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An operator theoretic approach is used to solve the linearized Vlasov–Maxwell equations for transverse plasma oscillations. In particular, the special cases of simple and second-order real zeros of the plasma dispersion function are treated and formulae for the amplitude of the plasma waves are presented. An existence and uniqueness theorem for the solution to the Vlasov–Maxwell transverse mode plasma equation is proved in an appendix. In a second appendix, a general characterization for the zeros of the plasma distribution function is presented for the case of any double humped equilibrium distribution.

I. INTRODUCTION

In a previous paper,[1] hereafter referred to as (I), the longitudinal oscillations of a neutral plasma were studied using a linearized Vlasov equation. In the present paper we extend this work to the case of the transverse modes.

The analysis of (I) differed from that previously published, notably by Van Kampen[2] and Case,[3] in that the "resolvent integration technique" introduced by Larsen and Habetler[4] for studying neutron transport problems was utilized rather than the standard method of singular eigenfunctions. This resolvent integration technique is more appealing to mathematicians because it does not require some of the heuristic arguments used when singular eigenfunction expansions are employed. More importantly, the calculation of stable oscillations was found to be given incorrectly by the singular eigenfunction method. [Since the publication of (I), Siewert[5] and Case[6] have developed the techniques for treating stable oscillations correctly by singular eigenfunction expansions, although for reasons of both taste and rigor we prefer the resolvent integration method.] Both in (I) and later in Ref. 6, the amplitudes of stable oscillations corresponding to degenerate eigenvalues were computed, and there is very good intuitive reason to suspect[1] that these can correspond to physical reality.

Many analyses of plasma stability, for example, those found in standard texts,[7] involve only studies of the zeros of the plasma dispersion function, $\Lambda(z)$. (If Λ has zeros with nonvanishing imaginary part, the plasma is, *ipso facto*, unstable.) The actual magnitude of the stable or unstable oscillations is not a factor in determining stability. This is true, of course, only for linear stability; but the question of nonlinear stability, specifically whether nonlinear effects can stabilize a linearly unstable plasma or destabilize a linear stable plasma, is of vital importance. One approach, as has been followed by Simon and Rosenbluth,[8] is to use a perturbative expansion about the linear solution. For this procedure, a complete solution of the linear equation seems necessary, and is a major practical motivation for our work. The mathematical motivation is, of course, to obtain a correct solution to the initial value problem for (linear) plasma waves.

Transverse oscillations in plasmas were first considered by Shure,[9] and later studied extensively by Felderhof.[10,11] In Refs. 9 and 10 the unperturbed plasma velocity distribution function $f_0(\mathbf{v})$ is assumed to be isotropic. However, it is easily shown[9] that for isotropic equilibrium distributions, the transverse plasma dispersion function, $\Lambda(z)$, has no zeros for $\omega > \omega_p$. Thus, transverse modes for $\omega > \omega_p$ are isotropically stable. For anisotropic equilibrium distributions, $\Lambda(z)$ may have real or complex zeros for $\omega > \omega_p$, and more interesting questions of stability arise.

We have tried to simplify the mathematical treatment we used in (I) by utilizing a Laplace transform technique, rather than the double resolvent integration technique previously employed.[1] (This was suggested by Case.)[12] In either case one integrates about the singularities of the resolvent operator, but in the present case, only a single, rather than a double, resolvent is needed for this procedure, which is sketched in Appendix A.

In Sec. II we write the relevant coupled equations for the transverse modes as derived by Felderhof[11] and define the quantities which appear. We then express the equation in operator form, and compute the resolvent of the operator $\mathbf{K}$. The resolvent involves the transverse dispersion function, $\Lambda(z)$, and although it has been studied for particular cases in the literature,[7,13,14] we consider it worthwhile to give a general analysis of its zeros, i.e., the eigenvalue spectrum of $\mathbf{K}$. This analysis is presented in Appendix B.

In Sec. III we invert the Laplace transform by integrating the resolvent around the spectrum of $\mathbf{K}$, and thereby obtain a solution of the initial value problem for transverse waves. While this has the form of the Shure,[9] Felderhof,[10,11] singular eigenfunction solution for continuum eigenvalues and simple discrete complex eigenvalues, we also consider the case of simple and second-order real eigenvalues.

Section IV contains some concluding remarks.

