Hindawi Publishing Corporation
Journal of Function Spaces
Volume 2016, Article ID 3970621, 15 pages
http://dx.doi.org/10.1155/2016/3970621

Research Article

A New Topological Degree Theory for Perturbations of Demicontinuous Operators and Applications to Nonlinear Equations with Nonmonotone Nonlinearities

Teffera M. Asfaw

Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, USA

Correspondence should be addressed to Teffera M. Asfaw; tefferam@yahoo.com

Received 30 June 2016; Accepted 18 October 2016

Academic Editor: Adrian Petrusel

Let X be a real reflexive locally uniformly convex Banach space with locally uniformly convex dual space X^*. Let $T : X \supseteq D(T) \to 2^{X^*}$ be maximal monotone of type Γ_d^ϕ (i.e., there exist $d \geq 0$ and a nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that $\langle v^*, x - y\rangle \geq -d\|x\| - \phi(\|y\|)$ for all $x \in D(T)$, $v^* \in Tx$, and $y \in X$), $L : X \supset D(L) \to X^*$ be linear, surjective, and closed such that $L^{-1} : X^* \to X$ is compact, and $C : X \to X^*$ be a bounded demicontinuous operator. A new degree theory is developed for operators of the type $L+T+C$. The surjectivity of L can be omitted provided that $R(L)$ is closed, L is densely defined and self-adjoint, and $X = H$, a real Hilbert space. The theory improves the degree theory of Berkovits and Mustonen for $L + C$, where C is bounded demicontinuous pseudomonotone. New existence theorems are provided. In the case when L is monotone, a maximality result is included for L and $L+T$. The theory is applied to prove existence of weak solutions in $X = L^2(0, T; H_0^1(\Omega))$ of the nonlinear equation given by $\partial u/\partial t - \sum_{i=1}^N ((\partial/\partial x_i) A_i(x, u, \nabla u)) + H_\lambda(x, u, \nabla u) = f(x, t)$, $(x, t) \in Q_T$; $u(x, t) = 0$, $(x, t) \in \partial Q_T$; and $u(x, 0) = u(x, T)$, $x \in \Omega$, where $\lambda > 0$, $Q_T = \Omega \times (0, T)$, $\partial Q_T = \partial\Omega \times (0, T)$, $A_i(x, u, \nabla u) = (\partial/\partial x_i)\rho(x, u, \nabla u) + a_i(x, u, \nabla u)$ $(i = 1, 2, \ldots, N)$, $H_\lambda(x, u, \nabla u) = -\lambda\Delta u + g(x, u, \nabla u)$, Ω is a nonempty, bounded, and open subset of $\mathbb{R}^N$ with smooth boundary, and $\rho, a_i, g : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy suitable growth conditions. In addition, a new existence result is given concerning existence of weak solutions for nonlinear wave equation with nonmonotone nonlinearity.

1. Introduction and Preliminaries

Throughout the paper, $(X, \|\cdot\|)$ denotes a real reflexive locally uniformly convex Banach space with locally uniformly convex dual space X^*. For $x \in X$ and $x^* \in X^*$, the duality pairing $\langle x^*, x\rangle$ denotes the value $x^*(x)$. Let $J : X \to 2^{X^*}$ be the normalized duality mapping given by

$$J(x) = \left\{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2,\ \|x^*\| = \|x\|\right\}. \quad (1)$$

It is well-known that $J(x) \neq \emptyset$ for all $x \in X$ because of the Hahn-Banach Theorem. Since X and X^* are locally uniformly convex reflexive Banach spaces, it is well-known that J is single valued and homeomorphism. For a multivalued operator T from X into X^*, the domain of T denoted by $D(T)$ is given as $D(T) = \{x \in X : Tx \neq \emptyset\}$. The range of T, denoted by $R(T)$, is given by $R(T) = \bigcup_{x\in D(T)} Tx$ and graph of T, denoted by $G(T)$, is given by $G(T) = \{(x, v^*) : x \in D(T),\ v^* \in Tx\}$. The following definition is used in the squeal.

Definition 1. A multivalued operator $T : X \supseteq D(T) \to 2^{X^*}$ is called

(i) "monotone" if, for all $x \in D(T)$, $y \in D(T)$, $v^* \in Tx$, and $u^* \in Ty$, we have $\langle v^* - u^*, x - y\rangle \geq 0$,

(ii) "maximal monotone" if T is monotone and $\langle u^* - u_0^*, x - x_0\rangle \geq 0$ for every $(x, u^*) \in G(T)$ implies $x_0 \in D(T)$ and $u_0^* \in Tx_0$. This is equivalent to saying that T is "maximal monotone" if and only if $R(T + \lambda J) = X^*$ for every $\lambda > 0$,

(iii) "coercive" if either $D(T)$ is bounded or there exists a function $\psi : [0,\infty) \to (-\infty,\infty)$ such that $\psi(t) \to \infty$ as $t \to \infty$ and

$$\langle y^*, x\rangle \geq \psi(\|x\|)\|x\| \quad \forall x \in D(T),\ y^* \in Tx. \quad (2)$$

It is the goal of the paper to develop a topological degree theory for classes of operators of the type $L+T+C$, where L, T, and C satisfy one of the following conditions:

(i) $L : X \supset D(L) \to X^*$ is linear, surjective, and closed such that $L^{-1} : X^* \to X$ is compact, $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone of type Γ_d^{ϕ}, and $C : X \to X^*$ is bounded demicontinuous operator.

(ii) $X = H$, a real Hilbert space, $L : H \supset D(L) \to H$ is linear, densely defined, self-adjoint, closed, and range closed such that $L^{-1} : R(L) \to H$ is compact, $T : H \supset D(T) \to 2^H$ is maximal monotone of type Γ_d^{ϕ}, and $C : H \to H$ is bounded demicontinuous operator.

The main reason for the need of such a theory is the existence of nonlinear problems (i.e., nonlinear equations and variational inequality problems) which cannot be addressed by the existing theories under minimal assumptions on L, C, and T. In addition, considering the classes of operators of the type $L + T + C$, it is an essential contribution to have a theory useful to drive existence theorems to treat larger class of problems. Therefore, Section 2 gives a preliminary lemma, which will be useful to extend the definition of pseudomonotone homotopy of maximal monotone operators initially introduced by Browder [1, 2]. Section 3 deals with the construction of the degree mapping along with basic properties and homotopy invariance results. The main contribution of this work is providing a new degree theory for treating nonlinear problems involving operators of type $L + T + C$, where L, T, and C satisfy condition (i) or (ii). In this theory, the operator L might not be pseudomonotone type and C is just bounded demicontinuous operator. The well-known degree for monotone type operators, which is attributed to Browder [1, 2], is for operators of type $T + C$, where $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone and $C : \overline{G} \to X^*$ is bounded demicontinuous operator of type (S_+). In view of this, the degree mapping constructed herein allows C to be bounded demicontinuous operator not necessarily compact, bounded of type (S_+), or pseudomonotone. To the best of the author's knowledge, this degree mapping is new and has the potential to address new classes of problems such as wave equations with nonmonotone nonlinearities. As a consequence of the theory, new existence results are given for the solvability of operator inclusions of the type $Lu + Tu + Cu \ni f^*$, $u \in D(L) \cap D(T)$. In the last section, examples are provided proving existence of weak solutions for nonlinear parabolic as well as hyperbolic problems in appropriate Sobolev spaces. For degree theories for bounded demicontinuous (S_+) perturbations of maximal monotone operators, the reader is referred to the papers of Browder [1, 2], Kobayashi and Ôtani [3], Hu and Papageorgiou [4], Berkovits and Mustonen [5, 6], Berkovits [7], Kartsatos and Skrypnik [8], and Kien et al. [9] and the references therein. For recent topological degree theories for bounded pseudomonotone perturbations of maximal monotone operators, the reader is referred to the recent papers of Asfaw and Kartsatos [10] and Asfaw [11]. Basic definitions, properties, and existence theorems concerning operators of monotone type can be found in the books of Barbu [12, 13], Pascali and Sburlan [14], and Zeidler [15].

2. A Preliminary Lemma

The following lemma is useful towards the extension of the definition of a pseudomonotone homotopy of maximal monotone operators introduced by Browder [1, 2].

Definition 2. A family $\{T^t\}_{t\in[0,1]}$ is said to be uniformly of type

(i) Γ_d^{ϕ} if there exist $d \geq 0$ and a nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(0) = 0$ and

$$\left\langle v^t, x - y\right\rangle \geq -d\,\|x\| - \phi\left(\|y\|\right) \tag{3}$$

uniformly for all $t \in [0,1]$ and $x \in D(T^t)$, $v^t \in T^t x$, and $y \in X$,

(ii) Γ_d if there exists $d \geq 0$ such that $\langle v^t, x\rangle \geq -d\|x\|$ uniformly for all $t \in [0,1]$ and $x \in D(T^t)$ and $v^t \in T^t x$.

If $T^t = T$ for all $t \in [0,1]$, then the operator T is said to be of type Γ_d^{ϕ} or Γ_d if it satisfies either (i) or (ii), respectively.

It is easy to see that a family of monotone operators $\{T^t\}_{t\in[0,1]}$ is uniformly of type Γ_d if $0 \in T^t(0)$ for all $t \in [0,1]$. It is also true that the class Γ_d includes the class Γ_d^{ϕ}. The lemma below is used in the construction of the degree.

Lemma 3. *Let $\{T^t\}_{t\in[0,1]}$ be a family of maximal monotone operators uniformly of type Γ_d. Then the following four conditions are equivalent:*

(i) *For any sequences t_n in $[0,1]$, $u_n \in D(T^{t_n})$ and $w_n^* \in T^{t_n}u_n$ such that $u_n \rightharpoonup u$ in X, and $t_n \to t_0 \in [0,1]$ and $w_n^* \rightharpoonup w_0^*$ in X^* as $n \to \infty$ with $\limsup_{n\to\infty}\langle w_n^*, u_n - u\rangle \leq 0$, it follows that $u \in D(T^{t_0})$, $w_0^* \in T^{t_0}u$, and $\langle w_n^*, u_n\rangle \to \langle w_0^*, u\rangle$ as $n \to \infty$.*

(ii) *For each $\varepsilon > 0$, the operator defined by $\psi(t,w) = (T^t + \varepsilon J)^{-1}w$ is continuous from $[0,1] \times X^*$ to X.*

(iii) *For each fixed $w \in X^*$, the operator defined by $\psi_w(t) = (T^t + \varepsilon J)^{-1}w$ is continuous from $[0,1]$ to X.*

(iv) *For any given pair $(x,u) \in G(T^{t_0})$ and any sequence $t_n \to t_0$ as $n \to \infty$, there exist sequences $\{x_n\}$ and $\{u_n\}$ such that $u_n \in T^{t_n}x_n$ and $x_n \to x$ and $u_n \to u$ as $n \to \infty$.*

Proof. The proof for the implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) follows from the result attributed to to Browder [1, 2] without requiring the condition $0 \in T^t(0)$ for all $t \in [0,1]$. Next we give the proof of the implication (i) $\Rightarrow$ (ii). Fix $\varepsilon > 0$. Let $(t_n, w_n^*) \in [0,1] \times X^*$ such that $t_n \to t_0$ and $w_n^* \to w_0^*$ as $n \to \infty$. For each n, let $u_n = (T^{t_n} + \varepsilon J)^{-1}w_n^*$ and $u_0 = (T^{t_0} + \varepsilon J)^{-1}w_0^*$. It follows that $w_n^* = z_n^* + \varepsilon J u_n$ for some $z_n^* \in T^{t_n}u_n$, for all n, and $w_0^* = z_0^* + \varepsilon J u_0$ for some

$z_0^* \in T^{t_0}u_0$. We shall show that $\{u_n\}$ is bounded. By applying Γ_d condition on the family $\{T^t\}_{t\in[0,1]}$, we see that

$$\begin{aligned}
&\langle w_n^* - w_0^*, u_n - u_0\rangle \\
&= \langle z_n^* + \varepsilon J u_n - z_0^* - \varepsilon J u_0, u_n - u_0\rangle \\
&= \langle z_n^* - z_0^*, u_n - u_0\rangle + \varepsilon \langle J u_n - J u_0, u_n - u_0\rangle \\
&\geq \langle z_n^* - z_0^*, u_n - u_0\rangle \\
&\quad + \varepsilon \left(\|u_n\|^2 - 2\|u_n\|\|u_0\| + \|u_0\|^2\right) \\
&= \langle z_n^* - z_0^*, u_n - u_0\rangle + \varepsilon (\|u_n\| - \|u_0\|)^2 \\
&= \langle z_n^*, u_n\rangle - \langle z_n^*, u_0\rangle - \langle z_0^*, u_n\rangle + \langle z_0^*, u_0\rangle \\
&\quad + \varepsilon (\|u_n\| - \|u_0\|)^2 \\
&\geq -d\|u_n\| - \|z_n^*\|\|u_0\| - \|z_0^*\|\|u_n\| - d\|u_0\| \\
&\quad + \varepsilon (\|u_n\| - \|u_0\|)^2 \\
&\geq -d(\|u_n\| + \|u_0\|) - (\|w_n^*\| + \varepsilon\|u_n\|)\|u_0\| \\
&\quad - \|z_0^*\|\|u_n\| + \varepsilon (\|u_n\| - \|u_0\|)^2 \quad \forall n;
\end{aligned} \tag{4}$$

that is, we get

$$\begin{aligned}
\varepsilon (\|u_n\| - \|u_0\|)^2 \leq d(\|u_n\| + \|u_0\|) \\
+ (\|w_n^*\| + \varepsilon\|u_n\|)\|u_0\| + \|z_0^*\|\|u_n\| \\
+ \|w_n^* - w_0^*\|\|u_n - u_0\| \quad \forall n.
\end{aligned} \tag{5}$$

