object in 2- or 3-dimensional Euclidean space, and *optimality* is with respect to some physical or geometrical constraints on the object. In nature, some organic bodies have been observed to change their shapes to adjust to their physical surroundings.

Optimal shape design problems have been widely studied among structural engineers. Their history goes back to the early part of 20th century. In practice, the need to design the shape of certain objects arises in the mechanical construction of structures using elastic or elasto–plastic materials; recent advances in composite and smart materials enables the implementation of optimally designed structural shapes.

The mathematical foundations, however, have matured only in the last two decades in accordance with the development of the theory of distributed parameter systems for partial differential equations and variational formulation techniques. Moreover, its practical implementation and numerical solutions have been accelerated by the rapid development of computer technologies and industrial needs.

Simply put, the optimal shape problem may be regarded as a branch of optimal control theory, which itself is based on the calculus of variations. However, there are some significant differences between the two. Unlike data or value controls such as external body forces, stresses, or boundary conditions in a fixed domain, shape optimization problems require the identification of the domain among a class of domains satisfying certain criteria. The crucial step in shape optimization is the shape sensitivity analyses and from it one often tends to derive the necessary optimality conditions leading to a numerical implementation. Hence the formulation has somewhat different characteristics.

The optimal shape problem can be composed of the following three corelated factors:

(i) $\mathcal{U}_{ad}$, a family of admissible domains Ω equipped with appropriate regularities.

(ii) A system of state equations which describes the physical or mechanical states of the system (usually described by partial differential equations with boundary conditions on each domain Ω in $\mathcal{U}_{ad}$).

(iii) A cost functional $\mathfrak{J}$ which depends on the domain through the solution of the well-posed state equations.

Then, the problem of determining optimal shapes is simply one of finding a domain Ω^* in $\mathcal{U}_{ad}$ so that $\mathfrak{J}(\Omega^*)$ is a minimum among all $\mathfrak{J}(\Omega)$, $\Omega \in \mathcal{U}_{ad}$, and such that the constraint equations are satisfied.

The purpose of this dissertation is to consider some mathematical issues in shape optimization in conjunction with the incompressible Navier–Stokes equations. It has long been believed that the Navier–Stokes equations describe general flows of fluids ranging from gas motions to the lubrication of ball bearings. Hence, optimal shape design problems associated with the Navier–Stokes equations, if settled successfully, have wide valuable, applications in aerodynamics and hydrodynamics such as designs of car hoods, airplane wings, forebody shapes of jet engines, *etc.* However, because of the present state of the study for the Navier–Stokes equations and the lack of mathematical structure that deals with rough shapes, these studies are in their infancy. At present, only a scant literatures are available. The first study toward the optimal shape design problems for the Navier–Stokes equations was attempted by Pironneau[100]. He tried to find a minimum drag profile submerged in a homogeneous, steady, viscous fluid by utilizing optimal control theories for distributed parameter systems due to Lions[90]. Glowinski and Pironneau[61] used a finite difference method in some computational experiments for minimal drag profiles and Prandtl's equations of laminar flows. Koda[85] tried to derive necessary conditions on the similar problem for the time–dependent Navier–Stokes equations. A successful application of optimal shape theory in fluid mechanics can be found in the design of riblets as a minimum drag device by considering a simplified boundary layer approximation of the Navier–Stokes equations (Arumugam *et al.*[6]). For drag reduction in Stokes flow, some rigorous mathematical results for the sensitivity analysis was done by Simon[119]. While the general criterion related to this field is drag reduction, one may also consider the shape related lift control problem. For example, the location of transitional points from laminar to turbulent layer closely depends on the wing shapes (c.f.[23]).

The major difficulties in the study of shape optimization problems associated with the Navier–Stokes equations arise from the following sources:

- Lack of comprehensive understanding of the Navier–Stokes equations.
- Lack of shape sensitivity analyses dealing with rough geometries.
- The adjoint equations induced from the shape sensitivity analysis have no physical meaning and it is usually hard to show the existence of its solution.
- Lack of computational efforts due to its massiveness and complexity.

Throughout this dissertation, we will deal with a rather specific model problem in two–dimensions. However, the approach to the problem will be discussed in general terms; our aim is to formulate the problem in a comprehensive manner. We show the existence of solutions for a drag minimization problem. The shape sensitivity analysis using the material derivative method and the Lagrange multiplier technique is presented. Finally, some aspects of the finite element approximation to the problem will be discussed.

In this chapter, we propose the model problem and present some necessary prerequisites for the mathematical formulations.

1.1. Model Problem

We consider two–dimensional incompressible flow of a viscous fluid passing through a channel having a finite depth (Figure 1).

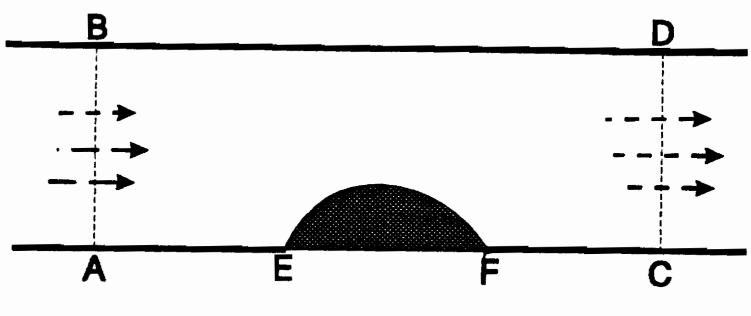

Figure 1: A 2-D flow through a channel with a bump

Let $\mathbf{g}_1$ and $\mathbf{g}_2$ be the preset velocities of the inflow and outflow along sides AB and CD, respectively. Along the bottom we have a bump which is to be determined, represented by the arc $\overset{\frown}{EF}$. One can examine several objectives for determining the shape of the bump, for example, the reduction of drag due to the viscosity or the identification of the velocity identification at a fixed vertical slit downstream of the bump.

Setting of $\mathcal{U}_{ad}$.

We cast the problem into the setting depicted in Figure 2.

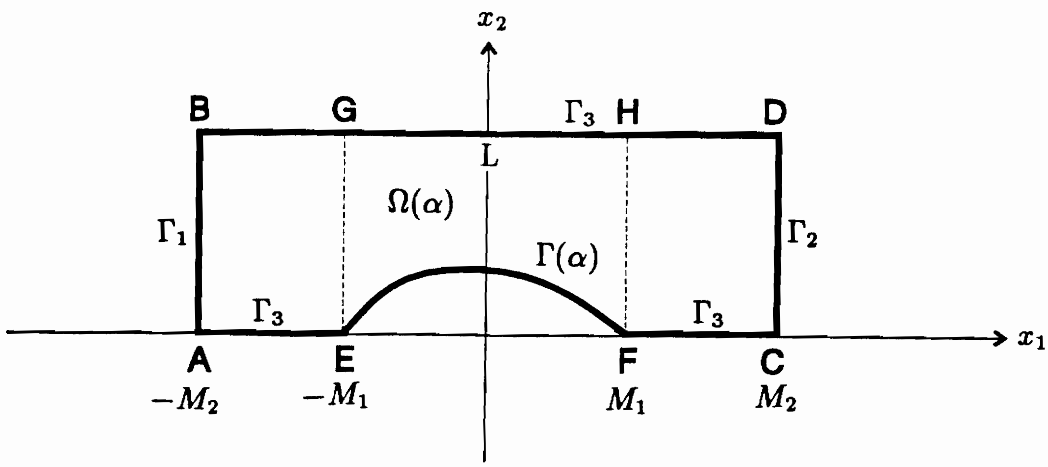

Figure 2: Setting of domain $\Omega(\alpha)$

Let the boundary shape corresponding to the bump is represented by the graph of a curve $\alpha : [-M_1, M_1] \to R$. The domain $\Omega(\alpha)$ is composed of the following parts: two fixed rectangles and a domain with a moving boundary. We assume that the domain $\Omega(\alpha)$ is determined by the shape of the moving boundary $\Gamma(\alpha)$, where

$$\Gamma(\alpha) = \{(x_1, x_2) \in [-M_1, M_1] \times [0, L] \mid x_2 = \alpha(x_1)\}.$$

Let $\Gamma = \partial\Omega(\alpha) = \cup_{i=1}^{3}\Gamma_i \cup \Gamma(\alpha)$, where Γ_1 is the side AB, Γ_2 the side CD and $\Gamma_3 = \Gamma - \Gamma_1 \cup \Gamma_2 \cup \Gamma(\alpha)$. Assume $\Gamma(\alpha) \subset [-M_1, M_1] \times [0, L]$ and that both end points of $\Gamma(\alpha)$ are fixed for all admissible domains. Since the domain $\Omega(\alpha)$ is determined by the shape of $\Gamma(\alpha)$, we may define the admissible family of curves defining $\Gamma(\alpha)$ as follows:

$$\mathcal{U}_{ad} = \{\alpha \in C^{0,1}([-M_1, M_1]) \mid 0 \leq \alpha(x_1) \leq L, |\alpha'(x_1)| \leq \beta, \forall x_1 \in [-M_1, M_1],$$
$$\alpha(-M_1) = \alpha(M_1) = 0\},$$

where the positive constant β is chosen in such a way that $\mathcal{U}_{ad} \neq \emptyset$. We denote the set of Lipschitz continuous functions in $[-M_1, M_1]$ by the symbol $\mathcal{C}^{0,1}([-M_1, M_1])$.

The condition $|\alpha'(x_1)| \leq \beta$ is set to prevent the blow-up of the boundary, *i.e.*, to suppress excessive oscillations of $\Gamma(\alpha)$. (Pironneau [101] provides an example such that: *when the boundaries are allowed to oscillate, the limit of a sequence that minimizes the objective functional may have nothing to do with the suggested problem.*)

Note that these restrictions on $\Omega(\alpha)$ play a central constraint role in the nonlinear program which is an essential part of numerical computations for shape optimization.

Finally, we assume that the graph of α for each $\alpha \in \mathcal{U}_{ad}$ lies in the rectangular region EFGH.

State Equation.

We consider the viscous, incompressible, steady–state Navier–Stokes equations in nondimensional form in $\Omega(\alpha)$ for each $\alpha \in \mathcal{U}_{ad}$.

Let $\mathbf{u} = (u_1, u_2)$ denote the velocity and p the pressure, where $u_i = u_i(x_1, x_2)$ and $p = p(x_1, x_2)$, $(x_1, x_2) \in \Omega(\alpha)$, for $i = 1, 2$. Then, we have

$$-\nu\, \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega(\alpha) \tag{1–1}$$

$$div\, \mathbf{u} = 0 \qquad \text{in } \Omega(\alpha) \tag{1–2}$$

with the Dirichlet boundary conditions

$$\mathbf{u} = \mathbf{g} = \begin{cases} \mathbf{g}_1 & \text{on } \Gamma_1 \\ \mathbf{g}_2 & \text{on } \Gamma_2 \\ \mathbf{0} & \text{on } \Gamma_3 \cup \Gamma(\alpha), \end{cases} \tag{1–3}$$

where $\mathbf{f}$ and $\mathbf{g}_i$, $(i = 1, 2)$ are given functions. Here, Δ, div and ∇ are usual linear differential operators. By the symbol Δ we denote the Laplacian operator in R^2, by

∇ the gradient operator, and by *div* the divergence operator. $\mathbf{f}$ denotes the given external force and ν the kinematic viscosity. In the nondimensional form of the Navier–Stokes equations, ν stands for the reciprocal of the Reynolds number Re.

Cost Functional.

The cost functional (or, *design performance functional* in the terminology of shape optimization) to be minimized is given by

$$\begin{aligned}\mathfrak{J}(\alpha) = J(\Omega(\alpha), \mathbf{u}(\alpha)) &= \nu \int_{\Omega(\alpha)} \nabla\mathbf{u} : \nabla\mathbf{u}\, dx \\ &= \nu \sum_{i,j=1}^{2} \int_{\Omega(\alpha)} (\frac{\partial u_i}{\partial x_j})^2\, dx, \end{aligned} \tag{1–4}$$

where $\mathbf{u}(\alpha)$ is a solution of (1-1)–(1–3) in $\Omega(\alpha)$. This functional represents the rate of the energy dissipation due to deformation. Physically, this term is equivalent to the viscous drag of the flow. (We could also consider other functionals such as the identification of the velocity at a location downstream of the bump.)

We then wish to find $(\alpha^*, \mathbf{u}(\alpha^*), p(\alpha^*))$, where $(\mathbf{u}(\alpha^*), p(\alpha^*))$ is a velocity and pressure pair which is a solution of the state equation over $\Omega(\alpha^*)$ and α^* is a solution of

$$\min_{\alpha \in \mathcal{U}_{ad}} \mathfrak{J}(\alpha) \ . \tag{1–5}$$

Success in solving this problem depends upon a suitable choice of formulation for the Navier–Stokes equations, the appropriate technical setting for the shape sensitivity analysis, and the application of a nonlinear programming method of optimization.

1.2. Preliminaries

In this section, we present some background material. Function spaces, norms, domains, and some variational principles are discussed. For a more detailed exposition, one may refer to Treves[131], Adams[2], Marti[92], Temam[127], and Dautray and Lions[37].

Let Ω be a open subset of R^n. We denote by $\mathcal{C}^m(\Omega)$ the set of m-times continuously differentiable functions on Ω. By $\mathcal{C}_0^m(\Omega)$, we denote the elements of $\mathcal{C}^m(\Omega)$ with compact support in Ω. We note that $\mathcal{C}^\infty(\Omega) = \bigcap_{m\geq 0} \mathcal{C}^m(\Omega)$ and $\mathcal{C}_0^\infty(\Omega) = \bigcap_{m\geq 0} \mathcal{C}_0^m(\Omega)$. The dual space of $\mathcal{C}_0^\infty(\Omega)$ is the space of the distributions $\mathfrak{D}'(\Omega)$. By $< \cdot, \cdot >$, we denote the duality map between test functions and their dual elements.

For Galerkin type variational formulations, we mainly depend on the separable Hilbert spaces $H^s(\Omega)$, for $s \in R$. For $s \in R$, each element of $H^s(\Omega)$ is the restriction of a certain element of $H^s(R^n)$ to Ω. When $\Omega = R^n$, $H^s(R^n)$ can be characterized by

$$H^s(R^n) = \{f \in L^2(R^n) \,|\, (1 + |\xi|)^{s/2}\widehat{f} \in L^2(R^n)\},$$

equipped with a norm $\|f\|_{H^s(R^n)} = \|(1 + |\xi|^2)^{s/2}\widehat{f}\|_{L^2(R^n)}$. Here $\widehat{f}$ denote the Fourier transform (its inverse if $s < 0$) between tempered distributions. Without the help of the Fourier transform, there is another intrinsic characterization of the fractional Sobolev space H^s employing L^2-Hölder continuity. For $s \geq 0$, using Parseval's theorem and Peetre's inequality for the principal symbol, it can be shown that the following is the square of a norm on H^s equivalent to $\|\cdot\|_{H^s}$:

$$\sum_{|\alpha|\leq[s]} \int |D^\alpha f|^2 + \sum_{|\alpha|=[s]} \iint \frac{|D^\alpha f(x) - D^\alpha f(y)|^2}{|x-y|^{n+2(s-[s])}}\,dxdy, \quad \forall x, y \in \Omega,$$

where $[s]$ denote the largest integer $\leq s$. Now, $H^s(\Omega) = \gamma_\Omega(H^s(R^n))$, where γ_Ω denote the restriction mapping of $H^s(R^n)$ to Ω. When $s < s'$, there is a continuous injection

$H^{s'}(\Omega) \subset H^s(\Omega)$ and when $P(D)$ is a differential operator of polynomial type of order $\leq k$, the map $f \mapsto P(D)f$ is a continuous linear map of $H^s(\Omega)$ into $H^{s-k}(\Omega)$.

When m=nonnegative integer, we easily deduce that

$$H^m(\Omega) = \{f \in L^2(\Omega) \,|\, \mathrm{D}^\alpha f \in L^2(\Omega), |\alpha| \leq m\},$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ is the multi-index and $\mathrm{D}^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$. The inner product over $H^m(\Omega)$ is given by

$$(f,g)_m = (f,g)_0 + \sum_{0<|\alpha|\leq m} (\mathrm{D}^\alpha f, \mathrm{D}^\alpha g)_0, \tag{1–6}$$

where $(f,g)_0 = \int_\Omega fg \, d\Omega$ denotes the inner product over $L^2(\Omega)$. Hence, we naturally associate the norm on $H^m(\Omega)$ with

$$\|f\|_m = (f,f)_m^{1/2}. \tag{1–7}$$

We will denote the inner product and norm by $(\cdot,\cdot)_{m,\Omega}$ and $\|\cdot\|_{m,\Omega}$, respectively, when their domain dependency needs to be emphasized.

There is another significant characterization for $H^m(\Omega)$. In 1964, it was shown that $H^m(\Omega)$ is equivalent to the completion of $\mathcal{C}^\infty(\Omega)$ with respect to the $\|\cdot\|_m$–norm ([2]). Many important properties for Sobolev spaces are derived from this feature. For example, H^{m+1} is densely injected into H^m. Let $H_0^m(\Omega)$ be the closure of $\mathcal{C}_0^\infty(\Omega)$ in $H^m(\Omega)$ with respect to the norm $\|\cdot\|_m$. For variational formulations, we take $H^{-m}(\Omega)$ as the dual space of $H_0^m(\Omega)$ whose norm is given by

$$\|p\|_{-m} = \sup_{0\neq f \in H_0^m(\Omega)} \frac{|<p,f>|}{\|f\|_m}. \tag{1–8}$$

Since $H^s(R^n) = H_0^s(R^n)$, the duality is naturally recovered in R^n, *i.e.*, the dual space of $H^s(R^n)$ is given by $H^{-s}(R^n)$. Each element in $H^{-m}(\Omega)$ can be represented in the form $\sum_{|\alpha|\leq m} D^\alpha f_\alpha$, for $f_\alpha \in L^2(\Omega)$.

Note that the *metaharmonic* operator $-\Delta+\lambda$, where $\lambda > 0$, provides the prototype for elliptic operators including the Stokes system. $-\Delta + \lambda$ yields an isomorphism between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$, which can be verified by the Lax–Milgram lemma. In some sense, this forms the cornerstone for variational formulations.

For our purpose we now turn our attention to the domain specific features of Sobolev spaces. The compact embedding, trace, and extension properties of Sobolev spaces are such features.

If we take $\alpha \in C^{0,1}([-M_1, M_1])$, then our domain $\Omega(\alpha)$ is Lipschitz continuous. Hence it excludes the possibility of domains $\{\Omega(\alpha)\}_{\alpha \in \mathcal{U}_{ad}}$ having a *cusp*. Let Γ denote the boundary of the domain Ω. Ω is said to have a *cusp* at $x \in \Gamma$ if no affine image in $\overline{\Omega}$ of a finite cone has a vertex at x. We observe that if a certain domain has the *cone property* (c.f. Adams[2]), then using the homogeneity along the line segment emanating from the vertex of the cone, a function with small support in a neighborhood of the vertex may be perturbed into the whole cone. Based on this fact, Chenais[29] showed the following.

Lemma 1.1. *The open sets satisfying the cone property are the uniform Lipschitz sets.* $\square$

For domain perturbations, the following extension property plays a central role.

Theorem 1.1. (Calderón's extension theorem)

For every Lipschitz domain Ω in R^n and every positive integer m there exists a linear continuous extension operator

$$P : H^m(\Omega) \longrightarrow H^m(R^n) \tag{1–9}$$

such that

$$\|Pf\|_{m,R^n} \le C\|f\|_{m,\Omega} \tag{1–10}$$

for each $f \in H^m(\Omega)$, where the positive constant C depends only on the cone embedded in Ω, but not on the domain Ω itself. $\square$

For the proof, see Marti[92] or Chenais[29].

Note that $\gamma_\Omega \circ P = id_{H^m(\Omega)}$=the identity map over $H^m(\Omega)$ for each m. For this reason, P is often called a *lifting*. For a domain $\Omega \subset R^n$, if there exists an extension operator P satisfying (1–9) and (1–10) for m, then the domain Ω is said to have an *m–extension property*. Theorem 1.1 states that Lipschitz continuous domains have the m–extension property for each m.

Hence we have the compact embedding property for bounded Lipschitz continuous domains.

Lemma 1.2. *For a bounded Lipschitz continuous domain Ω, the natural injection of $H^{m+1}(\Omega)$ into $H^m(\Omega)$ is compact.* $\square$

This can be proved using the Rellich's theorem for the compact embedding of $H_0^{m+1}(\Omega)$ into $H_0^m(\Omega)$ and an extension operator P. (For details, see Dautray and Lions[37]).

Finally, we consider the trace on the boundary. This justifies boundary values for the variational formulation for the Dirichlet problem. Using a local atlas and combining with the method of Treves[131] for the half plane, we can show the following.

Theorem 1.2. *For every Lipschitz domain Ω in R^n there exists a unique bounded trace operator*

$$\gamma_\Gamma : H^1(\Omega) \longrightarrow L^2(\Gamma) \tag{1–11}$$

such that $Ker\gamma_\Gamma = H_0^1(\Omega)$. $\square$

This trace operator satisfies $\gamma_\Gamma : H^1(\Omega) \longrightarrow H^{1/2}(\Gamma)$ if $\partial\Omega = \Gamma$ is piecewise continuously differentiable (see [37]). In this case, we may choose the norm on the boundary Γ to be

$$\|f\|_{1/2,\Gamma} = \inf_{\substack{v\in H^1(\Omega)\\ \gamma_\Gamma v=f}} \|f\|_{1,\Omega}\,, \tag{1–12}$$

and its dual norm to be

$$\|f^*\|_{-1/2,\Gamma} = \sup_{\substack{f\in H^{1/2}(\Gamma)\\ f\neq 0}} \frac{< f^*, f >}{\|f\|_{1/2,\Gamma}}\,, \tag{1–13}$$

where $(\cdot)^*$ denote the dual element.

For vector–valued functions and spaces, we use boldface notation. The previous discussions are naturally inherited. For example, $\mathbf{H}^s(\Omega) = [H^s(\Omega)]^n$ denotes the space of R^n-valued functions such that each component belongs to $H^s(\Omega)$.

For the domains of interest to us, we use the space $\mathbf{H}^1(\Omega) = [H^1(\Omega)]^2$ for the velocity equipped with the norm

$$\|\mathbf{v}\|_1 = \left(\sum_{i=1}^{2} \|v_i\|_1^2\right)^{1/2} . \tag{1–14}$$

We consider the semi–norm defined on $\mathbf{H}^1(\Omega)$:

$$|\!|\!|\mathbf{v}|\!|\!| = (\nabla\mathbf{v}, \nabla\mathbf{v})_0^{1/2} = \left(\int_\Omega \nabla\mathbf{v} \,:\, \nabla\mathbf{v}\, d\Omega\right)^{1/2} = \left(\sum_{i,j=1}^{2} \left\|\frac{\partial v_i}{\partial x_j}\right\|_0^2\right)^{1/2} . \tag{1–15}$$

Here $\nabla\mathbf{u} \,:\, \nabla\mathbf{v}$=trace of ${}^T(\nabla\mathbf{u})\nabla\mathbf{v} = \sum_{i,j=1,2}^{2} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}$ denotes the inner product between tensors, where ${}^T(\cdot)$ denote the transpose of $(\cdot)$. Then (1–14) can be simply written as $\|\mathbf{v}\|_1^2 = \|\mathbf{v}\|_0^2 + |\!|\!|\mathbf{v}|\!|\!|^2$.

Let Γ_s be a subset of Γ with nonzero measure. Let

$$\mathbf{H}^1_{\Gamma_s}(\Omega) = \{\mathbf{v} \in \mathbf{H}^1(\Omega)\,|\,\mathbf{v} = \mathbf{0} \text{ on } \Gamma_s\}\,.$$

Note that $\mathbf{H}^1_\Gamma(\Omega) = \mathbf{H}^1_0(\Omega)$.

By directly applying Korn's inequality, we obtain the following.

Lemma 1.3. *Let Ω be a Lipschitz continuous bounded domain and Γ_s be a subset of Γ with a positive measure. Then, there exists a positive constant C such that*

$$|||\mathbf{v}||| \geq C\|\mathbf{v}\|_1\,, \tag{1-16}$$

for all $\mathbf{v} \in \mathbf{H}^1_{\Gamma_s}(\Omega)$. $\square$

This implies that the semi-norm $|||\cdot|||$ is a norm which is equivalent to the norm $\|\cdot\|_{1,\Omega}$ on $\mathbf{H}^1_{\Gamma_s}(\Omega)$. Hence, if we take the inner product on $\mathbf{H}^1_{\Gamma_s}(\Omega)$ as

$$((\mathbf{u}, \mathbf{v}))_1 = (\nabla\mathbf{u}, \nabla\mathbf{v})_0\,,$$

then $|||\mathbf{u}||| = ((\mathbf{u}, \mathbf{u}))_1^{1/2}$.

Since the pressure is determined only up to a constant in the mathematical formulation of the Navier–Stokes equations with velocity boundary conditions, we take the space for the pressure to be

$$L^2_0(\Omega) = \left\{p \in L^2(\Omega) \,\Big|\, \int_\Omega p\, d\Omega = 0\right\}.$$

$L^2_0(\Omega)$ is isomorphic to $L^2(\Omega)/R$ whose norm is equipped with $\|[p]\|_{0,\Omega}$, where $[p]$ denote the equivalence class in $L^2(\Omega)/R$.

Throughout this dissertation, $\mathcal{I}$ will be used to denote the identity mapping, or the identity matrix which depends on context, and C a generic constant whose value also depends on context.

CHAPTER II

VARIATIONAL FORMULATIONS AND THE EXISTENCE OF OPTIMAL SOLUTIONS

The difficulties associated with the Navier-Stokes equations originate from three sources: the nonlinear convective term, the divergence free constraint, and the fact that it is parameter dependent, *i.e.*, the solution depends on the Reynolds number. The nonlinear convective term is introduced by the momentum balance law and the divergence free constraint enforces incompressibility. The Reynolds number is one of the determinining characteristics of the medium and the flow. The nonlinear convective term triggers the biggest difficulty. Since it originates from the natural mechanical balance equations, one cannot avoid it by a mere physical remodelling of the system.

Because of its complexity and its successes in describing a variety of fluid dynamical phenomena, a vast number of mathematical and numerical studies have been performed for the Navier–Stokes equations. However, many problems are still to be resolved (c.f. Heywood[79]). For example, global existence and generic properties still remain open.

The purpose of this chapter is to cast the state equation into a variational formulation and to show the existence of an optimal solution for our optimization problem (1–5). In Section 2.1, we give some additional facts concerning our cost functional and the Navier–Stokes equations. In Section 2.2, we briefly introduce some variational formulations and review existence and uniqueness results of the Navier–Stokes equations to motivate our choice of variational formulation and its continuing development. For this part, we mainly refer to Temam[126][127], Girault-Raviart[60], Lions[89], Foias *et al.*[54] and Gunzburger[64]. In Section 2.3, we reset the proposed problem into a penalty formulation. In Section 2.4, we show the existence of an optimal solution for the penalized problem.

2.1. Some Remarks on the Cost Functional and the Navier–Stokes Equations

Under the assumption of frame indifference, the constitutive equations for a Newtonian fluid are described in terms of the Cauchy stress $\mathbf{S}$ as

$$\mathbf{S}(\mathbf{u}, p) = -p\mathcal{I} + 2\mu\Upsilon(\mathbf{u}) \tag{2–1}$$

with the isochoric property $div\, \mathbf{u} = 0$, where p denotes a pressure and μ the dynamic viscosity of the fluid which depends on the temperature and chemical properties of the fluid, and $\Upsilon(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + {}^{T}\nabla\mathbf{u})$ is the deformation tensor. Here, $-p\mathcal{I}$ represents the stress for an ideal fluid and hence we can interpret (2–1) as the perturbation of the ideal fluid due to the shearing viscosity effect.

This point of view motivates our choice of the cost functional $\mathfrak{J}(\alpha)$ of (1–4) to be optimized. Physically the drag in the direction of the flow is furnished by

$$2\mu \int_{\Omega} \Upsilon(\mathbf{u}) : \Upsilon(\mathbf{u})\, d\Omega\ ,$$

whenever the fluid is isochoric and isotropic (see Gunz´ urger*et al.*[68]). In fact, the $\mathbf{e}$-component of the dissipative force exerted along the boundary is given by

$$-\left(\int_{\Gamma}(-p\mathcal{I} + 2\mu\Upsilon(\mathbf{u}))\cdot\mathbf{n}\, d\Gamma\right)\cdot\mathbf{e}\,,$$

where $\mathbf{e}$ is a unit vector. Take $\mathbf{e} = \dfrac{\mathbf{u}_\infty}{\|\mathbf{u}_\infty\|}$, where $\mathbf{u}_\infty$ denotes the uniform velocity of the flow (at infinity). Using integration by parts, this expression may be simplified to $\dfrac{2\mu}{\|\mathbf{u}_\infty\|}\displaystyle\int_\Omega \Upsilon(\mathbf{u}) : \Upsilon(\mathbf{u})\, d\Omega$ due to the incompressibility.

Since $\Upsilon(\mathbf{u}) : \Upsilon(\mathbf{u}) \leq \frac{1}{2}(\nabla\mathbf{u} : \nabla\mathbf{u})$, the drag in the flow is bounded above by

$$\mu\int_\Omega \nabla\mathbf{u} : \nabla\mathbf{u}\, d\Omega\,.$$

Hence, our choice of $\mathfrak{J}(\alpha)$ in our optimization problem (1–5) is tantamount to reduce the drag (or, the *dissipation energy*) by controlling the shape of $\Gamma(\alpha)$.

The state equations are given by conservation equations. Under the steady–state assumption, if the fluid is homogeneous, then (1–1) represents conservation of momentum and (1–2) the conservation of mass. The equation (1–2) is called the *continuity equation.* The kinematic viscosity is introduced by $\nu = \dfrac{\mu}{\rho}$, where ρ is the density of the fluid. We may choose $\rho = 1$, so that $\mu = \nu$.

2.2. Approach to the Variational Formulation

In this section, we discuss various aspects of variational formulations for the incompressible Navier–Stokes equations. To begin with, we assume $\mathbf{g} = \mathbf{0}$ on the boundary Γ of a Lipschitz continuous domain Ω to motivate the variational formulation for the problem:

Find $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ satisfying

$$\begin{aligned} -\nu\,\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega \\ \mathit{div}\,\mathbf{u} &= 0 && \text{in } \Omega \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega = \Gamma\,, \end{aligned} \tag{2–2}$$

in the sense of distributions, where $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$ is given.

Multiplying both sides of (2–2) by $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, it readily follows from Green's formula that

$$\nu((\mathbf{u},\mathbf{v}))_1 + ((\mathbf{u}\cdot\nabla)\mathbf{u},\mathbf{v})_0 - (p, div\,\mathbf{v})_0 = <\mathbf{f},\mathbf{v}>_{-1} , \tag{2–3}$$

where $(\cdot,\cdot)_0$ denotes the inner product in $L^2(\Omega)$ and $(\cdot,\cdot)_0$ in $\mathbf{L}^2(\Omega) = L^2(\Omega)^2$ *i.e.*, $(\varphi,\psi)_0 = \int_\Omega \varphi\psi\, d\Omega$ in $L^2(\Omega)$ and $(\boldsymbol{\Phi},\boldsymbol{\Psi})_0 = \sum_{i=1}^{2} \int_\Omega \phi_i\psi_i\, d\Omega$, where $\boldsymbol{\Phi} = (\phi_1,\phi_2)$ and $\boldsymbol{\Psi} = (\psi_1,\psi_2)$ are elements of $\mathbf{L}^2(\Omega)$. By $< \cdot,\cdot >_{-1}$, we denote the duality pairing between $\mathbf{H}^{-1}(\Omega)$ and $\mathbf{H}_0^1(\Omega)$.

Motivated by (2–3), we define the following forms:

$$a(\mathbf{u},\mathbf{v}) = ((\mathbf{u},\mathbf{v}))_1 \equiv \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, d\Omega = \sum_{i,j=1}^{2} \int_\Omega \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\, dx$$

$$c(\mathbf{w},\mathbf{u},\mathbf{v}) = ((\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{v})_0 = \int_\Omega (\mathbf{w}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\, d\Omega = \sum_{i,j=1}^{2} \int_\Omega w_j \frac{\partial u_i}{\partial x_j} v_i\, dx$$

$$b(\mathbf{v},p) = -\int_\Omega p\nabla\cdot\mathbf{v}\, d\Omega = -\sum_{i=1}^{2} \int_\Omega p\frac{\partial v_i}{\partial x_i}\, dx ,$$

for all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ $\in \mathbf{H}^1(\Omega)$ and $p \in L^2(\Omega)$.

Note that $a(\mathbf{u},\mathbf{v}) = \sum_{i=1}^{2} (\frac{\partial\mathbf{u}}{\partial x_i},\frac{\partial\mathbf{v}}{\partial x_i})_0$. Obviously $a(\cdot,\cdot)$ is a continuous bilinear form on $\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)$ and $c(\cdot,\cdot,\cdot)$ is a continuous trilinear form on $\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)\times\mathbf{H}^1(\Omega)$ which can be verified by the Sobolev embedding of $\mathbf{H}^1(\Omega)\subset\mathbf{L}^4(\Omega)$ and the Hõlder's inequality.

If we apply $q \in L_0^2(\Omega)$ to the second equation of (2–2), we get $b(\mathbf{u},q) = 0$ for every $q \in L_0^2(\Omega)$. Thus the weak formulation for (2–2) is given by:

Find $(\mathbf{u},p) \in \mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)$ satisfying

$$\begin{aligned} \nu a(\mathbf{u},\mathbf{v}) + c(\mathbf{u},\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &= <\mathbf{f},\mathbf{v}>_{-1}, \quad \forall\mathbf{v}\in\mathbf{H}_0^1(\Omega) \\ b(\mathbf{u},q) &= 0, \quad \forall q\in L_0^2(\Omega) . \end{aligned} \tag{2–4}$$

We can approach the solution of (2–4) by decoupling the velocity and pressure. This can be done by introducing appropriate Sobolev spaces. The basic idea is to impose the incompressibility constraint on the velocity.

We define the following spaces:

$$\begin{aligned}\mathcal{V}(\Omega) &= \{\mathbf{v} \in \mathcal{C}_0^\infty(\Omega)^2 \,|\, div\ \mathbf{v} = 0\}\\ \mathcal{L}(\Omega) &= \text{ the closure of } \mathcal{V}(\Omega) \text{ in } \mathbf{L}^2(\Omega)\\ \mathbf{V}(\Omega) &= \text{ the closure of } \mathcal{V}(\Omega) \text{ in } \mathbf{H}_0^1(\Omega)\\ &= \{\mathbf{u} \in \mathbf{H}_0^1(\Omega) \,|\, div\, \mathbf{u} = 0\}\end{aligned}$$

Then $\mathbf{V}(\Omega)$ is a closed subspace of $\mathbf{H}_0^1(\Omega)$ and the embeddings $\mathbf{V}(\Omega) \subset \mathcal{L}(\Omega) \subset \mathbf{V}(\Omega)^*$ are continuous.

Note that $\mathbf{V}(\Omega) = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) \,|\, b\ (\mathbf{v}, q) = 0 \text{ for all } q \in L^2(\Omega)\}$ and $c(\mathbf{w}, \mathbf{u}, \mathbf{v}) = -\,c(\mathbf{w}, \mathbf{v}, \mathbf{u})$ for all $(\mathbf{w}, \mathbf{u}, \mathbf{v}) \in (\mathbf{H}^1(\Omega) \cap \mathcal{L}(\Omega)) \times \mathbf{H}_0^1(\Omega) \times \mathbf{H}_0^1(\Omega)$.

We restrict $c(\cdot,\cdot,\cdot)$ over $\mathbf{V}(\Omega) \times \mathbf{V}(\Omega) \times \mathbf{V}(\Omega)$. Then (2–4) is turned into the problem of finding $\mathbf{u} \in \mathbf{V}(\Omega)$ such that

$$\nu a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) = <\mathbf{f}, \mathbf{v}>_{-1}, \quad \forall \mathbf{v} \in \mathbf{V}(\Omega). \tag{2–5}$$

Let $a_0(\mathbf{w}, \mathbf{u}, \mathbf{v}) = \nu a(\mathbf{u}, \mathbf{v}) + c(\mathbf{w}, \mathbf{u}, \mathbf{v})$. Then $a_0(\cdot,\cdot,\cdot)$ is clearly trilinear and continuous over $\mathbf{V}(\Omega) \times \mathbf{V}(\Omega) \times \mathbf{V}(\Omega)$. Moreover, since $c(\mathbf{v}, \mathbf{v}, \mathbf{v}) = 0$, $a_0(\cdot,\cdot,\cdot)$ is coercive over $\mathbf{V}(\Omega)$ by Lemma 1.3. Hence, applying compactness arguments(c.f. Girault–Raviart[60] or Temam[127]), we see that (2–5) has a solution $\mathbf{u}$ in $\mathbf{V}(\Omega)$.

At the present stage, one may ask: how can we recover the pressure p in $L_0^2(\Omega)$?

From (2–4), this question is equivalent to finding $p \in L_0^2(\Omega)$ such that

$$b(\mathbf{v}, p) = <\mathbf{f}, \mathbf{v}>_{-1} - \ a_0(\mathbf{u}, \mathbf{u}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega)\ , \tag{2–6}$$

where $\mathbf{u}$ is a solution of (2–5) in $\mathbf{V}(\Omega)$. The existence of such a p can be verified using de Rham's theorem (see Lions[89]). But on the basis of the mixed variational formulation, the existence of p is guaranteed if the following **inf–sup condition** is satisfied;

There is a positive constant C such that

$$\inf_{p \in L_0^2(\Omega)} \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{b(\mathbf{v}, p)}{\|\mathbf{v}\|_1 \|p\|_0} \geq C \, . \tag{2–7}$$

The inf–sup condition which is often called *Ladyzhenskaya-Brezzi-Babuska condition* or *div–stability condition* is an essential tool to maintain the stability of approximations. The inf–sup condition can be verified by using the following lemma (for the proof, see Girault–Raviart[60]);

Lemma 2.1. *Let $\mathbf{V}(\Omega)^{\perp}$ be the space of orthogonal components of $\mathbf{V}(\Omega)$ in $\mathbf{H}_0^1(\Omega)$, i.e., $\mathbf{H}_0^1(\Omega) = \mathbf{V}(\Omega) \oplus \mathbf{V}(\Omega)^{\perp}$. Then div is isometrically isomorphic from $\mathbf{V}(\Omega)^{\perp}$ onto $L_0^2(\Omega)$. Each element $\mathbf{v}$ of $\mathbf{V}(\Omega)^{\perp}$ is given in the form of $\mathbf{v} = (-\Delta)^{-1}(\nabla q)$ for some $q \in L^2(\Omega)$.* $\square$

This lemma implies that for each $p \in L_0^2(\Omega)$ there exists a $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ such that $div\, \mathbf{v} = -p$ in Ω. Moreover, $\|\mathbf{v}\|_1 \leq K \|p\|_0$ for some positive constant K. Hence for such a choice of $\mathbf{v}$,

$$b(\mathbf{v}, p) = -\int_{\Omega} p \, div\, \mathbf{v} \, d\Omega = \int_{\Omega} p^2 \, d\Omega = \|p\|_0^2 \geq \frac{\|p\|_0 \|\mathbf{v}\|_1}{K} \, ,$$

so that $C = \dfrac{1}{K} > 0$ is a constant satisfying the inf–sup condition (2–7).

Until now, we have observed the fundamental procedure for a mixed variational formulation of the incompressible Navier–Stokes equations and the search for a solution to the velocity and pressure pair when $\mathbf{g} = \mathbf{0}$ on the boundary. The major arguments for this procedure can be summarized as follows.

Find an appropriate pair of Sobolev spaces for the velocity and pressure so that coercivity for $a_0(\cdot,\cdot,\cdot) = \nu a(\cdot,\cdot) + c(\cdot,\cdot,\cdot)$ and the inf–sup condition for $b(\cdot,\cdot)$ are satisfied.

This principle is also naturally extended to the nonhomogeneous case.

Suppose $\mathbf{f}$ and $\mathbf{g} \neq \mathbf{0}$ are given in $\mathbf{H}^{-1}(\Omega)$ and $\mathbf{H}^{1/2}(\Gamma)$ respectively. We consider the problem of finding $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L^2_0(\Omega)$ such that

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} &\quad \text{in } \ \Omega \\ \mathit{div}\, \mathbf{u} = 0 &\quad \text{in } \ \Omega \\ \mathbf{u} = \mathbf{g} &\quad \text{on } \ \Gamma\,. \end{aligned} \tag{2–8}$$

The essential step is provided by the existence of $\mathbf{u}_0$ in $\mathbf{H}^1(\Omega)$ such that

$$\begin{aligned} &\mathit{div}\ \mathbf{u}_0 = 0 \ \text{ in } \ \Omega \qquad \text{and} \qquad \mathbf{u}_0 = \mathbf{g} \ \text{ on } \ \Gamma, \\ &|\, c(\mathbf{v}, \mathbf{u}_0, \mathbf{v})| \le \delta \|\mathbf{v}\|_1^2 \quad \text{for any } \delta > 0, \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega)\ , \end{aligned} \tag{2–9}$$

where $\mathbf{g}$ satisfies the *compatibility condition* $\int_\Gamma \mathbf{g}\cdot\mathbf{n}\, d\Gamma = 0$ (c.f. Girault–Raviart[60], Temam[129]). Then, the problem (2–8) is equivalent to finding $\mathbf{w} = \mathbf{u} - \mathbf{u}_0$ in $\mathbf{V}(\Omega)$ that satisfies the following variational formulation:

$$\nu a(\mathbf{w}, \mathbf{v}) + c(\mathbf{w}, \mathbf{u}_0, \mathbf{v}) + c(\mathbf{w}, \mathbf{w}, \mathbf{v}) + c(\mathbf{u}_0, \mathbf{w}, \mathbf{v}) = \, < \widetilde{\mathbf{f}}, \mathbf{v} >_{-1},\ \forall \mathbf{v} \in \mathbf{V}(\Omega)\,, \tag{2–10}$$

where $\widetilde{\mathbf{f}} = \mathbf{f} - \nu\Delta\mathbf{u}_0 - (\mathbf{u}_0\cdot\nabla)\mathbf{u}_0 \in \mathbf{H}^{-1}(\Omega)$.

If we take $\mathbf{w}$ in the place of $\mathbf{v}$, (2–10) is reduced to

$$a_0(\mathbf{w}, \mathbf{w}, \mathbf{w}) = \nu a(\mathbf{w}, \mathbf{w}) + c(\mathbf{w}, \mathbf{u}_0, \mathbf{w}) = \, < \widetilde{\mathbf{f}}, \mathbf{w} >_{-1}. \tag{2–11}$$

Then using (2–9), we easily see that $a_0(\cdot,\cdot,\cdot)$ is coercive over $\mathbf{V}(\Omega)$ and the solution $\mathbf{u} = \mathbf{u}_0 + \mathbf{w}$ of (2–8) exists. The inf–sup condition can be shown in the same manner as for the homogeneous case.

We state these results in the following theorem for later use.

Theorem 2.1. *Let Ω be a Lipschitz continuous bounded domain in R^n and let the data $(\mathbf{f}, \mathbf{g}) \in \mathbf{H}^{-1}(\Omega) \times \mathbf{H}^{1/2}(\Gamma)$ for the equations (2–8) be given. If $\mathbf{g}$ satisfies the compatibility condition $\int_\Gamma \mathbf{g} \cdot \mathbf{n}\, d\Gamma = 0$, we then have*

(I) *There exists at least one pair of solutions of (2–8); $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$.*

(II) *Let $\mathcal{S}$ be a set of solutions of (2–8) for the velocity. Then $\mathcal{S}$ is closed in $\mathbf{H}^1(\Omega)$ and is compact in $\mathbf{L}^2(\Omega)$.*

In addition, the uniqueness is secured under stronger conditions;

(III) *If $\nu > \nu_0(\Omega; \mathbf{f}, \mathbf{g})$ for some positive constant ν_0 which is determined by the given data, then $\mathcal{S}$ is composed of a single element.*

Proof: Since the coercivity of $a_0(\cdot,\cdot,\cdot)$ in (2–11) and the inf–sup condition of $b(\cdot,\cdot)$ are satisfied, the compactness argument (c.f.[60]) finishes the proof of (I). For (II), the closedness of $\mathcal{S}$ in $\mathbf{H}^1(\Omega)$ easily follows from the continuity of the operators Δ, ∇, *div* and trace operator. Since the continuous injection $\mathbf{H}_0^1(\Omega)$ into $\mathbf{L}^2(\Omega)$ is compact by Rellich's theorem, the closedness of $\mathcal{S}$ in $\mathbf{H}^1(\Omega)$ ensures the compactness of $\mathcal{S}$ in $\mathbf{L}^2(\Omega)$. It remains to show (III).

Let $\mathbf{u}$ be any element of $\mathcal{S}$ and $\mathbf{w} = \mathbf{u} - \mathbf{u}_0$. We first note that $\mathbf{w}$ in (2–11) is bounded in $\mathbf{H}^1(\Omega)$. Take δ in (2–11) small so that $\nu > \delta$. Then from (2–9) and (2–11),

$$\nu a(\mathbf{w}, \mathbf{w}) = <\widetilde{\mathbf{f}}, \mathbf{w}>_{-1} - \; c(\mathbf{w}, \mathbf{u}_0, \mathbf{w}) \leq \|\widetilde{\mathbf{f}}\|_{-1}\|\mathbf{w}\|_1 + \delta\, \|\mathbf{w}\|_1^2, \quad \text{and hence}$$

$$\|\mathbf{w}\|_1 \leq \frac{1}{\nu - \delta}\|\widetilde{\mathbf{f}}\|_{-1}\,. \tag{2–12}$$

Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be any two solutions of (2–8) and $\widetilde{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2$. Let $\mathbf{w}_1 = \mathbf{u}_1 - \mathbf{u}_0$ and $\mathbf{w}_2 = \mathbf{u}_2 - \mathbf{u}_0$, where $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$ is determined in (2–9). Then $\widetilde{\mathbf{u}} = \mathbf{u}_1 - \mathbf{u}_2 = \mathbf{w}_1 - \mathbf{w}_2$ is an element of $\mathbf{V}(\Omega)$. Plugging $\mathbf{w}_1$ and $\mathbf{w}_2$ in the place of $\mathbf{w}$ in (2–10), respectively, and subtracting the results yields

$$\nu a(\widetilde{\mathbf{u}}, \mathbf{v}) + c(\widetilde{\mathbf{u}}, \mathbf{u}_0, \mathbf{v}) + c(\widetilde{\mathbf{u}}, \mathbf{w}_1, \mathbf{v}) + c(\mathbf{w}_2, \widetilde{\mathbf{u}}, \mathbf{v}) + c(\mathbf{u}_0, \widetilde{\mathbf{u}}, \mathbf{v}) = 0, \;\; \forall \mathbf{v} \in \mathbf{V}(\Omega)\,.$$

Putting $\mathbf{v} = \widetilde{\mathbf{u}}$, this is simplified to

$$\nu a\left(\widetilde{\mathbf{u}}, \widetilde{\mathbf{u}}\right) = -c\left(\widetilde{\mathbf{u}}, \mathbf{u}_0, \widetilde{\mathbf{u}}\right) - c\left(\widetilde{\mathbf{u}}, \mathbf{w}_1, \widetilde{\mathbf{u}}\right).$$

Since $-c\left(\widetilde{\mathbf{u}}, \mathbf{w}_1, \widetilde{\mathbf{u}}\right) \le C\|\widetilde{\mathbf{u}}\|_1^2\|\widetilde{\mathbf{f}}\|_{-1}$ for some positive constant C, applying (2–9) and (2–12), we find that

$$\left(\nu - \delta - \frac{C}{\nu - \delta}\|\widetilde{\mathbf{f}}\|_{-1}\right)\|\widetilde{\mathbf{u}}\|_1^2 \le 0 .$$

Hence if we take ν large enough so that $\nu \ge \delta + \sqrt{C\|\widetilde{\mathbf{f}}\|_1}$, then (III) holds when we take $\nu_0(\Omega; \mathbf{f}, \mathbf{g}) = \delta + \sqrt{C\|\widetilde{\mathbf{f}}\|_1}$. $\square$

(III) also implies the uniqueness of the solution, if data $(\mathbf{f}, \mathbf{g}) \in \mathbf{H}^{-1} \times \mathbf{H}^{1/2}(\Gamma)$ is small.

REMARK 2.1: The operator $B = -\,div : \mathbf{H}_0^1(\Omega) \longrightarrow L_0^2(\Omega)$ is well defined by Green's formula and its adjoint operator $B^* : L_0^2(\Omega) \longrightarrow \mathbf{H}^{-1}(\Omega)$ corresponds to the gradient operator ∇ through the relation $(B\mathbf{u}, p)_0 = (\mathbf{u}, B^*p)_0$. Hence $L_0^2(\Omega)$ takes the role of a pivot space between B and its adjoint B^* and $\mathbf{V}(\Omega) = \mathbf{H}_0^1(\Omega) \cap Ker(B)$. $\mathbf{V}(\Omega)$ is often called the space of the *solenoidal vector fields.*

REMARK 2.2: Let Ω be smooth enough, *e.g.*, of class $C^2(\Omega)$. Consider $A = -\,\nu\Delta$ as an operator from $\mathcal{L}(\Omega)$ into $\mathcal{L}(\Omega)$ whose domain is $Dom(A) = \mathbf{H}^2(\Omega) \cap \mathcal{V}(\Omega)$. Then since A is a positive definite self–adjoint operator, it has discrete eigenvalues. Let $\{(\lambda_j, \mathbf{e}_j)\}_{j=1}^{\infty}$ denote the set of eigenpairs, where $\{\mathbf{e}_j\}_{j=1}^{\infty}$ is the orthogonal basis of $\mathcal{L}(\Omega)$ and $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots$, $\lambda_k \to \infty$ as $k \to \infty$. We set

$$\mathbf{H}_{,s}(\Omega) = \left\{ \mathbf{u} = \sum_{j=1}^{\infty} u_j\mathbf{e}_j,\ u_j \in R \,\middle|\, \|\mathbf{u}\|_{,s} = \sqrt{\sum_{j=1}^{\infty} \lambda_j^s |v_j|^2} < \infty \right\}, \quad \text{for any } s \in R.$$

Then $\|\cdot\|_{,s}$ is a norm on $\mathbf{H}_{,s}$ which is equivalent to the norm $\|\cdot\|_s$ on $\mathbf{H}^s(\Omega)$. This norm appears to be very useful to describe $\mathbf{V}(\Omega)$ and $\mathcal{L}(\Omega)$ and operators on it (c.f. Fursikov[59] and Foias *et al.* [51]). For example, $\{\mathbf{e}_j\}_{j=1}^{\infty} \cap \mathbf{H}^1(\Omega)$ forms a basis of

$\mathbf{V}(\Omega)$ and the continuous embeddings $\mathbf{V}(\Omega) \subset \mathcal{L}(\Omega) \subset \mathbf{V}(\Omega)^*$ can be rewritten in a more consistent way as $\mathbf{H}_{,1} \subset \mathbf{H}_{,0} \subset \mathbf{H}_{,-1}$. This approach seems to be especially valuable when we wish to construct periodic solutions of the incompressible Navier–Stokes equations.

REMARK 2.3: There are other ways to formulate the incompressible Navier–Stokes equations by using other function spaces. For the flows of an incompressible viscous fluid, the streamfunction–vorticity formulation is commonly used instead of the velocity–pressure formulation. This formulation is motivated by the following simple fact:

When Ω is a simply connected domain in R^2 with a suitable regularity,

$$\mathbf{v} \in \mathbf{V}(\Omega) \Longleftrightarrow \mathbf{v} = \mathbf{curl}\,\psi = (\frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1}), \quad \text{for some function } \psi \in \mathbf{H}_0^2(\Omega)\,.$$

Such a function ψ, which is called the *streamfunction* for the velocity, is uniquely determined. Combining with the *vorticity* of the velocity $\omega = \mathrm{curl}\,\mathbf{v} = \dfrac{\partial \mathrm{v}_2}{\partial \mathrm{x}_1} - \dfrac{\partial \mathrm{v}_1}{\partial \mathrm{x}_2}$ and

$$\mathbf{curl}\,(\,\mathrm{curl}\,\mathbf{v}) = -\,\Delta\mathbf{v} + \nabla(\mathit{div}\,\mathbf{v}), \quad \mathrm{curl}\,(\,\mathbf{curl}\,\psi) = -\,\Delta\psi\,,$$

we can exploit the streamfunction–vorticity formulation for (2–2) and (2–8). However, its relevant equation turns out to have a leading biharmonic term along with a natural boundary condition and hence a higher regularity assumption need be presumed. In shape perturbation problems, this may cause difficulties. Hence we do not consider this formulation.

As we have seen hither to, the two major factors determining the mathematical behavior of incompressible viscous fluids are the kinematic viscosity and physical constraints such as incompressibility.

The kinematic viscosity ν plays a central role determining the characteristics of the mixed variational formulation and its numerical schemes. If ν is quite large, then our variational formulation (2–4) is dominated by the symmetric elliptic term. Thus, major arguments for elliptic equations such as coercivity and continuity will be the keynotes in this case. However, if ν is small, we encounter the nonsymmetric convection dominated situation which often causes big difficulties with discretization methods.

The incompressibility constraint should be kept track of throughout the continuous and discrete formulations. Even though incompressibility simplifies the mathematical treatment of the incompressible Navier–Stokes equations, numerically it is a difficult problem to find stable finite element subspaces of $\mathbf{V}(\Omega)$, especially when the involved domains are changing.

To relax this constraints and regularity requirements for the solutions, the penalty method will be applied. The penalized version of the incompressible Navier-Stokes equations may be stated as:

Find $\mathbf{u}_\epsilon$ and p_ϵ satisfying

$$\begin{aligned} -\nu\,\Delta\mathbf{u}_\epsilon + (\mathbf{u}_\epsilon\cdot\nabla)\mathbf{u}_\epsilon + \nabla p_\epsilon = \mathbf{f} &\qquad \text{in} \quad \Omega \\ \mathit{div}\,\mathbf{u}_\epsilon = -\epsilon p_\epsilon &\qquad \text{in} \quad \Omega \\ \mathbf{u}_\epsilon = \mathbf{g} &\qquad \text{on} \quad \Gamma\,, \end{aligned} \tag{2–13}$$

where $\epsilon > 0$ is a given parameter.

The fundamental idea behind the penalty method is to introduce an artificial compressibility term $-\epsilon p_\epsilon$ in the incompressibility constraint and to expect *near incompressibility.* Simply put, we wish to approximate the solution $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ of (2–8) by $(\mathbf{u}_\epsilon, p_\epsilon) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ of (2–13). The major advantage of taking this formulation in our case is that we do not have to deal with the divergence free constraint and the pressure may be eliminated. This will reduce the problem size and may facilitate the complicated sensitivity analysis.

For $\epsilon > 0$, $p_\epsilon = -\dfrac{div\,\mathbf{u}_\epsilon}{\epsilon}$. Plugging into the first equation of (2–13), we get an equivalent formulation to (2–13) in which only the velocity is involved:

Find $\mathbf{u}_\epsilon \in \mathbf{H}^1(\Omega)$ satisfying

$$\begin{aligned} -\nu\,\Delta\mathbf{u}_\epsilon + (\mathbf{u}_\epsilon\cdot\nabla)\mathbf{u}_\epsilon - \frac{1}{\epsilon}\nabla(div\,\mathbf{u}_\epsilon) = \mathbf{f} \qquad &\text{in} \quad \Omega \\ \mathbf{u}_\epsilon = \mathbf{g} \qquad &\text{on} \quad \Gamma\ . \end{aligned} \tag{2–14}$$

After finding $\mathbf{u}_\epsilon$ from (2–14), the approximate pressure p_ϵ can be easily recovered from $p_\epsilon = -\dfrac{div\,\mathbf{u}_\epsilon}{\epsilon}$.

Fortunately, there are very reasonable approximation results as ϵ tends to zero (c.f. Temam[125], Bercovier *et al.*[16], Hughes *et al.*[83]). Thus, this mild relaxation of the divergence free constraint may be combined well with the domain identification problem.

REMARK 2.4: Temam[125] added additional stabilization term $\frac{1}{2}(div\,\mathbf{u}_\epsilon)\mathbf{u}_\epsilon$ to the left hand side of the first equation of (2–14) in the consideration of the Cauchy–Kowalevskaya theorem for the analyticity of the solutions. This results in a minor change to the trilinear form $c(\cdot,\cdot,\cdot)$.

The penalty parameter ϵ in our case can be chosen in various ways. It should be chosen small enough so that the compressibility and pressure errors are negligible, but not so small to avoid ill–conditioning. In the numerical process, its choice depends on the dynamic viscosity and the machine precision (c.f. Hughes *et al.*[83]).

2.3. The Penalized Variational Formulation

The purpose of this section is to reset the primal problem into a penalty formulation, to discuss regular branches of solutions of nonlinear problems, and to derive some existence and convergence results for the solution of the penalized variational formulation of the Navier–Stokes equations.

Some difficulties are interlaced with the shape optimization for the Navier–Stokes equations. As we have already noted in Theorem 2.1, the weak solution of the steady–state, incompressible Navier–Stokes equations is not unique unless the body force **f** is small enough or the kinematic viscosity ν is large enough. Hence, we have difficulties in chasing the natural argument for optimal shape problems:

$$\Omega \xrightarrow{\text{state} \quad \text{equation}} y_\Omega \xrightarrow{\text{cost} \quad \text{functional}} \mathfrak{J}(\Omega) = J(\Omega, y(\Omega)) \,,$$

where y_Ω is a solution of the well–posed state equation and $J(\Omega, y(\Omega))$ is a cost functional whose value is determined by a pair of admissible family of domains and a set of solutions of the state equation over a feasible domain.

To overcome this difficulty, we introduce a nonlinear functional setting of the Navier–Stokes equations involving branches of regular solutions and briefly review some well–known approximation results originally due to [20]. Motivated by these results, we recast the primal problem into a penalty formulation.

We consider the penalized steady–state Navier–Stokes equations:

Find $\mathbf{u}_\epsilon \in \mathbf{H}^1(\Omega(\alpha))$ satisfying

$$\begin{aligned} -\nu\,\Delta \mathbf{u}_\epsilon - \frac{1}{\epsilon}\nabla(div\ \mathbf{u}_\epsilon) + (\mathbf{u}_\epsilon \cdot \nabla)\,\mathbf{u}_\epsilon = \mathbf{f} \qquad & \text{in } \ \Omega(\alpha) \\ \mathbf{u}_\epsilon = \mathbf{g} \qquad & \text{on } \ \Gamma = \partial\Omega(\alpha)\,, \end{aligned} \tag{2–15}$$

in the sense of distribution, where $\mathbf{f} \in \mathbf{H}^{-1}(\Omega(\alpha))$ and $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$ are given. In this expression, $\epsilon > 0$ is an artificial parameter such that the solution **u** of the primal equations is expected to be obtained approximately, *i.e.*, the solution **u** of the primal state equations (2–8) lies in a neighborhood of the solution $\mathbf{u}_\epsilon$ of the penalized equations (2–15) in $\mathbf{H}^1(\Omega(\alpha))$ when ϵ is small. Physically, the associated errors from the penalization amounts to net fluid loss or gain.