II. COMPUTATION OF THE RESOLVENT

As in (I), we deal with equations which are Fourier transformed in space (with transform variable $\mathbf{k}$) in order to study wave propagation. Thus, the plasma distribution function is decomposed as $f = f_0(\mathbf{v}) + f_k(\mathbf{v}, t)$, where f_0 is the equilibrium distribution function and f_k is the deviation from equilibrium. f_k obeys the coupled Vlasov–Maxwell linearized equations[9-11]

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$$\frac{\partial f_k(\mathbf{v},t)}{\partial t}+\mathbf{v}\cdot\mathbf{k}f_k(\mathbf{v},t)+\frac{e}{mc}(\mathbf{v}\times\mathbf{B}_0)\frac{\partial f_k(\mathbf{v},t)}{\partial\mathbf{v}}$$

$$+\frac{n_0 e}{m}\left(E_k(t)+\frac{1}{c}\mathbf{v}\times\mathbf{B}_k(t)\right)\frac{\partial f_0(\mathbf{v})}{\partial\mathbf{v}}=0\,, \tag{1a}$$

$$\mathbf{k}\cdot\mathbf{B}_k=0\,, \tag{1b}$$

$$i\mathbf{k}\times\mathbf{E}_k=-\frac{1}{c}\frac{\partial\mathbf{B}_k}{\partial t}\,, \tag{1c}$$

$$i\mathbf{k}\times\mathbf{B}_k=\frac{4\pi e}{c}\int_{-\infty}^{\infty}\mathbf{v}f_k(\mathbf{v},t)\,d^3v+\frac{1}{c}\frac{\partial\mathbf{E}_k}{\partial t}\,. \tag{1d}$$

If the unperturbed distribution function were isotropic, that portion of the free field $\mathbf{B}_0$ due to f_0 would vanish. We view $\mathbf{B}_0$ to include not only a contribution from an anisotropic equilibrium distribution function but also, as usual, contributions from external currents not described by f_0. In any case, the free field $\mathbf{E}_0=0$ because of charge neutrality.

As in Ref. 11, one can rearrange Eqs. (1) to obtain two similar sets of these coupled equations for the transverse plasma modes which we write as

$$\frac{\partial\psi_\pm(u,t)}{\partial t}+ik\mathbf{K}_\pm\psi_\pm(u,t)=0\,, \tag{2a}$$

where

$$\psi_\pm(u,t)=\begin{bmatrix} f_x(u,t)\pm if_y(u,t)\\ E_x(t)\pm iE_y(t)\\ \pm(B_x(t)\pm iB_y(t))\end{bmatrix}\,, \tag{2b}$$

$$\mathbf{K}_\pm=\begin{bmatrix} u\pm u_c & -\frac{n_0 e}{ikm}F(u) & \frac{n_0 e}{ikm(ic)}G(u)\\ \frac{4\pi e}{ik}\int_{-\infty}^{\infty}du & 0 & -ic\\ 0 & ic & 0\end{bmatrix}\,, \tag{2c}$$

$u=v_z$ and $u_c=(eB_0/kmc)$,

$$F(u)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_0(\mathbf{v})\,dv_x\,dv_y\,, \tag{2d}$$

$$G(u)=uF(u)+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2}(v_x^2+v_y^2)f_0(\mathbf{v})\,dv_x\,dv_y\,, \tag{2e}$$

$$f_{\binom{x}{y}}(u,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}v_{\binom{x}{y}}f_k(\mathbf{v},t)\,dv_x\,dv_y\,. \tag{2f}$$

We will henceforth drop the $\pm$ subscripts of Eqs. (2a), (2b), and (2c) in the following analysis to simplify the notation. The Laplace transform of Eq. (2a) is then given by

$$(s+ik\mathbf{K})\hat{\psi}(u,s)=\psi(u,0)\,, \tag{3}$$

and the initial value problem is solved by $(s\to -ik\rho)$

$$\psi(u,t)=-\frac{1}{2\pi i}\int\exp(-ik\rho t)(\mathbf{K}-\rho)^{-1}\psi(u,0)\,d\rho\,. \tag{4}$$

Thus, an expression for the resolvent, $(\mathbf{K}-\rho)^{-1}$ must be derived. Further, the contour integral is around the spectrum of $\mathbf{K}$, instead of the usual line integral from $i\gamma-\infty$ to $i\gamma+\infty$. This modification is justified in Appendix A.

The computation of the resolvent is standard. We find

$$f=(\mathbf{K}-\rho)^{-1}g \tag{5a}$$

$$=\begin{bmatrix} \frac{1}{u+u_c-\rho}-\left(\frac{\omega_p}{k}\right)^2\frac{G(u)-\rho F(u)}{u+u_c-\rho}\frac{1}{\Lambda(\rho)}\int_{-\infty}^{\infty}du\frac{1}{u+u_c-\rho} & -\frac{n_0 e}{imk}\frac{G(u)-\rho F(u)}{u+u_c-\rho}\frac{1}{\Lambda(\rho)} & \frac{n_0 e}{mkc}H_1(u,\rho)\\ -\frac{\rho}{\Lambda(\rho)}\frac{4\pi e}{ik}\int_{-\infty}^{\infty}du\frac{1}{u+u_c-\rho} & \frac{\rho}{\Lambda(\rho)} & H_2(\rho)\\ -\frac{ic}{\Lambda(\rho)}\frac{4\pi e}{ik}\int_{-\infty}^{\infty}du\frac{1}{u+u_c-\rho} & \frac{ic}{\Lambda(\rho)} & H_3(\rho)\end{bmatrix}g\,, \tag{5b}$$

where

$$\Lambda(\rho)=c^2-\rho^2+\left(\frac{\omega_p}{k}\right)^2\int_{-\infty}^{\infty}\frac{G(s)-\rho F(s)}{s+u_c-\rho}ds\,, \tag{5c}$$

$$H_1(u,\rho)=\frac{1}{\Lambda(\rho)}\left[\frac{\rho G(u)-c^2F(u)}{u+u_c-\rho}+\frac{(\omega_{p/k})^2}{u+u_c-\rho}\left(G(u)\int_{-\infty}^{\infty}\frac{F(s)}{s+u_c-\rho}ds-F(u)\int_{-\infty}^{\infty}\frac{G(s)}{s+u_c-\rho}ds\right)\right]\,, \tag{5d}$$

$$H_2(\rho)=\frac{1}{ic}+\frac{1}{ic\Lambda(\rho)}\left[\rho^2+\left(\frac{\omega_p}{k}\right)^2\int_{-\infty}^{\infty}\frac{\rho F(s)}{s+u_c-\rho}ds\right]\,, \tag{5e}$$

$$H_3(\rho)=\frac{1}{\Lambda(\rho)}\left[\rho+\left(\frac{\omega_p}{k}\right)^2\int_{-\infty}^{\infty}\frac{\rho F(s)}{s+u_c-\rho}ds\right]\,, \tag{5f}$$

and $\omega_p=(4\pi n_0 e^2/m)^{1/2}$ is the plasma frequency.

The spectrum of $\mathbf{K}$, $\sigma(\mathbf{K})$, can be obtained from the singularities of $(\mathbf{K}-\rho)^{-1}$. Clearly, the continuous spectrum, $C\sigma(\mathbf{K})$, consists of $\mathbf{R}$ (the real line). The eigenvalues, or point spectrum $P\sigma(\mathbf{K})$ occur where $\Lambda(\rho)=0$. This portion of the spectrum consists of an isolated set of points, either real or appearing as complex conjugate pairs, as is discussed in Appendix B.

III. SOLUTION OF THE INITIAL VALUE PROBLEM

We refer to Eq. (4) and write

$$\psi(u,t)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\exp(-ik\rho t)[(\mathbf{K}-\rho)_{+}^{-1}-(\mathbf{K}-\rho)_{-}^{-1}]\psi(u,0)\,d\rho-\sum_{j}\frac{1}{2\pi i}\oint_{\Gamma_j}\exp(-ik\rho t)(\mathbf{K}-\rho)^{-1}\psi(u,0)\,d\rho\,. \tag{6}$$

Here, $(\mathbf{K}-\rho)_{\pm}^{-1}$ represent the limiting values of the resolvent as ρ approaches $\mathbf{R}$ from above and below, respectively, while Γ_j is a contour surrounding the eigenvalue ν_j. This formula applies if $\mathrm{Im}(\nu_l)\neq 0$. It must be modified if $\mathrm{Im}(\nu_l)=0$ (discrete eigenvalue imbedded in the continuum) to

$$\begin{aligned}\psi(u,t)=&\lim_{\epsilon_1\to 0}\frac{1}{2\pi i}\left(\int_{-\infty}^{\nu_l-\epsilon_1}+\int_{\nu_l+\epsilon_1}^{\infty}\right)\{\exp(-ik\rho t)[(\mathbf{K}-\rho)_{+}^{-1}-(\mathbf{K}-\rho)_{-}^{-1}]\psi(u,0)\}\,d\rho-\sum_{j\neq l}\frac{1}{2\pi i}\oint_{\Gamma_j}\exp(-ik\rho t)(\mathbf{K}-\rho)^{-1}\psi(u,0)\,d\rho\\&+\frac{1}{2\pi i}\int_{\Gamma_{l-}}\exp(-ik\rho t)(\mathbf{K}-\rho)_{-}^{-1}\psi(u,0)\,d\rho-\frac{1}{2\pi i}\int_{\Gamma_{l+}}\exp(-ik\rho t)(\mathbf{K}-\rho)_{+}^{-1}\psi(u,0)\,d\rho\,,\end{aligned} \tag{7}$$

$$(\mathbf{K}-\rho)_{\pm}^{-1}\psi(u,0)=\lim_{\epsilon\to 0+}\begin{bmatrix}\dfrac{\psi_1(u,0)}{u+u_c-\rho\mp i\epsilon}-\left(\dfrac{\omega_p}{k}\right)^2\dfrac{G(u)-\rho F(u)}{u+u_c-\rho\mp i\epsilon}M^{\pm}(\rho)-\dfrac{n_0 e}{ikm}\dfrac{G(u)-\rho F(u)}{u+u_c-\rho\mp i\epsilon}\dfrac{\psi_2(0)}{\Lambda^{\pm}(\rho)}+\dfrac{n_0 e}{mkc}H_1^{\pm}(u,\rho)\psi_3(0)\\ -\dfrac{\rho}{\Lambda^{\pm}(\rho)}\dfrac{4\pi e}{ik}M^{\pm}(\rho)+\dfrac{\rho}{\Lambda^{\pm}(\rho)}\psi_2(0)-H_2^{\pm}(\rho)\psi_3(0)\\ -\dfrac{e}{\Lambda^{\pm}(\rho)}\dfrac{4\pi e}{ik}M^{\pm}(\rho)+\dfrac{ic}{\Lambda^{\pm}(\rho)}\psi_2(0)+H_3^{\pm}(\rho)\psi_3(0)\end{bmatrix}, \tag{8}$$

where

$$M(\rho)=\frac{1}{\Lambda(\rho)}\int_{-\infty}^{\infty}\frac{\psi_1(u,0)}{u+u_c-\rho}\,du\,, \tag{9}$$

so that

$$M^{\pm}(\rho)=\frac{1}{\Lambda^{\pm}(\rho)}\left[P\int_{-\infty}^{\infty}\frac{\psi_1(u,0)}{u+u_c-\rho}\,du\pm i\pi\psi_1(\rho-u_c,0)\right], \tag{10a}$$

while, from Eq. (5c) for $\Lambda(\rho)$,

$$\Lambda^{\pm}(\rho)=c^2-\rho^2+\left(\frac{\omega_p}{k}\right)^2P\int_{-\infty}^{\infty}\frac{G(s)-\rho F(s)}{s+u_c-\rho}\,ds\pm i\pi\left(\frac{\omega_p}{k}\right)^2[G(\rho-u_c)-(\rho-u_c)F(\rho-u_c)]\,, \tag{10b}$$

$$\lambda(\rho)=c^2-\rho^2+\left(\frac{\omega_p}{k}\right)^2P\int_{-\infty}^{\infty}\frac{G(s)-\rho F(s)}{s+u_c-\rho}\,ds\,. \tag{10c}$$

The contours Γ_j and $\Gamma_{l\pm}$ are drawn in Fig. 1. The integral over the continuum, i.e., the first term on the right sides of Eqs. (6) and (7), are straightforward, if tedious exercises in complex variable theory. The details are similar to the evaluations of analogous integrals in (I), and are omitted. The integrals around the discrete eigenvalues are more or less trivial exercises in residue theory; we present the results for simple eigenvalues of the nonvanishing imaginary part, ν_j, and for single and two-fold degenerate real eigenvalues, ν_l and ν_m, respectively. The detailed calculations are presented elsewhere.[15]

$$\begin{aligned}\psi(u,t)=&\int_{-\infty}^{\infty}\phi^s(u)A(s)\exp(-ikst)\,ds+\sum_{j\neq l,m}A_j\phi_j(u)\exp(-ik\nu_j t)+A_l\phi_l(u)\exp(-ik\nu_l t)+A_{1m}\phi_m(u)\exp(-ik\nu_m t)\\&+A_{2m}[\phi_{2m}(u)-ikt\phi_m(u)]\exp(-ik\nu_m t)\,.\end{aligned} \tag{11a}$$

The continuum "eigenmode" and expansion coefficient are given by

$$\phi^s(u)=\begin{bmatrix}\left(\dfrac{\omega_p}{k}\right)^2P\dfrac{sF(u)-G(u)}{s-u_c-u}-\lambda(s)\delta(s-u_c-u)\\ [4\pi e/(ik)]s\\ [4\pi e/k]c\end{bmatrix}, \tag{11b}$$

and

$$A(s)=-\frac{M^+(s)-M^-(s)}{2\pi i}+\frac{n_0e}{ikm}\frac{G(s-u_c)-sF(s-u_c)}{\Lambda^+(s)\Lambda^-(s)}\psi_2(0)$$
$$+\frac{n_0e}{kmc}\frac{\psi_3(0)}{\Lambda^+(s)\Lambda^-(s)}\int_{-\infty}^{\infty}\left(\frac{\omega_p}{k}\right)^2\frac{P[G(s-u_c)F(\rho-u_c)-G(\rho-u_c)F(s-u_c)]}{\rho-s}d\rho. \tag{11c}$$

These are in agreement with results previously obtained in Ref. 11.

The eigenfunctions and expansion coefficients for the simple eigenvalues of nonvanishing imaginary part are

$$\phi_j(u)=\begin{bmatrix}\left(\frac{\omega_p}{k}\right)^2\frac{\nu_jF(u)-G(u)}{\nu_j-u_c-u}\\ (4\pi e/ik)s\\ (4\pi e/k)c\end{bmatrix}, \tag{11d}$$

and

$$A_j=\frac{1}{\Lambda'(\nu_j)}\left\{\int_{-\infty}^{\infty}\frac{\psi_1(s-u_c,0)}{s-\nu_j}ds-\frac{ik}{4\pi e}\psi_2(0)+\frac{k}{4\pi e}\frac{1}{c}\psi_3(0)\left[\nu_j+\left(\frac{\omega_p}{k}\right)^2\int_{-\infty}^{\infty}\frac{F(s-u_c)}{s-\nu_j}ds\right]\right\}. \tag{11e}$$

The eigenfunction and expansion coefficient for the simple real eigenvalues are

$$\phi_l(u)=\begin{bmatrix}\left(\frac{\omega_p}{k}\right)^2\frac{\nu_lF(u)-G(u)}{\nu_l-u_c-u}\\ (4\pi e/ik)\nu_l\\ (4\pi e/k)c\end{bmatrix}, \tag{11f}$$

$$A_l=\frac{1}{\Lambda'^+(\nu_l)\Lambda'^-(\nu_l)}\{\lambda'(\nu_l)\int_{-\infty}^{\infty}\frac{P\psi_1(s-u_c,0)}{s-\nu_l}ds-\pi^2\frac{\Lambda^{-\prime}(\nu_l)-\Lambda^{+\prime}(\nu_l)}{2\pi i}\psi_1(\nu_l-u_c,0)-\frac{ik}{4\pi e}\lambda'(\nu_l)\psi_2(0)+\frac{k}{4\pi ec}$$
$$\times[\lambda'(\nu_l)(\nu_l+\left(\frac{\omega_p}{k}\right)^2\int_{-\infty}^{\infty}\frac{PF(s-u_c)}{s-\nu_l}ds-\pi^2\left(\frac{\omega_p}{k}\right)^2\frac{\Lambda^{-\prime}(\nu_l)-\Lambda^{+\prime}(\nu_l)}{2\pi i}F(\nu_l-u_c)]\psi_3(0)\}. \tag{11g}$$

Finally, the eigenfunction and expansion coefficients for the second-order real eigenvalue are

$$\phi_{2m}(u)=\begin{bmatrix}\left(\frac{\omega_p}{k}\right)^2\frac{G(u)-(u+u_c)F(u)}{(\nu_m-u_c-u)^2}\\ (4\pi e/ik)\\ 0\end{bmatrix}, \tag{11h}$$

$$A_{1m}=2\mathrm{Re}\Bigg\{\{3[\Lambda''^+(\nu_m)]^2\}^{-1}\left[3\Lambda''^+(\nu_m)\left(\int_{-\infty}^{\infty}\frac{P\psi_1(s-u_c,0)}{(s-\nu_m)^2}ds+i\pi\psi_1(\nu_m-u_c,0)\right)-\Lambda'''^+(\nu_m)\left(\int_{-\infty}^{\infty}\frac{P\psi_1(s-u_c,0)}{s-\nu_m}ds\right.\right.$$
$$\left.\left.+i\pi\psi_1(\nu_m-u_c,0)\right)\right]\Bigg\}+\frac{ik}{4\pi e}\psi_2(0)2\mathrm{Re}\{\tfrac{1}{3}\Lambda'''^+(\nu_m)[\Lambda''^+(\nu_m)]^{-2}\}-\frac{k}{4\pi ec}\psi_3(0)2\mathrm{Re}(\{3[\Lambda''^+(\nu_m)]^2\}^{-1}\{\Lambda'''^+(\nu_m)$$
$$\times\left[\int_{-\infty}^{\infty}\frac{(\omega_p/k)^2PF(s-u_c)}{(s-\nu_m)}ds+i\pi\left(\frac{\omega_p}{k}\right)^2F(\nu_m-u_c)]\}-[\Lambda''^+(\nu_m)]^{-1}[1+\left(\frac{\omega_p}{k}\right)^2\int_{-\infty}^{\infty}\frac{PF(s-u_c)}{(s-\nu_m)^2}ds+i\pi\left(\frac{\omega_p}{k}\right)^2F'(\nu_m-u_c)]\right), \tag{11i}$$

$$A_{2m}=2\mathrm{Re}\left[[\Lambda''(\nu_m)]^{-1}\left(\int_{-\infty}^{\infty}\frac{P\psi_1(s-u_c,0)}{s-\nu_m}ds+i\pi\psi_1(\nu_m-u_c,0)\right)\right]-\frac{ik}{4\pi e}\psi_2(0)2\mathrm{Re}\{[\Lambda''^+(\nu_m)]^{-1}\}$$
$$+\frac{k}{4\pi ec}\psi_3(0)2\mathrm{Re}\{[\Lambda''^+(\nu_m)]^{-1}[\nu_m+\left(\frac{\omega_p}{k}\right)^2\int_{-\infty}^{\infty}\frac{PF(s-u_c)}{s-\nu_m}ds+i\pi\left(\frac{\omega_p}{k}\right)^2F(\nu_m-u_c)]\}. \tag{11j}$$

Here,

$$\psi(u,0)=\begin{bmatrix}\psi_1(u,0)\\ \psi_2(0)\\ \psi_3(0)\end{bmatrix}$$

is the initial condition.

IV. DISCUSSION

The results, given as an eigenfunction expansion, Eq. (11a), of our method agree with Felderhof's for the continuous modes, as well as for simple discrete, nonreal eigenvalues. We do obtain additional results however, mainly the magnitude of the stable modes, i.e., for real eigenvalues, both singly and doubly degenerate.

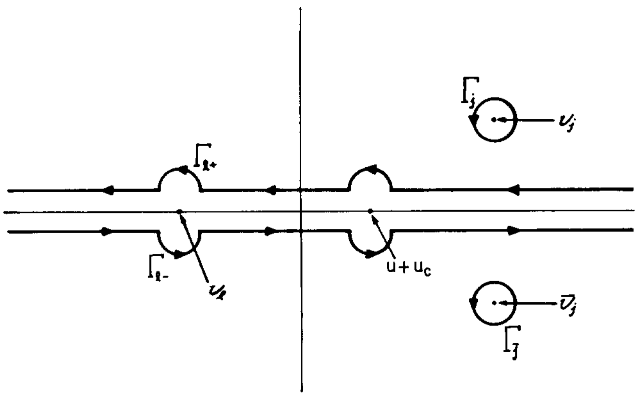

FIG. 1. The contours Γ_j, $\Gamma_{\bar{j}}$, and $\Gamma_{l\pm}$.

In addition, our method can easily be used to compute the magnitude for real or complex eigenvalues of higher degeneracy. It merely requires the use of the appropriate residue formula corresponding to the degree of the degeneracy. (There is also the slight additional complication of obtaining the correct power of t. The procedure used for the neutron transport equation[16] must be followed.) The general case is considered in Ref. 15.

We further note that when making relativistic considerations using cutoff equilibrium distributions which are identically zero for velocities larger than $c-u_c$ and requiring the initial value of the perturbed distribution function to vanish for $|u+u_c|>c$ (as in Ref. 11), the continuous spectrum of $\mathbf{K}$ becomes the finite interval $(-c,c)$. Any real zeros of Λ lying in this interval will need to be treated as we have previously treated real zeros; for real zeros of Λ lying outside this interval, the resolvent integral about such a zero will be identical in nature to the resolvent integral for isolated zeros of Λ with nonvanishing imaginary part.

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APPENDIX A

A rigorous derivation of the solution of Eq. (2a) presents certain technical problems for unbounded operators such as $\mathbf{K}$. In this particular case the spectrum, $\sigma(\mathbf{K})$, contains the real axis, and one would like to surround that part of the spectrum by the parallel lines $(\gamma\pm i\epsilon:\gamma\epsilon\mathbf{R})$ and take the limit $\epsilon\to 0$ in computing the contour integrals in Eq. (4). Indeed, in this case such an approach does lead to precisely the correct expressions, as will be indicated. However, to be certain in any given problem that an unbounded portion of the spectrum can be so treated requires a careful analysis of the unbounded operator.

A solution of the differential Eq. (2a) is understood to be a differentiable function $\psi(\mu,\cdot):\mathbf{R}\to X$, with values in the Banach space $X=\mathcal{L}^1(-\infty,\infty)\times\mathbb{C}\times\mathbb{C}$ with norm

$$|\varphi|=\int_{-\infty}^{\infty}|\varphi_1(u)|\,d\mu+|\varphi_2|+|\varphi_3|\,.$$

Using a semigroup approach, a necessary and sufficient condition for Eq. (2a) to have a unique solution is that the operator $-ik\mathbf{K}$ be the generator of a strongly continuous semigroup.[17] Let

$$-ik\mathbf{K}=-(ik\mathbf{K}_1+ik\mathbf{K}_2),$$

where

$$ik\mathbf{K}_1=\begin{pmatrix} ik(u+u_c) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

and

$$ik\mathbf{K}_2=\begin{bmatrix} 0 & (-n_0e/m)F(u) & [n_0e/m(ic)]G(u)\\ 4\pi e\int du & 0 & kc\\ 0 & -kc & 0\end{bmatrix}.$$

Since $ik\mathbf{K}_2$ is bounded in X, it is sufficient to show that $-ik\mathbf{K}_1$ is the generator of a strongly continuous semigroup.[18] By virtue of the Hille–Yosida–Phillips theorem,[19] $ik\mathbf{K}_1$ will be such a generator if, and only if,

(i) $(\gamma,\infty)\not\subset\sigma(ik\mathbf{K}_1)$

and

(ii) $|(\xi+ik\mathbf{K}_1)^{-n}|\leq M(\xi-\gamma)^{-n}\,\forall\,\xi>\gamma$,

where M and γ are positive constants. In fact, $\sigma(ik\mathbf{K}_1)=i\mathbf{R}$, so condition (i) is satisfied. Further, the estimate (ii) can be verified immediately. We conclude that $ik\mathbf{K}$ is indeed the generator of a strongly continuous semigroup.

Thus, the solution of Eq. (2a) may be represented by the Laplace transform

$$\psi(t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}(ik\mathbf{K}+s)^{-1}\psi(0)\exp(st)\,ds\,, \tag{A1}$$

and will be strongly continuously differentiable on $(0,\infty)$ (Ref. 15, p. 31). (In order to obtain strong differentiability at $t=0$, more delicate estimates would be necessary, e.g., absolute continuity of the semigroup.)

Since the spectrum of $ik\mathbf{K}$ consists (except for isolated poles) of the imaginary axis, the identity

$$\frac{1}{2\pi i}\int_{-\gamma+i\infty}^{-\gamma-i\infty}(ik\mathbf{K}+s)^{-1}\psi(0)\exp(st)\,ds=\frac{-1}{2\pi i}\sum_j\oint_{\Gamma_j}(ik\mathbf{K}+s)^{-1}\psi(0)\exp(st)\,ds\,, \tag{A2}$$

where Γ_j are contours about isolated poles in the left half-plane, follows immediately from the vanishing of the exponential at infinity. Combining Eqs. (A1) and (A2) gives Eq. (6).

An alternate approach, as developed in (I) is to deal not with the unbounded operator $\mathbf{K}$, but with some bounded function of it, say $\mathbf{S}=(\mathbf{K}-\xi)^{-1}$ for $\xi\not\in\rho(\mathbf{K})$. Then, a partition of the identity $[E(s):s\,\epsilon\,\sigma(\mathbf{S})]$ may be obtained for the bounded operator $\mathbf{S}$ by conventional resolvent

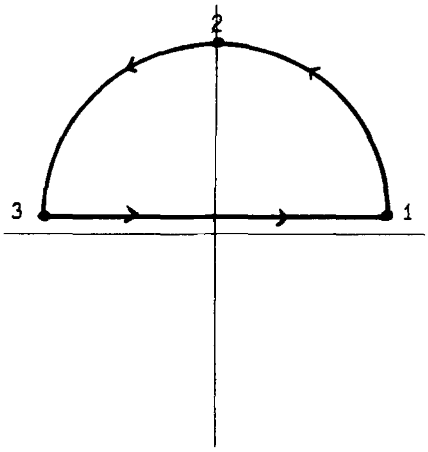

FIG. 2. The contour C.

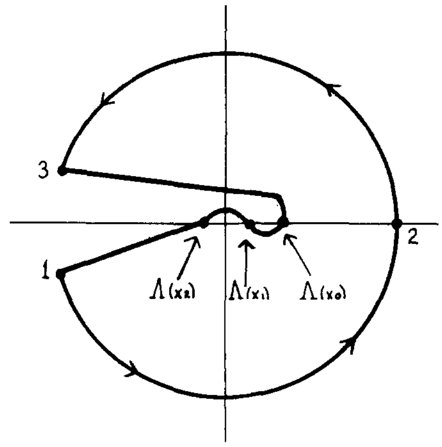

FIG. 3. The Nyquist diagram for case 2a.

integration techniques, and the behavior of **K** at infinity reduces to the study of the contour integrals in a neighborhood of the origin. Once the spectral projections $E(s)$ of **S** are constructed, it follows that **K** also is "spectral," and that

$$\mathbf{K} = \int_{\sigma(s)} \left(\xi - \frac{1}{s}\right) dE(s) . \tag{A3}$$

Finally, with an appropriate change of variable $s \to z = \xi - 1/s$, the results may be put in conventional eigenmode expansion form. However, we consider the Laplace transform a simpler approach.

APPENDIX B

For the analysis of the zeros of $\Lambda(z)$ we will consider the contour C in the upper half of the z plane drawn in Fig. 2. Since the zeros of Λ occur in complex-conjugate pairs, we will draw the Nyquist diagram for the contour of Fig. 2 to determine half the number of zeros of Λ. For the curved portion of the contour, $1-2-3$, we need the asyptotic value of Λ for large z. This is readily given by Eq. (5c)

$$\Lambda(z) \approx -z^2, \quad |z| \text{ large.} \tag{B1}$$

For the portion of the contour near the real axis, $3-1$, we need the limiting value of Λ, Λ^+, given by Eqs. (10b) and (10c)

$$\Lambda^+(x) = \lambda(x) + i\pi(\omega_p/k)^2 q(x) , \tag{B2}$$

where

$$q(x) = \frac{d}{dx} \int\int \frac{1}{2}(v_x^2 + v_y^2) f_0(v_x^2 + v_y^2,\ x - u_c)\, dv_x\, dv_y . \tag{B3}$$

If we regard the equilibrium distribution function as having two humps, then $q(x)$ will have three zeros, in increasing order: x_0, x_1, x_2. (This analysis is then applicable for two stream instability as well as bump on tail instability.)

We may now consider the Nyquist diagram for Λ, i.e., the image in the Λ plane of the contour drawn in Fig. 2. Since $q(x)$ has three zeros, the image contour must cross the real axis three times, and we have along the portion of the contour $3-1$,

$$q(x) > 0 \quad x < x_0 \quad \text{or } x_1 < x < x_2$$

and

$$q(x) < 0 \quad x_0 < x < x_1 \quad \text{or } x_2 < x ,$$

so that if

(1) $\lambda(x_0) < 0$, then Λ has one conjugate pair of zeros;

(2) $\lambda(x_0) > 0$ and (a) $\lambda(x_1)\lambda(x_2) < 0$, then Λ has one conjugate pair of zeros; (b) $\lambda(x_1)\lambda(x_2) > 0$, then Λ has no zeros.

As an example of these conclusions we show the Nyquist diagram for case 2(a) in Fig. 3. The other cases are fairly simple modifications of this diagram.

Real zeros occur if $\Lambda^+(x) = 0$. Clearly, these can only occur at x_0, x_1, and x_2 where the imaginary part of Λ^+ vanishes. Thus the condition for $\Lambda^+(x_j)$ to vanish is that $\lambda(x_j) = 0$, $j = 0, 1, 2$, and thus we see that there may exist one or two real roots. (A similar analysis for the case of the isotropic distribution shows that Λ has no zeros.)

[1] M. D. Arthur, W. Greenberg, and P. F. Zweifel, Phys. Fluids **20**, 1296 (1977).
[2] N. G. Van Kampen, Physica **21**, 949 (1955).
[3] K. M. Case, Ann. Phys. (N. Y.) **7**, 3 (1959).
[4] E. W. Larsen and G. J. Habetler, Commun. Pure Appl. Math. **26**, 529 (1973).
[5] C. E. Siewert, J. Math Phys. **18**, 2059 (1977).
[6] K. M. Case, Phys. Fluids **21**, 249 (1978).
[7] N. A. Krall and A. W. Trivelpiece, *Principles of Plasma Physics* (McGraw–Hill, New York, 1973).
[8] A. Simon and M. N. Rosenbluth, Phys. Fluids **19**, 1567 (1976).
[9] F. C. Shure, Ph D. dissertation, University of Michigan (1962).
[10] B. U. Felderhof, Physica **29**, 293 (1963).
[11] B. U. Felderhof, Physica **29**, 317 (1963).
[12] K. M. Case (private communication).
[13] R. Z. Sagdeev and V. D. Shafranov, Zh. Eksp. Teor. Fiz. **39**, 181 (1960) [Sov. Phys.-JETP **12**, 130 (1961)].
[14] E. G. Harris, Plasma Phys. **2**, 138 (1961).
[15] M. D. Arthur, Ph.D. dissertation, Virginia Polytechnic Institute and State University (1979).
[16] K. M. Case and P. F. Zweifel, *Linear Transport Theory* (Addison–Wesley, Reading, Mass., 1967), Appendix.
[17] S. G. Krein, *Linear Differential Equations in Banach Space*, (American Mathematical Society, Providence, R. I., 1971), p. 47.
[18] T. Kato, *Perturbation Theory for Linear Operators* (Springer–Verlag, New York, 1966), p. 495.
[19] M. Reed and B. Simon, *Methods of Modern Mathematical Physics* (Academic, New York, 1975), Vol. 2, p. 247.