Since $\{w_n^*\}$ is bounded, (83) implies the boundedness of the sequence $\{u_n\}$. Assume, without loss of generality, that $u_n \rightharpoonup u, Ju_n \rightharpoonup z$ as $n \to \infty$. Since $w_n^* = z_n^* + \varepsilon J u_n$, it follows that $z_n^* \rightharpoonup y_0^* = w_0^* - \varepsilon z$ as $n \to \infty$. By the condition in (i) and monotonicity of J and $w_n^* \to w_0^*$ as $n \to \infty$ and boundedness of $\{u_n\}$, we obtain that

$$\begin{aligned}
\limsup_{n\to\infty} \langle z_n^*, u_n - u\rangle &= \limsup_{n\to\infty} \langle w_n^* - \varepsilon J u_n, u_n - u\rangle \\
&= \limsup_{n\to\infty} \langle w_n^*, u_n - u\rangle \\
&\quad - \varepsilon \liminf_{n\to\infty} \langle J u_n - J u, u_n - u\rangle \\
&\quad - \varepsilon \liminf_{n\to\infty} \langle J u, u_n - u\rangle \leq 0.
\end{aligned} \tag{6}$$

Consequently, we arrive at

$$\limsup_{n\to\infty} \langle z_n^*, u_n\rangle \leq \langle y_0^*, u\rangle. \tag{7}$$

Thus, by using conditions in (i), it follows that $u \in D(T^{t_0})$, $y_0^* \in T^{t_0}u$, and $\langle z_n^*, u_n\rangle \to \langle y_0^*, u\rangle$ as $n \to \infty$, which implies

$$\limsup_{n\to\infty} \langle J u_n, u_n - u\rangle = 0. \tag{8}$$

Since J is of type (S_+) and continuous, we have $u_n \to u$ and $Ju_n \to Ju$ as $n \to \infty$, which implies $y_0^* = w_0^* - \varepsilon J u$; that is, $w_0^* \in (T^{t_0} + \varepsilon J)u$, implying in turn that $u = (T^{t_0} + \varepsilon J)^{-1} w_0^*$; that is, $\psi(t_n, w_n^*) \to \psi(t, w_0^*)$ as $n \to \infty$. This shows that ψ is continuous from $[0,1]\times X^*$ into X. Therefore, the equivalency of the four statements is proved. □

A larger class of pseudomonotone homotopies of maximal monotone operators is introduced below. The original definition of pseudomonotone homotopy of maximal monotone operators $\{T^t\}_{t\in[0,1]}$ is attributed to Browder [2] which requires the family to satisfy $0 \in T^t(0)$ for all $t \in [0,1]$.

Definition 4. A family $\{T^t\}_{t\in[0,1]}$ of maximal monotone operators uniformly of type Γ_d^ϕ is called a "pseudomonotone homotopy of type Γ_d^ϕ" if one of the equivalent conditions of Lemma 3 holds.

3. Degree Theory in Reflexive Banach Space with $R(L) = X^*$

The section deals with the main contribution of the paper. A new topological degree mapping is constructed for operators of type $T + C + L$, where $T : X \supset D(T) \to 2^{X^*}$ is maximal monotone of type Γ_d^ϕ, $C : X \to X^*$ is bounded demicontinuous operator, and $L : X \supset D(L) \to X^*$ is linear, surjective, and closed such that $L^{-1} : X^* \to X$ is compact. The construction is based on the Leray-Schauder degree mapping for the operator $I + L^{-1}(T_\varepsilon + C)$, where $T_\varepsilon : X \to X^*$ is the Yosida approximant of T. Since L is surjective, L^{-1} is compact, and T_ε is bounded continuous operator, it follows that $L^{-1}(T_\varepsilon + C) : X \to X^*$ is a well-defined compact operator. Next we prove the following theorem.

Theorem 5. *Let G be a nonempty, bounded, and open subset of X. Let $\{T^t\}_{t\in[0,1]}$ be a pseudomonotone homotopy of maximal monotone operators uniformly of type Γ_d^ϕ and $\{C^t = tC_1 + (1-t)C_2\}_{t\in[0,1]}$ with $C_i\,(i = 1,2) : X \to X^*$ is bounded demicontinuous operator and let $L : X \supset D(L) \to X^*$ be linear, surjective, and closed such that $L^{-1} : X^* \to X$ is compact. Assume, further, that $0 \notin (L+T^t+C^t)(D(L)\cap D(T^t)\cap \partial G)$ for all $t \in [0,1]$. Then there exists $\varepsilon_0 > 0$ such that*

$$d_{LS}\left(I + L^{-1}\left(T_\varepsilon^t + C^t\right), G, 0\right) \tag{9}$$

is well-defined and independent of $\varepsilon \in (0, \varepsilon_0]$ and $t \in [0,1]$, where d_{LS} denotes the Leray-Schauder degree mapping for compact displacement of the identity and $T_\varepsilon^t : X \to D(T^t)$ is the Yosida approximant of T^t.

Proof. Suppose the hypotheses hold. Assume to the contrary that there exist $\varepsilon_n \downarrow 0^+$, $x_n \in \partial G$, and $t_n \in [0,1]$ such that

$$x_n + L^{-1}\left(T_{\varepsilon_n}^{t_n} x_n + C^{t_n} x_n\right) = 0 \quad \forall n. \tag{10}$$

Since L is surjective, it follows that $x_n = -L^{-1}(T_{\varepsilon_n}^{t_n} x_n + C^{t_n} x_n) \in D(L)$ and $Lx_n + T_{\varepsilon_n}^{t_n} x_n + C^{t_n} x_n = 0$ for all n. The uniform boundedness of the family $\{C^t\}_{t\in[0,1]}$ implies the boundedness of $\{C^{t_n} x_n\}$. Since T^{t_n} is of type Γ_d^ϕ, let $d \geq 0$ and ϕ be as in

Definition 2. Let $v_n^* = T_{\varepsilon_n}^{t_n}x_n$. It is well-known that $J_{\varepsilon_n}^{t_n}x_n \in D(T)$, $v_n^* \in T^{t_n}(J_{\varepsilon_n}^{t_n}x_n)$, and $J_{\varepsilon_n}^{t_n}x_n = x_n - \varepsilon_n J^{-1}(v_n^*)$ for all n. For each $x \in X$, we see that

$$\begin{aligned}\langle Lx_n, x_n - x\rangle &= -\langle v_n^*, x_n - x\rangle - \langle C^{t_n}x_n, x_n - x\rangle \\ &= -\langle v_n^*, x_n - J_{\varepsilon_n}^{t_n}x_n + J_{\varepsilon_n}^{t_n}x_n - x\rangle \\ &\quad - \langle C^{t_n}x_n, x_n - x\rangle \\ &\le -\langle v_n^*, \varepsilon_n J^{-1}(v_n^*)\rangle - \langle v_n^*, J_{\varepsilon_n}^{t_n}x_n - x\rangle \\ &\quad + \|C^{t_n}x_n\|\,\|x_n - x\| \\ &\le -\varepsilon_n \|v_n^*\|^2 + d\,\|J_{\varepsilon_n}^{t_n}x_n\| + \phi(\|x\|) \\ &\quad + \|C^{t_n}x_n\|\,\|x_n\| \\ &\le -\varepsilon_n \|v_n^*\|^2 + d\,\|x_n\| + d\varepsilon_n \|v_n^*\| \\ &\quad + \phi(\|x\|) + \|C^{t_n}x_n\|\,\|x_n\| \\ &= \left(-\varepsilon_n \|v_n^*\|^2 + d\varepsilon_n \|v_n^*\|\right) + d\,\|x_n\| \\ &\quad + \phi(\|x\|) + \|C^{t_n}x_n\|\,\|x_n\| \\ &= \left(-\varepsilon_n z_n^2 + d\varepsilon_n z_n\right) + d\,\|x_n\| + \phi(\|x\|) \\ &\quad + \|C^{t_n}x_n\|\,\|x_n\| \\ &\le \frac{d^2\varepsilon_n}{4} + d\,\|x_n\| + \phi(\|x\|) \\ &\quad + \|C^{t_n}x_n\|\,\|x_n\| \le K_1(x),\end{aligned} \tag{11}$$

where $z_n = \|v_n^*\|$ for all n and K_1 is an upper bound for $\{d^2 + \phi(\|x\|) + d\|x_n\| + \|C^{t_n}x_n\|\|x_n\|\}$. Now, setting $x_n - x$ in place of x, we obtain that

$$\langle Lx_n, x\rangle = \langle Lx_n, x_n - (x_n - x)\rangle \le k + \phi(\|x_n - x\|) \tag{12}$$

for all n. Since ϕ is nondecreasing and $\{x_n\}$ is bounded, we see that

$$\langle Lx_n, x\rangle \le K_2(x), \tag{13}$$

where $K_2(x)$ is an upper bound for $\{k+\phi(\|x_n-x\|)\}$. By similar argument, setting $x_n + x$ in place of x, we get

$$\langle Lx_n, -x\rangle = \langle Lx_n, x_n - (x_n + x)\rangle \le k + \phi(\|x_n + x\|) \tag{14}$$

for all n; that is, $\langle Lx_n, x\rangle \ge -k - \phi(\|x_n + x\|) \ge -K_3(x)$, where $K_3(x)$ is an upper bound for $\{k + \phi(\|x_n + x\|)\}$. For each $x \in X$, combining these two inequalities shows that there exists $N(x) \ge 0$ such that $|\langle Lx_n, x\rangle| \le N(x)$ for all n. By applying the well-known uniform boundedness principle, we conclude that $\{Lx_n\}$ is bounded. Consequently, we obtain the boundedness of $\{v_n^*\}$. Since $\{C^{t_n}x_n\}$ is bounded and L^{-1} is compact, we assume without loss of generality that there exists a subsequence, denoted again by $\{L^{-1}(v_n^* + C^{t_n}x_n)\}$, such that $L^{-1}(v_n^* + C^{t_n}x_n) \to x_0$ as $n \to \infty$; that is, $x_n \to x_0 \in \partial G$ as $n \to \infty$. Assume without loss of generality that $t_n \to t_0 \in [0,1]$, $v_n^* \rightharpoonup v_0^*$, $Lx_n \rightharpoonup y_0^*$, and $C^{t_n}x_n \rightharpoonup C^{t_0}x_0$ as $n \to \infty$. Since $x_n \to x_0$, it follows that $J_{\varepsilon_n}^{t_n}x_n = x_n - \varepsilon_n J^{-1}(v_n^*) \to x_0$ as $n \to \infty$; that is,

$$\limsup_{n\to\infty} \langle v_n^*, x_n - x_0\rangle = 0. \tag{15}$$

Since $\{T^t\}_{t\in[0,1]}$ is a pseudomonotone homotopy of type Γ_d^{ϕ}, (iv) of Lemma 3 implies that $x_0 \in D(T^{t_0})$, $v_0^* \in T^{t_0}x_0$, and $\langle v_n^*, x_n\rangle \to \langle v_0^*, x_0\rangle$ as $n \to \infty$. Since L is closed, we conclude that $x_0 \in D(L)$ and $Lx_0 = y_0^*$, which implies that $0 \in (L + T + C)(D(L) \cap D(T) \cap \partial G)$. However, this is impossible. Therefore, there exists $\varepsilon_0 > 0$ such that $d_{\mathrm{LS}}(I + L^{-1}(T_\varepsilon^t + C^t), G, 0)$ is well-defined for all $\varepsilon \in (0, \varepsilon_0]$ and $t \in [0,1]$. Next we prove that there exists $\varepsilon_1 \in (0, \varepsilon_0]$ such that $d_{\mathrm{LS}}(I + L^{-1}(T_\varepsilon^t + C^t), G, 0)$ is independent of $\varepsilon \in (0, \varepsilon_1]$ and $t \in [0,1]$. Suppose this is false; that is, there exist $\varepsilon_n \downarrow 0^+$, $\delta_n \downarrow 0^+$, and $t_n \in [0,1]$ such that

$$\begin{aligned}&d_{\mathrm{LS}}\left(I + L^{-1}\left(T_{\varepsilon_n}^{t_n} + C^{t_n}\right), G, 0\right) \\ &\quad \ne d_{\mathrm{LS}}\left(L^{-1}\left(T_{\delta_n}^{t_n} + C^{t_n}\right), G, 0\right)\end{aligned} \tag{16}$$

for all n. For each n, we consider the homotopy

$$\begin{aligned}H_n(s,x) = x + L^{-1}\left(sT_{\varepsilon_n}^{t_n}x + (1-s)\,T_{\delta_n}^{t_n}x\right) + L^{-1}C^{t_n}x, \\ (s,x) \in [0,1] \times \overline{G}.\end{aligned} \tag{17}$$