For each $\alpha \in \mathcal{C}^{0,1}[-M_1, M_1]$, let $\Gamma_3 \cup \Gamma(\alpha) = \Gamma_0(\alpha)$ and $\Gamma_{\mathbf{g}} = \Gamma_1 \cup \Gamma_2$, so that $\partial\Omega(\alpha) = \Gamma_0(\alpha) \cup \Gamma_{\mathbf{g}}$. Since $\mathbf{u}_\epsilon = \mathbf{0}$ on $\Gamma_0(\alpha)$, we may take a solution $\mathbf{u}_\epsilon$ of (2–15) as an element in $\mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$, where

$$\mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)) = \{\mathbf{u} \in \mathbf{H}^1(\Omega(\alpha)) \,|\, \mathbf{u} = \mathbf{0} \ \text{ on } \ \Gamma_0(\alpha)\}\,.$$

$\mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$ is the space of $\mathbf{H}^1$–functions that vanish on $\Gamma_0(\alpha)$, *i.e.*, $\mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$ is the space on which the *homogeneous essential boundary condition* is imposed. Let $\mathbf{H}^{-1}_{\Gamma_0(\alpha)}(\Omega(\alpha))$ be the dual space of $\mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$. Note that $\mathbf{H}^{-1}_{\Gamma_0(\alpha)}(\Omega(\alpha))$ is a subspace of $\mathbf{H}^{-1}(\Omega(\alpha))$.

In terms of these function spaces, the corresponding variational formulation for the penalized state equations is turned into the problem of finding $\mathbf{u}_\epsilon \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$ such that for each $\alpha \in \mathcal{U}_{ad}$ and for given $(\mathbf{f}, \mathbf{g}) \in \mathbf{H}^{-1}_{\Gamma_0(\alpha)}(\Omega(\alpha)) \times \mathbf{H}^{1/2}_0(\Gamma_{\mathbf{g}})$,

$$\nu \int_{\Omega(\alpha)} \nabla \mathbf{u}_\epsilon : \nabla \mathbf{v}\, d\Omega + \int_{\Omega(\alpha)} (\mathbf{u}_\epsilon \cdot \nabla)\mathbf{u}_\epsilon \cdot \mathbf{v}\, d\Omega + \frac{1}{\epsilon}\int_{\Omega(\alpha)} div\, \mathbf{u}_\epsilon\, div \mathbf{v}\, d\Omega$$
$$- < \mathbf{t}_\epsilon, \mathbf{v} >_{-1/2,\Gamma_{\mathbf{g}}} = < \mathbf{f}, \mathbf{v} >_{-1,\Gamma_0(\alpha)}\,, \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_0}(\Omega(\alpha)) \quad (2\text{–}16)$$

and

$$< \mathbf{s}, \mathbf{u}_\epsilon >_{-1/2,\Gamma_{\mathbf{g}}} = < \mathbf{s}, \mathbf{g} >_{-1/2,\Gamma_{\mathbf{g}}}\,, \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_{\mathbf{g}})\,, \qquad (2\text{–}17)$$

where the duality between $\mathbf{H}^{-1}_{\Gamma_0(\alpha)}$ and $\mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$ is denoted by $< \cdot, \cdot >_{-1,\Gamma_0(\alpha)}$. Here we denote $\mathbf{H}^{1/2}(\Gamma_{\mathbf{g}}) = \{\mathbf{s} \in \mathbf{H}^{1/2}(\Gamma) \,|\, \mathbf{s} = 0 \ \text{ on } \ \Gamma_0(\alpha)\}$ and $\mathbf{H}^{-1/2}(\Gamma_{\mathbf{g}})$ its dual space and by $< \cdot, \cdot >_{-1/2,\Gamma_{\mathbf{g}}}$ we denote the duality pairing between $\mathbf{H}^{-1/2}(\Gamma_{\mathbf{g}})$ and $\mathbf{H}^{1/2}(\Gamma_{\mathbf{g}})$. We can show that

$$\mathbf{t}_\epsilon = \nu\,(\mathbf{n} \cdot \nabla)\,\mathbf{u}_\epsilon + \frac{1}{\epsilon}\,(div\, \mathbf{u}_\epsilon)\,\mathbf{n} \qquad \text{on } \ \Gamma_{\mathbf{g}}\,, \qquad (2\text{–}18)$$

where $\mathbf{n}$ denote the outward unit normal vector along $\Gamma_{\mathbf{g}}$. $\mathbf{t}_\epsilon$ actually represents the *traction force* (or the boundary stress) along the boundary $\Gamma_{\mathbf{g}} = \Gamma_1 \cup \Gamma_2$ (see Remark 2.5).

Since $\Gamma_{\mathbf{g}}$ is smooth, the trace mapping $\gamma_{\Gamma_{\mathbf{g}}} : \mathbf{H}^1(\Omega(\alpha)) \longrightarrow \mathbf{H}^{1/2}(\Gamma_{\mathbf{g}})$ is well-defined and $\mathbf{H}^{1/2}(\Gamma_{\mathbf{g}}) = \gamma_{\Gamma_{\mathbf{g}}}(\mathbf{H}^1_{\Gamma_{0(\alpha)}}(\Omega(\alpha))$ for each $\alpha \in \mathcal{U}_{ad}$. Hence the equation (2–17) is well justified. Since $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma_{\mathbf{g}})$, we assume $\int_{\Gamma_{\mathbf{g}}} \mathbf{g} \cdot \mathbf{n}\, d\Gamma = 0$ for the compatibility condition, so that $p_\epsilon \in L^2_0(\Omega)$. For forthcoming discussions, we define the following function space;

$$\mathbf{H}^{1/2}_0(\Gamma_{\mathbf{g}}) = \{\mathbf{s} \in \mathbf{H}^{1/2}(\Gamma_{\mathbf{g}}) \mid \int_{\Gamma_{\mathbf{g}}} \mathbf{s} \cdot \mathbf{n}\, d\Gamma = 0\}.$$

Now, let $\boldsymbol{\psi}$ be an element of $\mathbf{H}^{1/2}(\Gamma_{\mathbf{g}})$. It is well-known that $\mathbf{H}^{1/2}(\Gamma_{\mathbf{g}})$ is a Hilbert space with the norm

$$\|\boldsymbol{\psi}\|_{1/2,\Gamma_{\mathbf{g}}} = \inf_{\substack{\mathbf{v} \in \mathbf{H}^1_{\Gamma_{0(\alpha)}}(\Omega(\alpha)) \\ \gamma_{\Gamma_{\mathbf{g}}} \mathbf{v} = \boldsymbol{\psi}}} \|\mathbf{v}\|_{1,\Omega(\alpha)} \,.$$

Let $\boldsymbol{\psi}^*$ be the dual element of $\boldsymbol{\psi}$ in $\mathbf{H}^{-1/2}(\Gamma_{\mathbf{g}})$. By the definition of the dual norm, we note that

$$\|\boldsymbol{\psi}^*\|_{-1/2,\Gamma_{\mathbf{g}}} = \sup_{\substack{\boldsymbol{\psi} \in \mathbf{H}^{1/2}(\Gamma_{\mathbf{g}}) \\ \boldsymbol{\psi} \neq \mathbf{0}}} \frac{< \boldsymbol{\psi}^*, \boldsymbol{\psi} >_{-1/2,\Gamma_{\mathbf{g}}}}{\|\boldsymbol{\psi}\|_{1/2,\Gamma_{\mathbf{g}}}} \,.$$

We derive an equivalent norm to $\| \cdot \|_{-1/2,\Gamma_{\mathbf{g}}}$ for our practical use.

Lemma 2.2. *It holds that*

$$\|\boldsymbol{\psi}^*\|_{-1/2,\Gamma_{\mathbf{g}}} = \sup_{\substack{\mathbf{v} \in \mathbf{H}^1_{\Gamma_{0(\alpha)}}(\Omega(\alpha)) \\ \mathbf{v} \neq \mathbf{0}}} \frac{< \boldsymbol{\psi}^*, \gamma_{\Gamma_{\mathbf{g}}} \mathbf{v} >_{-1/2,\Gamma_{\mathbf{g}}}}{\|\mathbf{v}\|_{1,\Omega(\alpha)}} \quad \forall\, \boldsymbol{\psi}^* \in \mathbf{H}^{-1/2}(\Gamma_{\mathbf{g}}) \,. \tag{2–19}$$

For the proof, we introduce the space of tensors defined by

$$\mathbf{H}(div\,;\Omega) = \left\{\mathbf{S} = (S_{ij}) \in L^2(\Omega)^4 \mid div\, \mathbf{S} = \left(\textstyle\sum_{j=1}^2 \partial S_{ij}/\partial x_j\right) \in L^2(\Omega)^2\right\},$$

equipped with the norm

$$\begin{aligned}\|\mathbf{S}\|_{\mathbf{H}(div\,;\Omega)} &\equiv \left(\|\mathbf{S}\|^2_{L^2(\Omega)^4} + \|div\, \mathbf{S}\|^2_{L^2(\Omega)^2}\right)^{1/2} \\ &= \left(\textstyle\sum_{i,j=1}^2 \|S_{ij}\|_0^2 + \sum_{i=1}^2 \|\sum_{j=1}^2 \partial S_{ij}/\partial x_j\|_0^2\right)^{1/2} \,.\end{aligned}$$

For each $\alpha \in \mathcal{U}_{ad}$, the following Green's formula holds:

$$\int_{\Omega(\alpha)} \mathbf{S} \ : \ \nabla \mathbf{v}\, d\Omega + \int_{\Omega(\alpha)} div\, \mathbf{S} \cdot \mathbf{v}\, d\Omega = \int_{\Gamma_{\mathbf{g}}} (\mathbf{S} \circ \mathbf{n}) \cdot \mathbf{v}\, d\Gamma \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)) \ , \quad (2\text{–}20)$$

where $\mathbf{S} \circ \mathbf{n} = (\sum_{j=1}^{2} S_{1j} n_j, \sum_{j=1}^{2} S_{2j} n_j)$. By the operation $\circ$, we denote the product of a tensor and a vector.

Let us consider the problem of finding $\mathbf{w} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$ such that

$$\begin{aligned} -\Delta \mathbf{w} + \mathbf{w} = \mathbf{0} \qquad & \text{in } \ \Omega(\alpha) \\ \frac{\partial \mathbf{w}}{\partial \mathbf{n}} = \boldsymbol{\psi}^* \qquad & \text{on } \ \Gamma_{\mathbf{g}} \end{aligned} \qquad (2\text{–}21)$$

for a given $\boldsymbol{\psi}^* \in \mathbf{H}^{-1/2}(\Gamma_{\mathbf{g}})$. This can be written in a variational formulation as

$$\int_{\Omega(\alpha)} \nabla \mathbf{w} \ : \ \nabla \mathbf{v}\, d\Omega + \int_{\Omega(\alpha)} \mathbf{w} \cdot \mathbf{v}\, d\Omega = < \boldsymbol{\psi}^*, \gamma_{\Gamma_{\mathbf{g}}} \mathbf{v} >_{-1/2, \Gamma_{\mathbf{g}}} \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)) \, . \quad (2\text{–}22)$$

By the Lax–Milgram Lemma, it has a unique solution $\mathbf{w} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$ such that $(\mathbf{n} \cdot \nabla)\mathbf{w} \equiv \nabla \mathbf{w} \circ \mathbf{n} = \boldsymbol{\psi}^*$ on the boundary. Putting $\mathbf{S} = \nabla \mathbf{w}$, then $\mathbf{S}$ belongs to $\mathbf{H}(div\, ; \Omega(\alpha))$ by (2–21). Moreover, this $\mathbf{S}$ satisfies

$$\mathbf{S} \circ \mathbf{n} = \boldsymbol{\psi}^* \quad \text{and} \quad \|\mathbf{S}\|_{\mathbf{H}(div\, ; \Omega(\alpha))} = \|\mathbf{w}\|_{1, \Omega(\alpha)} \, . \qquad (2\text{–}23)$$

Next, let $\boldsymbol{\xi}$ be an element in $\mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$ such that $\gamma_{\Gamma_{\mathbf{g}}} \boldsymbol{\xi} = \boldsymbol{\psi}$ and

$$\int_{\Omega(\alpha)} \nabla \boldsymbol{\xi} \ : \ \nabla \boldsymbol{\eta}\, d\Omega + \int_{\Omega(\alpha)} \boldsymbol{\xi} \cdot \boldsymbol{\eta}\, d\Omega = 0 \quad \forall \boldsymbol{\eta} \in \mathbf{H}^1_0(\Omega(\alpha)) \, .$$

Clearly such a $\boldsymbol{\xi}$ is uniquely determined and by the definition of $\| \cdot \|_{1/2, \Gamma_{\mathbf{g}}}$, it follows that

$$\|\boldsymbol{\psi}\|_{1/2, \Gamma_{\mathbf{g}}} = \|\boldsymbol{\xi}\|_{1, \Omega(\alpha)} \, . \qquad (2\text{–}24)$$

We are now ready to prove Lemma 2.2.

(Proof of Lemma 2.2.): Using the continuity of the trace mapping, it follows that

$$\begin{aligned} < \boldsymbol{\psi}^*, \gamma_{\Gamma_{\mathbf{g}}} \mathbf{v} >_{-1/2, \Gamma_{\mathbf{g}}} &\le \|\boldsymbol{\psi}^*\|_{-1/2, \Gamma_{\mathbf{g}}} \|\gamma_{\Gamma_{\mathbf{g}}} \mathbf{v}\|_{1/2, \Gamma_{\mathbf{g}}} \\ &\le \|\boldsymbol{\psi}^*\|_{-1/2, \Gamma_{\mathbf{g}}} \|\mathbf{v}\|_{1, \Omega(\alpha)} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)) \, . \end{aligned}$$

Hence we obtain

$$\sup_{\mathbf{v}\in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))} \frac{<\boldsymbol{\psi}^*, \gamma_{\Gamma_\mathbf{g}}\mathbf{v}>_{-1/2,\Gamma_\mathbf{g}}}{\|\mathbf{v}\|_{1,\Omega(\alpha)}} \leq \|\boldsymbol{\psi}^*\|_{-1/2,\Gamma_\mathbf{g}} . \tag{2–25}$$

To show the other direction of the inequality, we choose $\boldsymbol{\xi} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$ satisfying (2–24). If we substitute $\mathbf{v} = \boldsymbol{\xi}$ into (2–22), it follows from (2–23) that

$$\begin{aligned}<\boldsymbol{\psi}^*, \boldsymbol{\psi}>_{-1/2,\Gamma_\mathbf{g}} = <\boldsymbol{\psi}^*, \gamma_{\Gamma_\mathbf{g}}\boldsymbol{\xi}>_{-1/2,\Gamma_\mathbf{g}} &= \int_{\Omega(\alpha)} (\nabla\mathbf{w} : \nabla\boldsymbol{\xi} + \mathbf{w}\cdot\boldsymbol{\xi})\, d\Omega \\ &\leq \|\mathbf{w}\|_{1,\Omega(\alpha)}\|\boldsymbol{\xi}\|_{1,\Omega(\alpha)} \\ &= \|\nabla\mathbf{w}\|_{\mathbf{H}(div;\Omega(\alpha))}\|\boldsymbol{\psi}\|_{1/2,\Gamma_\mathbf{g}} .\end{aligned} \tag{2–26}$$

If we next substitute $\mathbf{v} = \mathbf{w}$ into (2–22), then

$$<\boldsymbol{\psi}^*, \gamma_{\Gamma_\mathbf{g}}\mathbf{w}>_{-1/2,\Gamma_\mathbf{g}} = \|\mathbf{w}\|^2_{1,\Omega(\alpha)}$$

$$= \|\mathbf{w}\|_{1,\Omega(\alpha)}\|\nabla\mathbf{w}\|_{\mathbf{H}(div;\Omega(\alpha))} \tag{2–27}$$

$$\geq \|\nabla\mathbf{w}\|_{\mathbf{H}(div;\Omega(\alpha))}\|\gamma_{\Gamma_\mathbf{g}}\mathbf{w}\|_{1/2,\Gamma_\mathbf{g}} . \tag{2–28}$$

Hence, it follows from (2–26) and (2–28) that

$$\|\boldsymbol{\psi}^*\|_{-1/2,\Gamma_\mathbf{g}} = \|\nabla\mathbf{w}\|_{\mathbf{H}(div;\Omega(\alpha))} . \tag{2–29}$$

Finally, applying (2–29) to (2–27), we get

$$<\boldsymbol{\psi}^*, \gamma_{\Gamma_\mathbf{g}}\mathbf{w}>_{-1/2,\Gamma_\mathbf{g}} = \|\mathbf{w}\|_{1,\Omega(\alpha)}\|\boldsymbol{\psi}^*\|_{-1/2,\Gamma_\mathbf{g}} .$$

This implies that there exists $\mathbf{w} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$ such that

$$\|\boldsymbol{\psi}^*\|_{-1/2,\Gamma_\mathbf{g}} = \frac{<\boldsymbol{\psi}^*, \gamma_{\Gamma_\mathbf{g}}\mathbf{w}>_{-1/2,\Gamma_\mathbf{g}}}{\|\mathbf{w}\|_{1,\Omega(\alpha)}} .$$

Therefore, we come to the conclusion by (2–25). □

REMARK 2.5: Note that $div\,(\,^T(\nabla\mathbf{u}_\epsilon)\circ\mathbf{v}) = \nabla\mathbf{u}_\epsilon : \nabla\mathbf{v} + \Delta\mathbf{u}_\epsilon\cdot\mathbf{v}$. Since

$$\begin{aligned}-\int_{\Omega(\alpha)} \Delta\mathbf{u}_\epsilon\cdot\mathbf{v}\, d\Omega &= \int_{\Omega(\alpha)} \nabla\mathbf{u}_\epsilon : \nabla\mathbf{v}\, d\Omega - \int_\Gamma {}^T(\nabla\mathbf{u}_\epsilon)\circ\mathbf{v}\cdot\mathbf{n}\, d\Gamma \\ &= \int_{\Omega(\alpha)} \nabla\mathbf{u}_\epsilon : \nabla\mathbf{v}\, d\Omega - \int_\Gamma (\nabla\mathbf{u}_\epsilon)\circ\mathbf{n}\cdot\mathbf{v}\, d\Gamma\end{aligned}$$

and $\mathbf{v} = 0$ along $\Gamma_0(\alpha)$, we obtain the formula (2–18) for $\mathbf{t}_\epsilon$ using the Green's formula and integration by parts. Since $p_\epsilon = -\frac{1}{\epsilon}\, div\, \mathbf{u}_\epsilon$ in the penalty formulation, $\mathbf{t}_\epsilon$ can be rewritten as $\mathbf{t}_\epsilon = (-p_\epsilon \mathcal{I} + \nu \nabla \mathbf{u}_\epsilon) \circ \mathbf{n}$ over $\Gamma_\mathbf{g}$. As we have already discussed in Section 2.1, the Cauchy stress due to the deformation is given by $\mathbf{S}(\mathbf{u}, p) = -p\mathcal{I} + 2\nu\, \Upsilon(\mathbf{u})$ in nondimensional form. Since $div\, \Upsilon(\mathbf{u}) = \frac{1}{2}(\Delta \mathbf{u} + \nabla div\, \mathbf{u}) = \frac{1}{2} \Delta \mathbf{u}$ in incompressible flow, $\mathbf{t}_\epsilon$ is represented by $\mathbf{t}_\epsilon = \mathbf{S}(\mathbf{u}_\epsilon, p_\epsilon) \circ \mathbf{n}$ along $\Gamma_\mathbf{g}$, which can be interpreted as the traction force along the boundary $\Gamma_\mathbf{g}$ due to the penalized deformation.

$\mathbf{t}_\epsilon$ can also be regarded as a Lagrange multiplier since the domain perturbations affect the traction force along $\Gamma_\mathbf{g}$ to keep the mass balance. Gunzburger *et al.*[65] used a similar formulation in the boundary velocity control for the Dirichlet problem. Especially, for finite element approximations, decoupling the computation of $\mathbf{t}_\epsilon$ from the others in (2–16) will provide the same kind of a simplification suggested in [65].

To overcome some difficulties related to the well–posedness of the penalized Navier–Stokes equations (2–16)–(2–17), we examine the notion of branches of solutions for nonlinear problems. From this we get approximation results for our penalized problem and a modification for the feasible domains. The general discussion for the branch structure of nonlinear problems can be found in Brezzi *et al.*[20]. Related to the branch structure for the Navier–Stokes equations, we closely follow the exposition of Girault–Raviart[60] with some modification for our purpose. We also refer to [128] and [65].

The abstract structure of the parameter–dependent nonlinear problems we consider is of the form ;

$$F(\lambda, \psi) \equiv \psi + TG(\lambda, \psi) = 0\,, \tag{2–30}$$

where $T : Y \to X$ is a bounded linear mapping, $G : \Lambda \times X \to Y$ is a C^2–nonlinear mapping, X and Y are Banach spaces and Λ is a compact interval of R. Let the solution ψ of (2–30) depend on the parameter λ. We say that $\{(\lambda, \psi(\lambda)) \mid \lambda \in \Lambda\}$ is a branch of solutions of (2–30) if $\lambda \mapsto \psi(\lambda)$ is a continuous function from Λ into X such that $F(\lambda, \psi(\lambda)) = 0$. By $D_\psi F$, we denote the *Fréchet derivative* of $F(\cdot, \cdot)$ with respect to the second variable. If $D_\psi F(\lambda, \psi(\lambda))$ is an isomorphism from X into X for all $\lambda \in \Lambda$, then the branch $\lambda \mapsto \psi(\lambda)$ is called a *regular branch* (or nonsingular branch). Note that $D_\psi F(\lambda, \psi) = \mathcal{I} + T D_\psi G(\lambda, \psi)$ from (2–30). Hence, if we consider $D_\psi G(\lambda, \cdot)$ as a bounded linear mapping from X into Z, a subspace of Y, where the inclusion $Z \subset Y$ is a continuous embedding and $T|_Z : Z \to X$ is compact, then $D_\psi F$ appears to be a compact perturbation of the identity.

This structure is essential in the approximation theory for parameter–dependent nonlinear problems. For approximations, we introduce a subspace X^h of X and an approximating operator $T^h \in \mathcal{L}(Y, X^h)$, where $\mathcal{L}(\cdot, \cdot)$ denotes the bounded linear operators between Banach spaces. The approximation problem corresponding to the form (2–30) is to seek $\psi^h \in X^h$ such that

$$F^h(\lambda, \psi^h) \equiv \psi^h + T^h G(\lambda, \psi^h) = 0 \ . \tag{2–31}$$

The convergence to a regular branch of solutions of the approximation problem (2–31) is ensured under the following assumptions;

$$D_\psi G(\lambda, \psi) \in \mathcal{L}(X, Z) \quad \forall \lambda \in \Lambda \quad \text{and} \quad \psi \in X \ , \tag{2–32}$$

$$\lim_{h \to 0} \|(T^h - T)y\|_X = 0 \quad \forall y \in Y \tag{2–33}$$

and

$$\lim_{h \to 0} \|(T^h - T)\|_{\mathcal{L}(Z,X)} = 0 \ . \tag{2–34}$$

We now state the fundamental result concerning approximations of the regular branch. For the proof, refer to [60] and [20].

Theorem 2.2. *Assume that* $G : \Lambda \times X \to Y$ *is a* C^2*–nonlinear mapping and that the second Fréchet derivative* $D_{\psi\psi}G$ *is bounded on all bounded sets of* $\Lambda \times X$*. Assume that (2–32)–(2–34) hold and that* $\{(\lambda, \psi(\lambda)) \,|\, \lambda \in \Lambda\}$ *is a branch of regular solutions of (2–30). Then, there exists a neighborhood* $\mathcal{O}$ *of the origin in* X *and, for* $h \le h_0$ *small enough, a unique* C^2 *function* $\lambda \in \Lambda \mapsto \psi^h(\lambda) \in X^h$ *such that* $\{(\lambda, \psi^h(\lambda)) \,|\, \lambda \in \Lambda\}$ *is a branch of regular solutions of (2–31) and* $\psi^h(\lambda) - \psi(\lambda) \in \mathcal{O}$ *for all* $\lambda \in \Lambda$*. Furthermore, there exists a constant* $C > 0$ *which is independent of* h *and* λ*, such that*

$$\|\psi^h(\lambda) - \psi(\lambda)\|_X \le C\|(T^h - T)G(\lambda, \psi(\lambda))\|_X \quad \forall \lambda \in \Lambda \,. \tag{2–35}$$

The steady–state Navier–Stokes equations can be posed in the form of (2–30) via the Stokes operator and the parameter $\lambda = \dfrac{1}{\nu} =$ Reynold number Re. The fundamental idea of a regular branch in the study of solutions of the steady–state Navier–Stokes equations is based on the fact that bifurcation points and turning points are quite rare (c.f. Temam[128]). Theorem 2.2 also plays an essential role in error estimations for the finite element approximations to the Navier–Stokes equations.

We can apply this structure to the penalized Navier–Stokes equations (2–16)–(2–17) for the study of the convergence when ϵ tends to 0. Take $Y = \mathbf{H}^{-1}_{\Gamma_{\mathbf{o}(\alpha)}}(\Omega(\alpha)) \times \mathbf{H}^{1/2}_0(\Gamma_{\mathbf{g}})$ and $X = \mathbf{H}^{1}_{\Gamma_{\mathbf{o}(\alpha)}}(\Omega(\alpha)) \times L^2_0(\Omega(\alpha)) \times \mathbf{H}^{-1/2}(\Gamma_{\mathbf{g}})$. For the parameter, we take $\lambda = Re = \dfrac{1}{\nu} \in \Lambda \subset R_+$, where R_+ denotes the nonnegative real numbers and Λ is a compact interval in R_+. We also take $X^h = X$ and $Z = \mathbf{L}^{3/2}(\Omega(\alpha)) \times \{\mathbf{0}\}$ in the above discussion.

Let Y be the data space of a Stokes operator and X be its solution space. Let $T : Y \to X$ $\left((\widetilde{\mathbf{f}}, \widetilde{\mathbf{g}}) \mapsto (\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\mathbf{t}})\right)$ be the solution operator for the Stokes problem defined as follows:

$$a(\widetilde{\mathbf{u}},\mathbf{v})+b(\mathbf{v},\widetilde{p})-<\widetilde{\mathbf{t}},\mathbf{v}>_{-1/2,\Gamma_g}=<\widetilde{\mathbf{f}},\mathbf{v}>_{-1}\quad\forall\mathbf{v}\in\mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)),$$
$$b(\widetilde{\mathbf{u}},q)=0\qquad\forall q\in L^2_0(\Omega(\alpha)),\tag{2-36}$$
$$<\mathbf{s},\widetilde{\mathbf{u}}>_{-1/2,\Gamma_g}=<\mathbf{s},\widetilde{\mathbf{g}}>_{-1/2,\Gamma_g}\quad\forall\mathbf{s}\in\mathbf{H}^{-1/2}(\Gamma_g).$$

The nonlinearity of the Navier–Stokes equations is taken into account by the mapping $G:\Lambda\times X\to Y$ ($(\lambda,(\mathbf{w},q,\boldsymbol{\tau}))\mapsto(\boldsymbol{\eta},\boldsymbol{\kappa})$) defined by

$$<\boldsymbol{\eta},\mathbf{v}>_{-1}=\lambda\,c(\mathbf{w},\mathbf{w},\mathbf{v})-\lambda<\mathbf{f},\mathbf{v}>_{-1}\quad\forall\mathbf{v}\in\mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)),$$
$$<\mathbf{s},\boldsymbol{\kappa}>_{-1/2,\Gamma_g}=-<\mathbf{s},\mathbf{g}>_{-1/2,\Gamma_g}\quad\forall\mathbf{s}\in\mathbf{H}^{-1/2}(\Gamma_g),\tag{2-37}$$

where $(\mathbf{f},\mathbf{g})$ is given in $\mathbf{H}^{-1}_{\Gamma_0(\alpha)}(\Omega(\alpha))\times\mathbf{H}^{1/2}_0(\Gamma_g)$.

Since the weak formulation of the Navier–Stokes equations can be written by

$$a(\mathbf{u},\mathbf{v})+b(\mathbf{v},\lambda p)-<\lambda\mathbf{t},\mathbf{v}>_{-1/2,\Gamma_g}$$
$$=-\left[\lambda c(\mathbf{u},\mathbf{u},\mathbf{v})-\lambda<\mathbf{f},\mathbf{v}>_{-1,\Gamma_0(\alpha)}\right]\ \forall\,\mathbf{v}\in\mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)),$$
$$b(\mathbf{u},\lambda q)=0\quad\forall q\in L^2_0(\Omega(\alpha)),\tag{2-38}$$
$$<\mathbf{s},\mathbf{u}>_{-1/2,\Gamma_g}=-\left[-<\mathbf{s},\mathbf{g}>_{-1/2,\Gamma_g}\right]\quad\forall\mathbf{s}\in\mathbf{H}^{-1/2}(\Gamma_g),$$

and the mapping G corresponds to the weak formulation of

$$\begin{cases}\boldsymbol{\eta}=\lambda(\mathbf{w}\cdot\nabla)\mathbf{w}-\lambda\,\mathbf{f},\\ \boldsymbol{\kappa}=-\,\mathbf{g}.\end{cases}$$

Substituting $\mathbf{w}=\mathbf{u}$, we obtain from (2–38) that $q=\lambda p$, $\boldsymbol{\tau}=\lambda\mathbf{t}$ and $(\mathbf{u},\lambda p,\lambda\mathbf{t})=-TG(\mathbf{u},\lambda p,\lambda\mathbf{t})$. Hence,

$$F(\lambda,(\mathbf{u},p,\mathbf{t}))\equiv(\mathbf{u},\lambda p,\lambda\mathbf{t})+TG(\lambda,(\mathbf{u},\lambda p,\lambda\mathbf{t}))=0$$

is equivalent to the variarional formulation of (1–1)–(1–3).

Next, we associate $T^\epsilon:Y\to X$ ($(\widetilde{\mathbf{f}},\widetilde{\mathbf{g}})\mapsto(\widetilde{\mathbf{u}}_\epsilon,\widetilde{p}_\epsilon,\widetilde{\mathbf{t}}_\epsilon)$) with the penalized Stokes operator defined by

$$a(\widetilde{\mathbf{u}}_\epsilon,\mathbf{v})+b(\mathbf{v},\widetilde{p}_\epsilon)-<\widetilde{\mathbf{t}}_\epsilon,\mathbf{v}>_{-1/2,\Gamma_g}=<\widetilde{\mathbf{f}},\mathbf{v}>_{-1}\quad\forall\mathbf{v}\in\mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$$
$$b(\widetilde{\mathbf{u}}_\epsilon,q)=\epsilon\,(\widetilde{p}_\epsilon,q)_0\qquad\forall q\in L^2_0(\Omega(\alpha))\tag{2-39}$$
$$<\mathbf{s},\widetilde{\mathbf{u}}_\epsilon>_{-1/2,\Gamma_g}=<\mathbf{s},\widetilde{\mathbf{g}}>_{-1/2,\Gamma_g}\quad\forall\mathbf{s}\in\mathbf{H}^{-1/2}(\Gamma_g).$$

Then, the penalized Navier–Stokes equations (2–16)-(2–18) is equivalent to

$$F^\epsilon(\lambda,(\mathbf{u}_\epsilon,p_\epsilon,\mathbf{t}_\epsilon)) \equiv (\mathbf{u}_\epsilon,\lambda p_\epsilon,\lambda \mathbf{t}_\epsilon) + T^\epsilon G(\lambda,(\mathbf{u}_\epsilon,\lambda p_\epsilon,\lambda \mathbf{t}_\epsilon)) = 0 \ .$$

Now, we consider the main results for the existence and convergence of the solutions of the penalized Navier–Stokes equations. We first state the existence of a solution for the specified system (2–36) for the Stokes equations, whose existence and uniqueness depends on the following two lemmas.

Lemma 2.3. *Let $\alpha \in \mathcal{U}_{ad}$ be fixed. Let $(q,\boldsymbol{\xi}) \in L_0^2(\Omega(\alpha)) \times \mathbf{H}^{-1/2}(\Gamma_{\mathbf{g}})$. Then, there exists $\mathbf{w} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$ such that*

$$\begin{cases} div\, \mathbf{w} & = q \quad in\ \Omega(\alpha) \\ \mathbf{w} & = \mathcal{R}^{-1}(\boldsymbol{\xi}) \quad on\ \ \Gamma_{\mathbf{g}}\,, \end{cases}$$

where $\mathcal{R} : \mathbf{H}^{1/2}(\Gamma_{\mathbf{g}}) \longrightarrow \mathbf{H}^{-1/2}(\Gamma_{\mathbf{g}})$ is the inverse of the Riesz representation mapping, i.e.,

$$< \boldsymbol{\xi},\boldsymbol{\eta} >_{-1/2,\Gamma_{\mathbf{g}}} = < \mathcal{R}^{-1}(\boldsymbol{\xi}),\boldsymbol{\eta} >_{1/2,\Gamma_{\mathbf{g}}} \quad \forall \boldsymbol{\eta} \in \mathbf{H}^{1/2}(\Gamma_{\mathbf{g}})\,,$$

and $\quad \|\boldsymbol{\xi}\|_{-1/2,\Gamma_{\mathbf{g}}} = \|\mathcal{R}^{-1}(\boldsymbol{\xi})\|_{1/2,\Gamma_{\mathbf{g}}}\,.$

Moreover, there exists a positive constant C such that

$$\|\mathbf{w}\|_1 \leq C(\|q\|_0 + \|\boldsymbol{\xi}\|_{-1/2,\Gamma_{\mathbf{g}}})\,.$$

□

For the proof, refer to Temam[126] and Kikuchi *et al.*[84].

Let us state the augmented LBB condition coupling the pressure and the traction force along $\Gamma_{\mathbf{g}}$. For the notational convenience, let us denote $\mathcal{M} = L_0^2(\Omega(\alpha)) \times \mathbf{H}^{1/2}(\Gamma_{\mathbf{g}})$ and its dual space by $\mathcal{M}'$.

Lemma 2.4. *There exists a constant $C > 0$ such that*

$$C\|(q,\xi)\|_{\mathcal{M}'} \leq \sup_{\substack{\mathbf{w}\in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)) \\ \mathbf{w}\neq 0}} \frac{b(\mathbf{w},q) - <\xi,\mathbf{w}>_{-1/2,\Gamma_\mathbf{g}}}{\|\mathbf{w}\|_1}, \quad \forall (q,\xi)\in \mathcal{M}'. \qquad (2\text{–}40)$$

Proof: Let $(q,\xi) \in L^2_0(\Omega(\alpha)) \times \mathbf{H}^{-1/2}(\Gamma_\mathbf{g})$ be given. By Lemma 2.3, there exists $\mathbf{w} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$ such that $div\, \mathbf{w} = -q$ in $\Omega(\alpha)$, $\mathbf{w} = -\mathcal{R}^{-1}(\xi)$ on $\Gamma_\mathbf{g}$ and $\|\mathbf{w}\|_1 \leq C(\|q\|_0 + \|\tau\|_{-1/2,\Gamma_\mathbf{g}})$. Hence,

$$\begin{aligned} b(\mathbf{w},q) - <\xi,\mathbf{w}>_{-1/2,\Gamma_\mathbf{g}} &= \|q\|_0^2 + \|\mathcal{R}^{-1}(\xi)\|^2_{1/2,\Gamma_\mathbf{g}} \\ &= \|q\|_0^2 + \|\xi\|^2_{-1/2,\Gamma_\mathbf{g}} \\ &\geq C(\|q\|_0 + \|\xi\|_{-1/2,\Gamma_\mathbf{g}})\|\mathbf{w}\|_1 . \quad \square \end{aligned}$$

Let $\mathcal{B} : \mathbf{V} = \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)) \longrightarrow \mathcal{M} = L^2_0(\Omega(\alpha)) \times \mathbf{H}^{1/2}(\Gamma_\mathbf{g})$ be the bounded operator defined by

$$\begin{aligned} <\mathcal{B}\mathbf{w},(q,\xi)>_{\mathcal{M}\times\mathcal{M}'} &= <\mathbf{w},\ {}^T\mathcal{B}(q,\xi)>_{\mathbf{V}\times\mathbf{V}'} \\ &= b(\mathbf{w},q) - <\xi,\mathbf{w}>_{-1/2,\Gamma_\mathbf{g}} . \end{aligned} \qquad (2\text{–}41)$$

Then, Lemma 2.4 implies that

$$\|\ {}^T\mathcal{B}(q,\xi)\|_{\mathbf{V}'} \geq C\|(q,\xi)\|_{\mathcal{M}'} \quad \forall (q,\xi)\in\mathcal{M}',$$

whence $\|\ {}^T\mathcal{B}\|_{\mathcal{L}(\mathcal{M}';\mathbf{V}')} \geq C$. Using this relation, we easily conclude that $\mathcal{B}$ has a closed range in $\mathcal{M}$ and is surjective due to Lemma 2.3. Let us show the existence of the solution of the system (2–36).

Theorem 2.3. *Suppose $b(\cdot,\cdot)$ satisfies the LBB condition (2–7). Then given $(\mathbf{f},\mathbf{g}) \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)) \times \mathbf{H}^{1/2}(\Gamma_\mathbf{g})$, there exists a unique solution $(\mathbf{u},(p,\mathbf{t})) \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)) \times (L^2_0(\Omega(\alpha)) \times \mathbf{H}^{-1/2}(\Gamma_\mathbf{g}))$ satisfying*

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) - <\mathbf{t},\mathbf{v}>_{-1/2,\Gamma_\mathbf{g}} &= <\mathbf{f},\mathbf{v}>_{-1} \ \forall \mathbf{v}\in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)), \\ b(\mathbf{u},r) &= 0 \qquad \forall r \in L^2_0(\Omega(\alpha)), \end{aligned} \qquad (2\text{–}42)$$

$$< \boldsymbol{\eta}, \mathbf{u} >_{-1/2,\Gamma_g} = < \boldsymbol{\eta}, \mathbf{g} >_{-1/2,\Gamma_g} \quad \forall \boldsymbol{\eta} \in \mathbf{H}^{-1/2}(\Gamma_g).$$

Proof: Using (2–41), (2–42) can be rewritten as

$$\begin{aligned} & a(\mathbf{u}, \mathbf{v}) + < \mathcal{B}\mathbf{v}, (p, \mathbf{t}) > = < \mathbf{f}, \mathbf{v} >_{-1} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)), \\ & < \mathcal{B}\mathbf{u}, (r, \boldsymbol{\eta}) > = - < \boldsymbol{\eta}, \mathbf{g} >_{-1/2,\Gamma_g} \quad \forall (r, \boldsymbol{\eta}) \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)) \times \mathbf{H}^{-1/2}(\Gamma_g). \end{aligned} \tag{2–43}$$

Since $\mathbf{u} = \mathbf{0}$ on $\Gamma_0(\alpha)$, it is easy to check that $Ker\, \mathcal{B} \subset \mathbf{H}^1_0(\Omega(\alpha))$. Since $\sqrt{a(\cdot,\cdot)} = |\!|\!| \cdot |\!|\!|$ is equivalent to $\|\cdot\|_1$, it is obvious that $a(\cdot,\cdot)$ is coercive over $Ker\, \mathcal{B}$. So, combined with augmented LBB condition (2–40), the existence theorem for the abstract mixed formulation([60]) yields the result. □

From the well–posedness of (2–42), the following estimate immediately follows:

$$\|\mathbf{u}\|_1 + \|p\|_0 + \|\mathbf{t}\|_{-1/2,\Gamma_g} \le C(\|\mathbf{f}\|_{-1} + \|\mathbf{g}\|_{1/2,\Gamma_g}).$$

In the penalized problem corresponding to (2–42), Lagrange multipliers can be used to relax both constraints for the incompressibility and inhomogeneous boundary condition. Let us consider the following saddle point problem:

$$\inf_{\boldsymbol{\xi} \in \mathbf{H}^{1/2}(\Gamma_g)} \sup_{\mathbf{v} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))} \mathcal{E}(\mathbf{v}, \boldsymbol{\xi}),$$

where $\mathcal{E}(\cdot,\cdot)$ is a Lagrangian difined by

$$\mathcal{E}(\mathbf{v}, \boldsymbol{\xi}) = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) + \frac{1}{2\epsilon} d(\mathbf{v}, \mathbf{v}) - \mathbf{f}(\mathbf{v}) - \int_{\Gamma_g} \boldsymbol{\xi}(\mathbf{v} - \mathbf{g})\, d\Gamma.$$

Note that the coercivity of $a(\cdot,\cdot)$ and the augmented LBB condition (2-40) may guarantee the existence and uniqueness of the saddle point of the quadratic form $\mathcal{E}$ (refer to [84] for the discussion for the existence of the saddle point). The saddle point $(\mathbf{u}_\epsilon, \mathbf{t}_\epsilon)$ of $\mathcal{E}$ satisfies

$$\begin{aligned} & a(\mathbf{u}_\epsilon, \mathbf{v}) + \frac{1}{\epsilon} d(\mathbf{u}_\epsilon, \mathbf{v}) - < \mathbf{t}_\epsilon, \mathbf{v} > = \mathbf{f}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)), \\ & < \boldsymbol{\xi}, \mathbf{u}_\epsilon >_{\Gamma_g} = < \boldsymbol{\xi}, \mathbf{g} > \quad \forall \boldsymbol{\xi} \in \mathbf{H}^{-1/2}(\Gamma_g). \end{aligned} \tag{2–44}$$

Then the pressure can be recovered from $p_\epsilon = -\frac{1}{\epsilon}\, div\, \mathbf{u}_\epsilon$ and the Lagrange multiplier is given by $\mathbf{t}_\epsilon = (-p_\epsilon \mathcal{I} + \frac{\partial \mathbf{u}_\epsilon}{\partial \mathbf{n}}) \circ \mathbf{n}$. Hence the traction force plays the role of Lagrange multiplier to enforce the inhomogeneous boundary condition. For this reason, the variational formulation of the form (2–36) (or (2–39)) is called the *mixed Lagrangian formulation* for the (penalized) Stokes system incorporating a inhomogeneous boundary condition. Some implementation issues shall be discussed in Chapter IV.

We are now ready to show the existence and convergence for the solution for the penalized problem.

Theorem 2.4. *Let $\alpha \in \mathcal{U}_{ad}$ be fixed. Let $\{(\lambda, (\mathbf{u}(\lambda), \lambda p(\lambda), \lambda \mathbf{t}(\lambda)) \mid \lambda = \frac{1}{\nu} \in \Lambda\}$ be a branch of regular solutions of (2–8). Then, there exists a neighborhood $\mathcal{O}$ of the origin in $\mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)) \times L^2_0(\Omega(\alpha)) \times \mathbf{H}^{-1/2}(\Gamma_\mathbf{g})$ and for $\epsilon \le \epsilon_0$ small enough, a unique $\mathcal{C}^2$ branch $\{(\lambda, (\mathbf{u}_\epsilon(\lambda), \lambda p_\epsilon(\lambda), \lambda \mathbf{t}_\epsilon(\lambda)) \mid \lambda \in \Lambda\}$ of (2–16)–(2–18) such that $\mathbf{u}_\epsilon(\lambda) - \mathbf{u}(\lambda) \in \mathcal{O}$ for all $\lambda \in \Lambda$. Moreover, there exists a constant $C > 0$ which is independent of ϵ and λ, such that*

$$\|\mathbf{u}_\epsilon(\lambda) - \mathbf{u}(\lambda)\|_{1,\Omega(\alpha)} + \|p_\epsilon(\lambda) - p(\lambda)\|_{0,\Omega(\alpha)} + \|\mathbf{t}_\epsilon(\lambda) - \mathbf{t}(\lambda)\|_{-1/2,\Gamma_\mathbf{g}} \le C\epsilon \qquad \forall \lambda \in \Lambda. \quad (2\text{–}45)$$

Proof: We wish to apply Theorem 2.2. Let $\psi = (\mathbf{u}, p, \mathbf{t})$. Note that

$$D_\psi G(\lambda, (\mathbf{u}, p, \mathbf{t})) \cdot (\mathbf{v}, q, \mathbf{s}) = \lambda\, (((\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v}), \mathbf{0}) \in Y$$

and

$$D_{\psi\psi} G(\lambda, (\mathbf{u}, p, \mathbf{t})) \cdot ((\mathbf{v}, q, \mathbf{s}), (\widetilde{\mathbf{v}}, \widetilde{q}, \widetilde{\mathbf{s}})) = \lambda\, (((\mathbf{v} \cdot \nabla)\widetilde{\mathbf{v}} + (\widetilde{\mathbf{v}} \cdot \nabla)\mathbf{v}), \mathbf{0}) \in Y$$

for all $(\mathbf{v}, q, \mathbf{s})$, $(\widetilde{\mathbf{v}}, \widetilde{q}, \widetilde{\mathbf{s}}) \in X$. It is clear that G belongs to $\mathcal{C}^2$ and that $D_\psi G$ and $D_{\psi\psi} G$ are bounded on all bounded subset of $\Lambda \times X$ by the Sobolev embedding theorem. It should be noted that $(\mathbf{v}\cdot\nabla)\,\mathbf{u}+(\mathbf{u}\cdot\nabla)\,\mathbf{v}$ belongs to $\mathbf{L}^{3/2}(\Omega(\alpha))$ which is compactly embedded in $\mathbf{H}^{-1}(\Omega(\alpha))$. Since $Z=\mathbf{L}^{3/2}(\Omega(\alpha))\times\{\mathbf{0}\}$ is compactly embedded in $Y=\mathbf{H}^{-1}(\Omega(\alpha))\times\mathbf{H}_0^{1/2}(\Gamma_{\mathbf{g}})$, (2–34) follows from (2–33). To show (2–33), we consider (2–36) and (2–39). By Theorem 2.3, (2–36) has a unique solution $(\widetilde{\mathbf{u}},\widetilde{p},\widetilde{\mathbf{t}})$ in $X=\mathbf{H}^1_{\Gamma_{\mathbf{0}(\alpha)}}(\Omega(\alpha))\times L_0^2(\Omega(\alpha))\times\mathbf{H}^{-1/2}(\Gamma_{\mathbf{g}})$. Subtracting (2–36) from (2–39), we have

$$\begin{gathered}
a\,(\widetilde{\mathbf{u}}_\epsilon-\widetilde{\mathbf{u}},\mathbf{v})+b\,(\mathbf{v},\widetilde{p}_\epsilon-\widetilde{p})-<\widetilde{\mathbf{t}}_\epsilon-\widetilde{\mathbf{t}},\mathbf{v}>_{-1/2,\Gamma_{\mathbf{g}}}=0\ \forall\mathbf{v}\in\mathbf{H}^1_{\Gamma_{\mathbf{0}(\alpha)}}(\Omega(\alpha))\ ,\\
b\,(\widetilde{\mathbf{u}}_\epsilon-\widetilde{\mathbf{u}},q)=\epsilon\,(\widetilde{p}_\epsilon-\widetilde{p},q)_0+\epsilon\,(\widetilde{p},q)_0\ \ \forall q\in L_0^2(\Omega(\alpha))\ ,\\
<\mathbf{s},\widetilde{\mathbf{u}}_\epsilon-\widetilde{\mathbf{u}}>_{-1/2,\Gamma_{\mathbf{g}}}=0\quad\forall\mathbf{s}\in\mathbf{H}^{-1/2}(\Gamma_{\mathbf{g}})\ .
\end{gathered}\tag{2–46}$$

Taking $\mathbf{v}$ in $\mathbf{H}_0^1(\Omega(\alpha))$, then the first equation is reduced to

$$a\,(\widetilde{\mathbf{u}}_\epsilon-\widetilde{\mathbf{u}},\mathbf{v})+b\,(\mathbf{v},\widetilde{p}_\epsilon-\widetilde{p})=0\qquad\forall\mathbf{v}\in\mathbf{H}_0^1(\Omega(\alpha))\ .\tag{2–47}$$

Since $b\,(\cdot,\cdot)$ satisfy the inf–sup condition, (2–47) yields

$$\frac{1}{C}\|\widetilde{p}_\epsilon-\widetilde{p}\|_0\le\sup_{\substack{\mathbf{v}\in\mathbf{H}_0^1(\Omega(\alpha))\\ \mathbf{v}\neq\mathbf{0}}}\frac{b\,(\mathbf{v},\widetilde{p}_\epsilon-\widetilde{p})}{\|\mathbf{v}\|_1}\le\|\widetilde{\mathbf{u}}_\epsilon-\widetilde{\mathbf{u}}\|_1\ ,$$

for some positive constant C and hence

$$\|\widetilde{p}_\epsilon-\widetilde{p}\|_0\le C\|\widetilde{\mathbf{u}}_\epsilon-\widetilde{\mathbf{u}}\|_1\ .\tag{2–48}$$

By taking $\mathbf{v}=\widetilde{\mathbf{u}}_\epsilon-\widetilde{\mathbf{u}}$ and $q=\widetilde{p}_\epsilon-\widetilde{p}$ in (2–46) and (2–47), and by substituting the second equation of (2–46) to (2–47), we find

$$a\,(\widetilde{\mathbf{u}}_\epsilon-\widetilde{\mathbf{u}},\widetilde{\mathbf{u}}_\epsilon-\widetilde{\mathbf{u}})=-\epsilon\,(\widetilde{p}_\epsilon-\widetilde{p},\widetilde{p}_\epsilon-\widetilde{p})_0-\epsilon\,(\widetilde{p},\widetilde{p}_\epsilon-\widetilde{p})_0\le-\epsilon\,(\widetilde{p},\widetilde{p}_\epsilon-\widetilde{p})_0\ .$$

Therefore, combining (2–48) with (1–16) we obtain

$$\|\widetilde{\mathbf{u}}_\epsilon-\widetilde{\mathbf{u}}\|_1\le C\epsilon\,\|\widetilde{p}\|_0\qquad\text{and}\qquad\|\widetilde{p}_\epsilon-\widetilde{p}\|_0\le C^2\epsilon\,\|\widetilde{p}\|_0\ .$$

Finally, the first equation of (2–46) yields

$$< \widetilde{\mathbf{t}}_\epsilon - \widetilde{\mathbf{t}}, \gamma_{\Gamma_{\mathbf{g}}}\mathbf{v} >_{-1/2,\Gamma_{\mathbf{g}}} \le \|\widetilde{\mathbf{u}}_\epsilon - \widetilde{\mathbf{u}}\|_1 \|\mathbf{v}\|_1 + \|div\,\mathbf{v}\|_0 \|\widetilde{p}_\epsilon - \widetilde{p}\|_0$$
$$\le (\|\widetilde{\mathbf{u}}_\epsilon - \widetilde{\mathbf{u}}\|_1 + \|\widetilde{p}_\epsilon - \widetilde{p}\|_0)\|\mathbf{v}\|_1 \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)),$$

whence from Lemma 2.2, we obtain

$$\|\widetilde{\mathbf{t}}_\epsilon - \widetilde{\mathbf{t}}\|_{-1/2,\Gamma_{\mathbf{g}}} \le \|\widetilde{\mathbf{u}}_\epsilon - \widetilde{\mathbf{u}}\|_1 + \|\widetilde{p}_\epsilon - \widetilde{p}\|_0 \le (C + C^2)\epsilon\,\|\widetilde{p}\|_0 .$$

Therefore, we have shown

$$\lim_{\epsilon \to 0} \|(T^\epsilon - T)(\widetilde{\mathbf{f}}, \widetilde{\mathbf{g}})\|_X = 0 \quad \text{for all} \quad (\widetilde{\mathbf{f}}, \widetilde{\mathbf{g}}) \in \mathbf{H}^{-1}_{\Gamma_0(\alpha)}(\Omega(\alpha)) \times \mathbf{H}^{1/2}(\Gamma_{\mathbf{g}})$$

and Theorem 2.3 immediately follows from Theorem 2.2. □

Theorem 2.3 implies that branches of regular solutions of the penalized variational formulation are convergent to those of the primal variational formulations.

Since solutions of the Navier–Stokes equations are regular for almost all Reynolds numbers, the solutions of the Navier–Stokes equations are locally unique. Hence it is reasonable to revise the cost functional $\mathfrak{J}(\alpha)$ to

$$\mathfrak{J}_\epsilon(\alpha) = J_\epsilon(\Omega(\alpha), \mathbf{u}_\epsilon(\alpha)) = \nu \int_{\Omega(\alpha)} \nabla \mathbf{u}_\epsilon \;:\; \nabla \mathbf{u}_\epsilon \, d\Omega \,, \tag{2–49}$$

where $\mathbf{u}_\epsilon$ is a local solution of the penalized formulation (2–16)–(2–17). Our primal problem (1–5) is then turned into the problem of finding $\alpha^* \in \mathcal{U}_{ad}$ such that

$$\mathfrak{J}_\epsilon(\alpha^*) \le \mathfrak{J}_\epsilon(\alpha) \qquad \forall \alpha \in \mathcal{U}_{ad} \,. \tag{2–50}$$

However, the solution of (2–50) is coupled with the solution of the Navier–Stokes equations on the corresponding domain. Hence, it is natural to consider the setting of the coupled admissible family. This will be discussed in the next section in conjunction with the existence of optimal solutions.

2.4. The Existence of Optimal Solutions

The main purpose of this section is to show the existence of an optimal solution for the penalized problem. We first introduce some concepts dealing with convergence in function spaces and domains. Let X be a normed vector space. We use the notation "$x_n \rightharpoonup x$" to denote the weak convergence of a sequence $\{x_n\}$ in X to x, i.e.,

$$x_n \rightharpoonup x \Longleftrightarrow \ < f, x_n >_{X^*} \xrightarrow{(n\to\infty)} < f, x >_{X^*} \quad \text{for any } f \in X^* \ .$$

Let Y be a subspace of X. Y is called a *weakly closed subspace* of X if for every sequence in $\{x_n\}$ in Y, whenever $x_n \rightharpoonup x^*$ in X, we have $x^* \in Y$. In connection with optimal controls, the following Lemma is very useful for verifying the weak convergence of sequences.

Lemma 2.3. *Let X be a normed vector space. A sequence $\{x_n\}$ in X converges weakly to $x \in X$ if and only if $\sup_n \|x_n\|_X < \infty$ and $< f, x_n > \to < f, x >$ for each $f \in F$, where F is a linear span of a set which is dense in X^*. Moreover, if X is a reflexive Banach space, each bounded sequence in X contains a weakly convergent subsequence.* □

For the proof, one may refer to Taylor[124].

In general, the most crucial concept in optimization is semi–continuity, especially when the cost functional contains the gradient of the function. Let S be a subset of X and J be a real functional on S. We say that J is *(weakly) lower semi–continuous* if for every sequence $\{x_n\}$ in S,
whenever

$$x_n \longrightarrow x \quad (x_n \rightharpoonup x) \quad \text{in } X \ ,$$

we have

$$\liminf_{n\to\infty} J(x_n) \geq J(x) \ .$$

Note that the notion of (weak) lower semicontinuity is a local property.

To deal with domain optimization, we need to define an appropriate convergence criterion with respect to domains. Since domains and corresponding function spaces are changing, we need a fixed domain $\widehat{\Omega}$ such that $\cup_{\alpha \in \mathcal{U}_{ad}} \Omega(\alpha) \subset \widehat{\Omega}$ to discuss the convergence of domains and corresponding solutions of the state equations. We suppose hereafter that $\widehat{\Omega}$ is the interior of the rectangular region $ABCD$ and Ω_0 be the interior of the rectangular region $EFGH$ in (Figure 2) so that

$$\bigcup_{\alpha \in \mathcal{U}_{ad}} \Omega(\alpha) \subset \overline{\widehat{\Omega}} \qquad \text{and} \qquad \bigcup_{\alpha \in \mathcal{U}_{ad}} \Gamma(\alpha) \subset \overline{\Omega}_0 \, . \tag{2–51}$$

The domain class on which optimal shape problems usually have an optimal solution has been studied by Chenais[29] and Fujii[57]. It was shown by Chenais[29] that the set of domains with the cone property is compact for the strong $L^2(\widehat{\Omega})$–topology of the characteristic functions of its elements. Let χ_Ω denote the characteristic function of the domain Ω which is included in $\widehat{\Omega}$. The convergence of the sequence $\{\Omega_m\}$ of domains having the cone property may be defined by

$$\Omega_m \to \Omega \iff \int_{\widehat{\Omega}} |\chi_{\Omega_m} - \chi_\Omega|^2 \, d\Omega \to 0 \quad \text{as} \quad m \to \infty \, .$$

The method of convergence using characteristic functions is often used to solve some specific shape optimization problems such as the transmission problem governed by a pair of different elliptic equations over interconnecting regions (c.f. Sokolowski *et al.*[121], Pironneau[101] and Cèa[26]). However, since the convergence of characteristic functions does not preserve the regularity of domains, it is not appropriate for dealing with general shape optimization problems in which the regularity of domains is a concern.

In our case, domains $\{\Omega(\alpha)\}_{\alpha \in \mathcal{U}_{ad}}$ are determined by the variable part $\Gamma(\alpha)$ of the boundary Γ. Thus, it is more natural to define the convergence of domains in terms of a parameter α belonging to $\mathcal{U}_{ad}$. Let $\{\alpha_n\}$ be a sequence in $\mathcal{U}_{ad}$. Let $\Omega_n = \Omega(\alpha_n)$, for each $\alpha_n \in \mathcal{U}_{ad}$. We define the convergence of Ω_n to $\Omega(\alpha)$ by

$$\Omega_n \longrightarrow \Omega(\alpha) \Longleftrightarrow \|\alpha_n - \alpha\|_\infty \equiv \max_{-M_1 \le x \le M_1} |\alpha_n(x) - \alpha(x)| \to 0 \ . \tag{2–52}$$

REMARK 2.5: In general shape optimization problems, more stringent topologies are often introduced to enforce the convergence of geometrical elements (c.f. Pironneau[101] and Liu *et al.*[91]). When the inclusive relation between subdomains of R^n should be the main issue as in problems of domain identification, the topology induced by the following *Hausdorff metric* is widely used:

Let A and B be two closed subsets of R^n and define the Hausdorff metric δ by

$$\delta(A, B) = \max\{\rho(A, B), \rho(B, A)\}, \qquad \text{where } \rho(A, B) = \sup_{x \in A} \inf_{y \in B} |x - y|_{R^n} \, .$$

Then, the topology on the closed subsets of R^n is defined by

$$A_m \longrightarrow A \Longleftrightarrow \delta(A_m, A) \to 0 \, .$$

The most important property of this topology is that it preserves the relation of domain inclusions.

Another important topology can be used in conjunction with mapping techniques. Let D be a fixed domain in R^n. Suppose domain perturbations are described by the family of bijective mappings having some regularities, for example, $\mathfrak{F}^k = \{T(D) \mid T \in C^k \text{ and } T \text{ is bijective }\}$. Then, the convergence of domains can be explained using the minimal norm

$$\|T - \mathcal{I}\| + \|T^{-1} - \mathcal{I}\|$$

among the mappings T such that $T(D) \in \mathfrak{F}^k$.

Recall that $\Omega(\alpha)$ is a uniformly Lipschitz continuous domain for each $\alpha \in \mathcal{U}_{ad}$.

Hence it has the uniform extension property (refer to Section 1.2.). For any $\mathbf{v} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$, let $\tilde{\mathbf{v}}$ be its extension to $\mathbf{H}^1(\widehat{\Omega})$, *i.e.*,

$$\tilde{\mathbf{v}} = P_{\widehat{\Omega}}\mathbf{v} \ , \tag{2-53}$$

where $P_{\widehat{\Omega}}$ is the Calderón's extension of $\mathbf{v}$ defined on $\Omega(\alpha)$ to $\widehat{\Omega}$. Then, by Theorem 1.1, there exists some positive constant C such that $\|\tilde{\mathbf{v}}\|_{1,\widehat{\Omega}} \leq C\|\mathbf{v}\|_{1,\Omega(\alpha)}$, where C can be chosen to be independent of $\alpha \in \mathcal{U}_{ad}$.