Since L^{-1} is compact and $T_{\varepsilon_n}^{t_n}$, $T_{\delta_n}^{t_n}$, and C^{t_n} are bounded continuous operators, we observe that $\{H_n(s,\cdot)\}_{s\in[0,1]}$ is Leray-Schauder type homotopy. We shall show that $\{H_n(s,\cdot)\}_{s\in[0,1]}$ is an admissible homotopy for all large n; that is, for all large n, we have $0 \notin H_n(s, \partial G)$ for all $s \in [0,1]$. Suppose there exists a subsequence of $\{n\}$, denoted again by $\{n\}$, such that there exist $x_n \in \partial G$, $t_n \in [0,1]$, and $s_n \in [0,1]$ such that

$$x_n + L^{-1}\left(s_n T_{\varepsilon_n}^{t_n}x_n + (1 - s_n)\,T_{\delta_n}^{t_n}x_n + C^{t_n}x_n\right) = 0 \tag{18}$$

for all n; that is, we have

$$Lx_n + s_n T_{\varepsilon_n}^{t_n}x_n + (1 - s_n)\,T_{\delta_n}^{t_n}x_n + C^{t_n}x_n = 0 \tag{19}$$

for all n. Assume without loss of generality that $s_n \to s_0 \in [0,1]$ and $t_n \to t_0 \in [0,1]$ as $n \to \infty$. For each n, let $v_n^* = T_{\varepsilon_n}^{t_n}x_n$, $u_n^* = T_{\delta_n}^{t_n}x_n$, and $z_n^* = s_n v_n^* + (1 - s_n)u_n^*$. It is well-known that $v_n^* \in T^{t_n}(J_{\varepsilon_n}^{t_n}x_n)$, $u_n^* \in T^{t_n}(J_{\delta_n}^{t_n}x_n)$, $J_{\varepsilon_n}^{t_n}x_n = x_n - \varepsilon_n J^{-1}(v_n^*)$, and $J_{\delta_n}^{t_n}x_n = x_n - \delta_n J^{-1}(u_n^*)$ for all n. Let $x \in X$. By the definition of pseudomonotone homotopy of type Γ_d^{ϕ} and uniform boundedness condition on $\{C^t\}_{t\in[0,1]}$, we see that

$$\begin{aligned}
&\langle Lx_n, x_n - x\rangle = -s_n\langle v_n^*, x_n - x\rangle - (1-s_n)\langle u_n^*, x_n \\
&\quad - x\rangle - \langle C^{t_n}x_n, x_n - x\rangle = -s_n\langle v_n^*, x_n - J_{\varepsilon_n}^{t_n}x_n \\
&\quad + J_{\varepsilon_n}^{t_n}x_n - x\rangle - (1-s_n)\langle u_n^*, x_n - J_{\delta_n}^{t_n}x_n + J_{\delta_n}^{t_n}x_n \\
&\quad - x\rangle + \|C^{t_n}x_n\|\|x_n\| = -s_n\langle v_n^*, \varepsilon_n J^{-1}(v_n^*)\rangle \\
&\quad - s_n\langle v_n^*, J_{\varepsilon_n}^{t_n}x_n - x\rangle - (1-s_n)\langle u_n^*, \delta_n J^{-1}(u_n^*)\rangle \\
&\quad - (1-s_n)\langle u_n^*, J_{\delta_n}^{t_n}x_n - x\rangle + \|C^{t_n}x_n\| \\
&\quad \le \left(-s_n\varepsilon_n\|v_n^*\|^2 + s_n d\|J_{\varepsilon_n}^{t_n}x_n\| + \phi(\|x\|)\right) \\
&\quad + \left(-(1-s_n)\delta_n\|u_n^*\|^2 + (1-s_n)d\|J_{\delta_n}^{t_n}x_n\|\right. \\
&\quad \left.+ \phi(\|x\|)\right) + \|C^{t_n}x_n\| = \left(-s_n\varepsilon_n\|v_n^*\|^2\right. \\
&\quad \left.+ s_n d\varepsilon_n\|v_n^*\|\right) + \left(-(1-s_n)\delta_n\|u_n^*\|^2\right. \\
&\quad \left.+ (1-s_n)d\delta_n\|u_n^*\|\right) + 2\phi(\|x\|) + d\|x_n\| \\
&\quad + \|C^{t_n}x_n\| \le \left(-\varepsilon_n\|\sqrt{s_n}v_n^*\|^2 + d\varepsilon_n\|\sqrt{s_n}v_n^*\|\right) \\
&\quad + \left(-\delta_n\|\sqrt{1-s_n}u_n^*\|^2 + d\delta_n\|\sqrt{1-s_n}u_n^*\|\right) \\
&\quad + 2\phi(\|x\|) + d\|x_n\| + \|C^{t_n}x_n\| = \left(-\varepsilon_n\tau_n^2 + d\varepsilon_n\tau_n\right) \\
&\quad + \left(-\delta_n\lambda_n^2 + d\delta_n\lambda_n\right) + 2\phi(\|x\|) + d\|x_n\| + \|C^{t_n}x_n\| \\
&\quad \le \frac{d^2\varepsilon_n}{4} + \frac{d^2\delta_n}{4} + 2\phi(\|x\|) + d\|x_n\| + \|C^{t_n}x_n\| \\
&\quad \le 2(1+\phi(\|x\|)) + d\|x_n\| + \|C^{t_n}x_n\| = K_4(x),
\end{aligned} \tag{20}$$

where $\tau_n = \|\sqrt{s_n}v_n^*\|$, $\lambda_n = \|\sqrt{1-s_n}u_n^*\|$, and $K_4(x)$ is upper bound for $\{2(1+\phi(\|x\|)) + d\|x_n\| + \|C^{t_n}x_n\|\}$. By following the argument used in the first part of this proof, it follows that $\{Lx_n\}$ is bounded; that is, $\{z_n^*\}$ is bounded. By the compactness of L^{-1}, there exists a subsequence, denoted again by $\{x_n\}$, such that $x_n \to x_0$ as $n\to\infty$. Assume without loss of generality that $Lx_n \rightharpoonup h_0^*$ and $z_n^* \rightharpoonup z_0^*$ as $n\to\infty$. Since L is closed, we have $x_0 \in D(L)$ and $h_0^* = Lx_0$. Since $\{z_n^*\}$ and $\{C^{t_n}x_n\}$ are bounded and $x_n \to x_0$ as $n\to\infty$, we get $\langle z_n^*, x_n\rangle \to \langle z_0^*, x_0\rangle$ as $n\to\infty$. To complete the proof, we consider the following cases.

Case I. $\{s_n v_n^*\}$ is bounded. Since $\{z_n^*\}$ is bounded, $\{(1-s_n)u_n^*\}$ is also bounded. Since $\{T^t\}_{t\in[0,1]}$ is of type Γ_d^{ϕ} (i.e., of type Γ_d), it follows that $\langle v^t, x\rangle \ge -d\|x\|$ for all $x \in D(T^t)$ and $v^t \in T^t x$ for all $t\in[0,1]$. Let $(x,y)\in G(T^{t_0})$. Since $\{T^t\}_{t\in[0,1]}$ is a pseudomonotone homotopy of type $\Gamma_{\mathrm{d}}^{\phi}$, by (iv) of Lemma 3, there exists a sequence $(y_n, y_n^*) \in G(T^{t_n})$ such that $y_n \to x$ and $y_n^* \to y$ as $n\to\infty$. On the other hand, the monotonicity of T^{t_n} implies

$$\left\langle v_n^* - y_n^*, x_n - \varepsilon_n J^{-1}(v_n^*) - y_n\right\rangle \ge 0 \quad \forall n; \tag{21}$$

that is,

$$\begin{aligned}
\langle v_n^*, x_n\rangle &\ge \langle v_n^*, y_n\rangle + \langle y_n^*, x_n - y_n\rangle + \varepsilon_n\|v_n^*\|^2 \\
&\quad - \varepsilon_n\|y_n^*\|\|v_n^*\| \quad \forall n.
\end{aligned} \tag{22}$$

In a similar manner, we get

$$\begin{aligned}
\langle u_n^*, x_n\rangle &\ge \langle u_n^*, y_n\rangle + \langle y_n^*, x_n - y_n\rangle + \delta_n\|v_n^*\|^2 \\
&\quad - \delta_n\|y_n^*\|\|u_n^*\|
\end{aligned} \tag{23}$$

for all n. Multiplying (22) and (23) by s_n and $(1-s_n)$, respectively, and adding the resulting inequalities, we get

$$\begin{aligned}
\langle z_n^*, x_n\rangle &\ge \langle z_n^*, y_n\rangle + \langle y_n^*, x_n - y_n\rangle \\
&\quad + s_n\varepsilon_n\left(\|v_n^*\|^2 - \|y_n^*\|\|v_n^*\|\right) \\
&\quad + (1-s_n)\delta_n\left(\|u_n^*\|^2 - \|y_n^*\|\|u_n^*\|\right) \quad \forall n.
\end{aligned} \tag{24}$$

Since $\{z_n^*\}$ is bounded and $x_n \to x_0$, it follows that $\langle z_n^*, x_n\rangle \to \langle z_0^*, x_0\rangle$ as $n\to\infty$. Consequently, using (24), we obtain

$$\begin{aligned}
&\langle z_0^*, x_0\rangle = \lim_{n\to\infty}\langle z_n^*, x_n\rangle \\
&\ge \liminf_{n\to\infty}\left(\langle z_n^*, y_n\rangle + \langle y_n^*, x_n - y_n\rangle\right) \\
&\quad - \limsup_{n\to\infty}\left(s_n\varepsilon_n\|v_n^*\|\|y_n^*\| + (1-s_n)\delta_n\|y_n^*\|\|u_n^*\|\right) \\
&= \langle z_0^*, x\rangle + \langle y, x_0 - x\rangle
\end{aligned} \tag{25}$$

for all $(x,y)\in \mathrm{G}(T^{t_0})$, which yields $\langle z_0^* - y, x_0 - x\rangle \ge 0$. By the maximal monotonicity of T^{t_0}, we conclude that $x_0 \in D(T^{t_0})$ and $z_0^* \in T^{t_0}x_0$. Therefore, we obtain that $x_0 \in D(L)\cap D(T^{t_0})\cap \partial G$ and $z_0^* \in T^{t_0}x_0$ such that $Lx_0 + z_0^* + t_0C_1x_0 + (1-t_0)C_2x_0 = 0$. However, this is a contradiction.

Case II. Suppose $\{s_n v_n^*\}$ is unbounded. Then there exists a subsequence, denoted again by $\{s_n v_n^*\}$, such that $s_n\|v_n^*\| \to +\infty$ as $n\to\infty$. Then $\{(1-s_n)u_n^*\}$, $\{v_n^*\}$ and $\{u_n^*\}$ being unbounded. Assume without loss of generality that $\|v_n^*\| \to \infty$ and $\|u_n^*\| \to \infty$ as $n\to\infty$. If either $\{\varepsilon_n s_n\|v_n^*\|^2\}$ or $\{\delta_n(1-s_n)\|u_n^*\|^2\}$ is unbounded, (24) implies

$$\begin{aligned}
\langle z_n^*, x_n\rangle &\ge \langle z_n^*, y_n\rangle + \langle y_n^*, x_n - y_n\rangle \\
&\quad + s_n\varepsilon_n\|v_n^*\|^2\left(1 - \frac{\|y_n^*\|}{\|v_n^*\|}\right) \\
&\quad + (1-s_n)\delta_n\|u_n^*\|^2\left(1 - \frac{\|y_n^*\|}{\|u_n^*\|}\right).
\end{aligned} \tag{26}$$

Assuming $\varepsilon_n s_n \|v_n^*\|^2 \to \infty$ or $\delta_n(1-s_n)\|u_n^*\|^2 \to \infty$ and taking limits in (26) imply that

$$\langle z_0^* - y, x_0 - x \rangle \geq \infty, \tag{27}$$

which is impossible. Thus, $\{\varepsilon_n s_n \|v_n^*\|^2\}$ and $\{\delta_n(1-s_n)\|u_n^*\|^2\}$ are bounded. Consequently, we get

$$s_n \varepsilon_n \|v_n^*\| = \frac{s_n \varepsilon_n \|v_n^*\|^2}{\|v_n^*\|} \longrightarrow 0 \tag{28}$$

as $n \to \infty$. Similarly, we have $(1-s_n)\delta_n \|u_n^*\| \to 0$ as $n \to \infty$. In all cases, (24) and (25) yield a contradiction. Therefore, by using the compactness of L^{-1} and boundedness of $T_{\varepsilon_n}^{t_n}$ and $T_{\delta_n}^{t_n}$, we proved that the family $\{H_n(t,\cdot)\}_{s\in[0,1]}$ is an admissible homotopy of Leray-Schauder type; that is, $d(H_n(s,\cdot), G, 0)$ is independent of $s \in [0,1]$ for all large n; that is,

$$\begin{aligned} &d_{\mathrm{LS}}\left(I + L^{-1}\left(T_{\varepsilon_n}^{t_n} + C^{t_n}\right), G, 0\right) \\ &\quad = d_{\mathrm{LS}}\left(I + L^{-1}\left(T_{\delta_n}^{t_n} + C^{t_n}\right), G, 0\right). \end{aligned} \tag{29}$$

However, this is impossible. Therefore, there exists $\varepsilon_1 > 0$ such that $d_{\mathrm{LS}}(I + L^{-1}(T_\varepsilon^t + C^t), G, 0)$ is well-defined and independent of $\varepsilon \in (0, \varepsilon_1]$ and $t \in [0,1]$. The proof is completed. □

Next we give the definition of the required degree mapping.