If $\mathbf{v}_n \in \mathbf{H}^1_{\Gamma_0(\alpha_n)}(\Omega_n)$ and $\mathbf{v} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$, the convergence "$\mathbf{v}_n \to \mathbf{v}$" is defined by

$$\mathbf{v}_n \longrightarrow \mathbf{v} \Longleftrightarrow \tilde{\mathbf{v}}_n \equiv P_{\widehat{\Omega}}\mathbf{v}_n \rightharpoonup P_{\widehat{\Omega}}\mathbf{v} \equiv \tilde{\mathbf{v}} \quad \text{in} \ \ \mathbf{H}^1(\widehat{\Omega}) \ . \tag{2-54}$$

We now turn to the question of existence of optimal solutions. We will use what is called *a direct method* in the calculus of variations, *i.e.*, we will try to minimize the cost functional directly rather than to solve the Euler–Lagrange equations. As we have already indicated in the previous section, we reset the admissible family $\mathcal{U}_{ad}$ into the coupled admissible family $\widetilde{\mathcal{U}_{ad}}$ as follows:

$$\widetilde{\mathcal{U}_{ad}} = \{(\alpha, \mathbf{u}_\epsilon(\alpha)) \in \mathcal{U}_{ad} \times \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)) \mid J_\epsilon(\alpha, \mathbf{u}_\epsilon) < \infty, \ \text{and there exists a}$$
$$\mathbf{t}_\epsilon \in \mathbf{H}^{-1/2}(\Gamma_\mathbf{g}) \ \text{such that (2–16)–(2–17) are satisfied}\} \ . \tag{2-55}$$

Let us consider the modified optimal shape problem for the penalized Navier–Stokes equations:

$$\begin{aligned}
&\text{Find } (\alpha^*, \mathbf{u}_\epsilon(\alpha^*)) \in \widetilde{\mathcal{U}_{ad}} \text{ such that} \\
&J_\epsilon(\alpha^*, \mathbf{u}_\epsilon(\alpha^*)) \leq J_\epsilon(\alpha, \mathbf{u}_\epsilon(\alpha)) \quad \text{for all} \ \ (\alpha, \mathbf{u}_\epsilon(\alpha)) \in \widetilde{\mathcal{U}_{ad}} \\
&\text{where } \mathbf{u}_\epsilon(\alpha) \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)) \text{ is a solution of (2–16) and (2–17)} .
\end{aligned} \tag{2-56}$$

Fujii[57] has studied a certain class of functionals with the lower semi–continuity in domain optimization problems. We state one useful result.

Lemma 2.4. *Let Ω and $\{\Omega_m\}$ be bounded domains having the cone property. Let u and u_m be elements in $H^1(\Omega)$ and $H^1(\Omega_m)$, respectively. Assume that $f(p)$ is continuous, nonnegative, and convex for $p \in R^n$. Then, the inequality*

$$\int_\Omega f(\nabla u(x))\, d\Omega \leq \liminf_{m\to\infty} \int_{\Omega_m} f(\nabla u_m(x))\, d\Omega$$

holds. □

For the proof, see [57].

We are now in a stage to show the existence of an optimal solution for the problem (2–56).

Theorem 2.4. *There exists at least one optimal solution $(\alpha^*, \mathbf{u}_\epsilon(\alpha^*)) \in \widetilde{\mathcal{U}_{ad}}$ for the problem (2–56).*

Proof: The nonemptiness of $\widetilde{\mathcal{U}_{ad}}$ follows from Theorem 2.3 for the existence of regular branches of the penalized variational formulation. Note that

$$\begin{aligned} C\|\mathbf{u}_\epsilon(\alpha)\|^2_{1,\Omega(\alpha)} &\leq J_\epsilon(\alpha, \mathbf{u}_\epsilon(\alpha)) = \nu \, |\!|\!| \mathbf{u}_\epsilon(\alpha) |\!|\!|^2 \\ &\leq \nu \, \|\mathbf{u}_\epsilon(\alpha)\|^2_{1,\Omega(\alpha)} \qquad \text{for some constant } C > 0\,. \end{aligned} \tag{2–57}$$

The first inequality follows from the Korn's inequality (Lemma 1.3). So, $J_\epsilon(\alpha, \mathbf{u}_\epsilon(\alpha))$ is coercive and strongly continuous over $\mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$ for each $\alpha \in \mathcal{U}_{ad}$. Moreover, it readily follows that $J_\epsilon(\alpha, \mathbf{u}_\epsilon(\alpha))$ is convex with respect to $\mathbf{u}_\epsilon$. Hence $J_\epsilon(\alpha, \mathbf{u}_\epsilon(\alpha))$ is weakly lower semi–continuous by Lemma 2.4. We define $\mathbf{u}_\epsilon^{(n)} \equiv \mathbf{u}_\epsilon(\alpha_n)$, where $\{\alpha_n\}$ is a sequence in $\mathcal{U}_{ad}$. Let $\{(\alpha_n, \mathbf{u}_\epsilon^{(n)})\}$ be a sequence in $\widetilde{\mathcal{U}_{ad}}$. Since $J_\epsilon(\alpha, \mathbf{u}_\epsilon(\alpha))$ is obviously bounded from below for every $\alpha \in \mathcal{U}_{ad}$, there exists a minimizing subsequence, which is denoted by the same notation $\{(\alpha_n, \mathbf{u}_\epsilon^{(n)})\}$, *i.e.*, there exists a sequence $\{(\alpha, \mathbf{u}_\epsilon^{(n)})\} \in \widetilde{\mathcal{U}_{ad}}$ such that

$$\lim_{n\to\infty} J_\epsilon(\alpha_n, \mathbf{u}_\epsilon^{(n)}) = \liminf_{(\alpha, \mathbf{u}_\epsilon(\alpha)) \in \widetilde{\mathcal{U}_{ad}}} J_\epsilon(\alpha, \mathbf{u}_\epsilon(\alpha))\ .$$

Since $\Omega_n \equiv \Omega(\alpha_n)$ is contained in $\overline{\Omega}_0$ for $\{\alpha_n\} \subset \mathcal{U}_{ad}$, it is a family of uniformly bounded equicontinuous functions in $\overline{\Omega}_0$. Hence, by definition of $\mathcal{U}_{ad}$ and the Ascoli–Arzelà theorem, there exists a subsequence of $\{\alpha_n\}$, which we denote by the same notation $\{\alpha_n\}$ again, and $\alpha^* \in \mathcal{U}_{ad}$ such that $\alpha_n \to \alpha^*$ uniformly in $[-M_1, M_1]$. According to (2–55) and (2–57), there exists a positive constant K such that $\|\mathbf{u}_\epsilon^{(n)}\|_{1,\Omega_n} < K < \infty$ for all n. Furthermore, due to the uniform extension we can take a positive constant C to be independent of n such that

$$\|\widetilde{\mathbf{u}}_\epsilon{}^{(n)}\|_{1,\widehat{\Omega}} \leq C\|\mathbf{u}_\epsilon^{(n)}\|_{1,\Omega_n} \; .$$

Thus, $\|\widetilde{\mathbf{u}}_\epsilon^{(n)}\|_{1,\widehat{\Omega}}$ is uniformly bounded in $\mathbf{H}^1(\widehat{\Omega})$. Also, it is obvious that

$$\mathbf{t}_\epsilon^{(n)} = \nu\,(\mathbf{n}\cdot\nabla)\,\mathbf{u}_\epsilon^{(n)} + \frac{1}{\epsilon}\,(div\,\mathbf{u}_\epsilon{}^{(n)})\,\mathbf{n} \in \mathbf{H}^{-1/2}(\Gamma_\mathbf{g})$$

is uniformly bounded. Consequently, using the compactness of the continuous embedding $\mathbf{H}^1(\widehat{\Omega}) \subset \mathbf{L}^2(\widehat{\Omega})$ and $\mathbf{H}^{1/2}(\Gamma_\mathbf{g}) \subset \mathbf{L}^2(\Gamma_\mathbf{g})$, we may extract a subsequence of $\{(\widetilde{\mathbf{u}}_\epsilon^{(n)}, \mathbf{t}_\epsilon^{(n)})\}$ (denoted by $\{(\widetilde{\mathbf{u}}_\epsilon^{(n)}, \mathbf{t}_\epsilon^{(n)})\}$ again) in $\mathbf{H}^1(\widehat{\Omega}) \times \mathbf{H}^{-1/2}(\Gamma_\mathbf{g})$ such that

$$\widetilde{\mathbf{u}}_\epsilon^{(n)} \rightharpoonup \widetilde{\mathbf{u}}_\epsilon \quad \text{in} \quad \mathbf{H}^1(\widehat{\Omega}) \tag{2–58}$$

$$\widetilde{\mathbf{u}}_\epsilon^{(n)} \to \widetilde{\mathbf{u}}_\epsilon \quad \text{in} \quad \mathbf{L}^2(\widehat{\Omega}) \tag{2–59}$$

$$\mathbf{t}^{(n)} \rightharpoonup \mathbf{t}_\epsilon \quad \text{in} \quad \mathbf{H}^{-1/2}(\Gamma_\mathbf{g}) \tag{2–60}$$

$$\gamma_{\Gamma_\mathbf{g}}(\widetilde{\mathbf{u}}_\epsilon^{(n)}) \rightharpoonup \gamma_{\Gamma_\mathbf{g}}(\widetilde{\mathbf{u}}_\epsilon) \quad \text{in} \quad \mathbf{H}^{1/2}(\Gamma_\mathbf{g}) \tag{2–61}$$

$$\gamma_{\Gamma_\mathbf{g}}(\widetilde{\mathbf{u}}_\epsilon^{(n)}) \to \gamma_{\Gamma_\mathbf{g}}(\widetilde{\mathbf{u}}_\epsilon) \quad \text{in} \quad \mathbf{L}^2(\Gamma_\mathbf{g}), \tag{2–62}$$

for some $(\widetilde{\mathbf{u}}_\epsilon, \mathbf{t}_\epsilon) \in \mathbf{H}^1(\widehat{\Omega}) \times \mathbf{H}^{-1/2}(\Gamma_\mathbf{g})$. Now we define $\mathbf{u}_\epsilon(\alpha^*) = \widetilde{\mathbf{u}}_\epsilon\Big|_{\Omega(\alpha^*)}$. We wish to show that $\mathbf{u}_\epsilon(\alpha^*)$ is a solution of (2–16)–(2–17) over $\Omega(\alpha^*)$ and $\mathbf{t}_\epsilon$ satisfies

$$\mathbf{t}_\epsilon = \nu\,(\mathbf{n}\cdot\nabla)\,\mathbf{u}_\epsilon(\alpha^*) + \frac{1}{\epsilon}\,(div\,\mathbf{u}_\epsilon(\alpha^*))\,\mathbf{n} \quad \text{in} \quad \mathbf{H}^{-1/2}(\Gamma_\mathbf{g}) \, . \tag{2–63}$$

Let us define function spaces

$$\boldsymbol{W}_n = \{\varphi \in \mathcal{C}^\infty(\overline{\Omega_n})^2 \mid \varphi = \mathbf{0} \;\text{ in a neighborhood of }\; \Gamma_0(\alpha_n) = \Gamma_3 \cup \Gamma(\alpha_n)\}$$

and

$$\boldsymbol{W} = \{\varphi \in \mathcal{C}^\infty(\overline{\Omega(\alpha^*)})^2 \mid \varphi = \mathbf{0} \ \text{ in a neighborhood of } \ \Gamma_0(\alpha^*) = \Gamma_3 \cup \Gamma(\alpha^*)\}.$$

Then, it is clear that

$$\begin{aligned} \mathbf{H}^1_{\Gamma_0(\alpha_n)}(\Omega_n) &= \text{ the closure of } \boldsymbol{W}_n \text{ in } \mathbf{H}^1(\Omega_n) \quad \text{and} \\ \mathbf{H}^1_{\Gamma_0(\alpha^*)}(\Omega(\alpha^*)) &= \text{ the closure of } \boldsymbol{W} \text{ in } \mathbf{H}^1(\Omega(\alpha^*)) \ . \end{aligned}$$

We may consider $\mathbf{H}^1_{\Gamma_0(\alpha^*)}(\Omega(\alpha^*))$ as a closed subspace of

$$\mathbf{H}^1_L(\widehat{\Omega}) = \{\mathbf{u} \in \mathbf{H}^1(\widehat{\Omega}) \mid \mathbf{u}(x_1, 0) = \mathbf{0} = \mathbf{u}(x_1, L)\,\} \tag{2–64}$$

by extending all the elements of $\mathbf{H}^1_{\Gamma_0(\alpha^*)}(\Omega(\alpha^*))$ to be $\mathbf{0}$ in $\overline{\widehat{\Omega}} - \Omega(\alpha^*)$. Let us take $\varphi = (\varphi_1, \varphi_2) \in \boldsymbol{W}$. Since $\alpha_n \to \alpha^*$ uniformly, $\varphi \in \boldsymbol{W}_m$ for sufficiently large m (we suppose $m \geq m_0$, for example). We consider the equations (2–16)–(2–17) over Ω_m for $m \geq m_0$. If we substitute φ for $\mathbf{v}$, we obtain that

$$\begin{aligned} \nu \int_{\Omega_m} \nabla \mathbf{u}_\epsilon^{(m)} \, : \, \nabla \varphi \, d\Omega + \int_{\Omega_m} (\mathbf{u}_\epsilon^{(m)} \cdot \nabla)\, \mathbf{u}_\epsilon^{(m)} \cdot \varphi \, d\Omega + \frac{1}{\epsilon} \int_{\Omega_m} div\, \mathbf{u}_\epsilon{}^{(m)} \, div \varphi \, d\Omega \\ - < \mathbf{t}_\epsilon^{(m)}, \varphi >_{-1/2, \Gamma_\mathbf{g}} = < \mathbf{f}, \varphi >_{-1, \Omega_m} \end{aligned} \tag{2–65}$$

and

$$< \mathbf{s}, \mathbf{u}_\epsilon^{(m)} >_{-1/2, \Gamma_\mathbf{g}} = < \mathbf{s}, \mathbf{g} >_{-1/2, \Gamma_\mathbf{g}} \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_\mathbf{g})\,. \tag{2–66}$$

We take into account each term separately. We first note that

$$\begin{aligned} \int_{\Omega_m} \nabla \mathbf{u}_\epsilon^{(m)} \, : \, \nabla \varphi \, d\Omega &= \int_{\widehat{\Omega}} \nabla \tilde{\mathbf{u}}_\epsilon^{(m)} \, : \, \nabla \varphi \, d\Omega && \text{(by the extension of } \mathbf{u}_\epsilon^{(m)} \text{ to } \widehat{\Omega}) \\ &\overset{(m\to\infty)}{\longrightarrow} \int_{\widehat{\Omega}} \nabla \tilde{\mathbf{u}}_\epsilon \, : \, \nabla \varphi \, d\Omega && \text{(from (2–58))} \\ &= \int_{\Omega(\alpha^*)} \nabla \mathbf{u}_\epsilon(\alpha^*) \, : \, \nabla \varphi \, d\Omega && \text{(by the choice of } \varphi \text{ in } \boldsymbol{W}) \ . \end{aligned}$$

In a similar fashion, we can show that

$$\int_{\Omega_m} div\, \mathbf{u}_\epsilon{}^{(m)} \, div\, \varphi \, d\Omega \longrightarrow \int_{\Omega(\alpha^*)} div\, \mathbf{u}_\epsilon(\alpha^*) \, div\, \varphi \, d\Omega \ ,$$

as $m \to \infty$. Next, we estimate the nonlinear convective term. Since $\mathbf{u}_\epsilon^{(m)} = \mathbf{0}$ on $\Gamma_0(\alpha_m)$ for every m, using integration by parts, we have that

$$\int_{\Omega_m} (\mathbf{u}_\epsilon^{(m)} \cdot \nabla)\, \mathbf{u}_\epsilon^{(m)} \cdot \varphi \, d\Omega = \int_{\Gamma_\mathbf{g}} (\mathbf{u}_\epsilon^{(m)} \cdot \mathbf{n})\, \mathbf{u}_\epsilon^{(m)} \cdot \varphi \, d\Gamma$$
$$- \int_{\Omega_m} (div\, \mathbf{u}_\epsilon{}^{(m)})\, \mathbf{u}_\epsilon^{(m)} \cdot \varphi \, d\Omega - \int_{\Omega_m} (\mathbf{u}_\epsilon^{(m)} \cdot \nabla)\, \varphi \cdot \mathbf{u}_\epsilon^{(m)} \, d\Omega \,. \quad (2\text{–}67)$$

Note that the outward unit normal vector $\mathbf{n}$ along $\Gamma_\mathbf{g}$ is fixed through the domain perturbations because of the given boundary condition and our choice of function space. It follows from (2–61)–(2–62) that

$$\begin{aligned} \int_{\Gamma_\mathbf{g}} (\mathbf{u}_\epsilon^{(m)} \cdot \mathbf{n})\, \mathbf{u}_\epsilon^{(m)} \cdot \varphi \, d\Gamma &= \int_{\Gamma_\mathbf{g}} (\mathbf{g} \cdot \mathbf{n})\, \mathbf{g} \cdot \varphi \, d\Gamma \\ &= \int_{\Gamma_\mathbf{g}} (\widetilde{\mathbf{u}}_\epsilon^{(m)} \cdot \mathbf{n})\, \widetilde{\mathbf{u}}_\epsilon^{(m)} \cdot \varphi \, d\Gamma \\ &= \int_{\Gamma_\mathbf{g}} (\mathbf{u}_\epsilon(\alpha^*) \cdot \mathbf{n})\, \mathbf{u}_\epsilon(\alpha^*) \cdot \varphi \, d\Gamma \,. \end{aligned}$$

For the second and third terms of (2–67), we use the fact that every components φ_i and $(\nabla\varphi)_{ij}$ belong to $L^\infty(\Omega(\alpha^*))$. Since $\|div\, \widetilde{\mathbf{u}}_\epsilon{}^{(m)}\|_{L^2(\widehat{\Omega})} \le \|\widetilde{\mathbf{u}}_\epsilon^{(m)}\|_{1,\widehat{\Omega}} < \infty$ for all m, we may extract a subsequence, which is denoted by $div \widetilde{\mathbf{u}}_\epsilon^{(m)}$, such that

$$div\, \widetilde{\mathbf{u}}_\epsilon{}^{(m)} \rightharpoonup div\, \widetilde{\mathbf{u}}_\epsilon \qquad \text{in} \;\; L^2(\widehat{\Omega})\,. \quad (2\text{–}68)$$

We note that

$$\begin{aligned} &\int_{\widehat{\Omega}} (div\, \widetilde{\mathbf{u}}_\epsilon{}^{(m)})\, \widetilde{\mathbf{u}}_\epsilon^{(m)} \cdot \varphi \, d\Omega - \int_{\widehat{\Omega}} (div\, \widetilde{\mathbf{u}}_\epsilon)\, \widetilde{\mathbf{u}}_\epsilon \cdot \varphi \, d\Omega \\ &= \int_{\widehat{\Omega}} div\, \widetilde{\mathbf{u}}_\epsilon{}^{(m)} (\widetilde{\mathbf{u}}_\epsilon^{(m)} - \widetilde{\mathbf{u}}_\epsilon)\varphi \, d\Omega - \int_{\widehat{\Omega}} \widetilde{\mathbf{u}}_\epsilon (div\, \widetilde{\mathbf{u}}_\epsilon{}^{(m)} - div\, \widetilde{\mathbf{u}}_\epsilon)\varphi \, d\Omega \\ &\longrightarrow 0\,, \qquad \text{as } m \to \infty\,. \end{aligned}$$

Consequently, using (2–68) and the uniform extension, it holds that

$$\begin{aligned} \int_{\Omega_m} (div\, \mathbf{u}_\epsilon{}^{(m)})\, \mathbf{u}_\epsilon^{(m)} \cdot \varphi \, d\Omega &= \int_{\widehat{\Omega}} (div\, \widetilde{\mathbf{u}}_\epsilon{}^{(m)})\, \widetilde{\mathbf{u}}_\epsilon^{(m)} \cdot \varphi \, d\Omega \\ &\overset{(m \to \infty)}{\longrightarrow} \int_{\widehat{\Omega}} (div\, \widetilde{\mathbf{u}}_\epsilon)\, \widetilde{\mathbf{u}}_\epsilon \cdot \varphi \, d\Omega \\ &= \int_{\Omega(\alpha^*)} (div\, \mathbf{u}_\epsilon(\alpha^*))\, \mathbf{u}_\epsilon(\alpha^*) \cdot \varphi \, d\Omega \,. \end{aligned}$$

In a similar fashion, we get as $m \to \infty$,

$$\int_{\Omega_m} (\mathbf{u}_\epsilon^{(m)} \cdot \nabla)\, \varphi \cdot \mathbf{u}_\epsilon^{(m)}\, d\Omega \longrightarrow \int_{\Omega(\alpha^*)} (\mathbf{u}_\epsilon(\alpha^*) \cdot \nabla)\, \mathbf{u}_\epsilon(\alpha^*) \cdot \varphi\, d\Omega\ .$$

Therefore, combining all the results for the three terms in (2–67), we get as $m \to \infty$,

$$\begin{aligned}
&\int_{\Omega_m} (\mathbf{u}_\epsilon^{(m)} \cdot \nabla)\, \mathbf{u}_\epsilon^{(m)} \cdot \varphi\, d\Omega \\
&= \int_{\Gamma_\mathbf{g}} (\mathbf{u}_\epsilon^{(m)} \cdot \mathbf{n})\, \mathbf{u}_\epsilon^{(m)} \cdot \varphi\, d\Gamma - \int_{\Omega_m} (div\, \mathbf{u}_\epsilon{}^{(m)} \cdot \mathbf{n})\, \mathbf{u}_\epsilon^{(m)} \cdot \varphi\, d\Omega - \int_{\Omega_m} (\mathbf{u}_\epsilon^{(m)} \cdot \nabla)\, \varphi \cdot \mathbf{u}_\epsilon^{(m)}\, d\Omega \\
&\qquad \longrightarrow \int_{\Gamma_\mathbf{g}} (\mathbf{u}_\epsilon(\alpha^*) \cdot \mathbf{n})\, \mathbf{u}_\epsilon(\alpha^*) \cdot \varphi\, d\Gamma - \int_{\Omega(\alpha^*)} (div\, \mathbf{u}_\epsilon(\alpha^*))\, \mathbf{u}_\epsilon(\alpha^*) \cdot \varphi\, d\Omega \\
&\qquad\qquad - \int_{\Omega(\alpha^*)} (\mathbf{u}_\epsilon(\alpha^*) \cdot \nabla)\, \varphi \cdot \mathbf{u}_\epsilon(\alpha^*)\, d\Omega = \int_{\Omega(\alpha^*)} (\mathbf{u}_\epsilon(\alpha^*) \cdot \nabla)\, \mathbf{u}_\epsilon(\alpha^*) \cdot \varphi\, d\Omega\,.
\end{aligned}$$

Up to the present, we have shown that

$$\begin{aligned}
\nu \int_{\Omega(\alpha^*)} &\nabla \mathbf{u}_\epsilon(\alpha^*)\ :\ \nabla\varphi\, d\Omega + \int_{\Omega(\alpha^*)} (\mathbf{u}_\epsilon(\alpha^*) \cdot \nabla)\, \mathbf{u}_\epsilon(\alpha^*) \cdot \varphi\, d\Omega \\
&+ \frac{1}{\epsilon} \int_{\Omega(\alpha^*)} div\, \mathbf{u}_\epsilon(\alpha^*) div\, \varphi\, d\Omega - < \mathbf{t}_\epsilon, \varphi >_{-1/2,\Gamma_\mathbf{g}} = < \mathbf{f}, \varphi >_{-1,\Omega(\alpha^*)}
\end{aligned}$$

for any choice of $\varphi \in \boldsymbol{W}$. From this formulation, it follows that

$$< \mathbf{t}_\epsilon, \varphi >_{-1/2,\Gamma_\mathbf{g}} = < \nu(\mathbf{n} \cdot \nabla)\mathbf{u}_\epsilon(\alpha^*) + \frac{1}{\epsilon}(div \mathbf{u}_\epsilon(\alpha^*))\mathbf{n}\,, \varphi >_{-1/2,\Gamma_\mathbf{g}}\ .$$

Since $\boldsymbol{W}$ is dense in $\mathbf{H}^1_{\Gamma_0(\alpha^*)}(\Omega(\alpha^*))$, we can conclude that $\mathbf{u}_\epsilon(\alpha^*)$ is a solution of the penalized variational formulation of (2–16)–(2–17) over $\Omega(\alpha^*)$ and (2–63) immediately follows. Therefore, $(\alpha^*, \mathbf{u}_\epsilon(\alpha^*)) \in \widetilde{\mathcal{U}_{ad}}$ and this implies that $\widetilde{\mathcal{U}_{ad}}$ is weakly closed. Since $J_\epsilon(\cdot,\cdot)$ is weakly lower semi–continuous, the problem (2–56) has an optimal solution $(\alpha^*, \mathbf{u}_\epsilon(\alpha^*))$ belonging to $\widetilde{\mathcal{U}_{ad}}$. $\square$

It is clear from (2–45) that

$$|\mathfrak{J}_\epsilon(\alpha^*) - \mathfrak{J}(\alpha^*)| \longrightarrow 0 \qquad \text{as } \epsilon \text{ tends to } 0 \tag{2–69}$$

along the regular branch.

REMARK 2.7: Since the steady–state Navier–Stokes equations have multiple solutions for large Reynolds numbers, we cannot expect a unique optimal solution. Even in the case that the state equation has a unique solution, the optimal shape need not be unique. This was indicated in Begis *et al.*[12] in the optimal shape problem for elliptic state equations. The same argument can also be applied to our case.

CHAPTER III

SHAPE SENSITIVITY ANALYSIS AND THE OPTIMALITY SYSTEM

Generally speaking, the procedure of searching for an optimal shape design may include all or part of the the following steps:

(i) Determination of the involved factors for the physical problem to be studied. For example, the reduction of the drag or enhancement of lift, the Joukowsi condition and the curvature of the boundary surface should be considered in the design of aerofoils.

(ii) Setting the physical problem into a system of well-organized (tractible) partial differential equations by simplifying minor physical factors and choosing a design functional to fit into the task to be performed by imposing some necessary weights or constraints and by checking its validity.

(iii) Selection of an appropriate parametrization for the moving parts of domains.

(iv) Computation of variations of the cost functional and constraints with respect to design parameters.

(v) Setting adjoint state equations and determining necessary optimality conditions.

(vi) Selection of suitable devices for problem solution and convergence analysis. This may include discretizations for the state and adjoint state equations, sensitivity analysis for the discretized problem and adoption or development of effective optimization techniques.

(vii) Readjustment of the problem and data according to computed results.

and finally, if possible

(viii) Mathematical justifications for the optimal solution.

The core steps are in the computation of the shape gradient and in the determination of factors and criteria for the optimality conditions (steps (iv) and (v)).

In this chapter, we are mainly concerned with the shape sensitivity anlysis for the problem (2–56). Sensitivity analysis in a shape optimization problem is the study of the effects on the design functional and potential constraints due to variations of the design parameters. In the circle of structural engineering, shape optimization problems deal with the design of engineering structures to achieve optimal performance criteria and shape sensitivity analysis concerns the relationship between design variables available to engineers and structural responses or state variables that are determined by the laws of mechanics (c.f. Haug *et al.*[78]). Given any design, a sensitivity analysis determines if it is a stationary point in design space for the relevant optimization problem. Otherwise, one may try to improve the given design locally. Improvement of performance can be achieved iteratively by following the gradient of the cost functional. This is based on the existence of Gateaux derivatives of the cost functional and constraints in the direction of domain perturbations. The first result concerning the differentiability with respect to perturbations of a domain were obtained by Hadamard in 1907 for the first eigenvalue of a membrane. He also computed the derivative of the Green's function for the Laplace operator with respect to the normal variations of the domain. This technique, which is known as the *normal variation method*, is still widely used in many applications to compute the first order variation of the design parameters (c.f. [101][49][50]). The basic idea of the normal variation method is to consider the change of the design variables as a smooth function of the normal to the boundary of the domain to utilize the mean value theorem. Pironneau[101] developed the normal variation method further to obtain the necessary optimality conditions for shape optimizations of elliptic systems. He also examined some problems of fluid mechanics including Stokes and Navier–Stokes equations to find a minimum drag profile of a body embeded in the flow ([99][100][61]). Koda[85] took the similar technique to derive the optimality condition of a similar design performance functional for the evolutionary Navier–Stokes equations. However, this method requires sufficient regularities for the feasible domain and data of the state equations. Hence, it can not be generalized into general situations such as domains with piecewise smooth boundaries and data for the state equation with lesser regularities.

Zolésio[135] introduced a new technique that is called the *material derivative method* (or the speed method) to show the existence of a solution of a domain identification problem given a velocity field of domain perturbations. This method utilized the ideas of classical Eulerian and Lagrangian descriptions of the motion of fluid particles (or spatial and material descriptions in the motion of a body in continuum mechanics). He also developed the shape calculus which can be effectively used for the shape sensitivity analysis ([136]). There are numerous papers in the literature performing shape sensitivity analyses using the material derivative method. Standard guidelines for shape optimization were drawn by Cèa ([26][27][28]). General frameworks for the optimal control problems for the variational inequality were discussed by Mignot ([93][94][95]) and Lions ([89]). Haslinger and Sokolowski gave the mathematical foundation of the shape sensitivity analysis for optimal shape control problems governed by a variational inequality such as the unilateral problem dealing with the contact problem of elasto–plastic bodies and obstacle problems arising in continuum mechanics. In recent years, Delfour and Zolésio [45][47] have extended the shape calculus to obtain the *Shape Hessian* (or the second order variation of the design parameters) using the Lagrange multiplier technique associated with the state variables.

In our case, the domain and variables of the state equations are not smooth enough to accommodate the normal variation method. Moreover, the feasible domains should be perturbed in a special direction. We will use the material derivative method to describe the domain perturbation and to compute the shape gradient. The computation of the shape gradient and its treatment play a central role in the shape sensitivity analysis. We take the Lagrange multiplier technique associated with the state equation to obtain the necessary optimality condition leading to the optimality system. This appears to be an efficient tool to relax regularities for the feasible domain and state variables.

In Section 3.1, we briefly discuss the material derivative method and introduce some shape calculus. For this part, we mainly refer to Zolésio[136], Sokolowski *et al.*[121] and Delfour *et al.*[46]. In Section 3.2, we are concerned with the shape sensitivity analysis for the design functional. We employ the Lagrange multiplier technique to obtain the adjoint state equations. In Section 3.3, we show the existence of the Lagrange multiplier by applying a method from functional analysis. In Section 3.4, we discuss the optimality system and the regularity for the solution of the problem (2–51). The weak penalized optimality system is presented in Section 3.5.

3.1. The Material Derivative Method

How can we describe continuous variations of a shape? This question is equivalent to: how can we parameterize the deformation of a domain in Euclidean space? An answer to this question is found by considering the domain as a continuum medium. For the general discussion, we suppose Ω is a bounded simply connected domain in R^n. Since Ω is not an element of a vector space, we need a suitable descriptive tool for the domain perturbation to get the shape gradient with respect to domain. The material derivative method utilizes the conventional material and spatial descriptions for the deformation of a continuum medium in Euclidean coordinates.

We introduce a one parameter family of domains $\{\Omega_t\}_{t\geq 0}$ in the following way; Given a domain $\Omega \subset R^n$ and a (smooth) vector field $\mathcal{V}(t,\cdot)$ defined in a neighborhood of Ω, each point $\mathbf{p}$ of Ω is continuously transported in a one–to–one fashion onto a point $\mathbf{x}(t)$ at $t > 0$ through the following system of ordinary differential equations

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathcal{V}(t, \mathbf{x}(t)) \\ \mathbf{x}(0) = \mathbf{p}\,. \end{cases} \tag{3–1}$$

This induces a one–to–one transformation $\mathcal{F}_t : R^n \to R^n$ and $\sigma > 0$ such that

$$\mathcal{F}_t(\mathbf{p}) = \mathbf{x}(t) = \mathbf{p} + \int_0^t \mathcal{V}(s, \mathbf{x}(s))\, ds\,, \tag{3–2}$$

for $0 \leq t < \sigma$ and $\mathbf{p} \in \Omega \subset R^n$. From this, the domain perturbed in the $\mathcal{V}$–direction can be defined by $\Omega_t \equiv \mathcal{F}_t(\Omega)$, where $\Omega = \Omega_0 = \mathcal{F}_0(\Omega)$. Also, the boundary is preserved under the transformation $\mathcal{F}_t$, *i.e.*, $\Gamma_t \equiv \partial\Omega_t = \mathcal{F}_t(\Gamma)$ and $\Gamma = \Gamma_0 = \mathcal{F}_0(\Gamma)$.

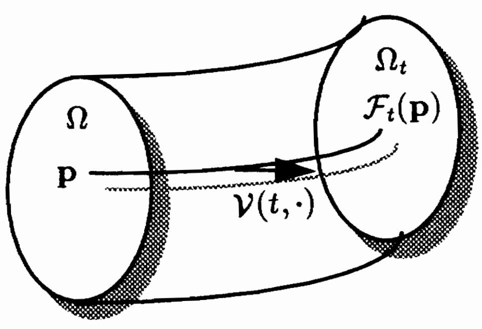

Figure 3: Transformation $\mathcal{F}_t$

In technical terms, $\mathcal{F}_t$ is called a local configuration, Ω a reference domain and Ω_t a domain of spatial fields. The map $t \mapsto \mathcal{F}_t(\mathbf{p})$ is called the *design trajectory* of $\mathbf{p}$ and $\mathcal{V}$ the *design velocity* (or simply, velocity). In case that the dependency of $\mathcal{F}_t$ on $\mathcal{V}$ should be emphasized, we will write $\mathcal{F}_t = \mathcal{F}_t(\mathcal{V})$. By the local existence and uniqueness theorem for the system of ordinary differential equations and the integral representation (3–2) for the trajectory, it is not difficult to show that

$$\mathcal{F}_{t_1+t_2}(\mathcal{V})(\mathbf{q}) = \mathcal{F}_{t_2}(\mathcal{V}_{t_1})(\mathcal{F}_{t_1}(\mathcal{V})(\mathbf{q})) \tag{3–3}$$

for all $t_1,\ t_2 \geq 0$ such that $0 \leq t_1,\ t_2,\ t_1 + t_2 < \sigma$ and all points $\mathbf{q}$ in a neighborhood $O_\mathbf{p}$ of $\mathbf{p}$, where $\mathcal{V}_{t_1}(s,\cdot) = \mathcal{V}(t_1 + s,\cdot)$.

For practical applications, we consider the case when all the perturbations of a domain are constrained in a fixed domain $\mathcal{Q}$. To be more precise, let us assume that $\mathcal{Q}$ is a bounded open set containing Ω and that $\mathcal{V} : [0,\widetilde{t}] \times \mathcal{Q} \longrightarrow R^n$ denotes a continuous vector field. Suppose that $t \mapsto \mathcal{V}(t,\mathbf{x})$ is continuous for each $\mathbf{x} \in \Omega_t$ and $\mathcal{V}(t,\cdot)$ is Lipschitz continuous, *i.e.*, there exists a positive constant c such that

$$|\mathcal{V}(t,\mathbf{x}_1) - \mathcal{V}(t,\mathbf{x}_2)| \leq C\,|\mathbf{x}_1 - \mathbf{x}_2|\,, \tag{3–4}$$

for $t \in [0,\widetilde{t}]$ and every $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathcal{Q}$. Then for any $\mathbf{p} \in \Omega$, there exists an $\widetilde{\sigma} \in (0,\widetilde{t})$, an open neighborhood $O_\mathbf{p}$ of $\mathbf{p}$ in $\mathcal{Q}$ and a one–to–one transformation

$$\mathcal{F}_t : O_\mathbf{p} \to \mathcal{F}_t(O_\mathbf{p}) \subset R^n \qquad \text{for} \ \ 0 \leq t < \widetilde{\sigma}\,,$$

such that $t \mapsto \mathcal{F}_t(\mathbf{p})$ is a unique solution of (3–1) for $0 \leq t < \widetilde{\sigma}$. Now, let us assume that

$$\bigcup_{\substack{O_\mathbf{p} \subset \mathcal{Q} \\ 0 \leq t < \widetilde{\sigma}}} \mathcal{F}_t(O_\mathbf{p}) \subset \mathcal{Q} \qquad \text{for every} \ \ \mathbf{p} \in \Omega\,. \tag{3–5}$$

This condition is needed to guarantee the existence of the inverse $\mathcal{F}_t^{-1}$ of $\mathcal{F}_t$ for $0 \leq t < \widetilde{\sigma}$, when $\mathcal{V}(t,\cdot)$ is defined on $\mathcal{Q}$ (see Lemma 3.1 and Remark 3.1).

Since $\overline{\Omega}$ is compact, there exists a finite open covering $\{O_i\}_{i=1}^m$ of $\overline{\Omega}$ in $\mathcal{Q}$ with correponding positive numbers $\sigma_1, \cdots, \sigma_m$ and transformations $\{\mathcal{F}_t^{(i)}\}$ such that $\mathcal{F}_t^{(i)}$: $O_i \to \mathcal{F}_t^{(i)}(O_i) \subset \mathcal{Q}$ is one-to-one for $0 \le t < \sigma_i$. Let us take $O = \bigcup_{i=1}^m O_i$ and $\sigma = \min\{\sigma_1, \cdots, \sigma_m\}$. Notice that by the uniqueness of the solution of the differential equations, $\mathcal{F}_t^{(i)}(\mathbf{q}) = \mathcal{F}_t^{(j)}(\mathbf{q})$ for $\mathbf{q} \in O_i \cap O_j$. So if we patch them together by defining

$$\mathcal{F}_t(\mathbf{q}) = \mathcal{F}_t^{(i)}(\mathbf{q}) \qquad \text{if } \ \mathbf{q} \in O_i\,,$$

then clearly $\mathcal{F}_t : \Omega \to \Omega_t \subset \mathcal{Q}$ is a one–to–one transformation for $0 \le t < \sigma$. The continuity of $\mathcal{F}_t(\cdot)$ for all $0 \le t < \sigma$ easily follows from the expression (3–2) of $\mathcal{F}_t$ and the Lipschitz continuity of $\mathcal{V}(t,\cdot)$. Furthermore, if $\mathcal{V}(t,\cdot)$ is of class $\mathcal{C}^k$ over $\mathcal{Q}$, from the classical regularity result (for example, see Lang[84] for a lucid exposition), it follows that $\mathcal{F}_t(\cdot)$ is also of class $\mathcal{C}^k$ over $\mathcal{Q}$. Clearly, $(0,\sigma) \ni t \mapsto \mathcal{F}_t(\mathbf{x})$ is continuously differentiable for each $\mathbf{x} \in \Omega_t$.

Next, we consider the inverse $\mathcal{F}_t^{-1}$ of $\mathcal{F}_t$. Note that if $t \mapsto \mathcal{V}(t,\cdot)$ is defined in a neighborhood $(-\sigma,\sigma)$ of 0 and $\dfrac{\partial \mathcal{V}}{\partial t} = \mathbf{0}$, then $\mathcal{F}_{t_1+t_2}(\mathbf{q}) = \mathcal{F}_{t_1}(\mathcal{F}_{t_2}(\mathbf{q}))$ for all t_1 and t_2 such that $-\sigma < t_1,\ t_2,\ t_1+t_2 < \sigma$ and all points $\mathbf{q}$ in a neighborhood $O_{\mathbf{p}}$ of $\mathbf{p}$. In this case, $\{\mathcal{F}_t\}_{-\sigma<t<\sigma}$ is a local one parameter group of transformation whose inverse is given by $\mathcal{F}_t^{-1} = \mathcal{F}_{-t}$ for $-\sigma < t < \sigma$.

Since this is not the case, to discuss its inverse $\mathcal{F}_t^{-1}$, we consider the following system of differential equations:

$$\begin{aligned} &\dot{\mathbf{p}}(s) = -\mathcal{V}(t-s, \mathbf{p}(s)) \quad 0 \le s \le t\,, \\ &\mathbf{p}(0) = \mathbf{x} = \mathcal{F}_t(\mathbf{p}(t)) \qquad \text{for } \ \mathbf{x} \in \Omega_t \subset \mathcal{Q}\,. \end{aligned} \tag{3–6}$$

This introduces a unique Lipschitzian solution $\mathcal{J}_t(\mathbf{x}) = \mathbf{p}(t)$.

Lemma 3.1. *Under the assumption (3–4) and (3–5), the transformation $\mathcal{J}_t$ induced from (3–6) is an inverse of $\mathcal{F}_t$. Moreover, if $\mathcal{V}(t,\cdot)$ is $\mathcal{C}^k(\mathcal{Q})$, so is $\mathcal{F}_t^{-1} = \mathcal{J}_t$.*

Proof: Consider the map $s \mapsto \mathcal{F}_{t-s}(\mathbf{p})$ for $0 \le s \le t$. Since $\mathcal{F}_{t-s}(\mathbf{p}) = \mathbf{x}(t-s) = \mathcal{V}(t-s, \mathbf{x}(t-s))$, $s \mapsto \mathcal{F}_{t-s}(\mathbf{p})$ is a solution (3–6), *i.e.*, $\mathcal{F}_{t-s} = \mathcal{J}_s$. Hence

$$\mathcal{J}_t(\mathcal{F}_t(\mathbf{p})) = \mathbf{p}(t) = \mathcal{F}_{t-t}(\mathbf{p}) = \mathbf{p}$$

whence $\mathcal{J}_t$ is a left inverse of $\mathcal{F}_t$. To show that $\mathcal{J}_t$ is also a right inverse of $\mathcal{F}_t$, we consider a function $\mathbf{y}(\xi) = \mathbf{p}(t-\xi)$. Since

$$\dot{\mathbf{y}}(\xi) = \frac{d}{d\xi}\mathbf{p}(t-\xi) = -\dot{\mathbf{p}}(t-\xi) = \mathcal{V}(t-(t-\xi), \mathbf{p}(t-\xi)) = \mathcal{V}(\xi, \mathbf{y}(\xi))$$

from (3–6), $\mathbf{y}(\xi)$ is a solution of

$$\dot{\mathbf{y}}(\xi) = \mathcal{V}(\xi, \mathbf{y}(\xi)), \qquad \mathbf{y}(0) = \mathbf{p}(t).$$

So, it follows that

$$\mathbf{x} = \mathbf{p}(0) = \mathbf{y}(t) = \mathcal{F}_t(\mathbf{p}(t)) = \mathcal{F}_t(\mathcal{J}_t(\mathbf{x}))$$

and hence that $\mathcal{J}_t = \mathcal{F}_t^{-1}$. It can be simply written as $\mathcal{F}_t(\mathcal{V})^{-1} = \mathcal{F}_t(-\mathcal{V}_t)$, where $\mathcal{V}_t(s,\cdot) = \mathcal{V}(t-s,\cdot)$. The second assertion can be proved in the same way as the regularity of $\mathcal{F}_t$ using the regularity of $\mathcal{V}_t(s,\cdot)$ and the equations (3–6). $\square$

Note that from (3–3), $\mathcal{F}_{t+h}(\mathcal{V}) - \mathcal{F}_t(\mathcal{V}) = (\mathcal{F}_h(\mathcal{V}_t) - \mathcal{I}) \circ \mathcal{F}_t(\mathcal{V})$ for t, $h > 0$. So, $\dfrac{\partial}{\partial t}\mathcal{F}_t(\mathcal{V})(\mathbf{p}) = \mathcal{V}(t, \mathcal{F}_t(\mathcal{V})(\mathbf{p}))$. Hence, it follows that

$$\mathcal{V}(t,\mathbf{x}) = \frac{\partial \mathcal{F}_t}{\partial t}(\mathcal{F}_t^{-1}(\mathbf{x})) \qquad \text{for every } \mathbf{x} \in \Omega_t,\ 0 \le t < \sigma. \tag{3–7}$$

The argument hitherto can be simply stated as follows: if $\mathcal{V}(t,\cdot)$ is of class $\mathcal{C}^k$ with $k \ge 0$ over $\mathcal{Q}$, there exists a $\mathcal{C}^k$–diffeomorphism $\mathcal{F}_t$ from Ω onto Ω_t and *vice versa*, if $\{\mathcal{F}_t\}_{0 \le t < \sigma}$ is a family of $\mathcal{C}^k$–diffeomorphisms, $\mathcal{V}$ can be recovered from (3–7) and $\mathcal{V}(t,\cdot)$ is also of class $\mathcal{C}^k$. (Note that if $\mathcal{V}(t,\cdot)$ is Lipschitz continous, so is $\mathcal{F}_t$, and *vice versa*.)

REMARK 3.1: Zolésio showed in his dissertation(1979) that for any domain D in R^n and a (smooth) velocity $\mathcal{V}: [0,\tau] \times \overline{D} \to R^n$ satisfying

$$\begin{aligned} \mathcal{V}(t,\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = 0 \qquad & \text{if the outward normal } \mathbf{n}(\mathbf{x}) \text{ is defined a.e. } \mathbf{x} \in \partial D \\ \mathcal{V}(t,\mathbf{x}) = 0 \qquad & \text{otherwise}, \end{aligned}$$

the solution $\mathcal{F}_t$ of (3–1) maps $\overline{D}$ into $\overline{D}$ for all $0 \le t \le \tau$ (c.f. Sokolowski *et al.*[121]).

Recently, Delfour and Zolésio[46][48] extended the class of $\mathcal{V}$ such that $\mathcal{F}_t(\overline{D}) \subset \overline{D}$, $\forall t \in [0,\tau]$, using the "viability theory" introduced by Aubin–Cellina[7]: The general motivation for the viability theory is to study the *viable* phenomena such that a trajectory $t \to x(t)$ belongs to a fixed closed subset K of a Hilbert space H. Let $T_K(x)$ be a *Bouligand contingent cone* to K at x which is characterized by

$$v \in T_K(x) \iff \liminf_{h \to 0^+} \frac{d_K(x+hv)}{h} = 0,$$

where $d_K(x) = \inf\{\|x-y\|_H \mid y \in K\}$. Obviously, $T_K(x)$ is a generalization of the tangent space to K at x, when K is a smooth manifold in R^n. The fundamental theorem for the study of the viability follows from the following version of Nagumo's theorem which can be found in [7]:

Nagumo's theorem: Under the above setting, let $f: K \to H$ be a continuous mapping satisfying the tangential condition

$$\forall x \in K, \quad f(x) \in T_K(x).$$

Then for all $x_0 \in K$, there exists $\tau > 0$ such that the differential equation $\dot{x}(t) = f(x(t))$, $x(0) = x_0$ has a viable trajectory on $[0,\tau]$. $\square$

Take $D \subset R^n$. Let us consider vector fields $\mathcal{V}$ which satisfy the following conditions:

(i) $\mathcal{V}(t,\cdot)$ is Lipschitz continuous,

(ii) $\mathcal{V}(t,\mathbf{x})$ and $-\mathcal{V}(t,\mathbf{x})$ belong to the Bouligand contingent cone $T_{\overline{D}}(\mathbf{x})$ to $\overline{D}$ at $\mathbf{x} \in \partial D$, for all $(t,\mathbf{x}) \in [0,\tau] \times \partial D$.

The condition (ii) is a double viability condition by Nagumo's theorem, which guarantees the the existence of a homeomorphism $\mathcal{F}_t : \overline{D} \to \overline{D}$ (for details, refer to Delfour *et al.*[43]).

We are now in a position to discuss the variation of a function due to the domain perturbation. Throughout this section, we assume

$$\bigcup_{t \in [0,\widetilde{t}]} \{t\} \times \Omega_t \subset [0,\widetilde{t}] \times \mathcal{Q}. \tag{3–8}$$

Let y_t be a regular function defined on $\Omega_t = \mathcal{F}_t(\Omega)$. Then the composite $y_t \circ \mathcal{F}_t$ is defined on a fixed reference domain Ω. The (*pointwise*) *material derivative* (or, Lagrangian derivative) of y_t at $\mathbf{p} \in \Omega$ in the $\mathcal{V}$–direction is defined by the following semi–derivative (if it exists):

$$\dot{y}(\mathbf{p};\mathcal{V}) = \frac{d}{dt}\, y_t(\mathcal{F}_t(\mathbf{p}))\bigg|_{t=0+}. \tag{3–9}$$

If $\{\Omega_t\}_{0 \le t \le \widetilde{t}}$ is a class of domains with the uniform extension property, we can consider y_t as a restriction of y to $\{t\} \times \Omega_t$, where y is defined globally in $[0,\widetilde{t}] \times \mathcal{Q}$, *i.e.*,

$$y(t,\mathbf{x}) = P_{\mathcal{Q}}(y_t(\mathcal{F}_t))(\mathcal{F}_t^{-1}(\mathbf{x})) \quad \text{and} \quad y_t(\mathbf{x}) = y(t,\mathbf{x}). \tag{3–10}$$

Then, using the chain rule, the material derivative (3–9) can be written as

$$\begin{aligned} \dot{y}(\mathbf{p};\mathcal{V}) &= \lim_{t \to 0+} \frac{y(t,\mathcal{F}_t(\mathbf{p})) - y(0,\mathbf{p})}{t} \\ &= \frac{\partial y}{\partial t}(0,\mathbf{p}) + (\nabla y \cdot \mathcal{V})(0,\mathbf{p}), \end{aligned} \tag{3–11}$$

where $\nabla y = (\frac{\partial y}{\partial x_1}, \cdots, \frac{\partial y}{\partial x_n})$. Similarly, if $\mathbf{u}_t$ is a vector–valued function defined in Ω_t and $\mathbf{u}(t,\cdot)$ is its extension to $\mathcal{Q}$, the material derivative of $\mathbf{u}_t$ can be written as

$$\dot{\mathbf{u}}(0,\mathbf{p}) = \frac{\partial \mathbf{u}}{\partial t}(0,\mathbf{p}) + (\mathcal{V}(0,\mathbf{p}) \cdot \nabla)\, \mathbf{u}(0,\mathbf{p}). \tag{3–12}$$

This concept can be naturally generalized into Sobolev spaces. For example, let $y_t \in H^m(\Omega_t)$ and $\mathcal{V}(t,\cdot) \in \mathcal{C}^k(\mathcal{Q}; R^n)$ for $0 \le t \le \tilde{t}$, where $0 \le k \le m$. Since $\mathcal{F}_t$ is $\mathcal{C}^k$–diffeomorphism, $y_t \circ \mathcal{F}_t \in H^k(\Omega)$, which can be verified by Leibniz's rule. In fact,

$$H^k(\Omega) = \{y_t \circ \mathcal{F}_t \mid \mathcal{V}(t,\cdot) \in \mathcal{C}^k(\mathcal{Q}; R^n) \text{ and } \ y_t \in H^k(\Omega_t)\}.$$

Then, $\dot{y} = \dot{y}(\Omega; \mathcal{V}) \in H^k(\Omega)$ is called the material derivative of y at $\Omega \subset \mathcal{Q}$ in the direction of $\mathcal{V}$ in the Sobolev space $H^k(\Omega)$ if

$$\lim_{t \to 0^+} \| \frac{y(t, \mathcal{F}_t(\mathbf{p})) - y(0, \mathbf{p})}{t} - \dot{y}(\Omega; \mathcal{V}) \|_{k,\Omega} = 0.$$

Notice that unless $k > \frac{n}{2}$, pointwise expressions such as (3–9) are meaningless. It makes sense only *a.e.* (*almost everywhere*). To avoid notational confusion, we write it by

$$\dot{y} = \dot{y}(\Omega; \mathcal{V}) = \lim_{t \to 0^+} \frac{y_t \circ \mathcal{F}_t - y \circ \mathcal{I}}{t},$$

where the limit is taken in $H^k(\Omega)$. The material derivative in the weak space can be defined in a similar manner via duality.

If $y \in H^k(\mathcal{Q})$ is a uniform extension of $y_t \in H^k(\Omega_t)$, the *shape derivative* $y'(\Omega; \mathcal{V})$ at $\Omega \subset \mathcal{Q}$ of the uniform extension $y \in H^k(\mathcal{Q})$ in the $\mathcal{V}$–direction is defined by

$$y'(\Omega; \mathcal{V}) = \dot{y}(\Omega; \mathcal{V}) - \nabla y(\Omega) \cdot \mathcal{V}(0). \tag{3–13}$$

Note that $y'(\Omega; \mathcal{V}) \in H^{k-1}(\Omega)$ from (3–13). If $k - 1 > \frac{n}{2}$, since $H^k(\mathcal{Q}) \subset \mathcal{C}^1(\overline{\mathcal{Q}})$, the shape derivative can be defined pointwise

$$\begin{aligned} y'(\Omega; \mathcal{V}) &= \frac{\partial y}{\partial t}(0, \mathbf{p}) \\ &= \dot{y}(0, \mathbf{p}; \mathcal{V}) - (\nabla y \cdot \mathcal{V})(0, \mathbf{p}). \end{aligned}$$

As we shall see later, the shape derivative is very crucial in the shape sensitivity analysis for the domain functional.

Next, we are concerned with the domain functional. Let $J(\Omega)$ be any domain functional on $\mathcal{Q}$ and let $j(t) = J(\Omega_t)$. Then the rate of variation of $J(\Omega)$ at the reference domain Ω with respect to the domain perturbation may be measured as a directional semi-derivative

$$dJ(\Omega\,;\,\mathcal{V}) = \lim_{t\to 0^+} \frac{J(\Omega_t) - J(\Omega)}{t} = \frac{d}{dt} J(\mathcal{F}_t(\mathcal{V})(\Omega))|_{t=0^+} = \dot{j}(0)\,.$$

The domain functional $J(\Omega)$ is said to be *shape differentiable* if

(a) $dJ(\Omega;\mathcal{V})$ exists for all directions $\mathcal{V}$

(b) $\mathcal{V} \mapsto dJ(\Omega;\mathcal{V})$ is linear and continuous over appropriate admissible vector fields.

If $J(\Omega)$ is shape differentiable, under the sophiscated structure, we can interpret $\mathcal{V} \mapsto dJ(\Omega;\mathcal{V})$ in the distribution sense:

$$dJ(\Omega;\mathcal{V}) = < \mathcal{G}(\Omega), \mathcal{V} > \,. \tag{3–14}$$

Then $\mathcal{G}(\Omega)$ appears to be a vector valued distribution of a finite order acting on the appropriate test function space which is determined by the regularity of the admissible domains. In this case, $\mathcal{G}(\Omega)$ is called the *shape gradient* of the domain functional $J(\Omega)$ and is usually written as

$$\mathcal{G}(\Omega) = grad\, J(\Omega)\,. \tag{3–15}$$

Then, the shape optimization problem is rendered into the problem of finding $\mathcal{G}(\Omega)$ so that $j(t) < j(0)$ for $t > 0$. The method of finding a shape gradient in this manner is called the *material derivative method.*

REMARK 3.2: In the above definition, condition (b) requires an appropriate topology for the admissible vector fields and the continuity of $\mathcal{V} \mapsto dJ(\Omega;\mathcal{V})$. For a vector field $\mathcal{V} \in \mathcal{C}^0([0,\tilde{t}];\mathcal{C}_0^k(\mathcal{Q};R^n))$, *i.e.*, $\mathcal{V}(t)(\cdot) = \mathcal{V}(t,\cdot)$ a vector valued $\mathcal{C}^k$–function with compact support in $\mathcal{Q}$, Delfour *et al.*[46] introduced the inductive limit topology to utilize the sheaf structure of the distribution:

Let $\mathfrak{L}^{m,k} = \{\mathcal{V} \in \mathcal{C}^m([0,\tilde{t}];\mathcal{C}_0^k(\mathcal{Q};R^n) \mid \mathcal{V}(t,\mathbf{x})$ and $-\mathcal{V}(t,\mathbf{x})$ belong to the Bouligand contingent cone to $\overline{\mathcal{Q}}$ at $\mathbf{x} \in \partial\mathcal{Q}\}$. Let $\mathfrak{L}_K^{m,k}$ denote the closed subspace of $\mathfrak{L}^{m,k}$ with $\mathcal{V}(t,\cdot) \in \mathcal{C}_0^k(K;R^n)$, where K is a relatively compact set in $\mathcal{Q}$. The inductive limit can be introduced by

$$\overrightarrow{\mathfrak{L}}_{\mathcal{Q}}^{m,k} = \varinjlim_{K}\{\mathfrak{L}_K^{m,k} \mid \forall K \Subset \mathcal{Q}\}\,,$$

where $\varinjlim$ denotes the inductive limit with respect to relative compact subsets of $\mathcal{Q}$ endowed with the natural inductive limit topology. Under this structure, $\overrightarrow{\mathfrak{L}}_{\mathcal{Q}}^{m,k} \ni \mathcal{V} \mapsto dJ(\Omega;\mathcal{V})$ is continuous (for details, refer to [46] and [48]).

For the structure of the shape gradient for a domain functional, we demonstrate fundamental properties for the shape gradient. This structure will be applied to the shape sensitivity analysis in the next section to obtain the shape gradient for the problem (2–56). Moreover, by taking this structure, we can relax the reguralty requirement for the admissible domains which is common in shape optimization problems. For the sake of completeness, we provide a brief proof. Our proof is based on the original thoughts of Zolésio's[132] and the concept of distribution which can be found in Hörmander[81]. One may also refer to [121] and [48]. In the sequel, we will denote the support of any function or functional by *supp* for simplicity.

Theorem 3.1. *We assume $\mathcal{V}$ belongs to a class of vector fields satisfying (3–4) and (3–5). Suppose $J(\Omega)$ is shape differentiable at $\Omega \subset \mathcal{Q}$. Then*

(I) $dJ(\Omega;\mathcal{V}) = dJ(\Omega;\mathcal{V}(0)), \quad \forall\,\mathcal{V} \in \mathcal{C}^0([0,\tilde{t}];\mathcal{C}_0^k(\mathcal{Q};R^n))\,.$ (3–16)

(II) $supp\,\mathcal{G}(\Omega) \subset \Gamma = \partial\Omega\,.$ (3–17)

(III) **(Hadamard's Structure Theorem)**

There exists a scalar distribution $g(\Gamma)$ of a finite order such that

$$dJ(\Omega;\mathcal{V}) = < \mathcal{G}(\Omega), \mathcal{V}(0) >_\Omega \tag{3–18}$$

$$= < g(\Gamma), \mathcal{V}(0)\cdot\mathbf{n} >_\Gamma , \tag{3–19}$$

where $\mathcal{V}(0)\cdot\mathbf{n}$ is the normal component of $\mathcal{V}(0)$ on Γ.