Definition 6. Let G be a nonempty, bounded, and open subset of X. Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone of type Γ_d^ϕ, $C : X \to X^*$ be bounded demicontinuous operator, and $L : X \supset D(L) \to X^*$ be linear, surjective, and closed such that $L^{-1} : X^* \to X$ is compact. Assume, further, that $f^* \notin (L+T+C)(D(T) \cap D(L) \cap \partial G)$. Then the degree of $L+T+C$ at $f^* \in X^*$ with respect to G is given by

$$\begin{aligned} &d\left(L+T+C, G, f^*\right) \\ &\quad = \lim_{\varepsilon \downarrow 0^+} d_{\mathrm{LS}}\left(I + L^{-1}\left(T_\varepsilon + C - f^*\right), G, 0\right), \end{aligned} \tag{30}$$

where d_{LS} denotes the Leray-Schauder degree mapping for compact perturbations of the identity and T_ε is the Yosida approximant of T.

The degree d satisfies the following basic properties and homotopy invariance result.

Theorem 7. *Let G be a nonempty, bounded, and open subset of X. Let $T : X \supset D(T) \to 2^{X^*}$ be maximal monotone of type Γ_d^ϕ, $L : X \supset D(L) \to X^*$ be linear, surjective, and closed such that $L^{-1} : X^* \to X$ is compact, and $C : X \to X^*$ be bounded demicontinuous operator. Then*

(i) *(normalization) there exists $\alpha > 0$ such that $d(L+\alpha J, G, 0) = 1$ if $0 \in G$ and $d(L+\alpha J, G, 0) = 0$ if $0 \notin \overline{G}$. If L is monotone, then $d(L+J, G, 0) = 1$ if $0 \in G$ and $d(L+J, G, 0) = 0$ if $0 \notin \overline{G}$;*

(ii) *(existence) if $0 \notin (L+T+C)(D(L) \cap D(T) \cap \partial G)$ and $d(L+T+C, G, 0) \neq 0$, then $0 \in (L+T+C)(D(L) \cap D(T) \cap G)$;*

(iii) *(decomposition) let G_1 and G_2 be nonempty and disjoint open subsets of G such that $f^* \notin (L+T+C)((\overline{G} \setminus (G_1 \cup G_2)))$. Then*

$$\begin{aligned} d\left(L+T+C, G, f^*\right) &= d\left(L+T+C, G_1, f^*\right) \\ &\quad + d\left(L+T+C, G_2, f^*\right); \end{aligned} \tag{31}$$

(iv) *(translation invariance) let $f^* \notin (L+T+C)(D(L) \cap D(T) \cap \partial G)$. Then we have*

$$d\left(L+T+C-f^*, G, 0\right) = d\left(L+T+C, G, f^*\right); \tag{32}$$

(v) *(homotopy invariance) let $\{T^t\}_{t\in[0,1]}$ be a pseudomonotone homotopy of maximal monotone operators uniformly of type Γ_d^ϕ and $C_i : X \to X^*$ $(i = 1,2)$ is bounded demicontinuous operator. Let*

$$H(t,x) = Lx + T^t x + tC_1 x + (1-t)C_2 x, \tag{33}$$

$(t,x) \in [0,1] \times (D(L) \cap D(T^t) \cap \overline{G})$. Then $d(H(t,\cdot), G, 0)$ is independent of $t \in [0,1]$ provided that $0 \notin H(t, D(L) \cap D(T^t) \cap \partial G)$ for all $t \in [0,1]$.

Proof. (i) Suppose the hypotheses hold. Since $L^{-1} : X^* \to X$ is continuous, there exists $d > 0$ such that $\|Lx\| \geq d\|x\|$ for all $x \in D(L)$. Let $\alpha \in (0, d)$ and $H(t,x) = x + tL^{-1}(\alpha Jx)$, $(t,x) \in [0,1] \times (D(L) \cap \overline{G})$. If there exist $t_0 \in [0,1]$ and $x_0 \in \partial G$ such that $0 = x_0 + t_0 L^{-1}(Jx_0)$, then it follows that $x_0 \in D(L)$ and $Lx_0 + t_0 \alpha J x_0 = 0$. Since $d\|x_0\| \leq \|Lx_0\| \leq t_0 \alpha \|x_0\| \leq \alpha \|x_0\|$ (i.e., $(d-\alpha)\|x_0\| \leq 0$), this gives $d \leq \alpha$ or $x_0 = 0$. But these are impossible because $0 \in G$ and $d > \alpha$. Since $\{tL^{-1}(\alpha J)\}_{t\in[0,1]}$ is a family of compact operators from $\overline{G}$ into X such that $\widetilde{H}(t,x) = tL^{-1}(\alpha Jx)$, $(t,x) \in [0,1] \times \overline{G}$, is uniformly continuous in $t \in [0,1]$ uniformly for all $x \in \overline{G}$, it follows that $\{I + tL^{-1}(\alpha J)\}_{t\in[0,1]}$ is an admissible homotopy of Leray-Schauder type; that is,

$$d\left(H(t,\cdot), G, 0\right) = d_{\mathrm{LS}}\left(I + t\mathrm{L}^{-1}(\alpha J), G, 0\right) \tag{34}$$

is independent of $t \in [0,1]$. Therefore, we obtain that $d(L + \alpha J, G, 0) = d(H(1,\cdot), G, 0) = d(I, G, 0) = 1$ if $0 \in G$ and $d(L+\alpha J, G, 0) = 0$ if $0 \notin \overline{G}$.

(ii) Suppose $0 \notin (L+T+C)(D(L) \cap D(T) \cap \partial G)$ and $d(L+T+C, G, 0) \neq 0$. By the definition of the degree mapping d, we see that $d_{\mathrm{LS}}(I + L^{-1}(T_\varepsilon + C), G, 0) \neq 0$ for all sufficiently small $\varepsilon > 0$; that is, for each $\varepsilon_n \downarrow 0^+$, there exists $x_n \in D(L) \cap D(T) \cap G$ such that

$$x_n + L^{-1}\left(T_{\varepsilon_n} x_n + Cx_n\right) = 0 \quad \forall n. \tag{35}$$

By the arguments used in the proof of Theorem 5, one can easily show that $0 \in (L+T+C)(D(L) \cap D(T) \cap G)$. The details are omitted here.

(iii) Suppose the hypotheses hold. The definition of the degree mapping d and decomposition property of the Leray-Schauder degree d_{LS} imply

$$\begin{aligned} d&\left(L+T+C,G,f^*\right)\\ &= d_{\mathrm{LS}}\left(I+L^{-1}\left(T_\varepsilon+C-f^*\right),G,0\right)\\ &= d_{\mathrm{LS}}\left(I+L^{-1}\left(T_\varepsilon+C-f^*\right),G_1,0\right)\\ &\quad + d_{\mathrm{LS}}\left(I+L^{-1}\left(T_\varepsilon+C-f^*\right),G_2,0\right)\\ &= d\left(L+T+C,G_1,f^*\right)+d\left(L+T+C,G_2,f^*\right) \end{aligned} \tag{36}$$

for all sufficiently small $\varepsilon > 0$. This completes the proof of (iii). The proof of (iv) follows from the translation invariance property of Leray-Schauder degree.

(v) Let $C^t(x) = tC_1x + (1-t)C_2x$, $(t,x) \in [0,1]\times\overline{G}$. The proof of Theorem 5 confirms the existence of $\varepsilon_1 > 0$ such that $d_{\mathrm{LS}}(I + L^{-1}(T^t_\varepsilon + C^t), G, 0)$ is well-defined and independent of $t \in [0,1]$ and $\varepsilon \in (0,\varepsilon_1]$; that is, by the definition of the degree, we get that

$$d\left(H(t,\cdot),G,0\right) = \lim_{\varepsilon\downarrow 0^+} d_{\mathrm{LS}}\left(I+L^{-1}\left(T^t_\varepsilon+C^t\right),G,0\right) \tag{37}$$

is well-defined and independent of $t \in [0,1]$. The details are omitted here. □

Consequently, we prove the following new existence result.

Theorem 8. *Let $T : X \supset D(T) \to 2^{X^*}$ be densely defined maximal monotone with $0 \in T(0)$ and of type Γ^ϕ_d, $L : X \supset D(L) \to X^*$ be linear monotone, surjective, and closed such that $L^{-1} : X^* \to X$ is compact, and $C : X \to X^*$ be bounded demicontinuous operator. Let $f^* \in X^*$. Assume, further, that there exists $R > 0$ such that*

$$\left\langle Lx + v^* + Cx - f^*, x\right\rangle > 0 \tag{38}$$

for all $x \in D(L)\cap D(T)\cap\partial B_R(0)$ and $v^* \in Tx$. Then $f^* \in (L+T+C)(D(L)\cap D(T)\cap B_R(0))$. In addition, $L+T+C$ is surjective if $L+T+C$ is coercive.

Proof. By the continuity of L^{-1}, there exists $d > 0$ such that $d\|x\| \le \|Lx\|$for all $x \in D(L)$. Let $\varepsilon \in (0,d)$ and

$$\begin{aligned} H(t,x) = Lx + Tx + t\left(Cx-f^*\right)+\varepsilon Jx,&\\ (t,x)\in[0,1]\times(D(L)\cap D(T)).& \end{aligned} \tag{39}$$

Since $0 \in T(0)$ and $L(0) = 0$, we see that $H(t,x)$ can be rewritten as

$$\begin{aligned} H(t,x) = t\left(Lx+Tx+Cx+\varepsilon Jx-f^*\right)&\\ +(1-t)\left(Lx+Tx+\varepsilon Jx\right),& \end{aligned} \tag{40}$$

$(t,x) \in [0,1]\times(D(L)\cap D(T))$. Since L and T are monotone with $0 \in T(0)$, we get

$$\begin{aligned} &\left\langle Lx+v^*+t\left(Cx-f^*\right)+\varepsilon Jx,x\right\rangle\\ &\quad= \left\langle t\left(Lx+v^*+Cx-f^*\right),x\right\rangle\\ &\qquad+\left\langle(1-t)\left(Lx+v^*\right)+\varepsilon Jx,x\right\rangle\\ &\quad\ge \left\langle(1-t)\left(Lx+v^*\right)+\varepsilon Jx,x\right\rangle \ge \varepsilon\|x\|^2>0 \end{aligned} \tag{41}$$

for all $t \in [0,1]$, $x \in D(L)\cap D(T)\cap\partial B_R(0)$, and $v^* \in Tx$; that is, we have

$$\begin{aligned} &\left\langle t\left(Lx+v^*+Cx+\varepsilon Jx-f^*\right)\right.\\ &\quad\left.+(1-t)\left(v^*+Lx+\varepsilon Jx\right),x\right\rangle>0 \end{aligned} \tag{42}$$

for all $x \in D(L)\cap D(T)\cap\partial B_R(0)$, $t \in [0,1]$, and $v^* \in Tx$; that is, $0 \notin H(t, D(L)\cap D(T)\cap\partial B_R(0))$ for all $t \in [0,1]$. Since $\{T^t = T\}_{t\in[0,1]}$ is a pseudomonotone homotopy of maximal monotone operators of type Γ^ϕ_d, (v) of Theorem 7 implies that

$$\begin{aligned} d&\left(L+T+C+\varepsilon J,B_R(0),f^*\right)\\ &= d\left(L+T+\varepsilon J,B_R(0),0\right). \end{aligned} \tag{43}$$

Next we show that $d(L+T+\varepsilon J, B_R(0), 0) = 1$. We consider

$$\begin{aligned} K(t,x) = Lx + tTx + \varepsilon Jx,&\\ (t,x)\in[0,1]\times\left(D(L)\cap D(T)\cap\overline{B}_R(0)\right).& \end{aligned} \tag{44}$$

Following the above arguments, it is not difficult to show that $0 \notin K(t, D(L)\cap D(T)\cap\partial B_R(0))$ for all $t \in [0,1]$. Since T is densely defined, it is well-known that $\{T^t = tT\}_{t\in[0,1]}$ is a pseudomonotone homotopy of type Γ^ϕ_d; that is, (v) of Theorem 7 gives

$$d\left(L+T+\varepsilon J,B_R(0),0\right) = d\left(L+\varepsilon J,B_R(0),0\right)=1. \tag{45}$$

Consequently, we get

$$\begin{aligned} d&\left(L+T+C+\varepsilon J,B_R(0),f^*\right)\\ &= d\left(L+T+\varepsilon J,B_R(0),0\right)=d\left(L+\varepsilon J,B_R(0),0\right)\\ &=1; \end{aligned} \tag{46}$$

that is, for each $\varepsilon_n \downarrow 0^+$, there exist $x_n \in D(L)\cap D(T)\cap B_R(0)$ and $v^*_n \in Tx_n$ such that

$$Lx_n + v^*_n + Cx_n + \varepsilon_n Jx_n = f^* \quad \forall n. \tag{47}$$

By using Γ^ϕ_d condition on T and boundedness of C, we can follow the arguments used in the proof of Theorem 5 to conclude that $f^* \in (L+T+C)(D(L)\cap D(T)\cap B_R(0))$. Furthermore, $L+T+C$ is surjective provided that $L+T+C$ is coercive. The proof is completed. □

Next we give the following important theorem on maximality of L and $L+T$ without requiring (i) T or L to be quasibounded and $0 \in D(L)\cap D(T)$ and (ii) $D(L)\cap \mathring{D}(T) \neq \emptyset$. The maximality condition (i) and (ii) are attributed to Browder and Hess [16] and Rockafellar [17], respectively.