Proof: (I); Let σ be a positive number found in (3–2). For $\mathcal{V} \in \mathcal{C}^0([0,\widetilde{t}];\mathcal{C}_0^k(\mathcal{Q};R^n))$, take a positive integer m_0 so large that $\dfrac{\widetilde{t}}{m_0} < \sigma$. Define $\mathcal{V}_m(t) = \mathcal{V}(\dfrac{t}{m})$ for $m \geq m_0$. Since *supp* $\mathcal{V}(t) \subset \mathcal{Q}$, there exists a compact subset K of $\mathcal{Q}$ such that

$$\bigcup_{t\in[0,\widetilde{t}]} supp\ \mathcal{V}_m(t) \subset K , \qquad \forall m \geq m_0 .$$

For every $|\alpha| \leq k$, since

$$\begin{aligned} &\sup_{\substack{0\leq t\leq\widetilde{t} \\ \mathbf{x}\in K}} |(\frac{\partial}{\partial\mathbf{x}})^\alpha(\mathcal{V}_m(t,\mathbf{x}) - \mathcal{V}(0,\mathbf{x}))|_{R^n} \\ &= \sup_{\substack{0\leq t\leq\widetilde{t} \\ \mathbf{x}\in K}} |(\frac{\partial}{\partial\mathbf{x}})^\alpha(\mathcal{V}(\frac{t}{m},\mathbf{x}) - \mathcal{V}(0,\mathbf{x}))|_{R^n} \longrightarrow 0 \quad \text{as} \quad m \to \infty , \end{aligned}$$

it follows that $dJ(\Omega;\mathcal{V}_m) \to dJ(\Omega;\mathcal{V}(0))$ and (3–16) follows from the uniqueness of the limit.

(II); In the sense of distribution, we note that

$$supp\ \mathcal{G}(\Omega) \cap\ supp\ \mathcal{V}(t,\cdot) = \emptyset \Longrightarrow < \mathcal{G}(\Omega), \mathcal{V} > = 0 . \tag{3–20}$$

Let $\mathcal{V}(t) \in \mathcal{C}_0^k(\mathcal{Q};R^n)$ be a vector field such that *supp* $\mathcal{V}(t) \cap \overline{\Omega} = \emptyset$. Since $\mathcal{V} = 0$ in Ω, (3–1) yields $\mathcal{F}_t(\mathcal{V}) = \mathcal{I}$. Hence we have $\mathcal{F}_t(\mathcal{V})(\Omega) \equiv \Omega_t = \Omega$ and $dJ(\Omega;\mathcal{V}) = 0$. So, it follows from (3–20) that *supp* $\mathcal{V}(t) \subset \Omega$ and

$$supp\ \mathcal{G}(\Omega) \subset \overline{\Omega} . \tag{3–21}$$

Next, we suppose that *supp* $\mathcal{V}(t) \subset \Omega$. Then, there exists an open set $\mathcal{O}$ and a one–to–one transformation $\widetilde{\mathcal{F}}_t$ such that $supp\mathcal{V}(t) \subset \mathcal{O} \subset \Omega$ and $\widetilde{\mathcal{F}}_t(\mathcal{V})(\mathcal{O}) = \mathcal{O}$, which can be obtained by using the similar technique we have already discussed in this section. We consider a one–to–one transformation $\mathcal{F}_t(\mathcal{V})$ defined on $\mathcal{Q}$ by

$$\mathcal{F}_t(\mathbf{p}) = \begin{cases} \widetilde{\mathcal{F}}_t(\mathcal{V})(\mathbf{p}), & \text{if } \mathbf{p} \in \overline{\mathcal{O}} \\ \mathbf{p}, & \text{if } \mathbf{p} \in \mathcal{Q} - \overline{\mathcal{O}}. \end{cases}$$

Then, clearly $\mathcal{F}_t(\mathcal{V})$ satisfies (3–1) and $\mathcal{F}_t(\Omega) = \Omega$. This implies that $\gamma_\Omega(\mathcal{G}(\Omega)) = 0$ and hence that *supp* $\mathcal{G}(\Omega) \subset \mathcal{Q} - \overline{\Omega}$. So, combined with (3–21), the result follows.

(III); (3–18) immediately follows from (3–16). Note that the distribution with a compact support has a finite order (c.f. Hörmander[81]). So, from (3–17), $g(\Gamma)$ has a finite order (if it exists). Let $\Omega \subset \mathcal{Q}$ be a domain of class $\mathcal{C}^k$, $(k \geq 1)$. Then the normal vector field $\mathbf{n}$ exists and belongs to $\mathcal{C}^{k-1}(\Gamma; R^n)$. Consider the continuous linear mapping

$$dJ(\Omega;\cdot) : \mathcal{C}^k(\overline{\mathcal{Q}}; R^n) \ni \mathcal{V} \longmapsto dJ(\Omega;\mathcal{V}) = < \mathcal{G}(\Omega), \mathcal{V}(0) > \in R.$$

Let $\mathcal{K}(\Omega) = \{\, \mathcal{V} \in \mathcal{C}^k(\overline{\mathcal{Q}}; R^n) \mid \mathcal{V}(0)\cdot\mathbf{n} = 0 \text{ on } \Gamma \,\}$. We first show that

$$\mathcal{K}(\Omega) \subset Ker\ dJ(\Omega;\cdot)\,; \tag{3–22}$$

Let $\mathcal{V}(0)\cdot\mathbf{n} = 0$ on Γ. Then $\mathcal{V}(0)$ is a tangent vector field to Γ. Hence transversality along Γ does not occur, *i.e.*, $\mathcal{F}_t(\mathcal{V})(\Omega) = \Omega$. Hence $dJ(\Omega;\mathcal{V}) = 0$, *i.e.*, $\mathcal{V} \in Ker\ dJ(\Omega;\cdot)$.

Therefore, there exists a continuous linear functional Λ on $\mathcal{C}^k(\overline{\mathcal{Q}}; R^n)/\mathcal{K}(\Omega)$ such that

$$dJ(\Omega;\mathcal{V}) = \Lambda \circ \pi\,,$$

where $\pi : \mathcal{C}^k(\overline{\mathcal{Q}}; R^n) \ni \mathcal{V}(0) \mapsto [\mathcal{V}(0)] \in \mathcal{C}^k(\overline{\mathcal{Q}}; R^n)/\mathcal{K}(\Omega)$ denotes the canonical projection. It is not difficult to show that $\mathcal{C}^k(\overline{\mathcal{Q}}; R^n)/\mathcal{K}(\Omega)$ is isomorphic to $\mathcal{C}^{k-1}(\Gamma)$

(see [48]). So, we can regard Λ as a continuous linear functional $dJ(\Gamma;\cdot)$ on $\mathcal{C}^{k-1}(\Gamma)$ and π as a projection map onto $\mathcal{C}^{k-1}(\Gamma)$ via $\pi(\mathcal{V}(0)) = [\mathcal{V}(0)\cdot\mathbf{n}]$. Hence there exists a distribution $g(\Gamma)$ of (k–1)–order in this case such that

$$dJ(\Omega;\mathcal{V}) = dJ(\Gamma;\mathcal{V}(0)\cdot\mathbf{n}) =< g(\Gamma), \mathcal{V}(0)\cdot\mathbf{n} >_\Gamma . \qquad \square$$

The representation (3–19) is usually called the *Hadamard formula* for the shape gradient of the domain functional $J(\Omega)$.

REMARK 3.3: From (3–22), $dJ(\Omega;\mathcal{V})$ can be written as

$$dJ(\Omega;\mathcal{V}) =< \mathcal{G}(\Omega), (\mathcal{V}(0)\cdot\mathcal{N})\mathcal{N} > ,$$

where $\mathcal{N}$ is a unitary extension of $\mathbf{n}$ to $\overline{\mathcal{Q}}$. Such an extension always exists if Γ is of class $\mathcal{C}^k$, $(k \geq 1)$, which can be verified by using the local atlas along the boundary of the domain and patching them using cutoff functions (see [121][136], for details). Hence $dJ(\Omega;\mathcal{V})$ can be written in the integral form as

$$dJ(\Omega;\mathcal{V}) = \int_{\mathcal{Q}} (\mathcal{G}(\Omega)\cdot\mathcal{N})(\mathcal{V}(0)\cdot\mathcal{N})\, d\Omega .$$

Consequently, in the representation of (3–18) and (3–19) for the shape gradient, $g(\Gamma)$ can be related to $\mathcal{G}(\Omega)$ via

$$g(\Gamma) = \gamma_\Gamma(\mathcal{G}(\Omega)\cdot\mathcal{N}) \tag{3–23}$$

and conversely,

$$G(\Gamma) = {}^T\gamma_\Gamma(g(\Gamma)\cdot\mathbf{n}) . \tag{3–24}$$

REMARK 3.4: Hadamard took the variation of the domain functional only in the normal direction to the boundary of the smooth domain to obtain necessary optimality conditions for specific problems (c.f.[48]). This method is usually called the *normal* *variation method*. By contrast, the material derivative method and the expression (3–19) for the shape gradient yields a generalization of the normal variation method.

For our work in the next section, we consider two standard examples for the domain functional;

$$J_1(\Omega_t) = \int_{\Omega_t} y_t \, d\Omega_t \qquad \text{and} \qquad J_2(\Omega_t) = \int_{\Gamma_t} y_t \, d\Gamma_t \,, \tag{3–25}$$

where $\Gamma_t = \partial\Omega_t$, y_t is a function defined on $\Omega_t \subset \mathcal{Q}$ and Γ_t, respectively. Let $\widetilde{y}$ be a uniform extension of y_t in $\mathcal{Q}$. Then, under some reasonable assumption on the regularity for the admissible domain and the class of functions, one can derive

$$dJ_1(\Omega; \mathcal{V}) = \int_\Omega \frac{\partial \widetilde{y}}{\partial t} \, d\Omega + \int_\Gamma y_0 \, (\mathcal{V}(0) \cdot \mathbf{n}) \, d\Gamma \tag{3–26}$$

and

$$dJ_2(\Omega; \mathcal{V}) = \int_\Gamma \left[\frac{\partial \widetilde{y}}{\partial t} + (\frac{\partial y_0}{\partial \mathbf{n}} + \kappa \, y_0) \, \mathcal{V}(0) \cdot \mathbf{n} \right] \, d\Gamma \,. \tag{3–27}$$

Here κ denotes the curvature of the boundary curve Γ when the spatial dimension of the domain is 2, and the mean curvature of the boundary surface Γ when the spatial dimension is 3. These formulations were induced by many authors. For demonstrations, one may refer to Zolésio[136], Sokolowski *et al.*[121], Haug *et al.*[78] and Rousselet[110].

In these two standard examples for the domain functional, $dJ_i(\Omega; \mathcal{V})$ consists of two main components; a linear term $\mathcal{V} \cdot \mathbf{n}$ on the boundary and a *shape derivative* term $\widetilde{y}\,' = \widetilde{y}\,'(\Omega; \mathcal{V}) = \dfrac{\partial \widetilde{y}}{\partial t}$. In order to obtain the shape gradients for J_1 and J_2, it should be justified that $\mathcal{V} \mapsto \dfrac{\partial \widetilde{y}}{\partial t}(\Omega; \mathcal{V})$ is linear and continuous over appropriate admissible vector fields. This implies that $\widetilde{y}\,'(\Omega; \mathcal{V})$ should be represented as a linear function of $\mathcal{V}$. A major step toward the shape sensitivity analysis is to give a sense to $\widetilde{y}'(\Omega;\mathcal{V})$ and to find an appropriate linear model for $\widetilde{y}'(\Omega;\mathcal{V})$. As a preliminary step, we demonstrate some basic properties of the shape derivative $\widetilde{y}'(\Omega;\mathcal{V})$.

Lemma 3.2. *Let $\mathcal{V}$ belong to a class of vector fields satisfying (3–4) and (3–5).*

(I) *$\widetilde{y}'(\Omega;\mathcal{V})$ is independent of the uniform extension of y_t to $\mathcal{Q}$.*

In addition, let $\mathcal{V} \mapsto \dot{\widetilde{y}}(\Omega;\mathcal{V})$ be linear and continuous. Then

(II) $\widetilde{y}'(\Omega;\mathcal{V}) = \widetilde{y}'(\Omega;\mathcal{V}(0))$.

(III) *For a vector field $\mathcal{V}$ with compact support in $\mathcal{Q}$ such that*

$$\mathcal{V}\cdot\mathbf{n} = 0 \qquad \text{on } \Gamma = \partial\Omega\,,$$

we have

$$\widetilde{y}'(\Omega;\mathcal{V}) = 0\,.$$

Proof: (I); Let $D\mathcal{F}_t$ denote the Jacobian matrix of the transformation $\mathcal{F}_t(\mathcal{V})$ and let y be an extension defined by (3–10). We first note that

$$\frac{d}{dt}\det(D\mathcal{F}_t)\Big|_{t=0^+} = div\ \mathcal{V}(0)\,. \tag{3–28}$$

For completeness, we provide a simple proof for (3–28). Let

$$\mathcal{F}_t : R^n \ni (p_1,\cdots,p_n) \longmapsto (x_1,\cdots,x_n) \in R^n, \qquad \text{where}$$

$$x_i = x_i(t,p_1,\cdots,p_n)\,, \qquad i = 1,\cdots,n\,.$$

We can write $\det(D\mathcal{F}_t) = \sum_{\sigma\in S_n}(sgn\sigma)\prod_{i=1}^{n}\frac{\partial x_i}{\partial p_{\sigma(i)}}$, where S_n denotes the permutations over $\{1,\cdots,n\}$ and $sgn\sigma$ the sign of a permutation σ.

$$\begin{aligned}\frac{d}{dt}\det(D\mathcal{F}_t)\Big|_{t=0^+} &= \sum_{j=1}^{n}\sum_{\sigma\in S_n}(sgn\sigma)\Big(\frac{\partial}{\partial p_{\sigma(j)}}\frac{\partial x_j}{\partial t}\prod_{i=1(,\neq j)}^{n}\frac{\partial x_i}{\partial p_{\sigma(i)}}\Big)\Big|_{t=0^+}\\ &= \sum_{j=1}^{n}\sum_{\sigma\in S_n}(sgn\sigma)\Big(\frac{\partial}{\partial p_{\sigma(j)}}\mathcal{V}_j(0)\prod_{i=1(,\neq j)}^{n}\delta_{i,\sigma(i)}\Big)\\ &= \sum_{j=1}^{n}\frac{\partial}{\partial p_j}\mathcal{V}_j(0) = div\ \mathcal{V}(0)\,,\end{aligned}$$

where $\delta_{i,j}$ denotes the Kronecker delta. Let ψ be a regular function defined in a neighborhood of $\widehat{\Omega}$. Then,

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t}\psi\, y_t\, d\Omega_t\Big|_{t=0+} &= \frac{d}{dt}\int_{\Omega}\psi\, \widetilde{y}(t,\mathcal{F}_t)\ \det(D\mathcal{F}_t)\, d\Omega\Big|_{t=0+}\\ &= \int_{\Omega}\psi\,(\widetilde{y}\,' + \nabla\widetilde{y}\cdot\mathcal{V}(0) + \widetilde{y}\, div\ \mathcal{V}(0))\, d\Omega\\ &= \int_{\Omega}\psi\,\widetilde{y}\,'\, d\Omega + \int_{\Omega}\psi\, div\ (\widetilde{y}\,\mathcal{V}(0))\, d\Omega\,.\end{aligned}$$

Since any two extensions coincide over Ω, (I) follows from the last equality.

(II); Consider the domain functional $J(\Omega_t) = \int_{\Omega_t}\psi\, y_t\, d\Omega$. Then, in the same manner as (I), we have

$$dJ(\Omega;\mathcal{V}) = \int_{\Omega}\psi\,(\dot{\widetilde{y}}(\Omega;\mathcal{V}) + \widetilde{y}\, div\ \mathcal{V}(0))\, d\Omega \tag{3–29}$$

Since $\mathcal{V} \mapsto \dot{\widetilde{y}}(\Omega;\mathcal{V})$ is linear and continuous, it readily follows that $J(\Omega)$ is shape differentiable. Hence, $dJ(\Omega;\mathcal{V}) = dJ(\Omega;\mathcal{V}(0))$ by Theorem 3.1. It follows from (3–29) that $\dot{\widetilde{y}}(\Omega;\mathcal{V}) = \dot{\widetilde{y}}(\Omega;\mathcal{V}(0))$ and the result is a consequence of the definition of the shape derivative.

(III); From (II), $\widetilde{y}\,'(\Omega;\mathcal{V}) = \widetilde{y}\,'(\Omega;\mathcal{V}(0))$. Since $\mathcal{V}\cdot\mathbf{n} = 0$, $\mathcal{V} \in \mathcal{K}(\Omega) \subset Ker\ dJ(\Omega,\cdot)$ by (3–22). Consequently, applying $dJ(\Omega;\mathcal{V}) = dJ(\Omega;\mathcal{V}(0)) = 0$ to (3–29), we get the result. □

3.2. Shape Sensitivity Analysis

The main objective of this section is to compute the shape gradient of the penalized cost functional (2–56) using the material derivative method. In particular, we wish to compute the shape gradient of J_ϵ at $\Omega(\alpha)$ in the direction of a specified vector field of deformation $\mathcal{V}$. As we have already mentioned, the major difficulty to overcome is a lack of regularity for the shape derivative of the state variables. For elliptic equations, the shape derivative of the state variables is usually expressed as a solution of a boundary value problem which depends on the state variables and the normal component of the design velocity along the boundary of the reference domain. If the existence of the solution for this boundary value problem is settled, this equation can be adopted as an adjoint equation to suppress the regularity requirement and to facilitate the computation of the shape gradient. However, for a nonlinear problem, it is rarely expected that one is able to resolve the equations containing the shape derivative.

To get around these difficulties, we will employ the Lagrange multiplier technique to attain the adjoint state equation. This can usually be done by introducing adjoint variables to combine the constraints with the domain performance functional. These adjoint variables, which are usually called *Lagrange multipliers*, play a central role in eliminating the cumbersome shape derivative of the state variables. Though direct computation is still widely used, the Lagrange multiplier technique seems to provide a more sound mathematical justification for the shape sensitivity analysis. In recent years, similar techniques have been systematically studied in a series of papers by Delfour and Zolésio ([41][42][44]) using the theory for the differentiability of the parametrized minmax function due to Correa and Seegel[35] and Dem'yanov[51]. However, their applications are mainly based on the uniqueness of the saddle points for the Lagrange formulation, which can hardly be expected in nonlinear problems such as ours. One can also find a rough framework using the Lagrange multiplier technique in Cèa[28]. For the historical background and simple applications in structural engineering, one may refer to Komkov[87] and some of the references cited therein.

In this approach, the price we should pay is, of course, showing the existence of stationary points for the Lagrange formulation. Leaving this nontrivial step to the next section, in this section we will mainly concentrate on computing the shape gradient straightforwardly and somewhat formally. We first note that the main contribution of the deformation to the shape gradient comes from the first order perturbation of the identity operator. In a sufficiently small neighborhood of $t = 0$, one can estimate the deformation at $\mathbf{p} \in \Omega$ as follows:

Using (3–1),

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{x}(0) + t\,\dot{\mathbf{x}}(0) + \mathbf{o}(t, \mathbf{x}(0)) \\ &= \mathbf{p} + t\,\mathcal{V}(0, \mathbf{p}) + \mathbf{o}(t, \mathbf{p}), \end{aligned}$$

where $\mathbf{o}(t, \mathbf{p})$ denotes the remainder function such that $\lim_{t \to 0^+} \frac{1}{t} |\mathbf{o}(t, \mathbf{p})|_{R^n} = 0$. Hence $\mathbf{p} + t\,\mathcal{V}(\mathbf{p})$ can be considered as a linear approximation of $\mathcal{F}_t(\mathbf{p})$, where $\mathcal{V}(\mathbf{p}) \equiv \mathcal{V}(0, \mathbf{p})$. Observe that $\mathcal{F}_t(\mathbf{p})$ and $\mathbf{p} + t\,\mathcal{V}(\mathbf{p})$ yield the same design velocity at $t = 0^+$. Hence, by Theorem 3.1 and Lemma 3.2, we can easily infer that they yield the same shape gradient and shape derivative. In this context, we may take

$$\mathcal{F}_t(\mathbf{p}) = \mathbf{p} + t\,\mathcal{V}(\mathbf{p}) = (\mathcal{I} + t\,\mathcal{V})(\mathbf{p}) \quad \text{for} \quad 0 \leq t < \sigma. \tag{3–30}$$

From the second expression for the deformation, we may regard $\mathcal{F}_t(\mathbf{p})$ as the first order perturbation of the identity operator over the reference domain.

REMARK 3.5: If $\sup_{\mathbf{p} \in R^n} \sum_{|\alpha| \leq 1} |D^\alpha \mathcal{V}(\mathbf{p})| < 1$, we easily see that $\mathcal{I} + \mathcal{V}$ and $D(\mathcal{I} + \mathcal{V}) = \mathcal{I} + D\mathcal{V}$ are invertible, where $D(\mathcal{I} + \mathcal{V})$ and $D\mathcal{V}$ denote the Jacobian matrices for the deformations. Simon[117] and Rousselet[111] used this feature to derive the design sensitivity analysis: If $\mathcal{V}$ is of class C^2 and $\sup_{\mathbf{p} \in R^n} \sum_{|\alpha| \leq 1} |D^\alpha \mathcal{V}(\mathbf{p})| < 1$, $\mathcal{I} + \mathcal{V}$ is C^2–diffeomorphism. Using this device, they derived a similar shape calculus to accommodate the Hadamard formula for the shape gradient.

The choice of $\mathcal{V}$ is very crucial in the shape sensitivity analysis. In our problem, we want to keep the variation of $\Gamma(\alpha)$ within the rectangular region $EFGH$ in Figure

2, *i.e.*,

$$\Gamma(\alpha) \subset \overline{\Omega}_0 \quad \text{for every} \quad \alpha \in \mathcal{U}_{ad}\,.$$

Such a restriction may arise from the need to observe the effect on the state variables over the unperturbed region when the design variables are allowed to change only in the restricted region. An appropriate choice for the velocity is thus to consider $\mathcal{V} = (0, \mathcal{V}_2)$. Utilizing the mapping technique, $\mathcal{V}_2$ can be determined as follows:

For a fixed $\alpha \in \mathcal{U}_{ad}$, we may associate a bijection

$$F_\alpha : \widehat{\Omega} \longrightarrow \Omega(\alpha) \ ((\widehat{x}_1, \widehat{x}_2) \mapsto (p_1, p_2)) \qquad \text{via}$$

$$p_1 = \widehat{x}_1\,,$$
$$p_2 = \begin{cases} \dfrac{(\widehat{x}_2 - L)(L - \alpha(\widehat{x}_1))}{L} + L\,, & \text{if} \quad -M_1 \le \widehat{x}_1 \le M_1 \\ \widehat{x}_2\,, & \text{otherwise}\,. \end{cases}$$

Let $\vartheta \in C^{0,1}([-M_1, M_1])$ such that $\vartheta(-M_1) = \vartheta(M_1) = 0$ and there exists $\sigma > 0$ such that the graph of $\alpha + t\vartheta$ lies in $\overline{\widehat{\Omega}}$ for $0 \le t < \sigma$. We may extend ϑ to $[-M_2, M_2]$ by defining $\vartheta = 0$ over $[-M_2, -M_1] \cup [M_1, M_2]$. If we consider a bijection

$$F_{\alpha + t\vartheta} : \Omega(\alpha) \longrightarrow \Omega(\alpha + t\vartheta) \ ((\widehat{x}_1, \widehat{x}_2) \mapsto (x_1, x_2))\,,$$

the composite $F_{\alpha+t\vartheta} \circ F_\alpha^{-1} : \Omega(\alpha) \longrightarrow \Omega(\alpha + t\vartheta) \ ((p_1, p_2) \mapsto (x_1, x_2))$ is given by

$$x_1 = p_1$$
$$x_2 = \begin{cases} p_2 + t\dfrac{(p_2 - L)\vartheta(p_1)}{(\alpha(p_1) - L)}\,, & \text{if} \quad -M_1 \le p_1 \le M_1 \\ p_2\,, & \text{otherwise}\,. \end{cases} \tag{3-31}$$

Since $0 \le \alpha(p_1) < L$ for all $p_1 \in [-M_1, M_1]$, the mapping (3–31) is well–defined and $(x_1, x_2) = (p_1, p_2) + t\,(0, \mathcal{V}_2(p_1, p_2))$, where

$$\mathcal{V}_2(p_1, p_2) = \begin{cases} \dfrac{(p_2 - L)\vartheta(p_1)}{(\alpha(p_1) - L)}\,, & \text{if} \quad -M_1 \le p_1 \le M_1 \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{3-32}$$

Hence, for the perturbation of the domain, it is reasonable to consider the transformation

$$\mathcal{F}_t(p_1,p_2) = (p_1,p_2) + t\,\mathcal{V}(p_1,p_2) = F_{\alpha+t\vartheta} \circ F_\alpha^{-1}(p_1,p_2)\,,$$

where $\mathcal{V} = (0,\mathcal{V}_2)$ is an autonomous vector field. Clearly, $\mathcal{F}_t$ is a one–to–one transformation from $\Omega(\alpha)$ onto $\Omega(\alpha+t\vartheta)$ whose inverse is given by $\mathcal{F}_t^{-1}(x_1,x_2) = (p_1,p_2)$, where

$$\begin{aligned} p_1 &= x_1 \\ p_2 &= \begin{cases} x_2 + t\dfrac{(L-x_2)\vartheta(x_1)}{(\alpha(x_1)-L+t\vartheta(x_1))}\,, & \text{if} \quad -M_1 \le x_1 \le M_1 \\ x_2\,, & \text{otherwise}\,. \end{cases} \end{aligned}$$

Note that $\mathcal{V}(p_1,\alpha(p_1)) = (0,\vartheta(p_1))$ for all $p_1 \in [-M_1,M_1]$. Thus, $\mathcal{V} = (0,\vartheta)$ along $\Gamma(\alpha)$ and $\mathcal{V} = \mathbf{0}$ along $\partial\Omega(\alpha) - \Gamma(\alpha)$. We define $\alpha_t = \alpha + t\,\vartheta$ for simplicity.

Now, we transform the problem into a Lagrangian formulation. The basic idea is to turn the constrained optimization problem into a unconstrained one by incorporating some adjoint variables, and to apply the *Lagrange principle* which can be precisely stated as follow: From the variables incorporated in the Lagrangian, one investigates subproblems determined by the Gateaux differentiability for the variables obtained by fixing all but one variable. These subproblems constitute the desired necessary optimality conditions for the original problem (c.f. Tikhomirov[130]).

This technique provides a direct bypass for the computation of the shape sensitivity without recourse to the equations dealing with the shape derivative.

We introduce the Lagrangian $L : \mathcal{U}_{ad} \times \mathbf{H}^1(\Omega(\alpha)) \times \mathbf{H}^1(\Omega(\alpha)) \times \mathbf{H}^{-1/2}(\Gamma) \longrightarrow R$

defined by

$$
\begin{aligned}
&L(\alpha, \mathbf{u}_\epsilon, \mathbf{q}_\epsilon, \boldsymbol{\eta}_\epsilon) \\
&= J_\epsilon(\alpha, \mathbf{u}_\epsilon(\alpha)) - \nu \int_{\Omega(\alpha)} \nabla \mathbf{u}_\epsilon : \nabla \mathbf{q}_\epsilon \, d\Omega - \int_{\Omega(\alpha)} (\mathbf{u}_\epsilon \cdot \nabla)\, \mathbf{u}_\epsilon \cdot \mathbf{q}_\epsilon \, d\Omega \\
&- \frac{1}{\epsilon} \int_{\Omega(\alpha)} div\, \mathbf{u}_\epsilon \, div\, \mathbf{q}_\epsilon \, d\Omega + < \mathbf{t}_\epsilon, \mathbf{q}_\epsilon >_{-1/2,\Gamma} + < \mathbf{f}, \mathbf{q}_\epsilon >_{-1} \\
&\quad - < \boldsymbol{\eta}_\epsilon, \mathbf{u}_\epsilon - \mathbf{g} >_{-1/2,\Gamma} ,
\end{aligned}
\tag{3–33}
$$

where $\mathbf{t}_\epsilon \in \mathbf{H}^{-1/2}(\Gamma)$ is given by (2–18) and $(\mathbf{q}_\epsilon, \boldsymbol{\eta}_\epsilon) \in \mathbf{H}^1(\Omega(\alpha)) \times \mathbf{H}^{-1/2}(\Gamma)$ is considered to be a formal Lagrange multiplier. We notice that finding a stationary point for the Lagrangian $L(\alpha, \mathbf{u}_\epsilon, \mathbf{q}_\epsilon, \boldsymbol{\eta}_\epsilon)$ is equivalent to finding a solution pair $(\alpha^*, \mathbf{u}_\epsilon(\alpha^*)) \in \widetilde{\mathcal{U}_{ad}}$ of (2–56). Formally, the minimum principle for optimal control problems may be employed to deduce the Euler–Lagrange equations for the problem.

Clearly, variations in the Lagrange multipliers $\mathbf{q}_\epsilon$ and $\boldsymbol{\eta}_\epsilon$ recover the constraints (2–16) and (2–17). From the variation in the state variable $\mathbf{u}_\epsilon$, one can derive the adjoint state equations. The Gateaux derivative

$$
\frac{d}{d\lambda} L(\alpha, \mathbf{u}_\epsilon + \lambda \mathbf{w}, \mathbf{q}_\epsilon, \boldsymbol{\eta}_\epsilon)\Big|_{\lambda=0} = 0 \qquad \text{for every } \mathbf{w} \in \mathbf{H}^1(\Omega(\alpha))
$$

yields the equations for $\mathbf{q}_\epsilon$ and $\boldsymbol{\eta}_\epsilon$. To assign a suitable boundary condition, we assume $\mathbf{q}_\epsilon = \mathbf{0}$ on $\Gamma = \partial\Omega(\alpha)$. Then, we obtain

$$
\begin{aligned}
&\nu \int_{\Omega(\alpha)} \nabla \mathbf{q}_\epsilon : \nabla \mathbf{w} \, d\Omega + \int_{\Omega(\alpha)} (\mathbf{w} \cdot \nabla)\, \mathbf{u}_\epsilon \cdot \mathbf{q}_\epsilon \, d\Omega + \int_{\Omega(\alpha)} (\mathbf{u}_\epsilon \cdot \nabla)\, \mathbf{w} \cdot \mathbf{q}_\epsilon \, d\Omega \\
&+ \frac{1}{\epsilon} \int_{\Omega(\alpha)} div\, \mathbf{q}_\epsilon \, div\, \mathbf{w} \, d\Omega + < \boldsymbol{\eta}_\epsilon, \mathbf{w} >_{-1/2,\Gamma} = 2\,\nu \int_{\Omega(\alpha)} \nabla \mathbf{u}_\epsilon : \nabla \mathbf{w} \, d\Omega
\end{aligned}
\tag{3–34}
$$

for every $\mathbf{w} \in \mathbf{H}^1(\Omega(\alpha))$ and

$$
< \mathbf{s}, \mathbf{q}_\epsilon >_{-1/2,\Gamma} = 0 \qquad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma) . \tag{3–35}
$$

To derive the equations for $\mathbf{q}_\epsilon$ and $\boldsymbol{\eta}_\epsilon$, we first note that

$$\begin{aligned}\int_{\Omega(\alpha)} (\mathbf{w}\cdot\nabla)\mathbf{u}_\epsilon\cdot\mathbf{q}_\epsilon\,d\Omega + \int_{\Omega(\alpha)} (\mathbf{u}_\epsilon\cdot\nabla)\mathbf{w}\cdot\mathbf{q}_\epsilon\,d\Omega &= \sum_{i,j=1,2}^{2}\int_{\Omega(\alpha)} \Big(w_j\frac{\partial u_{\epsilon i}}{\partial x_j}q_{\epsilon i} + u_{\epsilon j}\frac{\partial w_i}{\partial x_j}q_{\epsilon i}\Big)\,d\Omega \\ &= \sum_{i,j=1,2}^{2}\int_{\Omega(\alpha)}\bigg(q_{\epsilon j}\frac{\partial u_{\epsilon j}}{\partial x_i}w_i - u_{\epsilon j}\frac{\partial q_{\epsilon i}}{\partial x_j}w_i - \frac{\partial u_{\epsilon j}}{\partial x_j}q_{\epsilon i}w_i\bigg)\,d\Omega \\ &= \int_{\Omega(\alpha)}\bigg(\mathbf{q}_\epsilon\cdot{}^T(\nabla\mathbf{u}_\epsilon) - \mathbf{u}_\epsilon\cdot(\nabla\mathbf{q}_\epsilon) - (div\ \mathbf{u}_\epsilon)\,\mathbf{q}_\epsilon\bigg)\cdot\mathbf{w}\,d\Omega\,,\end{aligned}$$

using integration by parts and $\mathbf{q}_\epsilon = \mathbf{0}$ on Γ. Applying Green's formula, the formal computation of (3–34) and (3–35) yields

$$\begin{aligned}&-\nu\,\Delta\mathbf{q}_\epsilon + \mathbf{q}_\epsilon\cdot{}^T(\nabla\mathbf{u}_\epsilon) - \mathbf{u}_\epsilon\cdot(\nabla\mathbf{q}_\epsilon) - (div\ \mathbf{u}_\epsilon)\,\mathbf{q}_\epsilon \\ &\qquad -\frac{1}{\epsilon}\nabla(div\ \mathbf{q}_\epsilon) = -2\,\nu\,\Delta\mathbf{u}_\epsilon \quad \text{in}\ \ \Omega(\alpha)\,,\end{aligned} \tag{3–36}$$

$$\mathbf{q}_\epsilon = \mathbf{0} \qquad \text{on}\ \ \Gamma \tag{3–37}$$

and

$$\boldsymbol{\eta}_\epsilon = 2\,\nu\,\frac{\partial\mathbf{u}_\epsilon}{\partial\mathbf{n}} - \nu\,\frac{\partial\mathbf{q}_\epsilon}{\partial\mathbf{n}} - \frac{1}{\epsilon}(div\ \mathbf{q}_\epsilon)\,\mathbf{n} \quad \text{on}\ \ \Gamma\,. \tag{3–38}$$

Here we denote the normal derivative along the boundary Γ by the operator $\dfrac{\partial}{\partial\mathbf{n}} \equiv (\mathbf{n}\cdot\nabla)$. The equations (3–36)–(3–37) can be interpreted as a penalized version of linearized adjoint incompressible Navier–Stokes equations given by

$$\begin{aligned}-\nu\,\Delta\mathbf{q} + \mathbf{q}\cdot{}^T(\nabla\mathbf{u}) - \mathbf{u}\cdot(\nabla\mathbf{q}) + \nabla r = -2\,\nu\Delta\mathbf{u} \quad &\text{in}\ \ \Omega(\alpha)\,, \\ div\ \mathbf{q} = 0 \qquad &\text{in}\ \ \Omega(\alpha)\,, \\ \mathbf{q} = \mathbf{0} \qquad &\text{on}\ \ \Gamma\,,\end{aligned} \tag{3–39}$$

where $\mathbf{u}$ is a solution of the incompressible Navier–Stokes equations (1–1)–(1–3) and r corresponds to the adjoint version of the pressure p. The penalty term for (3–39) is introduced by $r_\epsilon = -\dfrac{1}{\epsilon}(\ div\ \mathbf{q}_\epsilon)$. In this case, $\boldsymbol{\eta}_\epsilon$ of (3–38) corresponds to

$$\boldsymbol{\eta} = 2\,\nu\,\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - \nu\,\frac{\partial\mathbf{q}}{\partial\mathbf{n}} + r\mathbf{n} \qquad \text{on}\ \ \Gamma\,. \tag{3–40}$$

Hence, (3–34)–(3–35) can be interpreted as an adjoint of the weak penalized formulation of the linearized equations (3–39) together with

$$< \boldsymbol{\eta}_\epsilon, \mathbf{s} >_{-1/2,\Gamma} = < 2\,\nu\,\frac{\partial \mathbf{u}_\epsilon}{\partial \mathbf{n}} - \nu\,\frac{\partial \mathbf{q}_\epsilon}{\partial \mathbf{n}} - \frac{1}{\epsilon}\,(div\ \mathbf{u}_\epsilon)\,\mathbf{n}, \mathbf{s} >_{-1/2,\Gamma} \quad \forall \mathbf{s} \in \mathbf{H}^{1/2}(\Gamma)\,. \quad (3\text{–}41)$$

It should be noted that the triplet $\{(\mathbf{u}, p, \mathbf{t})\}$ of velocity, pressure and traction force for the solutions of the state equations correspond to $\{(\mathbf{q}, r, \boldsymbol{\eta})\}$ for those of the adjoint equations, and the doublet $\{(\mathbf{u}_\epsilon, \mathbf{t}_\epsilon)\}$ to $\{(\mathbf{q}_\epsilon, \boldsymbol{\eta}_\epsilon)\}$ in the penalized equations. The existence of $(\mathbf{q}_\epsilon, \boldsymbol{\eta}_\epsilon)$ satisfying these relations will be shown at the next section. The relation between $(\mathbf{q}_\epsilon, \boldsymbol{\eta}_\epsilon)$ and $(\mathbf{q}, \boldsymbol{\eta})$ shall be discussed in Section 3.4.

We now consider the variation in the design control $\alpha \in \mathcal{U}_{ad}$. We notice that

$$\inf_{\alpha \in \mathcal{U}_{ad}} \mathfrak{J}_\epsilon(\alpha) = \inf_{\alpha \in \mathcal{U}_{ad}} L(\alpha, \mathbf{u}_\epsilon, \mathbf{q}_\epsilon, \boldsymbol{\eta}_\epsilon)\,,$$

whenever $(\mathbf{q}_\epsilon, \boldsymbol{\eta}_\epsilon)$ is a solution of (3–34)–(3–35). Thus, computations of the design sensitivity may involve the sensitivity of the state variables and adjoint state variables. Recall that $\mathcal{V} = (0, \vartheta)$ along $\Gamma(\alpha)$ for any $\vartheta \in \mathcal{C}^{0,1}([-M_1, M_1])$ such that $\vartheta(-M_1) = \vartheta(M_1) = 0$. Since the perturbation of a domain is determined by the variation of the boundary part $\Gamma(\alpha)$, for the computation of $\inf_{\alpha \in \mathcal{U}_{ad}} \mathfrak{J}_\epsilon(\alpha)$, we try to find a semi–derivative

$$d\mathfrak{J}_\epsilon(\alpha;\vartheta) \equiv \frac{d}{dt}\,\mathfrak{J}_\epsilon(\alpha + t\,\vartheta)\Big|_{t=0^+} = \lim_{t \to 0^+} \frac{\mathfrak{J}_\epsilon(\alpha_t) - \mathfrak{J}_\epsilon(\alpha)}{t}\,, \qquad (3\text{–}42)$$

where $\alpha_t = \alpha + t\,\vartheta$ for $\vartheta \in \mathcal{C}^{0,1}([-M_1, M_1])$. Then this will yield the information for the gradient of the design functional $\mathfrak{J}_\epsilon(\alpha)$. For this purpose, we assume $\mathbf{f} \in \mathbf{L}^2(\widehat{\Omega})$. This assumption is needed to guarantee the existence of weak shape derivative of $\mathbf{f}$ in the space of $\mathbf{H}^{-1}(\widehat{\Omega})$ and the regularity of the function space (see Section 3.4).

Hereafter, we are only concerned with the formal computation of the shape gradient rather than the consideration of issues about sufficient regularity to justify each computation. Some of these issues shall be discussed in Section 3.4. Let $\mathbf{u}_\epsilon(\alpha_t) \in \mathbf{H}^1(\Omega(\alpha_t))$ be a solution of the penalized incompressible Navier–Stokes equations over $\Omega(\alpha_t)$, which is represented by the following integral formulations:

$$\nu \int_{\Omega(\alpha_t)} \nabla \mathbf{u}_\epsilon(\alpha_t) : \nabla \mathbf{w}\, d\Omega_t + \int_{\Omega(\alpha_t)} (\mathbf{u}_\epsilon(\alpha_t)\cdot\nabla)\, \mathbf{u}_\epsilon(\alpha_t)\cdot \mathbf{w}\, d\Omega_t + \frac{1}{\epsilon}\int_{\Omega(\alpha_t)} div\, \mathbf{u}_\epsilon(\alpha_t)\, div\, \mathbf{w}\, d\Omega_t$$
$$- \nu \int_{\Gamma_t} \frac{\partial \mathbf{u}_\epsilon(\alpha_t)}{\partial \mathbf{n}_t} \cdot \mathbf{w}\, d\Gamma_t - \frac{1}{\epsilon}\int_{\Gamma_t} div\ \mathbf{u}_\epsilon(\alpha_t)(\mathbf{w}\cdot\mathbf{n}_t)\, d\Gamma_t = \int_{\Omega(\alpha_t)} \mathbf{f}\cdot\mathbf{w}\, d\Omega_t\,, \quad (3\text{–}43)$$

for $\mathbf{w} \in \mathbf{H}^1(\Omega(\alpha_t))$ and

$$\mathbf{u}_\epsilon(\alpha_t) = \mathbf{g}\,, \qquad \text{on } \Gamma_t\,. \qquad (3\text{–}44)$$

Here $\Gamma_t = \partial\Omega(\alpha_t)$, and $\mathbf{n}_t$ denotes the outward unit normal vector along Γ_t. The function space $\mathbf{H}^1(\Omega_t)$ is dependent on time t. To remove of this dependence, two methods are widely used (c.f.[48]):

(i) Using the homeomorphism $\mathcal{F}_t$, the situation can be transformed back onto the reference domain.

(ii) Using the extension property (under adequate regularity of the boundary of the domain), the situation can be considered to be a mere restriction of a function space which is defined on $\widehat{\Omega}$.

To be consistent with the argument adopted in the existence Theorem 3.1, we choose the function space extension method (method (ii)).

Let $\widetilde{\mathbf{u}}_\epsilon(t,\mathbf{x}) = P_{\widehat{\Omega}}(\mathbf{u}_\epsilon(\alpha_t)\circ\mathcal{F}_t)\circ\mathcal{F}_t^{-1}(\mathbf{x})$. Then $\widetilde{\mathbf{u}}_\epsilon(t,\cdot)$ is a uniform extension of $\mathbf{u}_\epsilon(\alpha_t)$ to $\widehat{\Omega}$ such that $\mathbf{u}_\epsilon(\alpha_t) = \widetilde{\mathbf{u}}_\epsilon\Big|_{\{t\}\times\Omega_t}$. We note that

$$\dot{\widetilde{\mathbf{u}}}_\epsilon = \widetilde{\mathbf{u}}_\epsilon{}' + (\mathcal{V}(0)\cdot\nabla)\,\widetilde{\mathbf{u}}_\epsilon\,, \qquad (3\text{–}45)$$

where $(\mathcal{V}(0)\cdot\nabla)\,\widetilde{\mathbf{u}}_\epsilon = (\nabla\widetilde{\mathbf{u}}_\epsilon)\circ\mathcal{V}(0)$. From (3–26), we have

$$d\mathfrak{J}_\epsilon(\alpha;\vartheta) = \frac{d}{dt}\nu\int_{\Omega(\alpha_t)} \nabla\mathbf{u}_\epsilon(\alpha_t) : \nabla\mathbf{u}_\epsilon(\alpha_t)\, d\Omega_t\Big|_{t=0+}$$
$$= 2\,\nu\int_{\Omega(\alpha)} \nabla\mathbf{u}_\epsilon(\alpha) : \nabla\widetilde{\mathbf{u}}_\epsilon{}'\, d\Omega + \nu\int_\Gamma \nabla\mathbf{u}_\epsilon(\alpha) : \nabla\mathbf{u}_\epsilon(\alpha)\,\mathcal{V}(0)\cdot\mathbf{n}\, d\Gamma\,,$$

where $\widetilde{\mathbf{u}}_\epsilon{}'$ denotes the shape derivative of an extension $\widetilde{\mathbf{u}}_\epsilon$.

Note that $\mathbf{n} = \left(\dfrac{\alpha'}{\sqrt{1+\alpha'^2}}, -\dfrac{1}{\sqrt{1+\alpha'^2}}\right)$ over $\Gamma(\alpha)$ and $d\Gamma = \sqrt{1+\alpha'^2}\,dx_1$, where $\alpha'(x_1) = \dfrac{d\alpha(x_1)}{dx_1}$. Since $\mathcal{V} = (0,\vartheta)$ on $\Gamma(\alpha)$ and $(0,0)$ otherwise,

$$\int_\Gamma \nabla\mathbf{u}_\epsilon(\alpha) \,:\, \nabla\mathbf{u}_\epsilon(\alpha)\,\mathcal{V}(0)\cdot\mathbf{n}\,d\Gamma = -\int_{-M_1}^{M_1} \nabla\mathbf{u}_\epsilon(x_1,\alpha(x_1)) \,:\, \nabla\mathbf{u}_\epsilon(x_1,\alpha(x_1))\,\vartheta(x_1)\,dx_1\,.$$

Thus,

$$\begin{aligned} d\mathfrak{J}_\epsilon(\alpha;\vartheta) = 2\,\nu \int_{\Omega(\alpha)} &\nabla\mathbf{u}_\epsilon(\alpha) \,:\, \nabla\widetilde{\mathbf{u}}_\epsilon{}'\,d\Omega \\ &- \int_{-M_1}^{M_1} \nabla\mathbf{u}_\epsilon(x_1,\alpha(x_1)) \,:\, \nabla\mathbf{u}_\epsilon(x_1,\alpha(x_1))\,\vartheta(x_1)\,dx_1\,. \end{aligned} \tag{3–46}$$

As we have mentioned in Section 3.1, $d\mathfrak{J}_\epsilon$ must depend linearly on $\mathcal{V}(0)\cdot\mathbf{n}$ to have a meaningful gradient. Since $d\mathfrak{J}_\epsilon(\alpha;\vartheta)$ contains a shape derivative term $\widetilde{\mathbf{u}}_\epsilon{}'$, we may use the state equations and its adjoint equations to eliminate it.

Let us consider the state equation (3–43). One may take $\mathbf{w} \in \mathbf{H}^1(\widehat{\Omega}) \cap \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$. Take the derivative of both sides of (3–43) with respect to time t. Since $\mathbf{f}$ and $\mathbf{w}$ are independent of t, by a similar computation to (3–26), we obtain the following equation at $t = 0^+$:

$$\begin{aligned} &\nu\int_{\Omega(\alpha)} \nabla\widetilde{\mathbf{u}}_\epsilon{}' : \nabla\mathbf{w}\,d\Omega + \int_{\Omega(\alpha)} (\widetilde{\mathbf{u}}_\epsilon{}'\cdot\nabla)\,\mathbf{u}_\epsilon(\alpha)\cdot\mathbf{w}\,d\Omega + \int_{\Omega(\alpha)} (\mathbf{u}_\epsilon(\alpha)\cdot\nabla)\,\widetilde{\mathbf{u}}_\epsilon{}'\cdot\mathbf{w}\,d\Omega \\ &\frac{1}{\epsilon}\int_{\Omega(\alpha)} div\ \widetilde{\mathbf{u}}_\epsilon{}'\,div\ \mathbf{w}\,d\Omega - \int_{-M_1}^{M_1} \left(\nu\nabla\mathbf{u}_\epsilon(\alpha) \,:\, \nabla\mathbf{w} + \frac{1}{\epsilon}\,div\ \mathbf{u}_\epsilon(\alpha)\,div\ \mathbf{w}\right)\vartheta(x_1)\,dx_1 \\ &\quad -\frac{d}{dt}\left(\nu\int_{\Gamma_t} \frac{\partial\mathbf{u}_\epsilon(\alpha_t)}{\partial\mathbf{n}_t}\cdot\mathbf{w}\,d\Gamma\right)\Bigg|_{t=0^+} - \frac{d}{dt}\left(\frac{1}{\epsilon}\int_{\Gamma_t} div\ \mathbf{u}_\epsilon(\alpha_t)(\mathbf{w}\cdot\mathbf{n}_t)\,d\Gamma_t\right)\Bigg|_{t=0^+} \\ &\qquad + \int_\Gamma ((\mathbf{u}_\epsilon(\alpha)\cdot\nabla)\,\mathbf{u}_\epsilon(\alpha)\cdot\mathbf{w})\,(\mathcal{V}(0)\cdot\mathbf{n})\,d\Gamma = \int_\Gamma (\mathbf{f}\cdot\mathbf{w})\,\mathcal{V}(0)\cdot\mathbf{n}\,d\Gamma\,. \end{aligned} \tag{3–47}$$

Since $\mathbf{w} = \mathbf{0}$ on $\Gamma_0(\alpha)$ and $\mathcal{V}(0) = \mathbf{0}$ on $\Gamma_{\mathbf{g}}$, we have

$$\int_\Gamma (\mathbf{f}\cdot\mathbf{w})\,\mathcal{V}(0)\cdot\mathbf{n}\,d\Gamma = \int_{\Gamma_{\mathbf{g}}} (\mathbf{f}\cdot\mathbf{w})\,\mathcal{V}(0)\cdot\mathbf{n}\,d\Gamma = 0\,.$$

Similarly, $\int_\Gamma ((\mathbf{u}_\epsilon(\alpha)\cdot\nabla)\mathbf{u}_\epsilon(\alpha)\cdot\mathbf{w})\,(\mathcal{V}(0)\cdot\mathbf{n})\,d\Gamma = 0.$

Next, we consider $\frac{d}{dt}\int_{\Gamma_t}\frac{\partial\mathbf{u}_\epsilon(\alpha_t)}{\partial\mathbf{n}_t}\cdot\mathbf{w}\,d\Gamma_t$ and $\frac{d}{dt}\int_{\Gamma_t} div\ \mathbf{u}_\epsilon(\alpha_t)\,(\mathbf{w}\cdot\mathbf{n}_t)\,d\Gamma_t$. For this, we need the surface measure of the transformation.

Lemma 3.3. *Let Ω_t be a domain in R^n which is transported by a one–to–one transformation $\mathcal{F}_t$ and let Γ_t be the boundary of Ω_t. If h is an integrable function defined on Γ_t, we have the following formula for the transformation of boundary integrals:*

$$\int_{\Gamma_t} h\,d\Gamma_t = \int_\Gamma (h\circ\mathcal{F}_t)\ \det(D\mathcal{F}_t)|^T(D\mathcal{F}_t^{-1})\mathbf{n}|_{R^n}\,d\Gamma\,. \tag{3–48}$$

□

For the proof, one may refer to Zolésio[136] or Sokolowski *et al.*[121].

Here, $\varpi(t) = \det(D\mathcal{F}_t)|\,^T(D\mathcal{F}_t^{-1})\mathbf{n}|_{R^n}$ is the cofactor of the Jacobian matrix $D\mathcal{F}_t$, and $\varpi(t)\,d\Gamma$ denotes the surface measure due to the transformation $\mathcal{F}_t$. It is easy to check that $\varpi(0) = 1$. For our purpose, we need the following fact.

Lemma 3.4. *If $[0,\sigma)\ni t\longmapsto\varpi(t)$ is differentiable,*

$$\varpi'(0) \equiv \frac{d}{dt}\varpi(t)\Big|_{t=0^+} = div\ \mathcal{V}(0) - (D\mathcal{V}(0)\mathbf{n})\cdot\mathbf{n}\,. \tag{3–49}$$

Proof: We first note that

$$D\mathcal{F}_t\Big|_{t=0^+} = \mathcal{I},\quad \frac{d}{dt}(D\mathcal{F}_t)\Big|_{t=0^+} = D\mathcal{V}(0)\ \ \text{and}\ \ \frac{d}{dt}(D\mathcal{F}_t^{-1})\Big|_{t=0^+} = -\,^T(D\mathcal{V}(0))\,.$$

It is clear that $\varpi(t)^2 = (\det D\mathcal{F}_t)^2(\,^T(D\mathcal{F}_t^{-1})D\mathcal{F}_t^{-1}\mathbf{n})\cdot\mathbf{n}$. Taking derivatives with respect to t and evaluating at $t = 0^+$, it follows from (3–28) that

$$\begin{aligned}2\varpi(0)\varpi'(0) &= 2div\ \mathcal{V}(0) - \Big((D\mathcal{V}(0))\mathbf{n} + \,^T(D\mathcal{V}(0))\mathbf{n}\Big)\cdot\mathbf{n}\\ &= 2div\ \mathcal{V}(0) - 2\Big((D\mathcal{V}(0))\mathbf{n}\Big)\cdot\mathbf{n}\,.\end{aligned}$$

Since $\varpi(0) = 1$, the result follows immediately. □

The expression (3–49) defines a differential operator on the boundary surface which is called the *tangential divergence*. This introduces an operator div_{Γ} on the boundary:

$$div_{\Gamma}\mathcal{V} = div\ \mathcal{V} - (D\mathcal{V}\mathbf{n}) \cdot \mathbf{n}\,. \tag{3–50}$$

REMARK 3.6: The corresponding (pseudo–) adjoint to div_{Γ} is the *tangential gradient* ∇_{Γ} which is defined by $\nabla_{\Gamma}\varphi = \nabla\varphi - \dfrac{\partial\varphi}{\partial\mathbf{n}} \cdot \mathbf{n}$, *i.e.*, ∇_{Γ} assigns φ the tangential component of its gradient. Combined with the following formula for the boundary integral

$$\begin{aligned}
&\int_{\Gamma} \nabla_{\Gamma} \cdot (\varphi\mathcal{V})\, d\Gamma \\
&= \int_{\Gamma} (div_{\Gamma}\mathcal{V})\varphi\, d\Gamma + \int_{\Gamma} \mathcal{V} \cdot \nabla_{\Gamma}\varphi\, d\Gamma = \int_{\Gamma} \kappa\varphi\mathcal{V} \cdot \mathbf{n}\, d\Gamma\,,
\end{aligned} \tag{3–51}$$

they are fundamental tools to deal with variational problems defined on the boundary surface of a domain (c.f. Zolésio[136] and Rousselet[112]).

We return to the computation for the boundary integrals. From (3–48), we have

$$\begin{aligned}
\int_{\Gamma_t} \frac{\partial\mathbf{u}_\epsilon(\alpha_t)}{\partial\mathbf{n}_t} \cdot \mathbf{w}\, d\Gamma_t &= \sum_{i,j=1}^{2} \int_{\Gamma_t} n_{jt} \frac{\partial u_{\epsilon i}(\alpha_t)}{\partial x_j} w_i\, d\Gamma_t \\
&= \sum_{i,j=1}^{2} \int_{\Gamma_{\mathbf{g}}} n_j \frac{\partial u_{\epsilon i}(\alpha_t)}{\partial x_j} w_i\, \varpi(t)\, d\Gamma\,,
\end{aligned}$$

for $\mathbf{w} = \mathbf{0}$ along $\Gamma_0(\alpha)$. Since $\mathbf{n}_t = \mathbf{n} =$ constant vector along $\Gamma_{\mathbf{g}}$ for all $0 \leq t < \sigma$,

$$\begin{aligned}
&\frac{d}{dt}\left(\sum_{i,j=1}^{2} \int_{\Gamma_{\mathbf{g}}} n_j \frac{\partial u_{\epsilon i}(\alpha_t)}{\partial x_j} w_i\, \varpi(t)\, d\Gamma \right)\Bigg|_{t=0+} \\
&= \sum_{i,j=1}^{2} \int_{\Gamma_{\mathbf{g}}} \left[n_j \left(\left(\frac{\partial \tilde{u}_{\epsilon i}}{\partial x_j}\right)' + \nabla\left(\frac{\partial u_{\epsilon i}(\alpha)}{\partial x_j}\right) \cdot \mathcal{V}(0) \right) w_j + n_j \frac{\partial u_{\epsilon i}(\alpha)}{\partial x_j} (\nabla w_i \cdot \mathcal{V}(0)) \right. \\
&\qquad\qquad \left. + \, n_j \frac{\partial u_{\epsilon i}(\alpha)}{\partial x_j} w_i\, div_{\Gamma}\mathcal{V}(0) \right] d\Gamma\,.
\end{aligned}$$

Since $\mathcal{V}(0) = \mathbf{0}$ along $\Gamma_{\mathbf{g}}$, this computation is reduced to

$$\frac{d}{dt} \int_{\Gamma_t} \frac{\partial\mathbf{u}_\epsilon(\alpha_t)}{\partial\mathbf{n}_t} \cdot \mathbf{w}\, d\Gamma_t \Bigg|_{t=0+} = \int_{\Gamma_{\mathbf{g}}} \frac{\partial\tilde{\mathbf{u}}_\epsilon{}'}{\partial\mathbf{n}} \cdot \mathbf{w}\, d\Gamma\,. \tag{3–52}$$

In a similar manner, we can show that

$$\frac{d}{dt}\int_{\Gamma_t} div\,(\mathbf{u}_\epsilon(\alpha_t))\,\mathbf{w}\cdot\mathbf{n}_t\,d\Gamma_t\bigg|_{t=0^+} = \int_{\Gamma_{\mathbf{g}}} div\,(\widetilde{\mathbf{u}}_\epsilon{}')\,\mathbf{w}\cdot\mathbf{n}\,d\Gamma\,. \tag{3–53}$$

Therefore, from (3–52)–(3–53), (3–47) is simplified to

$$\begin{aligned}
&\nu\int_{\Omega(\alpha)}\nabla\widetilde{\mathbf{u}}_\epsilon{}' : \nabla\mathbf{w}\,d\Omega + \int_{\Omega(\alpha)}(\widetilde{\mathbf{u}}_\epsilon{}'\cdot\nabla)\,\mathbf{u}_\epsilon(\alpha)\cdot\mathbf{w}\,d\Omega + \int_{\Omega(\alpha)}(\mathbf{u}_\epsilon(\alpha)\cdot\nabla)\,\widetilde{\mathbf{u}}_\epsilon{}'\cdot\mathbf{w}\,d\Omega\\
&\quad+\frac{1}{\epsilon}\int_{\Omega(\alpha)} div\ \widetilde{\mathbf{u}}_\epsilon{}'\,div\ \mathbf{w}\,d\Omega - \nu\int_{\Gamma_{\mathbf{g}}}\frac{\partial\widetilde{\mathbf{u}}_\epsilon{}'}{\partial\mathbf{n}}\cdot\mathbf{w}\,d\Gamma - \frac{1}{\epsilon}\int_{\Gamma_{\mathbf{g}}} div\ \widetilde{\mathbf{u}}_\epsilon{}'\,\mathbf{w}\cdot\mathbf{n}\,d\Gamma\\
&\qquad-\int_{-M_1}^{M_1}\Big(\nu\,\nabla\mathbf{u}_\epsilon(\alpha) : \nabla\mathbf{w} + \frac{1}{\epsilon}\,div\ \mathbf{u}_\epsilon(\alpha)\,div\ \mathbf{w}\Big)\,\vartheta(x_1)\,dx_1 = 0\,.
\end{aligned}\tag{3–54}$$

Next, we consider the adjoint equations (3–34)–(3–35). If we substitute $\mathbf{w} = \widetilde{\mathbf{u}}_\epsilon{}'$, then (3–34) may be written in the integral form :

$$\begin{aligned}
&\nu\int_{\Omega(\alpha)}\nabla\mathbf{q}_\epsilon : \nabla\widetilde{\mathbf{u}}_\epsilon{}'\,d\Omega + \int_{\Omega(\alpha)}(\widetilde{\mathbf{u}}_\epsilon{}'\cdot\nabla)\,\mathbf{u}_\epsilon(\alpha)\cdot\mathbf{q}_\epsilon\,d\Omega + \int_{\Omega(\alpha)}(\mathbf{u}_\epsilon(\alpha)\cdot\nabla)\,\widetilde{\mathbf{u}}_\epsilon{}'\cdot\mathbf{q}_\epsilon\,d\Omega\\
&\quad+\frac{1}{\epsilon}\int_{\Omega(\alpha)} div\ \mathbf{q}_\epsilon\,div\ \widetilde{\mathbf{u}}_\epsilon{}'\,d\Omega + \int_\Gamma \eta_\epsilon\cdot\widetilde{\mathbf{u}}_\epsilon{}'\,d\Gamma = 2\,\nu\int_{\Omega(\alpha)}\nabla\mathbf{u}_\epsilon(\alpha) : \nabla\widetilde{\mathbf{u}}_\epsilon{}'\,d\Omega\,.
\end{aligned}\tag{3–55}$$

By substituting $\mathbf{w} = \mathbf{q}_\epsilon$ into (3–54) and using the fact that $\mathbf{q}_\epsilon = \mathbf{0}$ along Γ, we get

$$\begin{aligned}
&\nu\int_{\Omega(\alpha)}\nabla\widetilde{\mathbf{u}}_\epsilon{}' : \nabla\mathbf{q}_\epsilon\,d\Omega + \int_{\Omega(\alpha)}(\widetilde{\mathbf{u}}_\epsilon{}'\cdot\nabla)\,\mathbf{u}_\epsilon(\alpha)\cdot\mathbf{q}_\epsilon\,d\Omega + \int_{\Omega(\alpha)}(\mathbf{u}_\epsilon(\alpha)\cdot\nabla)\,\widetilde{\mathbf{u}}_\epsilon{}'\cdot\mathbf{q}_\epsilon\,d\Omega\\
&+\frac{1}{\epsilon}\int_{\Omega(\alpha)} div\ \widetilde{\mathbf{u}}_\epsilon{}'\,div\ \mathbf{q}_\epsilon\,d\Omega - \int_{-M_1}^{M_1}(\nu\,\nabla\mathbf{u}_\epsilon(\alpha) : \nabla\mathbf{q}_\epsilon + \frac{1}{\epsilon}\,div\ \mathbf{u}_\epsilon(\alpha)\,div\ \mathbf{q}_\epsilon)\,\vartheta(x_1)\,dx_1\\
&= 0\,.
\end{aligned}\tag{3–56}$$

Hence, it follows from (3–55) and (3–56) that

$$\begin{aligned}
&2\,\nu\int_{\Omega(\alpha)}\nabla\mathbf{u}_\epsilon(\alpha) : \nabla\widetilde{\mathbf{u}}_\epsilon{}'\,d\Omega\\
&= \int_{-M_1}^{M_1}\Big(\nu\,\nabla\mathbf{u}_\epsilon(\alpha) : \nabla\mathbf{q}_\epsilon + \frac{1}{\epsilon}\,div\ \mathbf{u}_\epsilon(\alpha)\,div\ \mathbf{q}_\epsilon\Big)\,\vartheta(x_1)\,dx_1 + \int_\Gamma \eta_\epsilon\cdot\widetilde{\mathbf{u}}_\epsilon'\,d\Gamma\,.
\end{aligned}\tag{3–57}$$

For the computation of $\int_\Gamma \eta_\epsilon\cdot\widetilde{\mathbf{u}}_\epsilon{}'\,d\Gamma$, we note that $\widetilde{\mathbf{u}}_\epsilon = \mathbf{u}_\epsilon(\alpha) = \mathbf{g}$ on Γ, where $\mathbf{g}$ is given. So, $\dot{\widetilde{\mathbf{u}}}_\epsilon = \nabla\mathbf{g}\cdot\mathcal{V}(0)$ on Γ. Using $\widetilde{\mathbf{u}}_\epsilon{}' = \dot{\widetilde{\mathbf{u}}}_\epsilon - \nabla\widetilde{\mathbf{u}}_\epsilon\cdot\mathcal{V}(0)$, we get

$$\widetilde{\mathbf{u}}_\epsilon{}' = \nabla(\mathbf{g} - \mathbf{u}_\epsilon(\alpha))\cdot\mathcal{V}(0) \qquad \text{on } \Gamma\,.$$

Since $\mathbf{g}-\mathbf{u}_\epsilon(\alpha)=\mathbf{0}$ on Γ, the gradient of $\mathbf{g}-\mathbf{u}_\epsilon(\alpha)$ is parallel to the normal direction. Hence,

$$\widetilde{\mathbf{u}}_\epsilon{}' = \frac{\partial(\mathbf{g}-\mathbf{u}_\epsilon(\alpha))}{\partial \mathbf{n}}\,\mathbf{n}\cdot\mathcal{V}(0) \qquad \text{on} \ \ \Gamma. \tag{3–58}$$

Since $\boldsymbol{\eta}_\epsilon = 2\,\nu\,\dfrac{\partial \mathbf{u}_\epsilon(\alpha)}{\partial \mathbf{n}} - \nu\,\dfrac{\partial \mathbf{q}_\epsilon}{\partial \mathbf{n}} - \dfrac{1}{\epsilon}(div\ \mathbf{q}_\epsilon)\,\mathbf{n}$ along Γ,

$$\int_\Gamma \boldsymbol{\eta}_\epsilon\cdot\widetilde{\mathbf{u}}_\epsilon{}'\,d\Gamma = \int_\Gamma \frac{\partial(\mathbf{g}-\mathbf{u}_\epsilon(\alpha))}{\partial \mathbf{n}}\cdot\left(2\,\nu\,\frac{\partial \mathbf{u}_\epsilon(\alpha)}{\partial \mathbf{n}} - \nu\,\frac{\partial \mathbf{q}_\epsilon}{\partial \mathbf{n}} - \frac{1}{\epsilon}(div\ \mathbf{q}_\epsilon)\,\mathbf{n}\right)\,\mathcal{V}(0)\cdot\mathbf{n}\,d\Gamma.$$

However, since $\mathcal{V}(0)=\mathbf{0}$ along $\Gamma-\Gamma(\alpha)$ and $\mathbf{g}=\mathbf{0}$ on $\Gamma(\alpha)$, we obtain

$$\begin{aligned}&\int_\Gamma \boldsymbol{\eta}_\epsilon\cdot\widetilde{\mathbf{u}}_\epsilon{}'\,d\Gamma\\ &= \int_{-M_1}^{M_1}\left(\frac{\partial \mathbf{u}_\epsilon(\alpha)}{\partial \mathbf{n}}\cdot\left(2\,\nu\,\frac{\partial \mathbf{u}_\epsilon(\alpha)}{\partial \mathbf{n}} - \nu\,\frac{\partial \mathbf{q}_\epsilon}{\partial \mathbf{n}} - \frac{1}{\epsilon}\,(div\ \mathbf{q}_\epsilon)\,\mathbf{n}\right)\right)\,\vartheta(x_1)\,dx_1.\end{aligned} \tag{3–59}$$

Therefore, it follows from (3–46) and (3–57)–(3–59) that

$$\begin{aligned} d\mathfrak{J}_\epsilon(\alpha;\vartheta) &= 2\,\nu\int_{\Omega(\alpha)}\nabla\mathbf{u}_\epsilon(\alpha)\ :\ \nabla\widetilde{\mathbf{u}}_\epsilon{}'\,\mathcal{V}(0)\cdot\mathbf{n}\,d\Omega + \nu\int_\Gamma \nabla\mathbf{u}_\epsilon(\alpha)\ :\ \nabla\mathbf{u}_\epsilon(\alpha)\,\mathcal{V}(0)\cdot\mathbf{n}\,d\Gamma\\ &= \int_{-M_1}^{M_1}\Bigg[-\nu\,\nabla\mathbf{u}_\epsilon(\alpha)\ :\ \nabla\mathbf{u}_\epsilon(\alpha) + \left(\nu\,\nabla\mathbf{u}_\epsilon(\alpha)\ :\ \nabla\mathbf{q}_\epsilon + \frac{1}{\epsilon}\,div\ \mathbf{u}_\epsilon\ div\ \mathbf{q}_\epsilon\right)\\ &\qquad + \left(\frac{\partial \mathbf{u}_\epsilon(\alpha)}{\partial \mathbf{n}}\cdot\left(2\,\nu\,\frac{\partial \mathbf{u}_\epsilon(\alpha)}{\partial \mathbf{n}} - \nu\,\frac{\partial \mathbf{q}_\epsilon}{\partial \mathbf{n}} - \frac{1}{\epsilon}\,(div\ \mathbf{q}_\epsilon)\,\mathbf{n}\right)\right)\Bigg]\ \vartheta(x_1)\,dx_1.\end{aligned}$$

Recall that $\mathcal{V}(0)\cdot\mathbf{n}\,d\Gamma$ corresponds to $-\vartheta(x_1)\,dx_1$. Hence in the sense of (3–19), we may say that the shape gradient of the design functional $\mathfrak{J}_\epsilon$ is given by

$$\begin{aligned} g_\epsilon(\Gamma) \equiv grad\,\mathfrak{J}_\epsilon = \Bigg[&\nu\,\nabla\mathbf{u}_\epsilon(\alpha)\ :\ \nabla\mathbf{u}_\epsilon(\alpha) - \left(\nu\,\nabla\mathbf{u}_\epsilon(\alpha)\ :\ \nabla\mathbf{q}_\epsilon + \frac{1}{\epsilon}\,div\ \mathbf{u}_\epsilon\ div\ \mathbf{q}_\epsilon\right)\\ &- \left(\frac{\partial \mathbf{u}_\epsilon(\alpha)}{\partial \mathbf{n}}\cdot\left(2\,\nu\,\frac{\partial \mathbf{u}_\epsilon(\alpha)}{\partial \mathbf{n}} - \nu\,\frac{\partial \mathbf{q}_\epsilon}{\partial \mathbf{n}} - \frac{1}{\epsilon}\,(div\ \mathbf{q}_\epsilon)\,\mathbf{n}\right)\right)\Bigg]\end{aligned} \tag{3–60}$$

along the perturbed boundary, and $\mathbf{0}$ along the boundary of unperturbed region.

It is useful to summarize the above discussions in the following theorem.

Theorem 3.2. *Let* $(\alpha,\mathbf{u}_\epsilon(\alpha))\in\widetilde{\mathcal{U}_{ad}}$. *Let* $\mathfrak{J}_\epsilon(\alpha) = \nu\displaystyle\int_{\Omega(\alpha)}\nabla\mathbf{u}_\epsilon(\alpha)\ :\ \nabla\mathbf{u}_\epsilon(\alpha)\,d\Omega$ *be the design functional which represents the energy dissipation due to the flow. Under* *reasonable regularity assumptions for the data, the shape gradient of $\mathfrak{J}_\epsilon$ is given in the form of (3–60), where $\mathbf{q}_\epsilon$ is a solution of the adjoint equations (3–36)-(3–38) which represents the adjoint of the weak linearized incompressible Navier–Stokes equations.*

REMARK 3.7: In the computation of (3–52) and (3–53), the curvature κ as for (3–27) does not appear. This is due to the choice of a trial function $\mathbf{w}$ and $\mathbf{n}_t = \mathbf{n}$ along $\Gamma_{\mathbf{g}}$. For any unitary extension $\mathcal{N}$ of the normal vector field $\mathbf{n}$ on Γ, the curvature (or the mean curvature to the surface) is given by

$$\kappa = div_\Gamma \mathbf{n} = div \mathcal{N} .$$

In the $3-D$ case, the computation of the mean curvature to the surface is nontrivial.

In the formal computations, we implicitly assumed the existence of $\dot{\mathbf{u}}_\epsilon$. To justify these computations, we need two hypotheses (c.f.[136]);

[H1] The material derivative $\dot{\mathbf{u}}_\epsilon$ exists and $\mathcal{V} \longmapsto \dot{\mathbf{u}}_\epsilon(\Omega(\alpha); \mathcal{V})$ is linear and continuous.

[H2] Feasible domains $\{\Omega(\alpha)\}_{\alpha \in \mathcal{U}_{ad}}$ are regular enough so that there exists a linear continuous extension $P_{\widehat{\Omega}} : \mathbf{H}^m(\Omega(\alpha)) \longrightarrow \mathbf{H}^m(\widehat{\Omega})$ for positive integer m.

[H2] is true for uniform Lipschitz domains by the Calderón's extension Theorem (Theorem 1.1). Verification of **[H1]** is nontrivial. This may be verified by applying the Implicit Function Theorem to resolve the nonlinear structure of the problem. For the drag reduction problem in Stokes flow, Simon[119] showed the existence of the material derivative of the velocity (or, the *total variation* in his terms) using the implicit function theorem in somewhat different context.

Another possible way to perform this task is to reverse the process. We first find a meaningful boundary value problem for $\widetilde{\mathbf{u}}_\epsilon'\Big|_{\Omega(\alpha)}$. From it, one may understand $\dot{\mathbf{u}}_\epsilon$

through the relation

$$\dot{\mathbf{u}}_\epsilon = \widetilde{\mathbf{u}}_\epsilon{}' + \nabla \mathbf{u}_\epsilon \cdot \mathcal{V}\,.$$

From (3–54), if we take $\mathbf{w} = \varphi$ with compact support in $\Omega(\alpha)$, $\left.\widetilde{\mathbf{u}}_\epsilon{}'\right|_{\Omega(\alpha)}$ satisfies

$$\nu \int_{\Omega(\alpha)} \nabla \widetilde{\mathbf{u}}_\epsilon{}' : \nabla \varphi \, d\Omega + \int_{\Omega(\alpha)} (\widetilde{\mathbf{u}}_\epsilon{}' \cdot \nabla)\, \mathbf{u}_\epsilon \cdot \varphi \, d\Omega + \int_{\Omega(\alpha)} (\mathbf{u}_\epsilon \cdot \nabla)\, \widetilde{\mathbf{u}}_\epsilon{}' \cdot \varphi \, d\Omega$$
$$+ \frac{1}{\epsilon} \int_{\Omega(\alpha)} div \widetilde{\mathbf{u}}_\epsilon{}' \, div \varphi \, d\Omega = 0 \qquad (3\text{–}61)$$

Recall that

$$\widetilde{\mathbf{u}}_\epsilon{}' = \frac{\partial(\mathbf{g} - \mathbf{u}_\epsilon(\alpha))}{\partial \mathbf{n}} \mathbf{n} \cdot \mathcal{V}(0) \qquad \text{on } \Gamma\,.$$

Since $\mathbf{g} = \mathbf{0}$, on $\Gamma_0(\alpha)$ and $\mathcal{V} = \mathbf{0}$ along $\Gamma - \Gamma(\alpha)$,

$$\widetilde{\mathbf{u}}_\epsilon{}' = \begin{cases} -\dfrac{\partial \mathbf{u}_\epsilon(\alpha)}{\partial \mathbf{n}} \mathbf{n} \cdot \mathcal{V}(0), & \text{on } \Gamma(\alpha) \\ \mathbf{0}, & \text{on } \Gamma - \Gamma(\alpha). \end{cases} \qquad (3\text{–}62)$$

(3–61) represents the linearized version of a penalized incompressible Navier–Stokes equations. It can be shown that (3–61)-(3–62) has a solution for almost all Reynolds numbers. Since this is a linear equation whose boundary value is given by (3–62), $\left.\widetilde{\mathbf{u}}_\epsilon{}'\right|_{\Omega(\alpha)}$ depends linealy on $\mathcal{V}(0) \cdot \mathbf{n}$.

3.3. Justification of the Lagrange Multiplier

In the previous section, we have introduced a Lagrange multiplier $(\mathbf{q}_\epsilon, \boldsymbol{\eta}_\epsilon) \in \mathbf{H}^1(\Omega(\alpha)) \times \mathbf{H}^{-1/2}(\Gamma)$ for $\alpha \in \mathcal{U}_{ad}$. We have observed that the Lagrange multiplier plays a role of an adjoint variable in the derivation of the shape gradient (3–60). The major drawback in the formal discussion is the lack of mathematical justification for the choice of $\mathbf{q}_\epsilon$ and $\boldsymbol{\eta}_\epsilon$. Furthermore, we assumed that $\mathbf{q}_\epsilon = \mathbf{0}$ on Γ and observed that this obviously simplifies the expression for the shape gradient. How can we avoid such arbitrariness ?

In this section, we wish to verify the existence of the Lagrange multiplier. We will justify the existence of the Lagrange multiplier, when Reynolds number is small. Then, in conjunction with the well–posedness of the system, a rigrous sensitivity is derived. The general case may be verified using the similar technique. For our purpose, we will employ some techniques introduced by Gunzburger *et al.* [65] for boundary velocity control over a fixed domain and Tikhomirov[130] for minimal principle.

We introduce the following notations:

$$
\begin{aligned}
a(\mathbf{u},\mathbf{v}) &= \int_{\Omega(\alpha)} \nabla \mathbf{u} : \nabla \mathbf{v}\, d\Omega\,, \\
c(\mathbf{u},\mathbf{v},\mathbf{w}) &= \int_{\Omega(\alpha)} (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\, d\Omega\,, \\
d(\mathbf{u},\mathbf{v}) &= \int_{\Omega(\alpha)} div\ \mathbf{u}\, div\ \mathbf{v}\, d\Omega\,.
\end{aligned}
$$

Note that $a(\cdot,\cdot)$ and $d(\cdot,\cdot)$ are continuous bilinear functional on $\mathbf{H}^1(\Omega(\alpha))$, and $c(\cdot,\cdot,\cdot)$ is a continuous trilinear functional. Let $B_1 = \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)) \times \mathbf{H}^{-1/2}(\Gamma_\mathbf{g})$ and $B_2 = \mathbf{H}^{-1}_{\Gamma_0(\alpha)}(\Omega(\alpha)) \times \mathbf{H}^{1/2}_0(\Gamma_\mathbf{g})$.

Let us consider the nonlinear mapping $\mathcal{M} : B_1 \longrightarrow B_2$ ($\mathcal{M}(\mathbf{u}_\epsilon,\mathbf{t}_\epsilon) = (\mathbf{f},\mathbf{g})$) defined by the weak formulation of the penalized incompressible Navier–Stokes equations, where B_1 is a solution space and B_2 is a data space for the external force and the given boundary condition, *i.e.*, $\mathcal{M}(\mathbf{u}_\epsilon,\mathbf{t}_\epsilon) = (\mathbf{f},\mathbf{g})$ if and only if

$$
\begin{aligned}
\nu\, a(\mathbf{u}_\epsilon,\mathbf{v}) + c(\mathbf{u}_\epsilon,\mathbf{u}_\epsilon,\mathbf{v}) + \frac{1}{\epsilon}\, d(\mathbf{u}_\epsilon,\mathbf{v}) - &< \mathbf{t}_\epsilon,\mathbf{v} >_{-1/2,\Gamma_\mathbf{g}} \\
&=< \mathbf{f},\mathbf{v} >_{-1,\Gamma_0(\alpha)} \quad \forall\, \mathbf{v} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))\,, \quad (3\text{–}63) \\
< \mathbf{s},\mathbf{u}_\epsilon >_{-1/2,\Gamma_\mathbf{g}} =< \mathbf{s},\mathbf{g} >_{-1/2,\Gamma_\mathbf{g}} &\quad \forall\, \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_\mathbf{g})\,,
\end{aligned}
$$

where $\mathbf{t}_\epsilon = \gamma_{\Gamma_\mathbf{g}}\Big(\nu \dfrac{\partial \mathbf{u}_\epsilon}{\partial \mathbf{n}} + \dfrac{1}{\epsilon}(div\mathbf{u}_\epsilon)\,\mathbf{n}\Big) \in \mathbf{H}^{-1/2}(\Gamma_\mathbf{g})$ denotes a traction force along the boundary which completes the Navier–Stokes equations. By $\mathcal{M}'_{(\mathbf{u}_\epsilon,\mathbf{t}_\epsilon)}$, let us denote the Gateaux derivative of $\mathcal{M}$ in the direction of $(\mathbf{w}_\epsilon,\zeta_\epsilon) \in B_1$ at $(\mathbf{u}_\epsilon,\mathbf{t}_\epsilon)$, *i.e.*,

$\mathcal{M}'_{(\mathbf{u}_\epsilon,\mathbf{t}_\epsilon)}(\mathbf{w}_\epsilon,\zeta_\epsilon) = (\widetilde{\mathbf{f}},\widetilde{\mathbf{g}})$, where $(\mathbf{w}_\epsilon,\zeta_\epsilon) \in B_1$ and $(\widetilde{\mathbf{f}},\widetilde{\mathbf{g}}) \in B_2$, if and only if

$$\begin{aligned}
&\nu\, a(\mathbf{w}_\epsilon,\mathbf{v}) + c(\mathbf{w}_\epsilon,\mathbf{u}_\epsilon,\mathbf{v}) + c(\mathbf{u}_\epsilon,\mathbf{w}_\epsilon,\mathbf{v}) + \frac{1}{\epsilon}\, d(\mathbf{u}_\epsilon,\mathbf{v}) - <\zeta_\epsilon,\mathbf{v}>_{-1/2,\Gamma_\mathbf{g}} \\
&\qquad\qquad = <\widetilde{\mathbf{f}},\mathbf{v}>_{-1,\Gamma_0(\alpha)} \quad \forall\, \mathbf{v} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))\,, \qquad (3\text{–}64) \\
&\qquad <\mathbf{s},\mathbf{w}_\epsilon>_{-1/2,\Gamma_\mathbf{g}} = <\mathbf{s},\widetilde{\mathbf{g}}>_{-1/2,\Gamma_\mathbf{g}} \quad \forall\, \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_\mathbf{g})\,.
\end{aligned}$$

Then $\mathcal{M}'_{(\mathbf{u}_\epsilon,\mathbf{t}_\epsilon)}$ is a bounded linear operator from B_1 into B_2 for each $(\mathbf{u}_\epsilon,\mathbf{t}_\epsilon) \in B_1$.

Lemma 3.5. *For $(\mathbf{u}_\epsilon,\mathbf{t}_\epsilon) \in B_1$, the operator $\mathcal{M}'_{(\mathbf{u}_\epsilon,\mathbf{t}_\epsilon)}$ has a closed range.*

Proof: Let $\mathcal{S}_\epsilon : B_1 \longrightarrow B_2$ be an operator defined as follows:

$\mathcal{S}_\epsilon(\mathbf{w}_\epsilon,\zeta_\epsilon) = (\widetilde{\mathbf{f}},\widetilde{\mathbf{g}})$ if and only if

$$\begin{aligned}
&\nu\, a(\mathbf{w}_\epsilon,\mathbf{v}) + \frac{1}{\epsilon}\, d(\mathbf{w}_\epsilon,\mathbf{v}) - <\zeta_\epsilon,\mathbf{v}>_{-1/2,\Gamma_\mathbf{g}} = <\widetilde{\mathbf{f}},\mathbf{v}>_{-1,\Gamma_0(\alpha)} \quad \forall\, \mathbf{v} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))\,, \\
&\qquad <\mathbf{s},\mathbf{w}_\epsilon>_{-1/2,\Gamma_\mathbf{g}} = <\mathbf{s},\widetilde{\mathbf{g}}>_{-1/2,\Gamma_\mathbf{g}} \quad \forall\, \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_\mathbf{g})\,.
\end{aligned} \qquad (3\text{–}65)$$

Also, let $\mathcal{Q}_{\mathbf{u}_\epsilon} : \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)) \times \{\mathbf{0}\} \longrightarrow \mathbf{H}^{-1}_{\Gamma_0(\alpha)}(\Omega(\alpha)) \times \{\mathbf{0}\}$ be an operator defined by

$$\begin{aligned}
<\mathcal{Q}_{\mathbf{u}_\epsilon}(\mathbf{w}_\epsilon,\mathbf{0}),(\mathbf{v},\mathbf{0})> &= ((\mathbf{w}_\epsilon\cdot\nabla)\,\mathbf{u}_\epsilon + (\mathbf{u}_\epsilon\cdot\nabla)\,\mathbf{w}_\epsilon,\mathbf{v})_{0,\Omega(\alpha)} \\
&= c(\mathbf{w}_\epsilon,\mathbf{u}_\epsilon,\mathbf{v}) + c(\mathbf{u}_\epsilon,\mathbf{w}_\epsilon,\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))\,,
\end{aligned}$$

where $<\cdot,\cdot>$ denotes the pairing between $\mathbf{H}^{-1}_{\Gamma_0(\alpha)}(\Omega(\alpha))\times\{\mathbf{0}\} \subset B_1$ and $\mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))\times\{\mathbf{0}\} \subset {B_1}^*$. Then $\mathcal{M}'_{(\mathbf{u}_\epsilon,\mathbf{t}_\epsilon)} = \mathcal{S}_\epsilon + \mathcal{Q}_{\mathbf{u}_\epsilon}$. Note that (3–65) is completed if $\zeta_\epsilon = \gamma_{\Gamma_\mathbf{g}}\Big(\nu\,\dfrac{\partial\mathbf{w}_\epsilon}{\partial\mathbf{n}} + \dfrac{1}{\epsilon}\,(div\,\mathbf{w}_\epsilon)\,\mathbf{n}\Big)$, using integration by parts. Hence $\mathcal{S}_\epsilon$ is a weak formulation of the penalized Stokes equations. By the existence and uniqueness of the weak solution, $\mathcal{S}_\epsilon$ is a semi–Fredholm operator from B_1 into B_2 (see Girault–Raviart[60]). Obviously, $\mathcal{Q}_{\mathbf{u}_\epsilon}$ is a compact operator. Hence, $\mathcal{M}'_{(\mathbf{u}_\epsilon,\mathbf{t}_\epsilon)}$ is a compact perturbation of a semi–Fredholm operator, which is also a semi–Fredholm operator (c.f.[124]). Therefore, $\mathcal{M}'_{(\mathbf{u}_\epsilon,\mathbf{t}_\epsilon)}$ has a closed range in B_2. $\square$

Lemma 3.6. *When the Reynolds number is small, it holds that* $\mathcal{M}'_{(\mathbf{u}_\epsilon,\mathbf{t}_\epsilon)}(B_1) = B_2$.

Proof: Suppose that $\mathcal{M}'_{(\mathbf{u}_\epsilon,\mathbf{t}_\epsilon)}$ is not surjective. Then, by Lemma 3.5, $\mathcal{M}'_{(\mathbf{u}_\epsilon,\mathbf{t}_\epsilon)}(B_1)$ is a proper closed subspace of B_2. Hence, by the Hahn–Banach Theorem, there exists a nonzero element $\Psi \in B_2{}^*$ such that $\Psi(\mathcal{M}'_{(\mathbf{u}_\epsilon,\mathbf{t}_\epsilon)}(B_1)) = 0$.

Let $\Psi = (\boldsymbol{\xi}_\epsilon, \boldsymbol{\tau}_\epsilon) \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)) \times \mathbf{H}^{-1/2}(\Gamma_\mathbf{g})$ and let $(\widetilde{\mathbf{f}}, \widetilde{\mathbf{g}}) \in \mathcal{M}'_{(\mathbf{u}_\epsilon,\mathbf{t}_\epsilon)}(B_1)$. Then, we have $\ll (\boldsymbol{\xi}_\epsilon, \boldsymbol{\tau}_\epsilon), (\widetilde{\mathbf{f}}, \widetilde{\mathbf{g}}) \gg = 0$, where $\ll \cdot, \cdot \gg$ denotes the pairing between $B_2{}^*$ and B_2. So, it follows that $< \widetilde{\mathbf{f}}, \boldsymbol{\xi}_\epsilon >_{-1,\Gamma_0(\alpha)} + < \boldsymbol{\tau}_\epsilon, \widetilde{\mathbf{g}} >_{-1/2,\Gamma_\mathbf{g}} = 0$. Since $(\widetilde{\mathbf{f}}, \widetilde{\mathbf{g}}) \in \mathcal{M}'_{(\mathbf{u}_\epsilon,\mathbf{t}_\epsilon)}(B_1)$, there exists a nonzero element $(\mathbf{w}, \zeta)$ in B_1 which satisfies the following equations:

$$
\begin{aligned}
\nu\, a(\mathbf{w}, \mathbf{v}) + c(\mathbf{w}, \mathbf{u}_\epsilon, \mathbf{v}) + c(\mathbf{u}_\epsilon, \mathbf{w}, \mathbf{v}) + \frac{1}{\epsilon} d(\mathbf{w}, \mathbf{v}) - < \zeta, \mathbf{v} >_{-1/2,\Gamma_\mathbf{g}} & \\
= < \widetilde{\mathbf{f}}, \mathbf{v} >_{-1,\Gamma_0(\alpha)} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))\,, & \\
< \mathbf{s}, \mathbf{w} >_{-1/2,\Gamma_\mathbf{g}} = < \mathbf{s}, \widetilde{\mathbf{g}} >_{-1/2,\Gamma_\mathbf{g}} \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_\mathbf{g})\,. &
\end{aligned}
$$

Substituting $\mathbf{v} = \boldsymbol{\xi}_\epsilon$ and $\mathbf{s} = \boldsymbol{\tau}_\epsilon$, and using $< \widetilde{\mathbf{f}}, \boldsymbol{\xi}_\epsilon >_{-1,\Gamma_0(\alpha)} + < \boldsymbol{\tau}_\epsilon, \widetilde{\mathbf{g}} >_{-1/2,\Gamma_\mathbf{g}} = 0$, we obtain that

$$
\begin{aligned}
\nu\, a(\mathbf{w}, \boldsymbol{\xi}_\epsilon) + c(\mathbf{w}, \mathbf{u}_\epsilon, \boldsymbol{\xi}_\epsilon) + c(\mathbf{u}_\epsilon, \mathbf{w}, \boldsymbol{\xi}_\epsilon) + \frac{1}{\epsilon} d(\mathbf{w}, \boldsymbol{\xi}_\epsilon) & \\
- < \zeta, \boldsymbol{\xi}_\epsilon >_{-1/2,\Gamma_\mathbf{g}} + < \boldsymbol{\tau}_\epsilon, \mathbf{w} >_{-1/2,\Gamma_\mathbf{g}} = 0\,. &
\end{aligned}
$$

Since this holds for all $(\mathbf{w}, \zeta) \in B_1$, it follows that

$$
\begin{aligned}
\nu\, a(\mathbf{w}, \boldsymbol{\xi}_\epsilon) + c(\mathbf{w}, \mathbf{u}_\epsilon, \boldsymbol{\xi}_\epsilon) + c(\mathbf{u}_\epsilon, \mathbf{w}, \boldsymbol{\xi}_\epsilon) + \frac{1}{\epsilon} d(\mathbf{w}, \boldsymbol{\xi}_\epsilon) & \\
+ < \boldsymbol{\tau}_\epsilon, \mathbf{w} >_{-1/2,\Gamma_\mathbf{g}} = 0 \quad \forall \mathbf{w} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))\,, & \quad \text{(3–66)}
\end{aligned}
$$

and

$$
< \zeta, \boldsymbol{\xi}_\epsilon >_{-1/2,\Gamma_\mathbf{g}} = 0 \quad \forall \zeta \in \mathbf{H}^{-1/2}(\Gamma_\mathbf{g})\,. \quad \text{(3–67)}
$$

We note that $\boldsymbol{\xi}_\epsilon = \mathbf{0}$ on $\Gamma_\mathbf{g}$, and since $\boldsymbol{\xi}_\epsilon \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$, it follows that $\boldsymbol{\xi}_\epsilon = \mathbf{0}$ on Γ. It is easy to check that the system (3–66)–(3–67) is a weak formulation of the boundary value problem

$$-\nu\,\Delta\boldsymbol{\xi}_\epsilon + \boldsymbol{\xi}_\epsilon \cdot {}^T(\nabla \mathbf{u}_\epsilon) - \mathbf{u}_\epsilon \cdot (\nabla \boldsymbol{\xi}_\epsilon) - \frac{1}{\epsilon}\nabla(div\boldsymbol{\xi}_\epsilon) - (div\mathbf{u}_\epsilon)\,\boldsymbol{\xi}_\epsilon = \mathbf{0} \quad \text{in} \quad \Omega(\alpha)\,, \tag{3–68}$$

$$\boldsymbol{\xi}_\epsilon = \mathbf{0} \qquad \text{on} \quad \Gamma\,, \tag{3–69}$$

and

$$\boldsymbol{\tau}_\epsilon = -\nu\,\frac{\partial \boldsymbol{\xi}_\epsilon}{\partial \mathbf{n}} - \frac{1}{\epsilon}\,(div\boldsymbol{\xi}_\epsilon)\mathbf{n} \qquad \text{on} \quad \Gamma_{\mathbf{g}}\,. \tag{3–70}$$

If the Reynolds number is small, by taking $\mathbf{w} = \boldsymbol{\xi}_\epsilon$ in (3–66), we have

$$\nu\,a\,(\boldsymbol{\xi}_\epsilon, \boldsymbol{\xi}_\epsilon) + c\,(\boldsymbol{\xi}_\epsilon, \mathbf{u}_\epsilon, \boldsymbol{\xi}_\epsilon) + c\,(\mathbf{u}_\epsilon, \boldsymbol{\xi}_\epsilon, \boldsymbol{\xi}_\epsilon) + \frac{1}{\epsilon}\,d\,(\boldsymbol{\xi}_\epsilon, \boldsymbol{\xi}_\epsilon) = 0\,.$$

Since $\frac{1}{\epsilon}\,d\,(\boldsymbol{\xi}_\epsilon, \boldsymbol{\xi}_\epsilon) \geq 0$, using the Korn's inequality and the continuity of $c\,(\cdot,\cdot,\cdot)$ in the $\mathbf{H}^1$–space, it follows that $\boldsymbol{\xi}_\epsilon = \mathbf{0}$.

So, $\Psi = (\boldsymbol{\xi}_\epsilon, \boldsymbol{\tau}_\epsilon) = (\mathbf{0}, \mathbf{0}) \in B_2{}^*$, which contradicts the choice of a nonzero Ψ. □

This result may be generalized to include general ν by adding a natural boundary condition on a small piece of $\Gamma_{\mathbf{g}}$. Since the natural boundary condition does not affect the variational formulation, one may have the same formulation with two boundary conditions along the piece of $\Gamma_{\mathbf{g}}$, which makes it possible to extend the system to be $\mathbf{0}$ across the piece of $\Gamma_{\mathbf{g}}$. Since this extension includes a compact perturbation of the adjoint of the homogeneous penalized Stokes system which has a discrete spectrum, by considering an appropriate extension, if needed, it may be possible to show that $\frac{1}{\nu}$ is not an eigenvalue.

Now, we turn to the existence of a Lagrange multiplier. Unlike boundary velocity controls, shape controls have no known structure to verify the existence of Lagrange multipliers. In most cases, the technique of adjoint equations to derive the sensitivity of the design functional is usually used as a formal trick to get around the sensitivity of the state variables with respect to the design parameters. In the shape control problem, the difficulty in manipulating the Lagrange multiplier seems to originate from the following sources:

- For a fixed domain $\Omega(\alpha)$ for some $\alpha \in \mathcal{U}_{ad}$, we have no control for the state variables.
- The function spaces are changing according to the domain variations.

In our case, however, we may overcome these difficulties by extending the elements $\mathbf{u}_\epsilon(\alpha) \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$ to the elements $\widetilde{\mathbf{u}}_\epsilon(\alpha)$ of $\mathbf{H}^1_L(\widehat{\Omega})$ which are defined to be $\mathbf{0}$ over $\overline{\widehat{\Omega}} - \Omega(\alpha)$, where $\mathbf{H}^1_L(\widehat{\Omega})$ is defined in (2–64). Then, the mapping $\mathbf{u}_\epsilon(\alpha) \mapsto \widetilde{\mathbf{u}}_\epsilon(\alpha)$ is continuous from $\mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)) \longrightarrow \mathbf{H}^1_L(\widehat{\Omega})$ and

$$\left\|\widetilde{\mathbf{u}}_\epsilon(\alpha)\right\|_{1,\widehat{\Omega}} = \|\mathbf{u}_\epsilon(\alpha)\|_{1,\Omega(\alpha)} ,$$

for any $\alpha \in \mathcal{U}_{ad}$. We may set $B =$ the subspace of $\mathbf{H}^1_L(\widehat{\Omega})$ spanned by $\{\widetilde{\mathbf{u}}_\epsilon(\alpha)\}_{\alpha\in\mathcal{U}_{ad}}$, which can be regarded as a closed subspace of $\mathbf{H}^1_L(\widehat{\Omega})$. Note that

$$\mathbf{v} \in B \Longleftrightarrow \mathbf{v} = \widetilde{\mathbf{u}}_\epsilon(\alpha) \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)) \text{ on } \Omega(\alpha) \text{ and } \mathbf{v} = \mathbf{0} \text{ on } \widehat{\Omega} - \Omega(\alpha)\,,$$

for some $\alpha \in \mathcal{U}_{ad}$. We may set the definition of a local optimal solution among the admissible family $\{(\alpha, \mathbf{u}_\epsilon(\alpha))\} \subset \widetilde{\mathcal{U}_{ad}}$ which is defined in (2–55) as follows:

$(\alpha^*, \widetilde{\mathbf{u}}_\epsilon(\alpha^*)) \in \mathcal{U}_{ad} \times B$ is a local optimal solution of (2–55) if and only if there exist positive numbers δ_1 and δ_2 such that if $\|\alpha - \alpha^*\|_\infty < \delta_1$ and $\left\|\widetilde{\mathbf{u}}_\epsilon(\alpha) - \widetilde{\mathbf{u}}_\epsilon(\alpha^*)\right\|_{1,\widehat{\Omega}} < \delta_2$,

$$\int_{\widehat{\Omega}} \nabla\widetilde{\mathbf{u}}_\epsilon(\alpha^*) \,:\, \nabla\widetilde{\mathbf{u}}_\epsilon(\alpha^*)\, d\widehat{\Omega} < \int_{\widehat{\Omega}} \nabla\widetilde{\mathbf{u}}_\epsilon(\alpha) \,:\, \nabla\widetilde{\mathbf{u}}_\epsilon(\alpha)\, d\widehat{\Omega}\,.$$

Let $\widetilde{B_1} = B \times \mathbf{H}^{-1/2}(\Gamma_{\mathbf{g}})$ and $\widetilde{B_2} = (\mathbf{L}^2(\widehat{\Omega}) \cap \mathbf{H}^{-1}_L(\widehat{\Omega})) \times \mathbf{H}^{1/2}(\Gamma_{\mathbf{g}})$. To adjust the situation, we may extend the functionals $a(\cdot,\cdot)$, $c(\cdot,\cdot,\cdot)$ and $d(\cdot,\cdot)$ over $\mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))$ to the functionals $\widetilde{a}(\cdot,\cdot)$, $\widetilde{c}(\cdot,\cdot,\cdot)$ and $\widetilde{d}(\cdot,\cdot)$ over $\mathbf{H}^1_L(\widehat{\Omega})$. Moreover, we may reset the nonlinear mapping $\mathcal{M} : B_1 \longrightarrow B_2$ to $\widetilde{\mathcal{M}} : \widetilde{B_1} \longrightarrow \widetilde{B_2}$ which is defined as follows:

$\widetilde{\mathcal{M}}(\widetilde{\mathbf{u}}_\epsilon(\alpha), \widetilde{\mathbf{t}}_\epsilon) = (\mathbf{f}, \mathbf{g})$ if and only if

$$
\begin{aligned}
&\nu\, \widetilde{a}\,(\widetilde{\mathbf{u}}_\epsilon(\alpha), \mathbf{v}) + \widetilde{c}(\widetilde{\mathbf{u}}_\epsilon(\alpha), \widetilde{\mathbf{u}}_\epsilon(\alpha), \mathbf{v}) + \frac{1}{\epsilon}\widetilde{d}\,(\widetilde{\mathbf{u}}_\epsilon(\alpha), \mathbf{v}) - < \widetilde{\mathbf{t}}_\epsilon, \mathbf{v} >_{-1/2,\Gamma_\mathbf{g}} \\
&\qquad =< \chi_{\Omega(\alpha)}\mathbf{f}, \mathbf{v} >_{-1,\widehat{\Omega}} \quad \forall \mathbf{v} \in \widetilde{\mathbf{H}}^1_{\Gamma_0(\alpha)}(\widehat{\Omega}) \subset \mathbf{H}^1_L(\widehat{\Omega}), \qquad (3\text{–}71) \\
&< \mathbf{s}, \widetilde{\mathbf{u}}_\epsilon(\alpha) >_{-1/2,\Gamma_\mathbf{g}} =< \mathbf{s}, \mathbf{g} >_{-1/2,\Gamma_\mathbf{g}} \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_\mathbf{g}),
\end{aligned}
$$

where $\widetilde{\mathbf{H}}^1_{\Gamma_0(\alpha)}(\widehat{\Omega})$ denotes the space spanned by the extension $\widetilde{\varphi}$ when φ spans $\mathbf{H}^1_{\Gamma_0(\alpha)}$ $(\Omega(\alpha))$. We assume $\mathbf{f} \in \mathbf{L}^2(\widehat{\Omega}) \cap \mathbf{H}^{-1}_L(\widehat{\Omega})$ (see Lemma 3.8). Since the characteristic function over $\Omega(\alpha)$, $\chi_{\Omega(\alpha)} \in H^s(\widehat{\Omega})$ for $s < 1/2$, it follows that $\chi_{\Omega(\alpha)}\mathbf{f} \in \mathbf{H}^{-1}(\widehat{\Omega})$. Since the nonhomogeneous boundary $\Gamma_\mathbf{g}$ is invariant under the domain perturbation, we notice that the system (3–71) is actually equivalent to the system (3–60).

To show the existence of a Lagrange multiplier, let us consider the nonlinear mapping $\mathcal{K} : \widetilde{B}_1 \longrightarrow R \times \widetilde{B}_2$ defined by

$$
\mathcal{K}(\widetilde{\mathbf{u}}_\epsilon(\alpha), \widetilde{\mathbf{t}}_\epsilon) = (\widetilde{\mathfrak{J}}(\widetilde{\mathbf{u}}_\epsilon(\alpha)) - \widetilde{\mathfrak{J}}(\widetilde{\mathbf{u}}_\epsilon(\alpha^*)), \widetilde{\mathcal{M}}(\widetilde{\mathbf{u}}_\epsilon(\alpha), \widetilde{\mathbf{t}}_\epsilon)),
$$

where $\widetilde{\mathfrak{J}}(\widetilde{\mathbf{u}}_\epsilon(\alpha)) = \nu \int_{\widehat{\Omega}} \nabla\widetilde{\mathbf{u}}_\epsilon(\alpha) : \nabla\widetilde{\mathbf{u}}_\epsilon(\alpha)\, d\widehat{\Omega}$ and $(\alpha^*, \widetilde{\mathbf{u}}_\epsilon(\alpha^*)) \in \mathcal{U}_{ad} \times \mathbf{H}^1_L(\widehat{\Omega})$ is a local optimal solution. Let $\widetilde{\mathfrak{J}}'_{(\widetilde{\mathbf{u}}_\epsilon(\alpha), \widetilde{\mathbf{t}}_\epsilon)} : \widetilde{B}_1 \to R$ be an operator defined by $\widetilde{\mathfrak{J}}'_{(\widetilde{\mathbf{u}}_\epsilon(\alpha), \widetilde{\mathbf{t}}_\epsilon)}(\mathbf{w}, \zeta) = k \in R$ for some $\alpha \in \mathcal{U}_{ad}$ if and only if

$$
2\,\nu\, \widetilde{a}\,(\widetilde{\mathbf{u}}_\epsilon(\alpha), \mathbf{w}) = k\,.
$$

Obviously, $\widetilde{\mathfrak{J}}'_{(\widetilde{\mathbf{u}}_\epsilon(\alpha), \widetilde{\mathbf{t}}_\epsilon)}$ is a bounded linear operator. The operator $\mathcal{K}'_{(\widetilde{\mathbf{u}}_\epsilon(\alpha), \widetilde{\mathbf{t}}_\epsilon)} : \widetilde{B}_1 \longrightarrow R \times \widetilde{B}_2$ may be defined as follows:

$\mathcal{K}'_{(\widetilde{\mathbf{u}}_\epsilon(\alpha), \widetilde{\mathbf{t}}_\epsilon)}(\mathbf{w}, \zeta) = (k, \widetilde{\mathbf{f}}, \widetilde{\mathbf{g}})$ if and only if

$$
2\,\nu\, \widetilde{a}\,(\widetilde{\mathbf{u}}_\epsilon(\alpha), \mathbf{w}) = k\,, \qquad (3\text{–}72)
$$

$$
\begin{aligned}
&\nu\, \widetilde{a}\,(\mathbf{w}, \mathbf{v}) + \widetilde{c}(\mathbf{w}, \mathbf{u}_\epsilon, \mathbf{v}) + \widetilde{c}(\mathbf{u}_\epsilon, \mathbf{w}, \mathbf{v}) + \frac{1}{\epsilon}\widetilde{d}\,(\mathbf{w}, \mathbf{v}) - < \zeta, \mathbf{v} >_{-1/2,\Gamma_\mathbf{g}} \\
&\qquad =< \chi_{\Omega(\alpha)}\widetilde{\mathbf{f}}, \mathbf{v} >_{-1,\widehat{\Omega}} \quad \forall \mathbf{v} \in \widetilde{\mathbf{H}}^1_{\Gamma_0(\alpha)} \subset \mathbf{H}^1_L(\widehat{\Omega}), \qquad (3\text{–}73)
\end{aligned}
$$

$$
< \mathbf{s}, \mathbf{w} >_{-1/2,\Gamma_\mathbf{g}} =< \mathbf{s}, \widetilde{\mathbf{g}} >_{-1/2,\Gamma_\mathbf{g}} \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_\mathbf{g}). \qquad (3\text{–}74)
$$

The equations (3–73)–(3–74) correspond to a bounded linear operator $\widetilde{\mathcal{M}}'_{(\widetilde{\mathbf{u}}_\epsilon(\alpha),\widetilde{\mathbf{t}}_\epsilon)}$. It follows from Lemma 3.6 that $\widetilde{\mathcal{M}}'_{(\widetilde{\mathbf{u}}_\epsilon(\alpha),\widetilde{\mathbf{t}}_\epsilon)}$ is surjective. The existence of a Lagrange multiplier depends on the following Lemma.

Lemma 3.7. *$\mathcal{K}'_{(\widetilde{\mathbf{u}}_\epsilon(\alpha),\widetilde{\mathbf{t}}_\epsilon)}$ has a closed range but is not surjective.*

Proof: That $\mathcal{K}'_{(\widetilde{\mathbf{u}}_\epsilon(\alpha),\widetilde{\mathbf{t}}_\epsilon)}$ has a closed range can be shown in the same manner as that of Gunzburger *et al.*[65] (see also [130]). Let us show that it is not surjective. Suppose $\mathcal{K}'_{(\widetilde{\mathbf{u}}_\epsilon(\alpha),\widetilde{\mathbf{t}}_\epsilon)}$ is surjective. Then, since $\widetilde{\mathcal{M}}'_{(\widetilde{\mathbf{u}}_\epsilon(\alpha),\widetilde{\mathbf{t}}_\epsilon)}$ is surjective, $\widetilde{\mathfrak{J}}'_{(\widetilde{\mathbf{u}}_\epsilon(\alpha),\widetilde{\mathbf{t}}_\epsilon)}$ is also surjective. Hence using the implicit function theorem and the fact that

$$\|\alpha - \alpha^*\|_\infty \to 0 \Longrightarrow \widetilde{\mathbf{u}}_\epsilon(\alpha) \to \widetilde{\mathbf{u}}_\epsilon(\alpha^*) \quad \text{in } \mathbf{H}^1(\widehat{\Omega}),$$

which can be shown in the same manner as Theorem 2.4, we may choose positive numbers δ_1, δ_2 and $(\widetilde{\alpha}, \widetilde{\mathbf{u}}_\epsilon(\widetilde{\alpha})) \in \mathcal{U}_{ad} \times B$ such that $\|\widetilde{\alpha} - \alpha^*\|_\infty < \delta_1$, $\|\widetilde{\mathbf{u}}_\epsilon(\widetilde{\alpha}) - \widetilde{\mathbf{u}}_\epsilon(\alpha^*)\|_{1,\widehat{\Omega}} < \delta_2$ and $\int_{\widehat{\Omega}} \nabla\widetilde{\mathbf{u}}_\epsilon(\widetilde{\alpha}) : \nabla\widetilde{\mathbf{u}}_\epsilon(\widetilde{\alpha})\, d\widehat{\Omega} - \int_{\widehat{\Omega}} \nabla\widetilde{\mathbf{u}}_\epsilon(\alpha^*) : \nabla\widetilde{\mathbf{u}}_\epsilon(\alpha^*)\, d\widehat{\Omega} < 0$. This contradicts the hypothesis that $(\alpha^*, \widetilde{\mathbf{u}}_\epsilon(\alpha^*)) \in \mathcal{U}_{ad} \times B$ is a local optimal solution. □

Now, we are ready to show the existence of a Lagrange multiplier.

Theorem 3.3. *Let $(\alpha^*, \widetilde{\mathbf{u}}_\epsilon(\alpha^*)) \in \mathcal{U}_{ad} \times B$ be a local optimal solution. When the Reynolds number is small, there exists a nonzero Lagrange multiplier $(\widetilde{\mathbf{q}}_\epsilon, \widetilde{\eta}_\epsilon) \in \mathbf{H}^1_L(\widehat{\Omega}) \times \mathbf{H}^{-1/2}(\Gamma_\mathbf{g})$ such that*

$$-\widetilde{\mathfrak{J}}'_{(\widetilde{\mathbf{u}}_\epsilon(\alpha^*),\widetilde{\mathbf{t}}_\epsilon)}(\mathbf{w},\boldsymbol{\eta}) + \ll (\widetilde{\mathbf{q}}_\epsilon, \widetilde{\eta}_\epsilon), \widetilde{\mathcal{M}}'_{(\widetilde{\mathbf{u}}_\epsilon(\alpha^*),\widetilde{\mathbf{t}}_\epsilon)}(\mathbf{w},\boldsymbol{\eta}) \gg = 0\,, \tag{3–75}$$

for all $(\mathbf{w},\boldsymbol{\eta}) \in \widetilde{B_1}$.

Proof: Since $\mathcal{K}'_{(\widetilde{\mathbf{u}}_\epsilon(\alpha^*),\widetilde{\mathbf{t}}_\epsilon)}$ has a proper closed range in $R \times \widetilde{B_2}$, by the Hahn–Banach Theorem, there exists a nonzero element $(\widetilde{l}, \widetilde{\mathbf{q}}_\epsilon, \widetilde{\eta}_\epsilon) \in R \times (\widetilde{B_2})^*$ such that

$$\lll (\widetilde{l}, \widetilde{\mathbf{q}}_\epsilon, \widetilde{\eta}_\epsilon), (\widetilde{k}, \widetilde{\mathbf{f}}, \widetilde{\mathbf{g}}) \ggg = 0 \quad \forall\, (\widetilde{k}, \widetilde{\mathbf{f}}, \widetilde{\mathbf{g}}) \in \mathcal{K}'_{(\widetilde{\mathbf{u}}_\epsilon(\alpha^*),\widetilde{\mathbf{t}}_\epsilon)}(\widetilde{B_1})\,, \tag{3–76}$$

where $\ll \cdot,\cdot \gg$ denotes a duality pairing between $(R \times \widetilde{B_2})^*$ and $R \times \widetilde{B_2}$. Suppose $\widetilde{k} = 0$. Then, using the relation $< \chi_{\Omega(\alpha^*)}\widetilde{\mathbf{f}}, \widetilde{\mathbf{q}}_\epsilon >_{-1,\widehat{\Omega}} + < \widetilde{\eta}_\epsilon, \widetilde{\mathbf{g}} >_{-1/2,\Gamma_{\mathbf{g}}} = 0$, from (3–73)–(3–74), we may obtain the system equivalent to (3–66)–(3–67) with $\boldsymbol{\xi}_\epsilon = \mathbf{q}_\epsilon(\alpha^*)$ and $\boldsymbol{\tau}_\epsilon = \boldsymbol{\eta}_\epsilon$. Hence, from Lemma 3.6, we have $(\mathbf{q}_\epsilon(\alpha^*), \boldsymbol{\eta}_\epsilon) = (\mathbf{0}, \mathbf{0})$. Hence its extension $(\widetilde{\mathbf{q}}_\epsilon, \widetilde{\boldsymbol{\eta}}_\epsilon) = (\mathbf{0}, \mathbf{0})$, which contradicts the choice of a nonzero element. Thus, $\widetilde{k}$ is nonzero and we may assume that $\widetilde{k} = -1$ without loss of generality. By the definition of $\mathcal{K}'_{(\widetilde{\mathbf{u}_\epsilon(\alpha^*)}, \widetilde{\mathbf{t}_\epsilon})}$, (3–75) follows from (3–76). □

Using $\mathbf{q}_\epsilon(\alpha^*) = \mathbf{0}$ along $\Gamma = \Gamma_{\mathbf{0}}(\alpha) \cup \Gamma_{\mathbf{g}}$ (refer the Proof of Lemma 3.6), (3–75) leads to the adjoint system defined on $\Omega(\alpha^*)$. Hence if we drop $(\cdot)^*$, it is natural to seek a stationary points (or, decreasing points), *i.e.*, points such that $grad\mathfrak{J}_\epsilon = 0$, (or, ≤ 0), which is represented by state and its adjoint equations for each domain perturbation.

3.4. Some Remarks on Regularity and the Optimality System

In Sections 3.3–3.4, by introducing Lagrange multipliers, we have obtained the adjoint equations needed to derive the shape gradient. We may reverse this process: *A priori*, we may introduce the adjoint equations (3–36)–(3–38) and using them, we can obtain the shape gradient term (3–60). In the shape design problem, this is a rather popular process. It is interesting to notice that the adjoint equations have the similar form to those of Pironneau[100] for a simplified minimum drag profile problem.

The formal computations in Section 3.2 can be basically justified by some regularities for the domain (geometry) and solution. Moreover, in the finite element approximation of the problem, error estimates closely depend on the regularity of solutions of the optimality system. Let us begin by illustrating an example due to Serre[116]: Consider a driven cavity problem in a rectangular region. Suppose the flow is governed by the Navier–Stokes equations with uniform velocity along the top side. Then Serre showed that the solution $\mathbf{u}$ does not even belong to the $\mathbf{H}^1$–space.

The main cause of this irregularity is the existence of jumps of velocities around corners. We may dispense with this situation by imposing $\mathbf{g}_i$ to have compact support on Γ_i, $(i = 1, 2)$. Virtually, since $\mathbf{g}$ is a velocity of the continuous flow through the channel, we may regard $\mathbf{g}_i$, $(i = 1, 2)$, to be a parabolic flow along the inflow and outflow boundaries. Then the regularity problem related to the corners A, B, C and D in Figure 2 does not appear.

The basic requirement for the regularity of the external force field is that $\mathbf{f} \in \mathbf{L}^2(\widehat{\Omega})$. This requirement originates from the need to justify the material derivative involving $\mathbf{f}$. It also contributes to the regularity of the optimality system.

Lemma 3.8. *Let* $\mathbf{f} \in \mathbf{L}^2(\widehat{\Omega})$. *Then* $[0, \sigma) \ni t \longmapsto \mathbf{f} \circ \mathcal{F}_t \in \mathbf{L}^2(\widehat{\Omega})$ *is weakly differentiable in* $\mathbf{H}^{-1}(\widehat{\Omega})$.

Proof: When $\mathcal{V}$ is smooth enough, the proof can be found in Sokolowski *et al.*([120], p.71–p.75). In our case, the same principle can be applied since $\mathcal{F}_t \in \mathcal{C}^{0,1}(\widehat{\Omega})$, so that $\mathcal{F}_t$ is differentiable almost everywhere. □

Next, we consider the regularity of the domain. If the domain is non–smooth, rigorous computations of the boundary integral such as (3–47) may introduce extra jump states at the singular points of the boundary. For example, even though y and $\mathcal{V}$ are smooth in R^2, if $\partial\Omega = \Gamma$ has a piecewise smooth boundary, *i.e.*, Γ is smooth except at the points $\{a_1, \cdots, a_k\}$, then

$$\frac{d}{dt} \int_{\Gamma_t} y \, d\Gamma_t \bigg|_{t=0^+} = \int_{\Gamma} \left(\frac{\partial y}{\partial \mathbf{n}} + \kappa \, y \right) \mathcal{V}(0) \cdot \mathbf{n} \, d\Gamma + \sum_{i=1}^{k} \mathcal{V}(0, a_i)[\tau(a_i)] \, ,$$

where $\{[\tau(a_i)]\}$ = jump states of the tangents at the singular points a_i, $i = 1, \cdots, k$. However, if the domain has a Lipschitz boundary, a corner is no more a singular point (Simon[117]). Hence boundary integrals do not involve jump states. If $\Gamma(\alpha) \in \mathcal{C}^{0,1}([-M_1, M_1])$, an outward unit normal vector along the boundary $\Gamma(\alpha)$ exists almost everywhere and $\mathbf{n} \in \mathbf{L}^\infty(\Gamma; R^2)$. This implies that $\mathbf{n} = \left(\frac{\alpha'}{\sqrt{1+\alpha'^2}}, -\frac{1}{\sqrt{1+\alpha'^2}}\right) \in \mathbf{L}^\infty(\Gamma(\alpha); R^2)$ exists. We note that $\mathcal{V} = (0, \mathcal{V}_2)$ is continuous over $\overline{\Omega(\alpha)}$ and

$$\nabla \mathcal{V}_2 = \begin{cases} \left(\dfrac{(p_2 - L)\vartheta'(\alpha - L) - (\alpha' - L)(p_2 - L)\vartheta}{(\alpha - L)^2}, \dfrac{\vartheta}{(\alpha - L)}\right) & \text{if } -M_1 \le x_1 \le M_1\,, \\ \mathbf{0} \quad \text{otherwise}\,. \end{cases}$$

Hence $\mathcal{V} \in \mathbf{H}^1(\Omega(\alpha))$. So, if $\mathbf{u}_\epsilon$ is sought in $\mathbf{H}^1(\Omega(\alpha))$, since $\nabla \mathbf{u}_\epsilon \cdot \mathcal{V}$ belongs to $\mathbf{L}^2(\Omega(\alpha))$, $\dot{\mathbf{u}}_\epsilon$ or $\widetilde{\mathbf{u}}_\epsilon{}'\Big|_{\Omega(\alpha)}$ will belong to $\mathbf{L}^2(\Omega(\alpha))$. In this case, the computation discussed in Section 3.2 may not be justified. However, if we take ϑ to be of class $\mathcal{C}^{1,1}$ with smooth contacts at points $\pm M_1$, then all the regularity needed to justify the computations is secured.

Now, let us turn to the optimality system we have derived. The state equations and its adjoint have been used to derive the shape gradient "*grad* $\mathfrak{J}_\epsilon$" in the optimization process. "*grad* $\mathfrak{J}_\epsilon$" can be used to determine an optimal design points in the discrete problem, *i.e.*, the system of equations (2–15) and (3–36)–(3–37) plays the role of necessary conditions to seek an optimal position of the moving boundary. For this reason, this system of equations shall be called the *optimality system*. Using (2–15), we can replace the right hand side of the first equation of (3–39) by

$$\begin{aligned} -\nu\Delta\mathbf{q}_\epsilon + \mathbf{q}_\epsilon \cdot^T (\nabla\mathbf{u}_\epsilon) - \mathbf{u}_\epsilon \cdot \nabla\mathbf{q}_\epsilon - (div\mathbf{u}_\epsilon)\,\mathbf{q}_\epsilon& \\ -\frac{1}{\epsilon}\nabla(div\mathbf{q}_\epsilon) &= 2[\,\mathbf{f} - (\mathbf{u}_\epsilon \cdot \nabla)\,\mathbf{u}_\epsilon + \frac{1}{\epsilon}\nabla(div\mathbf{u}_\epsilon)\,]\,. \end{aligned}$$

This replacement facilitates the derivation of error estimates of finite element approximations and the analysis of penalty approximation to the primal problem, for the state and adjoint system corresponding to penalty problem employ the same penalty parameter. From this replacement, the optimality system (<u>penalized version</u>) consists of

$$-\nu\Delta\mathbf{u}_\epsilon - \frac{1}{\epsilon}\nabla(div\mathbf{u}_\epsilon) + (\mathbf{u}_\epsilon\cdot\nabla)\mathbf{u}_\epsilon = \mathbf{f} \quad \text{in} \quad \Omega(\alpha)\,, \tag{3–77}$$

$$\mathbf{u}_\epsilon = \mathbf{g} \quad \text{on} \quad \Gamma\,, \tag{3–78}$$

$$\begin{aligned} &-\nu\Delta\mathbf{q}_\epsilon + \mathbf{q}_\epsilon \cdot^T(\nabla\mathbf{u}_\epsilon) - \mathbf{u}_\epsilon\cdot\nabla\mathbf{q}_\epsilon - (div\mathbf{u}_\epsilon)\,\mathbf{q}_\epsilon - \frac{1}{\epsilon}\nabla(div\mathbf{q}_\epsilon) \\ &\qquad = 2\,[\,\mathbf{f} - (\mathbf{u}_\epsilon\cdot\nabla)\,\mathbf{u}_\epsilon + \frac{1}{\epsilon}\nabla(div\mathbf{u}_\epsilon)] \quad \text{on} \quad \Omega(\alpha)\,, \end{aligned} \tag{3–79}$$

and

$$\mathbf{q}_\epsilon = \mathbf{0} \quad \text{on} \quad \Gamma\,. \tag{3–80}$$

Equivalently, this can be rewritten in the form of

$$-\nu\Delta\mathbf{u}_\epsilon + \nabla p_\epsilon = \mathbf{f} - (\mathbf{u}_\epsilon\cdot\nabla)\mathbf{u}_\epsilon \quad \text{in} \quad \Omega(\alpha)\,, \tag{3–81}$$

$$div\mathbf{u}_\epsilon = -\epsilon\, p_\epsilon \quad \text{in} \quad \Omega(\alpha)\,, \tag{3–82}$$

$$\mathbf{u}_\epsilon = \mathbf{g} \quad \text{on} \quad \Gamma\,, \tag{3–83}$$

$$-\nu\Delta\mathbf{q}_\epsilon + \nabla\theta_\epsilon = 2\mathbf{f} - 2(\mathbf{u}_\epsilon\cdot\nabla)\mathbf{u}_\epsilon - \mathbf{q}_\epsilon \cdot^T(\nabla\mathbf{u}_\epsilon) + \mathbf{u}_\epsilon\cdot\nabla\mathbf{q}_\epsilon - \epsilon\, p_\epsilon\,(\mathbf{q}_\epsilon) \;\text{ on } \Omega(\alpha)\,, \tag{3–84}$$

$$div\mathbf{q}_\epsilon = -\epsilon\, r_\epsilon \quad \text{on} \quad \Omega(\alpha)\,, \tag{3–85}$$

and

$$\mathbf{q}_\epsilon = \mathbf{0} \quad \text{in} \quad \Gamma\,, \tag{3–86}$$

where $\theta_\epsilon = r_\epsilon + 2p_\epsilon \in L_0^2(\Omega(\alpha))$ for $\alpha \in \mathcal{U}_{ad}$. This latter expression makes the analysis for regularity and error estimates easier.

At this point, one may ask what correponds to the optimality system and the shape gradient for the primal problem? From (3–81)–(3–86), when ϵ tends to 0^+, it is natural to consider the following system (<u>primal version</u>):

$$-\nu\Delta\mathbf{u} + \nabla p = \mathbf{f} - (\mathbf{u}\cdot\nabla)\mathbf{u} \quad \text{in} \quad \Omega(\alpha)\,, \tag{3–87}$$

$$div\mathbf{u} = 0 \quad \text{in} \quad \Omega(\alpha)\,, \tag{3-88}$$

$$\mathbf{u} = \mathbf{g} \qquad \text{on} \quad \Gamma\,, \tag{3-89}$$

$$-\nu\Delta\mathbf{q} + \nabla\theta = 2\mathbf{f} - 2(\mathbf{u}\cdot\nabla)\mathbf{u} - \mathbf{q}\cdot^{T}(\nabla\mathbf{u}) + \mathbf{u}\cdot\nabla\mathbf{q} \quad \text{on} \quad \Omega(\alpha)\,, \tag{3-90}$$

$$div\mathbf{q} = 0 \quad \text{in} \quad \Omega(\alpha) \tag{3-91}$$

and

$$\mathbf{q} = \mathbf{0} \quad \text{in} \quad \Gamma\,. \tag{3-92}$$

Here $\theta = r + 2p \in L_0^2(\Omega(\alpha))$, where $r \in L_0^2(\Omega(\alpha))$ is given in (3–39). In this case, the corresponding shape gradient on $\Gamma(\alpha)$ will be given by

$$g(\Gamma) \equiv grad\,\mathfrak{J} = \nu\nabla\mathbf{u} \,:\, \nabla\mathbf{u} - \nu\nabla\mathbf{u} \,:\, \nabla\mathbf{q} - \frac{\partial\mathbf{u}}{\partial\mathbf{n}}\cdot\left(2\nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - \nu\frac{\partial\mathbf{q}}{\partial\mathbf{n}} + (\theta - 2p)\mathbf{n}\right), \tag{3-93}$$

for $\dfrac{1}{\epsilon}div\mathbf{u}_\epsilon\, div\mathbf{q}_\epsilon = -\epsilon p_\epsilon r_\epsilon$ in (3–60) tends to 0.

Let us first state the regularity result of the solutions of the system (3–87)–(3–92). From the result of Cattabriga[24], the regularity corresponding to the solution of the Navier–Stokes equations follows from that of the Stokes equations.

Theorem 3.4. *Suppose* $\mathbf{g} \in \mathbf{H}_0^{3/2}(\Gamma)$ *with* $\mathbf{g}_i$ *having compact support on* Γ_i, $(i = 1, 2)$. *Let* $\mathbf{f} \in \mathbf{L}^2(\widehat{\Omega})$. *Suppose* α *is of class* $\mathcal{C}^{1,1}$ *with smooth contacts at points* E *and* F. *If* $(\mathbf{u}, p, \mathbf{q}, r) \in \mathbf{H}^1(\Omega(\alpha)) \times L_0^2(\Omega(\alpha)) \times \mathbf{H}^1(\Omega(\alpha)) \times L_0^2(\Omega(\alpha))$ *denotes a solution of (3–87)–(3–92), we have that*

$$(\mathbf{u}, p, \mathbf{q}, r) \in \mathbf{H}^2(\Omega(\alpha)) \times (H^1(\Omega(\alpha)) \cap L_0^2(\Omega(\alpha))) \times \mathbf{H}^2(\Omega(\alpha)) \times (H^1(\Omega(\alpha)) \cap L_0^2(\Omega(\alpha)))\,.$$

Sketch of Proof: In the Stokes problem, the allowable domains Ω for the regularity in $H^2(\Omega)$ are those which are piecewise $\mathcal{C}^{1,1}$ with convex corners (see Girault–Raviart[60] and Grisvard[63]). Since $\Omega(\alpha)$ satisfy these requirement, we may allow the regularity of the velocity space up to $\mathbf{H}^2(\Omega(\alpha))$. Let us consider the right hand side of (3–87), $\widetilde{\mathbf{f}} \equiv \mathbf{f} - (\mathbf{u}\cdot\nabla)\mathbf{u}$. Since $\mathbf{u} \in \mathbf{H}^1(\Omega(\alpha))$, $\mathbf{u} \in \mathbf{L}^6(\Omega(\alpha))$ and $\dfrac{\partial \mathbf{u}}{\partial x_j} \in \mathbf{L}^2(\Omega(\alpha))$ for $j = 1,2$, we have that $(\mathbf{u}\cdot\nabla)\mathbf{u} \in \mathbf{L}^{3/2}(\Omega(\alpha))$ and hence $\widetilde{\mathbf{f}} \in \mathbf{L}^{3/2}(\Omega(\alpha)) \subset \mathbf{H}^{-1/2}(\Omega(\alpha))$. Since $\mathbf{g} \in \mathbf{H}^{3/2}(\Gamma)$, from the result of [125], the Stokes system (3–87)–(3–89) has a solution $(\mathbf{u},p) \in \mathbf{H}^{3/2}(\Omega(\alpha)) \times (H^{1/2}(\Omega(\alpha)) \cap L^2_0(\Omega(\alpha)))$. Repeating this argument with $\mathbf{f} \in \mathbf{L}^2(\widehat{\Omega})$, $\widetilde{\mathbf{f}} \in \mathbf{L}^2(\Omega(\alpha))$, so that $(\mathbf{u},p) \in \mathbf{H}^2(\Omega(\alpha)) \times (H^1(\Omega(\alpha)) \cap L^2_0(\Omega(\alpha)))$. If we next consider the right hand side of (3–90)–(3–92), then standard results also lead to $(\mathbf{q},r) \in \mathbf{H}^2(\Omega(\alpha)) \times (H^1(\Omega(\alpha)) \cap L^2_0(\Omega(\alpha)))$. □

REMARK 3.8: From the bootstrap argument, one can show that if the domain Ω is smooth enough and $(\mathbf{f},\mathbf{g}) \in \mathbf{H}^m(\Omega) \times (H^{m+3/2}(\Omega) \cap L^2_0(\Omega))$, then $(\mathbf{u},p,\mathbf{q},r) \subset \mathbf{H}^{m+2}(\Omega) \times H^{m+1}(\Omega) \times \mathbf{H}^{m+2}(\Omega) \times H^{m+1}(\Omega)$. However, if $\alpha \in C^{0,1}([-M_1,M_1])$, then even though $\mathbf{g} \in \mathbf{H}^{3/2}_0(\Gamma)$, losing the convexity and regularity of the domain will result in $(\mathbf{u},p) \in \mathbf{H}^{3/2-\delta}(\Omega(\alpha)) \times (H^{1/2-\delta}(\Omega(\alpha)) \cap L^2_0(\Omega(\alpha)))$ for some small $\delta > 0$.

3.5. The Weak Penalized Optimality System

Let us consider the approximation of the penalized system to the primal system as ϵ tends to 0^+. For this purpose, we pose (3–81)–(3–86) into a weak formulation as follows (<u>weak penalized version</u>):

$$\nu\, a(\mathbf{u}_\epsilon,\mathbf{v}) + b(\mathbf{v},p_\epsilon) - <\mathbf{t}_\epsilon,\mathbf{v}>_{-1/2,\Gamma_\mathbf{g}}$$
$$=<\mathbf{f},\mathbf{v}>_{-1} - c(\mathbf{u}_\epsilon,\mathbf{u}_\epsilon,\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)), \quad (3\text{–}94)$$

$$b(\mathbf{u}_\epsilon,q) = \epsilon\,(p_\epsilon,q)_0 \quad \forall q \in L^2_0(\Omega(\alpha)), \quad (3\text{–}95)$$

$$<\mathbf{s},\mathbf{u}_\epsilon>_{-1/2,\Gamma_\mathbf{g}} = <\mathbf{s},\mathbf{g}>_{-1/2,\Gamma_\mathbf{g}} \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_\mathbf{g}), \quad (3\text{–}96)$$

$$\nu\, a(\mathbf{q}_\epsilon,\mathbf{w}) + b(\mathbf{w},\theta_\epsilon) - <\tau_\epsilon,\mathbf{w}>_{-1/2,\Gamma} = 2<\mathbf{f},\mathbf{w}>_{-1} - 2c(\mathbf{u}_\epsilon,\mathbf{u}_\epsilon,\mathbf{w})$$
$$- c(\mathbf{w},\mathbf{u}_\epsilon,\mathbf{q}_\epsilon) - c(\mathbf{u}_\epsilon,\mathbf{w},\mathbf{q}_\epsilon) \quad \forall \mathbf{w} \in \mathbf{H}^1(\Omega(\alpha)), \quad (3\text{–}97)$$

$$b\,(\mathbf{q}_\epsilon,\rho)=\epsilon\,(\theta_\epsilon,\rho)_0-2\epsilon\,(p_\epsilon,\rho)_0 \quad \forall\,\rho\in L^2_0(\Omega(\alpha)) \tag{3–98}$$

and

$$<\mathbf{s},\mathbf{q}_\epsilon>_{-1/2,\Gamma}=\mathbf{0} \quad \forall\,\mathbf{s}\in\mathbf{H}^{-1/2}(\Gamma)\,, \tag{3–99}$$

where

$$\tau_\epsilon=\nu\,\frac{\partial\mathbf{q}_\epsilon}{\partial\mathbf{n}}-\theta_\epsilon\mathbf{n} \quad \text{and} \quad \mathbf{t}_\epsilon=\nu\,\frac{\partial\mathbf{u}_\epsilon}{\partial\mathbf{n}}-p_\epsilon\mathbf{n}\,. \tag{3–100}$$

In (3–98), we have used the relation $\theta_\epsilon=r_\epsilon+2p_\epsilon$.

Similarly, the corresponding (<u>weak primal version</u>) to (3–87)–(3–92) is given by

$$\begin{aligned}\nu\,a\,(\mathbf{u},\mathbf{v})+b\,(\mathbf{v},p)-&<\mathbf{t},\mathbf{v}>_{-1/2,\Gamma_\mathbf{g}}\\ &=<\mathbf{f},\mathbf{v}>_{-1}-c\,(\mathbf{u},\mathbf{u},\mathbf{v}) \quad \forall\,\mathbf{v}\in\mathbf{H}^1_{\Gamma_{0(\alpha)}}(\Omega(\alpha))\,,\end{aligned} \tag{3–101}$$

$$b\,(\mathbf{u},q)=0 \quad \forall\,q\in L^2_0(\Omega(\alpha))\,, \tag{3–102}$$

$$<\mathbf{s},\mathbf{u}>_{-1/2,\Gamma_\mathbf{g}}=<\mathbf{s},\mathbf{g}>_{-1/2,\Gamma_\mathbf{g}} \quad \forall\mathbf{s}\in\mathbf{H}^{-1/2}(\Gamma_\mathbf{g})\,, \tag{3–103}$$

$$\begin{aligned}\nu\,a(\mathbf{q},\mathbf{w})+b(\mathbf{w},\theta)-<\tau,\mathbf{w}>_{-1/2,\Gamma}&=2<\mathbf{f},\mathbf{w}>_{-1}-2c\,(\mathbf{u},\mathbf{u},\mathbf{w})\\ &-c\,(\mathbf{w},\mathbf{u},\mathbf{q})-c\,(\mathbf{u},\mathbf{w},\mathbf{q}) \quad \forall\mathbf{w}\in\mathbf{H}^1(\Omega(\alpha))\,,\end{aligned} \tag{3–104}$$

$$b\,(\mathbf{q},\rho)=0 \quad \forall\,\rho\in L^2_0(\Omega(\alpha)) \tag{3–105}$$

and

$$<\mathbf{s},\mathbf{q}>_{-1/2,\Gamma}=\mathbf{0} \quad \forall\mathbf{s}\in\mathbf{H}^{-1/2}(\Gamma)\,, \tag{3–106}$$

where

$$\tau=\nu\,\frac{\partial\mathbf{q}}{\partial\mathbf{n}}-\theta\,\mathbf{n} \quad \text{and} \quad \mathbf{t}=\nu\,\frac{\partial\mathbf{u}}{\partial\mathbf{n}}-p\,\mathbf{n}\,. \tag{3–107}$$

To study the approximation of the penalized optimality system to the primal system as ϵ tends to 0^+, we may again refer to the nonlinear functional settings discussed in

Section 2.3. Let

$$Y = \mathbf{H}^{-1}(\Omega(\alpha)) \times \mathbf{H}_0^{1/2}(\Omega(\alpha)) \times (\mathbf{H}^1(\Omega(\alpha)))^*,$$

$$X = \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)) \times L^2_0(\Omega(\alpha)) \times \mathbf{H}^{-1/2}(\Gamma_\mathbf{g}) \times \mathbf{H}^1(\Omega(\alpha)) \times L^2_0(\Omega(\alpha)) \times \mathbf{H}^{-1/2}(\Gamma),$$

$$Z = \mathbf{L}^{3/2}(\Omega(\alpha)) \times \{\mathbf{0}\} \times \mathbf{L}^{3/2}(\Omega(\alpha)).$$

Define the nonlinear mapping $G : \Lambda \times X \longrightarrow Y\,((\lambda, (\mathbf{u}, p, \mathbf{t}, \mathbf{q}, \theta, \boldsymbol{\tau})) \longmapsto (\boldsymbol{\eta}, \boldsymbol{\kappa}, \boldsymbol{\xi}))$ by

$$< \boldsymbol{\eta}, \mathbf{v} >_{-1} = \lambda c(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \lambda < \mathbf{f}, \mathbf{v} >_{-1} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)), \tag{3–108}$$

$$< \mathbf{s}, \boldsymbol{\kappa} >_{-1/2,\Gamma_\mathbf{g}} = - < \mathbf{s}, \mathbf{g} >_{-1/2,\Gamma_\mathbf{g}} \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_\mathbf{g}) \tag{3–109}$$

and

$$< \boldsymbol{\xi}, \mathbf{w} >_{-1} = 2\lambda c(\mathbf{u}, \mathbf{u}, \mathbf{w}) + \lambda\, c(\mathbf{w}, \mathbf{u}, \mathbf{q}) + \lambda\, c(\mathbf{u}, \mathbf{w}, \mathbf{q})$$
$$- 2\lambda < \mathbf{f}, \mathbf{w} >_{-1} \quad \forall \mathbf{w} \in \mathbf{H}^1(\Omega(\alpha)), \tag{3–110}$$

where the data $(\mathbf{f}, \mathbf{g}) \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)) \times \mathbf{H}_0^{1/2}(\Gamma_0(\alpha))$ is given.

Let the linear operator $T : Y \longrightarrow X((\widetilde{\mathbf{f}}, \widetilde{\mathbf{g}}, \widetilde{\mathbf{h}}) \longmapsto (\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\mathbf{t}}, \widetilde{\mathbf{q}}, \widetilde{\theta}, \widetilde{\boldsymbol{\tau}}))$ be defined in the following manner:

$$a(\widetilde{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, \widetilde{p}) - < \widetilde{\mathbf{t}}, \mathbf{v} >_{-1/2,\Gamma_\mathbf{g}} = < \widetilde{\mathbf{f}}, \mathbf{v} >_{-1} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)),$$
$$b(\widetilde{\mathbf{u}}, q) = 0 \quad \forall q \in L^2_0(\Omega(\alpha)), \tag{3–111}$$
$$< \mathbf{s}, \widetilde{\mathbf{u}} >_{-1/2,\Gamma_\mathbf{g}} = < \mathbf{s}, \widetilde{\mathbf{g}} >_{-1/2,\Gamma_\mathbf{g}} \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_\mathbf{g})$$

and

$$a(\widetilde{\mathbf{q}}, \mathbf{w}) + b(\mathbf{w}, \widetilde{\theta}) - < \widetilde{\boldsymbol{\tau}}, \mathbf{w} >_{-1,\Gamma} = < \widetilde{\mathbf{h}}, \mathbf{w} >_{-1} \quad \forall \mathbf{w} \in \mathbf{H}^1(\Omega(\alpha)),$$
$$b(\widetilde{\mathbf{q}}, \rho) = 0 \quad \forall \rho \in L^2_0(\Omega(\alpha)), \tag{3–112}$$
$$< \mathbf{y}, \widetilde{\mathbf{q}} >_{-1/2,\Gamma} = 0 \quad \forall \mathbf{y} \in \mathbf{H}^{-1/2}(\Gamma).$$

Analogously, the penalized operator $T^\epsilon \in \mathcal{L}(Y; X)$ is defined as follows:

$$a(\widetilde{\mathbf{u}}_\epsilon, \mathbf{v}) + b(\mathbf{v}, \widetilde{p}_\epsilon) - < \widetilde{\mathbf{t}}_\epsilon, \mathbf{v} >_{-1/2,\Gamma_\mathbf{g}} = < \widetilde{\mathbf{f}}, \mathbf{v} >_{-1} \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha)),$$
$$b(\widetilde{\mathbf{u}}_\epsilon, q) = \epsilon\,(\widetilde{p}_\epsilon, q)_0 \quad \forall q \in L^2_0(\Omega(\alpha)), \tag{3–113}$$
$$< \mathbf{s}, \widetilde{\mathbf{u}}_\epsilon >_{-1/2,\Gamma_\mathbf{g}} = < \mathbf{s}, \widetilde{\mathbf{g}} >_{-1/2,\Gamma_\mathbf{g}} \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_\mathbf{g})$$

and

$$
\begin{aligned}
a(\widetilde{\mathbf{q}}_\epsilon, \mathbf{w}) + b(\mathbf{w}, \widetilde{\theta}_\epsilon) - < \widetilde{\boldsymbol{\tau}}_\epsilon, \mathbf{w} >_{-1,\Gamma} = < \widetilde{\mathbf{h}}, \mathbf{w} >_{-1} \quad \forall \mathbf{w} \in \mathbf{H}^1(\Omega(\alpha))\,,& \\
b(\widetilde{\mathbf{q}}_\epsilon, \rho) = \epsilon\, (\widetilde{\theta}_\epsilon, \rho)_0 - 2\epsilon\, (\widetilde{p}_\epsilon, \rho)_0 \quad \forall \rho \in L^2_0(\Omega(\alpha))\,,& \qquad (3\text{–}114) \\
< \mathbf{y}, \widetilde{\mathbf{q}}_\epsilon >_{-1/2,\Gamma} = 0 \quad \forall \mathbf{y} \in \mathbf{H}^{-1/2}(\Gamma)\,.&
\end{aligned}
$$

Clearly, T and T^ϵ are composed of two Stokes operators. The weak penalized version (3–94)–(3–100) is equivalent to finding a solution of

$$
\begin{aligned}
F^\epsilon(\lambda, (\mathbf{u}_\epsilon, p_\epsilon, \mathbf{t}_\epsilon, \mathbf{q}_\epsilon, \theta_\epsilon, \boldsymbol{\tau}_\epsilon) = (\mathbf{u}_\epsilon, \lambda p_\epsilon, \lambda \mathbf{t}_\epsilon, \mathbf{q}_\epsilon, \lambda\theta_\epsilon, \lambda\boldsymbol{\tau}_\epsilon)& \\
+ T^\epsilon G(\lambda, (\mathbf{u}_\epsilon, \lambda p_\epsilon, \lambda \mathbf{t}_\epsilon, \mathbf{q}_\epsilon, \lambda\theta_\epsilon, \lambda\boldsymbol{\tau}_\epsilon)) = 0\,,&
\end{aligned}
$$

and its primal version (3–101)–(3–107) is equivalent to

$$
\begin{aligned}
F(\lambda, (\mathbf{u}, p, \mathbf{t}, \mathbf{q}, \theta, \boldsymbol{\tau}) = (\mathbf{u}, \lambda p, \lambda \mathbf{t}, \mathbf{q}, \lambda\theta, \lambda\boldsymbol{\tau})& \\
+ TG(\lambda, (\mathbf{u}, \lambda p, \lambda \mathbf{t}, \mathbf{q}, \lambda\theta, \lambda\boldsymbol{\tau})) = 0\,.&
\end{aligned}
$$

We are now ready to state the main result for the convergence of the weak penalized system to the primal one.

Theorem 3.5. *Assume that Λ be a compact set in R^+. Let $\alpha \in \mathcal{U}_{ad}$ be fixed. Let $\{(\lambda, (\mathbf{u}(\lambda), \lambda p(\lambda), \lambda \mathbf{t}(\lambda), \mathbf{q}(\lambda), \lambda\theta(\lambda), \lambda\boldsymbol{\tau}(\lambda)) \mid \lambda = \dfrac{1}{\nu} \in \Lambda\}$ be a branch of regular solutions of the primal optimality system (3–101)–(3–107). Then, for $\epsilon \le \epsilon_0$ small enough, there exists a unique $\mathcal{C}^2$–branch $\{(\lambda, (\mathbf{u}_\epsilon(\lambda), \lambda p_\epsilon(\lambda), \lambda \mathbf{t}_\epsilon(\lambda), \mathbf{q}_\epsilon(\lambda), \lambda\theta_\epsilon(\lambda), \lambda\boldsymbol{\tau}_\epsilon\,(\lambda)) \mid \lambda \in \Lambda\}$ of (3–94)–(3–100) in the neighborhood of $(\mathbf{u}(\lambda), p(\lambda), \mathbf{t}(\lambda), \mathbf{q}(\lambda), \theta(\lambda), \boldsymbol{\tau}(\lambda))$ in X and a positive constant C which is independent of ϵ and λ, such that*

$$
\begin{aligned}
&\|\mathbf{u}_\epsilon - \mathbf{u}\|_1 + \|p_\epsilon - p\|_0 + \|\mathbf{t}_\epsilon - \mathbf{t}\|_{-1/2,\Gamma_g} \\
&\qquad \|\mathbf{q}_\epsilon - \mathbf{q}\|_1 + \|\theta_\epsilon - \theta\|_0 + \|\boldsymbol{\tau}_\epsilon - \boldsymbol{\tau}\|_{-1/2,\Gamma} \le C\epsilon \quad \forall \lambda \in \Lambda\,. \qquad (3\text{–}115)
\end{aligned}
$$

Sketch of Proof: Let $\varphi = (\mathbf{u}, p, \mathbf{t}, \mathbf{q}, \theta, \boldsymbol{\tau})$. The Fréchet derivative $D_\varphi G(\lambda, \cdot)$ satisfies

$$
D_\varphi G(\lambda, (\mathbf{u}, p, \mathbf{t}, \mathbf{q}, \theta, \boldsymbol{\tau})) \cdot (\mathbf{v}, \rho, \mathbf{s}, \boldsymbol{\mu}, \phi, \boldsymbol{\xi}) = (\widetilde{\boldsymbol{\zeta}}, \mathbf{0}, \widetilde{\boldsymbol{\eta}})
$$

for $(\mathbf{v},\rho,\mathbf{s},\boldsymbol{\mu},\phi,\boldsymbol{\xi})\in X$ if and only if

$$< \widetilde{\zeta},\overline{\mathbf{v}} >_{-1} = \lambda c(\mathbf{u},\mathbf{v},\overline{\mathbf{v}}) + \lambda c(\mathbf{v},\mathbf{u},\overline{\mathbf{v}}) \quad \forall \overline{\mathbf{v}} \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha))\,,$$

$$\begin{aligned} < \widetilde{\boldsymbol{\eta}},\overline{\mathbf{w}} >_{-1} &= 2\lambda c(\mathbf{u},\mathbf{v},\overline{\mathbf{w}}) + 2\lambda c(\mathbf{v},\mathbf{u},\overline{\mathbf{w}}) + \lambda c(\overline{\mathbf{w}},\mathbf{v},\mathbf{q}) + \lambda c(\overline{\mathbf{w}},\mathbf{u},\boldsymbol{\mu}) \\ &\quad + \lambda c(\mathbf{v},\overline{\mathbf{w}},\mathbf{q}) - \lambda c(\mathbf{u},\overline{\mathbf{w}},\boldsymbol{\mu}) \qquad \forall \overline{\mathbf{w}} \in \mathbf{H}^1(\Omega(\alpha))\,. \end{aligned}$$

Using the continuity of $c(\cdot,\cdot,\cdot)$ over the $\mathbf{H}^1$–space, it follows that $D_\varphi G \in \mathcal{L}(X;Y)$. In the same manner, it can be shown that $D_{\varphi\varphi} G \in \mathcal{L}(X;Y)$. Moreover, since the embeddings $\mathbf{H}^1(\Omega(\alpha)) \subset \mathbf{L}^4(\Omega(\alpha)) \subset \mathbf{L}^2(\Omega(\alpha))$ are continuous, we can easily conclude that $D_\varphi G(\lambda,\cdot) \in \mathcal{L}(X;Z)$. Since Z is compactly embedded in Y, it is sufficient to check the assumption (2–33) to apply Theorem 2.2. From the result of Theorem 2.3, we only need to verify that

$$\|(\mathbf{q}_\epsilon,\theta_\epsilon,\boldsymbol{\tau}_\epsilon)-(\mathbf{q},\theta,\boldsymbol{\tau})\|_{\mathbf{H}^1\times L^2_0\times\mathbf{H}^{-1/2}(\Gamma)} \to 0\,, \quad \text{as} \quad \epsilon\to 0^+\,.$$

Subtracting (3–112) from (3–114), we have

$$\begin{aligned} a(\widetilde{\mathbf{q}}_\epsilon-\widetilde{\mathbf{q}},\mathbf{w}) + b(\mathbf{w},\widetilde{\theta}_\epsilon-\widetilde{\theta}) - < \widetilde{\boldsymbol{\tau}}_\epsilon-\widetilde{\boldsymbol{\tau}},\mathbf{w} >_{-1,\Gamma} &= 0 \quad \forall \mathbf{w}\in \mathbf{H}^1(\Omega(\alpha))\,, \\ b(\widetilde{\mathbf{q}}_\epsilon-\widetilde{\mathbf{q}},\rho) = \epsilon(\widetilde{\theta}_\epsilon - \widetilde{\theta},\rho)_0 &+ 2\epsilon(\widetilde{p}_\epsilon-\widetilde{p},\rho)_0 + 2\epsilon(\widetilde{p},\rho)_0 \\ &+\epsilon(\widetilde{\theta},\rho)_0 \quad \forall \rho\in L^2_0(\Omega(\alpha))\,, \\ < \mathbf{y},\widetilde{\mathbf{q}}_\epsilon-\widetilde{\mathbf{q}} >_{-1/2,\Gamma} &= 0 \quad \forall \mathbf{y}\in\mathbf{H}^{-1/2}(\Gamma)\,. \end{aligned} \tag{3–116}$$

We note that

$$\begin{aligned} a(\widetilde{\mathbf{q}}_\epsilon-\widetilde{\mathbf{q}},\widetilde{\mathbf{q}}_\epsilon-\widetilde{\mathbf{q}}) &= -b(\widetilde{\mathbf{q}}_\epsilon-\widetilde{\mathbf{q}},\widetilde{\theta}_\epsilon-\widetilde{\theta}) \\ &\le -2\epsilon(\widetilde{p}_\epsilon-\widetilde{p},\widetilde{\theta}_\epsilon-\widetilde{\theta})_0 - 2\epsilon(\widetilde{p},\widetilde{\theta}-\widetilde{\theta})_0 - \epsilon(\widetilde{\theta},\widetilde{\theta}_\epsilon-\widetilde{\theta})_0 \\ &\le 2M_\epsilon\|\widetilde{\theta}_\epsilon-\widetilde{\theta}\|_0\,, \end{aligned}$$

where $M_\epsilon = 2\|\widetilde{p}_\epsilon-\widetilde{p}\|_0 + 2\|\widetilde{p}\|_0 + \|\widetilde{\theta}\|_0 < \infty$. So, from Korn's inequality and the inf–sup condition for $b(\cdot,\cdot)$, we have $\|\widetilde{\mathbf{q}}_\epsilon-\widetilde{\mathbf{q}}\|_0 \le C\epsilon\,M_\epsilon$ and $\|\widetilde{\theta}_\epsilon-\widetilde{\theta}\|_0 \le C^2\epsilon\,M_\epsilon$. Finally, applying Lemma 2.2 to the first equation of (3–116), $\|\widetilde{\boldsymbol{\tau}}_\epsilon - \widetilde{\boldsymbol{\tau}}\|_{-1/2,\Gamma} \leq (C + C^2)\epsilon\, M_\epsilon$. Note that (3–115) is an immediate consequence of Theorem 2.2. □

For a final remark, let us consider the Lagrange multiplier $(\mathbf{q}_\epsilon, \boldsymbol{\eta}_\epsilon)$ defined in (3–36)–(3–38). Since

$$\begin{aligned}
\boldsymbol{\eta}_\epsilon &= 2\nu\,\frac{\partial \mathbf{u}_\epsilon}{\partial \mathbf{n}} - \nu\,\frac{\partial \mathbf{q}_\epsilon}{\partial \mathbf{n}} + r_\epsilon\,\mathbf{n} \\
&= 2\Big(\nu\frac{\partial \mathbf{u}_\epsilon}{\partial \mathbf{n}} - p\,\mathbf{n}\Big) - \Big(\nu\,\frac{\partial \mathbf{q}_\epsilon}{\partial \mathbf{n}} - (r + 2p)\,\mathbf{n}\Big) \\
&= 2\mathbf{t}_\epsilon - \boldsymbol{\tau}_\epsilon\,,
\end{aligned}$$

it follows from Theorem 3.5 that $\|(\mathbf{q}_\epsilon, \boldsymbol{\eta}_\epsilon) - (\mathbf{q}, \boldsymbol{\eta})\|_{\mathbf{H}^1(\Omega(\alpha))\times\mathbf{H}^{-1/2}(\Gamma)}$ tends to 0 as ϵ goes to 0^+.

CHAPTER IV

FINITE ELEMENT APPROXIMATIONS

In this chapter, we are concerned with the finite element approximation of the problem. Finite element discretizations for the domain and discretized description of the problem are given in Section 4.1. Discrete problem can be independently studied by setting the moving boundary nodes as parameters as in [123] or [27]. However, our main concern is how to utilize the design sensitivity analysis we have already derived. Two fundamental steps are composed of evaluation of optimality system using finite elements and the computation of the shape gradient at moving boundary nodes.

In Section 4.2, we discuss the approximation for the optimality system. To deal with inhomogeneous essential boundary condition, we will take the framework of the Lagrange multiplier technique. This method was primarily developed by Babuska[9] for the elliptic equations with constant coefficients and was adapted by several authors on somewhat different situations[10][18][53]. The main point of the implementation is to choose appropriate trial functions to enforce the boundary conditions and to approximate traction forces. To simplify our exposition, we proceed from the idea of Gunzburger *et al.*[67][53]. The main idea is to detach the computation of traction force from the computation of the velocity and pressure.

4.1. Finite Element Approximation

In this section, we shall focus our attention to the finite element discretizations of the problem. Instead of considering technical details, we intend to describe how the problem can be approximated using the standard finite element methods.

4.1.1. Basic Scheme

Based on the continuum approach and optimality criteria, we have derived an optimality system and a formula for the shape gradient. A rather important question at this point is how one can make use of this formula in connection with finite elements in $\Omega(\alpha)$ to build a better open set $\Omega(\alpha^*)$ to reduce the drag of the flow, *i.e.*, we need a design process to build an iterative sequence $\Omega(\alpha_n)$ so that the performance of the objective on $\Omega(\alpha_{n+1})$ is better than that on $\Omega(\alpha^n)$. With regard to extrema, mathematical programming techniques may be employed to solve the discretized model of the problem. Fundamental procedures for mathematical programming are based on the following iterations:

$$\mathbf{d}^{(j+1)} = \mathbf{d}^{(j)} + s_j \mathcal{V}^{(j)} \qquad j = 0, 1, 2, \cdots,$$

where $\mathbf{d}^{(j)}$ represents a design vector, s_j step length and $\mathcal{V}^{(j)}$ stands for a direction of search using the local behavior of the design functional and constraints. The efficiency of this procedure depends on available design conditions and shape sensitivity analyses. Its implementation, however, is not at all a straightforward job. Currently, many finite element systems only permit the analysis of a given structure, without providing any specific information about how to improve the design. Hence finding an efficient method to resolve this system is a rather important question. In this regard, some efficient methods to implement the present sensitivity analysis based on the material derivative method should be developed through additional computational efforts. For this reason, our approach to this problem cannot help but being somewhat restrictive.

The following algorithm based on the steepest descent method can be used as a computational method to find an optimal shape:

Step 1. Set an appropriate initial guess α_0 in $\mathcal{U}_{ad}$.

Step 2. Find a solution $\mathbf{u}_\epsilon(\alpha_n)$ (or $(\mathbf{u}_\epsilon(\alpha_n), p_\epsilon(\alpha_n), \mathbf{t}_\epsilon(\alpha_n))$) for the penalized Navier–Stokes equations (3–77)–(3–78) (or equivalently, (3–94)–(3–96)).

Step 3. Check if $(\alpha_n, \mathbf{u}_\epsilon(\alpha_n))$ is satisfactory to the expected performance. Otherwise, Goto Step 4.

Step 4. Find a solution $\mathbf{q}_\epsilon(\alpha_n)$ (or $(\mathbf{q}_\epsilon(\alpha_n), \theta_\epsilon(\alpha_n), \tau_\epsilon(\alpha_n))$) for the adjoint penalized equations (3–79)–(3–80) (or equivalently, (3–97)–(3–99)).

Step 5. Find a feasible height ϑ_n of perturbation. An appropriate candidate is to compute grad $\mathfrak{J}_\epsilon(\alpha_n)$ using the gradient formula (3–60) and set

$$\vartheta_n = \text{the projection of } \frac{grad\mathfrak{J}_\epsilon(\alpha_n)}{\|grad\mathfrak{J}_\epsilon(\alpha_n)\|} \text{ on the set of admissible controls}.$$

Step 6. Construct the new shape function

$$\alpha_{n+1} = \alpha_n + s_n\vartheta_n \in \mathcal{U}_{ad}\,, \text{for } s_n > 0\,. \tag{4–1}$$

Set n:=n+1 and Goto Step 2.

Note that since $d\mathfrak{J}_\epsilon(\alpha;\vartheta) = -\int_{-M_1}^{M_1}(grad\,\mathfrak{J}_\epsilon(\alpha))\,\vartheta\,dx_1$, in step 5, ϑ_n is taken to satisfy $d\mathfrak{J}_\epsilon(\alpha_n;\vartheta_n) \le 0$ for each n. In step 5, it is not easy to give a precise meaning for the projection. It can be understood to be a regularized process remeding the discontinuity coming from the computation of $grad\,\mathfrak{J}_\epsilon$ at moving nodes in the same sense as that in [109].

4.1.2. Discretization

The computation of $g_\epsilon(\alpha) = grad\, \mathfrak{J}_\epsilon(\alpha)$ involves the computation of the state variable $\mathbf{u}_\epsilon(\alpha)$ and the adjoint $\mathbf{q}_\epsilon(\alpha)$ for $\alpha \in \mathcal{U}_{ad}$. Since both variables are defined on the same domain $\Omega(\alpha)$, the same triangulation of the domain may be used to approximate them. In the discretization of the domain at each step, some strategy should be taken to keep track with the design boundary and to keep the domain from being flipped over. For this purpose, one may discretize the domain in the following manner (one may use it as a sample for the triangulation):

Let $h > 0$ be a parameter constricting the grid size in a triangulation of the domain into finite elements. The domain $\Omega(\alpha)$ is composed of two parts; $\Omega(\alpha) = \Omega_{fix} \cup \Omega_{mov}(\alpha)$ for each $\alpha \in \mathcal{U}_{ad}$, where Ω_{fix} denotes the interior of the rectangles AEBG and FCDH (Figure 2) and $\Omega_{mov}(\alpha) = \Omega(\alpha) - \overline{\Omega_{fix}}$.

Since domain perturbations are restricted only in the variable part, the mesh of Ω_{fix} may be fixed during the iterations. Since we are concerned with the shape of $\Gamma(\alpha^*)$, where α^* is an optimal solution to the problem, particular attention is paid to the variable part. Let $-M_1 = a_0 < a_1 < \cdots < a_{N+1} = M_1$ be a partition of $[-M_1, M_1]$ such that $\max_{1\leq i\leq N+1} |a_i - a_{i-1}| \leq h$. The discretized set of an admissible family can be defined by

$$\begin{aligned}\mathcal{U}_{ad}^h =&\{\alpha_h \in C^0([-M_1, M_1]) \mid \alpha_h\Big|_{[a_{i-1},a_i]} \in \mathcal{P}_1([a_{i-1}, a_i]),\ 0 \leq \alpha_h \leq L,\\ &\Big|\frac{\alpha_h(a_i) - \alpha_h(a_{i-1})}{a_i - a_{i-1}}\Big| \leq \beta, \alpha_h(-M_1) = \alpha_h(M_1) = 0, \quad i = 1 : N+1\},\end{aligned}$$

where $\mathcal{P}_k$ denotes the set of polynomials in x_1 and x_2 of degree $\leq k$. Note that each α_h in $\mathcal{U}_{ad}^h$ is completely characterized by a sequence $\{\alpha_j > 0\}_{1\leq j\leq N}$ of real numbers such that

$$\alpha_h(x_1) = \frac{a_{j+1} - x_1}{a_{j+1} - a_j}\alpha_j + \frac{x_1 - a_j}{a_{j+1} - a_j}\alpha_{j+1}\,, \quad \text{for } x_1 \in [a_j, a_{j+1}]\,,$$

and $\Gamma(\alpha_h)$ by N-nodes $\{(a_j, \alpha_j)\}$. This defines a polygonal domain

$$\Omega_{mov}(\alpha_h) = \{(x_1, x_2) \mid -M_1 < x_1 < M_1, \;\; \alpha_h(x_1) < x_2 < L\}.$$

Since domain changes occur only in the x_2-direction, one may attach the motion of these nodes to the vertical lines connected with fixed top boundary nodes $\{(a_j, L)\}_{1\leq j\leq N}$. We take a partition of x_2-axis into $(K+1)$–subintervals such that $b_0 = 0 < b_1 < \cdots < b_{K+1} = L$ and $\max_{1\leq j\leq K+1} |b_j - b_{j-1}| \leq h$. The triangulation of $\Omega_{mov}(\alpha_h)$ should be chosen not to be too flat. This can be achieved by attaching each node to the fixed nodes and design nodes in the following manner (c.f. Figure 4); Let D_{ij} denote vertices of a triangulation in $\Omega_{mov}(\alpha_h)$. The coordinates are given by

$$D_{ij} = (a_i, \alpha_i + \frac{L - \alpha_i}{M+1} j\,), \quad \text{for } i = 1 : N, \; j = 1 : M\,,$$

i.e., the x_2-coordinate of D_{ij} are given by equidistant partition of the vertical line segment from the design nodes $\{(a_i, \alpha_i)\}_{1\leq i\leq N}$ to the top boudary nodes $\{(a_i, L)\}_{1\leq i\leq N}$.

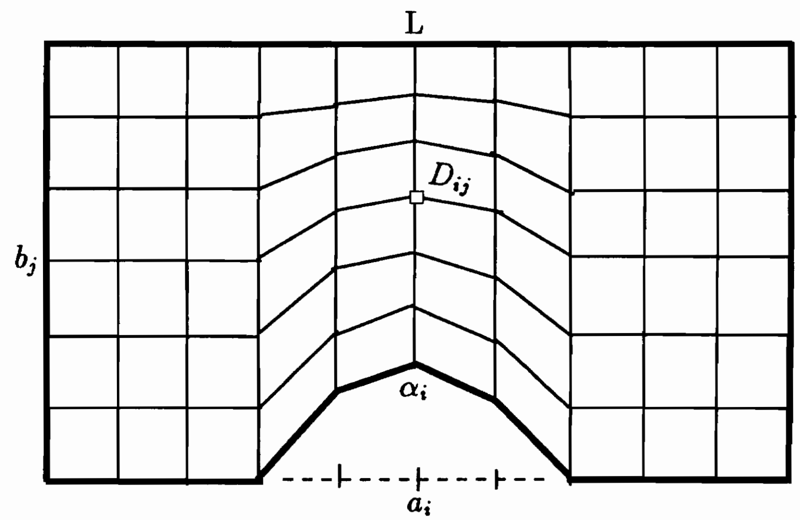

Figure 4: Discretization of domain $\Omega(\alpha_h)$.

Notice that the triangulation has been chosen in such a way that the triangles are not too much distorted.

REMARK 4.1: For each $\alpha \in \mathcal{U}_{ad}$, one can construct $\alpha_h \in \mathcal{U}_{ad}^h$ satisfying

$$\lim_{h \to 0^+} \alpha_h \longrightarrow \alpha \quad \text{uniformly}.$$

For instance, in the equidistant partition of $[-M_1, M_1]$, one can modify the scheme adopted by [12] as follows: Let us define $\alpha_h \in \mathcal{C}^0([-M_1, M_1])$ to be

$$\alpha_h \Big|_{[a_i, a_{i+1}]} \in \mathcal{P}_1 \qquad \forall i = 1 : N$$

and

$$\alpha_h(ih) = \frac{1}{h} \int_{-M_1+(i-\frac{1}{2})h}^{-M_1+(i+\frac{1}{2})h} \alpha(s)\, ds \quad \forall i = 1 : N .$$

Then, it is easy to check that $\alpha_h \in \mathcal{U}_{ad}$ and $\|\alpha_h - \alpha\|_\infty \le Ch$. In fact, since α_h is obtained by its values on partition points, it is sufficient to check that $\alpha_h \in \mathcal{U}_{ad}$.

Once a triangulation of the domain is defined, the finite element approximation of the optimality system (3–94)–(3–100) can be defined in the usual manner. Let us assume that $\alpha^h = \alpha$ so that $\Omega(\alpha^h) = \Omega(\alpha)$. One chooses families of finite dimensional subspaces $\mathbf{V}^h \subset \mathbf{H}^1(\Omega(\alpha_h)) \cap \mathcal{C}^0(\overline{\Omega(\alpha_h)})$ and $S^h \subset L^2(\Omega(\alpha^h))$, which are parametrized by the grid size $h > 0$ tending to zero.

For the velocity–pressure pair, we take $\mathbf{V}^h_{\Gamma_0(\alpha_h)} = \mathbf{V}^h \cap \{\mathbf{v}^h \in \mathbf{V}^h \mid \mathbf{v}^h = \mathbf{0} \text{ on } \Gamma_0(\alpha_h)\}$ and $S^h_0 = S^h \cap L^2_0(\Omega(\alpha_h))$. Also, we define $\mathbf{V}^h_0 = \mathbf{V}^h \cap \mathbf{H}^1_0(\Omega(\alpha_h))$. To represent the trace and the traction force along the boundary, we introduce $\mathbf{P}^h = \gamma_\Gamma(\mathbf{V}^h)$ and $\mathbf{P}^h_{\Gamma_\mathbf{g}} = \gamma_{\Gamma_\mathbf{g}}(\mathbf{V}^h_{\Gamma_0(\alpha_h)})$. Since $\mathbf{P}^h \subset \mathcal{C}^0(\overline{\Gamma})$ and $\mathbf{P}^h_{\Gamma_\mathbf{g}} \subset \mathcal{C}^0(\overline{\Gamma_\mathbf{g}})$, we have $\mathbf{P}^h \subset \mathbf{H}^1(\Gamma)$ and $\mathbf{P}^h_{\Gamma_\mathbf{g}} \subset \mathbf{H}^1(\Gamma_\mathbf{g})$.

Then, the finite element analogy corresponding to the penalized optimality system (3–94)–(3–100) reads:

Seek $(\mathbf{u}_\epsilon^h, p_\epsilon^h, \mathbf{t}_\epsilon^h, \mathbf{q}_\epsilon^h, \theta_\epsilon^h, \boldsymbol{\tau}_\epsilon^h) \in \mathbf{V}^h_{\Gamma_{\mathbf{0}(\alpha_h)}} \times S_0^h \times \mathbf{P}^h_{\Gamma_\mathbf{g}} \times \mathbf{V}^h \times S_0^h \times \mathbf{P}^h$ satisfying

$$\begin{cases} \nu\, a\,(\mathbf{u}_\epsilon^h, \mathbf{v}^h) + b\,(\mathbf{v}^h, p_\epsilon^h) - < \mathbf{t}_\epsilon^h, \mathbf{v}^h >_{\Gamma_\mathbf{g}} \\ \qquad\qquad = < \mathbf{f}, \mathbf{v}^h > - \, c\,(\mathbf{u}_\epsilon^h, \mathbf{u}_\epsilon^h, \mathbf{v}^h) \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h_{\Gamma_{\mathbf{0}(\alpha_h)}}, \\ b\,(\mathbf{u}_\epsilon^h, q^h) = \epsilon\,(p_\epsilon^h, q^h)_0 \quad \forall\, q^h \in S_0^h, \\ < \mathbf{s}^h, \mathbf{u}_\epsilon^h >_{\Gamma_\mathbf{g}} = < \mathbf{s}^h, \mathbf{g} >_{\Gamma_\mathbf{g}} \quad \forall\, \mathbf{s}^h \in \mathbf{P}^h_{\Gamma_\mathbf{g}}, \end{cases} \tag{4–2}$$

and

$$\begin{cases} \nu\, a\,(\mathbf{q}_\epsilon^h, \mathbf{w}^h) + b\,(\mathbf{w}^h, \theta_\epsilon^h) - < \boldsymbol{\tau}_\epsilon^h, \mathbf{w}^h >_\Gamma = 2 < \mathbf{f}, \mathbf{w}^h > -2c\,(\mathbf{u}_\epsilon^h, \mathbf{u}_\epsilon^h, \mathbf{w}^h) \\ \qquad\qquad - c\,(\mathbf{w}^h, \mathbf{u}_\epsilon^h, \mathbf{q}_\epsilon^h) - c\,(\mathbf{u}_\epsilon^h, \mathbf{w}^h, \mathbf{q}_\epsilon^h) \quad \forall\, \mathbf{w}^h \in \mathbf{V}^h, \\ b(\mathbf{q}_\epsilon^h, \rho^h) = \epsilon\,(\theta_\epsilon^h, \rho^h)_0 - 2\epsilon\,(p_\epsilon^h, \rho^h)_0 \quad \forall\, \rho^h \in S_0^h, \\ < \mathbf{s}^h, \mathbf{q}_\epsilon^h >_\Gamma = \mathbf{0} \quad \forall\, \mathbf{s}^h \in \mathbf{P}^h. \end{cases} \tag{4–3}$$

The discrete shape gradient would then be characterized by the function

$$g_\epsilon^h(\Gamma) = \Bigg[\nu \nabla \mathbf{u}_\epsilon^h : \nabla \mathbf{u}_\epsilon^h - \Big(\nu \nabla \mathbf{u}_\epsilon^h : \nabla \mathbf{q}_\epsilon^h - \epsilon p_\epsilon^h(\theta_\epsilon^h - 2p_\epsilon^h)\Big) \\ - \left(\frac{\partial \mathbf{u}_\epsilon^h}{\partial \mathbf{n}^h} \cdot \left(2\nu \frac{\partial \mathbf{u}_\epsilon^h}{\partial \mathbf{n}^h} - \nu \frac{\partial \mathbf{q}_\epsilon^h}{\partial \mathbf{n}^h} + (\theta_\epsilon^h - 2p_\epsilon^h)\mathbf{n}^h\right)\right)\Bigg]_{\Gamma(\alpha^h)}. \tag{4–4}$$

g_ϵ^h may be used to locate the height of the next design points. Then, as described in Section 4.1.1, the speed which defines a direction of steepest descent will be

$$\vartheta^h = -g_\epsilon^h / \|g_\epsilon^h\|_{0,\Gamma(\alpha^h)}. \tag{4–5}$$

However, the computation of g_ϵ^h usually yields a discontinuous design boundary. This follows mainly from the computation of $\nabla \mathbf{u}_\epsilon^h$ and $\mathbf{n}^h$ on $\Gamma(\alpha^h)$. Hence, smoothing out the perturbed boundary in the feasible set in conjunction with the regularity of resulting finite elements is yet to be solved. Even though technical details remained to be resolved, these problems of discontinuties at design points may be remedied by taking the domain smooth enough and by employing finite elements of degree large enough.

Remark 4.2: In this direction, some crude methods in simple situations were suggested by Rousselet[109], which are some sort of *ad hoc* averaging procedure to treat the discontinuity of the perturbed boundary. However, its efficiency cannot be certified until some computational work is done. In Section 3.2, we have specified the design velocity by parametrizing the moving boundary as the graph of a (smooth) curve. One may take another approach to get a continuous perturbed boundary at each steps as in [109]. Let the image of $\Gamma(\alpha^h) = \cup_{i=0}^{n}\Gamma^{(i)}$. Since the shape gradient is represented by the integral form

$$
\begin{aligned}
d\mathfrak{J}_\epsilon(\Omega;\mathcal{V}) &= \int_{\cup_{i=0}^{n}\Gamma^{(i)}} g_\epsilon(\Gamma)\,\mathcal{V}(0)\cdot\mathbf{n}\,d\Gamma \\
&= \sum_i \int_{\Gamma^{(i)}} g_\epsilon(\Gamma^{(i)})\,\mathcal{V}(0)\cdot\mathbf{n}\,d\Gamma\,, \qquad (4\text{–}6)
\end{aligned}
$$

one may exploit the triangulation described in Figure 4. Let $\{A_i = (a_i,\alpha_i)\}_{1\le i\le n}$ denote the design points. By taking the design velocity of the form $\mathcal{V} = (0, \sum_{i=1}^{n} c_i\phi_i)$ along the image of $\Gamma(\alpha)$, where c_i is a real number and ϕ_i denotes a (piecewise linear) shape function such that $\phi_i(A_j) = \delta_{ij}$, one gets $\mathcal{V}(A_i) = (0, c_i)$ and $\mathcal{V}\Big|_{\Gamma^{(i)}} = (0, c_i\phi_i + c_{i+1}\phi_{i+1})$. If we employ the $\mathbf{y}^{(i)}$–coordinate system along $\Gamma^{(i)}$, (4–6) can be written in the form of

$$
d\mathfrak{J}_\epsilon(\Omega,\mathcal{V}) = \sum_i \int_{\Gamma^{(i)}} \widehat{g_\epsilon}(\Gamma^{(i)})(c_i\phi_i + c_{i+1}\phi_{i+1})\, n_2^{(i)}\, d\mathbf{y}^{(i)}\,,
$$

where $\widehat{g_\epsilon}(\Gamma^{(i)})$ denotes the expression of $g_\epsilon(\Gamma^{(i)})$ in $\mathbf{y}^{(i)}$–coordinates and $\mathbf{n}^{(i)} = (n_1^{(i)}, n_2^{(i)})$ the outward normal to $\Gamma^{(i)}$. From this result, the shape gradient will have the form of $(0, \sum_i g_i\phi_i)$, where

$$
g_i = \int_{\Gamma^{(i)}} \widehat{g_\epsilon}(\Gamma^{(i)})\phi_i\, n_2^{(i)}\, d\mathbf{y}^{(i)} + \int_{\Gamma^{(i+1)}} \widehat{g_\epsilon}(\Gamma^{(i+1)})\phi_{i+1}\, n_2^{(i+1)}\, d\mathbf{y}^{(i+1)}\,.
$$

This obviously determines a new boundary which is piecewise linear and continuous (c.f.[109]).

The feasible choice for the speed ϑ^h can be obtained by projecting g_ϵ^h onto $\mathcal{U}_{ad}^h$ in the L^2–sense as follows:

$$\vartheta^h = -\mathcal{P}_{\mathcal{U}^h_{ad}}(g^h_\epsilon)/\|\mathcal{P}_{\mathcal{U}^h_{ad}}(g^h_\epsilon)\|_{0,\Gamma(\alpha^h)}\,, \tag{4–7}$$

where $\mathcal{P}_{\mathcal{U}^h_{ad}}$ denotes the L^2–projection onto $\mathcal{U}^h_{ad}$. Based on the scheme described in Section 4.1.1, this enables one to choose a feasible boundary. By selecting an appropriate step length $s_n > 0$, the shape function will be determined by the iteration corresponding to (4–1) such as

$$\alpha^h_{n+1} = \alpha^h_n + s_n\vartheta^h_n\,. \tag{4–8}$$

Note that if we follow the scheme for the triangulation described earlier, then the whole mesh can be automatically determined. Hence, the major factor to determine the optimal design is to determine the optimality system (or, the stationary conditions) in the appropriate spaces, and to find its solutions.

Finally, let us turn our attention to the penalized optimality system(s). There are two point of views in the penalty approach to the pressure. We first note from the second equation of (4–3) that the discrete pressure can be recovered by the formula $p^h_\epsilon = -\frac{1}{\epsilon}\mathcal{P}_{S^h_0}(div\, \mathbf{u}^h_\epsilon)$, where $\mathcal{P}_{S^h_0}$ denotes the orthogonal projection of L^2 onto S^h_0, *i.e.*, for any $\phi \in L^2(\Omega(\alpha))$, $(\phi - \mathcal{P}_{S^h_0}(\phi), \rho^h)_0 = 0$ for all $q^h \in S^h_0$. Similarly, θ^h_ϵ can be replaced by $\theta^h_\epsilon = -\frac{1}{\epsilon}\mathcal{P}_{S^h_0}(div\, \mathbf{q}^h_\epsilon + 2div\, \mathbf{u}^h_\epsilon)$. Using these terms, we can rewrite (4–3) and (4-4) as

$$\begin{cases} \nu\, a\,(\mathbf{u}^h_\epsilon, \mathbf{v}^h) + \frac{1}{\epsilon}(div\, \mathbf{v}^h, \mathcal{P}_{S^h_0}(div\, \mathbf{u}^h_\epsilon))_0 - < \mathbf{t}^h_\epsilon, \mathbf{v}^h >_{\Gamma_\mathbf{g}} \\ \qquad\qquad = < \mathbf{f}, \mathbf{v}^h > - c\,(\mathbf{u}^h_\epsilon, \mathbf{u}^h_\epsilon, \mathbf{v}^h)_0 \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h_{\Gamma_0(\alpha_h)}\,, \\ < \mathbf{s}^h, \mathbf{u}^h_\epsilon >_{\Gamma_\mathbf{g}} = < \mathbf{s}^h, \mathbf{g} >_{\Gamma_\mathbf{g}} \quad \forall\, \mathbf{s}^h \in \mathbf{P}^h_{\Gamma_\mathbf{g}}\,, \end{cases} \tag{4–9}$$

and

$$\begin{cases} \nu\, a\,(\mathbf{q}^h_\epsilon, \mathbf{w}^h) + \frac{1}{\epsilon}(div\, \mathbf{w}^h, \mathcal{P}_{S^h_0}(div\, \mathbf{q}^h_\epsilon + 2div\, \mathbf{u}^h_\epsilon))_0 - < \tau^h_\epsilon, \mathbf{w}^h >_\Gamma = 2 < \mathbf{f}, \mathbf{w}^h > \\ \quad -2c\,(\mathbf{u}^h_\epsilon, \mathbf{u}^h_\epsilon, \mathbf{w}^h) - c\,(\mathbf{w}^h, \mathbf{u}^h_\epsilon, \mathbf{q}^h_\epsilon) - c\,(\mathbf{u}^h_\epsilon, \mathbf{w}^h, \mathbf{q}^h_\epsilon) \ \ \forall\, \mathbf{w}^h \in \mathbf{V}^h\,, \\ < \mathbf{s}^h, \mathbf{q}^h_\epsilon >_\Gamma = 0 \quad \forall\, \mathbf{s}^h \in \mathbf{P}^h\,. \end{cases} \tag{4–10}$$

In this case, the corresponding shape gradient will be represented by the form:

$$g_\epsilon^h(\Gamma) = \Big[\nu\nabla\mathbf{u}_\epsilon^h : \nabla\mathbf{u}_\epsilon^h - \Big(\nu\nabla\mathbf{u}_\epsilon^h : \nabla\mathbf{q}_\epsilon^h + \frac{1}{\epsilon}\mathcal{P}_{S_0^h}(div\mathbf{u}_\epsilon^h)\mathcal{P}_{S_0^h}(div\mathbf{q}_\epsilon^h)\Big) - \Big(\frac{\partial\mathbf{u}_\epsilon^h}{\partial\mathbf{n}^h}\cdot\Big(2\nu\frac{\partial\mathbf{u}_\epsilon^h}{\partial\mathbf{n}^h} - \nu\frac{\partial\mathbf{q}_\epsilon^h}{\partial\mathbf{n}^h} + \mathcal{P}_{S_0^h}(div\mathbf{q}_\epsilon^h)\mathbf{n}^h\Big)\Big)\Big]_{\Gamma(\alpha^h)}. \tag{4–11}$$

On the other hand, direct discretization of the optimality system for the continuity penalty method yields

$$\begin{cases} \nu\, a\,(\mathbf{u}_\epsilon^h,\mathbf{v}^h) + \frac{1}{\epsilon}(div\,\mathbf{v}^h, div\,\mathbf{u}_\epsilon^h)_0 - <\mathbf{t}_\epsilon^h,\mathbf{v}^h>_{\Gamma_\mathbf{g}} \\ \qquad\qquad = <\mathbf{f},\mathbf{v}^h> - c\,(\mathbf{u}_\epsilon^h,\mathbf{u}_\epsilon^h,\mathbf{v}^h) \quad \forall\,\mathbf{v}^h \in \mathbf{V}^h_{\Gamma_0(\alpha_h)}, \\ <\mathbf{s}^h,\mathbf{u}_\epsilon^h>_{\Gamma_\mathbf{g}} = <\mathbf{s}^h,\mathbf{g}>_{\Gamma_\mathbf{g}} \quad \forall\,\mathbf{s}^h \in \mathbf{P}^h_{\Gamma_\mathbf{g}}, \end{cases} \tag{4–12}$$

and

$$\begin{cases} \nu\, a\,(\mathbf{q}_\epsilon^h,\mathbf{w}^h) + \frac{1}{\epsilon}(div\,\mathbf{w}^h, div\,\mathbf{q}_\epsilon^h + 2div\,\mathbf{u}_\epsilon^h)_0 - <\tau_\epsilon^h,\mathbf{w}^h>_\Gamma = 2<\mathbf{f},\mathbf{w}^h> \\ -2c\,(\mathbf{u}_\epsilon^h,\mathbf{u}_\epsilon^h,\mathbf{w}^h) - c\,(\mathbf{w}^h,\mathbf{u}_\epsilon^h,\mathbf{q}_\epsilon^h) - c\,(\mathbf{u}_\epsilon^h,\mathbf{w}^h,\mathbf{q}_\epsilon^h) \quad \forall\,\mathbf{w}^h \in \mathbf{V}^h, \\ <\mathbf{s}^h,\mathbf{q}_\epsilon^h>_\Gamma = 0 \quad \forall\,\mathbf{s}^h \in \mathbf{P}^h. \end{cases} \tag{4–13}$$

The discrete pressure is recovered by the form of $p_\epsilon^h\Big|_\mathcal{T} = -\frac{1}{\epsilon}div(\mathbf{u}_\epsilon^h\Big|_\mathcal{T})$ and its costate variable will be written by $\theta_\epsilon^h\Big|_\mathcal{T} = -\frac{1}{\epsilon}div\,\Big[(\mathbf{q}_\epsilon^h + 2\mathbf{u}_\epsilon^h)\Big|_\mathcal{T}\Big]$, where $\mathcal{T}$ denotes any element in the subdivision of $\Omega(\alpha^h)$ into finite elements. Since $\mathbf{q}_\epsilon^h$ is virtually sought in the space $\mathbf{V}_0^h(\Omega(\alpha^h)) \subset \mathbf{H}_0^1(\Omega(\alpha^h))$ and $\mathbf{g} \in \mathbf{H}_0^{1/2}(\Gamma_\mathbf{g})$ and $\mathbf{u}_\epsilon^h \in \mathbf{H}^1_{\Gamma_0(\alpha^h)}(\Omega(\alpha^h))$, it follows from the divergence theorem that p_ϵ^h and θ_ϵ^h belong to $S_0^h \subset L_0^2(\Omega(\alpha^h))$. Hence the pressure and its corresponding costate variable are implicitly defined in the $S_0^h \subset L_0^2(\Omega(\alpha^h))$. Obviously, both approaches provide some flexible choice for the pressure space while relaxing the choice among elements satisfying the discrete incompressibility constraints. However, the formal approach is preferred since it gives more flexibility in the choice of the pressure space, especially in the nonconforming approximation of the pressure (c.f.[84]).

The discrepancies between these two approaches have been observed in Cuvelier *et al.*[36], Sani *et al.*[113][114] and Gunzburger[64]. The formal approach (what is called *the discrete penalty method*) is to detach the pressure from the velocity after discretization while the latter (*the continuous penalty method*) detaching the pressure before discretization. Aside from this apparent distinction, there are different characteristics that are hidden until they are expressed in the matrix form (see, [36] and [64], for details).

In the numerical imprementation of the scheme, one should reflect these features on achieving improvement of the design performance.

4.2. Approximation to the Optimality System

In this section, we are mainly concerned with the approximation to the optimality system in the finite element framework. The major difficulties in the approximation follow from the existence of inhomogeneous boundary data $\mathbf{g}$, the nonlineality of the system and the penalized approximation to the pressure.

4.2.1. Split Formulation for the Traction Force

The space $\mathbf{H}^1_{\Gamma(\alpha)}(\Omega(\alpha))$ for the velocity appeared in the consideration of the physical boundary state and played a crucial role in showing the existence of the optimal shape design in Theorem 2.4, while it provides the traction force along the part $\Gamma_{\mathbf{g}}$ of the boundary. As we have already indicated, the traction force is an important physical quantity representing the forces exerted by the flow along the boundary. Even though this appearance raises additional computational efforts toward the computation for the traction force, it makes the system stable and physically meaningful. In the Dirichlet problem, this physical factor appeared in an effort to find an efficient method to implement the inhomogeneous essential boundary condition (Babuska[9] used the Lagrange multiplier technique to overcome the difficulty of finding stable finite element approximations satisfying the Dirichlet boundary data). In connection with the finite element methods and the variational principles, this appearance naturally led us to coupling the pressure and the traction force, which are Lagrange multipliers arising from the incompressibility constraint and the inhomogeneous boundary condition (see p.35 – p.37). Several different approaches have been studied to incorporate the traction force (or, the *boundary flux* specifically). Using the approximation of the boundary flux, Bramble[18] decomposed the Dirichlet problem with the inhomogeneous boundary condition into equations with natural boundary condition. In this case, the boundary flux was considered as a parameter to determine the velocity of the flow. In the consideration of the slip boundary condition, Verfürth[133] coupled the traction force with the pressure to attain the stability of the resulting mixed formulation. In our case, however, since the computation of the pressure can be detached from the velocity using the penalized formulation, it is natural to consider a method that uncouples the traction force from the others. This method of uncoupling the traction force was studied by Gunzburger *et al.*[67] (see also [53]). Fundamental steps include the suitable choice for the boundary interpolation for the boundary data and the decoupling of the traction force from the velocity.

To avoid unwanted irregularities in the boundary data, we assume $\mathbf{g}$ has a compact support in $\Gamma_{\mathbf{g}}$. We also assume $\alpha^h = \alpha$ for the polygonal choice for the feasible set. For the choice of the boundary interpolation, we first notice from the third equation of (4–8) that $< \mathbf{s}^h, \mathbf{u}_\epsilon^h - \mathbf{g} >_{1/2,\Gamma_{\mathbf{g}}} = 0$ for all $\mathbf{s}^h \in \mathbf{P}^h_{\Gamma_{\mathbf{g}}}$, *i.e.* $\mathbf{u}_\epsilon^h$ cannot exactly approximate the boundary data $\mathbf{g}$ along $\Gamma_{\mathbf{g}}$ by merely taking $\mathbf{u}_\epsilon^h\big|_{\Gamma_{\mathbf{g}}}$. This in general spoils the accuracy for the approximation. To circumvent it, we take $\mathbf{g}^h$ as the $\mathbf{L}^2(\Gamma_{\mathbf{g}})$–

projection of $\mathbf{g}$, *i.e.*, $\mathbf{g}^h = \mathcal{P}^h_{\Gamma_\mathbf{g}}(\mathbf{g})$, where $\mathcal{P}^h_{\Gamma_\mathbf{g}}$ denotes the $\mathbf{L}^2(\Gamma_\mathbf{g})$–projection from $\mathbf{H}^{1/2}(\Gamma_\mathbf{g})$ onto $\mathbf{P}^h_{\Gamma_\mathbf{g}}$ (c.f. Remark 4.4). Obviously, we have that

$$\int_{\Gamma_\mathbf{g}} (\mathcal{I} - \mathcal{P}^h_{\Gamma_\mathbf{g}})\, \mathbf{g} \cdot \mathcal{P}^h_{\Gamma_\mathbf{g}}(\gamma_{\Gamma_\mathbf{g}} \mathbf{v})\, d\Gamma = 0\,, \quad \forall\, \mathbf{v} \in \mathbf{H}^1_{\Gamma_{0(\alpha_h)}}(\Omega(\alpha^h))\,. \tag{4–14}$$

Before presenting the computational procedure, we make some assumptions for the approximation properties in the choice of spaces $\mathbf{V}^h$, S^h and $\mathbf{P}^h$. One may choose any pair of subspaces $\mathbf{V}^h$ and S^h that can be used to the finite element approximation of the Navier–Stokes equations. We first notice that since $\mathbf{V}^h_{\Gamma_{0(\alpha_h)}} \subset \mathbf{H}^1_{\Gamma_{0(\alpha_h)}}(\Omega(\alpha^h))$, $a(\cdot,\cdot)\Big|_{\mathbf{V}^h_{\Gamma_{0(\alpha_h)}} \times \mathbf{V}^h_{\Gamma_{0(\alpha_h)}}}$ is continuous and coercive. In order to attain the stability and convergence properties of the approximation (4–2) to the solution of the Navier–Stokes equations, we make use of a general result on the mixed formulations (4–2) known as *inf–sup condition* (or *LBB condition*): there exists a constant C which is independent of h such that

$$\inf_{0 \neq q^h \in S^h_0} \sup_{0 \neq \mathbf{v}^h \in \mathbf{V}^h_0} \frac{b(\mathbf{v}^h, q^h)}{\|\mathbf{v}^h\|_1 \|q^h\|_0} \geq C\,. \tag{4–15}$$

In the mixed formulation, this condition is needed to keep the balance between velocity and pressure, whence to allow the stability of the scheme. This assumption is rather stronger. In practical situations, the generic constant C of (4–15) can be taken to depend on the size of the mesh. Even in such cases, the inf–sup condition may still imply the convergence of the chosen elements, provided that the infimum of the constant $C(h)$ decays to zero not too fast (see, *e.g.*, [84]).

For general applications, we assume that $\mathbf{V}^h$ and S^h satisfy the following standard approximation properties: there exists an integer k and a constant C which is independent of h, $\mathbf{v}$ and q such that

$$\inf_{q^h \in S^h_0} \|q - q^h\|_0 \leq C h^{m-1} \|q\|_{m-1} \quad \forall\, q \in H^{m-1}(\Omega(\alpha^h)) \cap L^2_0(\Omega(\alpha^h)) \;\; 1 \leq m \leq k\,, \tag{4–16}$$

$$\inf_{\mathbf{v}^h\in\mathbf{V}^h} \|\mathbf{v}-\mathbf{v}^h\|_s \le C\, h^{m-s}\|\mathbf{v}\|_m \quad \forall\,\mathbf{v}\in\mathbf{H}^m(\Omega(\alpha_h)),\ s=0,1,\ 1\le m\le k\,, \tag{4–17}$$

and

$$\inf_{\mathbf{v}^h\in\mathbf{V}^h_{\Gamma_0(\alpha_h)}} \|\mathbf{v}-\mathbf{v}^h\|_s \le C\, h^{m-s}\|\mathbf{v}\|_m \quad \forall\,\mathbf{v}\in\mathbf{H}^m\cap\mathbf{H}^1_{\Gamma_0(\alpha_h)}(\Omega(\alpha_h)),\ s=0,1,\ 1\le m\le k\,. \tag{4–18}$$

Moreover, for $\mathbf{P}^h \subset \mathbf{H}^1(\Gamma)$ and $\mathbf{P}^h_{\Gamma_\mathbf{g}} \subset \mathbf{H}^1(\Gamma_\mathbf{g})$, we assume the following inverse inequalities:

$$\|\boldsymbol{\xi}^h\|_{s,\Gamma_\mathbf{g}} \le Ch^{t-s}\|\boldsymbol{\xi}^h\|_{t,\Gamma_\mathbf{g}} \quad \forall\,\boldsymbol{\xi}^h\in\mathbf{P}^h_{\Gamma_\mathbf{g}}\,,\ -1/2\le t\le s\le 1 \tag{4–19}$$

and

$$\|\boldsymbol{\xi}^h\|_{s,\Gamma} \le Ch^{t-s}\|\boldsymbol{\xi}^h\|_{t,\Gamma} \quad \forall\,\boldsymbol{\xi}^h\in\mathbf{P}^h_{\Gamma},\ -1/2\le t\le s\le 1\,, \tag{4–20}$$

where C is independent of h and $\boldsymbol{\xi}^h$. In (4–16)–(4–18), the integer k is related to the degree of the polynomial approximation.

Some remarks are in order for the use of (4–15) and (4–16). In the penalized formulation such as (4–13), the compatibility condition (4–15) between velocity and pressure rules out spurious modes such as the checkerboard modes in the computation of the pressure. Hence, as long as only the velocity is concerned, it plays no role in the stability or in the accuracy. In this case, the assumption (4–16) may be redundant, for it is surbodinate to the assumption (4–17) or (4–18).

Based on these structure, one can split the system (4–2)–(4–3) in the following manner (**split formulation**):

- Given $\mathbf{g}\in\mathbf{H}^{1/2}(\Gamma_\mathbf{g})$, evaluate $\mathbf{g}^h = \mathcal{P}^h_{\Gamma_\mathbf{g}}(\mathbf{g})$. (4–21)
- Solve for $(\mathbf{u}^h_\epsilon, p^h_\epsilon, \mathbf{q}^h_\epsilon, \theta^h_\epsilon)\in\mathbf{V}^h_{\Gamma_0(\alpha_h)}\times S^h_0\times\mathbf{V}^h\times S^h_0$ such that

$$\begin{cases} \nu\, a\,(\mathbf{u}^h_\epsilon,\mathbf{v}^h) + b\,(\mathbf{v}^h,p^h_\epsilon) = <\mathbf{f},\mathbf{v}^h> - c\,(\mathbf{u}^h_\epsilon,\mathbf{u}^h_\epsilon,\mathbf{v}^h) \quad \forall\,\mathbf{v}^h\in\mathbf{V}^h_0\,,\\ b\,(\mathbf{u}^h_\epsilon,q^h) = \epsilon\,(p^h_\epsilon,q^h) \quad \forall\,q^h\in S^h_0\,,\\ <\mathbf{s}^h,\mathbf{u}^h_\epsilon>_{\Gamma_\mathbf{g}} = <\mathbf{s}^h,\mathbf{g}^h>_{\Gamma_\mathbf{g}} \quad \forall\,\mathbf{s}^h\in\mathbf{P}^h_{\Gamma_\mathbf{g}}\,, \end{cases} \tag{4–22}$$

and

$$\left\{\begin{array}{l} \nu\, a\,(\mathbf{q}_\epsilon^h, \mathbf{w}^h) + b\,(\mathbf{w}^h, \theta_\epsilon^h) = 2 < \mathbf{f}, \mathbf{w}^h > -2c\,(\mathbf{u}_\epsilon^h, \mathbf{u}_\epsilon^h, \mathbf{w}^h) \\ \qquad\qquad\qquad -c\,(\mathbf{w}^h, \mathbf{u}_\epsilon^h, \mathbf{q}_\epsilon^h) - c\,(\mathbf{u}_\epsilon^h, \mathbf{w}^h, \mathbf{q}_\epsilon^h) \;\; \forall\, \mathbf{w}^h \in \mathbf{V}_0^h\,, \\ b\,(\mathbf{q}_\epsilon^h, \rho^h) = \epsilon\,(\theta_\epsilon^h, \rho^h) - 2\epsilon\,(p_\epsilon^h, \rho^h) \;\; \forall\, \rho^h \in S_0^h\,, \\ < \mathbf{s}^h, \mathbf{q}_\epsilon^h >_\Gamma = 0 \quad \forall\, \mathbf{s}^h \in \mathbf{P}^h\,. \end{array}\right. \tag{4–23}$$

– Solve for $\mathbf{t}_\epsilon^h \in \mathbf{P}_{\Gamma_\mathbf{g}}^h$ and $\boldsymbol{\tau}_\epsilon^h \in \mathbf{P}^h$ such that

$$< \mathbf{t}_\epsilon^h, \mathbf{v}^h >_{\Gamma_\mathbf{g}} = \nu a\,(\mathbf{u}_\epsilon^h, \mathbf{v}^h) + b\,(\mathbf{v}^h, p_\epsilon^h) + c\,(\mathbf{u}_\epsilon^h, \mathbf{u}_\epsilon^h, \mathbf{v}^h)$$

$$- < \mathbf{f}, \mathbf{v}^h > \quad \forall\, \mathbf{v}^h \in \mathbf{V}_{\Gamma_0(\alpha_h)}^h - \mathbf{V}_0^h\,, \tag{4–24}$$

and

$$< \boldsymbol{\tau}_\epsilon^h, \mathbf{w}^h > = \nu a\,(\mathbf{q}_\epsilon^h, \mathbf{w}^h) + b\,(\mathbf{w}^h, \theta_\epsilon^h) + 2c\,(\mathbf{u}_\epsilon^h, \mathbf{u}_\epsilon^h, \mathbf{w}^h) + c\,(\mathbf{w}^h, \mathbf{u}_\epsilon^h, \mathbf{q}_\epsilon^h)$$

$$+ c\,(\mathbf{u}_\epsilon^h, \mathbf{w}^h, \mathbf{q}_\epsilon^h) - 2 < \mathbf{f}, \mathbf{w}^h > \quad \forall\, \mathbf{w}^h \in \mathbf{V}^h - \mathbf{V}_0^h\,. \tag{4–25}$$

It is interesting that, despite the traction force not being a boundary condition for the problem considered, a natural postprocessing procedure for the traction force nevertheless ensues.

Note that using this split formulation, the weak primal version (3–101)–(3–107) can be rewritten in the form:

– Solve for $(\mathbf{u}, p, \mathbf{q}, \theta) \in \mathbf{H}_{\Gamma_0(\alpha_h)}^1 \times L_0^2(\Omega(\alpha^h)) \times \mathbf{H}^1(\Omega(\alpha^h)) \times L_0^2(\Omega(\alpha^h))$ such that

$$\left\{\begin{array}{l} \nu\, a\,(\mathbf{u}, \mathbf{v}) + b\,(\mathbf{v}, p) = < \mathbf{f}, \mathbf{v} >_{-1} - c\,(\mathbf{u}, \mathbf{u}, \mathbf{v}) \;\; \forall\, \mathbf{v} \in \mathbf{H}_0^1(\Omega(\alpha^h))\,, \\ b\,(\mathbf{u}, q) = 0 \;\; \forall\, q \in L_0^2(\Omega(\alpha^h))\,, \\ < \mathbf{s}, \mathbf{u} >_{\Gamma_\mathbf{g}} = < \mathbf{s}, \mathbf{g} >_{\Gamma_\mathbf{g}} \quad \forall\, \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_\mathbf{g})\,, \end{array}\right. \tag{4–26}$$

and

$$\left\{\begin{array}{l} \nu\, a\,(\mathbf{q}, \mathbf{w}) + b\,(\mathbf{w}, \theta) = 2 < \mathbf{f}, \mathbf{w} >_{-1} -2c\,(\mathbf{u}, \mathbf{u}, \mathbf{w}) \\ \qquad\qquad\qquad -c\,(\mathbf{w}, \mathbf{u}, \mathbf{q}) - c\,(\mathbf{u}, \mathbf{w}, \mathbf{q}) \;\; \forall\, \mathbf{w} \in \mathbf{H}_0^1(\Omega(\alpha^h))\,, \\ b\,(\mathbf{q}, \rho) = 0 \;\; \forall\, \rho \in L_0^2(\Omega(\alpha^h))\,, \\ < \mathbf{s}, \mathbf{q} >_\Gamma = \mathbf{0} \quad \forall\, \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma)\,. \end{array}\right. \tag{4–27}$$

– Solve for $\mathbf{t} \in \mathbf{H}^{-1/2}(\Gamma_\mathbf{g})$ and $\boldsymbol{\tau} \in \mathbf{H}^{-1/2}(\Gamma)$ such that

$$< \mathbf{t}, \mathbf{v} >_{\Gamma_\mathbf{g}} = \; \nu a\,(\mathbf{u}, \mathbf{v}) + b\,(\mathbf{v}, p) + c\,(\mathbf{u}, \mathbf{u}, \mathbf{v})$$

$$- < \mathbf{f}, \mathbf{v} >_{-1} \quad \forall\, \mathbf{v} \in \mathbf{H}_{\Gamma_0(\alpha_h)}^1(\Omega(\alpha^h)) - \mathbf{H}_0^1(\Omega(\alpha^h))\,, \tag{4–28}$$

and

$$< \boldsymbol{\tau}, \mathbf{w} >_\Gamma = \nu a(\mathbf{q}, \mathbf{w}) + b(\mathbf{w}, \theta) + 2c(\mathbf{u}, \mathbf{u}, \mathbf{w}) + c(\mathbf{w}, \mathbf{u}, \mathbf{q})$$

$$+ c(\mathbf{u}, \mathbf{w}, \mathbf{q}) - 2 < \mathbf{f}, \mathbf{w} >_{-1} \quad \forall \mathbf{w} \in \mathbf{H}^1(\Omega(\alpha^h)) - \mathbf{H}_0^1(\Omega(\alpha^h)) . \quad (4\text{–}29)$$

Our aim is to derive error estimates for this scheme.

4.2.2. Error Estimates

In this subsection, we are concerned with error estimates for the proposed split formulation. Since the stated approximation properties are fulfilled for the familiar regular finite elements (see, for example, [33][17][132]), the accuracy of the scheme mainly depends on how good an approximation $\mathbf{g}^h$ is to $\mathbf{g}$. For this purpose, we introduce the $\mathbf{H}^1_{\Gamma_{0(\alpha_h)}}$–projection $\mathcal{Q}^h$ from $\mathbf{H}^1_{\Gamma_{0(\alpha_h)}}(\Omega(\alpha^h))$ onto $\mathbf{V}^h_{\Gamma_{0(\alpha_h)}}$, *i.e.*, for $\mathbf{w} \in \mathbf{H}^1_{\Gamma_{0(\alpha_h)}}(\Omega(\alpha^h))$,

$$(((\mathcal{I} - \mathcal{Q}^h)\mathbf{w}, \mathbf{v}^h))_1 = a((\mathcal{I} - \mathcal{Q}^h)\mathbf{w}, \mathbf{v}^h) = 0 \quad \forall \mathbf{v}^h \in \mathbf{V}^h_{\Gamma_{0(\alpha_h)}} . \quad (4\text{–}30)$$

We need some preliminary results for the approximation to $\mathbf{g}$.

Lemma 4.1. *There exists a constant $C > 0$ such that*

$$\|\gamma_{\Gamma_g} \mathbf{v}\|^2_{0,\Gamma_g} \le C(\delta^{1/2} \|\!|\mathbf{v}|\!\|^2 + \delta^{-1/2} \|\mathbf{v}\|_0^2) , \quad (4\text{–}31)$$

for all $\mathbf{v} \in \mathbf{H}^1_{\Gamma_{0(\alpha_h)}}(\Omega(\alpha^h))$ and $0 < \delta < 1$.

Proof: This is an immediate consequence of Grisvard[63] (Theorem 1.5.1.10) and the continuity of the trace mapping. □

Lemma 4.2. *Let $\mathbf{w}$ be an element of $\mathbf{H}^1_{\Gamma_{0(\alpha_h)}}(\Omega(\alpha^h))$ with $\gamma_{\Gamma_g} \mathbf{w} = \mathbf{g}$. Then*

$$\left\| \gamma_{\Gamma_g} \Big((\mathcal{I} - \mathcal{Q}^h) \mathbf{w} \Big) \right\|_{0,\Gamma_g} \le C h^{1/2} \|(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}\|_1 . \quad (4\text{–}32)$$

Proof: For the proof, one may recourse to the so-called *Aubin–Nitsche trick* (see [33] or [98]). Let $\mathbf{e} = \mathbf{w} - \mathcal{Q}^h\mathbf{w}$. Since $\mathbf{V}^h_{\Gamma_{0}(\alpha_h)} \subset \mathbf{H}_{\Gamma_{0}(\alpha_h)}(\Omega(\alpha^h))$, $\mathbf{e} \in \mathbf{H}^1_{\Gamma_{0}(\alpha_h)}(\Omega(\alpha^h))$. Let us assume $\boldsymbol{\xi} \in \mathbf{H}^2 \cap \mathbf{H}^1_{\Gamma_{0}(\alpha_h)}(\Omega(\alpha^h))$ be the solution of the following auxiliary problem:

$$a(\boldsymbol{\xi}, \mathbf{v}) = (\mathbf{z}, \mathbf{v})_0 \quad \forall \mathbf{v} \in \mathbf{H}^1_{\Gamma_{0}(\alpha_h)}(\Omega(\alpha^h)), \tag{4–33}$$

where $\mathbf{z} \in \mathbf{H}^{-1}_{\Gamma_{0}(\alpha_h)}(\Omega(\alpha^h)) \cap \mathbf{L}^2$ is given. The regularity of (4–33) yields $\|\boldsymbol{\xi}\|_2 \leq C\|\mathbf{z}\|_0$. Now let us take $\mathbf{v} = \mathbf{e}$ in (4–32). Then, using the symmetry of $a(\cdot,\cdot)$ and (4–30), we have

$$\begin{aligned}\|\mathbf{e}\|_0 &= \sup_{\mathbf{z}} \frac{(\mathbf{z},\mathbf{e})_0}{\|\mathbf{z}\|_0}\\ &= \sup_{\mathbf{z}} \frac{a(\boldsymbol{\xi},\mathbf{e})}{\|\mathbf{z}\|_0}\\ &= \sup_{\mathbf{z}} \frac{a(\boldsymbol{\xi} - \Pi_h\boldsymbol{\xi},\mathbf{e})}{\|\mathbf{z}\|_0},\end{aligned}$$

where $\Pi_h : \mathbf{H}^1_{\Gamma_{0}(\alpha_h)}(\Omega(\alpha^h)) \to \mathbf{V}^h_{\Gamma_{0}(\alpha_h)}$ denotes the $\mathbf{V}^h_{\Gamma_{0}(\alpha_h)}$–interpolant. Hence,

$$\|\mathbf{e}\|_0 \leq C|||\boldsymbol{\xi} - \Pi_h\boldsymbol{\xi}||| \sup_{\mathbf{z}} \frac{|||\mathbf{e}|||}{\|\mathbf{z}\|_0}.$$

From the approximation assumption (4–18), we have

$$|||\boldsymbol{\xi} - \Pi_h\boldsymbol{\xi}||| \leq \|\boldsymbol{\xi} - \Pi_h\boldsymbol{\xi}\|_1 \leq Ch\|\boldsymbol{\xi}\|_2 \leq Ch\|\mathbf{z}\|_0,$$

whence

$$\|\mathbf{e}\|_0 = \|(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}\|_0 \leq Ch|||(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}||| \leq Ch\|(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}\|_1.$$

Applying the inequality (4–31) by taking $\delta = h^2$, it follows that

$$\left\|\gamma_{\Gamma_g}\left((\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\right)\right\|^2_{0,\Gamma_g} \leq C\left(h|||(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}|||^2 + h^{-1}\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_0^2\right) \leq Ch\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1^2.$$

□

REMARK 4.3: In the auxiliary problem (4–33), we assumed the $\mathbf{H}^2$–regularity for the solution $\boldsymbol{\xi}$. On a convex domain with $\mathcal{C}^{0,1}$–boundary, the elliptic problem always guarantees such an order of regularity by the classical regularity theory ([63]). For a nonconvex domain, however, this may not hold. One may get at most $\mathbf{H}^{1+\rho}$–regularity for some $0 < \rho < 1$. Nevertheless, the estimate (4–32) does not change, for (4–31) holds for all $\delta \in (0,1)$, *i.e.*, the minor adjusting of δ and the index of h will yield the same conclusion.

The preliminary estimates for the computation (4–21) can be obtained by refining the result of Lemma 4.2, using the inverse inequality (4–19). Our approach is illustrated in the following (non–commutative) diagram:

$$\begin{array}{ccc}
\mathbf{H}^1_{\Gamma_{\mathrm{O}(\alpha_h)}}(\Omega(\alpha^h)) & \xrightarrow{\mathcal{Q}^h} & \mathbf{V}^h_{\Gamma_{\mathrm{O}(\alpha_h)}} \\
\gamma_{\Gamma_{\mathbf{g}}}\big\downarrow & & \big\downarrow\gamma_{\Gamma_{\mathbf{g}}} \\
\mathbf{H}^{1/2}(\Gamma_{\mathbf{g}}) \subset \mathbf{L}^2(\Gamma_{\mathbf{g}}) & \xrightarrow{\mathcal{P}^h_{\Gamma_{\mathbf{g}}}} & \mathbf{P}^h_{\Gamma_{\mathbf{g}}}
\end{array}$$

Let $\mathbf{w}$ be an element of $\mathbf{H}^1_{\Gamma_{\mathrm{O}(\alpha_h)}}(\Omega(\alpha^h))$ such that $\gamma_{\Gamma_{\mathbf{g}}}\mathbf{w} = \mathbf{g}$. Then, preliminary result includes in the estimation of $\mathbf{g} - \mathcal{P}^h_{\Gamma_{\mathbf{g}}}\mathbf{g}$ in terms of $(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}$ (see also [18][17]). Once the basis of the discrete space $\mathbf{P}^h_{\Gamma_{\mathbf{g}}}$ is known, it is easy to build the approximation $\mathbf{g}^h$.

Lemma 4.3. *For a given* $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma_{\mathbf{g}})$, *let* $\mathbf{g}^h = \mathcal{P}^h_{\Gamma_{\mathbf{g}}}\mathbf{g} \in \mathbf{P}^h_{\Gamma_{\mathbf{g}}}$. *It holds that*

$$\|\mathbf{g} - \mathbf{g}^h\|_{0,\Gamma_{\mathbf{g}}} \leq Ch^{1/2}\|(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}\|_1\,, \tag{4–34}$$

$$\|\mathbf{g} - \mathbf{g}^h\|_{-1/2,\Gamma_{\mathbf{g}}} \leq Ch\|(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}\|_1 \tag{4–35}$$

and

$$\|\mathbf{g} - \mathbf{g}^h\|_{1/2,\Gamma_{\mathbf{g}}} \leq C\|(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}\|_1\,. \tag{4–36}$$

Proof: Note that $\int_{\Gamma_{\mathbf{g}}} (\mathbf{g} - \mathbf{g}^h) \cdot \boldsymbol{\phi}^h \, d\Gamma = 0 \quad \forall \boldsymbol{\phi}^h \in \mathbf{P}^h_{\Gamma_{\mathbf{g}}}$ by (4–14). Since $\mathbf{w} \in \mathbf{H}^1_{\Gamma_{0(\alpha_h)}}(\Omega(\alpha^h))$ is chosen so that $\gamma_{\Gamma_{\mathbf{g}}}\mathbf{w} = \mathbf{g}$ and $\gamma_{\Gamma_{\mathbf{g}}}(\mathcal{Q}^h\mathbf{w}) = \mathbf{g}^h \in \mathbf{P}^h_{\Gamma_{\mathbf{g}}}$, we have

$$\begin{aligned}
\|\mathbf{g} - \mathbf{g}^h\|^2_{0,\Gamma_{\mathbf{g}}} &= \int_{\Gamma_{\mathbf{g}}} (\mathbf{g} - \mathbf{g}^h) \cdot (\mathbf{g} - \gamma_{\Gamma_{\mathbf{g}}}(\mathcal{Q}^h\mathbf{w}) + \gamma_{\Gamma_{\mathbf{g}}}(\mathcal{Q}^h\mathbf{w}) - \mathbf{g}^h) \, d\Gamma \\
&= \int_{\Gamma_{\mathbf{g}}} (\mathbf{g} - \mathbf{g}^h) \cdot (\mathbf{g} - \gamma_{\Gamma_{\mathbf{g}}}(\mathcal{Q}^h\mathbf{w})) \, d\Gamma \\
&\leq \|\mathbf{g} - \mathbf{g}^h\|_{0,\Gamma_{\mathbf{g}}} \|\gamma_{\Gamma_{\mathbf{g}}}(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}\|_{0,\Gamma_{\mathbf{g}}} \, .
\end{aligned}$$

Hence (4–32) yields

$$\|\mathbf{g} - \mathbf{g}^h\|_{0,\Gamma_{\mathbf{g}}} \leq \|\gamma_{\Gamma_{\mathbf{g}}}(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}\|_{0,\Gamma_{\mathbf{g}}} \leq C h^{1/2} \|(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}\|_1 \, ,$$

so that (4–34) is obtained.

For the proof of (4–35), we need some preliminary facts: For each $\boldsymbol{\phi} \in \mathbf{H}^{1/2}(\Gamma_{\mathbf{g}})$, let $\mathbf{v}_{\boldsymbol{\phi}} \in \mathbf{H}^1_{\Gamma_{0(\alpha_h)}}(\Omega(\alpha^h))$ be a lifting of $\boldsymbol{\phi}$ for the trace, *i.e.*, $\gamma_{\Gamma_{\mathbf{g}}}\mathbf{v}_{\boldsymbol{\phi}} = \boldsymbol{\phi}$ and $\|\mathbf{v}_{\boldsymbol{\phi}}\|_1 \leq C\|\boldsymbol{\phi}\|_{1/2,\Gamma_{\mathbf{g}}}$. From the orthogonality of the projection $\mathcal{Q}^h$, we obtain

$$\|\mathbf{v}_{\boldsymbol{\phi}}\|_1^2 = \|(\mathcal{I} - \mathcal{Q}^h)\mathbf{v}_{\boldsymbol{\phi}}\|_1^2 + \|\mathcal{Q}^h\mathbf{v}_{\boldsymbol{\phi}}\|_1^2 \, .$$

Hence, it follows that

$$\|(\mathcal{I} - \mathcal{Q}^h)\mathbf{v}_{\boldsymbol{\phi}}\|_1 \leq \|\mathbf{v}_{\boldsymbol{\phi}}\|_1 \leq C\|\boldsymbol{\phi}\|_{1/2,\Gamma_{\mathbf{g}}} \, . \tag{4–37}$$

The inequality (4–35) is the composite result of (4–14), (4–34) and (4–37);

$$\begin{aligned}
\|\mathbf{g} - \mathbf{g}^h\|_{-1/2,\Gamma_{\mathbf{g}}} &= \sup_{0 \neq \boldsymbol{\phi} \in \mathbf{H}^{1/2}(\Gamma_{\mathbf{g}})} \frac{\int_{\Gamma_{\mathbf{g}}} (\mathbf{g} - \mathbf{g}^h) \cdot \boldsymbol{\phi} \, d\Gamma}{\|\boldsymbol{\phi}\|_{1/2,\Gamma_{\mathbf{g}}}} \\
&= \sup_{0 \neq \boldsymbol{\phi} \in \mathbf{H}^{1/2}(\Gamma_{\mathbf{g}})} \frac{\int_{\Gamma_{\mathbf{g}}} (\mathbf{g} - \mathbf{g}^h) \cdot (\boldsymbol{\phi} - \mathcal{P}^h_{\Gamma_{\mathbf{g}}}\boldsymbol{\phi}) \, d\Gamma}{\|\boldsymbol{\phi}\|_{1/2,\Gamma_{\mathbf{g}}}} \\
&\leq C\|\mathbf{g} - \mathbf{g}^h\|_{0,\Gamma_{\mathbf{g}}} \sup_{0 \neq \boldsymbol{\phi} \in \mathbf{H}^{1/2}(\Gamma_{\mathbf{g}})} \frac{\|\boldsymbol{\phi} - \mathcal{P}^h_{\Gamma_{\mathbf{g}}}\|_{0,\Gamma_{\mathbf{g}}}}{\|\boldsymbol{\phi}\|_{1/2,\Gamma_{\mathbf{g}}}} \\
&\leq C h^{1/2} \|(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}\|_1 \sup_{0 \neq \boldsymbol{\phi} \in \mathbf{H}^{1/2}(\Gamma_{\mathbf{g}})} \frac{h^{1/2}\|(\mathcal{I} - \mathcal{Q}^h)\mathbf{v}_{\boldsymbol{\phi}}\|_1}{\|\mathbf{v}_{\boldsymbol{\phi}}\|_1} \\
&\leq C h \|(\mathcal{I} - \mathcal{Q}^h)\mathbf{w}\|_1 \, .
\end{aligned}$$

In order to show (4–35), we notice that

$$\begin{aligned}\|\mathbf{g}-\mathbf{g}^h\|_{0,\Gamma_\mathbf{g}} &\le \|\mathbf{g}-\gamma_{\Gamma_\mathbf{g}}(\mathcal{Q}^h\mathbf{w})\|_{0,\Gamma_\mathbf{g}} = \|\gamma_{\Gamma_\mathbf{g}}(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_{0,\Gamma_\mathbf{g}}\\ &\le Ch^{1/2}\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1\,,\end{aligned}$$

which follows from the orthogonality of $\mathcal{P}^h_{\Gamma_\mathbf{g}}$ and (4–32). Then, using the inverse inequality (4–19) and the continuity of the trace mapping, we have

$$\begin{aligned}\|\mathbf{g}-\mathbf{g}^h\|_{1/2,\Gamma_\mathbf{g}} &\le \|\gamma_{\Gamma_\mathbf{g}}(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_{1/2,\Gamma_\mathbf{g}} + \|\gamma_{\Gamma_\mathbf{g}}(\mathcal{Q}^h\mathbf{w})-\mathbf{g}^h\|_{1/2,\Gamma_\mathbf{g}}\\ &\le C[\,\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1 + h^{-1/2}\|\gamma_{\Gamma_\mathbf{g}}(\mathcal{Q}^h\mathbf{w})-\mathbf{g}^h\|_{0,\Gamma_\mathbf{g}}\,]\\ &\le C[\,\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1 + h^{-1/2}(\|\mathbf{g}-\gamma_{\Gamma_\mathbf{g}}(\mathcal{Q}^h\mathbf{w})\|_{0,\Gamma_\mathbf{g}} + \|\mathbf{g}-\mathbf{g}^h\|_{0,\Gamma_\mathbf{g}})\,]\\ &\le C[\,\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1 + h^{-1/2}(\|\gamma_{\Gamma_\mathbf{g}}(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_{0,\Gamma_\mathbf{g}} + \|\mathbf{g}-\mathbf{g}^h\|_{0,\Gamma_\mathbf{g}})\,]\\ &\le C[\,\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1 + h^{-1/2}h^{1/2}\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1\,]\\ &\le C\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1\,. \qquad \square\end{aligned}$$

Using a similar technique, Lemma 4.3 can be generalized: Suppose $\mathbf{g}\in\mathbf{H}^s(\Gamma_\mathbf{g})$ for all $s\in[0,1/2]$ and $\mathbf{g}^h=\mathcal{P}^h_{\Gamma_\mathbf{g}}\mathbf{g}$. Let $\mathbf{v}_\mathbf{g}$ be a lifting of $\mathbf{g}$. Then, it follows

$$\|\mathbf{g}-\mathbf{g}^h\|_{0,\Gamma_\mathbf{g}} \le Ch^s\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{v}_\mathbf{g}\|_{s+1/2} \le Ch^s\|\mathbf{g}\|_s \quad \text{for } 0\le s\le 1/2. \tag{4–38}$$

Moreover, for $\phi\in\mathbf{H}^s(\Gamma_\mathbf{g})$, since

$$\begin{aligned}\int_{\Gamma_\mathbf{g}}(\mathcal{I}-\mathcal{P}^h_{\Gamma_\mathbf{g}})\mathbf{g}\cdot\phi\,d\Gamma &= \int_{\Gamma_\mathbf{g}}(\mathcal{I}-\mathcal{P}^h_{\Gamma_\mathbf{g}})\mathbf{g}\cdot(\mathcal{I}-\mathcal{P}^h_{\Gamma_\mathbf{g}})\phi\,d\Gamma\\ &\le \|(\mathbf{g}-\mathbf{g}^h)\|_0\|(\mathcal{I}-\mathcal{P}^h_{\Gamma_\mathbf{g}})\phi\|_0\\ &\le Ch^t\|\mathbf{g}\|_t h^s\|\phi\|_s \quad \text{for } 0\le s,\ t\le 1/2\,,\end{aligned}$$

we obtain

$$\|\mathbf{g}-\mathbf{g}^h\|_{-s,\Gamma_\mathbf{g}} = \sup_{\substack{0\ne\phi\in\mathbf{H}^s(\Gamma_\mathbf{g})\\ \|\phi\|_{s,\Gamma_\mathbf{g}}\le 1}}\int_{\Gamma_\mathbf{g}}(\mathcal{I}-\mathcal{P}^h_{\Gamma_\mathbf{g}})\mathbf{g}\cdot\phi\,d\Gamma \le Ch^{s+t}\|\mathbf{g}\|_t\,.$$

So, this naturally leads us to

$$\|\mathbf{g}-\mathbf{g}^h\|_{-s,\Gamma_\mathbf{g}} \le Ch^{1/2+s}\|\mathbf{g}\|_{1/2} \quad \text{for } 0 \le s \le 1/2\,. \tag{4–39}$$

Now, we state the main result for the estimation $\mathbf{g}^h$ of the $\mathcal{P}^h_{\Gamma_\mathbf{g}}$–projection of $\mathbf{g}$.

Theorem 4.1. *Let* $\mathbf{g}$ *and* $\mathbf{g}^h$ *be defined as in Lemma 4.3. Then, as* $h \to 0^+$,

$$h^{-1/2}\|\mathbf{g}-\mathbf{g}^h\|_{0,\Gamma_\mathbf{g}} \longrightarrow 0\,, \tag{4–40}$$

$$h^{-1}\|\mathbf{g}-\mathbf{g}^h\|_{-1/2,\Gamma_\mathbf{g}} \longrightarrow 0\,, \tag{4–41}$$

$$\|\mathbf{g}-\mathbf{g}^h\|_{1/2,\Gamma_\mathbf{g}} \longrightarrow 0\,. \tag{4–42}$$

Proof: Since $\mathbf{w}$ is a lifting of $\mathbf{g}$ taken arbitrary, (4–34) implies that

$$h^{-1/2}\|\mathbf{g}-\mathbf{g}^h\|_{0,\Gamma_\mathbf{g}} \le \inf_{\substack{\mathbf{w}\in\mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha^h))\\ \gamma_{\Gamma_\mathbf{g}}\mathbf{w}=\mathbf{g}}} \|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1\,.$$

Since $\mathcal{C}^\infty(\overline{\Omega(\alpha^h)})\cap\mathbf{H}^1_{\Gamma_0(\alpha_h)}(\Omega(\alpha^h))$ is dense in $\mathbf{H}^1_{\Gamma_0(\alpha_h)}(\Omega(\alpha^h))$, one can deduce (4–40) from the approximation property of $\mathbf{V}^h_{\Gamma_0(\alpha_h)}$. (4–41) and (4–42) can be shown in the similar manner. □

If sufficient regularity is allowed, the estimates of Lemma 4.3 for the approximation can be sharpened.

Theorem 4.2. *Suppose* $\mathbf{g}\in\mathbf{H}^{m-1/2}(\Gamma_\mathbf{g})$ *for* $1\le m\le k$ *and* $\mathbf{w}\in\mathbf{H}^m\cap\mathbf{H}^1_{\Gamma_0(\alpha_h)}(\Omega(\alpha^h))$ *be a lifting of* $\mathbf{g}$. *Then, under the same condition with Lemma 4.3, we have*

$$\|\mathbf{g}-\mathbf{g}^h\|_{0,\Gamma_\mathbf{g}} \le Ch^{m-1/2}\|\mathbf{w}\|_m\,, \tag{4–43}$$

$$\|\mathbf{g}-\mathbf{g}^h\|_{-1/2,\Gamma_\mathbf{g}} \le Ch^{m}\|\mathbf{w}\|_m\,, \tag{4–44}$$

$$\|\mathbf{g}-\mathbf{g}^h\|_{1/2,\Gamma_\mathbf{g}} \le Ch^{m-1}\|\mathbf{w}\|_m\,. \tag{4–45}$$

Proof: Since $\mathcal{Q}^h\mathbf{w}\in\mathbf{V}^h_{\Gamma_0(\alpha_h)}$, from the approximation property (4–18), (4–34) yields

$$\|\mathbf{g}-\mathbf{g}^h\|_{0,\Gamma_\mathbf{g}} \le Ch^{1/2}\|(\mathcal{I}-\mathcal{Q}^h)\mathbf{w}\|_1 \le Ch^{1/2}h^{m-1}\|\mathbf{w}\|_m = Ch^{m-1/2}\|\mathbf{w}\|_m\,,$$

so that (4–43) holds. Similarly, for (4–44) and (4–45). $\square$

REMARK 4.4: The regularity problem we encounter can be simply stated as follows; For $\mathbf{g} \in \mathbf{H}^{m-1/2}(\Gamma_{\mathbf{g}})$, can we find $\mathbf{w} \in \mathbf{H}^m \cap \mathbf{H}^1_{\Gamma_{\mathbf{0}}(\alpha_h)}(\Omega(\alpha^h))$ such that

$$\begin{cases} a(\mathbf{w}, \mathbf{v}) & = 0 \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega(\alpha^h)), \\ \mathbf{w}\Big|_{\Gamma_{\mathbf{g}}} & = \mathbf{g}. \end{cases}$$

This is always true on the smooth domain by the Lax–Milgram Lemma and the regularity results for elliptic problems (see [33]). On a polygonal domain illustrated in Figure 4, however, sufficient regularity may not be available (m will be at best 2 (c.f.[63])). That explains why we have taken the $\mathbf{L}^2$–projection for the approximation of the boundary data instead of boundary interpolants. Methods using conventional boundary interpolants do not ensure the convergence nor optimal $\mathbf{L}^2$–error estimates without supplying a sufficient regularity assumption for the solution.

We now wish to derive error estimates for the approximation $(\mathbf{u}^h_\epsilon, p^h_\epsilon, \mathbf{q}^h_\epsilon, \theta^h_\epsilon)$ to $(\mathbf{u}, p, \mathbf{q}, \theta)$, where $(\mathbf{u}^h_\epsilon, p^h_\epsilon, \mathbf{q}^h_\epsilon, \theta^h_\epsilon)$ and $(\mathbf{u}, p, \mathbf{q}, \theta)$ are solutions of the systems (4–22)–(4–23) and (4–26)–(4–27), respectively.

To study the approximation, we invoke the corresponding nonlinear function formulation as in Section 3.5. In fact, exploiting the similar structure, the study of the approximation can be reduced to the analysis of the corresponding approximation to the solution of the Stokes formulation whose traction force is decoupled from the velocity and pressure. To be more explicit, we take $X = \mathbf{H}^1_{\Gamma_{\mathbf{0}}(\alpha_h)}(\Omega(\alpha^h)) \times L^2_0(\Omega(\alpha^h)) \times \mathbf{H}^1(\Omega(\alpha^h)) \times L^2_0(\Omega(\alpha^h))$, $Y = \mathbf{H}^{-1}_{\Gamma_{\mathbf{0}}(\alpha_h)}(\Omega(\alpha^h)) \times \mathbf{H}^{1/2}_0(\Gamma_{\mathbf{g}}) \times (\mathbf{H}^1(\Omega(\alpha^h)))^*$ and $Z = \mathbf{L}^{3/2}(\Omega(\alpha^h)) \times \{\mathbf{0}\} \times \mathbf{L}^{3/2}(\Omega(\alpha^h))$. We define the solution operator $T \in \mathcal{L}(Y; X)$ for the Stokes equations by $T(\widetilde{\mathbf{f}}, \widetilde{\mathbf{g}}, \widetilde{\mathbf{h}}) = (\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\mathbf{q}}, \widetilde{\theta})$ if and only if

$$\begin{cases} a(\widetilde{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, \widetilde{p}) = < \widetilde{\mathbf{f}}, \mathbf{v} >_{-1} \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega(\alpha^h)), \\ b(\widetilde{\mathbf{u}}, q) = 0 \quad \forall q \in L^2_0(\Omega(\alpha^h)), \\ < \mathbf{s}, \widetilde{\mathbf{u}} >_{-1/2, \Gamma_{\mathbf{g}}} = < \mathbf{s}, \widetilde{\mathbf{g}} >_{-1/2, \Gamma_{\mathbf{g}}} \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_{\mathbf{g}}), \end{cases} \tag{4–46}$$

and

$$\begin{cases} a(\widetilde{\mathbf{q}}, \mathbf{w}) + b(\mathbf{w}, \widetilde{\theta}) =< \widetilde{\mathbf{h}}, \mathbf{w} >_{-1} \quad \forall \mathbf{w} \in \mathbf{H}_0^1(\Omega(\alpha^h)), \\ b(\widetilde{\mathbf{q}}, \rho) = 0 \quad \forall \rho \in L_0^2(\Omega(\alpha^h)), \\ < \mathbf{s}, \widetilde{\mathbf{q}} >_{-1/2,\Gamma} = 0 \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma). \end{cases} \tag{4–47}$$

Analogously, for the solution operator of the discrete penalized Stokes equations, we define $T_\epsilon^h(\widetilde{\mathbf{f}}, \widetilde{\mathbf{g}}, \widetilde{\mathbf{h}}) = (\widetilde{\mathbf{u}}_\epsilon^h, \widetilde{p}_\epsilon^{\,h}, \widetilde{\mathbf{q}}_\epsilon^{\,h}, \widetilde{\theta}_\epsilon^{\,h}) \in X^h = \mathbf{V}^h_{\Gamma_0(\alpha_h)} \times S_0^h \times \mathbf{V}_0^h \times S_0^h$ if and only if

$$\begin{cases} \widetilde{\mathbf{g}}^h = \mathcal{P}^h_{\Gamma_\mathbf{g}}(\widetilde{\mathbf{g}}), \\ a(\widetilde{\mathbf{u}}_\epsilon^h, \mathbf{v}^h) + b(\mathbf{v}^h, \widetilde{p}_\epsilon^{\,h}) =< \widetilde{\mathbf{f}}, \mathbf{v}^h > \quad \forall \mathbf{v}^h \in \mathbf{V}_0^h, \\ b(\widetilde{\mathbf{u}}_\epsilon^h, q^h) = \epsilon(\widetilde{p}_\epsilon^{\,h}, q^h)_0 \quad \forall q^h \in S_0^h, \\ < \mathbf{s}^h, \widetilde{\mathbf{u}}_\epsilon^h >_{\Gamma_\mathbf{g}} =< \mathbf{s}^h, \widetilde{\mathbf{g}}^h >_{\Gamma_\mathbf{g}} \quad \forall \mathbf{s}^h \in \mathbf{P}^h_{\Gamma_\mathbf{g}}, \end{cases} \tag{4–48}$$

and

$$\begin{cases} a(\widetilde{\mathbf{q}}_\epsilon^{\,h}, \mathbf{w}^h) + b(\mathbf{w}^h, \widetilde{\theta}_\epsilon^{\,h}) =< \widetilde{\mathbf{h}}, \mathbf{w}^h > \quad \forall \mathbf{w}^h \in \mathbf{V}_0^h, \\ b(\widetilde{\mathbf{q}}_\epsilon^{\,h}, \rho^h) = \epsilon(\widetilde{\theta}_\epsilon^{\,h}, \rho^h)_0 - 2\epsilon(\widetilde{p}_\epsilon^{\,h}, \rho^h)_0 \quad \forall \rho^h \in S_0^h, \\ < \mathbf{s}^h, \widetilde{\mathbf{q}}_\epsilon^{\,h} >_\Gamma = 0 \quad \forall \mathbf{s}^h \in \mathbf{P}^h. \end{cases} \tag{4–49}$$

Since X^h is a dense subspace of X, T_ϵ^h is a bounded lineal operator from Y to X. To cover the nonlinear part, we take Λ to be a compact subset of R^+ and define the nonlinear operator G from $\Lambda \times X$ to Y by $G(\lambda, (\mathbf{u}, p, \mathbf{q}, \theta)) = (\boldsymbol{\eta}, \boldsymbol{\kappa}, \boldsymbol{\xi})$, for $\lambda = \dfrac{1}{\nu} \in \Lambda$ if and only if

$$\begin{cases} < \boldsymbol{\eta}, \mathbf{v} >_{-1} = \lambda c(\mathbf{u}, \mathbf{u}, \mathbf{v}) - \lambda < \mathbf{f}, \mathbf{v} >_{-1} \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega(\alpha^h)), \\ < \mathbf{s}, \boldsymbol{\kappa} >_{-1/2,\Gamma_\mathbf{g}} = - < \mathbf{s}, \mathbf{g} >_{-1/2,\Gamma_\mathbf{g}} \quad \forall \mathbf{s} \in \mathbf{H}^{-1/2}(\Gamma_\mathbf{g}), \\ < \boldsymbol{\xi}, \mathbf{w} >_{-1} = 2\lambda c(\mathbf{u}, \mathbf{u}, \mathbf{w}) + \lambda c(\mathbf{w}, \mathbf{u}, \mathbf{q}) + \lambda c(\mathbf{u}, \mathbf{w}, \mathbf{q}) \\ \qquad\qquad\qquad - \lambda < \mathbf{f}, \mathbf{w} >_{-1} \quad \forall \mathbf{w} \in \mathbf{H}_0^1(\Omega(\alpha^h)). \end{cases} \tag{4–50}$$

Then, (4–26)–(4–27) and (4–22)–(4–23) can be written as

$$(\mathbf{u}, \lambda p, \mathbf{q}, \lambda\theta) + TG(\lambda, (\mathbf{u}, \lambda p, \mathbf{q}, \lambda\theta)) = 0$$

and

$$(\mathbf{u}_\epsilon^h, \lambda p_\epsilon^h, \mathbf{q}_\epsilon^h, \lambda\theta_\epsilon^h) + T_\epsilon^h(\lambda, (\mathbf{u}_\epsilon^h, \lambda p_\epsilon^h, \mathbf{q}_\epsilon^h, \lambda\theta_\epsilon^h)) = 0,$$

respectively.

It is easy to check that $D_{\boldsymbol{\phi}} G \in \mathcal{L}(X; Y)$ and $D_{\boldsymbol{\phi}\boldsymbol{\phi}} G \in \mathcal{L}(X; Y)$, where $\boldsymbol{\phi} = (\mathbf{u}, p, \mathbf{q}, \theta) \in X$. Moreover, Z is compactly embeded in Y (refer the proof of Theorem

3.5). Hence due to the Theorem 2.2, the analysis of the convergence turns into the analysis of the approximation of T_ϵ^h to T. Let $\mathbf{y} = (\widetilde{\mathbf{f}}, \widetilde{\mathbf{g}}, \widetilde{\mathbf{h}}) \in Y$ be given. The analysis of the approximation depends on the estimate for $\|(T_\epsilon^h - T)\mathbf{y}\|_X$. For this purpose, we introduce the discrete operator T^h of T defined by $T^h\mathbf{y} = (\widetilde{\mathbf{u}}^h, \widetilde{p}^h, \widetilde{\mathbf{q}}^h, \widetilde{\theta}^h)$ if and only if

$$\begin{cases} \widetilde{\mathbf{g}}^h = \mathcal{P}_{\Gamma_\mathbf{g}}^h(\widetilde{\mathbf{g}}), \\ a(\widetilde{\mathbf{u}}^h, \mathbf{v}^h) + b(\mathbf{v}^h, \widetilde{p}^h) = < \widetilde{\mathbf{f}}, \mathbf{v}^h > \quad \forall \mathbf{v}^h \in \mathbf{V}_0^h, \\ b(\widetilde{\mathbf{u}}^h, q^h) = 0 \quad \forall q^h \in S_0^h, \\ < \mathbf{s}^h, \widetilde{\mathbf{u}}^h >_{\Gamma_\mathbf{g}} = < \mathbf{s}^h, \widetilde{\mathbf{g}}^h >_{\Gamma_\mathbf{g}} \quad \forall \mathbf{s}^h \in \mathbf{P}_{\Gamma_\mathbf{g}}^h, \end{cases} \tag{4–51}$$

and

$$\begin{cases} a(\widetilde{\mathbf{q}}^h, \mathbf{w}^h) + b(\mathbf{w}^h, \widetilde{\theta}^h) = < \widetilde{\mathbf{h}}, \mathbf{w}^h > \quad \forall \mathbf{w}^h \in \mathbf{V}_0^h, \\ b(\widetilde{\mathbf{q}}^h, \rho^h) = 0 \quad \forall \rho^h \in S_0^h, \\ < \mathbf{s}^h, \widetilde{\mathbf{q}}^h >_\Gamma = 0 \quad \forall \mathbf{s}^h \in \mathbf{P}^h. \end{cases} \tag{4–52}$$

To analyze $\|(T_\epsilon^h - T)\mathbf{y}\|_X$, we first consider $\|(T_\epsilon^h - T^h)\mathbf{y}\|_X$.

Lemma 4.4. *Let $\mathbf{y} = (\widetilde{\mathbf{f}}, \widetilde{\mathbf{g}}, \widetilde{\mathbf{h}}) \in Y$. Let $(\widetilde{\mathbf{u}}_\epsilon^h, \widetilde{p_\epsilon}^h, \widetilde{\mathbf{q}_\epsilon}^h, \widetilde{\theta_\epsilon}^h)$ and $(\widetilde{\mathbf{u}}^h, \widetilde{p}^h, \widetilde{\mathbf{q}}^h, \widetilde{\theta}^h)$ be solutions of (4–48)-(4–49) and (4–51)-(4–52), respectively. Then, under the inf-sup condition (4–15), we have*

$$\|(T_\epsilon^h - T^h)\mathbf{y}\|_X \le C\epsilon,$$

where C is independent of $\epsilon > 0$.

Proof: From (4–48)-(4–49) and (4–51)-(4–52), $(T_\epsilon^h - T^h)\mathbf{y}$ yields

$$a(\widetilde{\mathbf{u}}_\epsilon^h - \widetilde{\mathbf{u}}^h, \mathbf{v}^h) + b(\mathbf{v}^h, \widetilde{p_\epsilon}^h - \widetilde{p}^h) = 0 \quad \forall \mathbf{v}^h \in \mathbf{V}_0^h, \tag{4–53}$$

$$b(\widetilde{\mathbf{u}}_\epsilon^h - \widetilde{\mathbf{u}}^h, q^h) = \epsilon(\widetilde{p_\epsilon}^h - \widetilde{p}^h, q^h)_0 + \epsilon(\widetilde{p}^h, q^h)_0 \ \forall q^h \in S_0^h, \tag{4–54}$$

$$a(\widetilde{\mathbf{q}_\epsilon}^h - \widetilde{\mathbf{q}}^h, \mathbf{w}^h) + b(\mathbf{w}^h, \widetilde{\theta_\epsilon}^h - \widetilde{\theta}^h) = 0 \quad \forall \mathbf{w}^h \in \mathbf{V}_0^h \tag{4–55}$$

and

$$b(\widetilde{\mathbf{q}_\epsilon}^h - \widetilde{\mathbf{q}}^h, \rho^h) = \epsilon(\widetilde{\theta_\epsilon}^h - \widetilde{\theta}^h, \rho^h)_0 - 2\epsilon(\widetilde{p_\epsilon}^h - \widetilde{p}^h, \rho^h)_0 + \epsilon(\widetilde{\theta}^h, \rho^h)_0 - 2\epsilon(\widetilde{p}^h, \rho^h)_0 \quad \forall \rho^h \in S_0^h. \tag{4–56}$$

As in Theorem 3.5, applications of (4–15) yield

$$\|\widetilde{p}_\epsilon^{\,h} - \widetilde{p}^h\|_0 \le C\|\widetilde{\mathbf{u}}_\epsilon^h - \widetilde{\mathbf{u}}^h\|_1 \qquad \text{and}$$
$$\|\widetilde{\theta}_\epsilon^{\,h} - \widetilde{\theta}^h\|_0 \le C\|\widetilde{\mathbf{q}}_\epsilon^{\,h} - \widetilde{\mathbf{q}}^h\|_1 .$$

Setting $\mathbf{v}^h = \widetilde{\mathbf{u}}_\epsilon^h - \widetilde{\mathbf{u}}^h$ and $q^h = \widetilde{p}_\epsilon^{\,h} - \widetilde{p}^h$ in (4–53), (4–54) yields

$$\begin{aligned}|\!|\!|\widetilde{\mathbf{u}}_\epsilon^h - \widetilde{\mathbf{u}}^h|\!|\!|^2 &\le -\epsilon\,(\widetilde{p}^h, \widetilde{p}_\epsilon^{\,h} - \widetilde{p}^h)_0 \le \epsilon\|\widetilde{p}_\epsilon^{\,h}\|_0\|\widetilde{p}_\epsilon^{\,h} - \widetilde{p}^h\|_0 \\ &\le C\epsilon\|\widetilde{\mathbf{u}}_\epsilon^h - \widetilde{\mathbf{u}}^h\|_1\|\widetilde{p}_\epsilon^{\,h}\|_0 .\end{aligned}$$

Here, we notice that $\widetilde{\mathbf{u}}_\epsilon^h - \widetilde{\mathbf{u}}^h$, $\widetilde{\mathbf{q}}_\epsilon^{\,h} - \widetilde{\mathbf{q}}^h \in \mathbf{V}_0^h \subset \mathbf{H}_0^1(\Omega(\alpha^h))$. Since the $|\!|\!|\cdot|\!|\!|$–norm in $\mathbf{H}_0^1(\Omega(\alpha^h))$ is equivalent to the $\|\cdot\|_1$–norm, we obtain

$$\|\widetilde{\mathbf{u}}_\epsilon^h - \widetilde{\mathbf{u}}^h\|_1 \le C\epsilon\|\widetilde{p}^h\|_0 .$$

Similarly, setting $\mathbf{w}^h = \widetilde{\mathbf{q}}_\epsilon^{\,h} - \widetilde{\mathbf{q}}^h$ and $\rho^h = \widetilde{\theta}_\epsilon^{\,h} - \widetilde{\theta}^h$ in (4–55), we obtain from (4–56) that

$$|\!|\!|\widetilde{\mathbf{q}}_\epsilon^{\,h} - \widetilde{\mathbf{q}}^h|\!|\!|^2 \le \epsilon M_h\|\widetilde{\theta}_\epsilon^{\,h} - \widetilde{\theta}^h\|_0 ,$$

where $M_h = 2\|\widetilde{p}_\epsilon^{\,h} - \widetilde{p}^h\|_0 + \|\widetilde{\theta}^h\|_0 + 2\|\widetilde{p}^h\|_0$.

Since $\|\widetilde{p}_\epsilon^{\,h} - \widetilde{p}^h\|_0 \le C\|\widetilde{\mathbf{u}}_\epsilon^h - \widetilde{\mathbf{u}}^h\|_1 \le C^2\|\widetilde{p}^h\|_0$, $M_h \le 2(C^2+1)\|\widetilde{p}^h\|_0 + \|\widetilde{\theta}^h\|_0$. Hence, it follows that

$$\|\widetilde{\mathbf{q}}_\epsilon^{\,h} - \widetilde{\mathbf{q}}^h\|_1 \le C\epsilon\,(\|\widetilde{p}^h\|_0 + \|\widetilde{\theta}^h\|_0) .$$

Therefore, using the fact that $\|\widetilde{\theta}^h\|_0$ and $\|\widetilde{p}^h\|_0$ are uniformly bounded in h, we have

$$\begin{aligned}\|(T_\epsilon^h - T^h)\mathbf{y}\|_X &\le \|\widetilde{\mathbf{u}}_\epsilon^h - \widetilde{\mathbf{u}}^h\|_1 + \|\widetilde{p}_\epsilon^{\,h} - \widetilde{p}^h\|_0 + \|\widetilde{\mathbf{q}}_\epsilon^{\,h} - \widetilde{\mathbf{q}}^h\|_1 + \|\widetilde{\theta}_\epsilon^{\,h} - \widetilde{\theta}^h\|_0 \\ &\le C\epsilon .\end{aligned}$$

Notice that the generic constant C has taken independent of ϵ. □

We now state estimates for $\|(T_\epsilon^h - T)\mathbf{y}\|_X$.

Lemma 4.5. *Let* $\mathbf{y} = (\widetilde{\mathbf{f}}, \widetilde{\mathbf{g}}, \widetilde{\mathbf{h}}) \in Y$. *Let* $(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\mathbf{q}}, \widetilde{\theta})$ *and* $(\widetilde{\mathbf{u}}^h, \widetilde{p}^h, \widetilde{\mathbf{q}}^h, \widetilde{\theta}^h)$ *be solutions of (4–46)-(4–47) and (4–51)-(4–52), respectively. Then, it holds that*

$$\|(T^h - T)\mathbf{y}\|_X \le C \inf_{(\check{\mathbf{u}}^h, \check{p}^h, \check{\mathbf{q}}^h, \check{\theta}^h) \in \widetilde{X}^h} \|(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\mathbf{q}}, \widetilde{\theta}) - (\check{\mathbf{u}}^h, \check{p}^h, \check{\mathbf{q}}^h, \check{\theta}^h)\|_X \,, \tag{4–57}$$

where $\widetilde{X}^h = \{(\check{\mathbf{u}}^h, \check{p}^h, \check{\mathbf{q}}^h, \check{\theta}^h) \in \mathbf{V}^h_{\Gamma_0(\alpha_h)} \times S^h_0 \times \mathbf{V}^h_0 \times S^h_0 \mid \gamma_{\Gamma_\mathbf{g}} \check{\mathbf{u}}^h = \widetilde{\mathbf{g}}^h\}$.

Proof: This follows directly from the double application of the result due to Gunzburger *et al.*[65] or [68]. □

One can combine (4–57) with the approximation result for $\widetilde{\mathbf{g}}^h$ to obtain the following result.

Lemma 4.6. *Under the same conditions with Lemma 4.5, we have*

$$\|(T^h - T)\mathbf{y}\|_X \le C \inf_{(\boldsymbol{\eta}^h, r^h, \boldsymbol{\phi}^h, \rho^h) \in X^h} \|(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\mathbf{q}}, \widetilde{\theta}) - (\boldsymbol{\eta}^h, r^h, \boldsymbol{\phi}^h, \rho^h)\|_X \,. \tag{4–58}$$

Furthermore, if we assume that $(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\mathbf{q}}, \widetilde{\theta}) \in X \cap (\mathbf{H}^m \times H^{m-1} \times \mathbf{H}^m \times H^{m-1})$ *for* $1 \le m \le k$, *then there exists a positive constant* C, *independent of* h, *such that*

$$\|(T^h - T)\mathbf{y}\|_X \le C h^{m-1} [\, \|\widetilde{\mathbf{u}}\|_m + \|\widetilde{p}\|_{m-1} + \|\widetilde{\mathbf{q}}\|_m + \|\widetilde{\theta}\|_{m-1}] \,. \tag{4–59}$$

Proof: We first show that

$$\begin{aligned} &\|(T^h - T)\mathbf{y}\|_X \\ &\quad \le C\,[\inf_{(\boldsymbol{\eta}^h, r^h, \boldsymbol{\phi}^h, \rho^h) \in X^h} \|(\widetilde{\mathbf{u}}, \widetilde{p}, \widetilde{\mathbf{q}}, \widetilde{\theta}) - (\boldsymbol{\eta}^h, r^h, \boldsymbol{\phi}^h, \rho^h)\|_X + \|\widetilde{\mathbf{g}} - \widetilde{\mathbf{g}}^h\|_{1/2, \Gamma_\mathbf{g}}] \,. \end{aligned} \tag{4–60}$$

From (4–30), we observe that

$$|\!|\!|(\mathcal{I} - \mathcal{Q}^h)\widetilde{\mathbf{u}}|\!|\!| = \inf_{\boldsymbol{\eta}^h \in \mathbf{V}^h_{\Gamma_0(\alpha_h)}} |\!|\!|\widetilde{\mathbf{u}} - \boldsymbol{\eta}^h|\!|\!| \le \inf_{\boldsymbol{\eta}^h \in \mathbf{V}^h_{\Gamma_0(\alpha_h)}} \|\widetilde{\mathbf{u}} - \boldsymbol{\eta}^h\|_1 \,.$$

Hence, from Korn's inequality we have

$$\|(\mathcal{I} - \mathcal{Q}^h)\widetilde{\mathbf{u}}\|_1 \le C\, |\!|\!|(\mathcal{I} - \mathcal{Q}^h)\widetilde{\mathbf{u}}|\!|\!| \le C \inf_{\boldsymbol{\eta}^h \in \mathbf{V}^h_{\Gamma_0(\alpha_h)}} \|\widetilde{\mathbf{u}} - \boldsymbol{\eta}^h\|_1 \,. \tag{4–61}$$

Let $\mathbf{v}^h \in \mathbf{V}^h_{\Gamma_0(\alpha_h)}$ be a lifting of $\widetilde{\mathbf{g}}^h - \gamma_{\Gamma_{\mathbf{g}}}(\mathcal{Q}^h\widetilde{\mathbf{u}})$. Take $\boldsymbol{\eta}^h = \mathbf{v}^h + \mathcal{Q}^h\widetilde{\mathbf{u}}$. Applying the continuity of the trace, (4–36) and (4–61), we deduce that

$$\begin{aligned}
\|\widetilde{\mathbf{u}} - \boldsymbol{\eta}^h\|_1 &\le \|\widetilde{\mathbf{u}} - \mathcal{Q}^h\widetilde{\mathbf{u}}\|_1 + \|\mathbf{v}^h\|_1 \\
&\le \|\widetilde{\mathbf{u}} - \mathcal{Q}^h\widetilde{\mathbf{u}}\|_1 + C\|\widetilde{\mathbf{g}}^h - \gamma_{\Gamma_{\mathbf{g}}}(\mathcal{Q}^h\widetilde{\mathbf{u}})\|_{1/2,\Gamma_{\mathbf{g}}} \\
&\le \|\widetilde{\mathbf{u}} - \mathcal{Q}^h\widetilde{\mathbf{u}}\|_1 + C(\,\|\widetilde{\mathbf{g}} - \widetilde{\mathbf{g}}^h\|_{1/2,\Gamma_{\mathbf{g}}} + \|\gamma_{\Gamma_{\mathbf{g}}}(\mathcal{I} - \mathcal{Q}^h)\widetilde{\mathbf{u}}\|_{1/2,\Gamma_{\mathbf{g}}}) \\
&\le C(\,\|\widetilde{\mathbf{u}} - \mathcal{Q}^h\widetilde{\mathbf{u}}\|_1 + \|\widetilde{\mathbf{g}} - \widetilde{\mathbf{g}}^h\|_{1/2,\Gamma_{\mathbf{g}}}) \\
&\le C(\,|||\widetilde{\mathbf{u}} - \mathcal{Q}^h\widetilde{\mathbf{u}}||| + \|\widetilde{\mathbf{g}} - \widetilde{\mathbf{g}}^h\|_{1/2,\Gamma_{\mathbf{g}}}) \\
&\le C\,[\inf_{\boldsymbol{\eta}^h \in \mathbf{V}^h_{\Gamma_0(\alpha_h)}} \|\widetilde{\mathbf{u}} - \boldsymbol{\eta}^h\|_1 + \|\widetilde{\mathbf{g}} - \widetilde{\mathbf{g}}^h\|_{1/2,\Gamma_{\mathbf{g}}}]\,.
\end{aligned}$$

Hence, (4–60) is obtained from (4–57). We further note from (4–36) and (4–61) that

$$\|\widetilde{\mathbf{g}} - \widetilde{\mathbf{g}}^h\|_{1/2,\Gamma_{\mathbf{g}}} \le C\|(\mathcal{I} - \mathcal{Q}^h)\widetilde{\mathbf{u}}\|_1 \le C \inf_{\boldsymbol{\eta}^h \in \mathbf{V}^h_{\Gamma_0(\alpha_h)}} \|\widetilde{\mathbf{u}} - \boldsymbol{\eta}^h\|_1\,. \tag{4–62}$$

Therefore, combined with (4–60), the estimate (4–58) is a composite result of (4–57) and (4–62).

Next, we turn to showing (4–59). From the regularity result for the Stokes operator([60][127]), we have; if $(\widetilde{\mathbf{u}}, \widetilde{p}) \in \mathbf{H}^m(\Omega(\alpha^h)) \times (H^{m-1} \cap L^2_0(\Omega(\alpha^h)))$ for $1 \le m \le k$, $\widetilde{\mathbf{g}} \in \mathbf{H}^{m-1/2}(\Gamma_{\mathbf{g}})$. So, it follows from (4–23) that $\|\widetilde{\mathbf{g}} - \widetilde{\mathbf{g}}^h\|_{1/2,\Gamma_{\mathbf{g}}} \le Ch^{m-1}\|\widetilde{\mathbf{u}}\|_m$. Then, applying the approximation properties (4–16)–(4–18) to (4–60), (4–59) is obtained. □

As a supplementary result to Lemma 4.4 and Lemma 4.6, by employing arguments similar to those used in the proof of Theorem 3.5, we can obtain the main estimates for $\|(\mathbf{u}^h_\epsilon, p^h_\epsilon, \mathbf{q}^h_\epsilon, \theta^h_\epsilon) - (\mathbf{u}, p, \mathbf{q}, \theta)\|_X$.

Theorem 4.3. *Assume that Λ be a compact subset of R^+. For a fixed $\alpha^h \in \mathcal{U}^h_{ad}$, let $X = \mathbf{H}^1_{\Gamma_0(\alpha_h)}(\Omega(\alpha^h)) \times L^2_0(\Omega(\alpha^h)) \times \mathbf{H}^1(\Omega(\alpha^h)) \times L^2_0(\Omega(\alpha^h))$.*

Suppose $\{(\lambda, (\mathbf{u}(\lambda), \lambda p(\lambda), \mathbf{q}(\lambda), \lambda\theta(\lambda)) \mid \lambda = \frac{1}{\nu} \in \Lambda\}$ *be a branch of regular solutions of (4–26)–(4–27). Then, for* $\epsilon \leq \epsilon_0$ *small enough, there exists a unique regular branch* $\{(\lambda, (\mathbf{u}^h_\epsilon(\lambda), \lambda p^h_\epsilon(\lambda), \mathbf{q}^h_\epsilon(\lambda), \lambda\theta^h_\epsilon(\lambda)) \mid \lambda \in \Lambda\}$ *of solutions of (4–22)–(4–23) in the neighborhood of* $(\mathbf{u}(\lambda), p(\lambda), \mathbf{q}(\lambda), \theta(\lambda))$ *in* X *and a positive constant* C*, independent of* ϵ *and* $\lambda \in \Lambda$*, such that*

$$\|(\mathbf{u}^h_\epsilon(\lambda), p^h_\epsilon(\lambda), \mathbf{q}^h_\epsilon(\lambda), \theta^h_\epsilon(\lambda)) - (\mathbf{u}(\lambda), p(\lambda), \mathbf{q}(\lambda), \theta(\lambda))\|_X \leq C\epsilon \tag{4–63}$$

as $h \to 0^+$*, uniformly in* $\lambda = \frac{1}{\nu} \in \Lambda$.

In addition, if we assume that $\{(\mathbf{u}(\lambda), p(\lambda), \mathbf{q}(\lambda), \theta(\lambda)) \mid \lambda \in \Lambda\}$ *belongs to* $X \cap (\mathbf{H}^m \times H^{m-1} \times \mathbf{H}^m \times H^{m-1})$ *for* $1 \leq m \leq k$*, there exists positive constants* C_1 *and* C_2 *which are independent of* $\lambda \in \Lambda$*,* ϵ *and* h *such that*

$$\begin{aligned}\|(\mathbf{u}^h_\epsilon(\lambda), p^h_\epsilon(\lambda), \mathbf{q}^h_\epsilon(\lambda), \theta^h_\epsilon(\lambda)) - (\mathbf{u}(\lambda), p(\lambda), \mathbf{q}(\lambda), \theta(\lambda))\|_X \\ \leq C_1\epsilon + C_2 h^{m-1}[\,\|\mathbf{u}(\lambda)\|_m + \|p(\lambda)\|_{m-1} + \|\mathbf{q}(\lambda)\|_m + \|\theta(\lambda)\|_{m-1}]\,,\end{aligned} \tag{4–64}$$

for all $\lambda \in \Lambda$*.* $\square$

We are now concerned with error estimates of the approximations (4–24)–(4–25) for the state and costate traction forces along the boundary. Some comments are in order for the choice of an approximation space $\mathbf{P}^h_{\Gamma_g}$ for the traction force (also, $\mathbf{P}^h$ for its costate variable). $\mathbf{P}^h_{\Gamma_g}$ has been chosen to accommodate both the trace of the velocity and the traction force. For strict computation, one may consider taking $\mathbf{P}^h_{\Gamma_g}$ independently of the velocity space; two different spaces $\mathbf{P}^{h_1}_{\Gamma_g}$ and $\gamma_{\Gamma_g}(\mathbf{V}^{h_2}_{\Gamma_0(\alpha_h)})$ with different meshes may be taken to approximate the traction force and the trace of the velocity. However, in order to sustain the stability of the scheme, this approach necessitates an additional requirement for meshes such that $h_1 \geq K h_2$, where K is a positive constant dependent on the domain (see [9] and [104]). This levies an additional difficulty of determining K. Our approach is simply $\gamma_{\Gamma_{\mathbf{g}}}(\mathbf{V}^h_{\Gamma_{0(\alpha_h)}}) = \mathbf{P}^h_{\Gamma_{\mathbf{g}}}$ and $\mathbf{P}^h_{\Gamma_{\mathbf{g}}}$, as an approximation space of the traction force, is understood to be embedded in $\mathbf{H}^{-1/2}(\Gamma_{\mathbf{g}})$. Pitkäranta[103] studied similar structure relating the interior mesh to the boundary for the Lagrange multiplier, which is represented by the normal derivative terms of state variables. However, his approach consists of taking the approximation space in $\mathbf{L}^2(\Gamma)$.

Let us begin our discussion by introducing new operator to interpret $\mathbf{P}^h_{\Gamma_{\mathbf{g}}}$ in $\mathbf{H}^{-1/2}(\Gamma_{\mathbf{g}})$. We consider the operator $\mathcal{R}^h_{\Gamma_{\mathbf{g}}} : \mathbf{H}^{-1/2}(\Gamma_{\mathbf{g}}) \to \mathbf{P}^h_{\Gamma_{\mathbf{g}}} \subset \mathbf{H}^{-1/2}(\Gamma_{\mathbf{g}})$ defined by

$$< \mathcal{R}^h_{\Gamma_{\mathbf{g}}}\boldsymbol{\xi}, \mathbf{v}^h >_{\Gamma_{\mathbf{g}}} = < \boldsymbol{\xi}, \mathbf{v}^h >_{\Gamma_{\mathbf{g}}} \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h_{\Gamma_{0(\alpha_h)}} . \tag{4–65}$$

It is clear that $\mathcal{R}^h_{\Gamma_{\mathbf{g}}}\Big|_{\mathbf{H}^s(\Gamma_{\mathbf{g}})} = \mathcal{P}^h_{\Gamma_{\mathbf{g}}}$ for all $s \geq \dfrac{1}{2}$. For our purpose, we call attention to particular properties of $\mathcal{R}^h_{\Gamma_{\mathbf{g}}}$.

Lemma 4.6. *The operator $\mathcal{R}^h_{\Gamma_{\mathbf{g}}}$ defined in (4–65) satisfies the following properties:*

(i) *$\mathcal{R}^h_{\Gamma_{\mathbf{g}}}$ is a bounded operator in $\mathbf{H}^{-1/2}_{\Gamma_{\mathbf{g}}}$.*

(ii) *For $\boldsymbol{\xi} \in \mathbf{H}^{-1/2}(\Gamma_{\mathbf{g}})$, we have*

$$\|(\mathcal{I} - \mathcal{R}^h_{\Gamma_{\mathbf{g}}})\boldsymbol{\xi}\|_{-1/2,\Gamma_{\mathbf{g}}} \to 0 \quad \text{as } h \to 0^+ . \tag{4–66}$$

(iii) *We suppose that $\boldsymbol{\xi} \in \mathbf{H}^s(\Gamma_{\mathbf{g}})$ for $-\dfrac{1}{2} \leq s \leq \dfrac{1}{2}$. Then we have*

$$\|(\mathcal{I} - \mathcal{R}^h_{\Gamma_{\mathbf{g}}})\boldsymbol{\xi}\|_{-1/2,\Gamma_{\mathbf{g}}} \leq C h^{s+1/2} \|\boldsymbol{\xi}\|_{s,\Gamma_{\mathbf{g}}} . \tag{4–67}$$

Analogous discussion can be found in Gunzburger *et al.*[67] (see also [17]).

Proof: (i); Let $\boldsymbol{\phi} \in \mathbf{H}^{1/2}(\Gamma_{\mathbf{g}})$ and $\mathbf{v}_{\boldsymbol{\phi}}$ be a lifting of $\boldsymbol{\phi}$ in $\mathbf{H}^1_{\Gamma_{0(\alpha_h)}}$ so that $\|\mathbf{v}_{\boldsymbol{\phi}}\|_1 \leq C\|\boldsymbol{\phi}\|_{1/2,\Gamma_{\mathbf{g}}}$. Using (4–36) and (4–37), we have

$$\begin{aligned} \|\mathcal{P}^h_{\Gamma_{\mathbf{g}}}\boldsymbol{\phi}\|_{1/2,\Gamma_{\mathbf{g}}} &\leq \|(\mathcal{I} - \mathcal{P}^h_{\Gamma_{\mathbf{g}}})\boldsymbol{\phi}\|_{1/2,\Gamma_{\mathbf{g}}} + \|\boldsymbol{\phi}\|_{1/2,\Gamma_{\mathbf{g}}} \\ &\leq C\|(\mathcal{I} - \mathcal{Q}^h)\mathbf{v}_{\boldsymbol{\phi}}\|_1 + \|\boldsymbol{\phi}\|_{1/2,\Gamma_{\mathbf{g}}} \\ &\leq C\|\boldsymbol{\phi}\|_{1/2,\Gamma_{\mathbf{g}}} . \end{aligned} \tag{4–68}$$

Since $\mathcal{R}^h_{\Gamma_g}\boldsymbol{\xi} \in \mathbf{P}^h_{\Gamma_g}$, we deduce from (4–14) that

$$\begin{aligned}
< \mathcal{R}^h_{\Gamma_g}\boldsymbol{\xi}, \boldsymbol{\phi} >_{\Gamma_g} &=< \mathcal{R}^h_{\Gamma_g}\boldsymbol{\xi}, (\mathcal{I} - \mathcal{P}^h_{\Gamma_g})\boldsymbol{\phi} + \mathcal{P}^h_{\Gamma_g}\boldsymbol{\phi} >_{\Gamma_g} \\
&=< \mathcal{R}^h_{\Gamma_g}\boldsymbol{\xi}, \mathcal{P}^h_{\Gamma_g}\boldsymbol{\phi} >_{\Gamma_g} \\
&=< \boldsymbol{\xi}, \mathcal{P}^h_{\Gamma_g}\boldsymbol{\phi} >_{\Gamma_g} \\
&\leq \|\boldsymbol{\xi}\|_{-1/2,\Gamma_g} \|\mathcal{P}^h_{\Gamma_g}\boldsymbol{\phi}\|_{1/2,\Gamma_g} \\
&\leq C\|\boldsymbol{\xi}\|_{-1/2,\Gamma_g} \|\boldsymbol{\phi}\|_{1/2,\Gamma_g} .
\end{aligned}$$

Therefore, it follows that

$$\|\mathcal{R}^h_{\Gamma_g}\boldsymbol{\xi}\|_{-1/2,\Gamma_g} = \sup_{\substack{0\neq\boldsymbol{\phi}\in\mathbf{H}^{-1/2}(\Gamma_g)\\ \|\boldsymbol{\phi}\|_{1/2,\Gamma_g}\leq 1}} < \mathcal{R}^h_{\Gamma_g}\boldsymbol{\xi}, \boldsymbol{\phi} >_{\Gamma_g} \leq C\|\boldsymbol{\xi}\|_{-1/2,\Gamma_g}$$

so that $\|\mathcal{R}^h_{\Gamma_g}\|_{\mathcal{L}(\mathbf{H}^{-1/2}(\Gamma_g))} \leq C$.

(ii); We use the fact that $\mathcal{C}^\infty(\Gamma_g)$ is dense in $\mathbf{H}^{-1/2}(\Gamma_g)$ and $\mathcal{R}^h_{\Gamma_g}(\mathcal{C}^\infty(\Gamma_g)) = \mathcal{P}^h_{\Gamma_g}(\mathcal{C}^\infty(\Gamma_g))$. For $\boldsymbol{\xi} \in \mathbf{H}^{-1/2}(\Gamma_g)$, take $\hat{\boldsymbol{\xi}} \in \mathcal{C}^\infty(\Gamma_g)$. Since $\mathcal{R}^h_{\Gamma_g}$ is bounded in $\mathbf{H}^{-1/2}(\Gamma_g)$, using (4–39), one can carry out

$$\begin{aligned}
\|(\mathcal{I} - \mathcal{R}^h_{\Gamma_g})\boldsymbol{\xi}\|_{-1/2,\Gamma_g} &= \|(\mathcal{I} - \mathcal{R}^h_{\Gamma_g})((\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}) + \hat{\boldsymbol{\xi}})\|_{-1/2,\Gamma_g} \\
&\leq \|\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}\|_{-1/2,\Gamma_g} + \|\mathcal{R}^h_{\Gamma_g}(\boldsymbol{\xi} - \hat{\boldsymbol{\xi}})\|_{-1/2,\Gamma_g} + \|(\mathcal{I} - \mathcal{P}^h_{\Gamma_g})\hat{\boldsymbol{\xi}}\|_{-1/2,\Gamma_g} \\
&\leq C[\, \|\boldsymbol{\xi} - \hat{\boldsymbol{\xi}}\|_{-1/2,\Gamma_g} + h\|\hat{\boldsymbol{\xi}}\|_{1/2,\Gamma_g}] .
\end{aligned}$$

Therefore, (4–66) follows from the denseness of $\mathcal{C}^\infty(\Gamma_g)$ in $\mathbf{H}^{-1/2}(\Gamma_g)$.

(iii); Employing interpolation arguments for the Sobolev space ([37]), (4–67) follows from (4–35) and (4–39). □

In the similar fashion, one can also show:

Lemma 4.7. *Let* $\mathcal{R}^h_\Gamma : \mathbf{H}^{-1/2}(\Gamma) \to \mathbf{P}^h \subset \mathbf{H}^{-1/2}(\Gamma)$ *be an operator defined by*

$$< \mathcal{R}^h_\Gamma\boldsymbol{\xi}, \mathbf{v}^h >_\Gamma =< \boldsymbol{\xi}, \mathbf{v}^h >_\Gamma \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h . \tag{4–69}$$

Then, $\mathcal{R}_\Gamma^h$ is a bounded operator in $\mathbf{H}^{-1/2}(\Gamma)$ satisfying:

(i) *For $\boldsymbol{\xi} \in \mathbf{H}^{-1/2}(\Gamma)$, $\|(\mathcal{I} - \mathcal{R}_\Gamma^h)\boldsymbol{\xi}\|_{-1/2,\Gamma} \to 0$ as $h \to 0^+$.*

(ii) *Supposing further $\boldsymbol{\xi} \in \mathbf{H}_\Gamma^s(\Gamma)$ for $-\dfrac{1}{2} \le s \le \dfrac{1}{2}$, it follows that*

$$\|(\mathcal{I} - \mathcal{R}_\Gamma^h)\boldsymbol{\xi}\|_{-1/2,\Gamma} \le C h^{s+1/2} \|\boldsymbol{\xi}\|_{s,\Gamma}\,. \tag{4–70}$$

□

We are particularly interested in the solution $(\mathbf{t}, \boldsymbol{\tau})$ of the system (4–28)–(4–29). For simplification, let us define $\mathcal{H} = \mathbf{H}^{-1/2}(\Gamma_\mathbf{g}) \times \mathbf{H}^{-1/2}(\Gamma)$. We recall that even though $\mathbf{g} \in \mathbf{H}^s(\Gamma)$ for $s \ge \dfrac{3}{2}$, if $\Omega(\alpha^h) \in \mathcal{C}^{0,1}$, the solution $(\mathbf{u}, p, \mathbf{q}, \theta)$ of the system (4–26)–(4–27) belongs to $X \cap (\mathbf{H}^{3/2-\delta} \times H^{1/2-\delta} \times \mathbf{H}^{3/2-\delta} \times H^{1/2-\delta})$ for some small $\delta > 0$ (c.f. Remark 3.8). We provide general perspectives for the operator $(\mathcal{R}_{\Gamma_\mathbf{g}}^h, \mathcal{R}_\Gamma^h)$.

Lemma 4.8. *Let $(\mathbf{t}, \boldsymbol{\tau}) \in \mathcal{H}$ be a solution of the system (4–28)–(4–29). The following properties hold for $(\mathcal{R}_{\Gamma_\mathbf{g}}^h, \mathcal{R}_\Gamma^h)$:*

(i) $\|(\mathbf{t}, \boldsymbol{\tau}) - (\mathcal{R}_{\Gamma_\mathbf{g}}^h \mathbf{t}, \mathcal{R}_\Gamma^h \boldsymbol{\tau})\|_\mathcal{H} \to 0$ *as* $h \to 0^+$. (4–71)

(ii) *If $\Omega(\alpha^h) \in \mathcal{C}^{0,1}$ and $\mathbf{g} \in \mathbf{H}^{3/2}(\Gamma_\mathbf{g})$,*

$$\|(\mathbf{t}, \boldsymbol{\tau}) - (\mathcal{R}_{\Gamma_\mathbf{g}}^h \mathbf{t}, \mathcal{R}_\Gamma^h \boldsymbol{\tau})\|_\mathcal{H} \le C h^{1/2-\delta} [\, \|\mathbf{u}\|_{3/2-\delta} + \|p\|_{1/2-\delta} + \|\mathbf{q}\|_{3/2-\delta} + \|\theta\|_{1/2-\delta} \,]\,, \tag{4–72}$$

for some small $\delta > 0$.

Under more assumed regularities for the domain, we suppose the solution $(\mathbf{u}, p, \mathbf{q}, \theta)$ of the system (4–26)–(4–27) belongs to $X \cap (\mathbf{H}^m \times H^{m-1} \times \mathbf{H}^m \times H^{m-1})$ for $1 \le m \le k$. Then, we have general estimates:

(iii) *If $1 \le m < 2$, we have*

$$\|(\mathbf{t}, \boldsymbol{\tau}) - (\mathcal{R}_{\Gamma_\mathbf{g}}^h \mathbf{t}, \mathcal{R}_\Gamma^h \boldsymbol{\tau})\|_\mathcal{H} \le C h^{m-1} [\, \|\mathbf{u}\|_m + \|p\|_{m-1} + \|\mathbf{q}\|_m + \|\theta\|_{m-1} \,]\,. \tag{4–73}$$

(iv) *If $m = 2$ and $\Omega(\alpha^h)$ is polygonal, it holds that*

$$\|(\mathbf{t}, \boldsymbol{\tau}) - (\mathcal{R}_{\Gamma_\mathbf{g}}^h \mathbf{t}, \mathcal{R}_\Gamma^h \boldsymbol{\tau})\|_\mathcal{H} \le C h^{1-\delta} [\, \|\mathbf{u}\|_2 + \|p\|_1 + \|\mathbf{q}\|_2 + \|\theta\|_1 \,]\,, \tag{4–74}$$

for all $0 < \delta < \dfrac{1}{2}$.

(v) *If* $2 \leq m \leq k$ *and* $(\mathbf{t}, \boldsymbol{\tau}) \in \gamma_{\Gamma_\mathbf{g}}(\mathbf{H}^1_{\Gamma_0(\alpha_h)}(\Omega(\alpha^h)) \cap \mathbf{H}^{m-1}) \times \gamma_\Gamma(\mathbf{H}^{m-1}(\Omega(\alpha^h)))$, *we obtain*

$$\|(\mathbf{t}, \boldsymbol{\tau}) - (\mathcal{R}^h_{\Gamma_\mathbf{g}}\mathbf{t}, \mathcal{R}^h_\Gamma \boldsymbol{\tau})\|_\mathcal{H} \leq C h^{m-1} \inf_{(\mathbf{v_t}, \mathbf{v}_\tau) \in \mathfrak{Z}} [\, \|\mathbf{v_t}\|_{m-1} + \|\mathbf{v}_\tau\|_{m-1} \,], \qquad (4\text{–}75)$$

where $\mathfrak{Z} = \{(\mathbf{v_t}, \mathbf{v}_\tau) \in (\mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha^h)) \cap \mathbf{H}^m) \times \mathbf{H}^m(\Omega(\alpha^h)) \mid (\gamma_{\Gamma_\mathbf{g}}\mathbf{v_t}, \gamma_\Gamma \mathbf{v}_\tau) = (\mathbf{t}, \boldsymbol{\tau})\}$.

Proof: (i) follows from Lemma 4.6 and 4.7.

(ii); Since $(\mathbf{u}, p, \mathbf{q}, \theta) \in X \cap (\mathbf{H}^{3/2-\delta} \times H^{1/2-\delta} \times \mathbf{H}^{3/2-\delta} \times H^{1/2-\delta})$, $(\mathbf{t}, \boldsymbol{\tau}) \in \mathcal{H} \cap (\mathbf{H}^{-\delta}(\Gamma_\mathbf{g}) \times \mathbf{H}^{-\delta}(\Gamma))$. Hence, from (4–67) and (4–70), we obtain

$$\begin{aligned}\|(\mathbf{t}, \boldsymbol{\tau}) - (\mathcal{R}^h_{\Gamma_\mathbf{g}}\mathbf{t}, \mathcal{R}^h_\Gamma \boldsymbol{\tau})\|_\mathcal{H} &\leq C h^{1/2-\delta}[\, \|\mathbf{t}\|_{-\delta,\Gamma_\mathbf{g}} + \|\boldsymbol{\tau}\|_{-\delta,\Gamma} \,] \\ &\leq C h^{1/2-\delta}[\, \|\mathbf{u}\|_{3/2-\delta} + \|p\|_{1/2-\delta} + \|\mathbf{q}\|_{3/2-\delta} + \|\theta\|_{1/2-\delta} \,].\end{aligned}$$

Similar argument also results in (iii).

(iv); Suppose $m = 2$ and the domain is polygonal. Since $(\mathbf{t}, \boldsymbol{\tau}) = (-p\mathbf{n} + \dfrac{\partial \mathbf{u}}{\partial \mathbf{n}}, -\theta\mathbf{n} + \dfrac{\partial \mathbf{q}}{\partial \mathbf{n}})$, $(\mathbf{t}, \boldsymbol{\tau})$ is not continuous along the boundary and $(\mathbf{t}, \boldsymbol{\tau}) \in \mathbf{H}^{1/2-\delta}(\Gamma_\mathbf{g}) \times \mathbf{H}^{1/2-\delta}(\Gamma)$ for $0 < \delta < \dfrac{1}{2}$. So, from (4–67) it follows that

$$\begin{aligned}\|(\mathbf{t}, \boldsymbol{\tau})\|_\mathcal{H} &\leq C h^{1-\delta}[\, \|\mathbf{t}\|_{1-\delta,\Gamma_\mathbf{g}} + \|\boldsymbol{\tau}\|_{1-\delta,\Gamma}] \\ &\leq C h^{1-\delta}[\, \|\mathbf{u}\|_2 + \|p\|_1 + \|\mathbf{q}\|_2 + \|\theta\|_1 \,].\end{aligned}$$

(v); Since $\mathbf{t} \in \mathbf{H}^s(\Gamma_\mathbf{g})$ and $\boldsymbol{\tau} \in \mathbf{H}^s(\Gamma)$ for $s \geq \dfrac{1}{2}$, $\mathcal{R}^h_{\Gamma_\mathbf{g}} = \mathcal{P}^h_{\Gamma_\mathbf{g}}$ and $\mathcal{R}^h_\Gamma = \mathcal{P}^h$ so that we obtain from (4–44)

$$\|(\mathbf{t}, \boldsymbol{\tau}) - (\mathcal{R}^h_{\Gamma_\mathbf{g}}\mathbf{t}, \mathcal{R}^h_\Gamma \boldsymbol{\tau})\|_\mathcal{H} \leq C h^{m-1}[\, \|\mathbf{v_t}\|_{m-1} + \|\mathbf{v}_\tau\|_{m-1}],$$

where $\mathbf{v_t}$ and $\mathbf{v}_\tau$ are liftings of $\mathbf{t}$ and $\boldsymbol{\tau}$, respectively. Since this holds for all $\mathbf{v_t}$ and $\mathbf{v}_\tau$, (4–74) follows. □

The main estimates for $\|(\mathbf{t}^h_\epsilon, \boldsymbol{\tau}^h_\epsilon) - (\mathbf{t}, \boldsymbol{\tau})\|_\mathcal{H}$ are found by Theorem 4.3 and Lemma 4.8.

Theorem 4.4. *Assume that Λ be a compact subset of R^+.*

Suppose $\{(\lambda, (\mathbf{u}(\lambda), \lambda p(\lambda), \mathbf{q}(\lambda), \lambda\theta(\lambda)) \mid \lambda = \dfrac{1}{\nu} \in \Lambda\}$ be a regular branch of solutions of (4–26)-(4–27) and let $(\mathbf{u}_\epsilon^h(\lambda), p_\epsilon^h(\lambda), \mathbf{q}_\epsilon^h(\lambda), \theta_\epsilon^h(\lambda)) \in X$ be the solution of (4–22)-(4–23) satisfying (4–63) in the neighborhood of origin in X. Let $(\mathbf{t}(\lambda), \tau(\lambda)) \in \mathcal{H}$ and $(\mathbf{t}_\epsilon^h(\lambda), \tau_\epsilon^h(\lambda)) \in \mathbf{P}_{\Gamma_\mathbf{g}}^h \times \mathbf{P}^h \subset \mathcal{H}$ be corresponding solutions of the system (4–28)-(4–29) and (4–24)-(4–25), respectively. Then, for $\epsilon \le \epsilon_0$ small enough, we have the following error estimates on the same branch :

(i) $\|(\mathbf{t}_\epsilon^h(\lambda), \tau_\epsilon^h(\lambda)) - (\mathbf{t}(\lambda), \tau(\lambda))\|_{\mathcal{H}} \le C(1 + \|\mathbf{u}\|_1 + \|\mathbf{q}\|_1)\,\epsilon$, *uniformly as $h \to 0^+$.*

(ii) *If $\Omega(\alpha^h) \in \mathcal{C}^{0,1}$ and $\mathbf{g} \in \mathbf{H}^{3/2}(\Gamma_\mathbf{g})$,*

$$\|(\mathbf{t}_\epsilon^h(\lambda), \tau_\epsilon^h(\lambda)) - (\mathbf{t}(\lambda), \tau(\lambda))\|_{\mathcal{H}} \le$$
$$(1 + \|\mathbf{u}\|_1 + \|\mathbf{q}\|_1)[\,C_1\epsilon + C_2 h^{1/2-\delta}(\|\mathbf{u}\|_{3/2-\delta} + \|p\|_{1/2-\delta} + \|\mathbf{q}\|_{3/2-\delta} + \|\theta\|_{1/2-\delta})\,],$$

for some small $\delta > 0$.

Under more assumed regularities for the domain, we suppose $(\mathbf{u}(\lambda), p(\lambda), \mathbf{q}(\lambda), \theta(\lambda)) \in X \cap (\mathbf{H}^m \times H^{m-1} \times \mathbf{H}^m \times H^{m-1})$ for $1 \le m \le k$ and let $(\mathbf{u}_\epsilon^h(\lambda), p_\epsilon^h(\lambda), \mathbf{q}_\epsilon^h(\lambda), \theta_\epsilon^h(\lambda))$ be corresponding approximate solution on the same branch satisfying (4–64). Then, the error estimates can be sharpened as followings :

(iii) *If $1 \le m < 2$, we have*

$$\|(\mathbf{t}(\lambda), \tau(\lambda)) - (\mathcal{R}_{\Gamma_\mathbf{g}}^h \mathbf{t}(\lambda), \mathcal{R}_\Gamma^h \tau(\lambda))\|_{\mathcal{H}} \le$$
$$(1 + \|\mathbf{u}\|_1 + \|\mathbf{q}\|_1)[\,C_1\epsilon + C_2 h^{m-1}(\|\mathbf{u}\|_m + \|p\|_{m-1} + \|\mathbf{q}\|_m + \|\theta\|_{m-1})\,].$$

(iv) *If $m = 2$ and $\Omega(\alpha^h)$ is polygonal, it holds that*

$$\|(\mathbf{t}(\lambda), \tau(\lambda)) - (\mathcal{R}_{\Gamma_\mathbf{g}}^h \mathbf{t}(\lambda), \mathcal{R}_\Gamma^h \tau(\lambda))\|_{\mathcal{H}} \le$$
$$(1 + \|\mathbf{u}\|_1 + \|\mathbf{q}\|_1)[\,C_1\epsilon + C_2 h^{1-\delta}(\|\mathbf{u}\|_2 + \|p\|_1 + \|\mathbf{q}\|_2 + \|\theta\|_1)\,],$$

for all $0 < \delta < \dfrac{1}{2}$.

(v) *If* $2 \leq m \leq k$ *and* $(\mathbf{t}(\lambda), \boldsymbol{\tau}(\lambda)) \in \gamma_{\Gamma_{\mathbf{g}}}(\mathbf{H}^1_{\Gamma_0(\alpha_h)}(\Omega) \cap \mathbf{H}^{m-1}) \times \gamma_{\Gamma}(\mathbf{H}^{m-1}(\Omega))$, *we obtain*

$$\|(\mathbf{t}(\lambda), \boldsymbol{\tau}(\lambda)) - (\mathcal{R}^h_{\Gamma_{\mathbf{g}}}\mathbf{t}(\lambda), \mathcal{R}^h_{\Gamma}\boldsymbol{\tau}(\lambda))\|_{\mathcal{H}} \leq$$
$$(1 + \|\mathbf{u}\|_1 + \|\mathbf{q}\|_1)[\, C_1\epsilon + C_2 h^{m-1} \inf_{(\mathbf{v_t}, \mathbf{v}_{\boldsymbol{\tau}}) \in \mathfrak{Z}} (\, \|\mathbf{v_t}\|_{m-1} + \|\mathbf{v}_{\boldsymbol{\tau}}\|_{m-1})\,]\,,$$

where $\mathfrak{Z} = \{(\mathbf{v_t}, \mathbf{v}_{\boldsymbol{\tau}}) \in \mathbf{H}^1_{\Gamma_0(\alpha)}(\Omega(\alpha^h)) \cap \mathbf{H}^m) \times \mathbf{H}^m(\Omega(\alpha^h)) \mid (\gamma_{\Gamma_{\mathbf{g}}}\mathbf{v_t}, \gamma_{\Gamma}\mathbf{v}_{\boldsymbol{\tau}}) = (\mathbf{t}, \boldsymbol{\tau})\}$. *Here,* C, C_1 *and* C_2 *are constants taken independent of* ϵ, h *and* $\lambda \in \Lambda$.

Proof: From the triangle inequality, we have

$$\|(\mathbf{t}, \boldsymbol{\tau}) - (\mathbf{t}^h_\epsilon, \boldsymbol{\tau}^h_\epsilon)\|_{\mathcal{H}} \leq \|(\mathbf{t}, \boldsymbol{\tau}) - (\mathcal{R}^h_{\Gamma_{\mathbf{g}}}\mathbf{t}, \mathcal{R}^h_{\Gamma}\boldsymbol{\tau})\|_{\mathcal{H}} + \|(\mathcal{R}^h_{\Gamma_{\mathbf{g}}}\mathbf{t}, \mathcal{R}^h_{\Gamma}\boldsymbol{\tau}) - (\mathbf{t}^h_\epsilon, \boldsymbol{\tau}^h_\epsilon)\|_{\mathcal{H}}\,. \quad (4\text{–}76)$$

Since the estimates for $\|(\mathbf{t}, \boldsymbol{\tau}) - (\mathcal{R}^h_{\Gamma_{\mathbf{g}}}\mathbf{t}, \mathcal{R}^h_{\Gamma}\boldsymbol{\tau})\|_{\mathcal{H}}$ are given in Lemma 4.8, it is sufficient to find estimates for $\|(\mathcal{R}^h_{\Gamma_{\mathbf{g}}}\mathbf{t}, \mathcal{R}^h_{\Gamma}\boldsymbol{\tau})\|_{\mathcal{H}}$. By the definition of $(\mathcal{R}^h_{\Gamma_{\mathbf{g}}}\mathbf{t}, \mathcal{R}^h_{\Gamma}\boldsymbol{\tau})$, we have

$$< \mathcal{R}^h_{\Gamma_{\mathbf{g}}}\mathbf{t}, \mathbf{v}^h >_{\Gamma_{\mathbf{g}}} = < \mathbf{t}, \mathbf{v}^h >_{\Gamma_{\mathbf{g}}} = \nu a\,(\mathbf{u}, \mathbf{v}^h) + b\,(\mathbf{v}^h, p)$$
$$+ c\,(\mathbf{u}, \mathbf{u}, \mathbf{v}^h) - < \mathbf{f}, \mathbf{v}^h >_{-1} \quad \forall\, \mathbf{v}^h \in \mathbf{V}^h_{\Gamma_{\mathbf{g}}}\,, \quad (4\text{–}77)$$

and

$$< \mathcal{R}^h_{\Gamma}\boldsymbol{\tau}, \mathbf{w}^h >_{\Gamma} = < \boldsymbol{\tau}, \mathbf{w}^h >_{\Gamma} = \nu a\,(\mathbf{q}, \mathbf{w}^h) + b\,(\mathbf{w}^h, \theta) + 2c\,(\mathbf{u}, \mathbf{u}, \mathbf{w}^h) + c\,(\mathbf{w}^h, \mathbf{u}, \mathbf{q})$$
$$+ c\,(\mathbf{u}, \mathbf{w}^h, \mathbf{q}) - 2 < \mathbf{f}, \mathbf{w}^h >_{-1} \quad \forall\, \mathbf{w}^h \in \mathbf{V}^h\,. \quad (4\text{–}78)$$

Since $\mathbf{V}^h \subset \mathbf{H}^1(\Omega(\alpha^h))$ and $(\mathbf{u}, p, \mathbf{q}, \theta) \in X$ is the solution of the system (4–26)–(4–27), (4–77) and (4–78) are justified in the same sense with (4–28) and (4–29), respectively. By subtracting (4–24) from (4–77), we obtain

$$< \mathcal{R}^h_{\Gamma_{\mathbf{g}}}\mathbf{t} - \mathbf{t}^h_\epsilon, \mathbf{v}^h >_{\Gamma_{\mathbf{g}}} = \nu a\,(\mathbf{u} - \mathbf{u}^h_\epsilon, \mathbf{v}^h) + b\,(\mathbf{v}^h, p - p^h_\epsilon) + c\,(\mathbf{u} - \mathbf{u}^h_\epsilon, \mathbf{u} - \mathbf{u}^h_\epsilon, \mathbf{v}^h)$$
$$+ c\,(\mathbf{u} - \mathbf{u}^h_\epsilon, \mathbf{u}, \mathbf{v}^h) + c\,(\mathbf{u}, \mathbf{u} - \mathbf{u}^h_\epsilon, \mathbf{v}^h)\,. \quad (4\text{–}79)$$

Let $\boldsymbol{\xi}^h = \mathcal{R}^h_{\Gamma_{\mathbf{g}}}\mathbf{t} - \mathbf{t}^h_\epsilon \in \mathbf{P}^h_{\Gamma_{\mathbf{g}}}$ and $\mathbf{v}^h_{\boldsymbol{\xi}^h}$ be a lifting of $\boldsymbol{\xi}^h$ such that $\|\mathbf{v}^h_{\boldsymbol{\xi}^h}\|_1 \le C\,\|\boldsymbol{\xi}^h\|_{1/2,\Gamma_{\mathbf{g}}} \le C\,h^{-1/2}\|\mathcal{R}^h_{\Gamma_{\mathbf{g}}}\mathbf{t} - \mathbf{t}^h_\epsilon\|_{0,\Gamma_{\mathbf{g}}}$, which is followed by the inverse inequality (4–19). Hence, by setting $\mathbf{v}^h = \mathbf{v}^h_{\boldsymbol{\xi}^h}$ in (4–78), we have that

$$\begin{aligned}\|\mathcal{R}^h_{\Gamma_{\mathbf{g}}}\mathbf{t} - \mathbf{t}^h_\epsilon\|^2_{0,\Gamma_{\mathbf{g}}} &\le (\nu\,\|\mathbf{u} - \mathbf{u}^h_\epsilon\|_1 + \|p - p^h_\epsilon\|_0 + \|\mathbf{u} - \mathbf{u}^h_\epsilon\|_1^2 + 2\|\mathbf{u}\|_1\|\mathbf{u} - \mathbf{u}^h_\epsilon\|_1)\|\mathbf{v}^h_{\boldsymbol{\xi}^h}\|_1\\ &\le Ch^{-1/2}\mathfrak{M}(\mathbf{u}, p, \mathbf{u}^h_\epsilon, p^h_\epsilon)\|\mathcal{R}^h_{\Gamma_{\mathbf{g}}}\mathbf{t} - \mathbf{t}^h_\epsilon\|_{0,\Gamma_{\mathbf{g}}}\,,\end{aligned}$$

where $\mathfrak{M}(\mathbf{u}, p, \mathbf{u}^h_\epsilon, p^h_\epsilon) = \nu\,\|\mathbf{u} - \mathbf{u}^h_\epsilon\|_1 + \|p - p^h_\epsilon\|_0 + \|\mathbf{u} - \mathbf{u}^h_\epsilon\|_1^2 + 2\|\mathbf{u}\|_1\|\mathbf{u} - \mathbf{u}^h_\epsilon\|_1$. Thus,

$$\|\mathcal{R}^h_{\Gamma_{\mathbf{g}}}\mathbf{t} - \mathbf{t}^h_\epsilon\|_{0,\Gamma_{\mathbf{g}}} \le Ch^{-1/2}\mathfrak{M}(\mathbf{u}, p, \mathbf{u}^h_\epsilon, p^h_\epsilon)\,. \tag{4–80}$$

Let $\boldsymbol{\phi} \in \mathbf{H}^{1/2}(\Gamma_{\mathbf{g}})$ and $\mathcal{P}^h_{\Gamma_{\mathbf{g}}}(\boldsymbol{\phi}) = \boldsymbol{\phi}^h$. Let $\mathbf{v}^h_{\boldsymbol{\phi}^h}$ be a lifting of $\boldsymbol{\phi}^h$ in $\mathbf{V}^h_{\Gamma_0(\alpha_h)}$ such that $\|\mathbf{v}^h_{\boldsymbol{\phi}^h}\|_1 \le C\|\mathcal{P}^h_{\Gamma_{\mathbf{g}}}\boldsymbol{\phi}\|_{1/2,\Gamma_{\mathbf{g}}} \le C\|\boldsymbol{\phi}\|_{1/2,\Gamma_{\mathbf{g}}}$, which is followed by (4–68). From (4–39), we also note that $\|(\mathcal{I} - \mathcal{P}^h_{\Gamma_{\mathbf{g}}})\boldsymbol{\phi}\|_{0,\Gamma_{\mathbf{g}}} \le Ch^{1/2}\|\boldsymbol{\phi}\|_{1/2,\Gamma_{\mathbf{g}}}$. Then, using (4–80), we have that

$$\begin{aligned}< \mathcal{R}^h_{\Gamma_{\mathbf{g}}}\mathbf{t} - \mathbf{t}^h_\epsilon, \boldsymbol{\phi} >_{\Gamma_{\mathbf{g}}} &=< \mathcal{R}^h_{\Gamma_{\mathbf{g}}}\mathbf{t} - \mathbf{t}^h_\epsilon, (\mathcal{I} - \mathcal{P}^h_{\Gamma_{\mathbf{g}}})\boldsymbol{\phi} + \mathcal{P}^h_{\Gamma_{\mathbf{g}}}\boldsymbol{\phi} >_{\Gamma_{\mathbf{g}}}\\ &=< \mathcal{R}^h_{\Gamma_{\mathbf{g}}}\mathbf{t} - \mathbf{t}^h_\epsilon, (\mathcal{I} - \mathcal{P}^h_{\Gamma_{\mathbf{g}}})\boldsymbol{\phi} >_{\Gamma_{\mathbf{g}}} + \nu a\,(\mathbf{u} - \mathbf{u}^h_\epsilon, \mathbf{v}^h_{\boldsymbol{\phi}^h}) + b\,(\mathbf{v}^h_{\boldsymbol{\phi}^h}), p - p^h_\epsilon)\\ &\quad + c\,(\mathbf{u} - \mathbf{u}^h_\epsilon, \mathbf{u} - \mathbf{u}^h_\epsilon, \mathbf{v}^h_{\boldsymbol{\phi}^h}) + c\,(\mathbf{u} - \mathbf{u}^h_\epsilon, \mathbf{u}, \mathbf{v}^h_{\boldsymbol{\phi}^h}) + c\,(\mathbf{u}, \mathbf{u} - \mathbf{u}^h_\epsilon, \mathbf{v}^h_{\boldsymbol{\phi}^h})\\ &\le \|\mathcal{R}^h_{\Gamma_{\mathbf{g}}}\mathbf{t} - \mathbf{t}^h_\epsilon\|_0\|(\mathcal{I} - \mathcal{P}^h_{\Gamma_{\mathbf{g}}})\boldsymbol{\phi}\|_0 + C\mathfrak{M}(\mathbf{u}, p, \mathbf{u}^h_\epsilon, p^h_\epsilon)\|\mathbf{v}^h_{\boldsymbol{\phi}^h}\|_1\\ &\le Ch^{-1/2}\mathfrak{M}(\mathbf{u}, p, \mathbf{u}^h_\epsilon, p^h_\epsilon)Ch^{1/2}\|\boldsymbol{\phi}\|_{1/2,\Gamma_{\mathbf{g}}} + C\mathfrak{M}(\mathbf{u}, p, \mathbf{u}^h_\epsilon, p^h_\epsilon)\|\boldsymbol{\phi}\|_{1/2,\Gamma_{\mathbf{g}}}\\ &\le C\mathfrak{M}(\mathbf{u}, p, \mathbf{u}^h_\epsilon, p^h_\epsilon)\|\boldsymbol{\phi}\|_{1/2,\Gamma_{\mathbf{g}}}\,.\end{aligned}$$

Thus,

$$\begin{aligned}\|\mathcal{R}^h_{\Gamma_{\mathbf{g}}}\mathbf{t} - \mathbf{t}^h_\epsilon\|_{-1/2,\Gamma_{\mathbf{g}}} &= \sup_{\substack{\boldsymbol{\phi}\in\mathbf{H}^{1/2}(\Gamma_{\mathbf{g}})\\ \|\boldsymbol{\phi}\|_{1/2,\Gamma_{\mathbf{g}}}\le 1}} < \mathcal{R}^h_{\Gamma_{\mathbf{g}}}\mathbf{t} - \mathbf{t}^h_\epsilon, \boldsymbol{\phi} >_{\Gamma_{\mathbf{g}}}\\ &\le C\mathfrak{M}(\mathbf{u}, p, \mathbf{u}^h_\epsilon, p^h_\epsilon)\,.\end{aligned} \tag{4–81}$$

In a similar fashion, one can show from (4–78) that

$$\|\mathcal{R}^h_\Gamma\boldsymbol{\tau} - \boldsymbol{\tau}^h_\epsilon\|_{-1/2,\Gamma} \le C\mathfrak{N}(\mathbf{q}, \theta, \mathbf{q}^h_\epsilon, \theta^h_\epsilon, \mathbf{u}, \mathbf{u}^h_\epsilon)\,, \tag{4–82}$$

where

$$\mathfrak{N}(\mathbf{q},\theta,\mathbf{q}_\epsilon^h,\theta_\epsilon^h,\mathbf{u},\mathbf{u}_\epsilon^h) = \nu\,\|\mathbf{q}-\mathbf{q}_\epsilon^h\|_1 + \|\theta-\theta_\epsilon^h\|_0 + 2\|\mathbf{u}-\mathbf{u}_\epsilon^h\|_1^2$$
$$+\, 2\|\mathbf{u}-\mathbf{u}_\epsilon^h\|_1\|\mathbf{q}\|_1 + 6\|\mathbf{u}-\mathbf{u}_\epsilon^h\|_1\|\mathbf{u}\|_1 + 2\|\mathbf{u}-\mathbf{u}_\epsilon^h\|_1\|\mathbf{q}-\mathbf{q}_\epsilon^h\|_1\,.$$

Now, let $\{\lambda, (\mathbf{u}(\lambda), \lambda p(\lambda), \mathbf{q}(\lambda), \lambda\theta(\lambda)) \mid \lambda = \dfrac{1}{\nu} \in \Lambda\}$ be a regular branch of solutions of the system (4–26)–(4–27). Let $(\mathbf{u}_\epsilon^h(\lambda), p_\epsilon^h(\lambda), \mathbf{q}_\epsilon^h(\lambda), \theta_\epsilon^h(\lambda)) \in X$ be the solution of (4–22)–(4–23) satisfying (4–63) for $\epsilon \le \epsilon_0$. Then, since $\nu = \dfrac{1}{\lambda} \in \Lambda \subset R^+$ is bounded, we obviously have in the neighborhood of origin of X,

$$\|(\mathcal{R}_{\Gamma_\mathbf{g}}^h \mathbf{t}(\lambda), \mathcal{R}_\Gamma^h \boldsymbol{\tau}(\lambda)) - (\mathbf{t}_\epsilon^h(\lambda), \boldsymbol{\tau}_\epsilon^h(\lambda))\|_{\mathcal{H}} \le C(\mathfrak{M}+\mathfrak{N})$$
$$\le C(\|\mathbf{u}(\lambda)\|_1 + \|\mathbf{q}(\lambda)\|_1)\|(\mathbf{u}(\lambda), p(\lambda), \mathbf{q}(\lambda), \theta(\lambda)) - (\mathbf{u}_\epsilon^h(\lambda), p_\epsilon^h(\lambda), \mathbf{q}_\epsilon^h(\lambda), \theta_\epsilon^h(\lambda))\|_X\,.$$

Hence, combined (4–76) with (4–63) and (4–71), (i) is obtained. The other estimates are also obtained in a similar manner. □

From the result of Theorem 4.4, we can conclude that the split formulation for the optimality system and projection techniques discussed in this section yield optimal error estimates. An obvious advantage of this approach can be realized when the boundary data with less regularity is dealt with.

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