Theorem 9. *Let $T : X \supset D(T) \to 2^{X^*}$ be densely defined maximal monotone of type Γ_d^ϕ and $0 \in T(0)$ and $L : X \supset D(L) \to X^*$ be linear monotone, surjective, and closed such that $L^{-1} : X^* \to X$ is compact. Then $L + T$ is maximal monotone.*

Proof. Let $f^* \in X^*$. By the boundedness and continuity of J and monotonicity of L and T with $0 \in T(0)$, it follows that there exists $R > 0$ such that $\langle Lx + v^* + Jx - f^*, x\rangle \geq \|x\|^2 - \|f^*\|\|x\|$ for all $x \in D(L) \cap D(T)$ and $v^* \in Tx$; that is, $L + T + J$ is coercive. By Theorem 8, we conclude that $R(L + T + J) = X^*$; that is, $L + T$ is maximal monotone. The maximality of L follows by setting $T = \{0\}$. □

4. Degree Theory in a Real Hilbert Space with $R(L) \neq H$

The content of this section outlines the construction of the degree mapping for operators of the type $L + T + C$ in the setting of a real Hilbert space, where T and C are as in Section 3 and $L : H \supseteq D(L) \to H$ is linear densely defined, self-adjoint, closed, and range closed. The closedness of $R(L)$ is achieved if we assume $R(L) = N(L)^\perp$, where $N(L)$ denotes the null space of L. Under this condition, one can easily see that the restriction of L to $D(L) \cap R(L)$ is one to one and onto $R(L)$. Let $P : H \to N(L)$ be the orthogonal projection onto $N(L)$. In addition, it is well-known that $H = N(L) \oplus R(L)$. For each $\varepsilon > 0$, it follows that $L_\varepsilon = \varepsilon P + L : H \supset D(L) \to H$ is linear and surjective. For further properties of operators of type L, the reader is referred to the paper by Brézis and Nirenberg [18]. In the following lemma, we shall show that $L_\varepsilon^{-1} : H \to H$ is compact for suitable $\varepsilon > 0$.

Lemma 10. *Let $L : H \supseteq D(L) \to H$ be linear, densely defined, and self-adjoint and $L^{-1} : R(L) \to H$ be compact. Then there exists $\alpha > 0$ such that, for each $\varepsilon \in (0, \alpha)$, the operator $L_\varepsilon : H \supset D(L) \to H$ is surjective and $L_\varepsilon^{-1} : H \to H$ is compact.*

Proof. By the property of the orthogonal projection P onto $N(L)$, it is well-known that P is nonexpansive; that is, $\|Px - Py\| \leq \|x - y\|$ for all $x \in H$ and $y \in H$. Since $L^{-1} : R(L) \to H$ is compact (i.e., it is continuous and linear), there exists $\alpha > 0$ such that

$$\|Lx - Ly\| \geq \alpha \|x - y\| \quad \forall x \in D(L), \ y \in D(L). \tag{48}$$

For each $\varepsilon \in (0, \alpha)$, we see that

$$\begin{aligned} \|L_\varepsilon x - L_\varepsilon y\| &= \|(Lx + \varepsilon Px) - (Ly + \varepsilon Py)\| \\ &= \|(Lx - Ly) + \varepsilon (Px - Py)\| \\ &\geq \|Lx - Ly\| - \varepsilon \|Px - Py\| \\ &\geq \alpha \|x - y\| - \varepsilon \|x - y\| \\ &= (\alpha - \varepsilon) \|x - y\| \end{aligned} \tag{49}$$

for all $x \in D(L)$ and $y \in D(L)$, which implies that L_ε is expansive; that is, $L_\varepsilon^{-1} : H \to H$ is continuous. Next we show that L_ε^{-1} is compact. Let $\{x_n\}$ be bounded in H and $y_n = (\varepsilon P + L)^{-1} x_n$ for all n; that is, $(\varepsilon P + L) y_n = x_n$ for all n; that is, $y_n = L^{-1}(x_n - \varepsilon P y_n)$ for all n. The boundedness of $\{y_n\}$ follows because of the expansiveness of $(\varepsilon P + L)^{-1}$. Since L^{-1} is compact, we assume by passing to a subsequence that $y_n = L^{-1}(x_n - \varepsilon P y_n) \to y_0$ as $n \to \infty$; that is, the compactness of L_ε^{-1} is proved. □

As a consequence of Lemma 10, it follows that $L_\varepsilon^{-1} B$ is a compact operator provided that $B : H \to H$ is a bounded operator. Theorem 11 gives analogous result like that of Theorem 5.

Theorem 11. *Let G be a nonempty, bounded, and open subset of H. Let $C^t x = tC_1 x + (1 - t)C_2 x$, $(t, x) \in [0, 1] \times \in \overline{G}$, where $C_i\,(i = 1, 2) : H \to H$ is bounded demicontinuous operator. Let $L : H \supset D(L) \to H$ be linear, densely defined, self-adjoint, closed, and range closed such that $L^{-1} : R(L) \to H$ is compact. Suppose $\{T^t\}_{t\in[0,1]}$ is a pseudomonotone homotopy of maximal monotone operators uniformly of type Γ_d^ϕ. Assume, further, that $0 \notin (L + T^t + C^t)(D(L) \cap D(T^t) \cap \partial G)$ for all $t \in [0, 1]$. Then degree $d_{LS}(I + L_\varepsilon^{-1}(T_\varepsilon^t + C^t), G, 0)$ is well-defined and independent of sufficiently small $\varepsilon > 0$ and $t \in [0, 1]$.*

Proof. Since L_ε^{-1} is compact and C and T_ε are bounded demicontinuous operators, it follows that $L_\varepsilon^{-1}(T_\varepsilon + C) : H \to H$ is a compact operator. Suppose there exist $x_n \in \partial G$ and $t_n \in [0, 1]$ such that $x_n + (\varepsilon_n P + L)^{-1}(T_{\varepsilon_n} x_n + C^{t_n} x_n) = 0$ for all n; that is, $\varepsilon_n P x_n + L x_n + T_{\varepsilon_n}^{t_n} x_n + C^{t_n} x_n = 0$ for all n. Since $\varepsilon_n P + L$ is surjective, it follows that $x_n \in D(L)$, $Lx_n \in R(L)$, and $\varepsilon_n P x_n + T_{\varepsilon_n}^{t_n} x + C^{t_n} x_n \in R(L)$ for all n; that is, we get

$$x_n + L^{-1} \left(\varepsilon_n P x_n + T_{\varepsilon_n}^{t_n} x + C^{t_n} x_n\right) = 0 \quad \forall n. \tag{50}$$

Since C_1, C_2, and P are bounded and $\{T^t\}_{t\in[0,1]}$ is uniformly of type Γ_d^ϕ, we can follow the arguments in the proof of Theorem 5 to conclude that $\{Lx_n\}$ and $\{T_{\varepsilon_n}^{t_n} x_n\}$ are bounded. As a result, the compactness of L^{-1} implies the existence of a subsequence, denoted again by $\{L^{-1}(\varepsilon_n P x_n + T_{\varepsilon_n}^{t_n} x + C^{t_n} x_n)\}$, such that

$$\begin{aligned} x_n = -L^{-1} \left(\varepsilon_n P x_n + T_{\varepsilon_n}^{t_n} x + C^{t_n} x_n\right) \longrightarrow \\ x_0 \in \partial G \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{51}$$

Since C and P are bounded and the family $\{T^t\}$ is uniformly of type Γ_d^ϕ, the proof can be completed by following exactly similar arguments as in the proof of Theorem 5. The details are omitted here. □

Based on Theorem 11, the definition of the degree mapping is given below.

Definition 12. Let G be a nonempty, bounded, and open subset of H. Let $T : H \supset D(T) \to 2^H$ be maximal monotone of type Γ_d^ϕ, $C : H \to H$ be bounded demicontinuous operator, and $L : H \supset D(L) \to H$ be linear, densely defined, self-adjoint, closed, and range closed such that $L^{-1} : R(L) \to H$ is compact. Assume, further, that $f^* \notin (L+T+C)(D(T) \cap D(L) \cap \partial G)$. Then the degree of $L+T+C$ at $f^* \in H$ with respect to G is defined by

$$d\left(L+T+C, G, f^*\right) = \lim_{\varepsilon \downarrow 0^+} d_{\mathrm{LS}}\left(I + L_{\varepsilon}^{-1}\left(T_{\varepsilon} + C - f^*\right), G, 0\right), \quad (52)$$

where $L_{\varepsilon} = \varepsilon P + L$, $P : H \to N(L)$ is the orthogonal projection onto $N(L)$, d_{LS} denotes the Leray-Schauder degree mapping for compact perturbations of the identity, and T_{ε} is the Yosida approximant of T.

The basic properties and homotopy invariance results like that of Theorem 5 and existence theorems analogous to Theorems 8 and 9 can be proved in Hilbert space setting by using the surjectivity of $\varepsilon P + L$ instead of the surjectivity of L. The degree satisfies the following properties.

Theorem 13. *Let G be a nonempty, bounded, and open subset of H. Let $T : H \supset D(T) \to 2^H$ be maximal monotone of type Γ_d^{ϕ}, $L : H \supset D(L) \to H$ be linear, densely defined, self-adjoint, closed, and range closed such that $L^{-1} : R(L) \to H$ is compact, and $C : H \to H$ be bounded demicontinuous operator. Then*

(i) *(normalization) there exists $\alpha > 0$ such that $d(L + \alpha J, G, 0) = 1$ if $0 \in G$ and $d(L + \alpha J, G, 0) = 0$ if $0 \notin \overline{G}$. If L is monotone, then $d(L + J, G, 0) = 1$ if $0 \in G$ and $d(L + J, G, 0) = 0$ if $0 \notin \overline{G}$;*

(ii) *(existence) if $0 \notin (L+T+C)(D(L) \cap D(T) \cap \partial G)$ and $d(L+T+C, G, 0) \neq 0$, then $0 \in (L+T+C)(D(L) \cap D(T) \cap G)$;*

(iii) *(decomposition) let G_1 and G_2 be nonempty and disjoint open subsets of G such that $f^* \notin (L+T+C)((\overline{G} \setminus (G_1 \cup G_2)))$. Then*

$$d\left(L+T+C, G, f^*\right) = d\left(L+T+C, G_1, f^*\right) + d\left(L+T+C, G_2, f^*\right); \quad (53)$$

(iv) *(translation invariance) let $f^* \notin (L+T+C)(D(L) \cap D(T) \cap \partial G)$. Then we have*

$$d\left(L+T+C-f^*, G, 0\right) = d\left(L+T+C, G, f^*\right); \quad (54)$$

(v) *(homotopy invariance) let $\{T^t\}_{t\in[0,1]}$ be a pseudomonotone homotopy of maximal monotone operators uniformly of type Γ_d^{ϕ} and $C_i : H \to H$ $(i = 1, 2)$ is bounded demicontinuous operator. Let*

$$H(t, x) = Lx + T^t x + tC_1 x + (1-t) C_2 x, \quad (55)$$

$(t, x) \in [0,1] \times (D(L) \cap D(T^t) \cap \overline{G})$. Then $d(H(t,\cdot), G, 0)$ is independent of $t \in [0,1]$ provided that $0 \notin H(t, D(L) \cap D(T^t) \cap \partial G)$ for all $t \in [0,1]$.

Proof. The proofs can be easily completed as in the arguments used in the proofs of Theorems 5 and 7. □

This part of the theory improves the degree theory developed by Berkovits and Mustonen [19] for operators of the type $L + C$, where C is bounded demicontinuous pseudomonotone. In the present paper, we only assumed that C is bounded demicontinuous operator. Berkovits and Mustonen [19, Theorem 10, p. 959], gave an existence result for solvability of operator equations of the type $Lx + cx + Cx = h$, where C is bounded demicontinuous operator with bounded range, $c > 0$ such that $-c \in \sigma(L)$ (where $\sigma(L)$ denotes the set of all eigenvalues of L), $cI + C$ is pseudomonotone, and the recession function (cf. Brézis and Nirenberg [18]) corresponding to C given by $J_C(u) = \liminf_{t\to\infty, v\to u}\langle C(tv), v\rangle$ satisfies $\langle h, v\rangle < J_C(v)$ for all $v \in \operatorname{Ker}(L + cI)$ with $\|v\| = 1$. However, for $c = 0$, these conditions on C and L exclude the possibility that $\operatorname{Ker} L = \{0\}$. If $R(C)$ is bounded and c is any constant, we can easily see that $cI + C$ is sublinear for all x satisfying $\|x\| \geq 1$ (i.e., $\|cx + Cx\| \leq c\|x\| + (d/\|x\|)\|x\| \leq (c+d)\|x\| = \tau\|x\|$ for all $\|x\| \geq 1$, where $\tau = c + d$ and $d = \sup\{\|Cx\| : x \in H\} < \infty$). As a result of Theorem 14 below, the surjectivity of $L + cI + C$ follows under mild assumption on the constant c omitting both conditions such that $cI + C$ is pseudomonotone and $-c \in \sigma(L)$.

Theorem 14. *Let $L : H \supseteq D(L) \to H$ be linear, densely defined, self-adjoint, closed, and range closed such that $L^{-1} : R(L) \to H$ is compact and $C : H \to H$ be bounded demicontinuous operator. Assume, further, that there exist nonnegative constants $\tau \in (0, \alpha)$ and θ such that $\|Cx\| \leq \tau\|x\| + \theta$ for all $x \in D(L)$ with sufficiently large $\|x\|$, where α is the largest positive constant satisfying $\alpha\|x\| \leq \|Lx\|$ for all $x \in D(L)$. Then $L + C$ is surjective. If a reflexive Banach space X is used instead of a real Hilbert space H, then the same conclusion holds provided that L is surjective.*

Proof. Let $f \in H$ and $\varepsilon \in (0, \alpha - \tau)$ (i.e., $\varepsilon \in (0, \alpha)$). Consider the homotopy equation given by

$$H(t, x) = Lx + t\left(Cx - f\right) + \varepsilon(1-t) Jx, \quad (t, x) \in [0,1] \times D(L). \quad (56)$$

Consequently, we have

$$\begin{aligned} \|H(t,x)\| &= \left\|Lx + t\left(Cx - f\right) + \varepsilon(1-t) Jx\right\| \\ &\geq \|Lx\| - t\left\|Cx - f\right\| - (1-t)\varepsilon\|Jx\| \\ &\geq \|Lx\| - \left\|Cx - f\right\| - \varepsilon\|x\| \\ &\geq \|Lx\| - \tau\|x\| - \theta - \left\|f\right\| - \varepsilon\|x\| \\ &= \|Lx\|\left(1 - \frac{(\tau+\varepsilon)\|x\|}{\|Lx\|} - \frac{\left\|f\right\| + \theta}{\|Lx\|}\right) \\ &= \|Lx\|\left(1 - \frac{\tau+\varepsilon}{\alpha} - \frac{\left\|f\right\| + \theta}{\|Lx\|}\right) \end{aligned} \quad (57)$$

for all $x \in D(L) \setminus \{0\}$ and $t \in [0,1]$. Since $\varepsilon \in (0, \alpha - \tau)$ (i.e., $1 > (\tau+\varepsilon)/\alpha$) and the right hand side of this inequality is independent of $t \in [0,1]$, letting $\|x\| \to \infty$ implies that there exists $R = R(f) > 0$ such that $\|H(t,x)\| > 0$ for all $x \in D(L) \cap \partial B_R(0)$ and $t \in [0,1]$. Consequently, by using (i) and (v) of Theorem 13, we conclude that

$$\begin{aligned} d\left(H(t,\cdot), B_R(0), 0\right) &= d\left(H(1,\cdot), B_R(0), 0\right) \\ &= d\left(L + C - f, B_R(0), 0\right) \\ &= d\left(H(0,\cdot), B_R(0), 0\right) \\ &= d\left(L + \varepsilon J, B_R(0), 0\right) = 1; \end{aligned} \tag{58}$$

that is, $f = Lx + Cx$ is solvable in $D(L)$. Since $f \in H$ is arbitrary, we conclude that $L + C$ is surjective. The proof is completed. □

In the case when $R(C)$ is bounded, we can apply Theorem 14 to conclude that $L + cI + C$ is surjective because of the sublinearity of $cI + C$ with $\tau = c + d \in (0,\alpha)$ and C is demicontinuous operator. As a result, it follows that c can be zero and $L + C$ is surjective if $\alpha > d$ without C being pseudomonotone. In [19], Berkovits and Mustonen gave an existence theorem for the surjectivity of operators of the type $L+C$, where L satisfies conditions of Theorem 14 and $C : H \to H$ is bounded demicontinuous pseudomonotone satisfying the following: (i) $\langle Lx, x\rangle \geq -(1/\alpha)\|Lx\|^2$ for all $x \in D(L)$, where α is the largest positive constant, (ii) there exist $c_1 > 0$ and $\eta > 0$ such that $\|Cx\| \geq \eta\|x\| - c_1$ for all $x \in D(L)$, and (iii) there exist $\gamma \in (0,\alpha)$ and $c_2 > 0$ such that $\langle Cx, x\rangle \geq (1/\gamma)\|Cx\|^2 - c_2$ for all $x \in D(L)$. By combining (ii) and (iii), we can easily see that

$$\|Cx\| \leq \gamma\|x\| + \gamma c_2 \tag{59}$$

for all x satisfying $\|x\| \geq (1 + c_1)/\eta$, which is a sublinearity condition used in Theorem 14. This shows that conditions (ii) and (iii) used by Berkovits and Mustonen [19, Theorem 8, p. 957] give stronger conditions on C as compared with the sublinearity of C. However, Theorem 14 does not need condition (ii) or (iii). The last but the main improvement of Theorem 14 over that of Berkovits and Mustonen [19] is dropping the requirement of C to be pseudomonotone. It is worth mentioning here that the same conclusion holds in Theorem 14 if the sublinearity condition on C holds for all $x \in D(L)$ with sufficiently large $\|x\|$. As a result, we get the following corollary.

Corollary 15. *Let $L : H \supseteq D(L) \to H$ be linear, densely defined, self-adjoint, closed, and range closed such that $L^{-1} : R(L) \to H$ is compact and $C : H \to H$ be bounded demicontinuous operator. Assume, further, that there exist nonnegative constants μ and β such that $0 \leq \mu \leq \beta\alpha - 1$ and*

$$\langle Cx, x\rangle \geq \beta\|Cx\|^2 - \mu\|x\| \quad \forall x \in H, \tag{60}$$

where $\alpha\beta \geq 1$ and α is the largest positive constant such that $\alpha\|x\| \leq \|Lx\|$ for all $x \in D(L)$. If $\|Cx\| \to \infty$ as $\|x\| \to \infty$, then $L + C$ is surjective. If a reflexive Banach space X is used instead of a real Hilbert space H, then the same conclusion holds provided that L is surjective.

Proof. By the side condition on C, we see that

$$\beta\|Cx\|^2 \leq \mu\|x\| + \langle Cx, x\rangle = \mu\|x\| + \|Cx\|\|x\| \quad \forall x \in H. \tag{61}$$

Consequently, we get

$$\|Cx\| \leq \frac{\mu\|x\|}{\beta\|Cx\|} + \frac{1}{\beta}\|x\| \quad \forall x \in H \setminus \{0\}. \tag{62}$$

Since $\|Cx\| \to \infty$ as $\|x\| \to \infty$, there exists $R > 0$ such that $\|Cx\| \geq 1$ for all $\|x\| \geq R$; that is, we have

$$\|Cx\| \leq \tilde{\tau}\|x\| \quad \forall x \text{ with } \|x\| \geq R, \tag{63}$$

where $\tilde{\tau} = (\mu + 1)/\beta$. Since $\tilde{\tau} \in (0,\alpha)$ by the hypotheses, we can apply Theorem 14 with $\theta = 0$ and $\tau = \tilde{\tau}$ to conclude that $L + C$ is surjective. The proof is completed. □

We can observe that the largest positive constant α used in the hypotheses of Theorem 14 satisfies the condition

$$\langle Lx, x\rangle \geq -\frac{1}{\alpha}\|Lx\|^2 \quad \forall x \in D(L). \tag{64}$$

In [18], Brézis and Nirenberg [18, Theorem III.2, p. 270] proved that $f \in R(L \mathring{+} C)$ provided that the following conditions hold:

(i) $\dim N(L) < \infty$ and there are positive constants $\gamma < \alpha$, μ, and λ such that

$$\langle Cx - f, x\rangle \geq \frac{1}{\gamma}\|Cx\|^2 - \mu\|x\| - \lambda\left(\|u_1\| + 1\right), \tag{65}$$

where $u = u_1 + u_2$, $u_1 \in N(L)$, and $u_2 \in R(L)$.

(ii) $J_C(v) > \langle f, v\rangle + c_0\|v\|$ for all $v \in N(L)$ with $v \neq 0$, where $c_0 > 0$, $\gamma < \alpha(1 + \mu/c_0)^{-1}$, and J_C is the recession function of C.

In view of these, Corollary 15 does not require $\dim N(L) < \infty$ or (ii). In conclusion, Theorem 14 and its corollary gave new surjectivity results with weaker assumptions on L and C. In addition, we note here that Berkovits and Mustonen [19] proved surjectivity of $L + C$ under weak coercivity condition of the type $\|Cx\| \geq \eta\|x\| - c_1$ for all $x \in D(L)$, for some $\eta > 0$ and $c_1 \geq 0$, and condition of type (i) with the possibility of having infinite dimensional null space of L.

5. Applications

In this section, we shall apply the abstract existence results to prove existence of weak solutions for nonlinear parabolic and hyperbolic problems such as wave and minimal surface equations. In these examples, the main contribution is that the Leray-Lion condition which guarantees pseudomonotonicity of the nonlinear term(s) is dropped. This will help to treat larger class of nonlinear equations and inequalities in appropriate Sobolev spaces. In the following examples, the norm of $u \in H$, where $H = L^2(0,T;V)$, where $V = H_0^1(\Omega)$, is denoted by $\|u\|$.

Example 16 (nonlinear parabolic equation). Let $H = L^2(0, T; V)$, where $V = H_0^1(\Omega)$. We prove the existence of weak solution for the nonlinear parabolic problem given by

$$\begin{aligned} \frac{\partial u}{\partial t} - \sum_{i=1}^{N} \left(\frac{\partial}{\partial x_i} A_i(x, u, \nabla u) \right) + H_\lambda(x, u, \nabla u) \\ = f(x,t) \quad \text{in } Q_T, \\ u(x,t) = 0 \quad (x,t) \in \partial Q_T, \\ u(x,0) = u(x,T) \quad x \in \Omega, \end{aligned} \tag{66}$$

where $Q_T = \Omega \times (0,T)$, Ω is a nonempty, bounded, and open subset of $\mathbb{R}^N$ with smooth boundary, $\partial Q_T = \partial\Omega \times (0,T)$, $\lambda > 0$, $A_i(x,u,\nabla u) = (\partial/\partial x_i)\rho(x,u,\nabla u) + a_i(x,u,\nabla u)$, $H_\lambda(x,u,\nabla u) = -\lambda\Delta u + g(x,u,\nabla u)$, and $b_i = \partial\rho(x,\eta,\xi)/\partial x_i$ $(i = 1,2,3,\ldots,N)$ for all $(x,\eta,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$. Suppose the following hypotheses are satisfied:

(i) a_i $(i = 1,\ldots,N)$, b_i $(i = 1,2,\ldots, N)$, and $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \mapsto \mathbb{R}$ are Carathéodory functions; that is, $x \mapsto a_i(x,\eta,\xi)$, $x \mapsto b_i(x,\eta,\xi)$, and $x \mapsto g(x,\eta,\xi)$ are measurable functions for almost all $(\eta,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and $(\eta,\xi) \mapsto a_i(x,\eta,\xi)$, $(\eta,\xi) \mapsto b_i(x,\eta,\xi)$, and $(\eta,\xi) \mapsto g(x,\eta,\xi)$ are continuous for almost all $x \in \Omega$.

(ii) There exist $k_1 \in L^2(Q)$ and $c_1 \geq 0$ such that

$$\begin{aligned} \max\{|\rho(x,\eta,\xi)|, |\nabla\rho(x,\eta,\xi)|\} \\ \leq k_1(x,t) + c_1(|\eta| + |\xi|), \\ \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\rho(x,\eta,\xi)\,\xi_i \geq |\eta|^2 \end{aligned} \tag{67}$$

for all $(x,\eta,\xi) \in \Omega \times \mathbb{R}$, where $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$ and

$$\nabla_x\rho(x,\eta,\xi) = \left(\frac{\partial\rho(x,\eta,\xi)}{\partial x_i} \right)_{i=1}^{N}. \tag{68}$$

(iii) There exist $k_2 \in L^2(Q)$ and $c_2 \geq 0$ such that

$$|a_i(x,\eta,\xi)| \leq k_2(x,t) + c_2(|\eta| + |\xi|) \tag{69}$$

for all $i = 1,\ldots,N$ and $(x,\eta,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$.

(iv) There exist $k_3 \in L^2(Q)$ and $c_3 \geq 0$ such that

$$|g(x,\eta,\xi)| \leq k_3(x,t) + c_3(|\eta| + |\xi|) \tag{70}$$

for all $(x,\eta,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$.

(v) There exists $k_0 \in L^2(Q)$ such that

$$\sum_{i=1}^{N} a_i(x,\eta,\xi)\,\xi_i \geq -k_0|\xi| \tag{71}$$

for all $(x,\eta,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, where $\xi = (\xi_i)_{i=1}^N$.

Let $L : H \supseteq D(L) \to H$ be given by $Lu = u' - \lambda\Delta u$, $u \in D(L)$, where $\lambda > 0$ and

$$\begin{aligned} D(L) = \{u \in L^2(0,T; H^2(\Omega) \cap H_0^1(\Omega)) : u' \\ \in L^2(Q),\ u(0) = u(T)\}; \end{aligned} \tag{72}$$

that is, we have

$$\langle Lu, \phi\rangle = \langle u', \phi\rangle + \lambda\int_Q \nabla u(x,t)\nabla\phi(x,t)\,dx\,dt, \tag{73}$$

$\phi \in H$, $u \in D(L)$, where u' is understood in the sense that

$$\int_0^T u'(t)\phi(t)\,dt = -\int_0^T u(t)\phi'(t)\,dt \tag{74}$$

for all $\phi \in C_0^\infty(0,T)$ and $<,>$ is the inner product in $L^2(Q)$. The norm of u in H is given by

$$\|u\| = \left(\int_0^T \|u(t)\|_V^2\,dt\right)^{1/2}, \tag{75}$$

where $\|u(t)\|_V = (\|u(t)\|_{L^2(\Omega)} + \|\nabla u(t)\|_{L^2(\Omega)})^{1/2}$, $t \in [0,1]$. It is well-known that L is surjective maximal monotone and $L^{-1} : H \to H$ exists and is continuous. As a consequence of the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$, it is not difficult to show that L^{-1} is compact. In addition, the maximal monotonicity of L implies the closedness of graph of L; that is, L is closed. Let $A : H \to H$ be given by

$$\langle Au, \phi\rangle = \int_Q \nabla\rho(x,u,\nabla u)\nabla\phi(x,t)\,dx\,dt, \tag{76}$$

$\phi \in H$, $u \in H$ and $B : H \to H$ be given by

$$\langle Bu, \phi\rangle = \sum_{i=1}^{N} \int_Q a_i(x,u,\nabla u)\frac{\partial\phi(x,t)}{\partial x_i}\,dx\,dt, \tag{77}$$

$\phi \in H$, $u \in H$. By conditions (i) through (iv), it well-known that A and B are bounded continuous operators; that is, $C : H \to H$ given by $C = A + B$ is bounded continuous operator. A weak solution in H of (66) is understood in the sense of the following definition.

Definition 17. Let $f \in L^2(Q)$. An element $u \in H$ is called a "weak solution" of (66) if

$$\langle Lu + Cu, \phi\rangle = \langle f, \phi\rangle \quad \forall\phi \in H; \tag{78}$$

that is, u satisfies the functional equation

$$\begin{aligned} \langle u', \phi\rangle + \int_Q \Bigg(\nabla u(x,t)\nabla\phi(x,t) \\ + \sum_{N}^{N} a_i(x,u,\nabla u)\frac{\partial\phi(x,t)}{\partial x_i} \Bigg)\,dx\,dt \\ = \int_Q (f(x,t) - g(x,u,\nabla u))\,\phi(x,t)\,dx\,dt \end{aligned} \tag{79}$$

for all $\phi \in H$. The following existence result holds.

Theorem 18. *Suppose conditions (i) through (v) are satisfied. Then, for each $f \in L^2(Q)$, (66) admits at least one weak solution in H.*

Proof. Let L, A, and B be as defined in (73), (76), and (77), respectively. It is well-known that $L : H \supseteq D(L) \to H$ is surjective maximal monotone (i.e., graph closed) and $L^{-1} : H \to H$ is compact and $C = A + B$ is bounded continuous operator. By applying Hólder's inequalities along with conditions (i) through (v), monotonicity of u', and coercivity of L we can see that

$$\begin{aligned}
\langle Lu + Cu - f, u\rangle &= \langle Lu, u\rangle + \langle Cu - f, u\rangle \\
&= \langle u', u\rangle - \int_Q \Delta u(x,t)\, u(x,t)\, dx\, dt \\
&\quad + \int_Q \left(\nabla\rho(x,u,\nabla u)\, \nabla u(x,t)\right) dx\, dt \\
&\quad + \sum_{i=1}^{N} \int_Q a_i(x,u,\nabla u) \frac{\partial u}{\partial x_i} dx\, dt \\
&\quad - \|f\|_{L^2(Q)} \|u\|_{L^2(Q)} \\
&\geq \int_Q |\nabla u(x,t)|^2\, dx\, dt + \int_Q |u(x,t)|^2\, dx\, dt \\
&\quad - \int_Q k_0 |\nabla u|\, dx\, dt - \|f\|_{L^2(Q)} \|u\|_{L^2(Q)} \\
&\geq \|u\|^2 - \|k_0\|_{L^2(Q)} \|\nabla u\|_{L^2(Q)} - \|f\|_{L^2(Q)} \|u\|_{L^2(Q)} \\
&\geq \|u\|^2 - \|k_0\|_{L^2(Q)} \|u\| - \|f\|_{L^2(Q)} \|u\| \\
&= \|u\|^2 \left(1 - \frac{\|f\|_{L^2(Q)} + \|k_0\|_{L^2(Q)}}{\|u\|}\right)
\end{aligned} \tag{80}$$

for all $u \in D(L) \setminus \{0\}$. Since the right side of the above inequality tends to ∞ as $\|u\| \to \infty$, there exists $R = R(f) > 0$ such that

$$\langle Lu + Cu - f, u\rangle > 0 \tag{81}$$

for all $u \in D(L) \cap \partial B_R(0)$; that is, the boundary condition in Theorem 8 is satisfied with linear operator L, maximal monotone operator $T = \{0\}$, and bounded continuous operator C. Therefore, we conclude that the problem $f^* = Lu + Cu$ is solvable in $D(L)$, where f^* is a functional on H generated by $f \in L^2(Q)$. Since $f \in L^2(Q)$ is arbitrary, there exists $u \in D(L)$ such that

$$\langle Lu + Cu, \phi\rangle = \langle f, \phi\rangle \quad \forall \phi \in H, \tag{82}$$

that is, (66) admits at least one weak solution in $D(L)$. The proof is completed. □

One of the main advantages of this theory concerning parabolic problems of type (66) is dropping the requirement for C to be pseudomonotone type, that is, dropping the condition

$$\sum_{i=1}^{N} \left(a_i(x,\eta,\xi') - a_i(x,\eta,\xi)\right)\left(\xi_i' - \xi_i\right) \geq 0; \tag{83}$$

for all (x,η,ξ') and (x,η,ξ) in $\Omega \times \mathbb{R} \times \mathbb{R}^N$.

Example 19 (minimal surface equation). Let $H, \Omega, Q_T = \Omega \times (0,T)$, and $\partial Q_T = \partial\Omega \times (0,T)$ be as in Example 16. Let $\beta_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be given by

$$\beta_i(x,\eta,\xi) = \frac{\xi_i}{\sqrt{1 + |\xi|^2}} \quad \forall (x,\eta,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}} \tag{84}$$

for all $i = 1, 2, \ldots, N$. It follows that β_i $(i = 1, 2, \ldots, N)$ satisfies (i) and $|\beta_i(x,\eta,\xi)| \leq |\xi| \leq |\eta| + |\xi|$, that is, condition (iv) of Example 16 with $k_3(x,t) = 0$ for all $(x,t) \in Q$ and $c_3 = 1$. We notice here that

$$\begin{aligned}
&-\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) \\
&\quad = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\frac{1}{\sqrt{1 + |\nabla u|^2}} \frac{\partial u}{\partial x_i}\right), \quad u \in H.
\end{aligned} \tag{85}$$

Let $L = \partial/\partial t - \lambda\Delta$, $\widetilde{A} : H \to H$, $\widetilde{B} : H \to H$, and $C = \widetilde{A} + \widetilde{B}$ be defined by

$$\begin{aligned}
\langle \widetilde{A}u, \phi\rangle &= \sum_{i=1}^{N} \int_Q \beta_i(x,u,\nabla u) \frac{\partial\phi(x,t)}{\partial x_i} dx\ dt, \\
\langle \widetilde{B}u, \phi\rangle &= \int_Q g(x,u,\nabla u)\, \phi(x,t)\, dx\, dt.
\end{aligned} \tag{86}$$

$u \in H$ and $\phi \in H$. By using the operators L and C and following analogous arguments used in the proof of Theorem 18, for each $f^* \in H$, we can show that $f^* = Lu + Cu$ is solvable in $D(L)$; that is, $u \in D(L)$ satisfies $\langle Lu + Cu, \phi\rangle = \langle f, \phi\rangle$ for all$\phi \in H$ provided that $f \in L^2(Q)$. Equivalently, we conclude that the minimal surface equation, given by

$$\begin{aligned}
&\frac{\partial u}{\partial t} - \lambda\Delta u - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) + g(x,u,\nabla u) \\
&\quad = f(x,t) \quad \text{in } \mathrm{Q}_T, \\
&u(x,t) = 0 \quad (x,t) \in \partial Q_T, \\
&u(x,0) = u(x,T) \quad x \in \Omega,
\end{aligned} \tag{87}$$

admits at least one weak solution in H provided that $\lambda > 0$, $f \in L^2(Q)$ and g satisfies conditions (i) and (iv) of Example 16, and $g(x,\eta,\xi)\eta \geq |\eta|^2$ for all $(x,\eta,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$.

Example 20 (nonlinear wave equation). Let $H = L^2(\Omega)$, $\Omega = (0, 2\pi) \times (0, \pi)$. We shall show existence of weak solutions in H of the wave equation given by

$$\begin{gathered} u_{tt} - u_{xx} + g(t, x, u) = f(x, t) \quad (t, x) \in \Omega, \\ u(t, 0) = u(t, \pi) = 0 \quad t \in (0, 2\pi), \\ u(x, 0) = u(x, 2\pi) \quad x \in (0, \pi), \\ u_t(x, 0) = u_t(x, 2\pi) \quad x \in (0, \pi), \end{gathered} \tag{88}$$

where $f \in L^2(\Omega)$ and

(a) $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is Carathéodory function;

(b) there exist $c_0 > 0$ and $k_0 \in L^2(\Omega)$ such that

$$|g(t, x, u)| \le k_0(t, x) + c_0 |u| \quad \forall (t, x) \in \Omega. \tag{89}$$

A weak solution $u \in H$ of (88) is understood in the sense of the following definition.

Definition 21. Let $f \in L^2(\Omega)$. An element $u \in H$ is called a "weak solution of (88)" if u satisfies

$$\langle u, \phi_{tt} - \phi_{xx} \rangle + \langle Cu, \phi \rangle = \langle f, \phi \rangle \quad \forall \phi \in C_\pi^2(\overline{\Omega}), \tag{90}$$

where

$$\begin{aligned} C_\pi^2(\overline{\Omega}) = \{ &\phi \in C^2(\overline{\Omega}) : \phi(0, t) = \phi(\pi, t) \\ &= 0, \ \phi(x, 0) = \phi(x, 2\pi), \ \phi_t(x, 0) = \phi_t(x, 2\pi) \}; \end{aligned} \tag{91}$$

that is,

$$\begin{aligned} &\int_0^{2\pi} \int_0^{\pi} u(\phi_{tt} - \phi_{xx})\, dx\, dt \\ &\quad + \int_0^{2\pi} \int_0^{\pi} g(t, x, u)\, \phi\, dx\, dt \\ &= \int_0^{2\pi} \int_0^{\pi} f(x, t)\, \phi\, dx\, dt \end{aligned} \tag{92}$$

for all $\phi \in C_\pi^2(\overline{\Omega})$.

Next we prove the following existence theorem.

Theorem 22. *Suppose g satisfies conditions (a) and (b), and let $C : H \to H$ be given by $Cu = g(\cdot, \cdot, u)$, $u \in H$. Then, for each $f \in H$, (88) admits at least one weak solution in H.*

Proof. By the sublinearity of g, we can easily see that $C : H \to H$ is bounded continuous operator and $\|Cu\| \le \|k_0\|_{L^2(\Omega)} + c_0\|u\|$ for all $u \in H$. The abstract representation of the wave operator $\partial^2/\partial t^2 - \partial^2/\partial x^2$ in $H = L^2(\Omega)$ is the linear operator $L : H \supseteq D(L) \to H$ given by

$$\begin{gathered} Lu = \sum_{m \in Z, n \in Z_+} (n^2 - m^2)\, u_{mn} \varphi_{mn}, \\ u = \sum_{m \in Z, n \in Z_+} u_{mn} \varphi_{mn}, \end{gathered} \tag{93}$$

where $u_{mn} = \langle u, \varphi_{mn} \rangle$, $\varphi_{mn}(t, x) = \pi^{-1} \exp(imt) \sin(nx)$, and

$$D(L) = \left\{ u \in H : \sum_{m \in Z, n \in Z_+} \left| n^2 - m^2 \right|^2 |u_{mn}|^2 < \infty \right\}. \tag{94}$$

If $u \in H^2(\Omega)$, then, by using successive integration by parts, we see that $u \in D(L)$ is a solution of $Lu + Cu = f$ if and only if u satisfies (90). It is well-known that L is linear densely defined, self-adjoint, and closed and $R(L) = N(L)^\perp$ (i.e., $R(L)$ is closed), the restriction of L to $D(L) \cap R(L)$, denoted again by L, is one to one onto $R(L)$, and $L^{-1} : R(L) \to H$ is compact. Since L^{-1} is continuous, there exists $d > 0$ such that $d\|u\| \le \|Lu\|$ for all $u \in D(L)$. Let $\alpha > 0$ be the largest positive constant to satisfy this condition. If $c_0 \in (0, \alpha)$, then, by applying Theorem 14 with $\theta = \|k_0\|_{L^2(\Omega)}$ and $\tau = c_0$, we conclude that there exists $u \in D(L)$ such that

$$\langle Lu + Cu, \phi \rangle = \langle f, \phi \rangle \quad \forall \phi \in H; \tag{95}$$

that is, (88) admits a weak solution in $D(L)$. The proof is completed. □

It is important to notice that the monotonicity assumption on g is not required to establish existence of weak solutions for (88). For existence of weak solutions under the requirement that g is monotone, the reader is referred to the papers by Rabinowitz [20], Brézis and Nirenberg [18, 21], Brezis [22], and Barbu and Pavel [23] and the references therein. In an attempt to remove the monotonicity assumption on g, Coron [24] used additional assumption on $N(L)$; that is, he assumed the existence of a closed subspace H_1 of H such that $N(L) \cap H_1 = \{0\}$ and H_1 is invariant under L and g. For further results on nonmonotone g, the reader is referred to the papers by Coron [24] and Hofer [25]. Consequently, Theorem 22 provides a new result concerning existence of weak solution for the nonlinear wave equation with lower order nonlinear part g satisfying only continuity and sublinearity conditions. In conclusion, we like to mention that various examples of pseudomonotone type operators under Leray-Lion type growth conditions along with (83) can be found in the papers of Landes and Mustonen [26], Mustonen [27], and Mustonen and Tienari [28] and the references therein. Existence results for perturbations of maximal monotone operator T by bounded demicontinuous operator of type (S_+) or bounded pseudomonotone can be found in the papers of Browder and Hess [16], Kenmochi [29], Kartsatos [30], Asfaw [31–33], and Le [34] and the references therein. For detailed study of ranges of sums of perturbed operators in Hilbert space and more examples and properties of elliptic, parabolic, and hyperbolic linear operators, we mention the papers by Brézis and Nirenberg [18] and Berkovits and Mustonen [19] and the references therein.

Competing Interests

The author declares that there are no competing interests regarding the publication of the paper.

Acknowledgments

The author is grateful to Virginia Tech for funding the article processing charge.

References

[1] F. E. Browder, "Degree of mapping for nonlinear mappings of monotone type," *Proceedings of the National Academy of Sciences*, vol. 80, no. 6, pp. 1771–1773, 1983.

[2] F. E. Browder, "Degree of mapping for nonlinear mappings of monotone type: densely defined mapping," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 80, no. 8, pp. 2405–2407, 1983.

[3] J. Kobayashi and M. Ôtani, "Topological degree for (S)+-mappings with maximal monotone perturbations and its applications to variational inequalities," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 59, no. 1-2, pp. 147–172, 2004.

[4] S. C. Hu and N. S. Papageorgiou, "Generalizations of Browder's degree theory," *Transactions of the American Mathematical Society*, vol. 347, no. 1, pp. 233–259, 1995.

[5] J. Berkovits and V. Mustonen, "Topological degree for perturbations of maximal monotone mappings and applications to a class of parabolic problems," *Rendiconti di Matematica*, vol. 12, pp. 597–621, 1992.

[6] J. Berkovits and V. Mustonen, "On the topological degree for mappings of monotone type," *Nonlinear Analysis*, vol. 10, no. 12, pp. 1373–1383, 1986.

[7] J. Berkovits, *On the degree theory for nonlinear mapping of monotone type [Dissertations]*, Annales Academiæ Scientiarum Fennicæ Series A1, 1986.

[8] A. G. Kartsatos and I. V. Skrypnik, "A new topological degree theory for densely defined quasibounded $(\tilde{S}+)$-perturbations of multivalued maximal monotone operators in reflexive Banach spaces," *Abstract and Applied Analysis*, vol. 2005, no. 2, pp. 121–158, 2005.

[9] B. T. Kien, M.-M. Wong, and N.-C. Wong, "On the degree theory for general mappings of monotone type," *Journal of Mathematical Analysis and Applications*, vol. 340, no. 1, pp. 707–720, 2008.

[10] T. M. Asfaw and A. G. Kartsatos, "A Browder topological degree theory for multivalued pseudomonotone perturbations of maximal monotone operators in reflexive Banach spaces," *Advances in Applied Mathematics*, vol. 22, pp. 91–148, 2012.

[11] T. M. Asfaw, "A new topological degree theory for pseudomonotone perturbations of the sum of two maximal monotone operators and applications," *Journal of Mathematical Analysis and Applications*, vol. 434, no. 1, Article ID 19800, pp. 967–1006, 2016.

[12] V. Barbu, *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff International Publishers, Leyden, The Netherlands, 1975.

[13] V. Barbu, *Nonlinear Differential Equations of Monotone Types in Banach Spaces*, Springer Monographs in Mathematics, Springer, New York, NY, USA, 2010.

[14] D. Pascali and S. Sburlan, *Nonlinear Mappings of Monotone Type*, Sijthoff and Noordhoof, Bucharest, Romania, 1978.

[15] E. Zeidler, *Nonlinear Functional Analysis and Its Applications, II/B*, Springer, New York, NY, USA, 1990.

[16] F. E. Browder and P. Hess, "Nonlinear mappings of monotone type in Banach spaces," *Journal of Functional Analysis*, vol. 11, no. 3, pp. 251–294, 1972.

[17] R. T. Rockafellar, "On the maximality of sums of nonlinear monotone operators," *Transactions of the American Mathematical Society*, vol. 149, no. 1, pp. 75–88, 1970.

[18] H. Brézis and L. Nirenberg, "Characterizations of the ranges of some nonlinear operators and applications to boundary value problems," *Annali della Scuola Normale Superiore di Pisa—Classe di Scienze*, vol. 5, no. 2, pp. 225–326, 1978.

[19] J. Berkovits and V. Mustonen, "An extension of Leray-Schauder degree and applications to nonlinear wave equations," *Differential Integral Equations*, vol. 3, no. 5, pp. 945–963, 1990.

[20] P. H. Rabinowitz, "Free vibrations for a semilinear wave equation," *Communications on Pure and Applied Mathematics*, vol. 31, no. 1, pp. 31–68, 1978.

[21] H. Brézis and L. Nirenberg, "Forced vibrations for a nonlinear wave equation," *Communications on Pure and Applied Mathematics*, vol. 31, no. 1, pp. 1–30, 1978.

[22] H. Brezis, "Periodic solutions of nonlinear vibrating strings and duality principles," *Bulletin of the American Mathematical Society*, vol. 8, no. 3, pp. 409–427, 1983.

[23] V. Barbu and N. H. Pavel, "Periodic solutions to nonlinear one dimensional wave equation with x-dependent coefficients," *Transactions of the American Mathematical Society*, vol. 349, no. 5, pp. 2035–2048, 1997.

[24] J. M. Coron, "Periodic solutions of a nonlinear wave equation without assumption of monotonicity," *Mathematische Annalen*, vol. 262, no. 2, pp. 273–285, 1983.

[25] H. Hofer, "On the range of a wave operator with nonmonotone nonlinearity," *Mathematische Nachrichten*, vol. 106, no. 1, pp. 327–340, 1982.

[26] R. Landes and V. Mustonen, "On pseudo-monotone operators and nonlinear noncoercive variational problems on unbounded domains," *Mathematische Annalen*, vol. 248, no. 3, pp. 241–246, 1980.

[27] V. Mustonen, "On elliptic operators in divergence form; old and new with applications," in *Proceedings of the Conference Function Spaces, Differential Operators and Nonlinear Analysis (FSDONA '04)*, pp. 188–200, May-June 2004.

[28] V. Mustonen and M. Tienari, "A remark on the Leray-Lions condition," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 49, no. 7, pp. 1015–1022, 2002.

[29] N. Kenmochi, "Monotonicity and compactness methods for nonlinear variational inequalities," in *Handbook of Differential Equations, IV*, pp. 203–298, Elsevier/North-Holland, Amsterdam, The Netherlands, 2007.

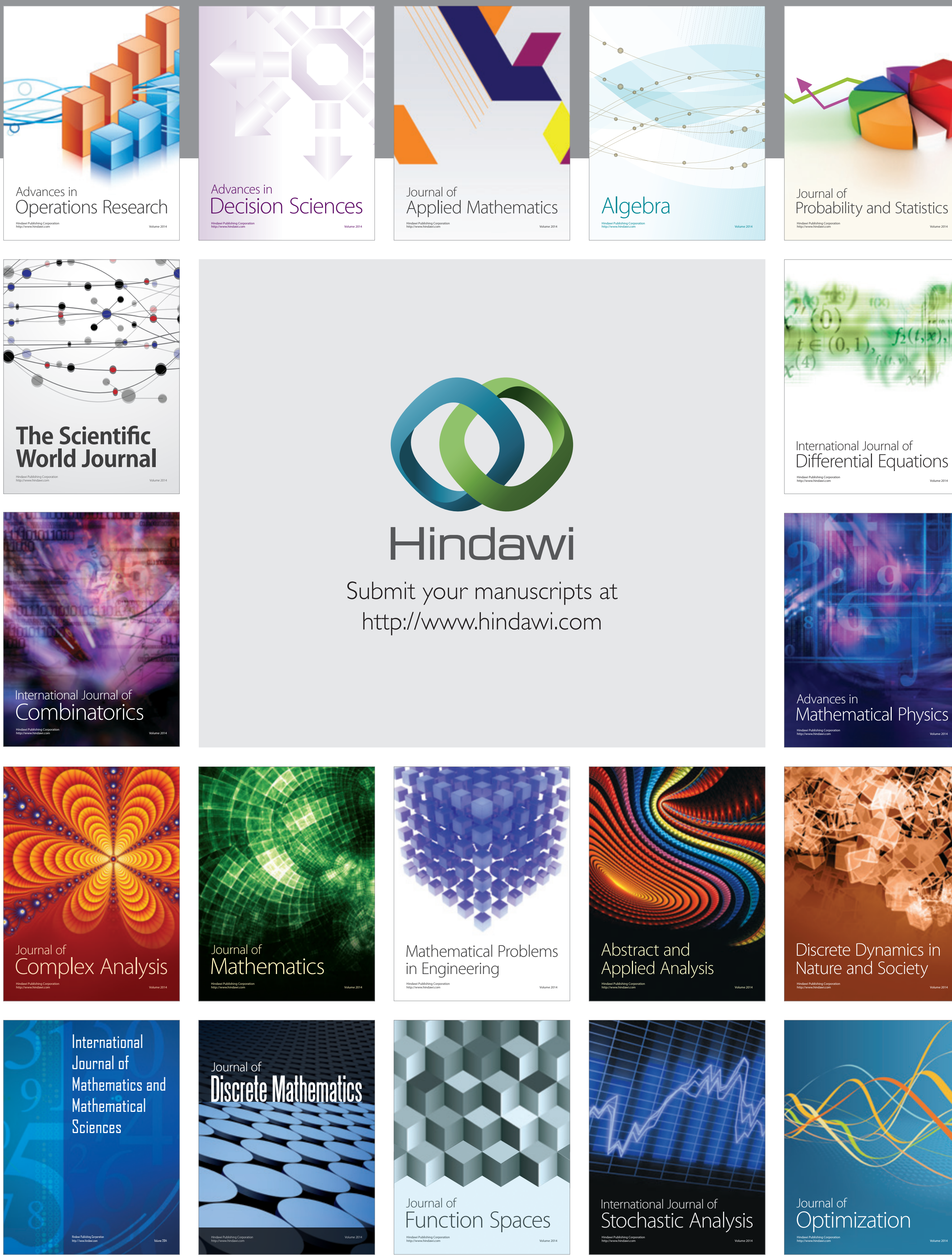
[30] A. G. Kartsatos, "New results in the perturbation theory of maximal monotone and M-accretive operators in banach spaces," *Transactions of the American Mathematical Society*, vol. 348, no. 5, pp. 1663–1707, 1996.

[31] T. M. Asfaw, *Topological degree and variational inequality theories for pseudomonotone perturbations of maximal monotone operators [Dissertations]*, ProQuest, Ann Arbor, Mich, USA, 2013.

[32] T. M. Asfaw, "New surjectivity results for perturbed weakly coercive operators of monotone type in reflexive Banach spaces," *Nonlinear Analysis, Theory, Methods and Applications*, vol. 113, pp. 209–229, 2015.

[33] T. M. Asfaw, "New variational inequality and surjectivity theories for perturbed noncoercive operators and application to nonlinear problems," *Advances in Mathematical Sciences and Applications*, vol. 24, pp. 611–668, 2014.

[34] V. K. Le, "A range and existence theorem for pseudomonotone perturbations of maximal monotone operators," *Proceedings of the American Mathematical Society*, vol. 139, no. 5, pp. 1645–1658, 2011.


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
International Journal of
Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization