On the Role of Student Understanding of Function and Rate of Change in Learning Differential Equations

George E. Kuster Jr.

Dissertation submitted to the faculty of the Virginia Polytechnic Institute and State University in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
In
Mathematics

Megan  Wawro, Committee Chair
Estrella Johnson
Catherine Ulrich
Joseph F. Wagner
Lizette Zietsman

June 30th, 2016
Blacksburg, VA

Keywords: Differential equations, Knowledge in Pieces, Undergraduate Mathematics Education

On the Role of Student Understanding of Function and Rate of Change in Learning Differential Equations

George E. Kuster Jr.

ABSTRACT

In this research, I utilize the theoretical perspective Knowledge In Pieces to identify the knowledge resources students utilize while in the process of completing various differential equations tasks. In addition I explore how this utilization changes over the course of a semester, and how resources related to the concepts of function and rate of change supported the students in completing the tasks. I do so using data collected from a series of task-based individual interviews with two students enrolled in separate differential equations courses. The results provide a fine-grained description of the knowledge students consider to be productive with regard to completing various differential equations tasks. Further the analysis resulted in the identification of five ways students interpret differential equations tasks and how these interpretations are related to the knowledge resources students utilize while completing the various tasks. Lastly, this research makes a contribution to mathematics education by illuminating the knowledge concerning function and rate of change students utilize and how this knowledge comes together to support students in drawing connections between differential equations and their solutions, structuring those solutions, and reasoning with relationships present in the differential equations.

## Dedication

This dissertation is dedicated to my brilliant and remarkably supportive wife and my ever-inquisitive son.

## Acknowledgements

I would like to express my deepest appreciation to my committee chair Megan Wawro, whose understanding, generous guidance, and support made completing this dissertation a reality. Her persistent help and counsel were sometimes all that kept me going, and she made me a better researcher, writer, and human being in the process. For that, I am forever grateful.

I would also like to thank my committee members, Estrella Johnson, Katy Ulrich, Joe Wagner, and Lizette Zietsman. Each of my committee members had a unique and special role in supporting me through not only my dissertation, but also the doctoral program. The advice and direction each of my committee members provided was invaluable.

In addition without my fellow graduate student and friend Alan Lattimer I don't think I would have made it though the graduate program. From late night homework sessions, to scheduling time devoted to working on our respective dissertations (and not to mention a few remodeling projects), I am convinced that Alan kept me sane during my tenure here at Virginia Tech.

Lastly, reflecting back on it all, the various forms of support, guidance and encouragement I received while in the process of completing this dissertation is simply astonishing. From the seemingly uncountable number of conversations with fellow graduate students and professors, to the tremendous amount of help provided by my family in the form of meals, daycare, and quiet places to work, it is not possible for me to convey my gratitude and appreciation. Surely, without the help provided by my family – especially my father – and friends this dissertation would not have been completed.

**Table of Contents**


**List of Figures** ..................................................................................................... **vii**
**List of Tables…** ..................................................................................................... **ixx**
**Chapter 1: Introduction** ..................................................................................................... **1**
1.1 Mathematics Background ..................................................................................................... 4
1.2 Mathematics Education Background ..................................................................................................... 7
1.3 Theoretical and Analytical Perspectives ..................................................................................................... 9
1.4 Summary ..................................................................................................... 11
**Chapter 2: Literature Review and Theoretical Perspective** ..................................................................................................... **13**
2.1 Brief Overview ..................................................................................................... 14
2.2 Student Understanding of Function and Rate of Change in Differential Equations ........ 15
2.2.1 Function ..................................................................................................... 16
2.2.2 Rate of Change ..................................................................................................... 25
2.2.3 Differential Equations as Functions ..................................................................................................... 30
2.3 Theoretical Perspective ..................................................................................................... 31
2.3.1 Assumptions Concerning Knowledge ..................................................................................................... 32
2.3.2 Knowledge Pieces ..................................................................................................... 33
2.3.3 Knowledge Activation ..................................................................................................... 36
2.3.4 Knowledge Development ..................................................................................................... 37
2.4 Summary of Chapter 2 ..................................................................................................... 38
**Chapter 3: Methods** ..................................................................................................... **38**
3.1 Participants ..................................................................................................... 40
3.2 Data Collection ..................................................................................................... 42
3.3 Interview Protocols ..................................................................................................... 45
3.4 Analytical Framework ..................................................................................................... 47
3.4.1 Identifying Resources ..................................................................................................... 52
3.4.2 Identifying Changes in the Utilization of Resources ..................................................................................................... 57
3.4.3 Identifying the Interpretations ..................................................................................................... 58
3.4.4 Documenting Resources and Interpretations ..................................................................................................... 59
3.4.5 Identifying Changes in The Sets of Resources ..................................................................................................... 60
**Chapter 4: Identification of Student Resources and Interpretations** ..................................................................................................... **62**
4.1 Analysis of Resources Identified for Hakeem ..................................................................................................... 65
4.1.1 Interview 1. ..................................................................................................... 65
4.1.2 Interview 2. ..................................................................................................... 92
4.1.3 Interview 3 ..................................................................................................... 116
4.1.4 Interview 4 ..................................................................................................... 140
4.1.5 Interview 5 ..................................................................................................... 157
4.2 Analysis of Resources Identified for Jordan ..................................................................................................... 179
4.2.1 Interview 1 ..................................................................................................... 180
4.2.2 Interview 2 ..................................................................................................... 210
4.2.3 Interview 3 ..................................................................................................... 238
4.2.4 Interview 4 ..................................................................................................... 263
4.2.5 Interview 5 ..................................................................................................... 285
4.3 Conclusions ..................................................................................................... 300

4.3.1 Comparison of Resource Between Hakeem and Jordan........................................ 300
4.3.2 Discussion of Hakeem's and Jordan's resources.................................................. 309
**Chapter 5: Changes in the utilization of resources and the interaction between function and rate of change... 327**
5.1 Changes in the Students' Utilization of the Resources................................................ 328
5.1.1 Hakeem............................................................................................................... 329
5.1.2 Jordan.................................................................................................................. 351
5.2 Interaction Between Function and Rate of Change ..................................................... 365
5.2.1 The Role of Function and Rate of Change Within the Interpretations.................. 366
**Chapter 6: Discussion of Findings, Implications, and Future Directions ......................... 378**
6.1 Summary of Findings ................................................................................................... 378
6.1.1 Resources............................................................................................................. 378
6.1.2 Changes in the Utilization of the Resources......................................................... 382
6.1.3 Interaction Between Function and Rate of Change .............................................. 385
6.1.4 Synthesis of Results.............................................................................................. 386
6.2 Contributions ................................................................................................................ 388
6.3 Practical implications.................................................................................................... 391
6.4 Constraints .................................................................................................................... 394
6.5 Future research.............................................................................................................. 396
**References………............................................................................................................ 398**
**Appendix ………............................................................................................................. 404**

## LIST OF FIGURES


Figure 2.1: Purpose and research questions for the proposed study. .......... 13
Figure 3.1: Timeline of data collection. .......... 43
Figure 4.1: Task one from interview 1 (I1-T1). .......... 66
Figure 4.2: The second task from interview one (I1-T2). .......... 72
Figure 4.3: Task 3 from Interview 1 (I1-T3). .......... 77
Figure 4.4: Hakeem's work showing his utilization of the method of integrating factors. .......... 78
Figure 4.5: Task 4 from Interview 1 (I1-T4). .......... 85
Figure 4.6: Hakeem correlated values of x and y with values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$. .......... 95
Figure 4.7: Second task from Interview 2 (I2-T2). .......... 101
Figure 4.8: Hakeem reasoned with values of P, P' and P'' to build $P(t)$. .......... 104
Figure 4.9: Task 1 Interview 3, large predator small prey (I3-T1). .......... 117
Figure 4.10: Task 2 from Interview 3 .......... 124
Figure 4.11: Task 3 from Interview 3 .......... 131
Figure 4.12: Task 2 from Interview 4 .......... 145
Figure 4.13: Task 3 from Interview 4 .......... 150
Figure 4.14: Task 2 from Interview 5 .......... 162
Figure 4.15: Hakeem utilized the method of eigenvalues to complete Task 2. .......... 163
Figure 4.16: Task 3 from Interview 5. .......... 168
Figure 4.17: Hakeem determined the behavior of the quantity of interest. .......... 170
Figure 4.18: Hakeem graphed $y = 6$ to show "the derivative is zero along it." .......... 173
Figure 4.19: Jordan substituted 10 for x and y to determine $\frac{dx}{dt}$ and $\frac{dy}{dt}$. .......... 186
Figure 4.20: Jordan's graph of the relationship between competing species. .......... 187
Figure 4.21: Graph of potential functions of P. (color added to help discern original graph from where Jordan scribbled over it) .......... 190
Figure 4.22: Slope field Jordan drew to discuss the relationship between values of P and $P'$ ... 191
Figure 4.23: Jordan's drawing of two fish populations with different initial conditions. .......... 199
Figure 4.24: Jordan's drawing of the solutions to the differential equation. .......... 205
Figure 4.25: Jordan only considered the sign of certain terms. .......... 215
Figure 4.26: Jordan used k to determine the relationship between x and y. .......... 216
Figure 4.27: Jordan's interpretation of the variables in the differential equation. .......... 218
Figure 4.28: Jordan's depiction of the behavior of P. .......... 219
Figure 4.29: Drawing P based on the differential equation. .......... 221
Figure 4.30: Task 4 from Interview 2. .......... 229
Figure 4.31: Jordan's work utilizing the method of integrating factors. .......... 230
Figure 4.32: Jordan attained an equation relating y to the independent variable $t$. .......... 250
Figure 4.33: Graphs illustrating the goal of the task. .......... 252
Figure 4.34: Jordan's goal was to find the equation for P in terms of $t$. .......... 253
Figure 4.35: Jordan's work while utilizing Euler's Method to complete the task. .......... 255
Figure 4.36: Graph showing the behavior of P. .......... 256
Figure 4.37: Task 2 from Interview 4. .......... 267
Figure 4.38: Jordan's work while constructing possible solution functions. .......... 270
Figure 4.39: Jordan rewrote the system of equations into a matrix equation. .......... 273
Figure 4.40: Jordan generating a system of equations from an $n^{th}$ order equation. .......... 277

Figure 4.41: Task 4 from Interview 4. ........................................................................ 278
Figure 4.42: Jordan constructed a graph of $\frac{dL}{dt}$ to construct solution functions. ......................... 281
Figure 4.43: Jordan justified $y = 6$ is a solution to $\frac{dy}{dt} = 6 - y$ because $\frac{dy}{dt} = 0$ when $y = 6$. ... 293
Figure 4.44: Task 2 from Interview 5. ........................................................................ 294
Figure 4.45: Jordan's work solving the initial value problem. .................................................. 295
Figure 4.46: Resources and interpretations expressed while completing I2-T1. ........................ 303
Figure 4.47: Resources and interpretations expressed while completing I3-T2. ........................ 305
Figure 4.48: Resources and interpretations expressed while completing I3-T3. ........................ 308
Figure 4.46: Jordan Coordinating values of P, P' and P'' .......................................................... 322
Figure 5.1: Interpretations Hakeem expressed for each task. .................................................... 331
Figure 5.2: Interpretations Jordan expressed for each task. ....................................................... 352
Figure 5.3: Jordan used resources related to function and rate of change. ................................ 369

## LIST OF TABLES

Table 3.1**:** Example table showing documentation of resources and interpretations. .................. 60
Table 4.1: Glossary of knowledge resources. ................................................................... 63
Table 4.2: Resources and interpretations Hakeem expressed in Interview 1. .............................. 91
Table 4.3: Resources and interpretations Hakeem expressed in Interview 2. ............................ 115
Table 4.4: Resources and interpretations Hakeem expressed in Interview 3. ............................ 139
Table 4.5: Resources and interpretations Hakeem expressed in Interview 4. ............................ 156
Table 4.6: Resources and interpretations Hakeem expressed in Interview 5. ............................ 178
Table 4.7: Resources and interpretations Jordan expressed in Interview 1. ............................... 209
Table 4.8: Resources and interpretations Jordan expressed in Interview 2. ............................... 237
Table 4.9: Resources and interpretations Jordan expressed in Interview 3. ............................... 262
Table 4.10: Resources and interpretations Jordan expressed while in Interview 4. ................... 284
Table 4.11: Resources and interpretations Jordan expressed in Interview 5. ............................. 299
Table 5.1: Resources supporting the interpretations Hakeem expressed in Interview 1. ........... 333
Table 5.2: Resources supporting the interpretations Hakeem expressed in Interview 2. ........... 337
Table 5.3: Resources supporting the interpretations Hakeem expressed in Interview 3. ........... 342
Table 5.4: Resources supporting the interpretations Hakeem expressed in Interview 4. ........... 346
Table 5.5: Resources supporting the interpretations Hakeem expressed in Interview 5. ........... 349
Table 5.6: Resources supporting the interpretations Jordan expressed in Interview 1. .............. 353
Table 5.7: Resources supporting the interpretations Jordan expressed in Interview 2. .............. 356
Table 5.8: Resources supporting the interpretations Jordan expressed in Interview 3. .............. 359
Table 5.9: Resources supporting the interpretations Jordan expressed in Interview 4. .............. 361
Table 5.10: Resources supporting the interpretations Jordan expressed in Interview 5. ............ 363
Table 5.11: Resources for function and resources for rate of change ........................................ 366

# CHAPTER 1: INTRODUCTION

Today, mathematics is used to model everything from the spread of disease to controlling the temperature in a room to smoothly flying the most technologically advanced, yet aerodynamically unstable fighter jets. More often than not, driving these models are differential equations, the language of interpreting the world around us. In their simplest form, differential equations are relationships between a quantity of interest and the quantity's rate of change (Kohler & Johnson, 2006). The predictive power afforded by differential equations gives them a key role within science, technology, engineering, and mathematics (STEM) fields. For these reasons, among others, it is immensely important that STEM majors leaving today's colleges and universities have a robust understanding of at least the fundamental properties of differential equations. Given the large number of STEM majors and recent efforts to increase their numbers (the *Engage to Excel* Report, PCAST, 2012), improving students' understanding of differential equations is becoming more and more important. As such so is research that helps illuminate and support student learning in differential equations.

Students majoring in a STEM discipline traditionally enroll in an introductory differential equations course after having completed a three-course calculus sequence (differential, integral and multivariable calculus). The content in a differential equations course, however, utilizes numerous other topics from many areas of math and science, such as linear algebra, matrix theory, physics, biology, and mechanics. Considering the mathematical content present in a typical differential equations course, the topic can be viewed as a culmination of multiple semesters of study that forms the foundation for exploring more advanced mathematics. That is, differential equations is a course that merges and builds on key mathematical ideas from other topics to create new mathematical ideas that pave the way for advancing students' mathematical understanding. Though multiple ideas from previous courses are coming together to form a cohesive whole, there is also a "fundamental leap in the thinking required" (Rasmussen, 2001, p. 67) to understand differential equations. While the specific context of Rasmussen's quote is the understanding of solutions to differential equations, the sentiment is similar for other aspects as well. For these reasons, the topic of differential equations provides a rich environment for studying student learning.

Research in student learning of differential equations is a relatively young field of study. Although multiple topics within differential equations have been investigated, there are still a large number of open questions concerning student learning. In a recent review of undergraduate mathematics education literature Rasmussen and Wawro (in press) were surprised to find fewer than 25 empirical studies on student learning in differential equations published in top tier journals between 2004 and 2014. I interpret this as a call for research on student learning in differential equations, one my dissertation responds to. Additionally, Rasmussen and Wawro (in press) note that function and rate of change are "ideas that are key to differential equations but also transcend the subject." Furthermore, they note a need for research that examines how students' understanding of these ideas grow and change across a differential equations course.

Considering the recent call for an examination of how these ideas change, and as will be outlined later in this chapter, I focus my attention on individual students' understanding and utilization of aspects of these topics when interpreting and completing differential equations tasks. Specifically, I investigated how students' notions of function and rate of change are utilized while they engage in completing tasks involving differential equations. To explore the nature and role of the students' notions of function and rate of change when learning differential equations, I raise the following research questions:

1. What resources concerning function and rate of change do students utilize to complete various differential equations tasks?
2. How do these resources change as the students progress through a differential equations course?
3. How do students' resources concerning rate of change and function influence one another during the development of their understanding of differential equations?

The three research questions are intended to come together and tell a single story concerning student learning in differential equations. For my purposes here, the term *resource* is defined as a small set of small-scale knowledge elements that have a productive role during the process of problem solving. My definition is commensurate with that of Black and Wittman who defined a resource as "an idea or a tool that one uses in problem solving" (p.2, 2009). The definition I pose here also encompasses Hammer's (2000) notion of conceptual resources, ideas "students bring to understanding physical phenomena and concepts" (p. 52) and Black and Wittman's (2009) procedural resources, which are procedural steps in a larger solution process. Although this study is concerned with the students' understanding of function and rate of change and various aspects thereof, the purpose is not to explicitly characterize their understanding of these concepts within the context of differential equations. Rather the purpose is to depict, in a high level of detail, the role various aspects of these concepts, in the form of resources, play in the development of the students' understanding of differential equations.

In the previous paragraphs I presented a general overview of why investigating student learning in differential equations is important enough to warrant study. In the remaining sections of this chapter I elaborate on the complexity involved in both the mathematics within differential equations and student learning of the subject. Specifically I briefly discuss some of the mathematics embedded within a typical ordinary differential equation and provide a general description of the research on student learning in differential equations. I then introduce the theoretical and analytical perspectives framing and driving the structure of the dissertation. I end the chapter with a summary and a framing for the remainder of the dissertation.

## 1.1 Mathematics Background

In this section I briefly outline some of the mathematical notions regarding the types of differential equations typically encountered in a sophomore level introductory course, such as linear ordinary differential equations of the form $y = p(t)y + g(t)$ where the functions $p(t)$ and $g(t)$ are continuous on the interval $(a, b)$. The following discussion is not intended to represent in any way, form, or manner the mathematical requirements for understanding differential equations. Rather this section is intended to aid the reader in situating, understanding, and appreciating the goals of this study. Specifically, there are three goals this section aims to achieve directly concerning the reader: to allow the reader to appreciate the mathematical sophistication present in differential equations, to situate the mathematics within the current research literature concerning mathematics education, and to situate my research questions within the mathematics.

Consider the somewhat standard ordinary differential equation $y' = y + t$, which is of the form $y' = p(t)y + g(t)$ with $p(t)$ and $g(t)$ continuous, in this case, over $\mathbb{R}$. The equation $y' = y + t$ encompasses numerous interrelated concepts, such as rate of change, derivative, function, and solution space. Given this equation, if one had access to corresponding values for $y$ and $t$, they could find the value of $y'$ by substituting the values of $y$ and $t$ into the equation. In this case, one can view the value of $y'$ as a function of both $y$ and $t$; in other words $y$ and $t$ are variables in the function $f(t, y) = y + t$. When solving a differential equation one aims to find a function that satisfies the differential equation; in this particular case, a solution would be a function $y(t)$ whose derivative function $y'$ is equal to $y + t$. In general, a differential equation of the form $y' = p(t)y + g(t)$ where $p(t)$ and $g(t)$ are continuous on the interval $(a, b)$, the initial value problem:

$$y' = p(t)y + g(t),\ \ y(t_o) = y_0 \text{ where } t_0 \in (a, b),$$

has a unique solution on the entire interval $(a, b)$. If not provided an initial condition, however, often differential equations have multiple solution functions that satisfy the differential equation itself; in fact, there is often an entire solution space of functions associated with a single differential equation as is the case with the differential equation $y' = y + t$. This means that the relationship $y' = y + t$ does not just hold for one function but is true for all of the functions in the solution space.

The differential equation $y' = y + t$ could also be written as $y'(t) = y(t) + t$, more clearly indicating the dependence of the functions $y$ and $y$ on the variable $t$. Written in this form it can be seen that $y$ is a function of $t$. As a result, $y$ can be thought of as both a function and a variable, and this provides numerous interpretations of the differential equation depending on the mathematical objects one is working with. For example, the equality could be thought of as a numerical equivalence between the value of $y'$ and the sum of the values of $y$ and $t$. In other words, the value of the quantity's rate of change is equal to the quantity's value plus the value of the quantity's independent variable. One could also think of the equation $y' = y + t$ as representing an equivalence between the functions $y'(t)$ and the function that results from summing $y(t)$ and $t$. Lastly, one can think of the derivative as an operator and the differential equation as indicating that taking the derivative of $y(t)$ is operationally equivalent to adding $y(t)$ and $t$. In general, while differential equations can be thought of as "rate of change equations," the various ways of interpreting the equivalence of the single equation is indicative of the various ways one can think of a differential equation.

Solutions to differential equations, such as the one above, are often found using graphical, analytical, or numerical methods. Each of these approaches has a separate yet related set of ideas associated with it. For instance, the method of integrating factors, an analytical solution method, makes use of integration techniques. Such techniques rely on relationships between differentiation and integration while operating under assumptions such as the solutions being differentiable and taking on a certain form. Other analytical methods such as those typically used when solving systems of equations make use of techniques involving eigenvalues and eigenvectors, which rely on topics from linear algebra. Graphical methods often represent the value of $y'(t)$ as slopes in a vector field. A vector field represents a relation between points in the $ty$-plane and the value of $y'(t)$ in the form of vectors tangent to points on each of the solution curves. Such representations allow for constructing approximations of the solutions by utilizing the tangent vectors to guide the path of the solution curves. Because $y' = f(t, y)$ the construction of a tangent vector field requires coordinating values for $y$ and $t$ to attain values for $y'$ (represented by vectors with a certain slope). In this way, the slope field is not simply an object generated from a differential equation; rather, it provides an additional representation of a differential equation. Much like graphs of linear functions can be thought of representations of a function, vector fields can be though of as representations of differential equations. Numerical methods include approximation methods such as Euler and Runge-Kutta. While the utilization of Euler's Method can be constructed graphically, one can also think of it as constructing the solution based on notions of differentials, typically introduced in calculus and motivated by the Taylor-series approximation. By keeping the time steps appropriately small, one can construct approximations of the value of the unknown solution at a particular value of $t$. In principle if one knows $y(t_0) = y_0$, $y'(t_0) = y'_0$, and a time interval for which the approximation is reasonable, one can approximate the value of the solution after that period of change. Each of these solution methods makes use of different representations and aspects of function and rate of change.

In this section I have only outlined some of the mathematics concerning topics in differential equations. As can be seen from a purely mathematical standpoint, the concepts of function and rate of change are heavily embedded in differential equations. For instance, differential equations represent both a relationship between the value of an unknown quantity $y$ and its rate of change at a certain time $t$, and the differential equation as a relationship between functions; from a mathematical standpoint both types of relationships are equally important. This study is specifically interested in the role the students' notion of function and rate of change play in their understanding of differential equations, and given the various ways one can interpret a differential equation these concepts may emerge in different ways.

In the following section I provide a brief overview of the current mathematics education literature that directly concerns student learning of differential equations in regard to their notion of function and rate of change. The purpose of this section is to allow the reader to better appreciate how the research questions are relevant to the mathematical foundations, as well as how they contribute to, and build on current literature. I provide a more in depth discussion of these topics in Chapter 2.

## 1.2 Mathematics Education Background

The notions of function and rate of change have long been topics of interest for mathematics education researchers, and numerous studies have been conducted on student learning of these concepts (e.g., Carlson, Jacobs, Coe, Larsen & Hue 2002; Tall & Vinner, 1981; Thompson, 1994; Zandieh, 2000). It is not surprising that many undergraduate mathematics education researchers have studied student learning of these topics as they both lie at the heart of learning calculus, and calculus has a central and fundamental role in the development of today's mathematics, including differential equations. Indeed, studies concerning student learning in differential equations have found that students' understanding of both function and rate of change are necessary and important for the development of their understanding of differential equations (Donovan, 2002, 2007; Rasmussen & Blumenfeld, 2007; Rasmussen & King, 2000; Rasmussen & Whitehead, 2003). These studies include, but are not limited to, documenting student difficulties concerning differential equations (Rasmussen, 2001), the importance of understanding a first order ordinary differential equation as a function itself (Donovan, 2007), graphically reasoning about functions to make sense of the phase plane (Keene, 2007), using the students' notion of rate of change to construct straight-line (Eigen) solutions (Rasmussen & Blumenfeld, 2007), and the role function and rate of change play when reasoning about slope fields (Rasmussen & Blumenfeld, 2007; Whitehead & Rasmussen, 2003). Additionally, in a recent study Kuster and Dominy (2015) found that students' notions of rate and slope could be leveraged to afford the students an understanding of an ordinary differential equation as a function of two variables. The concepts of function and rate of change have also been found to play a role in students' understanding of the existence and uniqueness theorems. Raychaudhuri (2008) found that students' understanding of function, differentiation, and integration play a crucial role when reasoning about solutions. These studies concern a wide array of topics within differential equations yet a common thread is that they each indicate having a robust understanding of function and rate of change is important.

Although these findings clearly demonstrate that a student's notion of function and rate of change play a crucial role in his or her ability to interpret, construct, and understand various topics central to differential equations, little is known about the dynamic nature of student understanding of these topics as students' experience with differential equations grows. Furthermore, it is unclear how the students' notions of these concepts interact with each other while learning about differential equations, or how students utilize different aspects or representations of these concepts across various topics/tasks within differential equations. Questions such as these are open areas in current mathematics education research, some of which this study illuminate. In order to investigate and characterize the dynamic nature of student understanding, it is necessary to utilize theoretical and analytical perspectives that afford such investigations. In the following section I introduce the theoretical perspective and analytical lens adopted for this study that frame the structure of the data collection and analysis phases, and provide a justification as to why they are appropriate for answering the research questions.

## 1.3 Theoretical and Analytical Perspectives

The research questions in this study are framed within the perspective of Knowledge in Pieces (diSessa, 1993; diSessa & Sherin, 1998; Hammer & Elby, 2003; Smith, diSessa & Rochelle, 1993; Wagner, 2006). Knowledge in Pieces (KiP) maintains that knowledge can be modeled as a dynamic system of elements and their connections. Knowledge is taken as context sensitive in that certain pieces of knowledge (or collections there of) may be considered to be applicable in some situations and not in others. For instance, the notion that continuous functions are differentiable holds in the context of polynomials, but not for all functions in general. Due to the contextually sensitive nature of knowledge, a KiP perspective demands focus on the contextual attributes a learner attends to and interprets in coordination with the knowledge they use in a given situation (diSessa & Wagner, 2005).

Knowledge in Pieces was originally developed and is still widely utilized in physics education research. Recently, researchers in the field of mathematics education have began to utilize KiP to construct detailed characterizations of learners' and teachers' cognition concerning topics such as area, probability, limits, proportion, and combinatorics (Adiredja, 2014; Campbell, 2011; Kapon, Ron, Hershkowitz, & Dreyfus, 2015; Orrill & Brown, 2012; Pratt & Noss, 2002; Wagner, 2006). This perspective is appropriate for the research questions within this study as "KiP is an epistemological perspective that models processes of conceptual change with high conceptual and temporal resolution" (p. 45, Kapon et al., 2015). Additionally given the numerous representations of rate of change (e.g., slope, derivative, ratio) and function (e.g., evaluated value, graphs) students may utilize across various tasks within differential equations, representing the students' knowledge as a dynamic system of elements is commensurate with the research goals.

Within the larger framing of KiP, Transfer in Pieces (Wagner, 2006) provides an analytical lens for documenting and characterizing the dynamic nature of knowledge systems with specific regard to how a single learner utilizes prior knowledge in new situations. In other words, KiP allows for documenting what students' knowledge systems look like and Transfer in Pieces (TiP) allows for characterizing the dynamic nature of that knowledge. Specifically, I interpret TiP as a lens for analyzing how the concepts such as function or rate of change develop to become applicable in more situations. This development of the concepts of function and rate of change may seem as though it is a question of abstraction, but adopting a context specific perspective on knowledge requires considering how students structure new situations in ways that afford the use of prior knowledge. According to Wagner (2006), abstraction is a consequence of transfer (incorporating prior knowledge into new situations), not the other way around.

## 1.4 Summary

There are multiple aspects of this study that contribute to the current mathematics education literature base. Primarily, this study contributes to the relatively unexplored area of student learning in differential equations; the literature pool for student learning in differential equations is relatively small when compared to other areas of mathematics such as differential calculus. Specifically, this study explores previously established findings on student learning in differential equations and addresses Rasmussen and Wawro's (in press) call for research on concepts that are important to and transcendent of differential equations. Secondly, this dissertation expands the utilization of the KiP perspective to longitudinal investigations of student understanding of differential equations. While KiP is well utilized in physics education, there is only a small number of studies utilizing KiP in mathematics education research, and as such this dissertation contributes to increasing its prevalence in mathematics education research. I now frame the remaining chapters.

A literature review of student learning in differential equations is located in Chapter 2. After a brief discussion of recent literature published in differential equations, I discuss the importance and centrality of understanding function and rate of change in understanding differential equations. Chapter 2 also houses a more detailed discussion of the theoretical perspectives utilized to document and analyze student thinking and presents definitions of relevant and important terms. The participants and methods of data collection and analysis methods are outlined in detail in Chapter 3, as is Transfer in Pieces the perspective informing the analytical lens utilized to answer the second research question. Chapter 4 contains the analysis of two students, Hakeem and Jordan (pseudonyms), leading to the identification of the resources they used to complete the various tasks; answering the first research question. The first research question is explicitly discussed in Chapter 4 which I conclude by presenting connections between the identified resources and current research on student understanding of function, rate of change and differential equations. This analysis also laid the foundation for answering the second and third research questions, and thusly encompasses a significant portion of the dissertation. In chapter 5 I address the second and third research questions by presenting analysis concerning the changes in the students utilization of the resources over a series of five interviews, and discussing how the students notions of function and rate of change came together within the patterns of the utilization of the resources across the different tasks. I conclude the dissertation with Chapter 6, where I discuss, limitations of the study, implications, and future research directions.

# CHAPTER 2: LITERATURE REVIEW AND THEORETICAL PERSPECTIVE

This chapter serves two main purposes: to situate the dissertation within the current literature on student learning in differential equations, and to provide a detailed description of the theoretical perspective utilized for the purposes of documenting student knowledge. These two sections are included in this chapter due to their interconnected nature. The literature review frames the relevance of the research questions in terms of the current literature, while the theoretical perspective describes how answering the research questions aims to illuminate these issues. In the first section of the literature review I provide a general overview of research on student understanding of differential equations. I do this for two reasons: to quickly introduce the reader to the array of topics researched, and to illuminate the importance and centrality of function and rate of change with regard to learning differential equations. I then present findings from studies whose results have implications for student understanding of function and rate of change, even though these were not the studies' main topics of investigation. I then transition to studies that focus specifically on the role of students' understanding of function and rate of change with regard to the learning of differential equations. The third section concludes the chapter with a detailed discussion of the theoretical perspective taken herein, Knowledge in Pieces. A detailed discussion of the analytical lens, Transfer in Pieces, is located in Chapter 3.

To remind the reader of the purpose and research questions for the proposed study, they are restated below in Figure 2.1.

**Purpose:** To investigate how students' notions of function and rate of change evolve as their experiences with differential equations become more sophisticated.
**Research Questions:**

1. What resources concerning function and rate of change do students utilize to complete various differential equations tasks?
2. How do these resources change as the students progress through a differential equations course?
3. How do students' resources concerning rate of change and function influence one another during the development of their understanding of differential equations?

*Figure 2.1: Purpose and research questions for the proposed study.*

## 2.1 Brief Overview

In this section I present a general description of current mathematics education literature on student learning of differential equations for the purpose of providing the reader with a general introduction to the field**.** Research in student learning of differential equations is a growing field of study and is an area that is still relatively young; Donovan noted that research on student learning in differential equations is still in its “infancy” (2002). Additionally, while multiple topics within differential equations have been investigated, it is still a quite open area of study. For instance, as noted in Chapter 1, in a recent review of undergraduate mathematics education literature, Rasmussen and Wawro (in press) noted their surprise in finding fewer than 25 empirical studies on student learning in differential equations published in top tier journals between 2004 and 2014. I now briefly introduce some of the literature concerning student learning.

Studies conducted on student learning in differential equations, including those prior to 2004, can be broadly placed into three categories: identifying student difficulties, examining individual student learning, and investigating classroom learning. Within these categories exist studies that investigate a wide range of topics encompassing student learning of differential equations. While most of these recent studies focus on student understanding of solutions to (systems of) differential equations (Arslan, 2010; Davis & Pearn, 2006; Keene, Rasmussen & Stephan, 2012; Mallet & McCue, 2009; Picard & Kidron, 2008; Rasmussen & Blumenfeld, 2007; Rasmussen & Rhodehamel, 2006; Raychaudhuri, 2008), collectively the topics studied are rather broad and include other topics such as students’ understanding of existence and uniqueness theorems (Raychaudhuri, 2007), ways students reason about differential equations and their solutions (Keene, 2007, 2008), classroom math practices involving ideas about differential equations (Stephan & Rasmussen, 2002), students' solution strategies (Habre, 2000), the construction of solution spaces (Stephan & Rasmussen, 2001), and student beliefs about mathematics (Ju & Kwon, 2007).

Although investigating student understanding of function and rate of change was not the focus of the aforementioned studies, a significant portion of them have implications concerning the role of the students' notions of function and/or rate of change in understanding differential equations. That is to say, after examining a large portion of the research studies across multiple areas of differential equations, students' notions of function and rate of change appear to be a theme in the vast majority of the studies. In particular, I interpret the reoccurrence of the students' notions of function and/or rate of change across the various studies as an indication of the importance of the students' understanding of these concepts within the broader setting of learning differential equations.

## 2.2 Student Understanding of Function and Rate of Change in Differential Equations

As mentioned in Chapter 1, mathematics education researchers have conducted studies specifically focusing on function and rate of change with regard to learning differential equations (Donovan, 2002, 2007; Rasmussen & Blumenfeld, 2007; Rasmussen & King, 2000; Rasmussen & Whitehead, 2003), though largely in regard to student understanding at a single instance in time. These studies provide great insight into how students utilize various notions of function and rate of change while making sense of differential equations tasks. In the following sections I outline the implications from studies concerning the role of students' notions of function and rate of change in student learning of differential equations. Connections to literature on student thinking about function and rate of change are made throughout the respective sections.

### 2.2.1 Function

Students' notions of function arise frequently in literature concerning student understanding in differential equations, and in this section I present related findings and implications. For ease of reading, this section is broken into the following subsections: solution, *y* as both a function and a variable, existence and uniqueness theorems, and the (ordinary) differential equation as a function.

**2.2.1.1 Solution.** As stated in the previous section multiple researchers have noted that students often have difficulty conceptualizing the idea of solutions to ordinary differential equations. One of the most common findings concerning the students' difficulty with understanding the concept of solution is that often students do not conceptualize solutions as functions (Arslan, 2010; Mallet & McCue, 2009; Rasmussen, 2001; Raychaudhuri, 2007, 2014). At least initially, students expect solutions to ordinary differential equations to be numbers (Arslan, 2010; Rasmussen, 2001; Raychaudhuri, 2008). This is precisely the "fundamental shift" Rasmussen (2001) states is required for learning differential equations, namely that solutions must now be thought of as (collections of) functions as opposed to numbers.

In his work on characterizing student difficulties and understandings of differential equations, Rasmussen (2001) found two major themes in the data. The theme relevant to this section is the *functions-as-solutions dilemma*. "The function-as-solution dilemma is a multifaceted theme that resonates with previously documented difficulties students have with function" (Rasmussen, 2001, p. 64), which are framed within the context of differential equations. The three facets that lie within the functions-as-solutions theme are *interpreting solutions*, *interpreting equilibrium solutions*, and *focusing on quantities*. These three facets respectively correspond to interpreting solutions to differential equations as functions that can be represented graphically, interpreting the special case of equilibrium solutions to differential equations as functions, and interpreting the quantitative significance represented by the solutions (Rasmussen, 2001).

Rasmussen connects these three student difficulties in understanding solutions to differential equations with those documented concerning function in general. With regard to the first facet, Rasmussen posits that students did not interpret the solution graph as a function because they did not have an explicitly defined equation or rule for it, which directly aligns with the findings of Leinhart, Zaslavsky and Stein (1990). Commensurate with the findings of multiple studies (e.g., Markovits, Eylon, & Bruckheimer, 1986; Marnyanskii, 1975; Vinner & Dreyfus, 1989) Rasmussen asserts that the students' difficulties with interpreting equilibrium solutions as functions is due to students' tendencies to not interpret constants as functions. This may be related to McDonald and Zandieh's (1999) finding that students often think of equilibrium solutions as disjoint from the larger set of solutions. The remaining facet refers to student tendencies to think of the graph as a pictorial representation of the modeled phenomenon, which has been noted in studies concerning function in other areas of mathematics (Monk, 1992; Leinhart, et. al., 1990). In this case, students lose sight of the quantities being represented in the graph and attend to shape of the curve. This could present significant issues for example when trying to interpret phase planes. Rasmussen's work is significant with regard to this dissertation because it connects student difficulties with function to their difficulties with understanding solutions as functions and provides insight into how students may utilize (or not) various aspects of the their understanding of function to interpret solutions to differential equations. These findings, however, are only a small set of the ways students use the concept of function to make sense of solutions.

The concept of function has also played a role in development of frameworks that categorize students' understanding of solutions to differential equations. Using the structure of the mathematical definition itself, Raychaudhuri (2008) created the context-entity-process-object framework for interpreting students' understanding of solutions to differential equations. It should be noted that in this framework, the process-object connection is not related to Sfard's (1992) process-object framework, APOS theory (Dubinsky, 1991), or Gray and Tall's (1994) notion of *Procept*. This framework:

> Portrays the mathematical entity (here, a function) that the solution has in the context (of a differential equation), but unlike its static counterpart there are now two mathematical processes that propagate (process mandates the desired property, or process generates the object) the function to the mathematical object, namely, solution (Raychaudhuri, 2008, p.163, parenthetical text in original).

For our purposes we can think of the process that mandates the desired property as differentiation and the process that generates the object as integration. In other words, from the perspective of the framework differentiating the solution determines the properties of the differential equation, and integrating the differential equation generates the solution. Although the way process was characterized is the result of the mathematical structure of the definition of solution, other researchers have found commensurate characterizations of student thinking with regard to solutions. For instance, Arslan (2010) found that students believed that "one might get the [differential equation] by integrating a function or one might get a solution by deriving [differentiating] the [differential equation]" (p. 884).

Of specific interest for this study is how student thinking about solutions as functions are classified using this framework. Broadly speaking, within the context-entity-process-object framework student thinking about solutions are categorized around a static definition and a dynamic definition of solution; specifically, understanding a solution to a differential equation as an entity (as a function that has a specific form), and as an object (as a function and a process that satisfies the differential equation). With regard to the first way of thinking, Raychaudhuri noted “our findings lead us to conclude that the students’ primary image of solution as an entity is that of a curve represented by an equation of the form $y = f(x)$” (p. 174). Though in her study, students also thought of solutions’ evaluated values as solutions. There are two images that comprise understanding the solution as an object: the solution is the object that generates the differential equation (through the process of differentiation), and the solution is an object that results from the process of integration. In either case, the students were working with functions.

Students’ notions about function have also been identified as impacting the ways students reason about solutions to differential equations. Keene (2007) identified six ways students reason with time as a parameter: making time an explicit quantity, using the metaphor of time as “unidimensional space,” using time to reason both quantitatively and qualitatively, using three-dimensional visualization of time related functions, fusing context and representation of time related functions, and using the fictive motion metaphor for function. Pertaining to this dissertation are the last three characterizations, or more importantly, the larger category Keene situates these ways of reasoning within: *imagining the motion of a function* with regard to solutions to systems of differential equations. Keene notes that “imagining the motion [of a function] contributes to students’ understanding of time as the independent variable in the solution functions of systems of differential equations and strengthens their dynamic reasoning” (2007, p. 240). This characterization of student reasoning builds on Janvier’s (1998) assertion that students imagine time in motion and extends it to ways in which students think about solutions to differential equations.

Imagining the motion of (solution) functions provided the students with support for visualizing solutions in the phase plane, comparing the behavior of the solution functions with their understanding of the situation being modeled, and deepening their understanding of a solution to differential equations. Keene's (2007) study is another example of work that indicates the importance of function in student learning of differential equations. Whereas Rasmussen's (2001) study connected student difficulties concerning understanding function to student difficulties with understanding solution, Keene (2007) discusses some of the ways students reason about function and connects them with how students reason about solutions (as functions) to differential equations (and systems thereof). I now transition to work discussing function in terms of making sense of what $y$ is in differential equations of the form $y' = f(t, y)$.

**2.2.1.2 $y$ as both a function and a variable.** Stephan and Rasmussen (2002) document the notion that $y$ in differential equations of the form $y' = f(t, y)$ is both a function and a variable, becoming taken-as-shared in a differential equations classroom. Rasmussen and Whitehead (2003) explain this as one of the most difficult images required for a robust understanding of differential equations and their solutions. This also fits well with Raychaudhuri's (2008) finding that students often fail to think of a differential equation as a relationship between an unknown function, the function's dependent variable, and one or more of its derivatives. If students do not think of $y$ as a function, it seems more than plausible that they will not think of differential equations as relationships between a function and its rate of change (function). Additionally without conceiving of $y$ as a function, it is improbable students will think of $t$ as a dependent variable for either $y$ or $y'$. Of particular interest here is the claim that students who understand $y$ as both a function and a variable put together images from both of these concepts to create new ways to reason (Rasmussen & Whitehead, 2003).

Specifically, Rasmussen and Whitehead note that students who understood $y$ to be both a function and a variable were able to reason about systems of differential equations by almost effortlessly transitioning between substituting values for $y$ and treating $y$ as a continuously changing function to glean information from the differential equations. These notions about functions, namely associating a function with a numerical value and thinking of a function as continuously changing, have been well documented in literature on student understanding of function using various characterizations (e.g., Dubinsky & Harel, 1992; Monk 1992; Sfard, 1992).

**2.2.1.3 Existence and uniqueness theorems.** Given that solutions to differential equations are functions, it is no surprise that the concept of function has such a significant role in students' understanding of solutions. Indeed, what may be surprising is the number of ways students' notions associated with function impact their understanding of the existence and uniqueness theorems for ordinary differential equations. Though in general there exists an entire space of solutions to the differential equations students encounter in an introductory course, there may be domains on which solutions may not be unique or exist at all. Therefore the purpose of the existence and uniqueness theorems is to provide conditions under which solutions to differential equations are guaranteed to exist and be unique. These conditions, by and large, are concerned with places of (dis)continuity of functions.

Raychaudhuri (2007) developed a framework for interpreting students' understanding and application of the existence and uniqueness theorems pertaining to first order ordinary differential equations. With regard to differential equations of the form $y' = p(t)y + g(t)$, Raychuadhuri found that most students "held the image that a unique solution to an [initial value problem] existed only where the coefficients $p(t)$ and $g(t)$ were continuous, based on their interpretations of the concepts of function, continuity and integration" (p. 378). In general the students expressed the idea that $p(t)$ and $g(t)$ had to be continuous in order to have a continuous solution function and to use solution methods employing integration as the students believed one can only integrate continuous functions. Additionally the students in the study tended to avoid using the theorems to determine where unique solutions may exist and instead preferred to solve the differential equation and then check the domain of the solution function, seemingly presuming one existed. What is most intriguing here is the students' notion that one can only integrate continuous functions. Specifically, students were utilizing knowledge about functions with regard to integration (and perhaps differentiation) to make such conclusions. In this case, the students' understanding of function played a role in the students reasoning about how to find solutions and when solutions existed.

In the previous sections I explored how the students' notions of function inform their understanding of various topics within differential equations. This culmination of findings strongly suggests that not only is the concept of function immensely important for understanding differential equations, but aspects of it are quite prevalent in students' understanding of differential equations as well. In the remaining subsection I discuss research on students' understanding of ordinary differential equations as functions themselves and how this relates to their understanding of the connections between the various representations of differential equations (e.g., algebraic equations and graphs of a vector field) and those of their solutions (Donovan, 2002, 2007).

**2.2.1.4 First order ordinary differential equations as functions.** Conceptualizing an ordinary differential equation as a function is reciprocally linked with some of the ideas from both the previous and coming sections, in that it both builds on and supports students' understanding of other aspects of differential equations, such as solution and rate of change. Donovan (2007) uses Sfard's (1991) reification theory to investigate students' cognitive development of the concept of a differential equation. Specifically, he used a card-sorting task to explore what it means to conceptualize a differential equation as an *object.* In doing so he addresses the three requirements for a *structural* conception, as posed by Sfard (1991), by showing what these may look like in the domain of differential equations. I now briefly discuss the developed framework in terms of its relevance to function and then discuss the implications concerning students' understanding of differential equations.

The first requirement for a structural conception is "seeing the same object under different disguises" (Sfard, 1992, p. 76). Donovan claims this is accomplished by seeing a differential equation as a function in its own right. In other words, the student must attend to the relationship between the solution's value and its rate of change across multiple representations. Second, students must see that the various representations are linked by a unifying abstract construct (Sfard, 1991), in this case the functional relationship between the variables in the differential equation. One such way to do this is to easily transition between the symbolic representation of a differential equation (a function) and its graphical representation or seeing a differential equation and its graph as the same thing (Sfard & Thompson, 1994). Separate yet related to this is the notion that students need to use both the symbolic (algebraic) and graphical registers to make sense of solutions to differential equations (Picard & Kidron, 2007). In other words, while transitioning between the symbolic and graphical representations is important for a structural understanding, it is also important in and of itself for students' understanding of differential equations. Lastly, for a structural conception of differential equations as a function, students need to be able to recognize the abstract quality present in the various representations quickly, without performing calculations. In other words, they must be able to reason the differential equation as a whole, where the operations used to analyze the differential equation have been interiorized. By asking students to sort cards with both graphical and symbolic representations of differential equations and their solutions, Donovan was able to determine a student's ability to accomplish this last requirement. If a student was able to quickly sort the cards representing the same differential equation using qualitative reasoning, they were said to meet this last requirement. Students that met all three of these requirements were said to understand a differential equation as an object.

In Donovan's (2002, 2007) comparisons of two students, Hassan, who was categorized as having a structural understanding of differential equations, was able to draw connections between differential equations and their solutions in ways that the student without the object understanding, Rich, was not. Donovan explained that Rich often suggested that the variables in differential equations had little meaning and that though Rich could treat differential equations as functions in some situations, the processes he utilized to do so were largely procedural. On the other hand Hassan's object view of differential equations as functions supported his ability to see them as connected to and informative of their solutions, treat $y$ as both a function and a variable in the differential equation, interpret $\frac{dy}{dt}$ as a variable, slope or derivative, and understand $t$ as an independent variable in both the differential equations and their solutions. Note that the way Hassan was able to think about various aspects of differential equations is directly in line with those reported in the previous sections. In short, Hassan was able to act on differential equations as a function, which afforded him a robust understanding of differential equations and their solutions.

Donovan's (2002, 2007) findings are immensely important concerning this literature review as they serve as a way to pull together the findings present in the various areas of research on student understanding of differential equations as discussed in the previous sections. Also, while Donovan's work contributes greatly to differential equations literature, little is still known about how students come to see an ordinary differential equation as a function in its own right. Recent work suggests students' notions of slope and rate can be leveraged to promote such an understanding (Kuster & Dominy, 2015), but more exploration is required. The notion that students utilize *dy/dt* as a variable, slope, derivative, and arguably a function (for those seeing a differential equation as such) serves as a transition to the following section on rate of change. It should be noted, however, that function and rate of change, despite being presented in separate sections in this chapter, are intrinsically linked in differential equations.

### 2.2.2 Rate of Change

Just as students' notions about function have a significant role in the way they think about differential equations, so do their notions about rate of change. For the purposes of this study I do not limit rate of change to any one specific way of thinking about or representing changing quantities such as covariational reasoning (Carlson et al., 2002) or any representations that could be associated with rate of change (e.g., derivative, slope and ratio). While studies exist that document students' use of various types of reasoning and/or representations of rate of change in differential equations (Donovan, 2002; Habre, 2000; Rasmussen & Blumenfeld, 2007; Rasmussen & Stephan, 2002; Whitehead & Rasmussen, 2003), I am unaware of any frameworks concerning students' understanding of rate of change in differential equations. Frameworks for rate of change, however, do exist in other areas of mathematics such as calculus (Carlson, et. al., 2002; Zandieh, 2000) and would be of great use for exploring the development of student understanding in differential equations. Throughout the following section, where appropriate, I draw connections between existing frameworks on rate of change and the findings of studies investigating student learning in differential equations. I do this for two reasons: to highlight the usefulness of such characterizations for the purpose of this study, and to illuminate the various ways in which rate of change emerges in differential equations.

**2.2.2.1 Reasoning with rate of change.** A research study with connections to multiple other studies concerning rate of change in differential equations is Keene's (2008) identification of ways in which students reason throughout an inquiry-oriented differential equations course. Keene identified the following ways of reasoning utilized by students: *dynamical reasoning*, *reasoning with prior knowledge*, *graphical reasoning and dynamic visualization*, *reasoning with the context*, and *algebraic reasoning*. Dynamic reasoning is defined as "developing and using conceptualizations about the dynamic quantity time as it implicitly or explicitly coordinates with other quantities to understand and solve problems" (Keene, 2008, p. 244). She noted that throughout the course students used time to reason about solutions as "graphs that were created as time passes"(p. 244). Whitehead and Rasmussen (2003) also noted this type of parametric reasoning in their discussion of students' understanding of systems of differential equations.

In their study, Whitehead and Rasmussen identified the theme of *rate use* in which students used rate as a tool to build images of population prediction and function. In their work they noted that students used rate as a quantity that determined the behavior of a function and as a comparison of changing quantities in the form of a ratio. By discretely coordinating changes in $x(t)$ and $y(t)$ compared to changes in *t*, students were able to talk about phase plane solutions as parameterized curves. In their discussion, Whitehead and Rasmussen directly connect their findings to level four and five in Carlson et al.'s (2002) covariational reasoning framework and Thompson's (1994) notion of rate as a ratio. Within Carlson et al.'s framework, level four is "coordinating the average rate of change of the function with uniform inputs of change in the input variable" (2002, p. 357), and level 5 is "coordinating the instantaneous change of the function with continuous changes in the independent variable for the entire domain of the function" (2002, p. 357). Although it is not clear whether or not the students in Whitehead's and Rasmussen's (2003) study were reasoning about the changes in $x(t)$ and $y(t)$ as average rates of change (level 4) or instantaneous rates of change (level 5), constructing the solution function, nevertheless, required simultaneously coordinating changes in two functions with changes in those functions' independent variable $t$, and then using that information to determine the nature of the solution function's behavior.

Returning to the work of Keene (2008), rate was also one of the pieces of prior knowledge students used to reason about differential equations. Keene noteed that students used rate to reason about the structure of solutions and did so before, during and after instruction on systems of differential equations. More specifically, students in the study used the notion of changing rates to qualitatively discuss solutions and reasoned using their notion of derivative (an instantaneous rate of change). These findings are also connected to various levels with Carlson et al.'s (2002) covariational reasoning framework. Keene noted that students often utilize the understanding that the sign of the function's rate of change (derivative) indicates whether or not solution functions were increasing or decreasing. Based on this we can see that the students were coordinating the direction of change of the solution function with changes in the input variable, which is directly related to level 2 from Carlson et al.'s (2002) framework.

Keene's (2008) study is not alone in documenting student reasoning in differential equations relating to rate of change. With the aid of a modified Toulmin's model of argumentation, Stephan and Rasmussen (2002) documented student reasoning in differential equations in terms of math practices existing at the classroom level. Their work uncovered multiple class math practices that emerged in an inquiry-oriented classroom, such as *creating and structuring a slope field as it relates to predicting*, and *reasoning with spaces of solution functions*. Each of these class math practices brought with them a set of ideas that became taken-as-shared in the classroom. The ideas associated with creating and structuring a slope field as it relates to predicting were: reasoning about the way in which slopes change over time, slopes are invariant horizontally for autonomous differential equations and infinitely many slopes are encountered in a slope field but only finitely many are visible. Each of these ideas relates to student thinking about slopes (a graphical representation of rate of change) and how they are useful in understanding solutions to differential equations.

The class math practice *reasoning with spaces of solution functions* brought with it ideas concerning students' reasoning with $\frac{dy}{dt}$ versus *y* graphs. This class math practice emerged as students began an instructional sequence culminating in the creation of bifurcation diagrams. The important aspect of this class math practice is that it encompassed ideas in which students reasoned about slope fields as a single dynamic entity. Students were not simply reasoning about how the value of $\frac{dy}{dt}$ changed at different values on the coordinate plane, they were reasoning about how slopes at every point in plane changed as parameter values in the differential equation changed. This requires coordinating a significant number of changes in quantities (i.e., changes in changes in *y* and changes in the parameter value and perhaps changes in *t*). It is worth noting that the student ideas associated with these class math practices are such that analysis by derivative frameworks (e.g., those of Carlson et al. (2002) or Zandieh (2000)) would be well-equipped to handle at the individual level. How exactly these ideas fit into the frameworks and whether or not new categories would need to be developed is a topic that warrants investigation.

The notion that students reason with rate in differential equations to construct new mathematical objects was discussed in detail by Rasmussen and Blumenfeld (2007) in their interview with one of the participants, Mario. In this study students were tasked with constructing the straight-line (Eigen) solutions of the following system of differential equations:

$$\frac{dx}{dt} = 2x + 2y$$
$$\frac{dy}{dt} = x + 3y$$

Importantly, students were asked do this without having learned the traditional technique of calculating the eigenvalues of the system first. During an interview where Mario was asked about straight-line solutions he explained that the ratio of $\frac{dy}{dt}$ to $\frac{dx}{dt}$ had to be the same as the ratio of *y* to *x* in order for the solution to have a constant slope. Furthermore, in his explanation he used the words "slope," "rate of change," and "derivative," indicating the various representations of rate within his understanding of straight-line solutions. The notion that the ratio between $\frac{dy}{dt}$ and $\frac{dx}{dt}$ have to be the same as that of *y* and *x* allows students to substitute appropriately and attain the equation $\frac{2x+2y}{x+3y} = \frac{y}{x}$. Students can then solve for *y* (obtaining two solutions) thus finding the straight-line solutions. In this method students utilize their notions of rate of change to not only reason about that of each straight-line solution, but the ratio between the rate of change of *x* and the rate of change of *y.* Although the technique is only used for systems with real and distinct eigenvalues, it shows precisely how powerful and useful the notion of rate of change can be in differential equations.

### 2.2.3 Differential Equations as Functions

Donovan (2007) noted that a student's ability to reason about a first order ordinary differential equation with an object understanding of function was immensely important in regard to his or her understanding of the relationship between differential equations and their solutions. Though it is unclear how exactly one can extend the notion of seeing a differential equation as an object (function) to topics such as systems of differential equations, the implications from his study are still relevant in these areas. Specifically, transitioning between algebraic and graphical representations of differential equations (e.g., slope fields), which based on the findings above encompass multiple aspects of students' notions of rate of change, supports students' understanding of differential equations and their solutions.

It should be noted that the vast majority of this literature review presents findings from studies that did not explicitly focus on student understanding of function or rate of change. Specifically, the creation of this literature review entailed examining research on student learning in all topics relevant to differential equations while keeping a keen eye for mentions of function and rate of change. I mention this so the reader is not misled to believing that function and rate of change have been well explored in the domain of differential equations. On the contrary, this literature review is designed to explicate how function and rate of change have been themes in the literature yet are still relatively unexplored in terms of the dynamical nature of student learning. In other words, while there are great number of studies that show how students utilize various notions of function and rate of change, it is still unclear how these understandings develop. Additionally given the few mentions of the role of context in student understanding, - Raychaudhuri's (2008) framework, and Keene's identification of reasoning with context (2008) - research sensitive to the role of context in student learning is still needed in differential equations. Based on this literature review it should be clear that students' notions of function and rate of change are not only essential for understanding differential equations, but are strongly interconnected. Due to the relationship between function, rate of change and differential equations, it seems appropriate to consider knowledge in this domain as an interconnected web of ideas. This requires adopting an epistemological perspective that affords such a characterization of knowledge. In the following section I describe the epistemological perspective of Knowledge in Pieces (diSessa, 1993), and the assumptions from which that underlie this dissertation.

### 2.3 Theoretical Perspective

This design of this study was influenced by the epistemological perspective of Knowledge in Pieces (diSessa, 1993; Smith et al., 1993), which, as Kapon, Ron, Hershkowitz, and Dreyfus (2015) have pointed out, fits under the larger umbrella of Piagetian Constructivism (von Glasersfeld, 1996). Within the KiP perspective knowledge is viewed as a dynamic system of elements and their connections, which is shaped by the learner's environment. These knowledge elements are context sensitive, and, as such, are thought of (perhaps unconsciously) as either productive or unproductive within a certain situation by the knower. This means in general that the knowledge elements themselves are not evaluated as correct or incorrect (Smith et al., 1993); the evaluation of correctness is typically only relevant to the application of the elements. That is not to say that knowledge in and of itself cannot be incorrect, but the focus of Knowledge in Pieces is on the organizational structures guiding the application of the knowledge. From a KiP standpoint, learning is viewed as the reorganization, contextualization and systematization of knowledge elements (diSessa & Sherin, 1998; Kapon et al., 2015; Wagner, 2006). Studies in which KiP is utilized are often designed for the identification of these elements and the mechanisms that afford their systematization (e.g., Adiredja, 2014; Kadron et.al, 2015).

### 2.3.1 Assumptions Concerning Knowledge

Knowledge in Pieces asserts that knowledge is context sensitive (diSessa & Sherin, 1998; diSessa & Wagner, 2005; Smith et al., 1993; Wagner, 2006). This means that the knower associates different sets of knowledge with the different situations in which that knowledge is considered to be useful. In terms of analysis, identifying the knowledge a student uses to solve certain problems without gleaning information regarding their attention to the contextual attributes, which afforded the student's perception of the applicability of that knowledge, only provides the researcher with an understanding of half of the story. In my experience as an educator I often came across situations where students would utilize knowledge in one situation but not in another, even when that same knowledge would have proved successful. As a learner in the field of mathematics I have done this quite a few times myself, often wondering "why didn't I think of that?" after realizing how "obvious" something was. The context sensitive nature of knowledge affords insight into situations such as these by providing a lens that not only allows for these occurrences, but also expects them (Wagner, 2010).

Knowledge in Pieces specifically emphasizes the importance of students' prior knowledge in the process of making the use of that prior knowledge more systematic across more situations, or in other words, its importance in the process of learning. Along these lines, utilizing the term "misconception" would be a misnomer when taking a KiP perspective. Smith et al. (1993) describe how the term misconception is inconsistent with constructivism, in that misconceptions literature poses students' misconceptions as interfering with learning, needing to be replaced, and resisting instruction. Specifically these notions of misconceptions directly go against a tenet of constructivism; that students learn by building on their prior knowledge. In fact "misconceptions" should be expected and thought of as productive:

> Errors are characteristic of initial phases of learning because students' existing knowledge is inadequate and supports only partial understandings. As their existing knowledge is recognized to be inadequate to explain phenomena and solve problems, students learn by transforming and refining that prior knowledge into more sophisticated forms. (Smith, et al., 1993, p. 123)

In this way, the authors reframe what others have often negatively referred to as "misconceptions," as valuable and necessary components of student learning in that these systematic patterns of errors drive the productive reorganization of knowledge structures. I interpret this as directly related to Wagner's (2006, 2010) assertions that the reorganization and systemization of knowledge does not occur as learners begin to overlook differences between similar objects (e.g., the various colors of apples); rather, it occurs as learners accommodate for these differences (e.g., classifying objects as an apple based on size, texture, color and smell *while* being cognizant of differences across these aspects). As learners become aware of faulty applications of knowledge or successful applications in new situations, they reorganize (as opposed to replace) their knowledge structures in ways that allow for remaining aware of differences as opposed to overlooking them.

### 2.3.2 Knowledge Pieces

There are a multitude of "pieces" used to model a learner's complex system of knowledge within the KiP perspective. Within this section I discuss only a few of the various terms existing in the KiP perspective, namely, *knowledge elements*, *knowledge resources* (this dissertation's main focus of interest), and *concept projections*. The term *knowledge elements* refers to any one of the various structures within the larger system of knowledge, which vary in size. For example they can be *phenomenological primitives* which are "as atomic and isolated a mental structure as one can find" (diSessa, 1993, p. 112), or they can be as large as entire *coordination classes* which are "complex knowledge systems (nets of knowledge elements) that provide a way to acquire certain classes of information from the world" (Kapon et. al, 2015, p. 46). Additionally, Smith et al. (1993) provide $F = Ma$ as an example of a knowledge element. The three examples of knowledge elements show the diversity within the pieces that make up the larger system of knowledge.

Though generally consistent in nature, different researchers have provided different characterizations of the elements known as knowledge resources. Widely speaking, knowledge resources are sets of one or more small-scale pieces utilized while engaging in problem solving. Wittmann (2006) noted the existence of four types of such small-scale pieces documented in the then current literature: *agents*, *phenomenological primitives*, *facets of knowledge,* and *intuitive rules.* Additional small-scale pieces of knowledge existing within the larger system, as addressed by Kapon and diSessa (2012), are *explanatory primitives.* Though each of these elements is classified as small-scale, their sizes slightly vary. For instance the "atomic" phenomenological primitive, force as a mover, which represents the notion that pushing an object makes it move in the direction of the push (diSessa, 1993), is smaller than Maria's intuitive rule that "the larger your sample, the closer you get to the expected value" (Wagner, 2006, p. 13). In general however, and consistent with the definition posed by Hammer (2000), and Black and Wittman (2009), I define knowledge resources as ideas consisting of small sets of small-scale knowledge elements that have a productive role during the process of problem solving. These resources can be procedural sets of mathematical actions, intuitive ideas, or definitions.

Classifying student understanding in terms all the ideas one has regarding an entire concept is far too large of a scale for KiP, as concepts encompass a large number of knowledge elements and resources all of which may be utilized in different situations at different times. For example, Adiredja (2014) identified nine knowledge resources utilized by calculus students to understand the limit of a function, such as *functional dependence* and *proportional variation*. Calculus, however, is not the only context in which the concept of limit exists and it may very well be the case that students' understanding of the concept of limit consists of other resources in the context of analysis. In this way, assessing a student's total understanding of limit would require determining all of the ideas they use to implement the concept in all of the different situations they perceive it as applicable. Given the rather large number of resources students may utilize in service to a single concept, and the context sensitivity of those resources, from a Knowledge in Pieces standpoint it is important to situate the resources within the context of their utilization. To discuss students' understanding of a certain concept in a certain situation diSessa and Wagner (2005) introduced the term *concept projection.* Wagner (2010) defines a concept projection as "a set of particular knowledge resources that enables the knower to attend to and interpret the available information necessary to 'perceive' or 'implement' a concept within a given situation" (p. 450). He further notes, "a single concept may be composed of a complex variety of knowledge resources, different combinations of which are required to compose different concept projections, as different contexts demand" (p. 450).

The literature review above indicates the great number of ways students broadly utilize function and rate of change across various topics in differential equations. Concept projections are a way of illuminating the specific notions about those concepts that students use when encountering certain tasks. In other words concept projections serve as a way to build on previous findings, and investigate the changes in resources and applicability of students' notions of function and rate of change.

### 2.3.3 Knowledge Activation

As a learner's experience grows, he or she may begin to associate certain situations and objectives with resources that have proven useful in those related situations. Encountering related or similar situations may then *cue* the sets of resources the learner associates with these situations and objectives, because they have proven useful in his or her previous experiences. For a resource to become cued means it becomes accessible for use. Knowledge resources that are more likely to be activated are said to have a higher cuing priority. This priority may manifest itself in two ways, knowledge may be activated due to its proven ability to be useful in a similar situation or knowledge can be *cued* or become accessible for use by the presence of other pieces of related knowledge that are already activated (diSessa, 1993; Wagner, 2010). An example of the former is Maria's (Wagner, 2006) use of the same rule concerning the law of large numbers across questions she saw as similar. Relating to the latter, the notion of tangent lines may have high cuing potential when a student's knowledge about functions and derivatives are activated. In reference to the activation of knowledge in certain situations, the perceived productivity of certain knowledge pieces can become more or less stable as the learner's experiences with that knowledge expands. For instance, cued knowledge that proves successful in various situations will likely be activated in those situations more reliably in the future. On the other hand, knowledge that proved unproductive may be deactivated in certain contexts (Kapon et al., 2015). Additionally, as is discussed in the next section, students' perceived utility of a particular knowledge resource can change.

### 2.3.4 Knowledge Development

At the heart of the changes within ones system of knowledge lie constructivist notions of *assimilation* and *accommodation* (von Glasersfeld, 1979). According to von Glasersfeld (1979), assimilation is “the application of an established invariant pattern or schema to a present experience regardless of discrepancies,” (p. 86) and “accommodation refers to discrepancies leading to the development of a new pattern.”(p. 86). Wagner (2010) develops these notions specifically with regard to Transfer in Pieces, an account of generalization and transfer built on the foundations of the Knowledge in Pieces perspective:

> The construction of new concept projections or the association of already existing ones becomes a cognitive mechanism for *accommodating* new contextual demands, and existing concept projections are the means by which other contextual circumstances are *assimilated* as familiar and recognized as similar to those encountered before. (p. 474, italics in original)

Specifically, as learners interact with their environment, assimilations or accommodations may be made based on feedback pertaining to the perceived productivity of certain activated resources in certain situations. There are two ways knowledge can develop that are of interest to this study. Namely, knowledge elements can be created or reorganized, and knowledge elements can become more or less applicable in new situations. In the case of the creation of a new concept projection, one can say the *span* (diSessa & Wagner, 2005) or set of situations in which the concept is applicable, has increased or broadened. In this case an accommodation was made as a result of generating a new set of resources to meet new contextual demands, which allowed for perceiving or implementing a certain concept in the new situation, one that previous concept projections did not allow for. In the case associating an already established set of resources to interpret or perceive a certain concept in a in a new situation, an assimilation was made in the form of structuring or recognizing a new situation in way that afforded the utilization of the existing concept projection.

### 2.4 Summary of Chapter 2

Within this chapter I have shown that function and rate of change play a key part in students' understanding of differential equations and have presented relevant findings of other researchers. Not only does this literature review serve as a way to situate my line of research within the broader mathematics education community, but it also helps inform my own perspectives and focus as a researcher. In addition I have outlined the perspective Knowledge in Pieces, which forms the foundation on which the analysis was conducted. This is discussed further in Chapter 3.

## CHAPTER 3:METHODS

In this chapter I describe the procedures I employed during the data collection and analysis phases of the study. Before discussing the methods concerning participant selection, data collection, and analysis, I briefly outline the specific theoretical assumptions that inform how the collection and analysis of data were conducted. The theoretical principles underlying Knowledge in Pieces (diSessa, 1993), the perspective taken herein, represent knowledge as an interconnected system of multiple, dynamic elements of varying complexity. Within this perspective knowledge is taken as context sensitive; the usefulness of certain knowledge elements at any given time depends on the contextual attributes one perceives and attends to, and how they are interpreted at that particular time. For any given person, at any given time, certain knowledge may be perceived as productive in one context but not in another. Further, one may come across a situation for which their knowledge does not support interpreting certain attributes in ways that would foster productive reasoning. In addition it is not unexpected for a student to use multiple seemingly contradictory (to an observer) pieces of knowledge at different times, and each may serve as productive (and even successful) within the contexts in which they are applied. For instance, the notions that equilibrium solutions are horizontal lines and that equilibrium solutions are circles or ellipses are in and of themselves contradictory. These pieces of knowledge, however, are both correct and productive when applied in certain mathematical contexts; graphs of individual solutions with respect to their independent variable, and phase plane representations of solutions to systems of two differential equations, respectively.

The construction of new knowledge or the modification of existing knowledge is then necessarily dependent on the perceived utility of that knowledge. Knowledge elements or their connections may be refined or outright created based, in part, on the contextual attributes a person is attending to and how they perceive them. Additionally, knowledge elements become applicable in more situations as one accommodates contextual differences between the different situations in which those sets of knowledge are applicable. In other words, knowledge may be constructed by modifying or creating elements or connections, or by modifying the contexts in which sets of knowledge elements are perceived as being useful. Thusly, a student's construction of knowledge can be evidenced by changes in the applicability of certain pieces of knowledge and/or changes in the elements themselves. Specifically, characterizing the structure of the elements, the dynamic nature of the elements, and their connections are of specific interest in this study. In terms of the research questions, this study investigates the sets of resources students' use while completing various differential equations tasks, how these sets of resources differ across different tasks, and how they are systematized over time – all of which first require the identification of students' resources.

I now describe the research setting and participants. Then, I outline the methods of data collection and I support the rationale for these methods relative to my theoretical perspective and research goals. I also provide the schedule followed pertaining to data collection. Finally, I describe the approaches I used to analyze the data relative to the theoretical perspective and how it was documented.

## 3.1 Participants

The participants in this study were undergraduates from a large research university. The participants were solicited from an honors differential equations course and a standard differential equations course (taught by two different faculty members, neither of whom was the author of this study). The mathematics discussed within these courses was commensurate with that of a typical sophomore level course on ordinary differential equations. Specifically, the courses each discussed topics concerning linear and non-linear $n^{\text{th}}$ order differential equations, analytical solution methods (e.g., separation of variables, methods of undetermined coefficients, and variation of parameters), numerical solution methods (e.g., Euler and Runge-Kutta), applications (e.g., mixing, cooling, and falling body problems), theoretical topics such as existence and uniqueness theorems, the law of superposition, and notions of fundamental sets of solutions. The honors course met for 75 minutes twice a week, and the regular course met for 50 minutes three times a week. While the goals of the two courses were similar, the honors course could be thought of as discussing the topics in a more conceptual manner and presented more applications with regard to modeling.

In terms of choosing the participants, Seidman (2006) noted that when conducting interviews for the purposes of qualitative research one should be hesitant to interview only individuals from any one particular category as it limits variation in the responses. With specific regard to the theoretical assumptions driving the methodology, seeking a broad range of participants from both an honors section and a regular section was intended to provide greater diversity within both the participant population and the content with which the participants were familiar. Moreover, the diversity of the participants was intended to promote richer opportunities for encountering students' use of different resources and different reasoning strategies, and thusly a greater opportunity for identifying and exploring diverse sets of student resources. For my purposes here the differences between the participants are not of primary interest, but their existence provided a greater chance of encountering rich and diverse data.

To recruit students, during the first week of the Spring 2015 semester I visited two course meetings, one for the regular section and one for the honors section. During the visits I explained the purpose of the study and the role of the participants. For the purposes of anonymity every student in the class was asked to complete a form indicating if they did or did not wish to participate in the study. Students that wished to participate were asked to include their name and an email address at which they could be contacted. Of the 22 students (total across both classes) that initially expressed interest in participating, 16 students replied to an initial email asking for a schedule of their availability. This list of 16 potential participants was narrowed to 8 based on a variety of factors. The goal from the beginning was to have four students from the honors section and four students from the regular section; only four students from the honors section expressed interest, thus each of them was selected. Of the 12 students enrolled in the regular section that continued to express interest, I asked the two female students to participate (only one accepted) and selected the remaining three students based largely on availability; many of the students had overlapping schedules or expressed being available for less than the required hour to conduct the interviews. Upon initially individually meeting with the students, I further explained their role as a participant in the study and introduced myself. The initial meeting took place within the second and third weeks of class during which time the students were familiarized with and asked to sign informed consent documents.

## 3.2 Data Collection

The primary method of data collection was one-on-one, semi-structured interviews (Clement, 2000). The interviews were task based, lasting roughly 60 minutes each and were spaced two to three weeks apart throughout the semester; there was some slight variation based on students scheduling needs. According to Clement (2000), "clinical interviews can give more information on depth of conceptual understanding, because oral and graphical explanations can be collected, and clarifications can be sought where appropriate" (p.548). Interviews of this nature (semi-structured) are well suited to studies utilizing a Knowledge in Pieces perspective because they support the researcher's goal of evoking highly detailed accounts of the students' mathematical understandings (Adiredja, 2014; Wagner, 2006). As this study aimed to capture the dynamic nature of student learning, each student participated in a five interviews (with the exception of one student that missed the fourth interview). Interview protocols were generated for each interview so that each of the participants was presented with the same tasks. It should be noted, however, that students were not always able to complete all of the tasks within the scheduled interview, so there was variation in the number of tasks completed across the participant population. While the tasks themselves were intended to draw out important aspects of the students' understanding, to better understand the students' mathematics (Steffe & Thompson, 2000) unplanned follow up questions were asked when warranted. These follow up questions were important for fully identifying and appreciating what each student was attending to within the task, the knowledge elements they were bringing to bear while completing each task, and the connections between these elements. A schedule of the data collection phase is given in Figure 3.1.

| Spring '15 Semester | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Week | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| Initial meeting | | | | | | | | | | | | | | | |
| Interviews | | | | | | | | | | | | | | | |

*Figure 3.1: Timeline of data collection.*

Throughout the semester of data collection, I attended multiple class sessions for both the honors and standard courses in an effort to keep track of the topics and ideas being presented during class time. This provided benefits in multiple ways. By attending class I was able to create and implement interview protocols that were reasonably inline with concepts and skills within the reach of the students. Attending the course meetings was also intended to inform the analysis. In terms of understanding the dynamic nature of knowledge elements and the contexts in which they are applicable, attending class provided insight concerning the perceived usefulness of certain resources. It is important to note that attending the course meetings was not intended to provide data directly concerning student learning; rather, it was intended to aid in the collection of meaningful data and the analysis of how the students were utilizing knowledge elements.

I was the researcher conducting the interviews, and each of the interviews was recorded using a video camera and a backup audio recorder. As noted by Rochelle (2000), the use of video recorded data can serve as important for the purpose of analysis. The video camera was positioned so that the participants and the work surface directly in front of them was visible. The tasks were printed individually on paper and provided to the students. The students worked on the tasks while sitting at a desk. Video of the interview provided multiple affordances audio alone would not, such as being able to: hear what students were saying while simultaneously watching what they were doing, analyze student gestures, and attend to the temporal order in which written work was created. In general the video data allowed for repeatedly experiencing the same event in a way that, for the purposes here, allowed for microanalysis of the data collected (Rochelle, 2000).

The written work the students generated while completing the interview tasks was collected as a secondary data source. The written work served as an additional way to document and analyze student thinking, and as such, provided an additional medium through which I could triangulate the students' utilization of certain knowledge elements. This is an important aspect of qualitative research as analysis of multiple data sources provides a way to ensure a more complete picture of the phenomenon being studied (Rossman & Rallis, 2003). The students' written work was collected for the same reasons here; it provided evidence of the utilization of certain knowledge resources which were analyzed and compared in conjunction with other evidence which supports or refutes the identification of these knowledge elements.

After conducting the interviews, during the summer of 2015 I selected four potential students for analysis and transcribed each of their interviews. These students were selected for various reasons. First, each of them articulated their thought process rather clearly during the interview process. Individually, I selected Hakeem (all student names are pseudonyms), because he expressed complex ways of reasoning while completing the tasks. Jordan was selected because during the interviews he often expressed that he felt he wasn't learning differential equations, rather he was only learning new applications of topics he learned in differential and integral calculus. Of the two other students, one was selected because his reasoning was rather rigid over the series of interviews, and the fourth student was selected because he had an uncanny ability to contextualize the differential equations and reason from that context as opposed to the differential equations themselves. After transcribing each of the five interviews for all four students, I then selected two students for analysis in this present study: Hakeem and Jordan. Reasons for their selection will be presented at the beginning of their respective sections in Chapter 4.

## 3.3 Interview Protocols

Over the course of the semester each interview protocol was designed to engage the participants in tasks centered on various topics relevant to differential equations so as to elicit a variety of knowledge resources and any discernable patterns in their utilization. One of the goals of this dissertation, to support answering the third research question, was to identify patterns in the students' application of resources that allowed them to complete a variety of tasks. As such, sets of tasks were chosen in an attempt to be somewhat consistent across the interviews but to also capture how students prior knowledge influenced the knowledge they applied while completing tasks involving more mathematically advanced topics. To accomplish this, successive interview protocols included, but were necessarily entirely composed of, a combination of tasks identical to those the students had encountered before and variations there of. In terms of the variation,n some tasks remained mathematically similar (e.g., an initial value problem involving a first order non-homogeneous differential equation) but were embedded in different situations (e.g., cooling metal object, population and position of an object). On the other hand, some tasks for instance remained context-free, but changed mathematically in terms of the topics they addressed (e.g., first order differential equation, second order differential equation and system of differential equations). This variation allowed for identifying a rich set of resources, patterns in their usage across different tasks, and how these resources supported the students' understanding of the various differential equations tasks and topics they encountered during the semester. The first three interview protocols can be found in Appendix A.

The specific tasks used in the interviews consisted of questions from existing literature, or modifications thereof, variants of questions the students encountered in their coursework, and tasks created to fit the goals of the study. Prior to their use in each of the actual interviews, I created a comprehensive protocol, which included justifications for the utilization of each task (with supporting research findings when appropriate), and a list of follow up questions and anticipated responses. A mathematics education researcher then reviewed the protocols in an effort to ensure the role of the questions was commensurate with the research goals. At times refinements were made when deemed appropriate.

With regard to the theoretical assumption that knowledge is context dependent, an additional goal of the interview protocols was to elicit evidence of the students' attention to contextual factors within the tasks. Here knowledge is only viewed as productive in the situations in which it is perceived (consciously or unconsciously) as useful, and therefore the contextual attributes students attend to and interpret in tasks are just as important as the knowledge students use to complete them. Drawing out the contextual attributes the students considered/attended to while completing the tasks allowed for analyzing how the students' various resources changed in terms of applicability, which represents a way in which knowledge can evolve; knowledge elements can become more or less productive in certain situations.

The choice of participants, the data collected, and the methods for data collection are commensurate with the research questions and the theoretical perspective taken in an effort to attain the research goals. The semi-structured, task-based interviews provided a medium for drawing out evidence of the resources students use and the patterns in which they use them, and the contextual factors that students perceive and coordinate that influence the utilization of these resources. Further, the length of the interviews, their frequency and spacing allowed for identifying changes in the students understanding.

## 3.4 Analytical Framework

This study aims to identify the sets of resources students use to complete various tasks and investigate how the students' resources are utilized, and systematized over time as the students' understanding of differential equations broadens or becomes more sophisticated. For this reason, I consider the goals of this study as fitting within the topic of transfer. In general, transfer research is concerned with how students utilize prior knowledge in new situations (Lobato, 2006). From a Knowledge in Pieces standpoint, where knowledge is viewed as context sensitive, focusing solely on the knowledge elements the students utilize and disregarding the situations in which the students find these elements productive would remove a mechanism by which their utility, applicability, and systematization can be observed. Additionally, careful analysis of the context sensitive nature of knowledge elements provides insight into the rationale for students' reasoning (Wagner, 2006) and in this way furnishes an additional and necessary lens for evaluating consistency in the students' reasoning. Examining the students' attention to context is commensurate with Labato's (1996, 2003) notion that careful inspection of transfer requires searching for the student's own perceptions of similarity as opposed to solely analyzing the students' normative reasoning patterns.

To accomplish this, I make use of the students' *interpretations* of the differential equations and their components as a way to document the organization and systemization of the sets of resources. I refer to an *interpretation* as the meaning the student assigns or associates with an object they perceive as important for completing a certain task. I contend the notion of interpretations align well with the Knowledge in Pieces perspective. In fact, I claim this interplay between sets of resources and a student's interpretations of relevant information is precisely the work of a concept projection. Wagner (2010) provided a characterization of the role of this particular knowledge element in terms of transfer:

> A concept projection (diSessa, 2004; diSessa & Wagner, 2005) is a set of particular knowledge resources that enables the knower to attend to and interpret the available information necessary to "perceive" or "implement" a concept within a given situation. In other words, a concept projection is a set of knowledge elements with which a knower assimilates and interprets that which is available to be perceived (what I refer to as the situation's affordances) in a particular, meaningful way. (p. 450).

Thusly the interpretations serve as an analytical tool for identifying the applicability of certain sets of resources as the students progressed through the course, while also remaining sensitive to their perceptions of similarity. Therefore the interpretations have a key role in answering the second and third research questions presented in Chapter 5. In addition, the interpretations provide a medium through which understanding the students' perceptions of similarity is possible while also accounting for why certain sets of resources were applicable in some situations but not in others.

Before I discuss the specific details of the analysis procedures I first explain how the interpretations emerged as an important aspect of the analysis. While in the process of identifying and documenting the resources the students used to complete the various tasks, nearly identical sets of resources and reasoning patterns began to emerge across the various tasks. Working under the assumption that students' utilization of resources is part of a consistent system, I began to seek a way to explain and support the similarities in the ways the students were reasoning across respective tasks. The variation in the tasks themselves, however, made definitively identifying specific attributes around which their knowledge utilization could be organized arduous. In the process of seeking the attributes, however, themes emerged in the students' responses regarding their interpretations of the differential equations and the components from which they were composed. These interpretations provided a way to explain, support, and account for the similarities and differences in the ways the students completed the different tasks while also accounting for the students' own perceptions of similarity and the meaning they associated with the attributes in the task.

According to Wagner (2006), when using a Knowledge in Pieces framework, "evidence is needed to demonstrate how it is possible that knowledge elements initially resulting in normatively inconsistent reasoning across contexts can be refined and reorganized in a manner that supports normatively consistent reasoning" (p. 8). To do this I documented the knowledge resources students utilized as well as the ways in which they interpreted the differential equation and its components for each task they encountered over the series of interviews. Documenting these resources and the interpretations of these attributes allowed for analyzing how the resources were refined, combined, and systematically utilized across and within the various tasks. I now address the data necessary for documenting their context sensitive nature.

Recall that according to Knowledge in Pieces, the perceived productivity of a piece of knowledge is dependent one's interpretation of the attributes in a given situation. This means that for the purposes of this study, the attributes students interpret within the task (and how they interpret them) that afford the student the perception that certain knowledge resources are relevant and useful for the completion of the task at hand must be documented. In this case, I take the differential equations and their components as aspects of the tasks that are available to be perceived. Documenting the resources students use in certain tasks informs what knowledge the students find useful for a given task, and documenting the interpretations of the salient features (in this case the differential equations and their components) the students attend to provides insight concerning why the students perceive that knowledge as useful in a particular situation. In this way, normatively consistent reasoning is not only concerned with the knowledge a student uses across different situations, it is also concerned with the student's interpretations that afforded using that knowledge at any given time. These interpretations and the sets of resources used in conjunction with the interpretations are precisely what I discuss while answering the second and third research questions. Now that I have addressed the type of data necessary when taking a Knowledge in Pieces perspective, I address the data necessary for using the related perspective, Transfer in Pieces (Wagner, 2006) which plays a role in the analysis presented in Chapter 5.

One of the fundamental assumptions underlying the notion of Transfer in Pieces is that a learner does not recognize the utility of a piece of knowledge in a novel situation until after she or he has structured the situation in a way that promotes the use of that knowledge. This perspective does not assume that students can immediately see the abstract nature of mathematical knowledge after it has been constructed and utilized. Generalizing the application of knowledge is a continual process that only occurs after a certain set of knowledge has been seen as useful across multiple experiences. For instance, researchers have investigated why students, after having learned differential and integral calculus, fail to utilize the notions of integration or differentiation in physics settings (Bajracharya & Thompson, 2014; Cui, Rebello, Bennett, 2005; Nguyen & Rebello, 2011; Thompson & Christensen, 2012). The answer a Transfer in Pieces framework would afford to this problem is that the current state of the students' knowledge does not support their interpretation of the contextual attributes in a way that promotes the utility of differentiation or integration techniques to solve the task. In other words, the contextual attributes the students are attending to and interpreting, and the knowledge they are coordinating with those attributes to make those interpretations are not affording the students with the ability to perceive differentiation and integration as useful in that particular setting. Thus, the student's apparent lack of ability to recognize the applicability of differentiation or integration is not just warranted, but expected until the student's knowledge system allows them to interpret the situation in a way that supports interpreting the contextual attributes as being associated with the use of integration or differentiation.

Furthermore, the "Pieces" part of Transfer in Pieces asserts that perceiving all of one's previously established knowledge as useful in new setting is not necessarily immediate or all inclusive. That is to say, generalizing the utility of large ideas such as differentiation in a certain task might require perceiving smaller bits of knowledge about differentiation as useful in the situation first. Returning to the example above, students may begin to relate velocity with rate of change within the task, but this may not be enough for them to see the utility of differentiation until rate of change in a physics context is connected with rate of change in mathematical context, and even this assumes the student associates rate of change with differentiation. While this particular example is indicative of "far" transfer (diSessa & Wagner, 2005), it helps illuminate the utility of Transfer in Pieces for this particular study.

In terms of the study I propose here, I see Transfer in Pieces as an analytical lens for effectively documenting how and why changes in the students' understanding of differential equations occur as they construct knowledge that accounts for contextual differences. In other words, Transfer in Pieces allows for analyzing how and why the students reorganize and systematize their knowledge as they accumulate more experience with differential equations. Additionally, the framework allows for describing the interactions between resources and the interpretations terms of learning.

### 3.4.1 Identifying Resources

In this section I outline how I identified the resources the students used while completing the various tasks and the patterns in their utilization. I then discuss the analysis process concerning changes in these items (i.e., the resources and their patterns of utilization).

Recall that resources small set of small-scale knowledge elements that have a productive role during the process of problem solving. For my purposes here, resources are small sets of knowledge that serve a productive role in the student's problem solving activity in a certain task. In addition I do not limit resources to only ideas that will lead to a correct answer. In this way I am operating with a broad definition of resource, commensurate with multiple physics and mathematics education researchers (Adiredja, 2014; Black &Wittman, 2009; Hammer, 2000; Smith & Wittman, 2008; Wagner, 2006; Wittman, 2006). In terms of the analysis here, however, to answer the research questions I focused on resources concerning, but not strictly limited to students' understanding of function, rate of change, and differential equations. Though the students most certainly utilized resources such as subtraction (Wittmann & Black, 2015) while in the process of solving the various differential equations tasks, I did not document the students' utilization of such resources.

With regard to the context in which a resource is perceived as useful, in general, evidence of the productivity of a piece of knowledge is that the student indeed used that knowledge to attain a solution (not necessarily a correct solution) to the problem posed. Though a great deal of resources may be identified based on analysis of what the students say in conjunction with what they do, resources need not be explicitly communicated by the students. In these cases, resources may be inferred based on a student's reasoning patterns or the steps they took during the solution process. Methodologically speaking however, when utilizing a KiP perspective, one must be careful in attributing knowledge to students; one cannot separate the resources from the situation in which they are productive. That is to say, identifying a student's utilization of a resource in one task and situation does not mean they will use that resource in another task, no matter how similar that task may seem. Thusly, one must be careful not to make a far-reaching inference and falsely identify a resource that was not actually used. For this reason while in the process of identifying and characterizing the resources a student utilized to complete a certain task (and even afterwards) one must remain open to the existence of alternative explanations, and actively seek such explanations.

My analysis approach shares characteristics with the iterative processes of Knowledge Analysis (diSessa 2008). As described by Levin (2011) and Wagner (2010), Knowledge Analysis is a set of analytical strategies which consists of describing the moment-by-moment actions of students engaged in the process of problem solving, involving careful consideration of the students' language and reasoning patterns, and iteratively seeking to improve these descriptions based on further evidence and potential alternative explanations.

To identify the resources the students' used while completing the various tasks, I considered the language each student used, the steps they completed in their overall reasoning process, and their patterns of argumentation when making *inferences*. By inference I mean a conclusion the student drew while in the process of problem solving which could be supported by logical (from the perspective of the student) reasoning or evidence. I identified individual resources by seeking students' statements and actions that provided evidence of the knowledge the students used to make these inferences. In general, describing the knowledge a student used requires determining what the result of the coordination of the knowledge was and what information the knowledge was being coordinated with. For example, during Interview 4 Hakeem made the following statement while considering the equation $\frac{dx}{dt} = 5x + 2xy$:

> "If $x$ is really small and $y$ is large or $y$ is much greater than $x$, then this term, or the magnitude of this term [-5x] will be much smaller than the magnitude of this term [2xy] and so you would get a positive value minus this value and you'd end up with a positive $\frac{dx}{dt}$. And so that means that ah, $x$ is increasing so I guess" (turn 10).

In this case Hakeem inferred that the value of $\frac{dx}{dt}$ would be positive, and he inferred that the value of $x$ would be increasing. To make the first inference, based on his statements he considered how certain values of $x$ and $y$ related to the value of $\frac{dx}{dt}$, considered the magnitudes each of the terms given the values of $x$ and $y$, and considered how each of the terms contributed the value of $\frac{dx}{dt}$. Each one of these pieces of information was utilized by Hakeem while in the process of making the inference that $\frac{dx}{dt}$ was positive, and as such provides evidence of the knowledge he was coordinating with this information to infer the value of $\frac{dx}{dt}$ was positive. The first piece of knowledge regards *functional dependence* (values of $x$ and $y$ determine the values of $\frac{dx}{dt}$), the second piece of knowledge regards the notion that *larger magnitudes have larger impacts* compared to smaller magnitudes ("the magnitude of this term will be much smaller than the magnitude of this term"), and third piece of knowledge regards the two terms on the right hand side coming together to form a single value for $\frac{dx}{dt}$ (*parts contribute to the whole*). He then used the first inference as information form which he could make the second inference. In this case he used information about the sign of the rate of change to make an inference abut the behavior of the solution. Indicating that he applied knowledge to the situation that allowed him to make an inference about the behavior of the solution based on the sign of the rate of change (*rates indicate behavior*). This example involved analyzing the explanations Hakeem provided when making the inferences (and thus consideration of the language he used), the information he was considering when making the inferences, and considering how his explanations and the information he used could be triangulated with the steps he took during the problem solving process.

This was an iterative process that required continual refinement of the identified resources. By comparing similar inferences (within and across tasks), one finds similarities (and differences) in the knowledge the students used to make inferences at different times; the identification of similar resources within similar sets of inferences supported the characterization of the corresponding resources (though did not guarantee it), while differences suggested either the existence of an additional or entirely different resource. This iterative process was not just with regard to the characterization of the knowledge resources but also required revisiting and reassessing the instances from the interviews in which the knowledge resources were identified.

As an example of the careful attention given to the students' language and reasoning patterns, students often made statements such as "as $t$ changes, $x$ changes", and "$x$ depends on

$t$." Both of these statements at face value could be reasonably interpreted by the researcher as a statement of the dependence of the value of $x$ on the value of $t$; this is implied in the former and explicit in the latter. In the case of the first statement, it might be the case (or not) that the student is using knowledge about the nature of the changes between the values of the quantities (*functional variation*) to make inferences as opposed to the nature of the role of $t$ in that relationship, namely determining $x$ (*functional dependence*). Thusly, analyzing the statements alone while ignoring the student's reasoning process, what the student is making the inference in reference to, and then subsequently using the inference for, would not afford parsing apart whether or not the statement "as $t$ changes, $x$ changes" is indicative of *functional variation* or *functional dependence*. As a result this might lead to falsely characterizing such statements as being indicative of certain resources, or in some cases failing to identify a new resource.

In instances when it was difficult to determine if a student was using a certain resource, I relied on two measures to determine if the knowledge they applied was not described by an existing resource. First I considered what the student was reasoning about before and after making the inference; for instance, using the example from the previous paragraph, was the student reasoning with the notion that the value of one variable determines the value of another, or was the student using the relationship between changing values to make inferences? Then I would determine what attributes the student was coordinating while making statements immediately preceding and following these inferences: was the student comparing different sets of values of $t$ to different sets of values of $x$, or did the student make the inference while reasoning about which variables determine the value of a function? If a distinction still could not be made, I sought relevant research literature. This process is precisely how the resource *functional variation* emerged.

Although the resources emerged as the result of comparing moment-by-moment inferences, identifying the student's reasoning process is an important part of identifying resources (Adiredja, 2014). The reasoning process is indicative of the nature of the resources the student utilized, the utility the resources served, and how the student perceived the attributes with which the resources were coordinated. By being sensitive to the purpose and utility of the resources and how they fit within the student's reasoning process, I was able to support the validity of the resources I identified. This was an important part of the analysis as it provided an additional way of creating alternative explanations.

### 3.4.2 Identifying Changes in the Utilization of Resources

To identify the changes in the student's utilization of the resources I first had to determine how the resources were organized. Due to the diverse set of tasks posed in the interviews, understanding how the student's knowledge was organized based on only a handful of tasks was initially extremely difficult to discern. Once resources were identified across various tasks, however, I then started analyzing for consistencies in how the student's interpreted certain attributes across sets of tasks where seemingly similar sets of resources emerged, and for differences in the student's interpretations of certain attributes across sets of tasks where different sets of resources emerged. As Adiredja (2014) noted, analyzing how resources were utilized across different situations is important for uncovering differences in the utility of different resources across different contexts. In doing so I sought to explain the similarities and differences in a student's utilization of the resources in terms of the differences and similarities in the student's perception of certain contextual attributes. This provided multiple affordances in terms of the analysis. This process provided a foundation on which comparisons could be made and allowed for identifying differences in the ways the student's perceived attributes across various tasks that had large impacts on their reasoning process. Additionally I leveraged students' statements concerning similarity or sameness across tasks; sometimes students would make declarations that certain tasks were the same. Such statements provided rich opportunities for identifying the attributes guiding the organization of their knowledge because such statements were often followed by descriptions of why the tasks were similar. In addition these statements also provided evidence of the students' perceptions of similarity, which as Lobato (1996, 2003) noted are an important aspect of transfer research. I now discuss some of the orienting questions that guided this process.

### 3.4.3 Identifying the Interpretations

In seeking to understand how the resources were organized in terms of contextual attributes, I followed a similar approach to that of identifying the resources. More specifically I considered the students' language, reasoning patterns, focus of interest, and in general sought to understand the differences and similarities in the emergence of resources in certain tasks but not in others. In doing so I leveraged the following guiding questions which, as discussed by Levin (2011), provide data about the organization of knowledge:

1. What were the students focusing their attention on while completing the tasks?
2. What appeared to be important for them to complete the task in the manner they did?
3. What were the inferences they were making about the task itself?
4. What information from the task were they reasoning with?

The process of using the guiding questions to determine the students' organization of knowledge followed a process consistent with open and axial coding (Strauss & Corbin, 2008). More specifically, by using the process of open coding with regard to the apparent salient aspects of the task (from the student's perspective), the information from the task the student's reasoning was focused on, and what the student did (holistically) to complete the task, I was able to determine different ways the student was interpreting attributes from the task, and the tasks themselves, in the form of axial codes. In addition these axial codes were able to account for the differences and similarities in the students' utilization of resources across different tasks and the different reasoning patterns the students used as well. More specifically, the axial codes emerged in the form of two aspects for each task; the students' *interpretation* of the differential equation, and the students' *interpretation of* the components of the differential equation. These interpretations are identified in the analysis presented in Chapter 4, but are not discussed in detail until Chapter 5. This is due to their role in identifying how the resources changed over the course of the interviews.

### 3.4.4 Documenting Resources and Interpretations

To document the resources and interpretations the students used to complete the tasks I constructed a table for each interview consisting of columns for the task, the student's interpretation of the differential equation, the resources they used while completing the task, and their interpretation of the components within the differential equation. An example of this for Hakeem (for one task from Interview 4) can be seen in Table 3.1. In the "task" column, I use "I4-T1" to indicate that the task being referred to in that row is Task 1 from Interview 4. In this example the table indicates that while completing the task Hakeem interpreted the differential equation as a description of the behavior of the quantity of interest and used the resources *parts contribute to the whole, larger magnitudes have larger impacts, functional dependence, functional mapping, unrestricted change,* and *rates indicate behavior.* In addition the last column shows that he interpreted the components in the differential equation $x$, $y$, $\frac{dx}{dt}$, and $\frac{dy}{dt}$ as population values, and their rates of change respectively.

*Table 3.1*: Example table showing documentation of resources and interpretations.

| Task | Interpretation of the differential equation | Resources used to complete the task | Interpretation of the components in the differential equation |
|---|---|---|---|
| I4-T1 Competing-cooperative species | Description of the behavior of the quantities of interest | *Parts contribute to the whole*<br>*Larger magnitudes have larger impacts*<br>*Functional dependence*<br>*Functional mapping*<br>*Unrestricted change*<br>*Rates indicate behavior* | Interpreted $x$ and $y$ as population values, and $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as the rates of change of the respective populations. |

### 3.4.5 Identifying Changes in The Sets of Resources

In terms of the changes in the sets of resources, new sets of resources can be generated to meet new contextual demands, which allows for interpreting or implementing a certain concept (in this case the concept of a differential equation) in a new situation. In addition one can utilize an existing set of resources to interpret or implement a certain concept in a in a new situation, by interpreting that new situation in way that afforded the utilization of the existing concept projection. In terms of the analysis here, the goal was to find instances in which the students utilized existing sets of resources in novel tasks and instances in which the students utilized new sets of resources in novel tasks. Additionally, it may be the case that the sets of resources the students apply in certain situations change over time.

Thusly by documenting the interpretations and the resources for each task in the form of a table, I was able to analyze across interviews and tasks to determine how the sets of resources were changing (presented in Chapter 5). More specifically I was able to find instances in which the students expressed a new interpretation and corresponding sets of resources, and instances in which students utilized existing interpretations and corresponding sets of resources. Lastly, by documenting both the sets of resources and the ways they were interpreting the differential equations and their components I was also able to identify instances when the students were using multiple sets of resources and interpretations in service to completing a single task.

The remaining chapters are devoted to presenting the analysis of Hakeem and Jordan, which lead to the identification of the resources students use to complete various differential equations tasks (Chapter 4), and discussing how the resources changed and how they interacted to support the student's understanding of differential equations (Chapter 5). I conclude the dissertation with Chapter 6 where I discuss the implications of the findings, limitations, and future directions.

# CHAPTER 4: IDENTIFICATION OF STUDENT RESOURCES AND INTERPRETATIONS

Chapter 4 has a significant and central role in this dissertation: in it I present analysis that answers the first research question and also lay the groundwork for answering the second and third research questions (Chapter 5). More specifically, in this chapter I present the analysis leading to the identification of the resources Hakeem and Jordan used to complete the various differential equations tasks and the ways in which they interpreted the differential equations (and their components) while doing so. I then outline the various ways the resources supported the students in completing the different tasks. Finally, I draw connections between the identified resources that students used to complete the tasks and the current literature on student understanding of function and rate of change.

To remind the reader, the research questions guiding this dissertation are:

1. What resources concerning function and rate of change do students utilize to complete various differential equations tasks?
2. How do these resources change as the students' progress through a differential equations course?
3. How do students' resources concerning rate of change and function influence one another during the development of their understanding of differential equations?

The glossary in Table 4.1 consists of the resources Jordan and Hakeem used while completing the various tasks they interacted with throughout the five interviews. The glossary is provided at the beginning of Chapter 4, before the analysis of the two students, to familiarize the reader with the resources and also locate the description of the resources in one location. While the glossary does represent all of the resources I identified, it should not be taken as an all-inclusive list of the possible resources the students used.

*Table 4.1: Glossary of knowledge resources.*

| Knowledge resource | Description |
|---|---|
| Carrying capacity | In general populations increase or decrease to some maximum or minimum values, respectively. The populations may stabilize or fluctuate between these values maximum and minimum values. |
| Continuity | The value of quantities change in a connected fashion, they do not jump around randomly, they change in a smooth manner. |
| Definition of cooperative systems | Growth or increases in the value of one quantity promotes growth or increase in value of the other quantity. |
| Definition of competing systems | Growth or increases in the value of one quantity promotes decay or decreases in the value of the other quantity |
| Definition of equilibrium solution | The value of the rate of change, derivative, slope or change in a quantity is zero. This may be applied in a continuous fashion to functions, or it may be applied discretely in the form of a vector field. |
| Definition of general solution | The general solution provides the form of the equation for all solutions to a differential equation. This is not restricted to algebraic forms of the solution and may apply to various forms of graphical representations (e.g., solutions graphed on a slope field) |
| Dependence | Characteristics of one mathematical entity (i.e., its value, behavior or form) are controlled or determined by other mathematical entities. This is a more general version of functional dependence. An application of this resource could include inferring that particular solutions depend on initial conditions, or that the shape of a graph is determined by an initial value. |
| Equality condition | For two mathematical entities to be or remain equal, they must satisfy certain conditions. This may be applied to equations such as $y = 3x$, or to a statement of equality involving a function and its derivative function (i.e., $y' = 0$). In the former, $x$ must be one-third the value of $y$ for the equality to remain true. In the latter, for $y$ to satisfy the differential equation it must be such that the value of its derivative is (identically) 0. |
| Function slots<br><br>(Adiredja, 2014) | As noted by Adiredja, (2014) "This resource supposes that when two quantities share a functional relationship, one quantity is the $x$ and the other is the $f(x)$ or the $y$" (p. 43). |
| Functional dependence | In a defined relationship between quantities (e.g., functions and equations) the value of one quantity is determined by the value of the other quantities in the relationship. |
| Functional variation | In a defined relationship between quantities (e.g., functions and equations) as the value of one or more of the quantities change the values of the other quantities in the relationship change as well. |
| Functional mapping | A defined relationship between quantities (e.g., functions and equations) of correlation between sets of values of the quantities in the relationship. |
| Larger magnitudes have larger impacts | This may be an application of Ohm's p-prim (diSessa, 1993) in the context of mathematics. This principle is often applied to the coefficients of terms to determine how much of an effect a particular term will have on the overall value of an expression. This could also be utilized on functions to determine the effect a particular set of values for the independent variable(s) would have on the dependent variable(s). For instance, in $\frac{dx}{dt} = 5x - 2xy$ a student may infer $5x$ has a larger effect on the value of $\frac{dx}{dt}$ than $-2xy$ because 5 has a larger magnitude than 2. |

*Table 4.l (Continued): Glossary of knowledge resources*

| | |
|---|---|
| Method of characteristic equation | Analytical solution method for second order differential equations of the form $ay'' + by' + cy = 0$, which supposes a solution of the form $y(t) = C_1e^{r_1t} + C_2e^{r_1t}$, where $r_1$ and $r_2$ are roots of the chatacteristic equartion $ar^2 + br + c = 0$. |
| Method of integrating factor | Analytical solution method in which one finds a function by which the differential equation is multiplied to make the differential equation integrable. |
| Method of eigenvalues | Analytical solution method for systems of differential equations consisting of writing the system in matrix form, determining the eigenvalues and eigenvectors of the coefficient matrix and then substituting the eigenvalues and eigenvectors into a known form of the solution (general solution). |
| Euler's method | Numerical or graphical solution method consisting of iteratively adding a quantity's differential to a known or approximated initial value to approximate successive values of the quantity. |
| Method of separation of variables | Analytical solution method consisting of rewriting a differential equation so that each of two variables occurs on a different side of the equation, and then integrating each side with respect to variable occurring on that side. |
| Opposite operations reverse | The idea that performing opposite operations on a single mathematical object will return the object. For instance, integration of $P'(t)$ reverses the differentiation and returns $P(t)$, addition reverses subtraction, etc. |
| Parts contribute to the whole | The idea that one can break expressions into "parts," and consider how those individual parts contribute to the value of the entire expression.<br>This may also be applied to defined relationships where multiple terms exist on one side of the equality. In this case one may break the expression on one side into pieces to see how the value of each of the pieces contributes to the value of the other side. |
| Positives add - negatives take away | In defined relationships between values of quantities, positive signs signify one value is adding to another where as negative signs signify one value is taking away from another.<br>This is often utilized with functional variation to infer how changes in the value of one quantity relate to changes in the value of another, or parts contribute to the whole to infer how the parts contribute to the whole. |
| Partial dependence | The idea that an equation can express multiple relationships, and one can explore those relationships depending on which quantities they consider. |
| Rates indicate behavior | One can use the relationship between a quantity and that quantity's rate of change to ascertain the behavior of that quantity. In this case, the value of the quantity's rate of change informs the behavior (i.e., increasing or decreasing, and how quickly) of the quantity. |
| Unrestricted change | The idea that in a defined relationship between quantities, sufficient change in the value of one quantity corresponds to sufficient change in the value of the other. That is, if increases in quantity A relate to increases in quantity B, quantity B will increase in an unlimited fashion as quantity A increases. This resource is often used to infer that increasing negative values will eventually become positive or decreasing positive values will eventually become negative. ("if $x$ increases enough $y$ will switch signs.") |
| Variable substitution | This resource supposes that given a defined relationship between quantities (e.g., equations, graphs, functions) one can substitute values for one set of quantities to determine the value of another quantity. |

## 4.1 Analysis of Resources Identified for Hakeem

In this section I present Hakeem's interactions with tasks over the course of five interviews. Hakeem was a freshman majoring in engineering and was enrolled in an honors differential equations course. I selected Hakeem for analysis because he demonstrated an ability to discuss differential equations in many different ways, using many different reasoning strategies. In addition, he demonstrated a strong mathematical ability, which in terms of the analysis provided a complex network of resources and ways of interpreting the differential equations and their components.

### 4.1.1 Interview 1

In this section I present the analysis of four of the five tasks Hakeem completed in interview 1. I omit the last task because it was not a task directly involving a differential equation. Over the course of the interactions with the tasks in interview 1 Hakeem utilized four distinct ways of interpreting the different differential equations: As an equation to be solved, as a relationship between the value of quantities and the value of their respective rates of change, the as description of the quantity of interest, and as a relationship between a function and its derivative function. As will be exemplified in the following sections these interpretations not only corresponded to sets of resources Hakeem utilized to complete the various tasks, but patterns in which he utilized those resources (more on this in Chapter 5).

**4.1.1.1 Interview 1 task 1.** Task 1 asked the students to reason about two non-linear systems of differential equations, one representing a competing species and the other representing a cooperative species (see Figure 4.1). At this point in the semester the students had not yet been introduced to systems of differential equations. While in the process of completing this task, Hakeem initially interpreted the differential equations as equations to be solved.

Specifically, Hakeem communicated his desire to utilize integration techniques to solve each of the differential equations. He was, however, unable to determine how to apply such a solution method, and as a result seemingly switched to interpreting the differential equation as a description of the behavior of the quantities of interest. Along these lines, Hakeem determined how changes in the values of $x$ and $y$ related to changes in the value of $\frac{dx}{dt}$ and $\frac{dy}{dt}$, and then used the changes in $\frac{dx}{t}$ and $\frac{dy}{dt}$ to make inferences about how the species interacted with one another. Hakeem utilized the following resources while completing this task: *functional dependence, functional variation, functional mapping, parts contribute to the whole, larger magnitudes have larger impacts, positives add- negatives subtract*, *rates indicate behavior*, *the definition of cooperative quantities* and *the definition of competitive quantities.*

TASK 1: In this task, we look at systems of rate of change equations designed to inform us about the future populations for two species that are either competitive (that is, both species are harmed by interaction), or cooperative (that is, both species benefit from interaction, for example bees and flowers). Which system of rate of change equations describes competing species and which system describes cooperative species? Explain your reasoning. (Rasmussen & Whitehead, 2003)

| **A** | **B** |
| --- | --- |
| $\frac{dx}{dt} = -5x + 2xy$ | $\frac{dx}{dt} = 3x - 2xy$ |
| $\frac{dy}{dt} = -4y + 3xy$ | $\frac{dy}{dt} = y - 4xy$ |

*Figure 4.1: Task one from interview 1 (I1-T1).*

I begin the presentation of the analysis leading to the identification of the resources with the opening segment of transcript from this task.

1 H: So for competing species, I guess the thing that I would be looking for would be that the rate of change would be negative like as time goes to, like gets larger and larger, the rate of change would be negative, while for a cooperative species their rate of change would be positive.

2 G: Ok so, um, so do you let me see if I have a good way of thinking about it, so the rate of change for a species or the rate of change...

3 H: Ah, for both.
4 G: Ok, so if they are cooperative, you're saying that you would expect the rates of change to be positive for both and if they are competing you would expect them to be negative for both?
5 H: Ah probably.

I then asked Hakeem how he intended to use the differential equations to get information about the rates of change. In reply to my question about using the differential equations to determine the rates of change of the species, he expressed that he would put the equations into a form that would allow him to isolate the variables $x$ and $y$; "I'd solve the equation for $x$ say over here, but I'm wondering what form I should get it into since it's not like a first order ordinary differential equation" (turn 15). Shortly before this, Hakeem noted that $\frac{dx}{dt}$ is "an equation using $x$, but there are only $x$'s and $y$'s, no $t$'s" (turn 9), and stated this is a problem because "when you do the integrating or differentiating, say I'm doing $\frac{dx}{dt}$, when I come across a $y$, like I don't know what $\frac{dy}{dt}$ is." Here Hakeem utilized *functional dependence* to infer that $y$ was dependent on $t$, and because there was no equation for $y(t)$ it could not differentiated or integrated. Based on these statements, I infer that Hakeem tried to separate the terms in the differential equation to put it into a recognizable form that would allow him to solve for $x$ or $y$ using an integration technique. This is indicative of the first way in which Hakeem interpreted the differential equation (which we will see again in Task 3 from this interview), as an indicator of a certain solution method. Hakeem abandoned this strategy, however, and after prompting (turn 24, transcript follows) he began to reason with relationship between the variables in the differential equations to determine the rates of change of the two species.

24 G: Ok, so if we went back to the idea that $\frac{dx}{dt}$ or the rates of change for this, the species had to be positive for cooperative and negative for competing, is there any way that you think you could get that information from the differential equations?

25 H: Well, probably. So for this [points to $\frac{dx}{dy} = -5x + 2xy$], it's a, as, like for example in this equation as $x$ say, gets larger the rate of change is decreasing from this part [points to $-5x$], but its also increasing from this part [points to $+2xy$], and it would be increasing from this part [points to $+2xy$] as well, because of $y$.

26 G: Ok, so what does that tell you?

27 H: So, if there is a high population in the group $y$, it, this rate of change would be positive.

28 G: Ok, so how are you thinking about $x$ and $y$? In other words, what are they and what do they represent in the differential equation?

29 H: The number of species in two different populations.

30 G: Ok, and when you are saying things like $x$ increases, or $y$ increases, what is that saying about $x$ or $y$?

31 H: As the population is growing.

32 G: Ok.

33 H: So, if I said for example, if $x$ is small and $y$ is large that, or if $y$ is increasing, then $x$ would be increasing here.

In the talk turns 24-33 Hakeem made use of the relationship between the value of the quantities and the value of the quantities' respective rates of change in two ways. In turn 24 Hakeem coordinated the resources *functional variation* and *parts contribute to the whole* to determine how changes in the values of the populations [$x$ and $y$] related to changes in the value of the rate of change of the populations [$\frac{dx}{dt}$]. Specifically, he used *functional variation* to infer that changes in $x$ and $y$ related to changes in $\frac{dx}{dt}$ and $\frac{dy}{dt}$, and parts contribute to the whole to characterize the nature of those changes. For example to determine "as $x$ say, gets larger the rate of change is decreasing" (turn 24), Hakeem utilized *parts contribute to the whole* to determine how *parts* of the right hand side of the differential equation contributed to the left hand side of the equation, the *whole;* this is evident in turn 25 when he said "the rate of change is decreasing from this part [first term], but it's also increasing from this part [second term], and it would be increasing from this part as well, because of $y$."

In turn 27 he then coordinated *parts contribute to the whole* and *larger magnitudes have larger impacts* to relate sets of $x$ and $y$ values to sets of values of $\frac{dx}{dt}$. Specifically, he inferred, "if there is a high population in the group $y$, it, this rate of change would be positive." In this case using the resources *larger magnitudes have larger impacts* and *parts contribute to the whole*, Hakeem reasoned that for large values of $y$, the impact of the second term would be larger than that of the first term resulting in a positive value for $\frac{dx}{dt}$. Additional support for this is provided in a subsequent paragraph.

Hakeem then went on to compare the two systems, and ultimately determine which system represented competing species and which represented cooperative species.

41 H: Yeah and then when you look at this group, it's kind of opposite, like here [points to System B] if $y$ is, ah the part of the equation is negative for these two parts [points to the second terms; $-2xy$ and $-4xy$], but here [points to System A] the $y$ part of the equation is positive for both parts [points to $+2xy$ and $+3xy$]. So, I think this one [points to System A] is actually the cooperative system and this [points to System B] is the competitive system.
42 G: Is that based on the plus signs and minus signs?
43 H: Yeah
44 G: Ok. So can you just talk about this one [points to System B] a little bit in the same way that you talked about this one [points to System A]?
45 H: Yeah so, for this one [points to $\frac{dx}{dt}$ equation in System B] when $y$ is increasing or the rate of growth is decreasing in the $x$ population so basically as $y$ grows $x$ has to go down. But then if $x$ grows $y$ has to go down thanks to this equation [points to $\frac{dy}{dt}$ equation in System B]. While this one [points to System A], they are lifting each other up, as $y$ grows, $x$ grows, and as $x$ grows here, $y$ grows. So, that's why I am saying this [points to System A] would be a cooperative system and this [points to System B] would be a competitive system.
46 G: Ok. I have a few questions let's see. How do you think of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ in these equations?
47 H: So they are derivatives. They are rates of change so, they are not necessarily mean that one thing is growing. If $\frac{dx}{dt}$ is positive it, I mean, yeah if $\frac{dx}{dt}$ is positive then $x$ is growing at that point in time, and if $\frac{dy}{dt}$ is positive then $y$ is growing at that point in time.

This segment of transcript is suggestive of the utilization of the resources *partial dependence*, *parts contribute to the whole*, *positives add- negatives subtract*, and *rates indicate behavior*. In talk turn 41, Hakeem compared the two systems by coordinating, by considering how the signs (*positives add- negatives subtract*) of the "parts" of the right hand side affect the whole, the left hand side (*parts contribute to the whole*). He did so, however, in a very specific way: he considered how the "parts" relating to one species, impacted the "whole" representing the other species (*partial dependence*). In other words he ignored how $x$ impacted $\frac{dx}{dt}$ and how $y$ impacted $\frac{dy}{dt}$, and only considered how changes in the value of the population of one species impacted the value of the rate of change of the other species. By coordinating these resources he determined that in System A, the value of $x$ was taking away from the rate of change of $y$ and the value of $y$ was taking away from the value of $\frac{dx}{dt}$, whereas in system B, the value of $x$ was adding to the rate of change of $y$ and the value of $y$ was adding to the value of $\frac{dx}{dt}$. With this inference, he was then able to use *rates indicate behavior* to determine how the changes in the value of one population related to changes in the value of the other population. To clarify this, consider talk turn 45 when Hakeem noted that in System B, "when $y$ is increasing or the rate of growth is decreasing in the $x$ population so basically as $y$ grows $x$ has to go down. But then if $x$ grows $y$ has to go down thanks to this equation [points to $\frac{dy}{dt}$ equation in System B]." This also supports Hakeem's utilization of *parts contribute to the whole* in talk turn 41. In addition his statements in turn 47, where Hakeem refers to $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as "rates of change" serve to support Hakeem's utilization of *rates indicate behavior* to infer that "as $y$ grows $x$ has to go down. But then if $x$ grows $y$ has to go down."

Lastly, after characterizing the relationships between the values of the populations and the values of their respective rates of change for each of the systems, he used the relationships to classify the systems of equations as competing and cooperative. To do so he used *the definition of cooperative systems* and *the definition of competing systems* resources. This is evident in turn 45 (transcript above). Specifically, Hakeem classified System A as a cooperative system because growth in the value of one population (quantity) promoted growth in value of the other (*the* *definition of cooperative systems*), and he classified System B as a competing system, because growth in the value of one population (quantity) promoted decay in the value of the other (*the definition of competing systems*).

In conclusion, in this task Hakeem interpreted $x$ and $y$ as population values, and $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as the rates of change of the respective populations. Additionally he interpreted the differential equations as describing the behavior of the quantities of interest. In doing so, he was able to utilize resources that enabled him to reason productively about the task. The resources *parts contribute to the whole* and *larger magnitudes have larger impacts* supported Hakeem in reasoning about how values of $x$ and $y$ related to values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$. The resources *parts contribute to the whole* and *functional variation* supported Hakeem in reasoning about how changes in $x$ and $y$ related to changes in $\frac{dx}{dt}$ and $\frac{dy}{dt}$. The resource *rates indicate behavior* supported Hakeem in then relating how the population of one species related to the growth or decay of the other. Lastly, *the definition of cooperative quantities* and *the definition of competing quantities* supported Hakeem in using the characterizations of the relationships between the population values to determine which system represented competing species and which system represented cooperative species.

**4.1.1.2 Interview 1 task 2.** Task 2 asked students how they thought about the differential equation $P' = 3P$ (see Figure 4.2). While completing this task Hakeem interpreted the differential equation as providing a model of the quantity of interest. In doing so he utilized a rule for solutions to differential equations of the form $P' = \mathrm{k}P$, where $k$ is any constant and provided a warrant for this rule. Providing the warrant lead him down a path of discussing rate constants and constants of integration in terms of the model provided by the differential equation. While these may seem as though they are tangential discussions, they in fact illuminate how Hakeem was reasoning with the differential equation, and its relation to the model it provides. I only present the first half of the interaction because one of my follow up questions prompted Hakeem to artificially shift the focus of the discussion. To complete task 2 Hakeem utilized his rule *differential equations of the form $P = \mathrm{k}P$ have solutions of the form $P = Ce^{\mathrm{k}t}$, rates indicate behavior, larger magnitudes have larger impacts, function slots,* and *functional dependence.*

| |
|---|
| TASK 2: What does the following differential equation mean to you?<br><br>$P' = 3P$ |

*Figure 4.2: The second task from interview one (I1-T2).*

Upon reading Task 2 Hakeem remembered seeing this type of question before; he immediately stated: “I remember going through this kind of thing in class.” The following except begins with Hakeem’s very next utterance. The numbers indicating the talk turn are continuous from one task to the next for comparative purposes; talk turn 63 starts just after Hakeem expressed recognizing the problem.

63 H: So when you have a, if we said say, $P$ was for a population then it’s basically a model for the growth or decay of that population. So the solution to the differential equation will end up being $P$ equals e to the say $3t$, something like that. Ah, I’m not exactly sure, but I mean if I solve it, it would be the integral of $P'$ over $p$, is equal to the integral of 3. And so you get the natural log of $P$ equals ah $3t$, so you have plus $c$, and then $P = Ce^{3t}$ and $C$ here would be like the initial population.

64 G: Ok. How do you know that $C$ would be the initial population?

65 H: Ah, because if you set, the initial population would be when $P'$ was zero, so let's say $P$ of zero becomes $e$ to the 0 and $e$ to the zero is one, so $P$ of zero is $c$. So you could say $P$ equals zero e to the three $t$. And this 3 here, is basically supposed to be ah, I guess a, well sort of, a sort of rate constant almost.

66 G: Ok when you say a rate constant, what does it influence, what does it do?

67 H: It influences how much it is changing as time goes on. So if this number was, if this number is really big then in short amounts of time the population goes through a lot of growth. But say if this was a fraction then it would be decaying I think. Or it wouldn’t be,

not if it is a fraction, if it was negative then the population would be decaying as time goes on.

In his initial statement Hakeem indicated differential equations provide a model of the quantity of interest noting that if "$P$ was for a population, then it's [$P$] basically a model for the growth or decay of that population" (turn 63). In this case he interpreted $P$ as representing population and proceeded to discuss the differential equation and it's variables in terms of population. Due to his ability to seemingly recognize the solution to the differential equation, "So the solution to the differential equation will end up being $P$ equals e to the say $3t$," I take turn 63 as indicating a rule regarding the form of models differential equations of the form $P' = \text{k}P$ provide. Namely, *differential equations of the form* $P = \text{k}P$ *have solutions of the form* $P = Ce^{\text{k}t}$. In addition to utilizing this rule, he expressed a warrant for it, showing that when you solve the differential equation by utilizing separation of variables, you arrive at the same conclusion, $P = Ce^{3t}$ where $C$ is the initial population. When asked how he knew $C$ was the initial population in turn 64, he used the resource *function slots* and *functional dependence* to show that $P(0) = Ce^0 = C$, by inferring a dependence relationship between $P$ and $t$ to show that when $t = 0$, $P$ must be equal to $C$. More specifically, in turn 65 he put 0 in the slot for $t$, and $C$ in the slot for $P(t)$ or $P$. In doing so, he inferred the existence of initial conditions for the solution and demonstrated the utilization of such conditions with the *function slots* resource.

In addition, Hakeem coordinated the resources *larger magnitudes have larger impacts* and *rates indicate behavior* to discuss the significance of the coefficient in the differential equation. Hakeem interpreted the coefficient 3 in the differential equation to be a "rate constant" which, in talk turn 67 he equated with how $P$ changes with respect to time; “if this number is really big then in short amounts of time the population goes through a lot of growth.” I interpret the statements in turn 67 as evidence of the resources; *larger magnitudes have larger impacts* (“if this number is really big…”) and *rates indicate behavior (“…*then the population goes through a lot of growth”). Together these resources, along with the inferences he made about the various parts of the differential equation (for instance $P$ being population and $P'$ being the rate of growth or decay), allowed him to reason about 3 as a rate constant that indicated whether the quantity was increasing or decreasing, and how quickly. I argue this does not indicate that Hakeem interpreted the differential equation as a relationship between a value of a quantity and the value of that quantity’s rate of change, as the emphasis is on the coefficient 3 and not on the relationship between the values of $P$ and $P'$. That is, Hakeem was considering how the value of the rate constant related to $P$, not how values of $P$ related to values of $P'$.

In this exchange, Hakeem interpreted the differential equation as providing a model of the quantity of interest (in this case population). While his initial response to reading the task was to provide an equation that was the same as the result of “solving” the differential equation (“but I mean if I solve it,” turn 63), I argue he was solving for the model. That is by solving the differential equation Hakeem was attaining the equation $P(t)$ that related the value of the population to $t$ (what Hakeem considered to be the model). In this case, for Hakeem the differential equation was not a model of the population, rather it provided away to get the model. The differential equation, however, was informative with regard to characteristics of the model; Hakeem volunteered statements about the rate constant impacting how quickly the population would increase.

I end the presentation of the resources Hakeem utilized in task 2 here and transition to how Hakeem was interpreting the terms in the equation.

In Hakeem's statements at the end of the interaction with Task 2 he expressed that he was unsure of what units to attribute to $P$ and $P'$. The following segments are statements made in response to a clarifying question I asked; Hakeem referred to $P'$ as "the rate" and I asked him what exactly he meant by that, to which he replied, "that'll be $P'$" (which, from my perspective was somewhat circular as I was aware that he was making reference to $P'$ when he said "the rate"). I then asked about the units associated with "the rate" because of the sophistication he expressed earlier (turn 67) with regard to conceptualizing the impact of the rate constant.

81 H: Well we don't have another unit, it could be time, like I guess we gave it units like a population over time. Like as an example…
82 G: Ok that's fine.
83 H: But it could have other, it could have other units too.
84 G: What is it that um makes you think it's not time, or that it could be other units perhaps.
85 H: Hmm.
86 G: It sounds like what you are saying is it not well defined based on the question and I am just curious...
87 H: I mean to be honest I don't really know, I can't really think of other examples with where it wouldn't be time but, cause usually like when we go through things like exponential growth or decay the independent variable is always time.

I then asked Hakeem how he thought about $P$.

89 H: Umm, well $P$ could be anything.
90 G: Meaning it doesn't have to represent population?
91 H: No like, this equation also works for compounding interest.

I provide these segments from the transcript to highlight the connection between the resources Hakeem used and the situation in which these resources are useful. At the very beginning, he started the conversation with "say $P$ was for a population." In light of the previously presented segments of transcript, this seemingly gave Hakeem a medium onto which he projected these resources so they could then be utilized; the interpretations Hakeem made about the variables and terms in the differential equation represented were all related to population (e.g., $P$ is a model for population, 3 was the rate constant, $P'$ was the rate, $C$ was the initial population). More specifically, by relating the differential equation to population, Hakeem was able to make interpretations about the various parts of the differential equation that allowed him to productively apply resources to complete the task; for example, by interpreting $P'$ as "the rate" he was able to utilize the resource *rates indicate behavior* to infer how the rate constant impacted the growth of $P$. Although it is impossible to tell how Hakeem would have handled this task differently had he not framed it around population, it is clear that projecting the task onto a context involving populations had a significant role in his reasoning. To be clear, I am arguing that "population" played a large role in the interpretations and inference Hakeem made, which he was then able to coordinate with the relevant resources to complete the task.

In summary, Hakeem used his rule for solutions of the form $P' = kP$ to determine the equation for $P(t)$ which represented a model of a population. He then used *function slots* and *functional dependence* to situate this model within certain initial conditions, the existence of which he supposed. Lastly, he reasoned about the coefficient in front of $P$ to make inferences about the behavior of the values of the population that the differential equation provided a model for. More specifically, he considered the coefficient to be a "rate constant" and, utilizing the resources *larger magnitudes have larger impacts* and *rates indicate behavior,* he inferred how fast or slow the population would grow was based on this rate constant; larger rate constants related to faster growth, smaller rate constants related to slower growth.

**4.1.1.3 Interview 1 task 3.** The analysis of Hakeem's interaction with Task 3 (see Figure 4.3) provides a great deal of insight into how he was interpreting not only the differential equation, but also the variables within it. This is of particular interest because while completing this task Hakeem seemingly expressed thinking of $P$ as both a dependent and independent variable. Additionally, though Hakeem never explicitly completed this task, his reasoning is commensurate with using the differential equation to find a solution function, and then using that solution function to find the value of the population at a given time. To do so Hakeem utilized *the method of integrating factors, functional dependence* and *opposites cancel*, and I argue he did so in service of interpreting the differential equation as an equation to be solved.

Within 30 seconds of reading the task Hakeem went to work "solving" the differential equation. The following transcript starts with Hakeem's initial statements concerning Task 3.

| |
|---|
| Suppose the equation $P' = 2P + 2t$ can be used to model the fish population $P$ in the campus Duck Pond. How might it be used to determine the number of fish in the pond at a given time $t$? |

*Figure 4.3: Task 3 from Interview 1 (I1-T3).*

97 H: So this is an ordinary, a first order ordinary linear differential equation.
98 G: Ok.
99 H: So you could easily solve for $P$.
100 G: So…
101 H: So yeah a solution would be $P$, some equation of $P(t)$, and that would basically give you what the population is at any given time, but the only thing you are missing is an initial value.
102 G: Ok.
103 H: So if I did solve it, it would be put it in this form [writes $P' - 2P = 2t$], then you can use an integrating factor method [he begins solving for $P(t)$].

To solve for $P$, Hakeem utilized the *method of integrating factor*; he found the integrating factor [$e^{-2t}$], multiplied both sides of the differential equation by this factor, and then integrated both sides of the differential equation; his work can be seen in Figure 4.4. After he completed the integration, I asked him why it was important to note that the differential equation was a linear first order differential equation. He replied, "cause then there is a method that always works …then I can just use a go to method like integrating factor" (turn 123). This indicates that

Hakeem was attending to the form of the differential equation to determine an appropriate solution method.

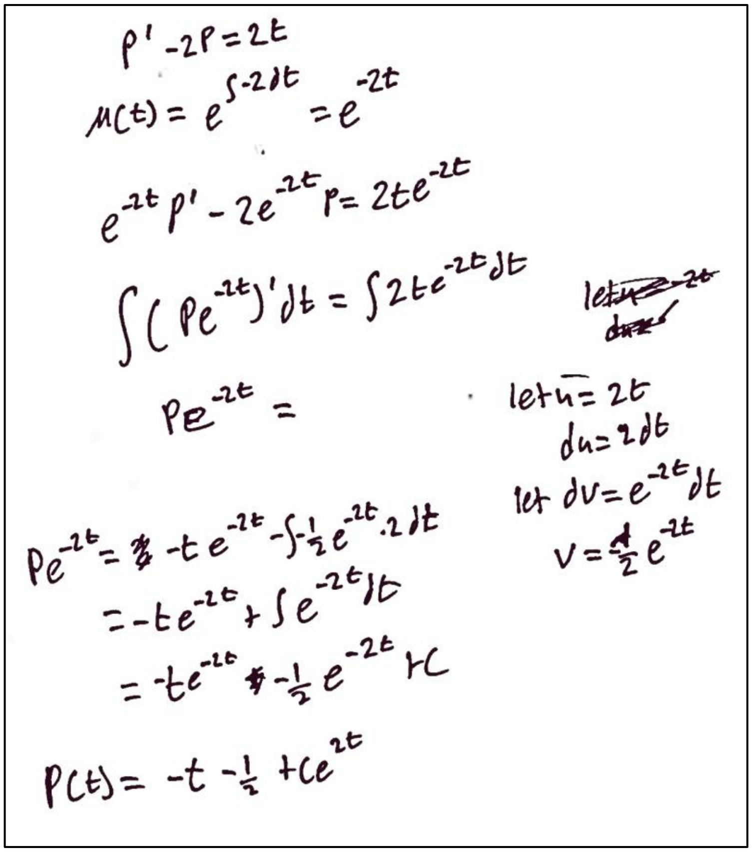



*Figure 4.4: Hakeem's work showing his utilization of the method of integrating factors.*

In addition to utilizing this solution method, he provided a justification for the effectiveness of the method and why it was appropriate in this situation. After Hakeem attained the solution $P(t) = -t - \frac{1}{2} + ce^{2t}$, I asked him why we integrate when solving the differential equation, to which he replied, "the integral is just the anti-derivative and in a differential equation you have a derivative, the $\frac{dP}{dt}$, so to get rid of that you would take the integral of it, and it becomes a function of just $t$" (turn 125). Here Hakeem used the resource *opposites operations cancel* while explaining that integrating is appropriate when "solving" differential equations because it "gets rid of" the derivative operation. That is, Hakeem used opposite operations to manipulate the differential equation into “a function of just $t$.” It is important to note that while using this resource he was interpreting integrals as “just the anti-derivative” and $\frac{dp}{dt}$ as “the derivative.” The correlation between the resources and interpretations supported Hakeem in being able to “get rid of that [$\frac{dp}{dt}$]”, leaving just a function of $t$.

After discussing why integrating is appropriate, I then prompted Hakeem to discuss the role of integrating in the solution process. In response, Hakeem discussed the relationship between the differential equation and its solution by utilizing the resource *functional dependence*. To do so Hakeem made use of the words independent and dependent variables quite frequently. Moreover, comparing such instances revealed that Hakeem used these terms in ways that could be interpreted as contradictory, though I claim should not be. As the following analysis shows, Hakeem thought of $P$ as both an independent and dependent variable, and he did so in systematic ways. This can be seen in the following segment in which Hakeem discusses solutions to differential equation in terms of dependent and independent variables.

130 G: And what we get by integrating is what?
131 H: Ah, well you don’t, well generally you don’t have any rates left, and you just have a, an equation of just your dependent variable and your independent variable. And you just solve it.
132 G: Ok, and what is that equation mean?
133 H: So for here, that would be this equation right here.
134 G: Ok and what does that equation give us?
135 H: Ah, that'll let you solve for $P$, or the dependent variable.

Considering this segment of transcript in conjunction with talk turn 101, for Hakeem solutions are equations “of just your dependent variable and your independent variable,” which you can then use to solve for the dependent variable. To make this inference he utilized the resource *functional dependence* with regard to the relationship between $P$ and $t$ when he notes that one can “solve for” (turn 135) or determine the dependent variable using the independent variable. While this is mathematically correct with regard to the equation resulting from the solution process, $P(t) = \frac{t-1}{2} + ce^{2t}$, he extended these characterizations to the differential equation itself as evidenced in talk turns 138-150 that follow.

138 G: And what is it the dependent variable of?
139 H: Ah so in this [differential] equation the dependent variable is the fish population, $P$.
140 G: Ok, so let me ask you this: $P' = 2p + 2t$, you said that the dependent variable is $P$?
141 H: Yeah, oh it's dependent on $t$, time.
142 G: Ok, let me think about this. The independent variable in this differential equation is $t$?
143 H: Yes, the dependent variable is $P$, the independent is $t$.
144 G: Ok. Is $P'$, are you thinking of $P'$ as a variable?
145 H: Well no, cause, $P'$ is a function, you could think of it as just a function of those two variables. I don't really think about it as a separate variable.
146 G: Ok, so this [$P$'] is a function of these two things? This guy is the independent variable [$t$], this guy is the dependent variable [$P$]?
147 H: yeah and, like, $P$ is, $P'$ is just the derivative of $P$. I don't, it's not, really a separate variable.
148 G: Ok
149 H: It's like saying, if you had like a $P^2$, the derivative is just another operation.
150 G: Ok, what makes you say that $t$ is independent variable and $P$ is the dependent variable?
151 H: Well ah, if you are just given the equation, the fact that they have a $P'$, or if this was also written to say $\frac{dP}{dt}$, then you know it's $P$ that is changing with $t$.
152 G: Ok
153 H: Then since we have this word problem here, we are given that it is used to model the fish population $P$ and at a time $t$. So, we are given what these variables mean, and we know that time is independent, population is dependent on time.

In turn 145 and 151 Hakeem utilized the resource *functional dependence* with respect to the differential equation, and the solution $P(t)$, respectively. In turn 145, referring to $P'$ he noted "you could think of it as just a function of those two variables," and in turn 151 he said, "or if this was also written to say $\frac{dp}{dt}$, then you know it's $P$ that is changing with $t$." More specifically, Hakeem used *functional dependence* to infer $P'$ depended on the variables $P$ and $t$, and that $P$ depended on the variable $t$. This helps explain why Hakeem was referring to $P$ as both an independent variable (it determined $P'$), and a dependent variable (it was determined by $t$). Additionally, in turn 153 Hakeem himself noted that he used the word problem to read out "that time is independent, population is dependent on time." While he notes that $P'$ is a function of $P$ and $t$, and that $P$ is dependent on $t$, in reference to the differential equation he refers to $P$ as the dependent variable and $t$ as the independent variable (talk turns 139 and 143).

At this point, Hakeem had expressed that $P$ and $t$ are inputs in the differential equation, dependent and independent variables in the differential equation, and dependent and independent variables in the function $P(t)$. During the interview, it was unclear how Hakeem was (if at all) differentiating between inputs and outputs, and independent and dependent variables, so I asked him about this explicitly. The following transcript begins just before I ask this question.

193 H: Because, like when say you take this equation [$P(t) = -t - \frac{1}{2} + ce^{2t}$], ah, $P$ changes with $t$, but depending, like that's why they are initial value problems, you have this constant $C$ here and that in this, this constant ah, depends on $P$. In other equations where you have $P = P_0$, like that one or $e$ to $kt$ I guess, like this one if more obvious that like the initial value of $P$ right here is what defines the equation almost.
194 G: So let me go to this for a second. Earlier you had said that you think of this [points to $P$] as the dependent variable and this [points to $t$] as the independent variable
195 H: Yes
196 G: And then just a few seconds ago or minutes ago, you said that you think of these two [points to $P$ and $t$] as inputs.
197 H: Yeah see you can think about it as both, the reason I say you can think about it as inputs is because, because we are not given an initial value of $P$, the initial value could be anything, like once we have an initial value then it [$P$] is pretty much set as a dependent variable.
198 G: So is it that um, I think I, let me see if I am thinking about this the same way you are. If I had $P'$, if I had $P$ like this [points to $P(t) = -t - \frac{1}{2} + ce^{2t}$], are you thinking you can take this $P$ and put it in here [points to the differential equation] and then this [$P'$] becomes dependent on time and that's why this is independent [$t$] and this is dependent [$P$]?
199 H: I mean when I was thinking of $P$ as an input, I was thinking that ah, almost separate from this equation [points to $P(t) = -t - \frac{1}{2} + ce^{2t}$].
200 G: Ok
201 H: I was thinking about it in terms of, we know that, we know that $P$ is the dependent variable so it is changing as time changes. But as $P$ changes it's also affecting its own rate because of this equation [points to differential equation].
202 G: so when you say it's affecting its own rate, um, is it still a dependent variable at that point?
203 H: ah I would say so.
204 G: I guess are you still thinking of it as a dependent variable?

205 H: Well yeah I am because I mean if you think about it, also say, it is affecting its own rate, but that's because $P$ itself is a function of $t$ so if we did plug this [the solution function $P(t)$] into this equation [the differential equation] you'd get you wouldn't have $P$ any more, it would like just be $P'$ as a function of $t$.
206 G: Ok
207 H: But it's just that this part is modeled as what $P$ is.

In turn 199 Hakeem noted that when he thinks of $P$ as an input, he thinks of it as almost being separate from the equation $P(t)$. In 201 he suggested that $P$ is an input in the differential equation because "as $P$ changes it's also affecting its own rate," which is indicative of Hakeem utilizing *functional dependence* with regard to the relationship between $P'$ and $P$ in the differential equation. In turn 197 he expressed $P$ being a dependent variable when referring to $P(t)$, which is found by using the initial value of $P$; here the function $P(t)$ was dependent on the initial condition (*dependence*) and $P$ was dependent on $t$ (*functional dependence*). To support this Hakeem noted, "once we have an initial value then it [$P$] is pretty much set as a dependent variable" (turn 197). I take the language "pretty much set" to indicate that the specific $P(t)$ equation depends on the initial conditions (*dependence*), which set the specific nature of $P$'s dependence on $t$ (*functional dependence*). Paraphrasing, Hakeem was arguing that without an initial condition there is no way to determine a value of $P$ given a value of $t$, and in this case $P$ is an input the differential equation. On the other hand, once $P(t)$ is set by the initial condition, $P$ is now a dependent variable as one can determine a value of $P$ given a value of $t$. This matches with his statement in turn 205 when he suggested that $P$ is a dependent variable "because $P$ itself is a function of $t$." In other words, for Hakeem $P$ is an input in reference to it's relation with $P'$, but a dependent variable in reference to it's relation with .

Putting this together allows us to characterize Hakeem's last statement "so if we did plug this [the solution function $P(t)$] into this equation [the differential equation] you'd get, you wouldn't have $P$ any more, it would like just be $P'$ as a function of $t$" (line 205). In this case, Hakeem expressed inputting a dependent variable $P$ into the differential equation to justify that $P$ is affecting its own rate. He used functional dependence to characterize the relationship between $P'$ and $t$, noting that part of the equation that defines the nature of that dependence is described by $P$: "this part is modeled as what $P$ is" (207). This demonstrates Hakeem's fairly complex understanding of the differential equation and the various ways the variables are related. In short, $P$ (and $t$) are inputs in the differential equation in the absence of an explicitly defined function $P(t)$, which depends on the initial condition. However, $P$ is a dependent variable with regard to $t$, (as is $P'$). Unpacking this complex network of dependence between the variables and the role of the initial condition in determining the specific equation relating $P$ to $t$ is important because it clarifies exactly what results from the process of solving the differential equation. Namely, "an equation of just your dependent variable and your independent variable" (turn 131) that lets you find the value of the dependent variable given a value for the independent variable ("you just solve it"; turn 131).

In summary, to complete Task 3 Hakeem made use of *the method of integrating factor, opposites cancel*, *functional dependence* and *dependence*. Hakeem utilized *the method of integrating factor* to find "a solution… some equation of $P(t)$" (turn 101), which was supported by the resource *opposite operations reverse,* which he used to justify why integration was appropriate with regard to solving differential equations. Hakeem applied the resource *functional dependence* to both the differential equation and the solution; he expressed interpreting $P'$ as a function of $P$ and $t$, and $P$ as a function of $t$. Moreover, Hakeem noted that he thought of $P$ as both a dependent variable, and though he did not explicitly utilize it as such, as an independent variable in the differential equation. In addition Hakeem's explanation of the solution included *dependence* where he outlines how initial values determine the exact solution function relating values of $P$ to values of $t$. Lastly, based on this, I infer that Hakeem interpreted the differential equation as an equation to be solved. These resources and interpretations of $P$ allowed Hakeem to reason both about the process of solving the differential equations, and the result of the process. For example, recall turn 131 when Hakeem explained the result of integrating: "well you don't, well generally you don't have any rates left, and you just have a, an equation of just your dependent variable and your independent variable. And you just solve it." In this case Hakeem used *opposites cancel* to reason that you are left with an equation for dependent and independent variables (*functional dependence*) that allows you to determine $P$.

**4.1.1.4 Interview 1 Task 4.** While completing Task 4 (see Figure 4.5), Hakeem used the differential equation as a relationship between a function and that function's derivative. This is indicated by his interpretation of $y'$ and $y$ as functions, and his coordination of the resources *dependence* and *equality condition.* More specifically, he expressed that the differential equation sets a condition that the function must satisfy. Further, he did so in a rather sophisticated way, arguing that not just any function will meet those conditions because a function's derivative is dependent on the function. In other words, Hakeem argued that different functions would have different derivative functions and that only certain functions would satisfy the differential equation. In addition I argue that Hakeem's understanding of what constitutes a differential equation expanded as a result of working on this task. When Hakeem started reasoning about the task, he was unsure of what differential equation would have the solution $y = 6$, despite being aware of and explicitly mentioning that "$y'$ has to be zero." He then convinced himself that $y' = 0$ is in fact a differential equation. I begin the presentation of this analysis with transcript of Hakeem's first statements.

TASK 4: Provide a differential equation which has a solution of $y = 6$

*Figure 4.5: Task 4 from Interview 1 (I1-T4).*

211 H: Well if the solution is $y = 6$ then we know the derivative, we'll just call it $y'$ has to be zero.
212 G: Ok, how are you thinking about what it means for $y$ to be equal to 6? In other words, how did you come to the conclusion that if $y = 6$, $y'$ must be zero? What did this tell you?
213 H: Ah well, you would say that $y$ doesn't depend on anything, and it's a constant value. So the rate is zero because $y$ isn't changing.

In talk turn 211 Hakeem considered the solution $y = 6$, immediately recognized $y'$ is 0; "we know the derivative… has to be zero." In turn 213 he justified this using the resource *functional dependence* (or the lack thereof) noting, "$y$ doesn't depend on anything … so the rate is zero because $y$ isn't changing" in turn 213. In this case Hakeem noted there is no dependence and as such the quantity does not change, from which he inferred "the rate is zero because $y$ isn't changing." He temporarily got stuck here, however, stating that he wasn't sure of any other way of "going back" to the solution $y = 6$ without using the equation $y' = 0$. It was unclear if Hakeem was thinking of $y' = 0$ as the differential equation with a solution of $y = 6$ or as a characteristic of such a solution, so I asked him about this:

220 G: Ok. Are you saying that this [points to $y' = 0$] is a differential equation that would have this [points to $y = 6$] as a solution or are you saying that this [points to $y' = 0$] is a consequence of this [points to $y = 6$]?
221 H: Well, at first I was thinking as this [points to $y' = 0$] as a consequence of this [points to $y = 6$], but at the same time it is technically a differential equation.
222 G: Ok
223 H: And yeah so if you set, yeah technically it is still a differential equation where if you solve it you take the integral of $y'$ is equal to the integral of zero, so you'd get $y$ equals some constant $c$.
224 G: Ok
225 H: And, $y = 6$ is a solution.
226 G: Ok. What does it mean for $y = 6$ to be a solution to a differential equation? Or, what does it mean for anything to be a solution to a differential equation?
227 H: Well it just means that, ah, so the thing about a differential equation is that it has $y'$ and $y$ in the same equation.

I interpret the phrase "yeah, technically it is still a differential equation" (talk turn 223) as indicating Hakeem's initial uncertainly regarding the equation $y' = 0$. More specifically, Hakeem was uncertain that $y' = 0$ was a differential equation because it did not have both $y$ and $y'$. This is supported by his statement "so the thing about a differential equation is that it has $y'$ and $y$ in the same equation" (talk turn 227). In talk turn 223, he sought to reconcile why $y' = 0$ is the only way of "going back" to the solution. To do so, he seemingly utilized the resource *opposite operations reverse* by integrating (anti-differentiating) "the derivative" $y' = 0$ to conclude that $y = 6$ is indeed a solution to the differential equation. He then went on to describe what it means to be a solution to a differential equation (transcript follows) and described solution as something that satisfies a condition of equality as set by the differential equation. I now focus the rest of the discussion on supporting this claim.

229 H: $y'$ and $y$ in the same equation. So, ah not every value, err, any $y$ equals, not any function can be used as $y$ because, for solving differential equations because, because it's like, well the derivative is an operation on $y$ so, for say $y'$ to equal $y$ for example, like there is only certain equations [for $y$] that have for which this is true. So in a sense the differential equation is almost like a condition.
230 G: A condition for what?
231 H: For what $y$ can be.

Further elaborating on what he means by "the differential equation is almost like a condition" Hakeem relates the differential equation $y' = y$ to the equation $x^2 = x$.

239 H: … Yeah $y'$ in a sense you could also call it, say that, well, you could say it depends on $y$, or basically like for every, for a certain $y$ it has its $y'$ or has its derivative, like you can't they are not independent of each other. Like there, because the derivative is just another operation on $y$, so $y'$ and $y$ are like, they have a certain relationship like you can't just ah, so it, it's almost like just a regular linear equation where we said $x^2 = x$, you cant just say like, or say $x^2 = x$, or just $x$, then you know $x = 1$.

In talk turns 229 and 239 Hakeem expressed that not just any function solves a differential equation: it must be a function for which the differential equation remains true. Further, Hakeem noted that $y'$ is dependent on $y$, "for a certain $y$ it has its $y'$ or has its derivative, like you can't they are not independent of each other," which is indicative of *dependence*; $y$'s derivative function is dependent on $y$. In turn 229 and 231 Hakeem also said, "the differential equation is almost like a condition" about what $y$ could be. In this case, Hakeem is arguing that for the differential equation to be satisfied (remain true) $y$ must satisfy certain conditions. Here Hakeem used the resource *equality condition* to infer that the differential equation only remains true for certain functions of $y$.

The coordination of the resources *dependence* and *equality condition* in this particular task indicates a very different interpretation of the differential equation from those in the previous tasks. In this instance Hakeem interpreted the differential equation as representing equality between functions. In talk turn 239 Hakeem used *equality condition* to explain why only certain functions are solutions to certain differential equations. Namely only certain functions satisfy the differential equation: "there is only certain equations [for $y$] that have for which this [differential equation] is true." In other words, Hakeem is applying the resource *equality condition* in a context where the differential equations are relating functions; more specifically, a context where the equation is relating a function and its derivative. This is supported by a later statement: "so *when you have this equation*, here, you're basically, *you basically have the restriction on what $y$ or $y'$ can be as a function of say $t$* or whatever the independent variable is" (line 257, italics added). In this case the equality is between a function and that function's derivative and "the condition" is that a function must satisfy the differential equation.

In summary Hakeem utilized the resource *functional dependence* to determine that for the solution $y = 6$, $y'$ was zero. He then verified this with the utilization of *opposite operations reverse* by integrating both sides of $y' = 0$ to show $y = 6$ is a solution. In addition once he convinced himself the differential equation $y' = 0$ was correct, he used *equality condition* and *functional dependence* to discuss why this was the appropriate differential equation. Furthermore, he utilized these resources while attending to $y'$ and $y$ as functions, where $y'$ was the derivative of $y$. In doing so interpreted the differential equation as a relationship between a function and its derivative function.

**4.1.1.5 Discussion of interview 1.** Over the course of the interactions with the tasks in Interview 1 Hakeem utilized four distinct ways of interpreting the various differential equations: the differential equation as an description of the behavior of the quantity of interest (I1-T1), the differential equation as providing a model of the quantity of interest (I1-T2), the differential equation as an equation to be solved (I1-T3), and the differential equation as a relationship between a function and the function's derivative (I1-T4). In addition, each of these interpretations corresponded to a set of resources that were utilized in conjunction with the interpretation to support Hakeem's completion of the task. Further, these resources and interpretations also corresponded to certain ways in which Hakeem was interpreting the various components from which the equations were composed. This is summarized in Table 4.2.

It is important to consider both the resources and with the ways in which Hakeem was interpreting the differential equation and the objects from which it was composed because they both inform different aspects of Hakeem's reasoning. For instance, in Task 3 Hakeem used *dependence* to relate initial conditions to the constant $C$ in the solution function that results from using the *method of integrating factors*, whereas in Task 4 *dependence* was used to justify why differential equations could be only be satisfied by certain functions. While the resource was the same, Hakeem utilized it on different entities. In Task 3 *dependence* was used to relate the value of the constant of integration in the solution function to the initial condition, but in Task 4 it was used to relate a function and its derivative function. As discussed further in Chapter 5, the interpretations serve as a way to identify the mathematical objects to which the applications apply (e.g., values, functions, and slopes). I now compare and contrast the relationship between the resources and the interpretations across the various tasks.

In the first task Hakeem interpreted the differential equation as a description of the behavior of the quantities of interest. In this case Hakeem interpreted $x$ and $y$ as population values, interpreted $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as the rates of change of the respective populations, and used the resources *functional variation, functional dependence, rates indicate behavior and parts contribute to the whole* (among others) to reason about his relationship to determine the behavior of the two populations. More specifically the resources supported Hakeem in determining how changes in the population of species $y$, related to changes in the population of species $x$. In doing so he relied on the relationship between the variables (i.e., $x$, $y$, $\frac{dx}{dt}$ and $\frac{dy}{dt}$) in the systems of differential equations. In the second task Hakeem interpreted the differential equation as providing a model of the quantity of interest and used resources that helped him determine an algebraic representation of this model, the function $P(t)$. When completing Task 2, Hakeem interpreted $P$ and $P'$, much as he did in Task 1, as representing the value of the population and the population's rate of change, respectively; however, Hakeem also expressed that $P$ was a function of $t$. The difference between the reasoning he used in the tasks lies in the relationship he was focusing on. In Task 1 Hakeem focused on relating values of $x$ and $y$ to values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$, whereas in Task 2, Hakeem only focused on characterizing the model of the population, $P(t)$. His reasoning in Task 2 did not rely on relating values of $P$ to $P'$ or changes in their values.

Although Hakeem utilized *rates indicate behavior,* the rate he was referring to was not the rate of change of the population $P'$, but rather the rate constant 3.

While completing Task 3, Hakeem interpreted the differential equation as an equation to be solved. This is indicated by the utilization of resources *functional dependence* and *opposite operations reverse* in conjunction with expressing that the components in the differential equation are functions. These resources and interpretations allowed Hakeem to reason about the process of solving the differential equation. That is, he saw integration as a way to get an equation that related independent and dependent variables which one could then use to find the solution. While completing Task 4, Hakeem interpreted the differential equation as a relationship between a function and that function's derivative. In addition to this he interpreted $y'$ and $y$ as functions and coordinated the resources *dependence* and *equality condition* with these functions to justify why $y' = 0$ was a differential equation with a solution of $y = 6$. In this case, Hakeem saw the differential equation as a condition functions must satisfy, and the resources Hakeem utilized supported him in reasoning about this condition to find a differential equation with the solution $y = 6$. In short, the resources Hakeem utilized in each of the tasks were largely tied to the meaning he attributed to the various differential equations and the mathematical objects from which they were composed. To complete the tasks Hakeem coordinated these interpretations and relevant resources to make inferences that supported his ability to successfully complete the tasks. In Task 1 the resources allowed him to characterize the behaviors of species $x$ and species $y$ to determine how they interacted, in Task 2 the resources supported his ability in finding the model, in Task 3 the resources enabled Hakeem to find an equation that would relate time to the fish population, and in Task 4 the resources enabled Hakeem in finding a differential equation that appropriately related the function $y = 6$ to its derivative function.

*Table 4.2: Resources and interpretations Hakeem expressed in Interview 1.*

| Task | Interpretation of the differential equations | Resources | Interpretation of the components in the differential equation |
|---|---|---|---|
| T1-I1 | Description of the behavior of the quantities of interest | *Functional dependence*<br>*Functional variation*<br>*Functional mapping*<br>*Parts contribute to the whole*<br>*Larger magnitudes have larger impacts*<br>*Positives add- negatives subtract*<br>*Rates indicate behavior*<br>*Definition of cooperative systems*<br>*Definition of competitive system.* | $x$ and $y$ - Population values, and $\frac{dx}{dt}$ and $\frac{dy}{dt}$ - Rates of change of the respective populations. |
| T2-I1 | Providing a model of the quantity of interest | *Rule for differential equations of the form* $P = kP$<br>*Rates indicate behavior*<br>*Larger magnitudes have larger impacts*<br>*Function slots*<br>*Functional dependence* | $P$ - Population value,<br>$P(t)$ - Equation that related the population values to $t$.<br>$P'$ - Rate of change of the population |
| T3-I1 | Equation to be solved | *Method of integrating factor*<br>*Dependence*<br>*Opposite operations reverse*<br>*Functional dependence* | $P(t)$ – Solution function<br>$P'$ – Function of $p$ and $t$ |
| T4-I1 | Relationship between a function and the function's derivative | *Functional dependence*<br>*Opposite operations reverse*<br>*Dependence*<br>*Equality condition* | $P$ and $P'$ - functions where $P'$ was the derivative of $P$ |

### 4.1.2 Interview 2.

In this interview Hakeem completed three tasks; the analysis of each of them is presented in this section. Hakeem expressed interpreting the differential equations in two ways while completing the three tasks, as a description of the behavior of the quantity of interest, and as relationship between the value of quantities and the value of the rate of change of the quantity of interest. I discuss the relationship between the resources and the interpretations in the discussion at the end of this section.

**4.1.2.1 Interview 2 task 1.** Task 1 from Interview 2 was the same as Task 1 from Interview 1. While completing this task for the second time Hakeem again interpreted the differential equations as a description of the behavior of the quantity of interest. I made this determination based on the resources Hakeem utilized and the meaning he associated with the components from the differential equation with which he was utilizing these resources. Namely he interpreted $x$ and $y$ as population values and $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as the rates of change of the populations at a certain instance, and he coordinated resources that allowed him to relate values of $x$ and $y$ with values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ to make inferences about the behavior of $x$ and $y$. Hakeem used the following resources to complete Task 1: *larger magnitudes have larger impacts*, *parts contribute to the whole, functional mapping, rates indicate behavior, unrestricted change, the definition of cooperative systems, positives add- negatives take away, functional variation,* and *the definition of competitive systems.* Further, although Hakeem may have been aware that $x, y, \frac{dx}{dt}$, and $\frac{dy}{dt}$ could be thought of as functions, he did not appear to utilize them as such while reasoning about the differential equation in this task. He utilized the variables as values of quantities or sometimes as sets of values of quantities with certain (perhaps ill-defined) characteristics: being large or small, and positive or negative.

In this interaction, Hakeem began by recalling this task from the first interview; however, the extent of his recall is not clear (e.g., remembering the solution, the approach, the wording of the task). The following transcript starts just as Hakeem finished reading the task out loud and ends with him determining system A is cooperative.

2 H: So, [long pause]. I think I remember this problem. So, I think the way you look at it is, when you separate the two terms and the coefficients, that's how you tell if it is cooperative or ah competitive. So, here, since there are two species $x$ and $y$, and to say, if they were competitive as $y$ grew, $x$ would have to, $x$ would be shrinking. And it would be visa versa, as $x$ grew $y$ would have to be shrinking and this would be for competitive. But then for cooperative species, if $y$ was benefiting and growing, then $x$ would start benefiting and growing, and if $x$ was benefiting and growing then $y$ would also start benefiting and growing. So, you, say for cooperative, we'd have as $y$ goes to infinity, $x$, I'm not going to say to infinity, but just as $y$ grows $x$ grows and as $x$ grows $y$ grows.

3 G: Ok.

4 H: Not necessarily to infinity but, so if we look at this differential equation here, $\frac{dx}{dt}$ is equal to $-5x + 2xy$, so the term that has just $x$ is negative and the term that has both $x$ and $y$ is positive. So, that means if $x$ was really large ah, the $\frac{dx}{dt}$ would probably be negative.

5 G: Ok.

6 H: But if say $x$ was small and $y$ was large, $\frac{dx}{dt}$ would be positive so say if we look at this equation [begins writing what he just said] if $x$ was how did I say, large, then $\frac{dx}{dt}$ would be negative, but otherwise $\frac{dx}{dt}$ would be positive. And so if $\frac{dx}{dt}$ is negative that means $x$ is shrinking. So say $x$ is a really big number, this equations means that it is going to start shrinking again, so, and it's the same thing for $\frac{dy}{dt}$, the term that is just $y$ is negative and the term that has both is positive.

7 G: Mhm.

8 H: So what that means is if $y$ is really large $\frac{dy}{dt}$ is going to be negative, but if $x$ is large, $\frac{dy}{dt}$ is going to be positive. So what that means is, when you put both of those equations together, if $x$ is really large and $y$ is really small $\frac{dx}{dt}$ is going to be negative and $\frac{dy}{dt}$ is going to be positive. And at that point that means $x$ is gonna start shrinking and $y$ is gonna start increasing, and if $y$ is really large and $x$ is really small $\frac{dy}{dt}$ is gonna be negative and $\frac{dx}{dt}$ is positive so then $y$ will start shrinking and $x$ will start growing. So, basically what that means if over time, like say you start off with $x$ is really large and $y$ is really small, [writing; see Figure 4.6] so start off with $x$ is large and $y$ is small so then $\frac{dx}{dt}$ will be negative and just like $\frac{dx}{dt}$ will be negative $\frac{dy}{dt}$ will be positive. So then what is going to happen is $x$ is gonna be shrinking and $y$ will be growing until $y$ is large and $y$ is small, and then at that point $\frac{dx}{dt}$ will be positive and $\frac{dy}{dt}$ will be negative. And then they would just reverse again and keep going

> back between $x$ being big and $y$ being small and then $y$ being big and $x$ being small. So that would make sense, $y$ growing, makes $x$ start growing. $y$'s growth benefits $x$ and vice versa. So that makes system A, a cooperative system.

In talk turn 4, Hakeem separated the terms and coefficients noting "$\frac{dx}{dt}$ is equal to $-5x + 2xy$, so the term that has just $x$ is negative and the term that has both $x$ and $y$ is positive. So, that means if $x$ was really large ah, the $\frac{dx}{dt}$ would probably be negative." Here he coordinated the resources *larger magnitudes have larger impacts* and *parts contribute to the whole* to infer that large values of $x$ corresponded to negative values of $\frac{dx}{dt}$. More specifically, when he considered the case of $x$ being really large he coordinated the two resources, *larger magnitudes have larger impacts* and *parts contribute to the whole* to determine that the $x$ "part" would have a larger impact on the "whole" ($\frac{dx}{dt}$) than the $xy$ "part", and inferred that $\frac{dx}{dt}$ would thusly be negative. I make these claims based his explicit attention to the various parts on the right hand side of the equation. In turn 6 Hakeem repeated this reasoning for a second case, "if say $x$ was small and $y$ was larger, $\frac{dx}{dt}$ would be positive." Having correlated the different cases of population size to values of $\frac{dx}{dt}$, his attention shifted to $\frac{dy}{dt}$. At the end of turn 6 Hakeem noted that the signs of the corresponding terms on the right hand side of the equations (for $\frac{dx}{dt}$ and $\frac{dy}{dt}$) were the same: "so, and it's the same thing for $\frac{dy}{dt}$, the term that is just $y$ is negative and the term that has both is positive." From this he inferred $\frac{dy}{dt}$ would be correlated to the two cases of population size in a way similar species $x$; "So what that means is if $y$ is really large $\frac{dy}{dt}$ is going to be negative, but if $x$ is large, $\frac{dy}{dt}$ is going to be positive" (turn 8). This reasoning process is also indicative of the resource *functional mapping*. More specifically, Hakeem mapped sets of values of $x$ and $y$ to sets of values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ (see Figure 4.6), which, as outlined in the following paragraph, had a central role in how Hakeem determined System A was cooperative.

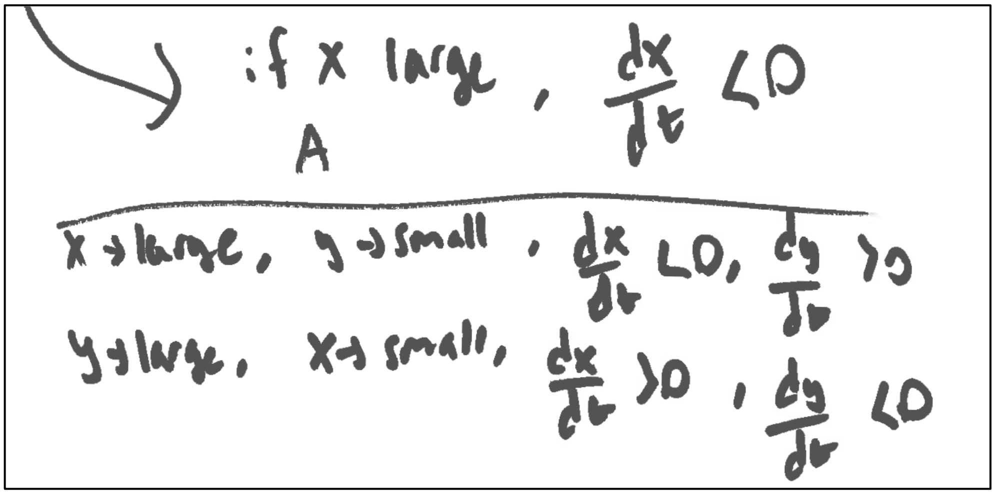



*Figure 4.6: Hakeem correlated values of $x$ and $y$ with values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$.*

Upon determining how the signs of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ correlated to different combinations of sizes of populations of $x$ and $y$, he then used *rates indicate behavior* and *functional mapping* to relate the population values to the behavior of the population for the two cases ($y$ being small and $x$ being large, and $y$ being large and $x$ being small). This can be seen in the following statement:

> "If $x$ is really large and $y$ is really small $\frac{dx}{dt}$ is going to be negative and $\frac{dy}{dt}$ is going to be positive. And at that point that means $x$ is gonna start shrinking and $y$ is gonna start increasing, and if $y$ is really large and $x$ is really small $\frac{dy}{dt}$ is gonna be negative and $\frac{dx}{dt}$ is positive so then $y$ will start shrinking and $x$ will start growing" (turn 8).

Here Hakeem reasoned with the sign of the rates of change $\frac{dx}{dt}$ and $\frac{dy}{dt}$ to make inferences about the behavior of $x$ and $y$ (*rates indicate behavior*) corresponding to each set of population values (*functional mapping*). That is, he inferred that (condensed for clarity):

> "If $x$ is really large and $y$ is really small … $x$ is gonna start shrinking and $y$ is gonna start increasing, and if $y$ is really large and $x$ is really small… then $y$ will start shrinking and $x$ will start growing" (turn 8).

Having correlated the two cases of $x$ and $y$ values to their behavior Hakeem then coordinated *unrestricted change* to infer that the population values would "keep going back between $x$ being big and $y$ being small and then $y$ being big and $x$ being small" (turn 8). This can be seen in the following excerpt from turn 8:

> "So, basically what that means if over time, like say you start off with $x$ is really large and $y$ is really small, [begins writing this], so start off with $x$ is large and $y$ is small so then $\frac{dx}{dt}$ will be negative and just like $\frac{dx}{dt}$ will be negative $\frac{dy}{dt}$ will be positive. So then what is going to happen is $x$ is gonna be shrinking and $y$ will be growing until $y$ is large and $y$ is small. And then at that point $\frac{dx}{dt}$ will be positive and $\frac{dy}{dt}$ will be negative. And then they would just reverse again and keep going back between $x$ being big and $y$ being small and then $y$ being big and $x$ being small."

Here Hakeem coordinated the mappings he created concerning the sizes of the population and their behavior. More specifically, he noted that if $x$ was really large (and $y$ was really small) it would be decreasing (and $y$ would be increasing), and used the idea that the population of $x$ would decrease to the point that it became really small (and $y$ would increase to the point that it became really large), at which point, as dictated by the correlation he established earlier (Figure 4.6), the signs of $\frac{dx}{dt}$ (and $\frac{dy}{dt}$) would switch. This would cause $x$ to increase to some maximum value (and $y$ to decrease to some minimum value), and the process would start again. In this case Hakeem utilized *unrestricted change* to infer that the populations would increase (or decrease) without restriction. The lack of a restriction is not on the values of $x$ and $y$, but rather on the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Unrestricted change, in this case is about the effect $y$ can have on $\frac{dx}{dt}$; $y$ can change "enough" so that $\frac{dx}{dt}$ can change so that it goes from being positive to being negative. He then attributed the oscillatory nature of the values of $x$ and $y$ to that of a cooperative system: "So that would make sense, $y$ growing, makes $x$ start growing, $y$'s growth benefits $x$ and vice versa. So that makes System A, a cooperative system" (turn 8). This is indicative of *his definition of cooperative systems.* He then switched his focus to system B.

18 H: So, if this was a competitive system, we look at it, again the term that has just $x$ is positive and the term that has both is negative. So, you can really just tell by these signs right here, the negative being in front of the $y$ term means that $y$'s growth is basically harmful to $x$, $y$ is like making $\frac{dx}{dt}$ more negative. And the same here, for $\frac{dy}{dt}$, $x$'s growth makes $\frac{dy}{dt}$ negative. So that's what makes them harmful to each other.

In turn 18, Hakeem utilized a different strategy than he used when reasoning about System A. Instead of relating the size of the population to the sign of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and then using the sign of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as a proxy for relating the values of the populations to their behavior, he directly attended to how the $y$ term impacted the value of $\frac{dx}{dt}$ and how the $x$ term impacted $\frac{dy}{dt}$. This is indicative of the resources *partial dependence*, *positives add – negatives take away*, and *functional variation.* Namely, Hakeem coordinated these three resources to determine how changes in $x$ related to changes in $\frac{dy}{dt}$ and how changes in $y$ related to changes in $\frac{dy}{dt}$. This coordination is evident in the statement, "the negative being in front of the $y$ term means that $y$'s growth is basically harmful to $x$, $y$ is like, making $\frac{dx}{dt}$ more negative*"* (turn 18). In this statement Hakeem only considered how $y$ impacted $\frac{dx}{dt}$ (*partial dependence*), used the negative sign in front of the $y$ term to infer that "$y$ is like making $\frac{dx}{dt}$ more negative" (*positives add, negatives take away*), and considered how change in $y$ related to change in $\frac{dx}{dt}$ (*functional variation*). Lastly, he coordinated *the definition of competing systems* with his determinations that the growth of one population harmed the other to characterize system B as competitive. This is evident in his last statement in turn 18 "so that's what makes them harmful to each other." I now briefly compare his reasoning with System A to that of System B.

Compared to the resources Hakeem utilized when he was considering System A, when reasoning with System B Hakeem no longer needed to unpack the how the parts of the right hand impacted the value of the left hand side (at the very least he no longer made these explicit); he used the sign in front of the terms to make inferences about how changes in the values of $x$ and $y$ related to changes in the values of $\frac{dy}{dt}$ and $\frac{dx}{dt}$, respectively. That is to say, Hakeem considered the sign of the variables to directly recognize (or figure out) how changes in one quantity related to changes in the other; "the negative being in front of the $y$ term means that $y$'s growth is basically harmful to $x$" (turn 18). In both cases, however, his reasoning is indicative of interpreting the differential equation as a description of the behavior of the quantities of interest. This is evident in a statement Hakeem made summarizing the process he went through to determine which system was competing and which was cooperative.

> "I was looking at what is the effect of $x$ and $y$ on $\frac{dx}{dt}$ but then after you figure that out, it's what the effect of $\frac{dx}{dt}$ on $x$. So then at $\frac{dx}{dt}$ being negative, means it's $[x]$ going from 10 to something lower than 10" (turn 67).

From excerpt this we see that Hakeem was utilizing $x$ and $y$ as values that determine the value of $\frac{dx}{dt}$ and then using the value of $\frac{dx}{dt}$ to make inferences about changes in the value of $x$. For Hakeem, while the equation relates $(-5x + 2xy)$ to $\frac{dx}{dt}$, the values of $x$ and $y$ are changing, which causes $\frac{dx}{dt}$ to change as well. In this way the differential equation is not just a statement of equality between the right and left hand side; rather, it's a relationship between variables (e.g., "$x$ was small and $y$ was large, $\frac{dx}{dt}$ would be positive" and "if $x$ is really large and $y$ is really small $\frac{dx}{dt}$ is going to be negative…" (turn 8) that describes the behavior of the quantities of interest ("…so then what is going to happen is $x$ is gonna be shrinking and $y$ will be growing"; turn 8).

Lastly I present a segment of transcript in which Hakeem expressed that differential equations are representations of functions, which may illuminate why some of the resources were applicable in the completion of this task.

40 G: Ok. Ok, Um, so let me just ask you a few questions about how you think about things in here. Um we talked about $\frac{dx}{dt}$ a little bit, um you said that you thought about plugging numbers in, so, as when $x$ is large and $y$ is small you can see what happens with this. Are you, how do you think about $x$ or $y$. What are $x$ and $y$?
41 H: So those are both the two different populations but when you are looking at it in this differential equation, I was just thinking of them as more as of just variables. So like you can plug in any number and you will get an output to this equation.
42 G: Ok. What is the, so the input it sounds like you are saying is the population, the number of species in that population. What is the output?
43 H: I guess the output would be the rate. Well, just in the way that this equation is set up you could say that $\frac{dx}{dt}$ is a function of $x$ and $y$.
44 G: Ok.
45 H: So you could say that this input, for this equation you could conceptualize it as the input being $x$ and $y$ and the output being the derivative. And that's basically how I approached it, so I thought of putting in a, like seeing what happens if, as if like $x$ is large and $y$ is small, what is the output it to, like what is the effect on the derivative.

In this segment of transcript Hakeem offered a characterization of the differential equations themselves in which he made use of *functional dependence*; "just in the way that this equation is set up you could say that $\frac{dx}{dt}$ is a function of $x$ and $y$" (turn 43). This has direct implications concerning the applicability of the resources Hakeem used to complete this task. Here Hakeem characterized the variables $x$ and $y$ as inputs in the differential equation and the variables $\frac{dx}{dt}$ and $\frac{dy}{dt}$ (their rates of change), as outputs; as is both identified in my analysis, and is reported by Hakeem (turn 45), this is precisely the relationship Hakeem was utilizing to determine the behavior of the populations of species $x$ and species $y$. This suggests that some of the resources may have been utilized because of their applicability (as seen by Hakeem) to functions.

In summary Hakeem interpreted the differential equations as a description of the behavior of the quantities of interest. In addition he thought of the variables $x$ and $y$ as the population values of the respective species, which were inputs in the differential equation, and he thought of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as the value of the rates of change of the species, which were outputs in the differential equation. Additionally he utilized the following resources to complete the task*: larger magnitudes have larger impacts, parts contribute to the whole, functional mapping, rates indicate behavior, unrestricted change the definition of cooperative systems, positives add-negatives take away, functional mapping,* and *the definition of competitive systems.*

Hakeem used the resources *larger magnitudes have larger impacts, parts contribute to the whole* and *functional mapping* when reasoning with System A to relate the values of the populations to their respective rates of change. He then used *rates indicate behavior* and *functional mapping* to determine the how the size of the populations related to their behavior. He then used *unrestricted change* to infer that populations would have oscillatory behavior indicating the growth of one species is beneficial to the other species, which was consistent with *the definition of cooperative systems*. When reasoning with System B Hakeem used the resources *partial dependence, positives add – negatives take away,* and *functional mapping* to determine that the growth of one species was harmful to the other species. He then used *the definition of competitive systems* to characterize the relationship between the two populations as competitive.

**4.1.2.2 Interview 2 task 2.** The second task (see Figure 4.7) in interview 2 asked students to interpret a second order differential equation. This interview occurred before the students were formally introduced to second order differential equations in class. Despite this, Hakeem was able to utilize multiple resources to successfully complete the task. While completing the task Hakeem interpreted $P$ as both a value and function, and $P'$ and $P''$ as the value of the rate of change of $P$ and $P'$ respectively. The resources Hakeem utilized and the ways he interpreted the components from the differential equation while completing this task indicate he interpreted the differential equation as a description of the behavior of the quantity of interest. In Hakeem's words, after considering the relationships between the quantities, "in the end you are getting just $P$ is a function of time" (turn 142) which "is not just like plug in a value you get a number back, I think of it as a, well almost like continuous" (turn 145). That is to say, Hakeem viewed the differential equation almost as a blueprint for building a function. The presentation of Hakeem's interaction with Task 2 is broken into two parts; first I discuss Hakeem's initial statements where he used the differential equation to characterize the behavior of $P$, and second I outline how Hakeem utilized the relationship between quantities and their rates of change to construct a graph of $P$.

TASK 2: What does the following differential equation mean to you?

$$P'' = 3P' + P$$

*Figure 4.7: Second task from Interview 2 (I2-T2).*

In the transcript that follows Hakeem utilized numerous resources in service of characterizing the behavior of $P$ by utilizing relationships present in the differential equation.

111 H: So yeah if you look at it like how we looked at that [points to Task 1], if you look at it in terms of the equations too, like I haven't thought much of the equation yet, but if I started plugging in, so if $P$ was, so you have $P$ is positive and $\frac{dP}{dt}$ is positive, $P''$ has to be positive. So that's going in general increase. But say both were negative, then so if both were negative at, so $P$ is negative and $\frac{dp}{dt}$ is negative, that means $\frac{dP}{dt}$ being negative means $P$ is also decreasing at that point in time, and this is also this would have to be negative so it would just keep decreasing to infinity. But if you had, say this was positive and this was negative but the total equation was such that $P''$ was positive, so at that point, at that specific point in time $P$ would be decreasing but then if $P''$ is positive $\frac{dP}{dt}$ is also increasing, so it would like

change from being negative to positive and at that point $P$ would start increasing again, possibly.

In this segment of transcript Hakeem considered three cases: When $P$ and $P'$ are both positive, when $P$ and $P'$ are both negative, and when $P$ and $P'$ are different signs. While doing so Hakeem utilized the resources: *variable substitution* ("if I start plugging in"), *parts contribute to the whole* ("so you have $P$ is positive and $\frac{dP}{dt}$ is positive," "say this [$P$] was positive and this [$P'$] was negative but the total equation was such that $P''$ was positive"), *rates indicate behavior* ("$\frac{dP}{dt}$ being negative means $P$ is also decreasing"), *unrestricted change* ("then if $P''$ is positive $\frac{dP}{dt}$ is also increasing, so it would like change from being negative to positive") and *functional mapping* to correlated the values of $P$, $P'$ with the same values of $t$ ("so if both were negative at, so $P$ is negative and $\frac{dp}{dt}$ is negative, that means $\frac{dp}{dt}$ being negative means $P$ is also decreasing at that point in time"). More specifically, Hakeem utilized the resources *variable substitution* and *parts contribute to the whole* to relate values of $P$ and $P'$ to values of $P''$, and he used *rates indicate behavior* and *unrestricted change* to relate the value of $P''$ to the behavior of $P'$ and the value of $P'$ to the behavior of $P$. Lastly, Hakeem utilized *functional mapping* to consider these values as being relevant to the same values of $P$ and $t$. I now elaborate on this with an example.

His reasoning was consistent across the three cases, so I use the case when $P'$ and $P$ were different signs as an example. In this case he considered a situation in which "[$P$] was positive and [$P'$] was negative, but the total equation was such that $P''$ was positive" (turn 111, *variable substitution,* and *parts contribute to the whole*) and noted that $P$ would be decreasing (presumably because $P'$ was negative), and that $\frac{dP}{dt}$ (Hakeem switched notations, from $P'$ to $\frac{dP}{dt}$) was increasing because $P''$ was positive (*rates indicate behavior).* He then used *unrestricted change* to infer that $P'$ would become positive because it was increasing, or in other words, that $P'$ would increase enough to become positive. He then inferred that $P$ would start increasing (*rates indicate behavior*). Based on this, Hakeem reasoned with the differential equation to characterize the behavior of the quantity of interest, $P$; this is supported by the fact that in turn 111 Hakeem expressed that he was looking at this task in the same way as Task 1.

In the next few talk turns Hakeem described the nature of the relationship between $P$, $P'$ and $P''$. The following except of transcript demonstrates this relationship.

113 H: Yeah another way to look, so if you think about it ah, so $\frac{dP}{dt}$ has like a direct relationship with $P$ and, $\frac{d^2P}{dt^2}$ has a direct relationship with $\frac{dP}{dt}$ and to that effect it also has a relationship with $P$.

114 G: Ok, can you tell me a little bit about what that relationship is?

115 H: All right, so, $\frac{dp}{dt}$ dictates whether $P$ is just increasing or decreasing, or how much it's increasing how much its decreasing. But, it can, whether it's increasing or decreasing, I mean depending on if this, this is the value of *P''* is changing *P'* so at certain points in time, $P$ is gonna be increasing or decreasing at a certain magnitude based on this, $\frac{dp}{dt}$. But then at a different point in time, ah, this $\frac{dP}{dt}$ will be different it's not going to be the same value thanks to this part of the equation *P''*. So then $P$ might not be increasing or decreasing in the same way as it was at the earlier point in time, because we have another part of the equation that is changing $\frac{dP}{dt}$.

On the surface this may not seem much different than what he stated earlier in turn 111. In turn 113, however, Hakeem not only expressed that the values of the quantities will change, but he discussed the interconnected nature of those changes. In doing so he coordinated the resources *rates indicate behavior* and *functional variation.* Paraphrasing, Hakeem noted that $P''$ is changing the value of $P'$, which in turn is changing the value of $P$ (*rates indicate behavior),* but by the value of $P$ and $P'$ changing, the value of $P''$ changes (*functional variation*) which in turn causes the value of $P'$ to change, and the process continues.

Hakeem then went on to discuss using this relationship to build a graphical representation of the function $P(t)$. To do so, Hakeem created graphical representations of the values of $P$ and $P'$ and then used the values of $P'$ and $P''$ to estimate what the value of $P$ would be at a larger value of $t$; furthermore, he was able to do so in a continuous manner. This construction can be seen in Figure 4.8; the transcript of the statements he made while explaining the construction of this drawing immediately follows Figure 4.8.

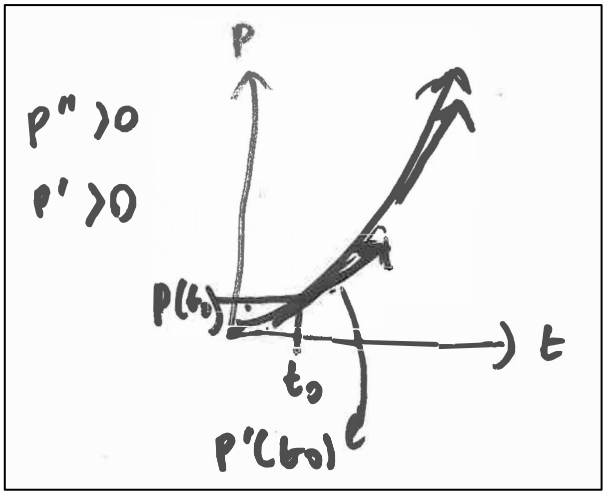



*Figure 4.8: Hakeem reasoned with values of $P$, P' and P'' to build $P(t)$.*

121 H: All right so, let's say, so at a specific point in time $P$ has a specific value, this would be $P$, I mean whatever at whatever time, $P$ of $t_0$. And this is $t_0$. At that moment of time, it's increasing, so another thing you could say is that this is also the tangent to this, if you draw that tangent then, at this direction of increase and this specific slope would be ah $P'$ of zero, would be this thing. And so we have $P'(t_0)$, so at that specific moment in time it dictates the direction, like if it is either increasing or decreasing and how much its increasing or decreasing based on the slope of this tangent line.
122 G: Mhm.
123 H: Ah if you are looking you also have different snapshots in time, this tangent is also changing and this change is represented by *P''*.
124 G: Ok, so.
125 H: So you o could also like draw a graph of like $P'$ of $t$, and it would be like a line [linear graph].
126 G: Ok. And so when you were talking about all of these things kinda effecting each other in a relationship, are you imagining this thing building, this $P$ of $t$ function building and so, know now what p is and then you can find this thing [$P$] and that tells you kind of how $P$ is changing and so then you move this way and then you can figure out what $\frac{d^2p}{dt^2}$ is and that tells you how the rate of change is changing, and then all of these things allow you to build like as you know how the numerical values of p are changing you can kind of calculate these guys and see in which direction $P$ is moving.
127 H: Yeah, basically so, if you know $\frac{dP}{dt}$ you know where to start going, if $P$ is either going up or down, over time. So like instantaneously $P$ would be going in like this direction [draws

increasing slope] but, this is also $\frac{d^2P}{dt^2}$, is changing $\frac{dP}{dt}$ so what does is, what that gives you is knowing how this changes $\frac{dP}{dt}$, or knowing this, means you know how $\frac{dP}{dt}$ changes. So, you know that it might not, the tangent isn't always going to be at this direction right here. Like you see, it's like changing from this direction to this direction.

128 G: Mhm.

129 H: So when you get the change in $\frac{dP}{dt}$, you also know that, that $P$, the direction of $P$'s change is also changing and the magnitude of $P$'s change is also changing.

130 G: Ok.

131 H: So it's like, it's like another piece of information that tells you that basically guides you about where $P$ is going.

Based on the drawing containing different tangent lines and how Hakeem mentioned "different snapshots in time" (turn 123), he seems to be considering iteratively updated values of $P$ and $P'$ in order to build the function $P(t)$. In short, Hakeem was discussing how $P''$ changes the value of $P'$ which changes the value of $P$. These new values of $P$ and $P'$ then change the value of $P''$ and the process starts again. In term of the resources, Hakeem utilized *functional mapping* to correlate the values of $P$ and $P'$ to the same point in the $P, t$ plane, and used *rates indicate behavior* to infer the value of $P$ at a later time based on the value of $P'$, and similarly the value of $P'$ from the value of $P''$. He then used *functional variation* with regard to the relationship between the variables in the differential equation, to relate the changes in the values of $P$ and $P'$ to changes in the value of $P''$, and repeated the process.

During this process, he constructed the graph by representing the values of $P$ and $P'$ as points and vectors tangent to those points (functional mapping). In addition he noted in turn 123 that "this tangent is also changing and this change is represented by $P''$"(*rates indicate behavior*). In other words, Hakeem constructed the graph by representing the changing values of $P'$ (as indicated by the value of $P''$) as tangent vectors which correlated to values of $P$ at "different snapshots in time" which together indicated future values of $P$. Here, Hakeem reasoned with the relationship between the values of $P$, $P'$ and $P''$ in the differential equation to determine the behavior of the quantity of interest.

In summary, Hakeem utilized the differential equation as a relationship between values to construct the function $P(t)$. He did so by utilizing the resources *variable substitution, parts contribute to the whole*, *rates indicate behavior, unrestricted change, functional mapping, and functional variation.* Specifically, Hakeem utilized the resources *variable substitution* and *parts contribute to the whole,* to relate values of $P$ and $P'$ to values of $P''$. He used *rates indicate behavior* and *unrestricted change* to infer the behavior of $P'$ from the value of $P''$, and the behavior of $P$ from the value of $P'$. In addition he used *functional mapping* to correlate the values of $P$, $P'$ and $P''$ to the same value of $t$ ("snapshot in time"), and *functional variation* to relate the changes in $P$ and $P'$ to changes in the value of $P''$. Further, Hakeem thought of $P$ as both a function and a value; he utilized $P$ as a value in the differential equation, but graphed the function on a $P, t$ axis. Hakeem also interpreted $P'$and $P''$ as values of the rate of change of $P$ and $P'$, respectively. When "building" the graphical representation of $P(t)$ Hakeem utilized the resource functional mapping to correlate values of $P$, and $P'$ to the same point on the $P, t$ plane, and *rates indicate behavior* by interpreting the value of $P'$ as a vector which dictated the direction of $P$. In short Hakeem interpreted the differential equation as a description of the behavior of the quantity of interest.

**4.1.2.3 Interview 2 task 3.** While completing Task 3 (see Appendix A) Hakeem utilized the resources *functional mapping*, *functional variation*, *functional dependence*, and *parts contribute to the whole*. These resources were coordinated with both the slope fields and the differential equations; I elaborate on this in the final paragraph of this subsection. Additionally, after (correctly) matching the first slope field with an equation Hakeem's approach to completing the remaining two matches changed. Specifically, he made use of the resource *functional mapping* to compare across the two representations while working with the first slope field by determining which equation related $t$ and $y$ values to values of $\frac{dy}{dt}$ in the same way the vector field related $t, y$ points to slope vectors. After determining which equation matched slope field A, however, his reasoning then centered around the utilization of *functional variation* to make the remaining matches. More specifically, he switched focus to matching the nature of how the value of $\frac{dy}{dt}$ changed with changes in the values of $y$ and $t$ (from the differential equations) with the nature of how the slopes changed as the points moved the $t, y$ plane (from the slope field). In either case, however, I claim that Hakeem interpreted each of the differential equations and the vector fields as a relationship between the values of quantities and the rate of change of the quantity of interest.

I start the discussion of the resources Hakeem utilized with transcript that begins with Hakeem reading the task.

153 H: All right. [reading] Task 3, below are three different tangent vector fields and six rate of change equations, without using technology determine which differential equation is the best match for each tangent vector field. You also have three rate of change equations left over. Explain your reasoning. So without looking at these equations, we have these fields. So what these tell us is the slope or the rate of change basically $\frac{dy}{dt}$ at specific points of $t$ and $y$. So if you put in, so you can almost think about it $\frac{dy}{dt}$, this is almost kind of like what I was doing in the first task. So $\frac{dy}{dt}$ would be a function of $t$ and $y$. So if you plug in some value of $t$ and $y$ you are getting which you get this slope value, but at the same time $\frac{dy}{dt}$ is a relationship for $y$'s so like it tells you have $y$ is changing with $t$ at that time. So what you have with the slope fields is you are given a bunch of slopes at specific $t$ and $y$, and what this tells you, you can basically draw out $y(t)$. So for example if you just started from here, you know at this point the slope is going down, but then the slope is also changing as $y$ and $t$ are changing. And as the, so the slope is changing as $y$ and $t$ are changing but then, but as the slope is changing it's giving you the direction $y$ is going. And that's how you get the curve $y(t)$. And so, so the way to approach this, we know that, so these slopes are basically $\frac{dy}{dt}$, they are giving information about what this function of $t$ and $y$ is. By just seeing these slopes. So, one thing I have noticed about the slope field for A is that its

autonomous, and what that means is, that $\frac{dy}{dt}$ is just a, since A is autonomous, you just have $\frac{dy}{dt}$ is equal to $f(y)$. You can say that because, the slope value is constant for a specific $y$ value. So, $\frac{dy}{dt}$ is only changing with $y$ and not with $t$ in this equation or in this slope field.

154 G: So just a second ago you had told me that $\frac{dy}{dt}$ changes as $y$ changes and as $t$ changes.

155 H: Ah generally, but in this specific example, or this specific slope field [slope field A], you are given that $t$ has no effect on $\frac{dy}{dt}$ in this specific example. But generally you could have that, so what that gives you is that the $t$ term in this equation is zero.

156 G: So what you were saying before about this, ah happens in general but doesn't happen to apply to this example.

157 H: Yes.

158 G: Ok.

159 H: So, since this one is only changing with $y$, or since the slope is only changing with $y$ here, you can look for that in these equations. So it has to be either this one [points to $\frac{dy}{dt} = 1 - y^2$] or this one [points to $\frac{dy}{dt} = 1 - y$].

160 G: Ok.

161 H: And, if you look at this equation [points to $\frac{dy}{dt} = 1 - y^2$], generally if $\frac{dy}{dt}$ is most likely going to be negative in this equation, because if $y$ was a positive value that was greater than one then $\frac{dy}{dt}$ would be negative. But it was also a negative value, if $y$ was negative, $\frac{dy}{dt}$ would still be negative because you have a $y^2$, but for this equation, $y'$ can either be positive or negative, and then when you look at this slope field, it's negative for a large $y$ value and positive for a negative $y$ value or $y$ values that are less than one. So this one [points to slope field A] you know corresponds to this equation [points to $\frac{dy}{dt} = 1 - y$]. So you could say that this [points to $\frac{dy}{dt} = 1 - y$] is equation A.

After reading the task, Hakeem explained how he interpreted slope fields. In turn 153, Hakeem utilized the resource *functional mapping* to explain that the vectors in a slope indicate the rate of change or the value of $\frac{dy}{dt}$ at certain points in the $t$-$y$ plane. That is, Hakeem expressed the idea that the vector field indicates a correspondence between two sets (*functional mapping*): "$\frac{dy}{dt}$ at specific points of $t$ and $y$. " Further he indicated that the values of $\frac{dy}{dt}$ are dependent on $t$ and $y$, "if you plug in some value of $t$ and $y$ you are getting, which, you get this slope value." This statement is indicative of the resource *functional dependence* because slope values depend on values of the $t$ and $y$. Additionally Hakeem noted that the values of "the slope are changing as $t$ and $y$ are changing.” The idea that the value of one quantity changes as the value of another quantity changes is indicative of coordinating changes in the quantities, and as such represents the resource *functional variation.* He then moves on to discuss slope field A.

While considering slope field A, Hakeem said, “$\frac{dy}{dt}$ is just a, since A is autonomous, you just have $\frac{dy}{dt}$ is equal to $f(y)$.” He made this inference by utilizing *functional variation*: “you can say that because, the slope value is constant for a specific $y$ value. So, $\frac{dy}{dt}$ is only changing with $y$ and not with $t$ in this equation or in this slope field.” That is, Hakeem related changes in the value of the slope with changes in the value of $y$ by iteratively holding multiple values of $y$ constant and running through values on the $t$ axis. After noting the values of the slope did not change as $t$ changed, he determined the $\frac{dy}{dt}$ was only dependent on $y$ and not on $t$. In turn 159 Hakeem then used *functional dependence* to narrow down the choices by looking for equations with $y$ on the right hand side but not $t$. With two potential matches identified [equations 2 and 4], in turn 161 he used the resource *functional mapping* to determine how the equations mapped values of $t$ and $y$ to $\frac{dy}{dt}$. For example when considering $\frac{dy}{dt} = 1 - y^2$ he inferred “if $y$ was a positive value that was greater than one, then $\frac{dy}{dt}$ would be negative. But it was also a negative value, if $y$ was negative, $\frac{dy}{dt}$ would still be negative.” With the mappings present in each of the equations identified, he went on to use the resource *parts contribute to the whole* to make the final determination about which equation matched the slope field: “then when you look at this slope field, it’s negative for a large $y$ value and positive for a negative $y$ value, or $y$ values that are less than one. So this one [points to slope field A] you know corresponds to this equation [points to $\frac{dy}{dt} = 1 - y$]*”* (turn 161). Namely Hakeem looked at the part of the vector field that corresponded to the large $y$ values, looked at the part of the vector field that corresponded to $y$ values less than one, determined that both parts matched the mapping present in $\frac{dy}{dt} = 1 - y$, and determined that whole vector field matched the equation. Hakeem then repeated a similar process for the remaining two vector fields: first for slope field C and then for slope field B.

To determine which equation matched vector field C he used the resource *functional variation* to infer that the slopes are dependent on $t$; "$\frac{dy}{dt}$ is a constant for every $y$ value and you can see that for each point in time. So for this equation you are getting a $\frac{dy}{dt}$ is equal to $f(t)$" (turn 163). He then went on to determine the potential matches by looking for the equations that only had $t$'s on the right hand side and used *functional variation* to determine which equation matched the vector field; "and what is happening here [points to slope field B] is that as $t$ is growing, $\frac{dy}{dt}$ is going from positive to negative. So $t$ has a negative effect on $\frac{dy}{dt}$, so that's when you know it's this one, $1 - t$" (turn 165). In this case he inferred that $\frac{dy}{dt}$ was going from positive to negative as $t$ increased, and associated that relationship with the equation $\frac{dy}{dt} = 1 - t$.

Hakeem then repeated this process one last time on the final slope field. He used *functional variation* to determine which variables the value of the slope depend on; "this one isn't like the other two in that, it's changing with both $y$ and $t$, the slope is. So that's when this equation like, you get an equation of $y$ and $t$" (turn 171). Hakeem then used *functional dependence* to infer that $\frac{dy}{dt} = \mathrm{y}^2 - \mathrm{t}^2$ and $\frac{dy}{dt} = \mathrm{t}^2 - \mathrm{y}^2$ were the two equations which potentially matched the vector field. To determine which of the two equations matched the vector field Hakeem utilized *functional variation* to determine how the values of both $y$ and $t$, varied with the quantity $\frac{dy}{dt}$ in the equations. He did so in two steps, first he fixed the value of $y$ and varied $t$; “the best way to put it as $t$ is large and $y$ is zero, so you can look at what just the effect of $t$ is” (turn 173). In doing so he inferred “as $t$ is really negative… $\frac{dy}{dt}$ stays large but it’s decreasing then it starts increasing to infinity again” (turn 173) Hakeem then fixed $t$, and varied $y$, “and for $y$ you have the opposite effect almost like it’s [$\frac{dy}{dt}$] negative but then it starts increasing but then it starts decreasing again” (turn 173). From this Hakeem was able to make the determination “so, that would mean it’s this equation [points to $\frac{dy}{dt} = \mathrm{t}^2 - \mathrm{y}^2$], because $t$ has a positive effect on $\frac{dy}{dt}$ and $y$ has a negative effect on $\frac{dy}{dt}$.”

In summary Hakeem utilized the resource *functional variation* to determine which variables $\frac{dy}{dt}$ was dependent upon. After making this determination, Hakeem then used *functional dependence* to determine which of the 6 equations shared the same dependence relation as the slope field in question. That is, by looking at how the slopes changed along the $t$ and $y$ axes, Hakeem determined which variables effected the value of the slope, and used this to narrow down the choices of which differential equation matched the slope field. While working with slope field A, to a match the potential equations to the slope field, Hakeem used *functional mapping*. That is he compared how the value of $\frac{dy}{dt}$ corresponded to values of $t$ and $y$ in both the equations and the slope field looking for which correspondences matched. While working with slope fields B and C however, Hakeem utilized *functional variation* to match the differential equations to the slope field. He did this by checking to see if the behavior of $\frac{dy}{dt}$ matched that of the slope from the slope field. In other words, he varied the values of $t$ and $y$ in the equations to see how the values of $\frac{dy}{dt}$ changed, and then compared this to how the slopes changed when varying the location of the points in the $t, y$ plane.

Lastly, while completing this task Hakeem applied many of the same resources to both the graphical representations (slope fields) and symbolic representations of the differential equation. For instance Hakeem utilized *functional mapping* in commensurate ways in relation to both the slope field and the algebraic form of the differential equations; "its negative for a large $y$ value and positive for a negative $y$ value or $y$ values that are less than one. So this one [points to slope field A] you know corresponds to this equation [points to $\frac{dy}{dt} = 1 - y$]" (turn 161). I argue that Hakeem was able to do this because he interpreted the corresponding components from each of the representations (e.g., $\frac{dy}{dt}$, slope vectors, $t, y$ points, the variables $t$ and $y$) in ways that were commensurate for him. For example, Hakeem in turn 153 said "So what these [the vectors in the slope field] tell us is the slope or the rate of change basically $\frac{dy}{dt}$ at specific points of $t$ and $y$," indicating the multiple ways in which Hakeem attended to what the vector represented. More specifically, knowing that the vector field provided $\frac{dy}{dt}$ at specific values of $t$ and $y$, he could coordinate these representations and apply resources that allowed him to compare the appropriate characteristics of the equations and slope fields. That is, Hakeem attended to the $t$ and $y$ points on the graph, but could coordinate those points with values being input into the differential equation for $t$ and $y$.

**4.1.2.4 Discussion of interview 2.** During this interview Hakeem expressed two ways of interpreting the differential equations: As a description of the behavior of the quantity of interest (I2-T1 and I2-T2), and as a relationship between the value of quantities and the value of the rate of change of the quantity of interest (I2-T3). In addition the set of resources Hakeem utilized to complete Tasks 1 and 2 are largely similar (see Table 4.3), and both are quite different from the set of resources he utilized to complete Task 3. The set of resources Hakeem utilized in

Tasks 1 and 2 supported his ability to determine the behavior of the quantities of interest, and the resources he used to complete Task 3 supported his ability to compare the relationships between the values of variables across different representations of differential equations. I will now discuss similarities and differences across the sets of resources and interpretations Hakeem utilized with regard to the support they provided Hakeem in completing the tasks.

Collectively speaking, while completing the tasks Hakeem used the resources *parts contribute to the whole, larger magnitudes have larger impacts*, and *positives add – negatives take away* in the process of correlating values of the quantities to values of the rates of change. In doing so, these resources supported Hakeem's ability to utilize *functional variation* and *functional mapping.* These resources were used in service to determining how changes in the value of the quantities related to changes in the corresponding rates of change. Similarly these same resources supported Hakeem in determining how sets of values of certain quantities related to sets of values of their rates of change. To complete the three tasks, these correlations were utilized in different ways.

In Tasks 1 and 2 Hakeem utilized *functional variation* and *functional mapping* in conjunction with rates indicate behavior and unrestricted change to determine the behavior of the quantities of interest. More specifically by correlating the values of a quantity and values of its rate of change Hakeem could then determine future values of the quantity. A lucid example of this would be using the known position and velocity of a vehicle to predict future positions of the vehicle. Using this example, the resources Hakeem utilized in Tasks 2 and 3 allowed him to construct different combinations of positions and velocities to discuss different paths the vehicle might take. This is particularly evident in Task 2 when Hakeem related different sets of values of $P$ and $P'$ to different values of $P''$, each of which provided a different behavior of $P(t)$.

To complete Task 3 Hakeem made use of *functional mapping* and *functional variation* in a different way, not to build the behavior of a quantity, but to compare correlations between the values of quantities and rates of change. That is, *functional mapping* afforded Hakeem in comparing how algebraic representations of differential equations mapped values of $y$ and $t$ to $\frac{dy}{dt}$ with how graphical representations mapped $t, y$ points to slope values. Similarly Hakeem utilizes *functional variation* to see how changes in these components were related across the representations.

Lastly, Hakeem was able to utilize the resources across multiple representations and interpretations of the various mathematical components from of the differential equation. For example, In Task 2 Hakeem was able to think of $P$ as both a function and a value. While completing the same task Hakeem thought of $P'$ as a value and as representative of a tangent line, both of which he utilized in conjunction with the resource *rates indicate behavior* and *functional mapping.* In Task 3 he expressed that the vectors in a slope field represented rates of change at a specific instance, the value of the slope of a function, and symbolically as the value of $\frac{dy}{dt}$. Further, he was able apply the resources to the variables $t$ and $y$ as both $(t, y)$ coordinate points and as corresponding values in the differential equation. In this way Hakeem expressed a new way of interpreting the variables in a differential equation, as points in plane; prior to this interaction he had only expressed them as numbers, values and functions. In general, Hakeem expressed great flexibility in how he was able to reason with and about the various parts of the differential equations. As a result Hakeem was supported in utilizing resources across multiple representations.

*Table 4.3: Resources and interpretations Hakeem expressed in Interview 2.*

| Task | Interpretation of the differential equation | Resources | Interpretation of the components in the differential equation |
|---|---|---|---|
| I2-T1 | As a description of the behavior of the quantity of interest | *Definition of competitive systems*<br>*Definition of cooperative systems*<br>*Functional mapping*<br>*Functional variation*<br>*Larger magnitudes have larger impacts*<br>*Parts contribute to the whole*<br>*Positives add- negatives take away*<br>*Rates indicate behavior*<br>*Unrestricted change* | $x$ and $y$ - Population values<br>$\frac{dx}{dt}$ and $\frac{dy}{dt}$ - Rates of change of the respective populations |
| I2-T2 | As a description of the behavior of the quantity of interest | *Functional mapping*<br>*Functional variation*<br>*Parts contribute to the whole*<br>*Rates indicate behavior*<br>*Unrestricted change*<br>*Variable substitution* | $P$ – Function values<br>$P'$ - Rate of change of $P$<br>$P''$ - Rate of change of $P'$ |
| I2-T3 | As a relationship between the value of a quantity and the value of the rate of change of the quantity of interest | *Functional dependence*<br>*Functional mapping*<br>*Functional variation*<br>*Parts contribute to the whole* | $t$, and $y$ - Points on the slope field and values in the differential equation.<br><br>$y'$- Slope of $y$ |

### 4.1.3 Interview 3

During Interview 3 Hakeem completed three tasks; the analysis of all of the tasks is presented in this section. Two of the three tasks were seemingly novel to Hakeem while the second task, which presented a fairly standard second order differential equation, was in a form Hakeem had at least seen in Interview 2. While each of the tasks in the interview were different from each other, there were aspects of Hakeem's problem solving activity that were similar across the three tasks. For example, in each of the three tasks Hakeem reasoned from the differential equation to determine the behavior of the quantity of interest. The reasoning pattern and the set of resources Hakeem used to characterize the behavior of the quantity of interest by reasoning from the differential equation were indicative of interpreting the differential equation as a description of the behavior of the quantity of interest.

Additionally, in the second two tasks Hakeem utilized an analytical solution method to determine a specific equation that related the value of the quantity of interest to some independent variable. He used this equation to get more specific information about the behavior of the quantity of interest. The reasoning pattern and set of resources that are associated with this overall approach are indicative of interpreting the differential equation as providing a model. Lastly, he reasoned with the relationship between the variables in the differential equation to collect information about the behavior of the solution function, then solved the differential equation for that solution, and coordinated the two equations to build a more comprehensive model of the quantity of interest. In this way, while the two interpretations relate to different sets of resources and reasoning patterns Hakeem utilized, he coordinated the results of the two approaches to attain a more complete description of the quantity of interest. In other words, each of these sets of resources and the interpretations associated with them each supported the completion of the task in different ways.

**4.1.3.1 Interview 3 task 1.** This task (see Figure 4.9) asked students to distinguish between two systems of differential equations, one representing small predators and large prey, the other representing large predators and small prey. This task was utilized in lieu of the competing/cooperative species task (I1-T1 and I2-T1) because after completing Interviews 1 and 2, it was apparent that the participants were coordinating the values of multiple quantities, but it was not clear how they were doing so to complete the task. The nature of this task –specifically asking students to distinguish the size of the predators and prey in each system - prompted students to more explicitly coordinate the values of the quantities to complete the task. As such, this task was useful in helping determine how students were completing tasks of this nature (I1-T1, I2-T1, I3-T1, and I4-T1), and the resources they utilized while doing so.

TASK 1: Consider the following systems of rate of change equations:

System A

$$\frac{dx}{dt} = 3x(1 - \frac{x}{10}) - 20xy$$

$$\frac{dy}{dt} = -5y + \frac{xy}{20}$$

System B

$$\frac{dx}{dt} = 0.3x - \frac{xy}{100}$$

$$\frac{dy}{dt} = 15y(1 - \frac{y}{17}) + 25xy$$

In both of these systems, $x$ and $y$ refer to the number of two different species at time $t$. In particular, in one of these systems the prey are large animals and the predators are small animals, such as humans and piranhas. The other system has very large predators and very small prey.

Which system represents the small prey and large predator? Which system represents the large prey and small predator? Explain your reasoning.

Adapted from the IO-DE curriculum materials (Rasmussen and Kwon, 2007)

*Figure 4.9: Task 1 Interview 3, large predator small prey (I3-T1).*

To complete this task Hakeem used the resources *partial dependence, parts contribute to the whole, larger magnitudes have larger impacts, functional variation, rates indicate behavior,* *unrestricted change*, *functional mapping* and *positives add-negatives take away*. He utilized these resources to determine which species were predators and which were prey (in both systems), and then to determine which were large and which were small. Hakeem utilized these resources in a way that indicates he interpreted the differential equations as relationships between quantities and their respective rates of change. Additionally he utilized different combinations of these resources during different parts of the problem solving process. Hakeem began by considering System A, as can be seen in the following transcript.

14 H: So, in the $\frac{dx}{dt}$ equation, $y$ has a pretty large effect on $\frac{dx}{dt}$ because the coefficient on the $y$ term is -20, but for $\frac{dy}{dt}$ the $x$ term has a, I guess a really small impact or on $\frac{dy}{dt}$. So, I guess what that means is the, whatever value $x$ has, like the number of, so since $x$ represents the number of species, that number doesn't have much of a significant to the rate that $y$ changes. Or, I guess it has a significance, but its ah, say if $x$ was increasing, like I am just going to ignore $\frac{dx}{dt}$ for now, for system A. If $x$ was increasing then, $\frac{dy}{dt}$ would I guess increase at a small rate. But, so yeah the rate that $\frac{dy}{dt}$ increases would be pretty small, and for $\frac{dx}{dt}$, if $y$ was say increasing or something $\frac{dx}{dt}$ would be shrinking at a pretty fast rate, or decreasing at a pretty fast rate. Cause of this -20.

15 G: Mhm.

16 H: So yeah an increase in $y$ actually means a decrease in $x$ to an extent. But an increase in $x$ means an increase in $y$. And I've also noticed that there is an $x$ squared term, a negative $x$ squared term in $\frac{dx}{dt}$.

17 G: Mhm.

18 H: So I guess large values of $x$ also have a, make $\frac{dx}{dt}$ go negative. So in this system in general it looks like, $\frac{dx}{dt}$ is gonna be, it looks like $\frac{dx}{dt}$ is gonna like maybe be negative for ah, like it has, I don't want to say like it's always negative, it just looks like it has a lot of things that would make it negative.

19 G: Sure.

20 H: And, where as $\frac{dy}{dt}$ would be positive I guess for any increasing $x$ or $y$. Well for an increasing $x$, but for an increasing $y$ $\frac{dy}{dt}$ would actually be negative. Cause there is negative $5y$.

21 G: Ok.

22 H: And if $y$ was, you can't have a negative population so yeah $\frac{dy}{dt}$ would be negative but $y$ would never be negative, so, so this term is always negative so yeah all these terms would be negative, so. Yeah it seems like $\frac{dx}{dt}$ would I guess be purely negative most of the

time in this system I don't know how to translate that into what it would be, like that has to be large or small.

In turn 14 Hakeem coordinated the resources *partial dependence* and *larger magnitudes have larger impacts* to compare the effect $y$ and $x$ have on $\frac{dx}{dt}$ and $\frac{dy}{dt}$ respectively. Specifically he used the resource *partial dependence* to relate values of $y$ to values of $\frac{dx}{dt}$ and values of $x$ to values of $\frac{dy}{dt}$, and the resource *larger magnitudes have larger impacts* to characterize the size of the impacts the variables $x$ and $y$ have on $\frac{dy}{dt}$ and $\frac{dx}{dt}$ based on the coefficients in front of each term: "$y$ has a pretty large effect on $\frac{dx}{dt}$ because the coefficient on the $y$ term is -20. But for $\frac{dy}{dt}$ the $x$ term has a, I guess a really small impact on $\frac{dy}{dt}$." He then coordinated the resources *partial dependence*, *functional variation,* and *larger impacts have larger magnitudes* to determine that if "$x$ was increasing then, $\frac{dy}{dt}$ would I guess increase at a small rate… if $y$ was say increasing or something $\frac{dx}{dt}$ would be shrinking at a pretty fast rate, or decreasing at a pretty fast rate, cause of this -20." More specifically, in this statement Hakeem related changes in the value of $y$ to changes in the value of $\frac{dx}{dt}$, and changes in the value of $x$ to changes in the value of $\frac{dy}{dt}$ (*partial dependence* and *functional variation*), and used *larger magnitudes have larger impacts* to determine "$\frac{dx}{dt}$ would be decreasing at a pretty fast rate cause of this -20."

In turn 16 Hakeem noted, "an increase in $y$ actually means a decrease in $x$ to an extent. But an increase in $x$ means an increase in $y$." On the surface it may seem as though Hakeem is conflating decreases in $\frac{dx}{dt}$ with decreases in $x$ (and increases in $\frac{dy}{dt}$ with increases in $y$). I do not interpret his statement as indicating such a conflation, rather I argue this inference is the result of coordinating the resources: *unrestricted change* and *rates indicate behavior*. Recall that in turn

14 Hakeem used *functional variation* to infer that increases in $y$ cause $\frac{dx}{dt}$ to decrease and that increases in $x$ cause $\frac{dy}{dt}$ to increase. I claim that Hakeem used this inference along with the resource *unrestricted change* in turn 16 to determine that if $y$ increased enough $\frac{dx}{dt}$ would decrease to the point that it becomes negative and then coordinated this with the resource *rates indicate behavior* to determine that $x$ would decrease (because $\frac{dx}{dt}$ was negative). He used similar reasoning to determine that increases in $x$ relate to increases in $y$.

The identification of the resource *unrestricted change* in Hakeem's reasoning in turn 16 was made in consideration of statements he made later in the interview. In turn 18 Hakeem said "large values of $x$ make $\frac{dx}{dt}$ go negative" and in turn 32 Hakeem noted, "so growth in $y$ is detrimental to growth in $x$, so it would make $\frac{dx}{dt}$ negative." The language he used seems indicative of the resource *unrestricted change*. That is, they seem indicative of Hakeem expressing the idea that decreasing values eventually become negative. For instance, "large values of $x$ make $\frac{dx}{dt}$ go negative" implies that small values of $x$ correspond to positive values of $\frac{dx}{dt}$, and that $\frac{dx}{dt}$ *goes* negative when $x$ gets large enough.

Along with the resource *unrestricted change*, in turn 18 Hakeem used the resources *functional dependence* and *functional mapping*. After considering how $x$ and $y$ impacted $\frac{dx}{dt}$, in turn 18 Hakeem said "I don't want to say like it's [referring to $\frac{dx}{dt}$] always negative, it just looks like it has a lot of things that would make it negative." His language indicates his attention to $\frac{dx}{dt}$ depending on $x$ and $y$ (*functional dependence*), and suggests he is considering all of the possible values of $x$ and $y$ that would make $\frac{dx}{dt}$ negative. In doing so he drew a correspondence between the set of the possible $x$ and $y$ values and the set of values where $\frac{dx}{dt}$ is negative (*functional mapping*). Specifically, he considered the possible conditions in a domain that would make certain conditions true in the range ($\frac{dx}{dt}$ being negative). He then used the inference that $\frac{dx}{dt}$ is almost always negative to determine that $x$ is prey in turn 24.

After he determined that in System A species $x$ represented prey and species $y$ represented predators, I asked him if he knew which were large and which were small. He replied, “I guess I'd have to look at the other system to compare it,” (turn 28) and proceeded to determine which species were predators and which were prey in system B.

32 H: See, I guess ah, so for this system, ah well first go back to look at this system only, ah, positive $x$ or growth in $x$, means a growth in $y$ because there is a plus $25x$ term in the $\frac{dy}{dt}$. Whereas there is a minus $xy$ in the $\frac{dx}{dt}$ so I guess like growth in $y$ is detrimental to growth in $x$, so it’d make $\frac{dx}{dt}$ negative.

33 G: Ok.

34 H: So I guess first to identify which one is predator and which one is prey I would say that $y$ is again the predator and $x$ is the prey, ok. So we have prey and predator. So it looks in both systems the $x$ species is the prey and the $y$ species is the predator. But the difference is in system B, the prey is, like $\frac{dx}{dt}$ isn’t as negative as $\frac{dx}{dt}$ in System A. Cause this one had like the $y$ term in $\frac{dx}{dt}$ is ah, negative $xy$ over 100 for system B, but for System A its $-20xy$. And what that means is that System B $y$'s effect on $\frac{dx}{dt}$ is a lot smaller than System A. So, I guess perhaps that would mean that this is the large prey small predator. So System B is large prey small predator and System A would be small prey large predator.

To discern which was predator and which was prey in System B Hakeem inferred “positive $x$ or growth in $x$, means a growth in $y$” (turn 32) with the resources *partial dependence, positives add - negatives take away, functional variation,* and *unrestricted change.* More specifically, Hakeem inferred “positive $x$ or growth in $x$, means a growth in $y$”, by noting the positive sign in front of the $xy$ term (*positives add – negatives subtract*) in the equation $\frac{dy}{dt} = 15y\left(1-\frac{y}{17}\right)+25xy$ indicates that increases in $x$ (*partial dependence*) will relate to increases in

$\frac{dy}{dt}$ (*functional variation*), and that increases in $dy/dt$ implies $y$ will increase (*unrestricted change*). Summarizing, Hakeem considered the sign in front of the $xy$ term in each equation and used p*ositives add - negatives take away* along with *partial dependence* and *functional variation* to determine how changes in the value of one of the variables impacted the rate of change of the other. He then used *unrestricted change* to infer that the direction in which the value of the rate of change was changing was the direction in which the value of the quantity was changing.

Having determined which species represented predator and which represented prey in both systems, in turn 34 Hakeem then began to compare the two systems to figure out which species are small and which are large. He did this by comparing across the two systems, specifically focusing on how $y$ affected $\frac{dx}{dt}$ in the respective equations. To characterize the differences between the equations in the two systems Hakeem coordinated the resources, *partial dependence, parts contribute to the whole,* and *larger magnitudes have larger impacts.* These resources are present in the statement from turn 34:

> "But the difference is in System B, the prey is, like $\frac{dx}{dt}$ isn't as negative as $\frac{dx}{dt}$ in System A. Cause this one had like the $y$ term in $\frac{dx}{dt}$ is ah, $\frac{-xy}{100}$ for System B, but for System A its $-20xy$. And what that means is that System B's $y$'s effect on $\frac{dx}{dt}$ is a lot smaller than System A."

Here Hakeem coordinated *parts contribute to the whole* and *larger magnitudes have larger impacts* when he considered how the magnitudes of the parts $-\frac{xy}{100}$, and $-20xy$ contributed to their respective wholes, $\frac{dx}{dt}$. Specifically, by utilizing these resources Hakeem inferred that -20 would have a larger impact than $-\frac{1}{100}$. Lastly, Hakeem used the resource *partial dependence* to relate the value of $y$ in each of the terms to the value of $\frac{dx}{dt}$ in each of the respective equations ("System B $y$'s effect on $\frac{dx}{dt}$ is a lot smaller than System A"). Noting the impact each of the $y$ terms had on the value of $\frac{dx}{dt}$, Hakeem then concluded that "System B is large prey small predator and System A would be small prey large predator." It is important to note that Hakeem was not comparing how changes in $y$ related to changes in $\frac{dx}{dt}$ across the two systems, rather he was comparing the magnitude of the impact $y$ had on decreasing the value of $\frac{dx}{dt}$. This is supported by the fact that in turn 34 Hakeem was talking about how negative $\frac{dx}{dt}$ was in the two systems ("in System B, the prey is, like $\frac{dx}{dt}$ isn't as negative as $\frac{dx}{dt}$ in System A"). That is, Hakeem's attention is focused how the value of $y$ relates to the rate at which $x$ is decreasing, not the rate at which $\frac{dx}{dt}$ is decreasing.

In conclusion, to complete this task, Hakeem utilized a large number of resources; this is not surprising as, mathematically speaking, this is a fairly complex task. These resources supported his ability to determine which species were predators and which were prey in the two systems, and to then compare the rates at which the prey would be decreasing in the two systems. To do this Hakeem utilized *partial dependence, larger magnitudes have larger impacts*, and *functional variation* to correlate sets of values of $x$ and $y$ to sets of values of $\frac{dy}{dt}$ and $\frac{dx}{dt}$ respectively. He then utilized *unrestricted change* and *rates indicate behavior* to determine the behavior of the populations of species $x$ and species $y$, from which he concluded $y$ was predator and $x$ was prey. To determine which species was large and which was small, he used *partial dependence, parts contribute to the whole*, and *larger magnitudes have larger impacts*, to infer that $y$ in System B had a larger impact on $\frac{dx}{dt}$ then $y$ from System A. In addition he expressed that $x$ and $y$ were values of the population of species $x$ and $y$, and that $\frac{dx}{dt}$ and $\frac{dy}{dt}$ were the values of the populations' rate of change. Lastly while completing this task Hakeem interpreted the differential equation as a description of the behavior of the quantity of interest. This is indicated by his work discerning which species was predator and which was prey. More specifically, Hakeem utilized the rates of change to characterize the behavior of the populations; he determined increases in $x$ related to increases in $y$, and increases in $y$ related to decreases in $x$.

**4.1.3.2 Interview 3 task 2.** The second task in Interview 3 (see Figure 4.10) asked Hakeem to interpret a second order differential equation. A variant of this task was presented in Interview 2; the task was the same except the coefficients in the equation were different. In this interaction Hakeem utilized the relationship between the variables in the differential equation to reason about how $P$ changed. Hakeem also solved the differential equation utilizing the roots of the corresponding characteristic equation and then drew a parallel between $P(t)$ – the solution function he found – and the differential equation. Namely he said "this equation [$P(t) = C_1e^{-t} + C_2e^{-5t}$] and the differential equation in general are both just ways to represent the way that $P$ is changing I guess over I guess time" (turn 83). That is, Hakeem interpreted the differential equation as a relationship between quantities and their respective rates of change and as an equation that provides a model for the behavior of the quantity of interest. In the following paragraphs I present the analysis of the resources Hakeem utilized while completing this task, and support my claim about his interpretation of the differential equation. Specifically I discuss the resources Hakeem utilized to complete Task 2 and how these resources supported his reasoning.

| TASK 2: What does the following differential equation mean to you? $$P'' = -6P' - 5\text{P}$$ |
|---|

*Figure 4.10: Task 2 from Interview 3*

To interpret the differential equation as a description of behavior of $P$ Hakeem utilized the resources *functional dependence*, *functional variation*, *rates indicate behavior*, *unrestricted* *change*, and *parts contribute to the whole.* I start the presentation of the resources Hakeem utilized with transcript from the beginning of his interaction with Task 3.

60 H: So here, *P* represents anything and it's changing with respect to I guess some other variable, so it could be like a population changing over time. So, ah, and I guess what this equation represents is that, so *P"* is I guess decreasing as I guess *P'* and *P* are increasing.
61 G: Ok.
62 H: And I guess what that means is that if *P"* is decreasing as *P'* and *P* are increasing, by *P"* decreasing, would cause I guess *P'* and *P* to both decrease. And when both of these are negative *P"* would be positive. So it's a, it cycles back and forth.
63 G: Ok, what is the 'it' that cycles back and forth?
64 H: I guess the nature of $P$. Or well, at least *P"*, its going positive and negative. So we have and if *P"* is going between positive and negative then that means *P* and *P'* are both going to be increasing and decreasing that same way.
65 Interviewer: Ok, and so, $P$ is oscillatory is what you are saying?
66 H: Yes.
67 G: Ok.
68 H: Um I can guess, solve this. The characteristic equation would be [writing] $\lambda^2 + 6\lambda + 5 = 0$. And… [inaudible: solving the characteristic equation for roots]
69 H: So lambda... So I guess for, I'll just call it $P(t)$, I'm not really sure, $e^{-t} + C_2\, e^{-5t}$.
70 G: Ok.
71 H: All right so, so as when you're in this form, one thing I guess falls under this is that as $t$ goes to infinity, you'd ah, wait, as yeah as $t$ goes to infinity $P(t)$, would approach zero.

In turn 60 Hakeem utilized the resource *functional dependence* to infer that *P* was a quantity that is dependent on "some other variable." He then coordinated *functional variation, parts contribute to the whole, and positives add- negative take away* to deduce that the differential equation means that " *P"* is I guess decreasing as I guess *P'* and *P* are increasing" (turn 60). That is Hakeem considered how the parts $-6$*P'* and $-5P$ contributed to whole *P"* as values of *P'* and *P* increased. He then coordinated this inference (that *P"* was increasing as *P* and *P'* both increased) with the resources *rates indicate behavior* and *unrestricted change* to infer that "if *P"* is decreasing as *P'* and *P* are increasing, by *P"* decreasing, would cause I guess *P'* and *P* to both decrease" (turn 62). More specifically, Hakeem utilized *rates indicate behavior* to infer that *P'* would decrease because *P"* was negative and then *unrestricted change* to extend decreases in *P'* to *P'* eventually becoming negative. Hakeem then used these same resources (*rates indicate behavior* and *unrestricted change*) to infer that *P'* being negative means *P* decreases to the point that it becomes negative. Hakeem then repeated this reasoning process to determine that *P'* and *P* becoming negative would mean *P''* will be positive, presumably causing *P'* and *P* to increase again. This explanation is in line with Hakeem's statement in turn 64 where he noted the value of $P$ would "cycle back and forth." Based on the fact that he used the differential equation to characterize $P$ as cycling back and forth, Hakeem was interpreting the differential equation as a description of the quantity of interest.

Before continuing the discussion of the resources Hakeem utilized in later talk turns, I first support my claim about the coordination of the resources *unrestricted change* and *rates indicate behavior* in turn 62. The following transcript comes after Hakeem solved the differential equation, and resulted from a planned follow up question in which Hakeem was asked how he thought about $P$, $P'$ and $P''$.

76 G: All right, you talked a little about this but how do you think about $P$, $P'$ and $P''$?
77 H: So, *P* is a variable, it could be anything, a population or anything. Like one of the equations we looked at in class would be like the motion of an object on a spring. Where so *P* would be like the displacement of the object from the springs equilibrium point and if we used that example then *P'* would be, so it's the rate of change of $P$. So in that example *P'* would be the velocity, and *P''* would be acceleration.
78 G: Ok.
79 H: And then I was looking at, when I was looking at the signs here, I was saying that if *P'* and *P* are positive, then *P''* is negative so that means if the velocity and the displacement are both positive, the acceleration is negative. But a negative acceleration would eventually lead to a negative velocity and a negative velocity would lead to a negative displacement, so if we had both negative velocity and negative displacement then you would have a positive acceleration, and that would in turn have an increase in velocity which would then in turn have an increase in displacement. So that was where I was getting the oscillatory nature of $P$.

In turn 77 Hakeem provided instantiations of $P$, $P'$, and $P''$ and explained that $P$, $P'$ and $P''$ could represent displacement, velocity and acceleration, respectively. In turn 79 he explained how these examples relate to his previous reasoning about the differential equation:

“When I was looking at the signs here, I was saying that if $P'$ and $P$ are positive, then $P''$ is negative so that means if the velocity and the displacement are both positive, the acceleration is negative.” Through his next statement we see a vivid example of coordinating the resources *rates indicate behavior* and *unrestricted change*: “but a negative acceleration would eventually lead to a negative velocity and a negative velocity would lead to a negative displacement.” *Rates indicate behavior* was used to determine the direction of the quantities’ change (displacement and velocity in this example) and *unrestricted change* was used to determine that the change in these quantities was such that the values would eventually switch signs. I now return to Hakeem’s discussion about solving the differential equation.

Upon deducing that $P$ would oscillate between positive and negative values, Hakeem said he could solve the differential equation. The affect behind his statement seemed to be that he could solve it to check if his determination about the oscillatory nature was correct. To solve the differential equation he wrote out the corresponding characteristic equation $\lambda^2 + 6\lambda + 5 = 0$, found the values of lambda that satisfied the characteristic equation ($\lambda = -1, \lambda = -5$), and concluded that $P(t) = C_1e^{-t} + C_2e^{-5t}$. In doing so he utilized the *method of characteristic equations.* Upon finding $P(t)$, he immediately used *functional variation* to infer the behavior of $P$ as $t$ approached infinity: “all right so as, when you're in this form, one thing I guess falls under this is that as $t$ goes to infinity, you'd ah, wait, as yeah as $t$ goes to infinity $P(t)$ would approach zero” (turn 72). Here Hakeem used the solution $P(t)$ to characterize the nature of the quantity of interest, this is an import distinction between interpreting the differential equation as an equation that provides a model, and interpreting the differential equation as a description of the quantity of interest. More specifically, the model provided by the differential equation is the function that results from solving the differential equation (which also represents the quantity of interest), where as the description of the behavior is an inherent property of the differential equation itself.

I now transition to Hakeem's discussion of the relationship between the solution and the differential equation. This is particular important for identifying how Hakeem was interpreting the differential equation. To do so, I now shift focus to a later portion of the interview where Hakeem voluntarily drew parallels between $P(t)$ and the differential equation.

80 G: Gotcha, so you had said that *P* is a variable. Can you talk to me a little bit about what you mean by that?

81 H: So, ah well I guess this equation [points to $P(t) = C_1e^{-t} + C_2e^{-5t}$] and the differential equation in general are both just ways to represent the way that *P* is I guess changing over I guess time. So, what this equation, this [points to $P(t) = C_1e^{-t} + C_2e^{-5t}$] equation I guess is a little more specific in which in what is happening to $P$, but it's also kind of incomplete right because there is two constants here, that you have to figure out, and you can't do that without an initial values, and I guess that could also affect, like these constants could affect the, what is actually happening to $P$. Like say if one was positive and one was negative.

82 G: How does, so you said that both of these equations tell you what's happening to $P$. How does this one [points to $P'' = -6P' - 5P$] tell you what's happening to $P$?

83 H: Well that one, was kinda just, what I just explained earlier about the oscillatory motion.

84 G: Ok.

85 H: I guess.

86 G: It's telling you what's happening to *P* because you can figure out…?

87 H: Cause you have *P'* and *P''* and those are both related to *P* in that they tell you how *P* is varying.

88 [G: Gotcha. So what is this actually? How do you think about this thing here? [points to $P(t) = C_1e^{-t} + C_2e^{-5t}$] Like the whole thing?

89 H: So, I am just using $t$ as, I guess $t$ for time, so I guess it's just, so *P* is a variable that's changing with say respect to time, then this is just like explaining how exactly it's changing with respect to time. So, I'm looking at I guess like $t$ as another variable here, and the way that $t$ changes its effecting what's happening to $P$. So when I look say here, when $t$ goes to infinity, this eventually goes to zero.

In turn 83 Hakeem is explicit about the connection between the role of $P(t)$ and that of the differential equation $P'' = -6P' - 5P$: "this equation [points to $P(t) = C_1e^{-t} + C_2e^{-5t}$] and the differential equation in general are both just ways to represent the way that *P* is I guess changing over I guess time." While these equations each provide a description of how *P* is changing, Hakeem utilized them in different ways. Specifically, Hakeem used the relationship between the values of the variables in the differential equation to characterize the general behavior of $P$ (turn 87), and then reasoned from the solution $P(t)$ to characterize the specific way $P$ varies with $t$ (turn 89). Based on the resources Hakeem used while completing this task, the ways in which he reasoned about the differential equation with these resources, and the ways he expressed thinking about the differential equation and its components, I claim Hakeem interpreted the differential equation in two ways while working on this task: as a description of the quantity of interest, and as an equation that provides a model of the quantity of interest. While one could argue that these two interpretations are actually the same because both supported Hakeem in attaining a description of the behavior of $P$, I disagree because the resources and the reasoning across the two interpretations are significantly different.

At the beginning of the interview Hakeem reasoned with the relationships between $P$, *P'* and *P''* to construct the behavior of $P$. This was indicative of interpreting the differential equation as a description of the behavior of the quantity of interest. To determine the behavior of $P$, he used the resources *functional variation, parts contribute to the whole,* and *positives add - negative take away* to determine how changes in the values of $P$ and *P'* related to changes in the values of *P''* and then made use of *rates indicate behavior* and *unrestricted change* to relate the values of *P''* and *P'* to the behavior of $P$. As a result, he determined that $P$ would oscillate between positive and negative values. In this case he utilized $P$, *P'*, and *P''* as dynamic values in the differential equation values to infer the behavior of *P,* which represented, as he said in the beginning of the interview, "anything, and it's changing with respect to I guess some other variable" (turn 60).

Immediately after determining the behavior of $P$, he switched interpretations and made use of the *method of characteristics* to find the roots of the characteristic equation $\lambda^2 + 6\lambda +$

$5 = 0$ and used them to determine the solution function $P(t) = C_1 e^{-t} + C_2 e^{-5t}$. He then used *functional variation* to make inferences about the behavior of $P(t)$ as $t$ approached infinity. In this case he did not explicitly utilize the relationships between the variables; rather he utilized their coefficients to construct the characteristic equation. In addition, $P(t)$ was a result of the solution method, and his attention was not focused on $P'$ or $P''$. In this case Hakeem interpreted the differential equation as a providing a model of the quantity of interest.

**4.1.3.3 Interview 3 task 3.** Hakeem's approach to Task 3 (see Figure 4.11) was very similar to that of Task 2. He initially suggested interpreting the differential equation as a description of the behavior of the quantity of interest; he reasoned with the relationship between $P$ and $P'$ in the differential equation to correlate changes in the value of $P$ to changes in the value of $P'$, and then relate these changes to the behavior of $P$. After determining that $P$ was generally decreasing, Hakeem utilized an analytical solution method to determine specific equations (one for each initial condition) relating $P$ and $t$. He then considered the long-term behavior of each of these solution functions as $t$ approached infinity. This was suggestive of interpreting the differential equation as providing a model of the quantity of interest. In the process of doing so, however, Hakeem demonstrated significant interplay between these two approaches. More specifically, Hakeem coordinated information from specific solutions to further reason about the differential equation. In terms of the analysis, the presentation of Hakeem's work on this task demonstrates the interaction between two interpretations but also delineates the different affordances provided by the corresponding sets of resources.

TASK 3: A group of biologists are making predictions about the spotted owl population in a forest in the Pacific Northwest. The differential equation the scientists use to model the spotted owl population is $\frac{dP}{dt} = \frac{P}{2}\left(1 - \frac{P}{5}\right)$, where $P$ is in hundreds of owls and $t$ is in years. The problem is that the current number of owls is only approximately known.

Because they can only get an estimate of the population, they make long-term predictions of the owl population for various initial conditions. Two of these initial conditions are $P = 3.0$ and $P = 5.0$. Determine the long-term predictions for these initial conditions based on the differential equation.

Adapted from the IO-DE curriculum materials (Rasmussen and Kwon, 2007)

*Figure 4.11: Task 3 from Interview 3*

I begin the presentation with transcript from the beginning of Hakeem's interaction with Task 3.

101 H: Well, first just in general, it looks like this owl population is just decreasing over time.
102 G: Ok, how do you know?
103 H: Because there is a $P - P^2$ term and $P$ is always positive, and it's in the hundreds of owls, so like if you plugged in any value of $P$, you'd end up with $\frac{dp}{dt}$ being negative. Well, not any value, but, plug in large values.
104 G: Ok.
105 H: So in general it looks like the population would be decreasing unless the population was like really small and looked kind of plateau in a sense cause you'd have cause this one is ah divided by 10 and this one is h, the $P^2$ term is divided by 10 and the $P$ term is divided by 2, so when $P$ is really small, this term [$P$], would be greater than this term [$P^2$], so $\frac{dP}{dt}$ would end up being positive.
106 G: Ok.
107 H: And I guess I could try and solve this equation and…
108 G: So, to be able to make long-term predictions, you want to solve for $P$?
109 H: Yeah, because if I solve for $P$, with respect to $t$, I'll end up with an equation with some constants which I can figure out using the initial conditions and the equation itself should be able to, it like I said in the example before, this [the differential equation] is one way like I guess to model how $P$ is changing, but like the equation is like another way. If I get $P(t)$, its another way and its more specific in how $P$ is changing with respect to $t$ and I can also then do the same thing I did last time, like look at how as $t$ approaches infinity what's happening to $P$.

In turn 101, Hakeem declared that the owl population appears to be "just decreasing over time." He then provided an explanation for this claim in turns 103 and 105, in which he utilized the resources *variable substitution, functional mapping, rates indicate behavior, parts contribute to the whole,* and *larger magnitudes have larger impacts.* Specifically, in turn 103 he noted that if you "plugged in any value for *P*" you would in turn "end up with $\frac{dp}{dt}$ being negative" (*variable substitution*), and in turn 105 he used the resource *rates indicate behavior* to relate $\frac{dx}{dt}$ being negative with the population decreasing. Additionally he then refined his original claim that the population would be decreasing for any value of $P$, noting this is true "unless the population was like really small" (turn 105). He supported this new assertion by explaining that if the value of $P$ is really small, $\frac{p}{2}$ would be larger than $\frac{P^2}{10}$, making $\frac{dP}{dt}$ positive. To do this he coordinated *larger magnitudes have larger impacts* and *parts contribute to the whole by* comparing the magnitudes of the two terms and then determining how each of those contributed to the value of the $\frac{dP}{dt}$. In addition he coordinated this with *functional mapping* to correlate sets of $P$ values to sets of *P'* values; large values of $P$ relate to $\frac{dP}{dt}$ being negative, small values of $P$ correlated to $\frac{dP}{dt}$ being positive. In short, Hakeem reasoned with the relationship between the values of and *P'* and the values of $P$ in the differential equation using *variable substitution, parts contribute to the whole*, and *larger magnitudes have larger impacts*, and he used *rates indicate behavior* to infer the behavior of *P* based on the value of *P'*. This set of resources is indicative of Hakeem interpreting the differential equation as a description of the behavior of the quantity of interest. After determining that $\frac{dP}{dt}$ would be negative except for small values of $P$, Hakeem's focus shifted to analytically solving the differential equation.

Hakeem noted in 109 that he wanted to solve the differential equation for $P(t)$ because "it's more specific in how $P$ is changing with respect to $t$ and I can also then do the same thing I did last time [referring to Task 2], like look at how as $t$ approaches infinity what's happening to $P$." It is important to note that Hakeem expressed that reasoning from the differential equation "is one way like I guess to model how $P$ is changing," but that "$P(t)$, its another way and it's more specific in how $P$ is changing with respect to $t$." In this case Hakeem explicitly expressed two ways of reasoning with the differential equation, which as the analysis will show corresponds to two ways of interpreting the differential equation: as a description of the behavior of the quantity of interest (described in the previous paragraphs) and as providing a model of the quantity of interest (described in the following paragraphs). Though he referred to them both as ways of modeling $P$, for Hakeem the latter provides a model that is "more specific in how $P$ is changing with respect to $t$" (turn 109). Namely it provides an equation that relates specific values of $t$ to specific values of $P$.

Hakeem went on to solve the differential equation by performing a change of variables, a method he stated he learned in class, which "says what you need to do is, you do a substitution here… and that [points to the differential equation] will become a linear differential equation that I can solve" (turns 120 and 122; transcript follows) Hakeem remained virtually silent during this process, making it difficult to identify the resources he utilized to find the general solution. For this reason this portion of the interaction is not presented.

119 G: So tell me what you are doing here.
120 H: So this is the way that we learned to solve these types of equations in class, the non linear equations. It says what you need to do is, you do a substitution here, so here my substitution would be $p = v^{-1}$ and with that substitution I think basically get this differential equation in terms of v instead of in terms of $P$.
121 G: Ok.
122 H: And that will become a linear differential equation that I can solve [silently writing for roughly 6 minutes].
123 G: Gotcha.
124 H: So that is one equation for $P$ based on the first initial condition.
125 G: Ok.
126 H: So it looks like, based on the first initial condition that as $t$ goes to infinity $P$ would just ah plateau off to this value of 5.
127 G: Ok.
128 H: So there would be 5 hundred owls I guess. And that's like a constant value almost. I guess that makes sense when you go back earlier, is that I guess then there is a lot of owls

they decrease, so then there is a lot more than 500, this $P^2$ term would, I guess cause this $\frac{dP}{dt}$ to be negative so it would decrease, but then once it reaches like a certain point, so even if I plugged in 5, it'd be 5 over 2 minus, 25 over 10 minus 2.5, which is zero. So at that point $\frac{dp}{dt}$ is zero, which is constant so there is no change at that point.

129 G: Ok.

130 H: So that I guess makes sense. If I wanted to solve for the other initial condition that would be $P(0) = 5$, and you'd have $P(t)$ equals … [inaudible, begins solving for $P(t)$].

131 H: And so basically this initial condition, it means that if, going back to this initial equation, so the owl population would plateau off at 500 and that's when there really wouldn't be a change cause that's when $\frac{dP}{dt}$ is zero. And so if you are starting off at 500, so that's why when you put in that initial condition it doesn't change with respect to $t$.

After finding the general solution $P(t) = \left(\frac{1}{5} + Ce^{\frac{-t}{2}}\right)^{-1}$, he used the initial condition $P(0) = 3$, to attain a particular solution. To do so he utilized the resources *dependence, function slots,* and *the definition of general solution*. More specifically, he used the *definition of general solution* to infer that $P(t) = \left(\frac{1}{5} + Ce^{\frac{-t}{2}}\right)^{-1}$ would be the form of all of the solutions to the differential equation, he expressed that the value of *C* in the general solution was determined by the initial condition (*dependence*), and he used *function slots* to fill the "slots" for *P* and $t$ in the general solution using the initial conditions. That is, he used the initial condition $P(0) = 3$ to replace $P(t)$ with 3 and $t$ with 0 in the general solution to determine that $C = \frac{2}{15}$ and that the corresponding solution was $P(t) = \frac{1}{5} + \frac{2}{5}e^{-\frac{t}{2}}$.

Upon determining this solution he used *functional variation* to determine the behavior of $P(t)$ as $t$ went to infinity and discovered that the population approached 5: "based on the first initial condition that as $t$ goes to infinity *P* would just ah plateau off to this value of 5" (turn 126). He then coordinated this finding with the differential equation at the end of turn 128, where he used the resource *variable substitution* to substitute the value of 5 in for *P* in the differential equation to determine that $\frac{dp}{dt}$ is zero when $P = 5$. Having compared the behavior of *P* represented by the particular solution ($P$ approaches 5) to that of the differential equation *(*using *parts contribute to the whole* and *larger magnitudes have larger impacts*), Hakeem noted their agreement by saying, "so that I guess makes sense" (turn 130).

Hakeem then repeated this process for the second initial condition and used the same resources (*dependence* and *function slots*) to find its corresponding solution function. Using the resource *function slots* he attained the particular solution $P(t) = 5$ for the initial condition $P(0) = 5$. He then compared the behavior of *P* as indicated by the function $P(t) = 5$ to that indicated by the differential equation; "that's when there really wouldn't be a change cause that's when $\frac{dp}{dt}$ is zero. And so if you are starting off at 500, so that's why when you put in that initial condition it doesn't change with respect to $t$." This statement is indicative of Hakeem's utilization of *functional mapping* ("that's when $\frac{dp}{dt}$ is zero"), *rates indicate behavior* ("there wouldn't be any change cause … $\frac{dp}{dt}$ is zero"), and *functional dependence* "(it doesn't change with respect to $t$"). When Hakeem notes that *P* doesn't change with respect to $t$, he is providing a warrant for *P* not depending on $t$ in the second particular solution.

Although Hakeem was able to determine the general behavior of *P* from the differential equation, he utilized the particular solutions to gather more detailed information about *P,* which he then coordinated with the differential equation. This coordination supported the development of a cohesive prediction about the long-term behavior of $P$. For instance, Hakeem used the first particular solution to determine the value of the long term owl population, which he then used in the differential equation to support/verify the particular solution $P(t) = 5$. This helps illuminate what Hakeem meant by the solution being more specific about the behavior of *P* with respect to

$t$; it provided Hakeem with a specific value the population approached, whereas his reasoning with the differential equation only provided a general behavior.

In summary while completing this task Hakeem interpreted the differential equation in two ways: as a description of the behavior of the quantity of interest and as providing a model of the quantity of interest. These two interpretations are signified by the two approaches Hakeem took to predict the long-term behavior of the owl populations and the distinct set of resources associated with the approaches. While interpreting the differential equation as a description of the quantity of interest Hakeem utilized *variable substitution, rates indicate behavior, functional mapping, parts contribute to the whole,* and *larger magnitudes have larger impacts* to determine the behavior of the quantity of interest. Namely he used *variable substitution, functional mapping, parts contribute to the whole,* and *larger magnitudes have larger impacts* to determine how the values of $P$ related to $P'$, and then *rates indicate behavior* to determine that $P$ would generally decrease, unless $P$ was small. To determine the exact behavior of $P$, Hakeem used an analytical solution method to find the general solution. Once Hakeem attained the general solution, he used *dependence* and *function slots* to place the initial conditions into the general solution and find the solution pertaining to the specific initial conditions. He then applied *functional variation* to the solution function to determine the behavior of $P(t)$ as $t$ approached infinity. Thus, his first approach provided him with general characteristics of the behavior of $P$, (it would decrease except for small values of $P$), and his second approach allowed him to determine the exact nature of the populations for each initial condition.

**4.1.3.4 Discussion of interview 3.** Though the tasks in this interview were largely different, and there were clear differences in the resources Hakeem utilized to complete the tasks (see Table 4.4), his overall approach to completing each of them overlapped. Specifically, in each of the tasks Hakeem utilized the relationship between the variables in the differential equations to correlate changes in the value of the quantities with changes in the value of the rate of change, and then used the value of the rate of change to determine the behavior of the quantity of interest. While there were differences in the ways in which this occurred across the different tasks, generally speaking Hakeem utilized different combinations of the resources *variable substitution, functional mapping, functional variation, parts contribute to the whole, larger magnitudes have larger impacts,* and *positives and negatives take away* to characterize how the values of the quantities related to the values of the rate of change. He then used a combination of *rates indicate behavior*, and *unrestricted change* to determine how the values of the rate of change related to the behavior of the quantity. This general reasoning pattern and set of resources is indicative of interpreting the differential equation as a description of the behavior of the quantity of interest.

Additionally, in Tasks 2 and 3, Hakeem utilized the differential equation to determine a specific solution function that represented the equations that modeled the quantities of interest. Again, while there are slight differences in the ways Hakeem accomplished this across the two tasks, in general he made use of an analytical solution technique and then utilized *functional variation* by considering what happened to the value of the solution function as the independent variable approached infinity. By doing so Hakeem was able to use the model to then make inferences about the specific nature of the quantity of interest. This general reasoning pattern and the resources that support it are indicative of interpreting the differential equation as providing a model of the quantity of interest.

While these two approaches both provided Hakeem with a way of characterizing the values of the quantities in each task, each supported his ability to do so in different ways. More specifically the result of each reasoning pattern provided Hakeem with different information about the quantity of interest. Further, as seen in Task 3 Hakeem utilized information gleaned from the solution functions to reason about the behavior of the quantity of interest with the relationship between the variables in the differential equation. For instance, after determining that the value of the first particular solution approached 5 as $t$ approached infinity, Hakeem then substituted 5 for $P$ in the differential equation. In doing so he determined that when $P = 5$, $\frac{dP}{dt} = 0$, which he claimed "makes sense." In addition while completing Task 2 Hakeem utilized the relationship between the variables to infer that the value of $P$ would oscillate between positive and negative values. He then, however, utilized an analytical solution technique to find an equation relating values of $P$ to values of $t$, and determined that $P$ approached 0 as $t$ approached infinity. In doing so Hakeem was able to build a more complete description of the behavior of the quantity of interest ($P$).

*Table 4.4: Resources and interpretations Hakeem expressed in Interview 3.*

| Task | Interpretation of the differential equation | Resources | Interpretation of the components in the differential equation |
|---|---|---|---|
| I3-T1 | Description of the behavior of the quantities of interest | *Partial dependence*<br>*Parts contribute to the whole*<br>*Larger magnitudes have larger impacts*<br>*Functional variation*<br>*Rates indicate behavior*<br>*Unrestricted change*<br>*Functional mapping*<br>*Positives add-negatives take away* | $x$ and $y$ - population values, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ -rates of change of the respective populations. |
| I3–T2 | Description of the behavior of the quantity of interest | *Functional variation*<br>*Parts contribute to the whole*<br>*Positives add - negative take away*<br>*Rates indicate behavior*<br>*Unrestricted change* | *P*- Variable that changed with another variable<br>*P'* – Rate of change<br>*P"*- Rate of change of the rate of change |
| | Providing a model of the quantity of interest | *Method of characteristics*<br>*Functional variation* | $P(t)$ - Equation that related the population values to $t$. |
| I3-T3 | Description of the behavior of the quantities of interest | *Variable substitution*<br>*Functional mapping*<br>*Rates indicate behavior*<br>*Parts contribute to the whole*<br>*Larger magnitudes have larger impacts* | *P* – Value of population<br>*P'* – Value of the rate of change of the population |
| | Providing a model of the quantity of interest | *The definition of general solution*<br>*Dependence*<br>*Function slots*<br>*Functional variation* | $P(t)$ – Equation which related $t$ values to $P$ values. |

### 4.1.4 Interview 4

In interview 4 Hakeem completed three tasks; the analysis of each of them is presented in this section. The first task in this interview was the same as the first task from Interviews 2 and 3. The second task presented a graphical representation of an autonomous differential equation, and the third task presented a system of three differential equations. To complete Task 1 Hakeem utilized much of the same reasoning as he did while completing the task in the first two interviews. To complete the second and third tasks Hakeem made use of two sets of resources, each corresponding to a certain reasoning pattern. Each of these sets of resources and the pattern of reasoning associated with them supported Hakeem in completing the tasks in different ways.

**4.1.4.1 Interview 4 task 1.** The discussion of Hakeem's completion of Task 1 is relatively concise, as it is very similar to the approach from Interviews 1 and 2. The resources Hakeem used to complete Task 1 during Interview 4 are *parts contribute to the whole, larger magnitudes have larger impacts, functional dependence, functional mapping, unrestricted change,* and *rates indicate behavior.* In this interaction Hakeem interpreted the differential equation as a function that related quantities and their respective rates of change. Further, with regard to the differential equation he thought of the value of the quantities as inputs and the value of the quantities' respective rates of change as the output. Further, Hakeem was able to reason with this relationship over a continuum of values, which enabled him to not only reason about the behavior of the rates of change, but to characterize the behavior of the quantities themselves.

Hakeem started this task by considering the terms on the right side of each equation, and using them to determine when $\frac{dx}{dt}$ and $\frac{dy}{dt}$ would be positive, and when they would be negative. He did so by coordinating the resources *parts contribute to the whole, larger magnitudes have* *larger impacts,* and *functional dependence, and functional mapping.* For instance this can be seen in line 10 from the following transcript.

10 H: Yeah, and so if $x$ is really small and $y$ is large or $y$ is much greater than $x$, then this term, or the magnitude of this term will be much smaller than the magnitude of this term and so you would get a positive value minus this value and you'd end up with a positive $\frac{dx}{dt}$. And so that means that ah, $x$ is increasing so I guess.

11 G: The positive $\frac{dx}{dt}$?

12 H: Yeah. Yeah the positive $\frac{dx}{dt}$ means that $x$ is increasing with time

13 G: Ok.

14 H: And the same can be said for $\frac{dy}{dt}$ it is positive when $x$ is much larger than $y$ and its negative when $y$ is much larger than $x$. So what this means is that given some starting point and say we start with a pretty large value of $y$ and a small value of $x$, what's gonna happen is that $\frac{dx}{dt}$ will be positive and $\frac{dy}{dt}$ will be negative so $x$ will start increasing and $y$ will start decreasing and once and if they get to a point where $x$ is, like if they continue on that trend to a point where $x$ is large and $y$ is small it'll reverse so $x$ will be decreasing and $y$ will be increasing. So it'll basically like oscillate the values will just be going back and forth. So it'll be sustained but like as $x$ increases $y$, or a large value of $x$ is beneficial to, so like these are species, so $x$, I guess larger population for $x$ is beneficial for the growth of $y$. And a larger population for $y$ is beneficial for the growth of the $x$ species. And that the larger population of one species makes the derivative or the growth rate positive for the other species. And for system B, ah, it's the opposite.

In this case he considered how each $x$ and $y$ impacted $\frac{dx}{dt}$ (*functional dependence*), by looking at how the magnitudes of each of the parts contributed to the whole (*parts contribute to the whole* and *larger magnitudes have larger impacts*); "the magnitude of this term will be much smaller than the magnitude of this term and so you would get a positive… $\frac{dx}{dt}$." He then went on to do the same for $\frac{dy}{dt}$, and in doing so he correlated sets of values of $x$ and $y$ to sets of values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$, which is indicative of *functional mapping*. After which he coordinated the resources *rates indicate behavior* and *unrestricted change* to determine that System A is cooperative. This can be seen in turn 14 when Hakeem utilized the resource *rates indicate behavior* to determine "what's gonna happen is that $\frac{dx}{dt}$ will be positive and $\frac{dy}{dt}$ will be negative so $x$ will start increasing and $y$ will start decreasing," and then coordinated the resource *unrestricted change* ("if they continue on that trend to a point where $x$ is large and $y$ is small it'll reverse") to determine "that the larger population of one species makes the derivative or the growth rate positive for the other species." That is to say, by coordinating *rates indicate behavior* and *unrestricted change*, Hakeem determined the nature of the relationship between $\frac{dx}{dt}$ and $\frac{dy}{dt}$, and $x$ and $y$. Namely he was able to determine "the larger population of one species makes the derivative or the growth rate positive for the other species."

After making a determination about System A, Hakeem then moved to System B:

> "So for system B, $\frac{dx}{dt}$ is positive when $x$ is larger than $y$ and it's negative when $y$ is larger , much larger than $x$. And you could say the same for $\frac{dy}{dt}$. So what that means is ah, that a large value of $y$ will only make $x$ decrease, the population of the $x$ species decrease further. So the larger population of the $y$ species is harmful to the growth of $x$. And the other way around, a larger population of the $x$ species is harmful to the growth of $y$" (turn 16).

To reason about System B Hakeem expressed coordinating *functional mapping, functional dependence* and *rates indicate behavior* to determine that large values of one species are detrimental to the growth of the other species. For example, Hakeem reasoned that the values of $x$ and $y$ determined the value of $\frac{dx}{dt}$ (*functional dependence*), that larger values of $y$ would cause $\frac{dx}{dt}$ to be negative (*functional mapping*), and negative values of $\frac{dx}{dt}$ indicate a decrease in $x$ (*rates indicate behavior*). By coordinating these resources (for both equations in the system) Hakeem determined that larger populations of one species were harmful to the other species.

Having considered both systems, Hakeem then characterized the two systems, as can be seen in the following transcript.

18 H: So this one would be cooperative [points to System A] and this one [points to System B] would be competitive.
19 G: Ok.

20 H: And yeah cause in this one the species are competing because ah, one is ah, ah, basically the one that's growing is the one that will like eventually ah, the other one would eventually die out
21 G: Gotcha. Ok so let me go back to this, cause you have little bit more written out here.
22 H: Yes.
23 G: Um, so when you say $y$ is much much greater than $x$, or $x$ is much, much greater than $y$, are you thinking about these as changing things? As changing values or are you thinking about these in just instances when this one is greater than this one?
24 H: Yeah I am thinking of them as instances, that's sort of like initial conditions maybe. So if we start off with just a really large value of $y$ and small value of $x$, so I am thinking of plugging in different values.
25 G: Gotcha.
26 H: And then so yeah I am like plugging in values in this equation where the setup of this equation is that $x$ and $y$ are like inputs and the derivative is the output.
27 G: Gotcha.
28 H: So I am just thinking of the instances that would make this positive or this negative, and then once you get that this will tell you this like trend or in the growth of $x$ or the trend in the growth of $y$.

Based on the justification he provided in turn 28, Hakeem utilized two steps to determine which system was cooperative and which was competitive. First he determined the correlations between the sets of values of $x$ and $y$, and sets of values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ , and then he used these correlations to determine the behavior of the species. This can be seen in turn 26 for example, "the setup of this equation is that $x$ and $y$ are like inputs and the derivative is the output." Further elaborating on this in turn 28 he said, "so I am just thinking of the instances that would make this [points to $\frac{dx}{dt}$] positive or this negative" (*functional mapping*). Considering these statements in conjunction with those in turns 14 and 16, Hakeem was using the differential equation as a way to correlate sets values of $x$ and $y$ with values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ to make determinations about the behavior of the species from he could infer which systems represented the competing and cooperative species. In doing so he expressed interpreting each of the differential equations as a description of the behavior of the quantity of interest.

In summary Hakeem utilized the resources *parts contribute to the whole, larger magnitudes have larger impacts, functional dependence, functional mapping, unrestricted change,* and *rates indicate behavior* to complete this task. Specifically, the resources *functional mapping, functional dependence, parts contribute to the whole, larger magnitudes have larger impacts* supported Hakeem in determining how the value of the quantities related to the values of their rates of change. In addition the resources *rates indicate behavior* and *unrestricted change* supported Hakeem in determining how the sets of values of $x$ and $y$ related to their growth. In short, these resources afforded Hakeem with the ability to describe how the values related to each other, and then make use of that relation to determine their behavior. It was from their behavior that Hakeem was able to determine which system represented the cooperative species and which represented the competitive species.

**4.1.4.2 Interview 4 task 2.** While completing Task 2 (see Figure 4.12) Hakeem, determined an algebraic representation of the differential equation and then used an analytical solution method to determine an equation for the quantity of interest. After expressing this provided him a limited amount of information, he then used the differential equation to relate values of $y$ to their respective values of $\frac{dy}{dt}$, and used the values of $\frac{dy}{dt}$ (in this case as slopes) to determine the behavior of the quantity of interest. In completing this task Hakeem collectively utilized the resources *rates indicate behavior, variable substitution, functional mapping, continuity, functional dependence, separation of variables*, and *functional variation.* These resources were utilized by Hakeem to productively reason with the different representations of the differential equation, and its solutions. These resources align with two interpretations of the differential equation, as a description of the behavior of the quantity of interest, and as an equation that provides a model of the quantity of interest. The following paragraphs will support this claim.

TASK 2: Below you are provided with a graph of a rate of change equation rather than the equation itself (Note that $dy/dt$ depends only on $y$). Figure out the long-term behavior of possible solution functions, illustrate your conclusions with a suitable graph or graphs, and state your conclusions about the long-term behavior of these solutions.

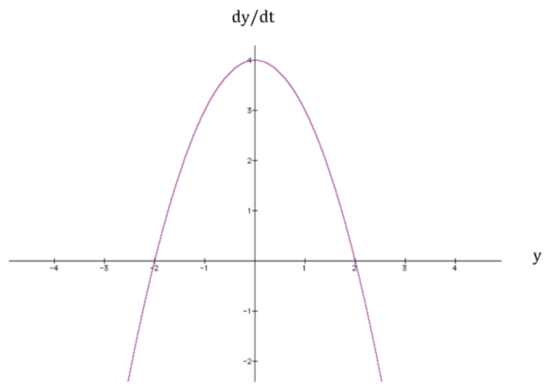



*Figure 4.12: Task 2 from Interview 4*

I start the discussion with transcript that begins with Hakeem's first statements.

44 H: So, if $\frac{dy}{dt}$ only depends on time, say, what this looks like is just a parabola with... [writes $\frac{dy}{dt} = -y^2 + 4$]. Looking at the graph you can derive this equation for $\frac{dy}{dt}$. And, you could solve this for $y$.
45 G: Ok
46 H: So if you had, [begins solving it the way he solves I3-T2, silent for 3 minutes]
47 H: So in this form of the equation I have it in function of $y$ in terms of a function of $t$. So it's, $f(y) = g(t)$, as to the long term behavior of the function if we look as $t$ approaches infinity this side goes to zero, the limit as $t$ approaches infinity of I'll just call it $f(y)$ will be zero. so then $y - 4$ would be zero, so $y$ would equal 4. So basically the solution, wherever, it would plateau off at $y = 4$.
48 G: Ok.
49 H: And at $t = 0$, you have $y - 4 = c1 + 4c$,
50 G: So what are you trying to figure out now?
51 H: Well right now I was trying to figure out I guess, cause I know that, like towards, at the end, it plateaus off, but I guess I'm trying to picture how $y$ changes with $t$, like direct of function. So I guess like how it would look in a graphical sense.
52 G: And what are you using to try and figure that out?
53 H: At first I was just thinking about plugging in different values.
54 G: For?
55 H: I guess $t$ and $y$.

56 G: Into where?

57 H: Into this equation [points to the equation for $\frac{dy}{dt}$], but you could also use the $\frac{dy}{dt}$ graph here.

In turn 45 Hakeem coordinated the resources *functional dependence* and *functional mapping* to determine how the values on the $y$-axis related to values on the $\frac{dy}{dt}$-axis. More specifically he created an algebraic representation of the graph by utilizing these resources. That is, Hakeem considered $\frac{dy}{dt}$ to be dependent on $y$, and figured out a way to algebraically represent the mapping present in the graph ("what this looks like is just a parabola…"). He determined the equation which represented the graph was $\frac{dy}{dt} = -y^2 + 4$, and began to solve it analytically using *separation of variables*; he remained silent for this process. As a result of this process he attained the solution $\frac{y-4}{y+4} = -Ce^{-8t}$(this is not the correct solution, but is later corrected by Hakeem), and in turn 47 he utilized *functional variation* to determine the behavior of $y$ as $t$ approached infinity. Specifically he determined that the solutions would tend to 4 as $t$ approached infinity. In turn 51 he expressed that this process provided him with the value the solutions would tend to, but not "what it would look in a graphical sense." To build a more complete picture, he expressed a desire to use the differential equation to determine what the solutions would look like graphically. Specifically, he was trying to see how $y$ changed with $t$ at various locations along the solution curve, and to do that he was considered "plugging in different values" (*variable substitution*) for $y$ and $t$ in the differential equation to get corresponding values of $\frac{dy}{dt}$. However instead of using the algebraic form of the differential equation Hakeem decided to use the graphical representation presented in the task. This is depicted in the following transcript.

59 H: Ah, so this is basically just, these tangent lines are the slopes at different points, when at different values of $y$. So, when $y$ is zero the slope is ah, the slope is 4. So if you had a I

guess a graph of [draws $y$ and $t$ axis] so at $y = 0$ , the slope is pretty high, so if you drew like a slope field, you'd have a high slope at $y = 0$ [marks this on the slope field].

60 G: Ok.

61 H: At $y = 2$ and -2 you'd have a since this is independent of $t$ they are autonomous, so the slope would be the same throughout the $t$ axis [draws equal slopes across the $t$-axis].

62 G: Ok.

63 H: And at $y = 2$ and -2, the slope is zero [drawing slope field]. So I guess, you could say that solution curves look like this in a sense. Start from like -2 and I guess end at 2.

64 G: How do you know that that is what it looks like?

65 H: Um, well, when you draw it in terms of $y$ and $t$, but then put in the slopes, when you plug in, so if you plug in values for $y$ and $t$, or in this case just $y$ because its autonomous.

66 G: Mhm.

67 H: You get the slope at the point, and the slope is basically the direction the instantaneous direction of the curve. So, if you have different slopes, you can, they'll tell you exactly how the curve is going to move.

In turn 59 Hakeem noted that the graph of the differential equation represented the slopes at different $y$ values, and used this to create a slope field. That is, Hakeem utilized the correlation between values of $y$ and values of $\frac{dy}{dt}$ to draw appropriate tangent vectors at specific $y$ values. Here Hakeem expressed the resource *functional mapping*. That is he expressed that for each $y$ value there is a corresponding slope value. In addition, he utilized the resource *functional dependence* to infer that $\frac{dy}{dt}$ does not depend on $t$ in turn 61. With this inference he then drew equal slopes across the values of $t$ to construct the slope field.

After drawing the slopes along $y$ values of -2, 0, and 2 Hakeem then in turn 63 reasoned with the slope field to draw the solution function between -2 and 2. In turn 67 he explained that the slopes “tell you exactly how the curve is going to move.” This is indicative of the resource *rates indicate behavior*. I claim however that he also made inferences about the behavior of the solution functions by coordinating the resources *continuity,* and *functional variation*. This can be seen in the following transcript, which is the result of questioning Hakeem with regard to how he knew the shape of the solutions.

68 G: Ok, so I understand how you got the zero here [points to $y = 2$], the zero here [points to $y = -2$], and the four [points to $y = 0$], for your slopes here, I'm just, I'm not disagreeing that this is what it looks like, I'm just wondering how you filled in between zero and negative 2 and 0 and positive 2. Like, how did you know that's what it would look like?

69 H: Ah, well you can also, I guess plug in multiple points here, at one we have a slope of ah, about, you have a slope of 3, so, so at 1 the slope is still pretty steep but not as steep. So you can still like connected in a sort of sense and basically for this graph as the, as $y$ goes from zero to negative two, the slope is decreasing but it's still positive until it gets to negative 2.

In turn 69 Hakeem utilized *variable substitution*, and *functional mapping* to show that $y$ values of 1 correlate to $\frac{dy}{dt}$ values of 3. However, I take the phrase "well you can also…" as indicating that this was not the way in which he reasoned about the solution curves originally. That is, I claim these resources were only utilized as an additional way to justify his reasoning. On the other hand I take the phrase, "so you can still like…" as indicating reasoning that was representative of the original reasoning he used. In this case, Hakeem coordinated the resources *functional variation* and *continuity* to infer that "as $y$ goes from zero to negative two, the slope is decreasing but it's still positive until it gets to negative 2." More specifically, he reasoned that as $y$ decreased to -2, the value of the slope was decreasing in a smooth fashion; "like connected in a sort of sense." That is, Hakeem inferred that the value of the slopes would be decreasing, and connected these decreasing slopes to generate the solution functions. Hakeem then used similar reasoning to fill in the solutions above 2 and below -2.

In the analysis presented in the previous paragraphs, I identified the resources Hakeem utilized while completing the task. These resources coincided with two reasoning patterns, first he used separation of variables to find an equation which related the quantity of interest to an independent variable, and then applied functional variation to the solution to determine the value the quantity of interest approached as the independent variable approached infinity. In this case, Hakeem interpreted the differential equation as providing a model; that is, an equation that specifies how the quantity of interest relates to an independent variable. After noting the equation did not provide him with "what it would look in a graphical sense," he began reasoning directly from the relationship between $\frac{dy}{dt}$ and $y$ in the graphical representation of the differential equation. To do so he utilize the resources, *variable substitution, functional mapping,* and *functional dependence* to determine how the values of $y$ related to values of $y'$ correlating them graphically on a slope field. He then used the *rates indicate behavior* and *continuity* to make inferences about the nature of the graphs of the solution functions. This reasoning pattern and corresponding resources are indicative of interpreting the differential equation as a description of the quantity of interest (in this case $y$).

In summary Hakeem used the graphical representation presented in the task to generate an algebraic representation, which he then solved using analytical methods (*separation of variables*). This provided Hakeem with a way to determine the long-term behavior of the solution function(s) but admittedly little insight concerning the graphical nature of the solutions. To determine the graphs of the solutions Hakeem reasoned with the graphical representation of the differential equation to determine the local behavior of the solutions. He did this with the aid of a slope field, which he generated by reasoning with the graph presented in the task. In addition, the resources *functional mapping,* and *functional dependence* afforded Hakeem with the ability to generate both representations of the differential equation (i.e., the slope field and algebraic expression). The resources *rates indicate behavior, functional variation,* and *continuity* supported Hakeem in using the slope field to generate the graphs of the solutions, and *functional variation* supported Hakeem in determining the long-term behavior of the quantity of interest from the equation which resulted from the analytical solution method.

**4.1.4.3 Interview 4 Task 3.** This task asked Hakeem to interpret a system of three differential equations (see Figure 4.13). While completing the task Hakeem utilized the resources *functional dependence, variable substitution, rates indicate behavior, functional variation,* and *equality condition.* Hakeem utilized two combinations of these resources while completing this task, each representing a different interpretation of the differential equation and its components. Based on his usage of these resources I assert Hakeem interpreted the differential equations in two ways, as a relationship between the value of a quantity and the value of the quantities rate of change, and as a relationship between a function and its derivative function. I begin to support my claim of his utilization of these resources with a segment of transcript from the beginning of the interaction.

TASK 3: What does the following mean to you?

$$y_1{}' = 3y_1 + 3y_2 - 6y_3$$
$$\mathrm{y_2}{}' = 5\mathrm{y_1} - 8\mathrm{y_2} + \mathrm{y_3}$$
$$y_3' = -10y_1 + y_2 - 4y_3$$

*Figure 4.13: Task 3 from Interview 4*

144 H: All right, so this is another system of equations. Like, so I am going to look at it like this, I'll just write it out as this, $y'$ is equal to Ay. *S*o you could write $y_1{}'$ , $y_2{}'$, $y_3{}'$ equals, 3, 3, -6, 5, -8, 1 10, 11, -4. And basically this it's just like in the first task so it's a system where, you have basically three different variables $y_1$, $y_2$, $y_3$, they could represent anything be it species, or I mean I don't know what else, and but, it's like in task one where, cause task one was a two species $x$ and $y$, but they weren't independent of each other. Ones value effected the growth of another one and it's the same here. So, when $y_1{}'$, or the derivative of $y_1$, has $y_2$ and $y_3$ in it, so that means different values of $y_2$ and $y_3$ will give you, will also affect $y_1$, and that in turn will also affect $y_2$ and $y_3$ because their derivatives are affected by values of $y_1$.

145 G: Mhm. Ok so it sounded like you were talking about a relationship between $y_1$, $y_2$, and $y_3$, there.

146 H: Yeah.

147 G: Um is there a relationship between $y_1$, $y_2$, and $y_3$, prime. Each of those prime?

148 H: Ah, well, I guess the way I start off as looking at it is, first I look at so what are the values of, like if I just start off like I started with task one, so I plug in values and see, given certain $y_1$, $y_2$, $y_3$ I'd get a value of $y_1{}'$ but that value of $y_1{}'$, since these are all changing as time, when you plug in certain values that's just what $y_1{}'$ is at an instance, but that instance, what $y_1{}'$ at that instance is, is it's telling us how it's going to change so that at a different time, it's telling you that you are going to have a different value of $y_1$. The way, so then you are gonna have basically I guess say if $y_1{}'$ was positive then at a later time $y_1$ would be greater since the value of $y_1$ is changed, you know when you plug that into $y_3{}'$ say, you are

gonna get a certain value of $y_3'$. so $y_1'$ effects $y_1$, $y_2'$ effects $y_2$, $y_3'$ affects $y_3$, and in turn since $y_1$, $y_2$, $y_3$ are in each equation, they are gonna effect how ah, I guess the other $y$'s.

Hakeem began by coordinating the resources *functional dependence* and *rates indicate behavior.* The utilization of *functional dependence* is evident when Hakeem inferred the value of $y_1'$ is dependent on the values of $y_2$ and $y_3$; "So, when $y_1'$, or the derivative of $y_1$, has $y_2$ and $y_3$ in it…" He coordinated this with the resource *rates indicate behavior* when he went on to say, "…so that means different values of $y_2$ and $y_3$ will give you, will also affect $y_1$." In this portion of his statement Hakeem utilized the resource *rates indicate behavior* by suggesting the value of $y_1$ (which is dependent on $y_2$ and $y_3$) will change because of $y_1'$. In addition this segment of transcript is also indicative of the resources *functional variation*; this is perhaps more clear in the end of his statement "and that in turn will also affect $y_2$ and $y_3$ because their derivatives are affected by values of $y_1$." In other words, Hakeem coordinated the resources *functional dependence*, *rates indicate behavior,* and *functional variation* to infer that the values of the quantities affect the value of the derivative (*functional dependence*), which then cause the value of each of the quantities to change (*rates indicate behavior*, *functional variation*).

After discussing the relationship between the terms in the differential equation, I asked Hakeem what it meant to be a solution to the system of equations in turn 149.

149 G: Ok, last question. What does it mean to be a solution to this system?
150 H: Like it's a condition where this is true, so in this one [points to Task 2] it was a condition where $\frac{dy}{dt}$ is negative $y^2 + 4$, but in this sense its a little I guess more complex because you have three different ah equations. So the solution is going to be ah, basically it's just gonna be a condition where, so basically you have ah, your gonna, since you have three equations you are gonna have basically these are three conditions you have to satisfy. So, a solution will be in terms of an equation where $y_1 = f(t)$, $y_2 = g(t)$, $y_3 = h(t)$, but each, but when you, your gonna have to be able to put all of these together to satisfy all three of these conditions for it to be a valid solution.
151 G: Ok. Before, just a clarifying question. You had said the word condition before and it sounded like you were referring to the solutions?
152 H: Yeah that's, pretty much

153 G: In other words yeah. Here it sounds like you are referring to these [the differential equations] as the conditions, so I'm just wondering. If you could just clarify that for me,
154 H: Well you can think about it both ways, as one being the condition or the other being the condition.
155 G: Ok.
156 H: So, the like, for this [Task 2] one I was looking at, well, this will if you look at say a single solution here [points to slope field] you are gonna get $y = f(t)$, and when you have the derivative you have $\frac{dy}{dt}$ is equal to I guess I'll say $g(y)$.
157 G: Ok.
158 H: So you could say that this is a condition that $y$ has to satisfy, to be a valid solution or you could say that I guess you could still say that the solution itself is in a sense a condition but I guess it's, like, I guess to correct myself I 'd say it's probably better to say, I think I meant this equation [the differential equation] is the condition because ah, you are looking, like if this is a condition, what you try to find a solution you are looking to satisfy the condition. And whether you satisfy it or not, that is what determines if the solution is valid or not.
159 G: Ok, so these are the conditions.
160 H: Yeah when you have three equations, you could say that you have three different conditions that you have to satisfy.

Here Hakeem utilized the resources *variable substitution*, *functional dependence*, and *equality condition* while explaining characteristics of solutions to differential equations. In addition this segment indicates that Hakeem was interpreting the differential equation as a relationship between functions and their derivative functions. To support this, consider the statement he made in turn 156, regarding the exact condition solutions must satisfy: "if you look at say a single solution here you are gonna get $y = f(t)$, and when you have the derivative you have $\frac{dy}{dt}$ is equal to I guess I'll say $g(y)$." Here Hakeem was indicating that solutions ($y = f(t)$) must be such that $\frac{dy}{dt} = g(y)$. I take this to mean that, $\frac{dy}{dt}$ must equal $g\big(f(t)\big)$ for solutions to satisfy the differential equation. I make this determination based on Hakeem's language in turn 150, where he discussed needing to "put all of these [solutions] together to satisfy all three of these conditions." Here Hakeem was discussing putting the solutions into the differential equations (*variable substitution*) to verify the equality in the differential equations is retained (*equality condition*). In addition his statement in turn 156 is indicative of *functional dependence*;

the notation he used implies $f$ depends on $t$ and that $g$ depends on $y$. This interpretation of Hakeem's utilization of functional dependence is not made strictly based on the mathematical meaning associated with the notation. Hakeem discussed this explicitly while discussing Task 2.

132 H: So a function is like, you have an input and an output, or you have one input or multiple inputs, and an output. When I was looking at, ah, $\frac{dy}{dt} = f(y)$, and like I got $\frac{dy}{dt} = -y^2 + 4$, I was looking at this as a function where and this is basically how I drew the direction field, slope field, I drew, I was looking at this [points to $\frac{dy}{dt}$] as an output and this, $y$, as an input.

In summary, Hakeem used two different combinations of the resources *functional dependence*, *variable substitution*, *rates indicate behavior*, *functional variation and equality condition* while reasoning with the systems of differential equations. These two combinations of resources and the patterns associated with them indicate he interpreted the differential equation as a relationship between the value of the quantities of interest and the values of their respective rates of change, and as a relationship between a function and its derivative function. Hakeem initially was discussing the relationship between the values of the quantities and the value of their respective rates of change; "I plug in values and see, given certain $y_1$, $y_2$, $y_3$ I'd get a value of $y_1{}'$"(turn 148). While doing so he used the resources *functional dependence*, *variable substitution* and *functional variation* to determine how the values of $y_1$, $y_2$, and $y_3$, related to values of their respective rates of change, $y_1{}'$, $y_2{}'$, and $y_3{}'$. In addition he used *rates indicate behavior* to make inferences about how the values of $y_1$, $y_2$, and $y_3$ would change based on the values of $y_1{}'$, $y_2{}'$, and $y_3{}'$. After I asked him to discuss the solution to differential equations his language indicated he thought of the variables as functions. In addition Hakeem utilized function notation when discussing how $y_1$, $y_2$, and $y_{3,}$ were related to $\frac{dy}{dt}$. Within this interpretation the resources *equality condition, functional variation, and variable substitution* supported Hakeem in inferring that variables in the differential equation were functions that had to be such that they retained the equality in the differential equation.

**4.1.4.4 Discussion of interview 4.** Over the course of the interview Hakeem utilized a large number of resources (see Table 4.5), however, not all of these resources were utilized while completing each task. That is, Hakeem utilized different sets of resources while completing the various tasks, and furthermore, used those sets of resources in conjunction with different reasoning patters. Each of these sets of resources and corresponding reasoning patterns is indicative of interpreting the differential equation in a certain way.

For instance in the first two tasks Hakeem utilized different combinations of *variable substitution, functional mapping, functional dependence,* and *parts contribute to the whole* to reason about the relationships between the variables in the differential equations and then coordinated *rates indicate behavior* to make inferences about the behavior of the quantity of interest bases on the values of their rates of change. In this case Hakeem interpreted the differential equation as a description of the behavior of the quantity of interest. In the first task, this was expressed verbally (e.g., "that a large value of $y$ will only make $x$ decrease...so the larger population of the $y$ species is harmful to the growth of $x$"), Task 2 this was expressed graphically through the construction of the slope field and the corresponding graphs of solution functions.

In addition, in the second task Hakeem also made use of an analytical solution method (*separation of variables*) to find the equation that related the quantity of interest to its independent variable, (in this case $\frac{y-4}{y+4} = -Ce^{-8t}$). Hakeem then coordinated *functional variation* with this equation to determine the value $y$ approached as its independent variable approached infinity. This supported Hakeem in completing the task, because he was able to determine the long term behavior of the solutions (approaching 4), but left him dissatisfied because he did not know what the solution looked like graphically; this highlights the differences in the ways the interpretations supported Hakeem in completing the different tasks. In the case of utilizing the resources *separation of variables* and *functional variation* in the pattern as outlined in this paragraph, Hakeem interpreted the differential equation as providing a model of the quantity of interest; specifically, $\frac{y-4}{y+4} = -Ce^{-8t}$.

While completing Task 3, Hakeem expressed working with similar resources but in a slightly different way. Initially he used the resources *functional dependence*, *variable substitution*, *rates indicate behavior*, and *functional variation* to determine how the values of the quantities of interest were related to the values of the quantities' rates of change. It is important to note that while Hakeem utilized rates indicate behavior, he did not do so to characterize the behavior of the quantities of interest, rather he utilized it to connect changes in the value of the $y_1'$, to changes in the values of $y_2$ and $y_3$:

> "So, when $y_1'$, or the derivative of $y_1$, has $y_2$ and $y_3$ in it, so that means different values of $y_2$ and $y_3$ will give you, will also affect $y_1$, and that in turn will also affect $y_2$ and $y_3$ because their derivatives are affected by values of $y_1$" (turn 144).

In addition, to discuss what it meant to be a solution to a differential equation Hakeem expressed interpreting the differential equation as a relationship between a function and its derivative function. That is, Hakeem utilized *functional variation*, *variable substitution*, and *equality condition* to discuss the differential equations, as being conditions the solution functions must satisfy.

*Table 4.5: Resources and interpretations Hakeem expressed in Interview 4.*

| Task | Interpretation of the differential equation | Resources used to complete the task | Interpretation of the components in the differential equation |
|---|---|---|---|
| I4-T1 | Description of the behavior of the quantities of interest | *Parts contribute to the whole*<br>*Larger magnitudes have larger impacts*<br>*Functional dependence*<br>*Functional mapping*<br>*Unrestricted change*<br>*Rates indicate behavior* | $x$ and $y$ - Population values<br>$\frac{dx}{dt}$ and $\frac{dy}{dt}$ -2 rates of change of the respective populations. |
| I4 –T2 | Description of the behavior of the quantities of interest | *Rates indicate behavior*<br>*Variable substitution*<br>*Functional mapping*<br>*Continuity*<br>*Functional dependence* | $y$ – value of the quantity of interest,<br>$\frac{dy}{dt}$ - rate of change of the quantity of interest |
| | Providing a model of the quantity of interest | *Separation of variables*<br>*Functional variation* | $y(t)$ – Equation relating the quantity of interest to the value of an independent variable |
| I4 –T3 | Relationship between the value of the quantity of interest and the rate of change of the quantity of interest | *Functional dependence*<br>*Variable substitution*<br>*Rates indicate behavior*<br>*Functional variation* | $y_1, y_2, y_3$ – Values of the quantities of interest<br>$y_1', y_2', y_3'$ – Values of the rate of change of the quantities of interest |
| | Relationship between a function and the function's derivative | *Functional variation*<br>*Variable substitution*<br>*Equality condition* | $y_1, y_2, y_3$– Solution functions<br>$y_1', y_2', y_3'$ – Derivatives of solution functions |

### 4.1.5 Interview 5

With the exception of the first task, Interview 5 was used as an opportunity to re-present the participants with certain tasks that were only presented once during the first four interviews. In doing so the aim of this interview was to collect data that would allow for identifying any observable changes in the sets of resources or interpretations the students applied to the tasks. Hakeem completed six tasks during this interview; I report analysis on the first five.

As the analysis will show Hakeem demonstrated an admirable ability to interpret the differential equations and their components in numerous ways. Additionally each of these interpretations corresponded to a set of resources that supported Hakeem in completing the various tasks; some tasks he completed in multiple ways, in others the multiple interpretations supported each other or provided him with different information. Lastly, Hakeem was able to apply great deal of the resources he expressed during this interview to different mathematical objects (e.g., values, functions, derivatives, slopes) and to different representations of them (e.g., slope field, graph of $\frac{dy}{dt}$ with respect to $y$). In doing so Hakeem demonstrated great complexity in organizational structure of these resources, in terms of their applicability and utility.

**4.1.5.1 Interview 5 task 1.** The first task in Interview 5 solicited students' conceptualizations of differential equations by prompting them to "tell me what it means for something to be a differential equation." While this task did not present the students with a problem to solve, and was not very effective in eliciting resources (Hakeem expressed *functional dependence*, *functional mapping*, *variable substitution*, and *rates indicate behavior*), presenting the analysis of this task is important because it provides a great deal of insight into the various ways Hakeem interpreted differential equations. While completing Task 1, Hakeem expressed three ways of interpreting differential equations, as a relationship between the value of the quantities and the value of the rate of change of the quantity of interest, as a description of the behavior of the quantity of interest, and as providing a model of the quantity of interest. His response is also indicative of the ways in which he interpreted the different variables in the differential equation. Thusly, presenting this task is beneficial for identifying aspects that influenced the resources he coordinated to complete the various tasks presented in the interview.

In responding to the prompt, Hakeem began by providing characteristics of differential equations. This can be seen in the portion of transcript that follows

4 H: So, I guess for something to be a differential equation the variables would have to be, you'd have to have some variable in terms of another variable so like $y$ in terms of $t$ for example.
5 G: Mhm.
6 H: And then you'd they would have to have its derivative so $\frac{dy}{dt}$ in that same equation. So it's an equation that has um, I guess there's a lot of different ways to write it. Like one would be the derivative as a function of $y$ and $t$. So if you have a $\frac{dy}{dt}$ equals…
7 G: You can write them if you want if that…
8 H: Yeah there's, so with the ah, independent variable, so that means $y$ is a function of $t$, and then it's basically an equation that has $y$, $t$, and $\frac{dy}{dt}$ in it. So one way to write it would be $\frac{dy}{dt}$, is a function of $y$, $t$ [writes, $\frac{dy}{dt} = f(y, t)$].

In this segment Hakeem stated that differential equations "have to have some variable in terms of another variable so like $y$ in terms of $t$ for example" (turn 4), "and then you'd they would have to have its derivative so $\frac{dy}{dt}$ in that same equation," and lastly that "it's basically an equation that has $y$, $t$, and $\frac{dy}{dt}$ in it. So one way to write it would be $\frac{dy}{dt}$, is a function of $y$, $t$." Here Hakeem utilized *functional dependence* in two ways; first he noted that $y$ is dependent on $t$, and second he noted that $\frac{dy}{dt}$ is dependent on $y$ and $t$. As will be discussed later by Hakeem (see turns 22 – 28; follows), here Hakeem was interpreting the differential equation as a relationship between the values of quantities and the value of the rate of change of the quantity of interest.

More specifically Hakeem's statements in turns 4, 6, and 8 indicate that he was thinking of the differential equation as a relationship between the values of $t$ and $y$, and the value of $\frac{dy}{dt}$.

After turn 8 Hakeem provided two examples of differential equations of the form $\frac{dy}{dt} = f(y, t)$, which formed the foundation for some of his statements later in the discussion about this task; $\frac{dy}{dt} = \sin(t) + \frac{\ln(y)}{t}$ and $\frac{dy}{dt} = ky + ct$, where $k$ and $c$ are constants. In his discussions about these differential equations he made use of the resources *functional dependence*, and *rates indicate behavior*. In the transcript that follows Hakeem utilized these examples to answer my question, "are you thinking of $y$ as um, a value, as a number, or are you thinking of it as something that is also in terms of $t$." I asked him this question because he referred to differential equations as functions and as having "some variable in terms of another variable" (turn 4), for this reason it wasn't clear how he was thinking of $\frac{dy}{dt}$ or $y$.

22 H: So, like what I was saying earlier is that if you look back to this equation [points to $\frac{dy}{dt} = f(y, t)$], whatever you put in for $y$ and $t$ you get something for $\frac{dy}{dt}$ in that so you can look at it as its own separate variable, that's a function of the other two variables. But at the same time $\frac{dy}{dt}$ is almost not another variable in a true sense because it's, it's like, it's an operation on $y$. The derivative is just another operation. So like it'd be like saying $\sin(y)$ is equal to a function of $y$ and $t$.
23 G: Mhm.
24 H: So, while you can treat it as its own variable, it's not really separated from $y$ in that it's really just an operation on $y$, and I guess when you solve the differential equation that's what lets you go back and get like a function of $y$. Like when you solve this you get to $y$ as a function of $t$.
25 G: Ok.
26 H: So to answer your question about whether I think of $y$ as just value or like a function of $t$, like, written in this style, [points to $\frac{dy}{dt} = ky + ct$] I'd say that if $t$ was just a value in that, these are all different variables, but when you solve it, it lets you get to write it in a form of $y$ as a function of $t$.
27 G: Ok.
28 H: So it is a function of $t$, but I just look at it as its own value until I solve it.
29 G: Ok. Cool. Is there anything else about differential equations in general before we move on?

30 H: Ah, I guess one thing to note is that a differential equation I guess it's all about solving to find $y$ as a function of $t$, so it really is just like another way of writing ah, so when you write $y = f(t)$ what this is , basically it defines a curve.

31 G: Mhm.

32 H: So, ah, $\frac{dy}{dt}$ so like when you have $\frac{dy}{dt}$ is a function of $y$ and $t$, that's another way of I guess describing the curve, in that you are describing, it's the given equation for like the slope of the curve.

In turn 22 Hakeem noted, "whatever you put in for $y$ and $t$ you get something for $\frac{dy}{dt}$," which is indicative of the resources *functional dependence* and *variable substitution*. This supports my claim that in turns 4-8 Hakeem was interpreting the differential equation as a relationship between the values of quantities the value of the rate of change of the quantity of interest, or as he stated in turn 4, as "some variable in terms of another variable." In turn 26 Hakeem noted that he thought of $y$ as a value in the differential equation, but also as a function whose algebraic form results from solving the differential equation. That is, he suggested that he is aware that $y$ represents a function, but he utilizes it as a value within the differential equation. These statements relate to his statement in turn 30 when he noted that differential equations are "all about finding $y$ as a function of $t$." This is commensurate with reasoning patterns associated with interpreting the differential equation as providing a model of the quantity of interest.

Additionally, in turn 32, Hakeem coordinated the resources rates *indicate behavior* and *functional mapping* to explain that differential equations are "when you have $\frac{dy}{dt}$ is a function of $y$ and $t$, that's another way of I guess describing the curve, in that you are describing, it's the given equation for like the slope of the curve." In this case I argue that Hakeem was interpreting the differential equation as a description of the behavior of the quantity of interest. More specifically, Hakeem was explaining that one utilizes the value of the function and the value of its derivative to determine the behavior of the solution function.

In summary, Hakeem utilized the resources *functional dependence, variable substitution, functional mapping* and *rates indicate behavior* while discussing characteristics of differential equations. The resources *functional dependence, variable substitution* supported Hakeem in working with the differential equation as a relationship the value of quantities and the values of the rate of change of the quantity of interest, while *functional mapping* and *rates indicate behavior* supported Hakeem in utilizing the differential equation as "another way of I guess describing the curve." Additionally, Hakeem provided insight into how and when he thinks of $y$ as a value as opposed to $y$ as a function. Hakeem expressed an awareness of $y$ representing a function, though he noted with regard differential equations he thinks of $y$ as a value in a differential equation but as a function that the differential equation is describing, or results from solving the differential equation.

**4.1.5.2 Interview 5 task 2.** While completing Task 2 (see Figure 4.14) Hakeem utilized the resources *functional dependence, functional variation, functional mapping, variable substitution, rates indicate behavior, equality condition* and *the method of eigenvalues.* More specifically, Hakeem utilized these resources while solving the system of differential equations, and while discussing the significance of the solutions in terms of their relation to the system of differential equations. In this way, these resources are not only indicative of how Hakeem solved the differential equation and graphed the solutions, but also of how Hakeem conceptualized solutions to differential equations (and their graphs) in general. Additionally, Hakeem expressed two ways of representing the solutions to systems of differential equations; graphing each of the solutions $x(t)$ and $y(t)$ "in terms of $t$" and also being able to "graph $y$ in terms of $x$." These two representations were closely tied to the two ways in which Hakeem attended to the system of equations; as a system and as a matrix equation. This is important to note because Hakeem specifically noted a correlation between the two representations of the differential equation, and the two representations of the solution. Namely he associated graphing $x(t)$ and $y(t)$ separately with the system of differential equations, and the phase plane with the matrix representation. In doing so Hakeem suggested that solutions to the differential equations and the differential equations themselves are characterizations of the same thing, the behavior of the quantities of interest. This is discussed in the paragraphs that follow.

TASK 2: The system of differential equations below can be used to model a spring mass damper system, where $x$ is the displacement of an object attached to the end of the spring. Provide a graph of the solution(s) to this system of differential equations.

$$\frac{dx}{dt} = y$$
$$\frac{dy}{dt} = -2x - 3y$$

*Figure 4.14: Task 2 from Interview 5*

Upon receiving the task, Hakeem said, "whenever I see a system like this, I guess the first thing I try to do is solve it" (turn 39), and proceeded to do precisely that, solve the system of equations. To solve the system, Hakeem utilized *the method of eigenvalues*. Much like the resource *method of characteristics*, identified in the second interview, the resource *method of eigenvalues* consists of a set of procedures, which Hakeem outlined and then systematically utilized to find a solution to the system of differential equations. His work can be seen in Figure 4.15. While Hakeem's work is indicative of *the method of eigenvalues*, I use a quote (which aligns precisely with his work from task) from his interaction with Task 6 to support his utilization of this resource in task 2:

> "Basically what I do is find the eigenvalues of the matrix, and find the eigenvectors that are linearly independent, so it'd have, so then I would be able to go from there to the fundamental solution set, or fundamental solution because I would be able to write $y = c_1 e^{\lambda t}$ times the eigenvector, and then you would just plug in the initial conditions to solve the $c_1$ and $c_2$." (turn 132)

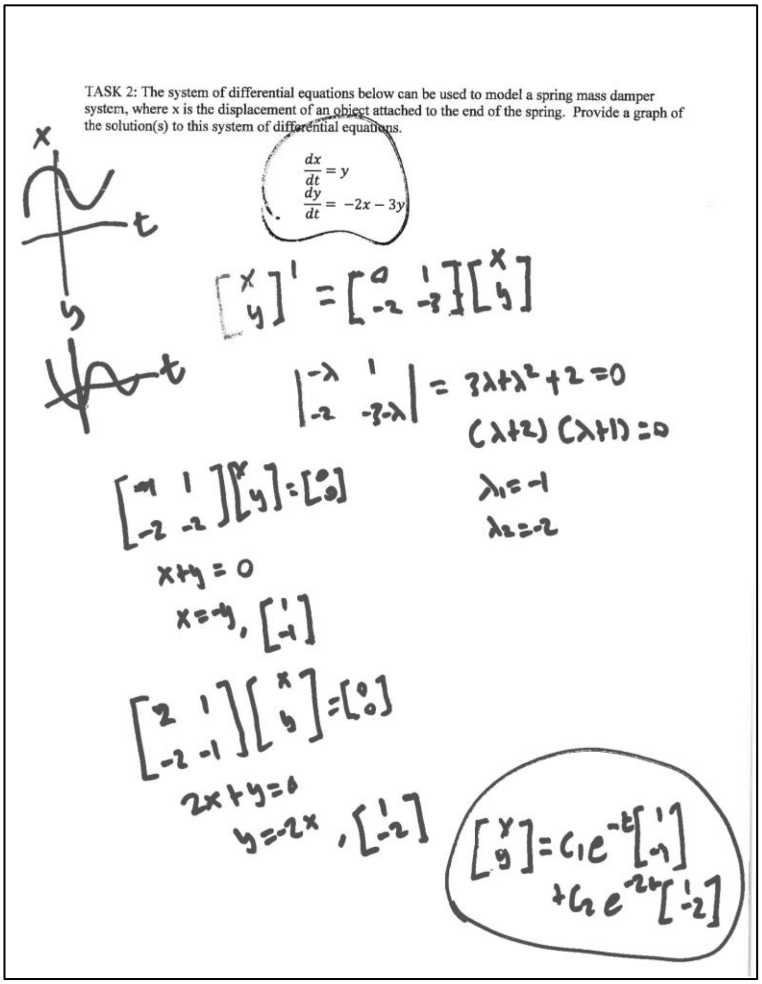



*Figure 4.15: Hakeem utilized the method of eigenvalues to complete Task 2.*

The utilization of this process yielded the solution $\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. After solving the system, I asked Hakeem how his work was related to graphing some of the solutions – the point of the task. This was a non-trivial question because depending on Hakeem's conceptualization of "solution" (e.g., $x(t)$ and $y(t)$ each being a different solution, $x(t)$ and $y(t)$ collectively being a single solution, graphing them on the same set of axes, different sets of axes, or graphing them as parametric equations) the work he performed may have served different purposes. In response Hakeem noted he could make two graphs, "since I have $x$ in terms of $t$, and $y$ in terms of $t$ with the solution I can, make two graphs, one of a graph with $x$ and $t$, and a graph of $y$ and $t$," (turn 44). He supported the construction of these graphs in turn 46 by saying “that’s what I do given this type of solution in this form.” He then said “one of the other things you can do I, ah, make a graph of $y$ in terms of $x$” (turn 46). In these statements Hakeem made use of the resources *functional dependence* and *functional mapping* to describe how his equations afforded the construction of the graphs. That is to say, he could depict “$x$ in terms of $t$,” “$y$ in terms of $t$,” “$y$ in terms of $x$” and correlate their values graphically.

This set of resources and the pattern in which he utilized them are indicative interpreting the differential equation as providing a model of the quantity of interest; Hakeem utilized the *method of eigenvalues* to determine the solution function, and then began discussing how that function could be used to represent the values of $x$ and $y$, both with respect to $t$, and each other. I then asked Hakeem if he could explain the differences between the two types of graphs.

51 G: So I guess, can you help me understand how you think about those two things, in other words, graphing these on separate axes, and graphing the phase plane?
52 H: Well I guess I mean I haven’t really thought about it much but I mean at first I would say at first glance I would say that here since I have two variables, ah two variables that are both functions of $t$ and they are sort of like separated from each other here, so one would be $x$ as a function of time and one is $y$ is a function of time, and I just, they're pretty much separate from each other. but when you are graphing $y$ in terms of $x$, you are showing how they are interrelated I guess.
53 G: Ok.
54 H: And so yeah given the solution you just see that these are two functions of time, but, I guess the phase plane graph is sort of it keeps in the relationship between $y$ and $x$. Cause they do have a relationship cause here we have $\frac{dx}{dt}$ is equal to $y$, and we have an $x$ term in $\frac{dy}{dt}$. So, a change in $y$ is gonna effect, when $y$ changes you are gonna have a different value for $\frac{dx}{dt}$ and a different value for $\frac{dx}{dt}$ means that $x$ is gonna change differently.

In turn 52 Hakeem expressed functional variation when discussing that both $x$ and $y$ depend on $t$, referring to each of them as “functions of time.” He then said that the phase plane shows how they are interrelated, and elaborated on this in turn 54 utilized *functional variation* (“when $y$ changes you are gonna have a different value for $\frac{dx}{dt}$”), *functional mapping* (“it keeps in the relationship between $y$ and $x$”), and *rates indicate behavior* (“a different value for $\frac{dx}{dt}$ means that $x$ is gonna change differently") to relate the relationship between the variables in the differential equation to that of the phase plane. This is indicative of Hakeem reasoning about the relationship between the values of $x$ and $y$, and the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

In turn 57 I asked Hakeem to describe how he was thinking about $x, y, \frac{dx}{dt}$ and $\frac{dy}{dt}$. In responding to the question (turn 58) he made a comparison between the solution to the matrix equation and the system of differential equations. This response is related to this statements in turn 54, explicates additional resources Hakeem used while completing this task, and shows how the solution function and the behavior described by the differential equation provide Hakeem with different information about the quantity of interest.

57 G: So can you tell me how you are thinking about $x, y, \frac{dx}{dt}$ and $\frac{dy}{dt}$?

58 H: I mean pretty much almost I guess like two pictures in my head right now. One is ah, like this here, [circles the system of equations] and one is this here [circles the solution of the matrix equation]. So in this here, [points to the solution of the matrix equation] like all you see is that you have $x$ as a function of time and $y$ as a function of time, so $t$ effects $x$ and $t$ affects $y$. But over here [points to the system of differential equations], you have here it's telling you that $y$ is gonna ah effect $\frac{dx}{dt}$ and $x$ is gonna affect $\frac{dy}{dt}$. And so here when I look at, I'm looking at $y$ and $x$ they are still both functions of time that are, they are changing over time, but we are given extra information in that cause the derivatives tell us the rate. So it's how they are changing over time. How fast or how slow they are changing, and so, these are basically like rate equations [they tell you about the solutions through the rate of change], so ah, when you look at say this first one, $\frac{dx}{dt} = y$, so $x$ is a function of time $y$ is a function of time. So, I'm just gonna I guess draw something like if you have $x$ and $t$ and just had I guess in terms of any, like not ah this graph but any graph if you had $x$ and if you have $y$ and $t$, so, $x$ and $t$, or $x$ and $y$ both change with time but at a specific instance in time say for this graph at a specific instance in time, $y$ will have a certain value and that value can be plugged into that equation and that will give you the answer to $\frac{dx}{dt}$ and what that tells you is how $x$ is changing at that moment in time.

While discussing how he thought about the variables in the differential equation, Hakeem coordinated the resources *functional dependence* ("$y$ is gonna ah effect $\frac{dx}{dt}$ and $x$ is gonna effect $\frac{dy}{dt}$"), *functional variation* ($y$ and $x$… are changing over time), *functional mapping* ("a specific instance in time say for this graph at a specific instance in time, $y$ will have a certain value”) *variable substitution* (“that value can be plugged into that equation and that will give you the answer to $\frac{dx}{dt}$”), and then *rates indicate behavior* (“that will give you the answer to $\frac{dx}{dt}$ and what that tells you is how $x$ is changing at that moment in time”) to discuss how $x$ and $y$ interacted in the differential equtions. This set of resources and the pattern in which they were utilized, signifies that Hakeem was interpreting the differential equation as a description of the quantity of interest. Additionally, Hakeem expressed that the differential equation provides extra information about the curve when compared to the solution functions. Specifically, Hakeem expressed that the differential equation tells you how $x$ and $y$ are changing with $t$, whereas with the solution functions “all you see is that you have $x$ as a function of time and $y$ as a function of time.” This explicates an additional difference between the interpretations for Hakeem, particularly in the case of systems of differential equations; while the differential equations provide model(s) of the quantities of interest, the model(s) only provides information about the quantity of interest, whereas the differential equation describes their interaction.

In summary, Hakeem collectively utilized the resources *functional dependence, functional variation, functional mapping, variable substitution, rates indicate behavior,* and *the method of eigenvalues* to complete Task 2. In addition these resources were utilized in two different combinations. Hakeem used *functional dependence, functional variation, functional mapping, variable substitution, rates indicate behavior* to discuss how the differential equations provided a description of how the $x$ and $y$ interacted. Further he used *functional dependence, functional mapping,* and *the method of eigenvalues* to find a solution to the differential equation and discuss the significance of the graph of $x(t)$ and $y(t)$. In both cases, as he said him self in Task 1 Hakeem reasoned about the variables in the differential equation as values.

**4.1.5.3 Interview 5 task 3.** To complete Task 3 (see Figure 4.16) Hakeem utilized the resources *functional variation*, *functional dependence*, *rates indicate behavior*, *variable substitution*, *functional mapping*, and *continuity*. Unlike when he encountered this task in Interview 4, Hakeem did not analytically solve the differential equation for an equation relating the quantity of interest to an independent variable; he reasoned solely with the relationship between the values of the variables in the differential equation to determine the long term behavior of the solution functions. This is significant because in Hakeem's first statement (transcript follows) he inferred that "the overall goal is to find that solution curve, what is $y$ in terms of $t$" (turn 76). To complete this task Hakeem utilized the graph to construct a slope field, which he then used to construct graphs of the possible solution functions. Hakeem utilized the majority of the resources identified in this task while constructing a slope field. This process is best summarized using Hakeem's own words "so, this [points to graph of $\frac{dy}{dt}$ with respect to $y$] is a graph of the derivative, but you use that to draw the derivative of this [points to graphs of the solution curves] graph" (turn 92). While completing this task Hakeem interpreted the differential equation as a description of the behavior of the quantity of interest. This process is present in the following segment of transcript.

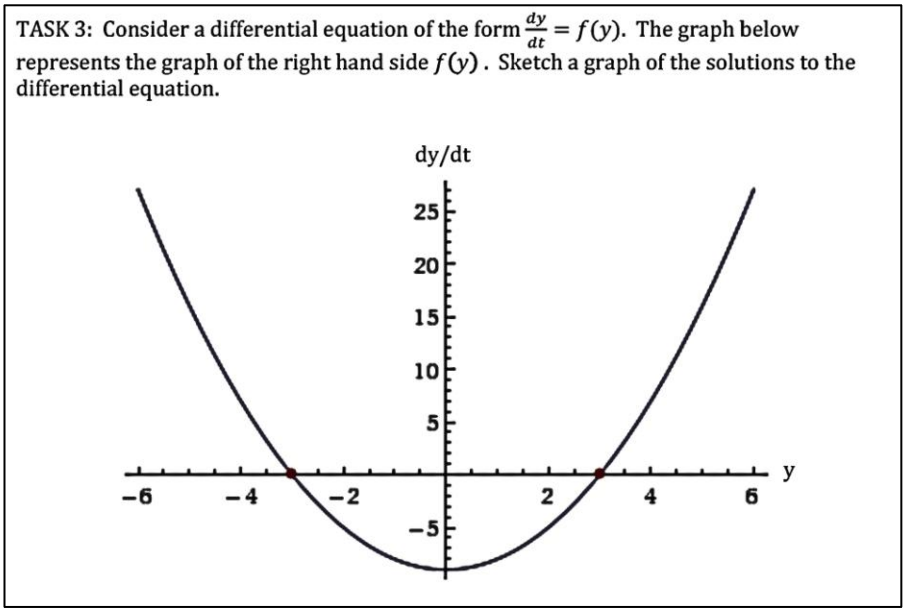



*Figure 4.16: Task 3 from Interview 5.*

76 H: So, $\frac{dy}{dt}$ is in terms of, equals $f(y)$, ah, so I guess the first thing I see when I look at that is, so $y$ is a function of $t$ and the overall goal is to find that solution curve, what is $y$ in terms of $t$. But this basically ah, gives you information about that curve because $\frac{dy}{dt}$ is telling you how $y$ is changing with $t$ and so basically at, so since its written in this form where $\frac{dy}{dt}$ is equal to $f(y)$, if you plug in any $y$ value, you are gonna get a $y'$, value, or $\frac{dy}{dt}$ value. So, and at that point you know that, so you know what the derivative is at that point. And that lets you, and when you, so that basically let’s you draw the slope field.[begins drawing slope field]. So also one thing to not is the equation is autonomous, in that there is no $t$ value in the $\frac{dy}{dt}$ equation so um, when you solve for $\frac{dy}{dt}$ at a point given certain values of $y$, that is the derivative for all values of $t$. So here you have at $y = 3$, $\frac{dy}{dt} = 0$ and at $y = -3$, $\frac{dy}{dt}$ is 0. So, here you can just draw these, I guess slope vectors we'll call those, at 3 and -3.

77 G: Mhm.

78 H: So what that tells you is that however the graph is going as it gets to ah this portion of the graph its going at a slope of zero so it's just going in a direction this way, ah, no, and you can basically plug in multiple points until you get, and once you have enough slope vectors you know what the slope is doing, or you know what the curve is doing.

79 G: Ok, so could you quickly fill in…

80 H: Yeah.

81 G: I mean you don’t have to do it in great detail, just…

82 H: So at $y = 0$, $\frac{dy}{dt}$ is negative, basically $\frac{dy}{dt}$ is really negative, and so from -3 to 3 the slope is going to be always negative so, it'll be closer to zero when it is closer to zero. So I can already fill in this middle part with a solution, which would, can I use a different color?

83 G: Sure.

84 H: So here the solution would look, in this part, something like this.

85 G: Ok.
86 H: So here, basically it is asymptotically approaches 3 and -3 when it goes towards -infinity and positive infinity.
87 G: Mhm.
88 H: And then outside of these two the slope is always positive so you are gonna get something looking like this [draws slopes above and below 3 and -3] so the solution would look like.

The first half of turn 76 Hakeem utilized *functional variation* to read out that $y$ depended on $t$, and that he was tasked with finding the equation for $y$ in terms of $t$. In addition he coordinated the resource *functional dependence* with the graph to determine that, "$\frac{dy}{dt}$ is in terms of, equals $f(y)$,", expressed *rates indicate behavior* to explain that "$\frac{dy}{dt}$ is telling you how $y$ is changing with $t$," and *functional mapping* to interpret the graph as providing "what the derivative is at that point. Coordinating these resources with the information he read out allowed Hakeem to determine that constructing a slope field would be productive for graphing the solution curves, and he proceeded to do so toward the end of turn 76.

To construct the slope field (see Figure 4.17) from the graph of $\frac{dy}{dt}$ with respect to $y$, Hakeem utilized *functional dependence* ("since its written in this form where $\frac{dy}{dt}$ is equal to $f(y)$" ), *variable substitution* ("if you plug in any $y$ value, you are gonna get a $y'$, value, or $\frac{dy}{dt}$ value"), and *functional mappin*g. More specifically Hakeem utilized *variable substitution* to determine values of $\frac{dy}{dt}$ for certain values of $y$, *functional dependence* to determine that "when you solve for $\frac{dy}{dt}$ at a point given certain values of $y$, that is the derivative for all values of $t$," and *functional mapping* to actually map the values of $\frac{dy}{dt}$ (in the form of slope vectors) to values of $y$ and $t$ (in the form of the location of the slope vectors on the $t, y$ plane.).

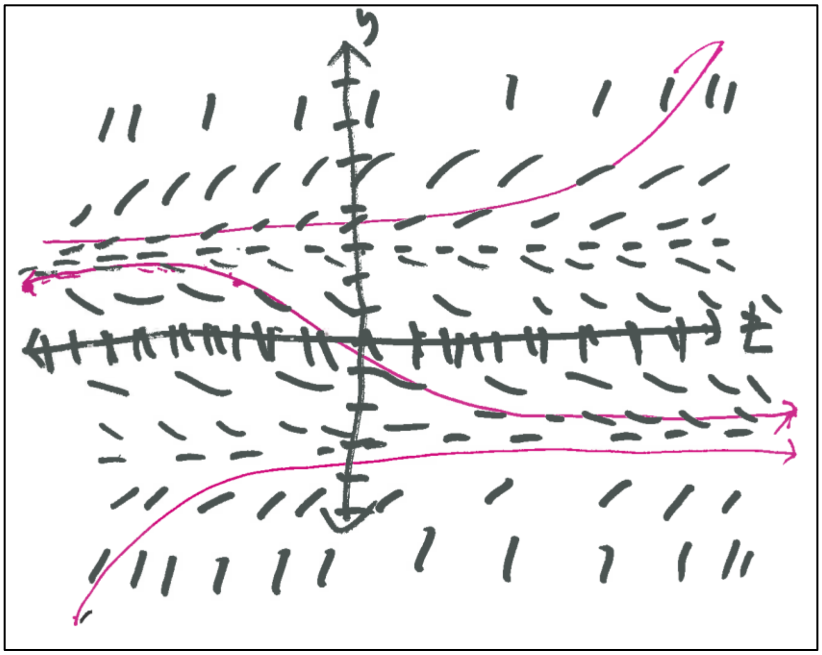



*Figure 4.17: Hakeem determined the behavior of the quantity of interest.*

With regard to using the slope field to construct the solutions, in turn 78 Hakeem discussed how one “can basically plug in multiple points” (*variable substitution*) until you get “enough slope vectors to… know what the curve is doing.” Hakeem however, did not actually plug in multiple values to determine “what the curve is doing.” Instead, he coordinated the resources *functional variation* and *continuity* to infer the slope values between those vectors he already drew. This is evident in turn 82; “and so from -3 to 3 the slope is going to be always negative so, it'll be closer to zero when [$y$] is closer to zero.” To be more specific, instead of plugging in multiple values to determine the value of $\frac{dy}{dt}$ for $y$ values between -3 and 3, Hakeem inferred the values of $\frac{dy}{dt}$ decrease, and take on all values between 0 and “really negative” (turn 82) as $y$ moved from -3 to 0. Similarly he inferred that $\frac{dy}{dt}$ would increase and take on all values between “really negative” (turn 82) and 0 as $y$ moved from 0 to 3. He then coordinated this inference with the resource *rates indicate behavior* to construct the graph of the solution functions. That is, by coordinating the resources *functional variation* and *continuity*, Hakeem was able to get "enough slope vectors…" with which he would coordinate with *rates indicate behavior* "…know what the curve is doing." Hakeem then used similar reasoning to fill in the solution curves above $y = 3$ and below $y = -3$.

While completing Task 3 Hakeem interpreted the slope field and the provided graph as both relating values of $\frac{dy}{dt}$ to values of $y$ and $t$, and Hakeem interpreted the slope vectors as indicating the direction in which the solution curve was moving. To construct the slope field from the provided graph Hakeem utilized the resources *functional dependence*, *functional mapping*, and *variable substitution* and to construct graphs of the possible solution functions Hakeem utilized the resources *rates indicate behavior*, *functional variation* and *continuity*. Based on the resources Hakeem utilized to complete the task and the pattern in which he utilized them Hakeem interpreted the differential equation as a description of the behavior of the quantity of interest.

**4.1.5.4 Interview 5 task 4.** Hakeem had first interacted with this task in Interview 1 (I1-T4). During this interaction (in Interview 5) Hakeem interpreted the task as asking him to find the condition (i.e., the differential equation) the solution $y = 6$ satisfied. While completing the task in Interview 5 Hakeem utilized the resources *equality condition,* and *opposite operations reverse* to show that the derivative of $y$, $\frac{dy}{dt} = 0$ is an appropriate differential equation. In this case his reasoning was primarily composed of coordinating the resource *equality condition* with recognized properties of $y = 6$. In doing so I argue that he interpreted $y = 6$ as a function, and the differential equation as a relationship between a function and it's derivative function. In addition Hakeem proposed an alternative approach to completing the task where he utilized *functional variation*, *functional mapping* and *equality condition* to show that the values of $\frac{dy}{dt} = 0$

at all locations on the curve $y = 6$, thus concluding that $\frac{dy}{dt} = 0$ was a differential equation that had the solution $y = 6$. In doing so he expressed interpreting the differential equation as a relationship between the value of a quantity and the value of the quantities rate of change.

Hakeem quickly completed this task; the following segment of transcript is indicative of the resources Hakeem utilized to complete the task.

101 H: I guess a simplest approach would be, if you ah, $\frac{dy}{dt}$, is basically just to take the derivative of this, which is zero. So if you set $\frac{dy}{dt} = 0$, then when you integrate that you'd get $y$ is equal to a constant. So it could equal, and $y = 6$ is one of the constants that would work. So I guess another way to look at it is if you graph, $y$ and $t$ and you graph the line $y = 6$, the derivative is zero along it. So we are able to, for any solution is any curve and, so basically the goal would be to find a differential equation so that when it give you the solution, I guess it would be going backwards so, like a solution is, cause there is multiple solutions to a differential equation, so its ah, finding something a solution that, basically, so what I was trying to say is that normally that when you do a differential equation you are given $\frac{dy}{dt}$ and you are trying to find the solution $y$. so this is like a condition that a solution has to solve, but when you are given $y = 6$, you are given the solution, so its just what has that condition, or what is the condition for, or what is a condition almost.

102 G: Ok. Um, when you see $y = 6$, I mean you drew the graph here so I know some information about how you are thinking about it, can you just tell me how you are thinking about $y = 6$ and what that means.

103 H: Well, it says what would have a solution of $y = 6$. So $y = 6$ is a solution to the differential equation, it doesn't have to be the only solution so, ah, basically, like even in this differential equation when you get an answer to this, you'd end up with $y = c$, and 6 is only an instant, is only one solution that solves this differential equation.

Working with the interpretation of $\frac{dy}{dt}$ as the derivative of $y$, in turn 89 Hakeem noted that the "simplest approach would be… basically just to take the derivative of this [points to $y = 6$], which is zero." He then sets out to show that $\frac{dy}{dt} = 0$ is indeed the solution to the differential equation. To do so, Hakeem coordinated the resources *equality condition* and *opposite operations reverse* with $\frac{dy}{dt} = 0$. Namely Hakeem used the resource *opposite operations reverse* to find the functions that have that condition $\frac{dy}{dt} = 0$ by integrating, and *equality condition* when he claimed that $y = 6$ is one such function for which $\frac{dy}{dt} = 0$ "would work." He then proposed what he considered to be an alternative way of completing the task in which he utilized *functional mapping*, *equality condition* and *functional variation*. Specifically he used *functional mapping* to graph the line $y = 6$ (see Figure 4.18) in terms of $y$ and $t$, and then coordinated *functional variation* and *equality condition* with the graph to explain, "the derivative is zero along it." With this alternate approach, the condition "the derivative is zero along it" is the condition that must be captured by the differential equation. In this alternative approach of Hakeem related the values of the quantity of interest ($y = 6$) to values of its rate of change $y' = 0$. In this case he noted that $y'$ is always zero, concluding that $y = 6$ satisfied the condition set by the differential equation $\frac{dy}{dt} = 0$.

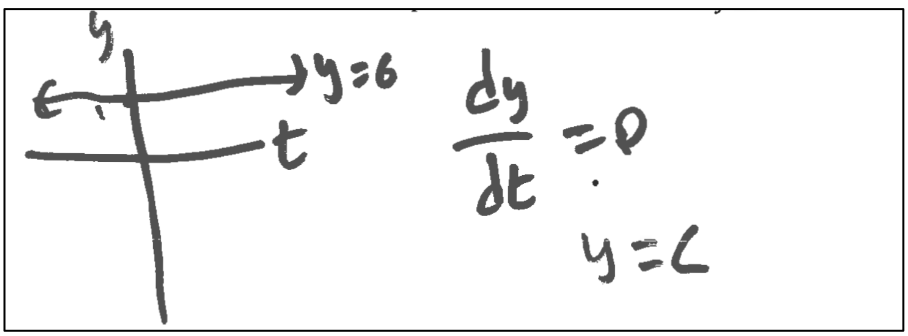



*Figure 4.18: Hakeem graphed $y = 6$ to show "the derivative is zero along it."*

In summary while completing the task Hakeem utilized the resources *opposite operations reverse, equality condition, functional mapping,* and *functional variation*. The resources *opposite operations reverse* and *equality condition* were used to verify that a proposed differential equation did indeed have the solution function $y = 6$, and the resources *equality condition, functional mapping* and *functional variation* were utilized to show that the equation $y = 6$ had the condition that $\frac{dy}{dt} = 0$ . In either case the resource *equality condition* was utilized in service to justifying that the differential equation $\frac{dy}{dt} = 0$ and function $y = 6$ shared the necessary conditions for $y = 6$ to be a solution to $\frac{dy}{dt} = 0$. His first approach was indicative of interpreting the differential equation as a relationship between a function and it's derivative function, and his second approach was indicative of interpreting the differential equation as a relationship between the value of a quantity ($y = 6$) and the value of its rate of change ($y' = 0$).

**4.1.5.5 Interview 5 task 5.** This task asked students to match three vector fields to their respective differential equations. This was the second time Hakeem interacted with this task, the first being during Interview 2 (I2-T3). As indicated by the concise segment of transcript that follows, Hakeem's completion of this task was quite systematic. I make this claim because his reasoning was quite organized and focused from the very beginning; there is little evidence of Hakeem struggling to figure out an approach to take. Though his reasoning appeared more directed, it was very similar to the way he completed the task during Interview 2. Therefore my presentation of Hakeem's interaction with this task is somewhat abbreviated.

While completing the task Hakeem utilized the resources *functional variation, functional dependence*, and *functional mapping*. Additionally, he interpreted the differential equation across the various representations (i.e., the slope field and the symbolic equations) as a relationship between the value of quantities and value of the rate of change of the quantity of interest. More specifically, in this case he interpreted $\frac{dy}{dt}$ as a function of $y$ and $t$, and attended to these variables as correlated values. The identified resources are evident in the following segment of transcript.

118 H: So the first thing I do is ah, so the other, slope field, so what they do is ah, they tell you how solutions work, but also when you are given just a slope field, you know what $\frac{dy}{dt}$ is, because you are given what the slope is at multiple different points, and I guess you can go backwards from that to an equation to $\frac{dy}{dt}$. So here for example one of the first things I see is that $\frac{dy}{dt}$ is constant along different values of $y$, so it doesn't change with $t$. so for this one you have $\frac{dy}{dt}$ is equal to $f(y)$. So when I was looking for this one I know that A is either this one, either the first equation or the second, no last equation. First equation or last equation,

cause the, or, it's $f(y)$, my bad, so its either the second equation or the fourth equation cause those are the only ones that don't have a $t$ in them. And I guess the, to solve it specifically another thing to note is that $\frac{dy}{dt}$ is 0 at $y = 1$ and that is true for the well that's true for both of these. But for this one, $\frac{dy}{dt}$ would also be zero at -1, but it's not so we know that this, for this one, I guess by omission, $\frac{dy}{dt} = 1 - \mathrm{y}$.

119 G: Ok cool.

120 H: And for C it's the same thing except $\frac{dy}{dt} = \mathrm{g(t)}$ instead of a function of $y$ it's a function of $t$, so, I'd look for an equation that only has $t$'s in it so it's either the first or the last.

121 G: Mhm.

122 H: And, here $\frac{dy}{dt}$ is 0 at 1 and here it is zero at -1. But like the graph shows it being zero at one, so that means this equation here, the first equation so, $\frac{dy}{dt} = t - 1$.

123 G: What told you that the graph in C is zero at one?

124 H: Ah, the slope field, so, if you look at one, I guess kind of extrapolate it, like the slope goes from a pretty low positive value to a pretty low negative value and so in between is zero and that's approximately at one.

Upon receiving the task, he interpreted the slope fields as telling you "how solutions work" because, as he explained, "you know what $\frac{dy}{dt}$ is, because you are given what the slope is at multiple different points and I guess you can go backwards from that to an equation to $\frac{dy}{dt}$." Here Hakeem utilized *functional mapping, and functional dependence* to infer that the slope field can be used to get an equation for $\frac{dy}{dt}$. In this case Hakeem interpreted the value of the slope as the value of $\frac{dy}{dt}$ and the "multiple different points" as indications of the values of $t$ and $y$. Working with the slope field, he then used the resource *functional dependence* ("$\frac{dy}{dt}$ is equal to $f(y)$") and *functional variation* ("so it doesn't change with $t$") to determine which of the equations matched the slope field. More specifically, by checking if $\frac{dy}{dt}$ changed as $t$ changed and then as $y$ changed (*functional variation*), Hakeem was able to determine that "$\frac{dy}{dt}$ is constant along different values of $y$." He associated the dependence of $\frac{dy}{dt}$ only on $y$ with the second and the fourth equations.

With the identification of potential matches, Hakeem then used the resource *functional mapping* to determine which of the two differential equations correctly matched the slope field. In particular, Hakeem noted "$\frac{dy}{dt}$ is 0 at $y = 1$" and "$\frac{dy}{dt}$ would also be zero at -1," and used these characteristics to correctly match the slope field with the differential equation. This reasoning pattern was utilized by Hakeem to match all three of the slope fields to their respective equations. That is, Hakeem determined which variables $\frac{dy}{dt}$ was dependent on by varying the values of $t$ and $y$ to see how $\frac{dy}{dt}$ changed (*functional variation*), then used functional dependence to determine potential differential equations, and lastly, he used *functional mapping* to check which equation correctly related the different values.

In summary, much like Hakeem's first interaction during interview 2 Hakeem utilized *functional mapping*, *functional variation* and *functional dependence* to match the slope fields to their respective differential equations. He used the resource *functional mapping* to see how the values of $t$ and $y$ related to the value of $\frac{dy}{dt}$, in both the slope field and the equations. Additionally he utilized *functional variation* and *functional dependence* to determine which variables impacted the values of $\frac{dy}{dt}$, which supported Hakeem in narrowing down possible matches. Lastly Hakeem was supported in utilizing these resources as he interpreted the slopes in the slope field as representing values of $\frac{dy}{dt}$, and the $(t, y)$ point at which these slopes were located as representing values of $t$ and $y$. As such I claim that Hakeem interpreted the differential equation as a relationship between the value of quantities and value of the rate of change of the quantity of interest.

**4.1.5.6 Discussion of interview 5.** Over the course of the entire interview Hakeem utilized a rather large list of resources. While the cumulative list of resources is informative of the ideas Hakeem made use of while completing the tasks, a more revealing story emerges when one considers the patterns in which combinations of the resources were utilized. Among the five tasks discussed in this interview Hakeem utilized four interpretations, and further many of the tasks were completed using multiple interpretations (see Table 4.6). Additionally, each of these sets of resources and the patters in which they were utilized corresponded to certain ways of interpreting the differential equation and its components.

While there is some variation in the sets of resources corresponding to the same interpretations across the tasks, there are some resources that appear to have a central role in the reasoning process of each of the interpretations. For instance, Hakeem utilized some combination of *functional dependence, functional mapping* and *functional variation* to relate the values of the quantities to the value of the rate of change of the quantity of interest in each of the three tasks which correspond to interoperating of the differential equation as a Relationship between the value of quantities and the values of the rate of change of the quantity of interest.

Additionally, with the exception of Task 1, Hakeem utilized *functional dependence, functional variation, functional mapping, variable substitution,* and *rates indicate behavior* in two tasks where he expressed interpreting the differential equation as a description of the quantity of interest. Further, in each of these tasks Hakeem utilized a combination of *functional dependence, functional variation, functional mapping, variable substitution* to characterize the nature of the relationship between the variables in the differential equation, and then *rates indicate behavior* to draw conclusions about the behavior of the quantity of interest.

*Table 4.6: Resources and interpretations Hakeem expressed in Interview 5.*

| Task | Interpretation of the Differential Equation | Resources | Interpretation of the components in the differential equation |
|---|---|---|---|
| I5-T1 | Relationship between the value of quantities and the values of the rate of change of the quantity of interest | *Functional dependence*<br>*Variable substitution* | $y$ and $t$<br>"Whatever you put in for $y$ and $t$ you get something for $\frac{dy}{dt}$" |
| | Description of the behavior of the quantity of interest | *Functional mapping*<br>*Rates indicate behavior* | "When you have $\frac{dy}{dt}$ is a function of $y$ and $t$, that's another way of I guess describing the curve" |
| | Providing a model of the quantity of interest | N/A | "All about finding $y$ as a function of $t$" |
| I5-T2 | Description of the behavior of the quantity of interest | *Functional dependence*<br>*Functional variation*<br>*Functional mapping*<br>*Variable substitution*<br>*Rates indicate behavior* | $x$, $y$ $\frac{dx}{dt}$, and $\frac{dy}{dt}$ - Values in the differential equation which describes<br>"how [$x$ and $y$] are interrelated I guess" |
| | Providing a model of the quantity of interest | *The method of eigenvalues*<br>*functional dependence*<br>*Functional mapping* | Functions- "$x$ in terms of $t$, and $y$ in terms of $t$ with the solution I can, make two graphs" |
| I5-T3 | Description of the behavior of the quantity of interest | *Functional dependence,*<br>*Functional mapping*<br>*Variable substitution*<br>*Rates indicate behavior*<br>*Functional variation*<br>*Continuity* | $y$, $t$ and $\frac{dy}{dt}$ - values in the differential equation which "is telling you how $y$ is changing with $t$" |
| I5-T4 | Relationship between a function and the function's derivative | *Opposite operations* reverse<br>*Equality condition* | $\frac{dy}{dt}$ and $y$- Functions: "set $\frac{dy}{dt} = 0$, then when you integrate that you'd get $y$ is equal to a constant." |
| | Relationship between the value of quantities and the values of the rate of change of the quantity of interest | *Equality condition*<br>*Functional mapping*<br>*Functional variation* | $y$, constant value<br>$\frac{dy}{dt}$ values of the slope of $y$ |
| I5-T5 | Relationship between the value of quantities and the values of the rate of change of the quantity of interest | *Functional mapping*<br>*Functional variation*<br>*Functional dependence* | Values - "given what the slope is at multiple different points" |

Lastly, while completing the tasks in this interview, Hakeem displayed a rather impressive ability to interpret the differential equations (and their components ) in each task in multiple ways. For instance, in Tasks 2 and 5 Hakeem expressed thinking of the variables in the differential equations as representing both values and functions. While he often utilized the variables as values in the differential equation, his ability to attend to them as functions supported his ability to both determine their behavior and discuss them as being a representation of a quantity that changes with some other variable. The latter is significant because Hakeem often expressed the sentiment that "a differential equation I guess it's all about solving to find $y$ as a function of $t$" (turn 30).

## 4.2 Analysis of Resources Identified for Jordan

Like Hakeem, Jordan completed 5 interviews over the course of the semester. Each of the interviews were spaced roughly three weeks apart, though the fifth interview occurred only two weeks after Interview 4 due to final exams. Jordan was a sophomore engineering major taking differential equations for the first time. In addition Jordan was enrolled in a regular (as opposed to honors or accelerated) differential equations course where the material, which consisted mostly of analytical solution methods was taught in a traditional manner. The decision to analyze Jordan's interactions with the tasks was made for two reasons. First, and most importantly, Jordan seemed to utilize similar resources (and sets thereof) to those of Hakeem but with less flexibility across the tasks. Second, during the third and fourth interviews Jordan expressed frustration with the content of the course; on multiple occasions he expressed feeling as though he was not learning new content, and was only applying ideas from differential and integral calculus to memorize processes for solving differential equations. Though the purpose of this dissertation is far from analyzing students' classroom experiences, his expressed frustration suggested the existence of a gap between what he thought differential equations were, and the resources he used while completing tasks concerning differential equations. I will address this is more detail in Chapter 5 when I present findings related to how the resources changed and supported Jordan's ability to complete the tasks over the course of the 5 interviews.

### 4.2.1 Interview 1

Jordan completed five tasks during Interview 1. I discuss the analysis of the first four tasks; the fifth task is omitted because it did not directly involve differential equations. Over the course of completing the four tasks Jordan expressed three ways of interpreting the differential equations: As a relationship between the value of quantities and the value of the rate of change of the quantity of interest, (I1-T1and I1-T4), as an equation to be solved (I1-T2), and as a description of the behavior of the quantity of interest, (I1-T2 and I1-T3). In this case Jordan interpreted Task 2 in two ways as is indicated by the language he used, the distinct reasoning patterns he expressed, and the ways he utilized the resources while reasoning in these two distinct ways.

**4.2.1.1 Interview 1 Task 1.** Task 1 presented students with two systems of differential equations. The students were told one system represented competing species while the other represented cooperative species, and they were tasked with determining which system represented competing species and which system represented cooperative species. While completing the task Jordan utilized the resources *functional dependence, partial dependence, functional variation, variable substitution, function mapping,* and his *definition of competing species.* Additionally he expressed that the variables $x$ and $y$ represented the population values of the respective species, and that $\frac{dx}{dt}$ and $\frac{dy}{dt}$ were the values of the rates of change of the respective species. Jordan used these resources and interpretations to make judgments about how changes in the population of one species in the system affected the rate of change of the population of the other species in the system. More specifically, Jordan used each of the differential equations in the system to correlate the value of quantities with the value of the quantities' respective rates of change. This combination of resources, and the pattern in which they were used is indicative of interpreting the differential equation as a relationship between the value of quantities and the value of the rate of change of the quantity of interest.

I start the analysis with a segment of that transcript begins just after Jordan finished reading the task. The purpose of this segment is to outline how Jordan conceptualized competing and cooperative species.

1 J: Ok so I have to figure out competing or cooperating species.
2 G: Mhm.
3 J: Ok, hmm. So if they are cooperating I want to see some sort of positive trend for an increase of both species, but for competing I want to find, I mean the opposite. Like as one species grows the other diminishes.
4 G: Um so you said some sort of increasing trend, increasing trend in what?
5 J: Um hm. Well I guess if we are looking at rate of change then if there are two equal competing species, the rate of change will be relatively constant cause they won't be really increasing, like there won't be like a largely positive rate of change of population or there might be a negative change in population because they are competing for like a food source or something but if they are cooperating then… Let's see, how can I better describe this.

In turn 5 Jordan characterized the nature of the relationships between competing and cooperative species; "if they are cooperating I want to see some sort of positive trend for an increase of both species" and "but for competing I want to find, I mean the opposite. Like as one species grows the other diminishes." In turn 7 Jordan suggested he would to use the rate of change to determine which system is competing and which is cooperative, noting that for competing species he expects to see "the rate of change will be relatively constant cause they won't be really increasing, like there won't be like a largely positive rate of change of population or there might be a negative change." While in the process of reasoning about the systems, his characterization of cooperative species seemingly matched the result of his reasoning, but he struggled to match his expectations for the competitive species with the results of reasoning. I then asked him how he was going to use the differential equations to get this information.

10 G: What is it that you are going to look for, if anything, in the systems or the equations that are going to tell you something about the rate of change?
11 J: Um I think I am going to look for some sort of correlation between the change in $x$ and the change in $y$. So either one on their own I don't think will really tell me much.
12 G: Either equation on their own?
13 J: Yeah.
14 G: Ok, so you have to look at the information that both of them are telling you? Is that what you are thinking?
15 J: Yeah. I guess that may not be true because they both have functions of either species within them. Like so, I think $B$ could be competing species because as species $y$ increases the rate of change of $x$ would decrease so it would be as more of $y$, takes more food and harms the growth of species $x$, it would reduce species $x$'s rate of change. Rate of growth, decay, no matter what it would be going down. And it would be the same vice, versa for the lower equation. So, based off of that, I don't know.
16 G: I want to go back to something you had said before. You said that these equations, and I might not have the words that you used exactly right, but functions of both species within them.
17 J: Yeah.
18 G: What did you mean by that?
19 J: I mean both equations are dependent on both species. So I guess it looks like they both have, like in this one they both have a function of $x$ times the current population of $y$. So I guess I was trying to say that they are both dependent on both species and its just if it's a positive or negative dependence that I am looking for
20 G: So what is it, one other thing I should have said is that I might ask questions that seem like they have obvious answers but I want to make sure that I am thinking about the things within here in the same way that you are. When you say that it's dependent on species $x$ and species $y$ what is "it"?
21 J: Um, both of the separate species' either growth or decline. Their change. So how fast their increasing or decreasing their species is dependent on the other species.

This segment of transcript is indicative of Jordan's utilization of *functional dependence, partial dependence* and *functional variation.* Based on turn 11 Jordan planned to use the differential equations to "look for some sort of correlation between the change in $x$ and the change in $y$." In turn 15 Jordan used *functional dependence* to infer that the value of both populations determines the value of each of the rates of change, however, he only used one of the variables ($x$ and $y$) in each equation. More specifically Jordan utilized *partial dependence* to infer how the population of species $x$ related to the rate of change of species $y$ and how the population of species $y$ related to the rate of change of species $x$. This can be seen in turn 15:

> "Like so, I think $B$ could be competing species because as species $y$ increases the rate of change of $x$ would decrease so it would be as more of $y$, takes more food and harms the growth of species $x$, it would reduce species $x$'s rate of change. Rate of growth, decay, no matter what it would be going down. And it would be the same vice, versa for the lower equation. So, based off of that, I don't know."

In this case, Jordan considered the first equation and coordinated *partial dependence* and *functional variation* to determine that increases in $y$ make the rate of change of $x$ decrease, and plausibly applied this same reasoning pattern to the second equation to infer that increases in $x$ related to decreases in the rate of change of species $y$. It is important to note that I do not take Jordan's statement "more of $y$, takes more food and harms the growth of species $x$" in turn 15 to be synonymous with more of $y$, takes more food and makes the population of $x$ decrease, rather I take it as further evidence of functional variation, where the "growth of species $x$" represents the value of $\frac{dx}{dt}$. While the difference seems subtle, this had a central role in identifying the reasoning pattern Jordan was using, and how he was interpreting the differential equation and the various entities from which it was composed.

To further support this analysis consider Jordan's response to my question in turn 18 when I asked Jordan what he meant by each equation having "functions of either species within them." He replied, "so I guess I was trying to say that they [the species' rates of change] are both dependent on both species and it's just if it's a positive or negative dependence that I am looking for" (turn 19). I interpret his statement about classifying the nature of the dependence ("it's just if it's a positive or negative dependence that I am looking for") as referring to how changes in $x$ relate to changes in $\frac{dy}{dt}$, and how changes in $y$ relate to changes in $\frac{dx}{dt}$ (*partial dependence, functional variation*). Clarifying this in turn 21 he said, "so how fast they're increasing or decreasing their species is dependent on the other species" (*partial dependence*). That is, these statements are indicative of a coordination of *partial dependence,* and *functional variation*. Again, it is important to note that I take "how fast they're increasing or decreasing" to be about the value of $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

The topic of the conversation then switched focus to how Jordan was interpreting the various components of the differential equations. While Jordan was seemingly aware of $x$ and $y$ representing functions – he referred to $x$ and $y$ as functions in turn 15 ("they [indicating both differential equations in System B] both have functions of either species within them") and later in turn 39 provided a graph of what he expected populations of competing species to look like – he reasoned with $x, y$, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as values. When asked about this explicitly in turn 24 he replied, "So like $x$ is the current, like the actual population number of species $x$ and $y$ is the actual population of species $y$. And then the $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are changes in population at time, the rate of change" (turn 25). In addition he provided a vivid example of this in turn 29:

> "Um, really I, all I am thinking of right now is, mainly because I just did a bunch of problems on it, is like ah, like the flow in and the flow out, they are dependent on q, which is the current, the quantity of salt in the solution. So that q is a quantity of salt and we are trying to find the equation for the rate of salt increase or decrease the rate of change of the salt concentration. So if q is just the actual concentration of the salt in the tank at the one point that you are trying to figure out, then I am assuming that $x$ and $y$ would be, if $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are the rates of change of the species, like the rate of change of the concentration of salt, then the $x$'s and the $y$'s, I mean if that [$\frac{dx}{dt}$] is a rate of change then that [$x$] has to be the actual number."

Recall that despite finding that increases in $y$ were detrimental to the growth of species $x$ in turn 15, Jordan was still not sure if System B represented competitive species; "So based off of that, I don't know." After providing the example in turn 29 I then returned the conversation to task at hand, classifying the two systems. Other than mentioning System B being competitive one time in turn 15, Jordan had not explicitly classified the two systems, so in turn 30, I attempted to verify my interpretation of how he was classifying them. This seemingly prompted him to began substituting values for $x$ and $y$ into the differential equation to determine specific values of $\frac{dx}{dt}$and $\frac{dy}{dt}$. This is evident in the following segment of transcript.

30 G: Ok, I think I understand what you are saying. So going back to it, if there was nothing else you were going to say, don't let me jump forward… [pause]. So going back to it, you thought these were the competing species [points to System B] and these were the cooperative species [points to System A], is that right?

31 J: Possibly, yes that is what I think initially. I don't know. I think what I would do next is I would start plugging in numbers for each species, so if there was like 10 of $x$ and 10 of $y$ how would they respond in each, like what would be the rate of change in all of these.

32 G: Ok so let's try it.

33 J: Ok, so they are both growing. Ok so at each of these species has 10, I've got these numbers, both of these species have a rate that is declining and both of these have a rate that is increasing. So I guess next I would try increasing $x$ and decreasing $y$ to see how the dependence on the two work.

In turn 31 and 33 Jordan utilized *variable substitution* to determine what the signs of the rates of change would be for species $x$ and species $y$ if both species had a population value of 10. His work can be seen in Figure 4.19. In doing so he determined that when $x$ and $y$ were both 10, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ were both negative in System B, and both positive in System A. He then noted that his next step was to increase the value of $x$ and decrease the value of $y$ to see how the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ would change. This process is indicative of utilizing the resources *variable substitution* and *functional mapping* that is, correlating sets of values of $x$ and $y$ to sets of values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ by substituting values for $x$ and $y$ into the differential equation. Jordan did not actually complete this step, however, as my question (turn 34; transcript follows) prompted him to discuss how substituting values in for the variables $x$ and $y$ was related to completing the task.

As a result however, and for no certain reason, in the process of describing why his work was relevant to completing the task he became convinced that System B was indeed competitive.

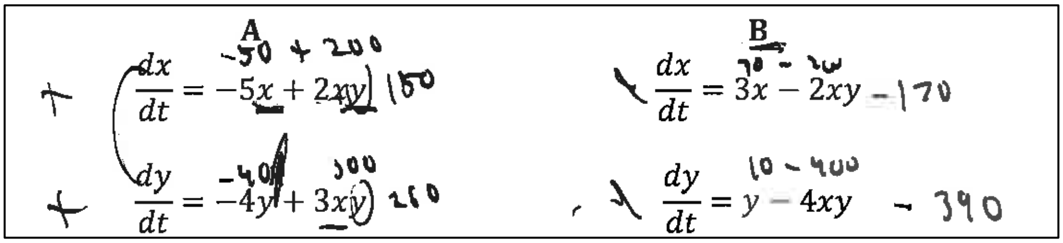



*Figure 4.19: Jordan substituted 10 for x and y to determine* $\frac{dx}{dt}$ *and* $\frac{dy}{dt}$.

34 G: So it sounds to me like you are saying there is some relationship between $\frac{dx}{dt}$ and $\frac{dy}{dt}$.
35 J: Yeah.
36 G: Could you talk about how you think about that relationship?
37 J: I think for a competing species they are probably related to being the opposite of each other, so like if one species has a positive increase then I guess it is likely that, I guess I just proved that that doesn't always happen, but it is somehow inversely proportional to the other species. So if you have as one, for the competing species, as one has a positive and one has a negative if you extend out through a period of time I guess they will exponentially increase and decrease. Because you've got, let's say if one was positive you would have it here at whatever population and the negative here at this population and so you kind of follow those out, and if they are competing and as this species declines in species, the rate of this species and the actual species there would be increase until you hit some limiting factor.

In turn 37 Jordan noted that he associated competing species with each having rates of change that are opposite signs of each other or “somehow inversely proportional.” Though it did not seem to be the case in turn 33 that Jordan was sure System B represented competing species, at the end of turn 37 he was seemingly convinced that System B was competitive. While he did not provide a justification for this, he did draw a graph to illustrate the inversely proportional relationship he was talking about (see Figure 4.20).

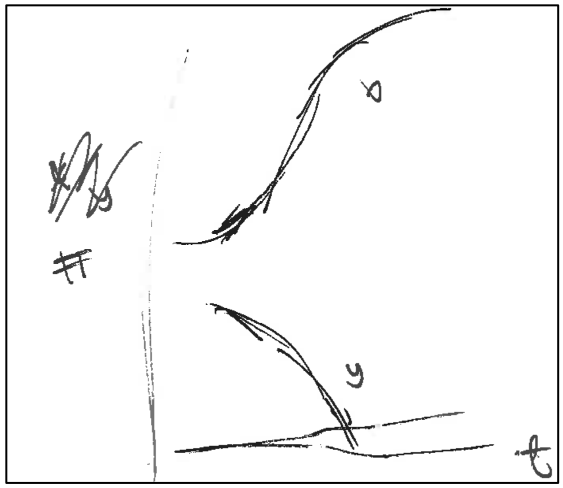



*Figure 4.20: Jordan's graph of the relationship between competing species.*

Jordan described the graph in the following transcript:

38 G: So did you um, so ah, is this a graph with respect to time?
39 J: Yeah this is a graph with respect to time and I guess $x$ slash $y$. I mean it's not really, it's just the number of species. Population
40 G: And you graphed, so you graphed both $x$ and $y$.
41 J: Yeah so this, like if this is $x$ this would be $y$ [labels graphs]. If you have some starting point like an initial condition when there is more of $x$ than $y$, then if they are competing $y$ will eventually die out because of the competition with $x$.

It is not clear if or how Jordan saw this type of behavior present in System B, as he did not discuss it, however, at the end of his interaction with this task he did ask if his answer was correct, indicating some level of uncertainty about System B being competitive and System A being cooperative. To offer a potential explanation for how Jordan made this classification of the two systems, note that the behavior he determined to be present in System A, namely both rates of change being positive, matched with his *definition of cooperative species*, "some sort of positive trend for an increase of both species" (turn 5). Having determined System A represented cooperative species, he could then use a process of elimination to infer System B was competitive. Nonetheless, it should be noted that while reasoning with the differential equations in each of the systems Jordan related values of $x$ and $y$ to values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

In summary, while completing this task Jordan used the resources *functional dependence, functional variation, partial dependence, functional mapping, variable substitution,* and his *definition of cooperative species.* While utilizing these resources Jordan interpreted the components within the differential equations as values of quantities. Namely he interpreted $x$ and $y$ as the value of the populations of species $x$ and $y$, respectively and, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as the value of the respective populations' rates of change. He used the resource *functional dependence* to infer that the value of $\frac{dx}{dt}$ depended on $y$ and that the value of $\frac{dy}{dt}$ depended on the value of $x$. Additionally, while initially reasoning with system B he utilized *functional variation* to determine how changes in the value of $y$ related to changes in the value of $\frac{dx}{dt}$. That is, he used *functional variation* to see how increasing $y$ related to changes in $\frac{dx}{dt}$. In his second attempt to reason with System B he utilized *variable substitution* to determine the signs of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ supposing that the value of the populations of species $x$ and species $y$ were both 10.The language and resources Jordan utilized while completing the task indicate he was interpreting the differential equation as a relationship between the value of quantities and the value of the rate of change of the quantity of interest.

**4.2.1.2 Interview 1 task 2.** This task presented Jordan with the differential equation $P' = 3P$ and asked him to explain what it meant to him. While completing the task Jordan utilized two sets of resources, each indicating a different interpretation of the differential equation. Jordan utilized the resources *functional variation, functional dependence and rates indicate behavior* to construct graphical representations of the quantity of interest. By inferring that the value of $P'$ increased as the value of $P$ increased, and using the value of $P'$ to indicate the behavior of $P$, Jordan was able to graph the behavior of $P$. In this way he interpreted the differential equation as a description of the behavior of the quantity of interest. In addition he also solved the differential equation using *the method of integrating factors* and *dependence.* That is he found the symbolic form of the equation for $P$ and discussed the role of initial conditions with regard to finding coefficients for particular solutions. In this way Jordan also expressed interpreting the differential equation as an equation to solve.

Upon reading the task Jordan provided details about how he was interpreting the variables in the differential equation and their relationship both verbally and graphically (see Figure 4.2.3; Jordan later scribbled over the graph upon determining it was not correct), which is evident in the following transcript.

63 J: $P$, I guess I Immediately, I'm still stuck on the last question so, $P$, population possibly, as the actual quantity of something increases, the rate that it increases is squared. So like, this [begins graphing a series of vertically aligned quadratic functions] would be the function of $P$over I'll go with $t$ because it doesn't specify. So I mean if I integrated this it would be [writes $\frac{3}{2}\ P^2 + C$], so it would be some sort of quadratic function of $P$ at an initial value. So I guess as $P$ increases, or as $t$ or whatever it is related to increases the rate of change increases so its exponential or quadratic I guess it's $x$ squared.

64 G: Ok, $P'$ is the rate of increase?

65 J: I mean it could be decrease I guess, like if it was negative once you'd start seeing the reverse. I mean I haven't dealt with a differential equation that is not like dependent on time so I have never really dealt, like everything I have seen similar to this has been some sort of positive population or concentration with a positive time. So I haven't really dealt with anything outside of that yet, so I have just been making assumptions as to what it could be.

In turn 63 Jordan utilized *functional variation* to infer that "as the actual quantity of something increases, the rate that it increases is squared." Here Jordan utilized *functional variation* to infer that increases in the value of $P$ corresponded to increases in the value of $P'$. In doing so he interpreted $P$ as "the actual quantity of something" and $P'$ as the rate at which it was increasing (or decreasing; turn 63). From this inference Jordan graphed a sequence of quadratic functions, which were vertically aligned (see Figure 4.21). He then attempts to verify that the behavior in the graphs are correct by integrating the differential equation with respect to $P$

(*opposite operations reverse*), but one minute later, and without prompting him, Jordan realized his reasoning was incorrect as he "completely ignore[d] everything and consider[ed] that [points to $P$] like an $x$" (turn 69). This can be seen in talk turns 69-73 (follows).

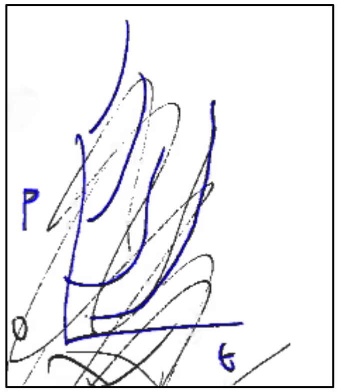



*Figure 4.21: Graph of potential functions of $P$. (color added to help discern original graph from where Jordan scribbled over it)*

69 J: I mean, I guess to think about it more in terms so I don't do exactly what I just did and completely ignore everything and consider that like an $x$. So I guess I'm just going to write this as $y' = 3y$, its completely, it doesn't depend on $x$ at all, so forget that, that was me just misreading things. So if it's the slope due to whatever it's completely dependent on $y$ so its homogeneous I guess. So if it's $y$ is one and then as it increases.
70 G: As what increases?
71 J: As $y$ or I guess in this form $P$ as $P$ increases the slope of $P$ increases [begins drawing slope field].
72 G: And what does that mean, how do you think about it? What does it mean for as $y$ increases or as $P$ increases $P'$ increases?
73 J: Um I mean I guess just kind of like a quadratic growth. Like as one increases the other increases and it s kinda just continuous, they both increase or decrease together.

In talk turn 69, in an attempt to avoid making another mistake Jordan re-wrote the differential equation as $y' = 3y$, and then utilized *functional dependence* to infer that "the slope… it's completely dependent on $y$ so it's [the differential equation] homogeneous I guess." In turn 71 he expressed interpreting $P'$ as "the slope" and said, "as $y$ or I guess in this form $P$, as $P$ increases the slope of $P$ increases" and described this by drawing a slope field; (see Figure 4.22). In doing so he utilized the resource *functional variation* to show how changes in the value of $P$ related to changes in the value of the slope and expressed interpreting $P'$ as representing a slope. It is important to note that his construction of the slope field is not indicative of *functional mapping* as he did not use the slope field to correlate specific values of $P$ and $P'$, he used the slope field as a way to represent how changes in $P$ related to changes in $P'$.

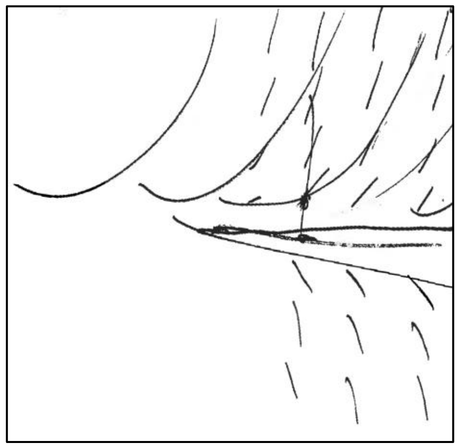


*Figure 4.22: Slope field Jordan drew to discuss the relationship between values of $P$ and $P'$*
.

After drawing the slope field Jordan went on to explain why his initial graph and the reasoning that supported it were incorrect. This is significant because while doing so he presumably relied on what he considered to be the correct line of reasoning, and it serves to explicate how different sets of resources and interpretations not only supported Jordan in completing the task, but in overcoming and correcting a mistake he made. To explain why his original reasoning was flawed Jordan utilized the resources *functional dependence*, *rates indicate behavior*, and *dependence*.

76 G: So you were integrating it like it was a function of $x$?
77 J: Yeah and that is not what it is.
78 G: Ok, why not?
79 J: Um, because it is completely dependent only on that initial $P$ [draws middle curve on slope field]. So its actual $x$ or $t$ or whatever this $P'$ is taken with respect to is completely irrelevant in terms of the behavior of the function. So if it's just the only thing that it [referring to $x$ or $t$] would be useful for is like say with this slope field, if I wanted to find an initial condition then it would follow that.
80 G: That would be the only thing that $t$ would be useful for? Is that what you mean?
81 J: yeah, I mean it's gonna be the same behavior of the equation no matter where you start it [draws left and right curves on slope field], it's just if you are gonna actually include any sort of $t$ or $x$ it has to be relative to an initial condition that you have to be given. Because,

that's why your slope field, because there is really no dependence the way it currently stands without a [pause].
82 G: When you say, the last thing you said. You are going to have a $t$, what did; ah I'm trying to think about how you just said it. If you are given a $t$.
83 J: $t$ only matters for a specific solution.

In this segment Jordan suggested that integrating the differential equation with respect to $t$ ("treating it like an $x$") isn't appropriate because $P'$ does not depend on $t$; "$t$ only matters for a specific solution" (turn 83). I asked him to clarify what he meant by $t$ being "irrelevant in terms of the behavior of the function", and "$t$ only matters for a specific solution" and after a series of exchanges he provided a lucid example in turn 91. His responses help illuminate how Jordan was interpreting the differential equation and serves to illuminate one of the major characteristics of interpreting the differential equation as a description of the behavior of the quantity of interest.

91 J: Like if you have some equation. Like if this [points to $P^{'} = 3P$] was kinda like that population growth thing [T1-I1]. Like yes, populations are gonna increase over time but you need that initial point to take every other point on that curve relative to that initial point. Like one of the things we actually did in class in section 12.5, we did yesterday, was the ah, finding the initial condition that is simple to find for a homogeneous equation and then just literally sliding it over to the point that you need. So it's completely not dependent on time for the behavior of it, but it is for specific solutions.
92 G: When you say it's not dependent on time for the behavior of it what is the 'it'.
93 J: Um the curve, $y'$ the slope.
94 G: So the time, all of these curves have the same behavior is that what you are saying?
95 J: Yeah they all have identical behavior they're all parallel they all, if they all started from the same initial point.

In turn 79 Jordan noted, "$x$ or $t$ or whatever this $P'$ is taken with respect to is completely irrelevant in terms of the behavior of the function." In turn 81 he elaborated on this by saying, "it's gonna be the same behavior of the equation no matter where you start it", which he justified by drawing multiple curves with the same behavior on the slope field and by noting there is no dependence on $t$ present in the slope field. Additionally in turns 81 and 91 Jordan suggested that solutions are determined by initial conditions. Collectively, these statements suggest that for Jordan the differential equation describes the behavior of the solutions (as opposed to their value), which in this case is not dependent on $t$ (or $x$); their behavior is only dependent on the value of $P$. In this case, for Jordan the function is constructed by knowing where it started and its behavior. This is distinctly different from constructing the function based on correlating values of $p$ with values of $t$. This supports why Jordan claimed all of the solutions were parallel, namely because they all had the same (parallel) slopes at each $P$ value and illuminates why integrating with respect to $t$ (or $x$), is not appropriate. Namely, integrating is not appropriate because the functions behavior does not depend on $x$ or $t$, rather it only depends on where the function starts. In this case Jordan is not attending to the value of $P$ being dependent on $t$ but rather he was focused on the shape of the graph and where it was located on the $p, t$-plane, not how values of $t$ correlated to values of $P$. I now discuss the resources supporting these inferences.

I interpret these statements and actions as representing a coordination of *rates indicate behavior*, *functional dependence* and *dependence*. Specifically Jordan inferred that the behavior of a specific solution function is dependent on its value and initial condition. Specifically, while considering the differential equation he utilized *functional dependence* to determine that *P'* does not depend on the value of "$x$ or $t$ or whatever this $P'$ is taken with respect to", rather it only depends on the value of $P$. He used *rates indicate behavior* to infer the behavior of $P$ from the value of $P'$, and lastly, he used *dependence* to infer that the specific solutions were determined by an initial condition. This interpretation is supported by statements he made in turn 91 when he talked about the importance of having an initial point to find specific solutions; "you need that initial point to take every other point on that curve relative to that initial point."

Jordan paused after turn 95, and interpreting this as a cue to move on to the next task, but also wanting to make sure Jordan was provided with opportunities to discuss other relevant aspects of the differential equation, I asked him "are there any other ways you think of, any other things that you see or think of or relate to, or any other ways you think of $P' = 3P$" (turn 96)? To which he replied "how to solve it"(turn 97). What transpired as Jordan proceeded to solve the differential equation is significant in terms of the analysis because he expressed a different interpretation of the differential equation in that he was now focused on finding a solution to the differential equation, namely an equation for $P$. To do so Jordan utilized an analytical solution method and went on to discuss the role of initial conditions with regard to finding specific solutions. Namely that initial conditions do not contribute directly to the behavior of the curve, but that they do determine the specific algebraic form of the solution curve.

In turn 98 I asked, "how would you solve it" to which he replied:

> "I mean put it in the standard thing then, integrating factor. I mean I guess this one's an easy one, so couldn't I just do ah, $P$ equals $C\ e$ to the negative $3t$, I skipped a lot of steps, I think that was the shortcut. Actually, no that's negative, so that's positive so, it's just $e$ to the $3t$.

This statement is indicative of the resource *integrating factor method*. As evidence this resource I take the statement "I think that was the shortcut" as evidence of this method. After attaining the solution, I asked Jordan what the $C$ in his solution represented. He initially said that it comes from the integrating factor method, and went on to say:

> "$C$ is the product of the integration it's a constant that is determined by whatever initial condition you are given. Just like with a simple you are given a function $y$ of $x$, um, when you integrate it unless you are doing a specific integration between closed points if you want to just integrate that graph and plot another graph, you need that plus $C$" (turn 103).

It is important to note that Jordan is discussing the role of integrating the differential equation to attain an algebraic representation of the solution. I take this as evidence supporting that Jordan is now operating with a different interpretation of the differential equation. With regard to the transcript, Jordan utilized *dependence* when he explained that the value of $C$ is determined by the initial conditions you are given. Further, he suggested that if you want to plot a graph with a given set of initial conditions “you need that plus $C$.” In turn 105 he said however that $C$ “doesn’t really contribute to the direct behavior of the graph, its whatever the initial condition is… So it’s just, its where to start and it's just there because this [points to the $C$] is just what gives you all possibilities.” Elaborating on this in turn 113 he said, “cause if you get rid of the $C$, you only have one solution and that’s a solution to a specific differential equation and a specific situation.” In this case, Jordan was interpreting the differential equation as an equation to be solved, where solving it meant finding a specific algebraic form of the solution.

In summary Jordan utilized the resources *functional dependence*, *functional variation*, *rates indicate behavior* and *dependence* while interpreting the differential equation as a describing the behavior of the quantity of interest. Jordan utilized the resource *functional dependence* to infer that the rate of change of the quantity was only impacted the value of the quantity and the resource *functional variation* to determine how the value of the rate of change related to changes in the value of the quantity. Moreover he used *rates indicate behavior to* construct the graph of possible solution curves from the slope field. In addition Jordan coordinated *rates indicate behavior* with *functional dependence* to infer that the behavior of the quantity was not impacted by *P'*s independent variable. Additionally he utilized the *method of integrating factors* and *dependence* while interpreting the differential equation as an equation to be solved. To solve the differential equation Jordan utilized the analytical solution method to find an algebraic presentation of $P$ with respect to $t$, and discussed the importance of the $C$ in the solution. More specifically, Jordan utilized *dependence* to discuss how the different solution equations are determined by the different initial conditions.

**4.2.1.3 Interview 1 task 3.** This task presented Jordan with a first order differential equation said to represent the fish population in the campus duck pond, and asked that he find the fish population with respect to time. Jordan did not explicitly complete this task, though his reasoning indicated he used the resources *functional dependence, functional variation, rates indicate behavior* and *variable substitution* to describe the general characteristics of the fish population. Jordan used these resources to relate the value of the fish population to the value of its rate of change, from which he reasoned about the behavior of the population. In doing so he interpreted the differential equation as describing the behavior of the quantity of interest.

As Jordan read the task he underlined various words and phrases (i.e., fish population, $P$, number of fish, pond, given time $t$), and after discussing how he would solve the task in turn 120 I asked him why he underlined them. I present his reply (turn 124) before discussing turn 120 because it provides insight into how he was interpreting the various components of the differential equation and thusly illuminates both how he was interpreting the differential equation and the reasoning he utilized while doing so. The following is taken from his reply to my question, "so you underlined a few things in here, why did you underline them?"

> "So, it's kind of just a reminder, like fish population, $P$, that is what $P$ is. $P'$, change in population. Number of fish, ok like how to actually like think of it and if I do get a solution, if I get the solutions a couple million fish I probably need to rethink things. Pond, just because I don't know, I underlined it because I have been dealing with a lot of water in the past couple of hours I guess. So, I mean if it's just a closed pond, there is no sort of change in volume, there is no answer, there is no sort of correlation to the constraints of the size or like water within the pond, it's purely population and time [draws a box around the differential equation $P' = 2P + t$]" (turn 124)

In this statement we can see that Jordan expressed that $P$ represented the number of fish in the population and $P'$ represented the change in the population (both values). Also his last statement "it's purely population and time," is indicative of Jordan inferring which of the aspects of the problem impact the change in the population, which I take as evidence of his utilization of *functional dependence*. I now identify the resources Jordan utilized to complete the task.

After reading Task 3 Jordan compared it to Task 2, and began discussing (turn 120) how he would use the differential equation to determine the fish population.

> "Ok, it's the same as the last one. If you know, you solve the differential equation, you get the rate of change of the population, which will be with respect to time but you get the rate of change and you plug in however much you start with. If you know the duck pond had 1,000 fish in it yesterday I could solve that out to a given $t$ and find out how much fish it has left in like a week (turn 120).

It was not immediately apparent what Jordan means by "solve the differential equation" because in Task 2, Jordan had discussed both analytically finding a solution that satisfied the differential equation, and using a slope field to construct such a solution. After analyzing the transcript however, I interpret "solve the differential equation" to mean that Jordan would utilize, what an observer would consider to be some variation of Euler's Method. As evidence, Jordan said, "…you plug in however much you start with. If you know the duck pond had 1,000 fish in it yesterday I could solve that out to a given $t$." As such, this segment of transcript is indicative of the resources *variable substitution* ("plug in however much you start with") and *rates indicate behavior* ("I could solve that out to a given $t$"). That is, Jordan expressed utilizing a starting point and the behavior of the population to determine an ending point. This was a theme in Jordan's subsequent explanations.

Inviting Jordan to elaborate on this more, in turn 127 I asked Jordan what information about the fish population the differential equation provided. His reply indicated he was utilizing the differential equation to relate population values to the population's rate of change.

128 J: Um the more fish there are in the pond the more fish there will be in the pond.
129 G: Ok, what tells you that?
130 J: Um, there is a positive correlation between the population of fish and the rate of change. So the more fish there are the faster the fish population will grow.

In turn 128 Jordan used the differential equation to infer "the more fish there are in the pond the more fish there will be in the pond." In making this statement Jordan did not simply mean the fish population was increasing; in turn 130 he noted, “the more fish there are the faster the fish population will grow.” In other words, Jordan inferred the fish population was growing at an increasing rate; “there is a positive correlation between the population of fish and the rate of change (turn 130).” To make this inference he coordinated *functional variation* and *rates indicate behavior*. That is, he utilized *functional variation* to determine that increases in the population related to increases in the rate of change of the population, and *rates indicate behavior* to conclude increases in the rate of change meant the fish population would grow faster. I then asked Jordan how this helped him complete the task. The following segment of transcript is representative of his reply.

138 J: Like if you have two fish you are gonna get like another fish in however long. If you have 100 fish, you're gonna get another 50 to 100 or quite a bit more fish per a period of time. At a specific point there is a much higher birth rate. Birth rate over death rate, that whole thing. The death rate is relatively constant no matter how many fish are in the pond, fish a have a life span. The birth rate, if there is a lot more fish then they are going to make a lot more fish.

139 G: Can you go back and maybe draw an example of what you just said. If there, these numbers might not be exactly correct, but you said of there is 10 fish in the pond, then the increase you are gonna get is smaller than the increase you get, you said, I think you said in the same amount of time or same something. I’m just wondering if you could draw that or elaborate on how you or what that looks like.

140 J: So like population of the fish and time [draws and labels set of axes]. So if you start with a really low population they are gonna have kind of like a slower like initial growth rate but like once they as there is more fish to make more fish the population is gonna continue to grow [draws lower graph] so if you have this same situation where you are starting with a lot more fish [draws upper graph], you're already gonna be kind of at the same point of increase, that same rate of change [draws tangent on lower graph] you are already starting with here [draws tangent on upper graph].

In turn 138 Jordan compared the rate of increase for different populations. That is, Jordan compared how values of $P$ related to values of $P'$; “if you have two fish you are gonna get like another fish in however long. If you have 100 fish, you're gonna get another 50 to 100 or quite a bit more fish per a period of time” (turn 138). From this he inferred that “if there is a lot more fish then they are going to make a lot more fish” (turn 138). Which is again indicative of coordinating the resources *functional variation* and *rates indicate behavior*. In turn 140 I asked Jordan to elaborate on this by drawing an example as during the interview I was uncertain of how to interpret his statements in turn 138 and how they related to completing the task. In turn 140 he drew two graphs representing two different fish populations, each having different initial conditions (see Figure 4.23).

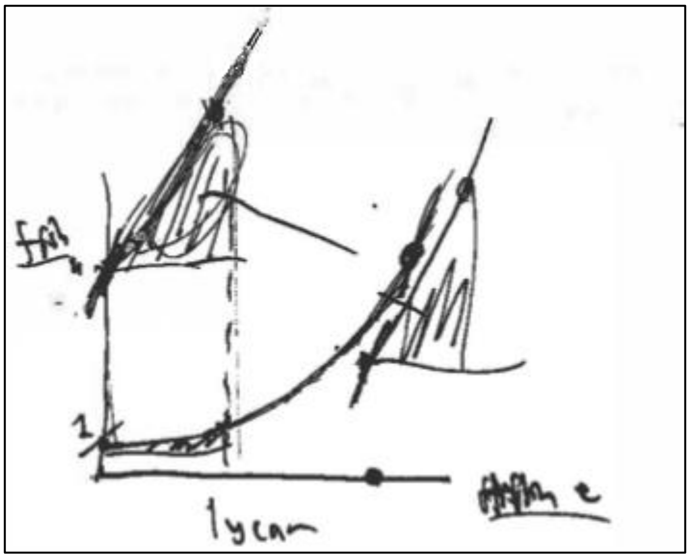



*Figure 4.23: Jordan's drawing of two fish populations with different initial conditions.*

Using the graph, he proceeded to compare the instantaneous rates of change across the two graphs at various points. To summarize, Jordan argued that lower populations will have lower rates of change, "if you start with a really low population they are gonna have kind of like a slower like initial growth rate", and there are different points were the two population graphs will have the same rate of change ("you're already gonna be kind of at the same point of increase, that same rate of change, you are already starting with here"). These statements are indicative of his utilization of *functional variation, rates indicate behavior* and *functional mapping.* More specifically, Jordan used *functional mapping* to correlate values of the population to values of the rate of change of the population, *functional variation* to determine how increases in the population related to increases in the rate of change of the population, and *rates indicate* *behavior* to determine how the values of the rate of change related to the behavior of the populations.

In an effort to understand how this was related to using the differential equation to find the population at a given value of $t$, I started asking him questions about the graph he drew. This can be seen in the following portion of transcript.

143 G: So are these, what are these. You just drew these two lines, what are those two lines?
144 J: Ah the rate of change. The slope of the fish over time, so it's the $\frac{dP}{dt}$.
145 G: Ok and you are saying that if these populations are the same these slopes are the same?
146 J: Ah no, because there is also a time dependence, but they are correlated.
147 G: Ok how about these ones [points to tangent lines drawn on vertical axis]? Are these slopes the same?
148 J: No.
149 G: Ok.
150 J: Because it's, I mean it's dependent on population and time. So if we start or we keep the time consistent so you put these at zero time if this is at one, the rate of change would be one and if this was at four, it would be four. So there's, as there is more fish there is a faster rate of change.

In turn 146 Jordan noted that because the rate of change of the population also depends on time, the rates of change of the two curves would not necessarily be the same when their populations are equal. Here he utilized *functional dependence* to justify why the populations would have different initial rates of change. Additionally, in turn 150 Jordan utilized *variable substitution* and *rates indicative behavior* to show that the value of the rate of change of the lower population would be smaller than the value of the rate of change of the larger population.

To complete this task Jordan did not explicitly find a solution (graphical or otherwise) representing the number of fish in the pond at a given time. The graph Jordan provided was the result of me prompting him to explain what he meant with regard to different populations having different rates of change based on their population values. There were, however, numerous times when Jordan discussed using the initial conditions to determine the rate of change of the population (turns 138, 140, and 150), and how these rates of change would impact future values of the population (turns 128, 130, 138 and 150). As such I interpret Jordan's statements as being commensurate with interpreting the differential equation as describing the behavior of the quantity of interest. Namely, his statements and actions are commensurate with using the initial condition and information about the rate of change of the population to determine future values of the population. Or, in his words "if you know the duck pond had 1,000 fish in it yesterday I could solve that out to a given $t$ and find out how much fish it has left in like a week" (turn 120). Here Jordan used the differential equation to infer the general behavior of the fish population.

While completing this task Jordan utilized *functional dependence, functional variation, rates indicate behavior* and *variable substitution.* Also he expressed that $P$ was the number of fish in the population, and $P'$ was the rate of change of the fish population. To reason about the task he utilized *functional dependence* to reason about how values of $P'$ related to $P$ and $t$. For example he used *functional dependence* to determine that equal populations would not relate to equal rates of change because "there is also a time dependence" turn (146). Additionally, he made use of *functional variation* to determine how the value of $P'$ changed as $P$ changed, and *rates indicate behavior* to relate those changes to the behavior of the populations. Lastly he used *variable substitution* to determine specific values of $P'$ given value of $P$ and $t$, which he used to discuss characteristics of the populations behavior. Based on the set of resources Jordan utilized to complete the task and they way he interpreted the various parts of the differential equation, I conclude that Jordan interpreted the differential equation as describing the behavior of the quantity of interest.

**4.2.1.4 Interview 1 task 4.** Task 4 asked Jordan to find a differential equation that had a solution of $y = 6$. Jordan completed this task by finding a differential equation that correlated a $y$ value of 6 to a $y'$ value of 0 ($y' = y - 6$). In this case Jordan expressed thinking of $y = 6$ as a function but with regard to the differential equation he utilized $y$ as a value and $y'$ as the value of the slope (rate of change) of $y$. Lastly, to construct the differential equation Jordan utilized, *functional mapping*, *equality condition*, *functional dependence* and *rates indicate behavior*. In doing Jordan interpreted the differential equation as a relationship between the value of a quantity and the value of the rate of change of the quantity.

The following transcript starts with Jordan's initial statements.

165 J: you know what I am just gonna I'm not thinking straight so I am gonna do this [draws constant function at $y = 6$ on a $y, t$ axis], this is at 6. [Writes $y' = 0$] Yeah I'm gonna go with that [writes to $\frac{dy}{dt} = y - 6$].
166 G: Ok so tell me um, so you drew this or wrote this [$y' = 0$] down and then said I'm not thinking straight so I am gonna do this [points to graph]. What was this that you were trying to, how did this help you verify to yourself maybe that what you were doing was the right thing?
167 J: Ok, so I mean so if we are gonna find a differential equation that has a solution $y$ equals 6, that means so $y$ equals 6, so $y$ prime equals to the slope of this, so if I take the derivative it equals zero.
168 G: And you took the derivative of this [$y = 6$]
169 J: Yeah of this. So I have to find a differential equation so at 6 the slope is zero.
170 G: Ok.
171 J: So at $y = 6$ [substitutes 6 into the differential equation for $y$, gets $\frac{dy}{dt} = 0$],
172 G: Now when you say at $y$ equals 6 the derivative equals zero, how are you thinking of $y$ equals 6?
173 J: As a function of whatever its relating to, because its, there is no dependence it is not $y = 6x$ or anything it could just be, it could just be a one dimensional space of $y$ and $y = 6$. I just visualize things by habit in a 2D plane [pointing to the graph he constructed].

In turn 167 Jordan utilized the resource *rates indicate behavior* to infer that $y' = 0$; he expressed that $y'$ was the slope of $y$ which was zero. Though he said, "take the derivative" in turns 167 and 169, his language suggests that he used the slope of $y = 6$ to infer that $y' = 0$. In turn 169 Jordan coordinated *functional mapping* and *equality condition*. Namely he correlated $\frac{dy}{dt}$ being 0 with $y$ being 6 and then used *equality condition* to infer he needed a "differential equation so at 6 the slope is zero." In other words, Jordan used the fact that when $y$ was 6 the value of $\frac{dy}{dt}$ was (*functional mapping*) to create a differential equation [$\frac{dy}{dt} = y - 6$] that remained true under the condition, when $y = 6$, $\frac{dy}{dt} = 0$ (*equality condition*). In turn 171, he utilized *variable substitution* to confirm that the equation $\frac{dy}{dt} = y - 6$ provided a slope of 0 at $y = 6$. I then asked Jordan how he was thinking about $y = 6$ in the differential equation; due to $y$ being a constant function this was not immediately obvious. Before concluding, I discuss how Jordan was thinking about $y = 6$.

In turn 173, Jordan said that he thinks of $y = 6$ as a function and then used *functional dependence* ("there is no dependence it is not $y$ equals $6x$ or anything") to explain why the graph he drew consisted of a straight line. Jordan went on to elaborate on this, but Jordan's consistent use of the language "at $y = 6$" seemed contradictory to referring to $y = 6$ as a constant function. More specifically, during the interview it seemed as though "at $y = 6$" indicated that $y$ could be different values, but then Jordan talked about $y = 6$ being a constant function. So I asked him about this explicitly in turn 181. His response (follows) clarified this seemingly contradictory language use. Jordan expressed the existence of multiple solutions, any one of which had different slopes at different $y$ values. The solution he was interested in ($y = 6$), was the one that had a slope value of 0 at $y = 6$, which indicated a constant function.

182 G: So I am confused ,or not umm. There seems to be a bit more under the surface in the way you are thinking about these because you say, you are saying things like, and this isn't me saying this isn't sufficient, I am just trying to understand how you are pulling things apart, or how you are thinking about things. You are saying 'at $y$ equals 6', like $y$ can be different values, in other words maybe it's because you are thinking about it in this space [Jordan had mentioned thinking about $y = 6$ in a one dimensional space in turn 173], and you are saying when $y$ equals 6 it doesn't change any more. So I guess I am trying to figure out, you are saying at $y$ equals 6, $\frac{dy}{dx}$, $dt$ d whatever it is, is zero. In other words $y$ doesn't change?

183 J: The 'at $y$ equals 6', I am just kind of throwing that in there because we are thinking about it in the context of a differential equation. So if I actually wanted to draw out that differential equation that I wrote, it would have that equilibrium solution at 6 [draws line $y = 6$ on new set of axes] but once you get to something, like lets say I have $y = 0$, it

would actually have [draws negative slopes across $y = 0$], there would be plenty of other different solutions but for what I am actually trying to solve it would look like something like that [points to $y = 6$]. But at the equilibrium solution $y$ equals 6, at $y$ equals 6 you would get equilibrium solution of zero, which is $\frac{dy}{dt}$ equals 0. I just wanted to find something with an equilibrium solution at $y$ equals 6 because that to me just looks like the easiest way to solve that.

In this segment of transcript Jordan expressed using the resources *rates indicate behavior,* and *functional mapping* while reasoning about the task. Specifically used the resource *functional mapping* to show that different solutions will have different slope values at different $y$ values (see Figure 4.24); "like let's say I have $y = 0$, it would actually have [draws negative slopes across $y = 0$], there would be plenty of other different solutions but for what I am actually trying to solve it would look like something like that [points to $y = 6$]" (turn 182). Additionally, using the slopes he drew potential solution functions, indicating the utilization of the resources *rates indicate behavior*. In this segment of transcript Jordan's reasoning is consistent with attending to different values of $y$ relating to different slope values (*functional mapping*), but also that at $y = 6$ the slope was zero, thus indicating in a constant function (*rates indicate behavior*). In other words, Jordan attended to the existence of multiple solutions, and expressed that those solutions would have different slope values for different values of $y$, but that at $y = 6$ the slope had to be zero. In other words, Jordan was aware that $y$ represented a function, but within the differential equation $y$ and $y'$ were values.

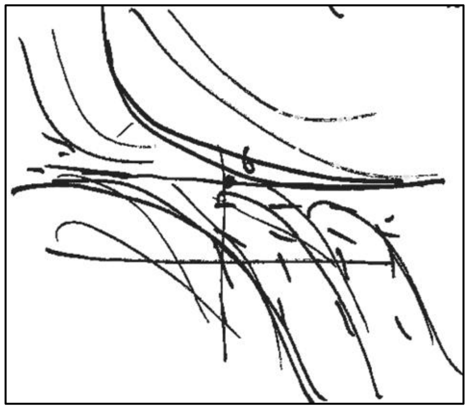



*Figure 4.24: Jordan's drawing of the solutions to the differential equation.*

In summary while completing Task 4 Jordan utilized *functional mapping*, *equality condition*, *functional dependence,* and *rates indicate behavior*. While utilizing these resources Jordan interpreted $y$ as a function, and $y'$ as the rate of change of $y$. In this case, he interpreted $y = 6$ as a solution function with a (invariant) slope of 0. Jordan utilized *functional dependence* to infer the equation $y = 6$ was referring to a constant function and *rates indicate behavior* to infer that the slope of $y = 6$ is zero. He then used this pair of values ($y = 6$, $y' = 0$) in coordination with *functional mapping* and *equality condition* to relate the $y$ value of 6 with the $y'$ (slope) value of 0 to construct a differential equation such that $\frac{dy}{dt} = 0$ when $y = 6$; "I have to find a differential equation so at 6 the slope is zero" (turn 169) In addition, while Jordan expressed thinking of $y = 6$ as a function, within the differential equation he utilized $y$ and $y'$ as values. This is evidenced by his construction of the slope field, which was the result of explaining that $y'$ would be different values for different values of $y$. Lastly, considering the resources Jordan utilized and the ways he attended to the various parts of the differential equation, he interpreted the differential equation as a relationship between the value of a quantity and the value of the quantity's rate of change.

**4.2.1.5 Discussion of interview 1.** While completing the four tasks in Interview 1 Jordan expressed three ways of interpreting the differential equations; as a relationship between the values of quantities and the value of the respective quantity's rate of change (T1-I1, T4-I1), as a description of the behavior of the quantity of interest (T2-I1, T3-I1), and as an equation to be solved (T2-I1). Each of the sets of resources associated with them (see Table 4.7) were utilized to support Jordan in completing the respective task. In Task 1 he utilized the resources *Functional dependence, partial dependence, functional variation, variable substitution,* and *functional mapping* to relate values of $x$ to values of $\frac{dy}{dt}$ (rate of change of $y$) and values of $y$ to values of $\frac{dx}{dy}$ (rate of change of $x$). More specifically the resources *partial dependence* and *variable substitution* supported Jordan in determining the exact nature of the relationship between the values of the variables. While *functional dependence* and *functional mapping* supported Jordan in being able to consider relationships between changes in sets of the values and correlations in the sets of values, respectively.

Similarly in Task 4 Jordan used *functional mapping, equality condition and functional dependence* to relate values of $y$ to values of $y'$. The resource *functional mapping* supported Jordan in interpreting the differential equation as a relationship between the value of a quantity and the value of the quantity's rate of change. The resource *functional dependence* supported Jordan in constructing a graph of the solution function $y = 6$, and from this graph Jordan was able to utilize *rates indicate behavior* to determine the corresponding value of $y'$. It is important to note that Jordan did not use rates indicate behavior to determine the behavior of the quantity of interest from the value of the quantity's rate of change, rather in this case he used the resource to infer the value of the rate of change from the behavior. By coordinating his interpretation of the differential equation and the entities from which it was composed ($y$ and $y'$) with *equality condition* Jordan was able to construct a differential equation that represented a mapping that satisfied the condition $\frac{dy}{dt} = 0$ when $y = 6$.

There are aspects of Jordan's reasoning with Task 2 worthy of note as they explicate the importance of the different interpretations with regard to student reasoning and the resources they use in conjunction with that reasoning. Recall that when completing Task 2 Jordan's initial reasoning lead him to draw an incorrect set of solution curves though he quickly determined that he was incorrect and began to explain why. In his explanation he noted that increasing values of the $P$ related to increasing values of $P'$, which he graphically represented on a slope field by applying *functional mapping* and *functional dependence.* He then used this slope field to construct graphs which represented different curves of $P$ with the resource *rates indicate behavior*. More importantly, however, he also utilized the resource *functional dependence* to correctly infer that the value of $P'$ depended only on the value of $P$. This inference however had a rather surprising consequence; while this same inference supported Jordan in realizing his initial reasoning was incorrect, it also lead him to incorrectly infer that one could not integrate the differential equation with respect to the quantity's independent variable to find a solution because $P'$ did not depend on that same independent variable. From this Jordan discussed that the solution curves only depended on the initial value of $P$, which is very much correct in the sense of building a solution curve based on knowing its initial condition and rate of change, but also very incorrect in the sense that the variables the differential equations represent functions.

This highlights one of the aspects of interpreting the differential equation as a description of the behavior of the quantity of interest; namely that the focus is on building a solution curve by correlating the value of the quantity and its rate of change, not on building a graph (or other representation) from a known correlation between the quantity and its independent variable. Thusly, while intriguing, it's not necessarily unexpected that Jordan then went on to use an analytical solution method (*method of integrating factors*) that made use of an integration technique in a manner he, just minutes prior, said was not appropriate. That is to say, this seemingly contradictory statement is actually the result of switching interpretations where the solution was no longer the result of building a description of the quantities behavior from values and its rates of change; the solution was now an equation that directly related the quantity to its independent variable. In this case, integration (in some form, e.g., numerical, analytical) is entirely appropriate, if not entirely necessary.

*Table 4.7: Resources and interpretations Jordan expressed in Interview 1.*

| Task | Interpretation of the differential equation | Resources | Interpretations of the components |
|---|---|---|---|
| I1-T1 | As a relationship between the values of quantities and the value of the respective quantity's rate of change. | *Functional dependence*<br>*Partial dependence*<br>*Functional variation*<br>*Variable substitution*<br>*Function mapping*<br>*Definition of competing systems* | $x$ and $y$ were the value of the species populations<br>$\frac{dx}{dt}$ and $\frac{dy}{dt}$ were the values of the respective rates of change |
| I1-T2 | As a description of the behavior of the quantity of interest | *Functional dependence*<br>*Functional variation*<br>*Rates indicate behavior* | $P$ was the value of the population,<br>$P'$ -value of the rate of change of $P$ |
| | As an equation to be solved | *Dependence*<br>*Method of integrating factors* | $P(t)$ – an equation that results from the solution process |
| I1-T3 | As a description of the behavior of the quantity of interest. | *Functional dependence*<br>*Functional variation*<br>*Rates indicate behavior*<br>*Variable substitution* | $P$ was the fish population<br>$P'$ was the populations rate of change |
| I1-T4 | As a relationship between the value of a quantity and the value of the quantity's rate of change. | *Functional mapping*<br>*Equality condition*<br>*Functional dependence*<br>*Rates indicate behavior* | $y$ - represented the value of a function,<br>$y'$-value of the slope (rate of change) of a function |

### 4.2.2 Interview 2

During Interview 2 Jordan completed a total of 4 tasks, all of which are presented in this section. To complete each of the tasks Jordan utilized distinct (though overlapping) sets of resources. Further, Jordan utilized the components from which the differential equations were composed as different mathematical objects (e.g., functions and values) at different times. For instance while completing the first task Jordan used the differential equation to correlate the value of the quantities with the values of their respective rates of change, whereas while completing the third task Jordan sought an equation that related the quantity to some independent variable. These different sets of resources and the different ways in which Jordan expressed conceptualizing the components of the differential equation were indicative of the different ways in which he was interpreting the differential equation. More specifically, across Interview 2 Jordan expressed sets of resources and ways of interpreting the components of the differential equations that allowed him to determine how values of the quantities related to their respective rates of change, describe the behavior of the quantities, and determine an equation that related the value of a quantity to an independent variable.

**4.2.2.1 Interview 2 task 1.** This was the second time Jordan interacted with this exact task; the task was initially presented in Interview 1 (I1-T1) To complete this task Jordan utilized the resources *partial dependence, functional variation, definition of competing species, parts contribute to the whole* and *positives add – negatives take away.* He utilized these resources to complete the task by characterizing System B competitive based on how increases in $y$ related to increases in $\frac{dx}{dt}$ and increases in $x$ related to increases in $\frac{dy}{dt}$. In addition he utilized $x$ and $y$ as the value of the populations of species $x$ and $y$, and $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as the rate of change of the two populations. Though Jordan was aware of $x$ and $y$ being functions he utilized these entities as values while reasoning with the differential equation. Lastly, as the analysis will show Jordan interpreted the differential equations as a relationship between the value of a quantity and the value of the quantity's rate of change.

I begin the presentation of Jordan's interaction with transcript that starts with Jordan's first uteracnces after reading the task. It should be noted that Jordan's initial statements in turn 17 seem as though he was incorrectly reasoning about System A. Analysis of the video data suggests he was recalling the behavior of System B and incorrectly attributing it to system A. That is, the data as suggesta he was recollecting his work from Systen B and not re-working the problem. In support of this claim, while he was pointing toward system A, it did not appear as though Jordan was actually reading from the paper, rather his attention seemed focused on the interviewer. Further, in turn 15 Jordan correctly identified System A as cooperative and System B as competative thussly I interpret Jordans initial statements as being relavent to System B.

13 J: You changed it up on me.
14 G: What did I do?
15 J: Last time it was a simple sign change. Like one was positive negative, ah wait now. Wasn't like negative competitive, or like positive, like they worked together, cooperative, like cooperative [points to System A] competitive [points to System B], but let's see. Change in $x$ Change in $t$ [pointing at $\frac{dx}{dt}$], more $y$, more positive change [pointing to $\frac{dx}{dt}$ equation in System A], More $y$ more $x$ more positive change [pointing to $\frac{dy}{dt}$ equation in System A]. More $y$, more negative change [pointing to $\frac{dx}{dt}$ equation in System B], More $x$, more negative change [pointing to $\frac{dy}{dt}$ equation in System B]. Ok, so yeah competitive [circles system B], cooperative [circles System A].
16 G: And so can you work through some of the things you were doing in your head. You were saying things like more $x$ or more $y$ and then you said positive change or negative change.
17 J: Ok [points to System A] so as $y$ increases, $\frac{dx}{dt}$, so the change in $x$ decreases, so that would be a competitive nature, to where more of one species like lowers the change, the rate of change of the other species. So like let's say $x$ is a growing species, but then a ton more of $y$ show up, then $x$ could become a decreasing species, it could be dying out depending on the balance of it. It could become less positive, like the rate of change could be less positive, but it's a negative correlation between the two. The same with $\frac{dy}{dt}$, as $x$ increases $\frac{dy}{dt}$ decreases. And over here [points to System B], actually wait I think I just mixed that up.

> Yeah I just mixed that up, everything I just said goes to these [System B]. So yeah, as over here, for actual A, I'm a little low on sleep, so yeah the ones with the actual positive sign as $x$ increases, um, $\frac{dy}{dt}$ increases and as $y$ increases $\frac{dx}{dt}$ increases.

In turn 15 Jordan looked at each of the four equations and considered how increasing the value of one species impcted the other's rate of change (e.g., "more $x$ more positive change [points to $\frac{dy}{dt}$].") It is important to note that Jordan did not vary the values of both $x$ and $y$ in the equations, he only varied the values of the variables that afford him the ability to characterize the relationship of interest. More specifically Jordan coordinated the resources *partial dependence* and *functional variation* to characterize how changes in the value of $y$ related to changes in the value of $\frac{dx}{dt}$ and how changes in the value of $x$ related to changes in the value of $\frac{dy}{dt}$. He utilized *partial dependence* to discern which variables he was going to consider and functional variation to the relationship between their changes. In turn 17, Jordan provided a little more detail with regard to his reasoning during turn 15.

His first statement in turn 17 is indicative of his *definition of competing species*, which he both utilized to characterize System B, ("as $y$ increases, $\frac{dx}{dt}$, so the change in $x$ decreases, so that would be a competitive nature") and defined ("to where more of one species like lowers the change, the rate of change of the other species"). From this and his statements in turn 15 it is evident that Jordan was considering how changes in the population of one species impacted the rate of change of the other. How he actually made this determination from the equations is illuminated at the end of turn 17 after he realized he incorrectly attributed System A with competing species. Specifically he said "so yeah the ones with the actual positive sign as $x$ increases, um, $\frac{dy}{dt}$ increases and as $y$ increases $\frac{dx}{dt}$ increases." Jordan is talking about the positive sign in front of the $xy$ term in the equation for $\frac{dy}{dt}$ and the positive sign in front of the $xy$

term in the equation for $\frac{dx}{dt}$ (both in system A). This is indicative of the resource *positives add-negatives subtract.* This also provides insight with regard to Jordan's comments in turn 15; "wasn't like negative competitive, or like positive, like they worked together, cooperative." For Jordan while the relationship between changes in the value of one species' population and the rate of change of the other is important, it is the sign in front of the $xy$ term that characterizes this relationship. To alleviate any uncertainty that may stem from Jordan's mix-up in turn 17, I further support this analysis by presenting the analysis of transcript immediately following this interaction, which is also indicative of the same reasoning.

In trying to elicit information about why looking at the behavior of the rates of change was important, I asked Jordan "What does it mean for $\frac{dx}{dt}$ to increase" in turn 18. His reply not only provided great insight into how he was thinking about rates of change as they relate to population, but lead him to further elaborate on why he only considered changing values of $x$ in the equation for $\frac{dy}{dt}$ and changing values of $y$ in the equation for $\frac{dx}{dt}$. Moreover, he explicitly discussed utilizing the sign in front of the $xy$ terms to determine the nature of the relationship. Transcript of this discussion follows.

19 J: Um if it is currently a growing population it grows faster, if it is currently a dying out population it dies out faster.
20 G: Ok.
21 J: So the rate of change, like the birth date over death rate. If it's [the rate of change] positive it's a higher birth rate, if it [the rate of change] is negative it's a higher death rate.
22 G: Ok.
23 J: So, like doing over here [points to System A], because these are both positive, it would increase the birth rate, because it would make $\frac{dx}{dt}$, it wouldn't technically, it might not increase the birthrate, it might decrease the death rate, but either one, its making that relationship more positive towards increasing. Right, so over here [points to System B], adding more of the opposite species takes away from it and adds more to the death rate side of the ratio.
24 G: Ok, um, what is it about the rate of change, so I am trying to make sense of a few things that you had said, make sense isn't the right word, keep track of some of the things you were

saying. It seems like there is a lot of things changing and so one of the things you were talking about is as $y$ increases or as $x$ increases, so that is one change, and then another thing you were saying is that this thing [$\frac{dx}{dt}$] changes.

25 J: Yeah.

26 G: And as these things change [points to $\frac{dx}{dt}$ and $\frac{dy}{dt}$] it causes some other stuff to change, so can you try to help me make sense of all of these things changing and how they are kind of related.

27 J: Ok, so like, cause and effect. So we are just gonna ignore all of these factors [crosses out $x$'s in equations for $\frac{dx}{dt}$ and $y$'s in equations for $\frac{dy}{dt}$], cause these are just the effects of its current population on its own growth so that doesn't really help figure out anything about how they interact with each other.

28 G: Ok.

29 J: So I have just been ignoring like this factor (the single $x$ and $y$ terms). But these $x$'s and $y$'s that are multiplied together are basically the cause, it's the independent variable. So, because like again like the $x$'s are not really what we are looking at, cause it's the change of itself by itself and that doesn't say much about the relationship between each other. So if $\frac{dx}{dt}$, is the effect, this is the final result [writes $\frac{dx}{dt} = ky$], it is some relation to $y$.

30 G: Ok.

31 J: So this is what I am looking at as to whether they are competitive or cooperative, so if this $k$ is positive, which for both of these is a 2 and a 3, then ah, an increase in population $y$ would increase the population growth or decrease the population death, it would increase this change [points to $\frac{dx}{dt}$ and $\frac{dy}{dt}$].

32 G: Ok.

33 J: So and the same goes for switching the $x$'s and the $y$'s and then over here [points to System B] it is the opposite with the negative $k$'s.

In turn 21 Jordan expressed thinking about $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as some combination of a birthrate and deathrate. This way of thinking about the rate of change is indicative of the resource *parts contribute to the whole*, where the whole rate of change is a composite of the birth and death rates. In turn 23 Jordan discussed how adding more of one species in System A was equivelent to either increasing the birthrate or decreasing the deathrate of the other species, whereas in System B, adding more of one species was equivelent to increasing the deathrate of the other species. This was a somewhat unexpected response, and in turns 24 and 26 I asked Jordan to help me keep track of the relationships between all of the changing values. In turn 29 he discussed ignoring the entire first term in all of the differential equations, and only looking at the second term (which is commensurte with his reasoning from turns 15 and 17). With regard to this second term, however, he ignored the variables that represent "the change of itself by itself" (turn 29), or more specifically, the $x$ variable in the $\frac{dx}{dt}$ equation, and the $y$ variablle in the $\frac{dy}{dt}$ equation. Jordan actually crossed these terms and variables out (see Figure 4.25) while noting they do not "say much about the relationship between each other." That is, he considered the role of each variable in the equation, and removed the ones that did not contribute to the relationship in question, which is indicative of the resource *partial dependence.*

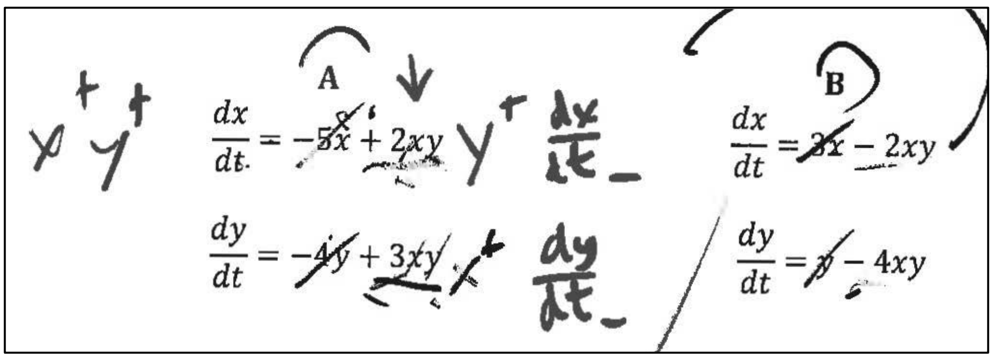


*Figure 4.25: Jordan only considered the sign of certain terms.*

In turn 31 he went on to discuss how he used the truncated equation, a single remaining term (see Figure 4.26 for an example of this) to characterize the relationship between $\frac{dx}{dt}$ and $y$. Specifically he used *positives add and negatives take away* when he considered the sign of the remaining coeffient; it either increased or decreased the overall rate of change. Jordan is explicielt about this in turns 31 and 33. His utilization of this resource can be seen in turn 31 when he said, "so this is what I am looking at as to whether they are competitive or cooperative, so if this $k$ is positive… an increase in population $y$ would increase… this change [points to $\frac{dx}{dt}$ and $\frac{dy}{dt}$]," and 33 when he said, "then over here it is the opposite with the negative $k$'s."

$$\frac{dx}{dt} = +ky$$

*Figure 4.26: Jordan used k to determine the relationship between $x$ and $y$.*

In summary Jordan used *partial dependence* to determine which variabels from each of the differential equations to consider. Specifically, to detremine how the change in one population impacted the rate of change of the other, he inferred that he only needed to consider the $y$ variables in the equations for $\frac{dx}{dt}$ and only the $x$ variables in the $\frac{dy}{dt}$ equations. He then coordinated the resources *functional variation* and *positives add–negatives take away* to determine how increases in one population related to changes in the other. For example knowing that coeffeceient in front of the $y$ term in the equation $\frac{dx}{dt} = 2y$ is positive, Jordan was able to infer that increases in $y$ would relate to an increase the rate of change of $x$. Lastly Jordan used his *definition of competing systems* to infer that System B was competing because "as $y$ increases, $\frac{dx}{dt}$, so the change in $x$ decreases, so that would be a competitive nature." In addition he utilized $x$ and $y$ as population values, and $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as the rate of change of the population, which were themselves composed of the sum of a birthrate and death rate. Based on the resources Jordan utilized and the ways in which he utilized the various entities in the differential equation, he interpreted the differential equation as a relationship between the value of quantities and the values of their respective rates of change.

**4.2.2.2 Interview 2 task 2.** This task presented Jordan with a second order differential equation, and asked what it meant to him. Jordan used the differential equation to describe the behavior of the solution functions based on different sets of values of $P$ and $P'$. To do this Jordan utilized the resources *functional dependence, functional mapping, dependence, rates indicate* *behavior, parts contribute to the whole, variable substitution, positives add – negatives take away*, and *larger magnitudes have larger impacts*. Additionally he expressed that $P$ represented numbers, $P'$ represented be the rate of change of $P$, and $P''$ represented the change in the rate of change. Moreover, he thought of these values as "snapshots" that supported the building of the function $P(t)$, or in Jordan's words, the "complete picture" (turn 129). Based on the resources Jordan employed and the ways in which he reasoned with the various parts of the differential equation to construct a graph of $P(t)$, Jordan interpreted the differential equation as a description of the behavior of the quantity of interest. I start the presentation of the analysis with transcript of Jordan's initial statements which are indicative of how he was interpreting the differential equation and using $P$, $P'$ and $P''$.

93 J: All right, what does the following differential equation mean to me? Bunch of $P'$s in it. Um, all right. So, $P$, number. $P'$, rate of change. $P''$, change of rate of change [writes these descriptions on task sheet]. So, I see double prime and really the only thing I have never been taught to do with that is concavity.

94 G: Ok.

95 J: I guess, whatever this is modeling, cause again really all I know about differential equations is it's for modeling things, so whatever this is modeling, it kind of pushes it to extremes I guess. Like as $P$ is positive, $P'$, I don't actually know how $P'$ is related to $P$, I'll just say that it's positive for the time being. So the change of $P$would be positive which would then force this change in the rate of change to be positive, so it's just a huge exponential growth in either a positive or negative direction. If this $P'$ is a positive relationship. I guess I have really no idea what this could model or where this is useful, but I just, I guess I could see it being just this massive exponential growth [gesturing an exponentially increasing function with his hand].

In turn 93 Jordan said he thinks of $P$ as a number (value), $P'$ as the rate of change of $P$ (value), and $P''$ as "the change of the rate of change," (value) which he marked on the paper as well (see Figure 4.27). This is important to note because, as will be discussed in subsequent paragraphs he discussed utilizing $P$ as a number to construct a graph of $P(t)$. This also serves to show that for Jordan, while $P$ was a function, with regard to the differential equation itself $P$ represented a number. In turn 95 Jordan noted that the differential equation is for "modeling things," and that this particular differential equation depicts exponential growth. In doing so he suggested that he was interpreting the differential equation as a description of the behavior of the quantity of interest. That is, Jordan appeared to be using the relationships present between the variables in the differential equation to make inferences about the behavior of $P(t)$. The presentation now shifts focus to identifying the resources that supported Jordan in making such inferences.

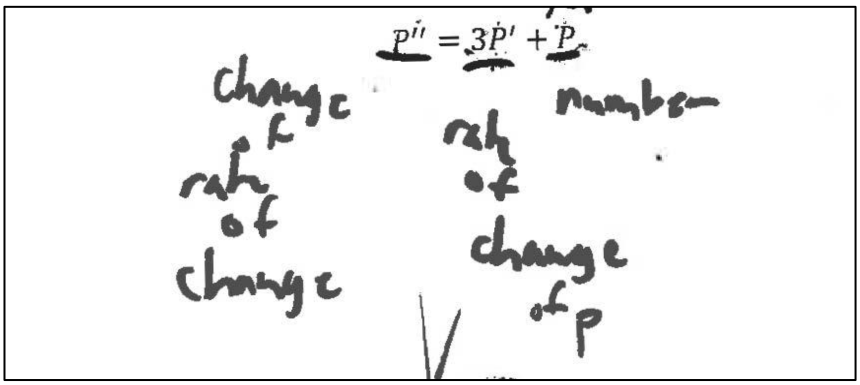



*Figure 4.27: Jordan's interpretation of the variables in the differential equation.*

In turn 98 I asked Jordan to provide an example of what he was gesturing. His response is included in the segment of transcript following this paragraph. I claim this transcript is indicative of the rather large set of resources *functional dependence, functional mapping, dependence, rates indicate behavior, parts contribute to the whole, variable substitution* and *larger magnitudes have larger impacts*.

98 G: So could you give me an example of how you see $P$, I mean you are kind of gesturing, maybe you could draw it for me?

99 J: Ok, so $P$ we are given no other variables, so I am just going to say $t$, and $P$ will have this independent variable. So, if we are given an initial value for this, well say at time equals zero, whatever this $P$ is, is like one.

100 G: Ok.

101 J: At this initial $P$ [draws point on $P$ axis], if this very first step is positive at all, it'll force $P''$ to be positive, that's that upward concavity. And it'll just gradually just grow [draws increasing function].

102 G: Ok, so let me ask you...

103 J: This concavity actually gets more and more positive so it might be more than exponential, I don't know.

104 G: What's telling you that it is getting more and more positive?
105 J: Um $P''$ is positive and $P'$ is positive and if $P$ initial is positive, there is nothing bringing it [makes motion tracing over graph of $P$] back it will just continually grow.

In turn 99 Jordan utilized *functional dependence* to (rather subtly) infer that $P$ had an independent variable $t$ (the importance of this will be discussed in a subsequent paragraph). In addition he coordinated *parts contribute to the whole* and *dependence* to construct a graph of $P$ which was determined by a set of initial values for $P$ and $P'$ in turn 101. Specifically, Jordan drew a point on the positive $P$ -axis, and then discussed that if $P'$ was positive, "if this very first step is positive at all," that $P''$ would be positive, and drew an increasing function (see Figure 4.28) noting that "it'll just gradually just grow." I take this as Jordan reasoning that $P$ and $P'$ being positive values, make $P''$ positive (*functional dependence,* and *parts contribute to the whole*), and that the initial values of $P$ and $P'$ determine the behavior of $P$ (*dependence*). In this case, the positive values of $P$ and $P'$ determine that $P$ will be an increasing function.

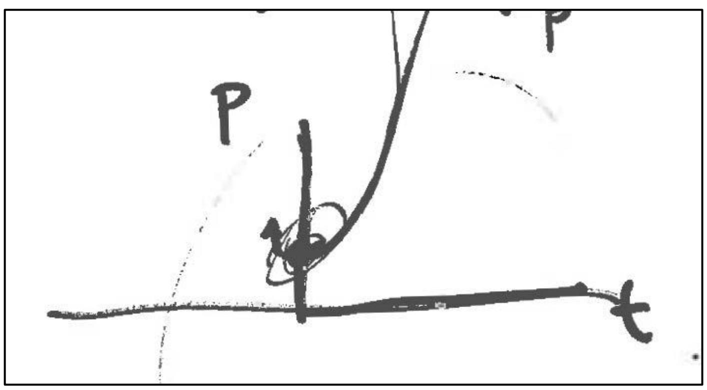



*Figure 4.28: Jordan's depiction of the behavior of P.*

In turn 105 he discussed that under these conditions, "there is nothing bringing it [$P$] back it will just continually grow." Elaborating on this later in the interview he said, "it's like a chain reaction that is feeding back into itself" (turn 115). I take these statements as indicating Jordan's utilization of *rates indicate behavior* and *variable substitution*. More specifically, in turn 105 I take the statement "there is nothing bringing it back it will just continually grow" as evidence that Jordan inferred when $P$, $P'$ and $P''$ are positive, $P'$ and $P''$will both always be positive, thus indicating the continual growth of $P$. In other words, he reasoned that because $P'$ was positive, $P$ would increase (*rates indicate behavior*), and because $P''$ was positive, $P'$ would increase (*rates indicate behavior*) and thusly each would remain positive. He then inferred that putting larger values of $P$ and $P'$ into the differential equation would yield larger values of $P''$(*variable substitution, larger magnitudes have larger impacts*) resulting in “massive exponential growth”(turn 97). This is commensurate with his statement in turn 115; the “chain reaction” is represented by $P$ and $P'$ being positive, causing $P''$ to be positive (parts contribute to the whole), which causes $P'$ to increase (*rates indicate behavior*), which in turn causes $P$ to increase (*rates indicate behavior*), and “the feeding back into itself” is represented by taking the now larger values of $P$ and $P'$ and putting them into the differential equation (*variable substitution, larger magnitudes have larger impacts*) to get a larger value of $P''$. As part of this process, Jordan correlated values of $P$, $P'$ and $P''$ this is also indicative *functional mapping.* Later in the interview Jordan discussed this process in greater detail, showing exactly how correlating the values of $P$, $P'$ and $P''$ allowed him to make the inferences about the behavior of $P$.

In turn 122 I began asking Jordan questions about the seemingly contradictory ways in which he was taking about $P$; as a number and as a function. In his response he discussed the process he went through to construct $P(t)$, the solution function, from values of $P$, $P'$ and $P''$. The process he outlined is commensurate with my analysis of “the chain reaction” he discussed in turn 113 and is representative of his utilization of the additional resource *functional mapping*. It should also be noted that the statement “when I was kind of explaining this earlier I was kinda of just explaining like we are just going to ignore $t$ right now” (transcript follows) suggests Jordan left this portion of his reasoning out of his explanation during turns 97 – 107.

122 G: Ok, so now the other thing I am curious to know a little about, is you said that you think of $P$ as a number and $P'$ as a rate of change. Rate of change of what?
123 J: Of $P$.
124 G: Of, the numbers that...
125 J: Of the numbers yeah
126 G: Ok, but here [points to graph he drew] $P$ is a function, here you graphed it as a thing, like as a function.
127 J: Yea.
128 G: But here you are thinking of it as numbers. Can you talk to me a little bit about that. Like the distinction you are making between knowing that $P$ is a function, but also saying that it is a number.
129 J: Ok, so because up here [points to differential equation], we are really not given any other truly independent variable, like a $t$ or anything like that. So I'm just, it’s kinda the whole like snapshot versus complete picture. When I was kind of explaining this earlier [referring to turns 97-107] I was kinda of just explaining like we are just going to ignore $t$ right now, at $P$, like we just ignore this $t$ axis, at $p$ if we are positive above zero we would be here. The rate of change if it was positive, $P$ would be positive, and this concavity, that would force $p''$ to be positive. And it would be and it would be kind of sloping up. So at whatever it is, if they are positive, if $p$ and $p'$ are positive, you get this shape [points to graph]. If you throw in a relation to $t$ or $x$ or any other independent variable, that just allows these [$P$, $P'$ and $P''$] to actually relate back to each other. Cause it was to be a $P'$ of something, $P'$ of $t$, $P''$ of like $t$, so if whatever independent variable this is relating back to, it as that variable increases, if you have that initial positive, it will be this number at this point. I mean its number, change concavity. This number change and concavity you can put at every point relating to $t$ and you would just get, you would get this graph [marks points with slopes and concavities, then draws a new graph, see Figure 4.29].

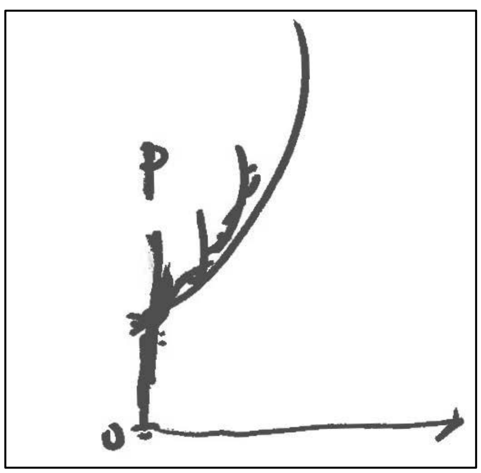


*Figure 4.29: Drawing P based on the differential equation.*

In turn 129 he said that he ignored the $t$-axis, but used “the relation to $t$ or $x$ or any other independent variable” to relate the values of $P$, $P'$ and $P''$ back to one another to draw the graph.

In other words Jordan utilized *functional dependence* to infer that the value of $P$, $P'$ and $P''$ all corresponded to the same $t$ value. Knowing that the values of $P$, $P'$ and $P''$ all related to the same $t$ value, Jordan coordinated *functional mapping* and *positives add – negatives take away* ("The rate of change if it was positive, $P$ would be positive, and this concavity, that would force $P''$ to be positive.") with *rates indicate behavior* to infer the next value of $P$ and $P'$; "the number change [$P'$] and concavity [$P''$] you can put at every point relating to $t$ and you would just get, you would get this graph" (turn 129). This process can be seen in the graph Jordan constructed in Figure 4.29 where various $P$ values are marked with a slope and concavity, which Jordan used to determine the behavior of the graph. This is commensurate with the iterative process of determining values of $P$, $P'$ and $P''$ and then using them to construct the graph of $P$ Jordan discussed in turn 115.

Jordan was also able to utilize this same process to discuss how different initial conditions for the value of $P$ and $P'$ determined the behavior of the solution $P(t)$.

139 G: Gotcha, but this is just , like you said before, I think you said this, like an estimate.
140 J: Yeah, if there is a possible initial value where you get like a positive $P$ and a positive $P'$ you could get a shape something like that. If you get a negative $P$ and a negative $P'$, you could get something like this. If you get a negative $P$ and a positive $P'$, then it kind of depends on the relationship, like if you get 3 and 1 so $P$ could start down here and it would be 3, but if you start with a $P'$ of 1, no we'll go with negative one, ah well go with 1. So it's a $P'$ of 1, but then you have like no concavity, so it gets a little hazy. Like I could think through it a little more I guess but I really depends on what this initial $p$ and $p'$ are where this graph will go.

In summary Jordan utilized the resources *functional dependence, dependence, rates indicate behavior, parts contribute to the whole, larger magnitudes have larger impacts, and variable substitution*. He used *functional dependence* to infer the existence of an independent variable, which he coordinated with *functional mapping* to infer the values of $P$, $P'$, and $P''$ all corresponded to the same value of $t$. Jordan used *dependence* to infer the solution functions were determined by the initial value of $P$ and $P'$, and *parts contribute to the whole* to determine how the values of $P$ and $P'$ related to the value of $P''$. In addition, he used *rates indicate behavior* to determine new values of $P$ based on the values of $P'$ and $P''$. Lastly he coordinated *larger magnitudes have larger impacts, variable substitution, and parts contribute to the whole* to infer that larger values of $P$ and $P'$ would lead to larger values of $P''$. In short, Jordan used these resources to discuss the process of using values of $P$, $P'$ and $P''$ to build specific solution functions. While doing se he utilized the differential equation to relate values of $P$, $P'$ and $P''$ to construct the solution function $P$. Based on the resources he utilized, the patterns in which he reasoned with them, and the ways in which he interpreted $P$, $P'$ and $P''$ in the differential equation, Jordan interpreted the differential equation as description of the behavior of the quantity of interest.

**4.2.2.3 Interview 2 task 3.** This task provided Jordan with three slope fields and six differential equations and asked him to match the slope field to the three corresponding equations. Jordan's reasoning with this task was markedly similar to that of Hakeem's during his first encounter with the task. More specifically, in the process of narrowing down and selecting an equation that matched with each vector field, both Hakeem and Jordan utilized *functional dependence* to infer which equations were possible matches for each slope field, *functional mapping* to relate specific values of $y$ and $t$ to values of $\frac{dy}{dt}$ and *functional variation* to determine which variables the value of $\frac{dy}{dt}$ depended on, and to verify the value of $\frac{dy}{dt}$ (in the equations) in the same way the value of slope changed in the slope field. The only notable difference between Jordan's reasoning and that of Hakeem is that Jordan also utilized *the definition of equilibrium solutions* (when such solutions existed) during the process of determining a match. While completing this task Jordan interpreted the differential equation as a relationship between the value of quantities and the value of the respective quantity's rate of change.

Jordan's reasoning was quite consistent across the three vector fields and in an effort to avoid presenting redundant analysis, the focus of the analysis presented regarding Jordan's interaction with Task 3 is on Jordan's reasoning with Slope field A; though I also briefly summarize the interaction prior to his discussion of Slope field A as well. His reasoning with slope field A is representative of the similarity with Hakeem's reasoning, and is also illustrative of Jordan's usage of *equilibrium solutions.*.

Task three begins with talk turn 157, Jordan's analysis of slope field A starts with turn 176. Summarizing the events occurring between these two talk turns, Jordan initially considered slope field C, and determined that equation 1 was "a candidate for C" (turn 160). He said he determined this because "no matter like what the $y$ value is at a specific $t$, it's always the same" (turn 162). As can be seen in turn 164, the "its" refers to "the rate of change, these vectors, the tangent lines" from the slope field. Justifying the potential match further, in turn 166 he said "plus it also helps that at $t = 1$ it's [points to the slopes on the slope field] zero, about." I then asked him how he knew the slope was zero at $t = 1$ to which he replied, "slightly positive [points to the left of $t = 1$] slightly negative [points to the right of $t = 1$]" (turn 172), and followed this up by saying "so, unless there is some weird jump which I doubt there is based off of any of these equations, it's gonna have to cross through zero" (turn 174). Without having determined a definite match for slope field C, in turn 176 he transitioned to looking at slope field A, unprompted. The following transcript begins as Jordan made this transition, and ends with Jordan discussing what it means to be an equilibrium solution, and how he knew equation 1 matched slope field A.

176 J: So, let's see. We'll go to this one [slope field A]. Well actually this one is completely independent on, of time also. So it has to be one [points to the equations] that doesn't have a $t$.

177 G: Ok.

178 J: Or I guess I mean the other way, wait. Yeah this [points to slope field A] one is independent on $t$ where this one [pints to slope field C] is independent on $y$. So there is no $y$ on this one [labels C with “no $y$”], and there is no $t$ relation in this one [labels A with “no $t$”].

179 G: Ok.

180 J: they could be somewhere in there and end up getting canceled out or something but most likely given what I am seeing there is no $y$ here, there is no $t$ here. This one [points to slope field B] if you go across the $t$ axis there is change, if you go up the $y$-axis there is change so this one depends on both.

181 G: When you say there is change, change in what?

182 J: The slope.

183 G: Change in the slope, so as you, in other words... Could you give me an example of what you mean?

184 J: Ok so let's go, I'll go to $t = 1$. Like around this area, if $y$ is zero we have like this slightly positive slope. But if $y = 1.5$ we have a negative slope.

185 G: On the same $t$?

186 J: On the same $t$. So at a specific time, the $\frac{dy}{dt}$ is dependent on $y$. And it goes the same way like if I am here [point on positive $x$ axis] it's slightly positive but if I go back to zero [points to origin] it’s about zero.

187 G: Gotcha.

188 J: So they change based off each other.

189 G: Ok, with ya.

190 J: so let's see. For this one [slope field A] I will look for ones that don’t have a $t$ and then see if I can find an equilibrium solution that fits this and then go off that.

191 G: Ok.

192 J: So [pointing to equation 2] one minus one, if $y$ is one, this is zero [points to $\frac{dy}{dt}$]. So, let's see what it is at $y = 2$. At $y = 2$ it [points to $-y^2$] would be a negative 4. So that would be 4 so it would be 1-4 so it would be negative 3. So the slope would be negative 3. So it’s, that could be about negative three [checking if the slope is negative 3 at $y = 2$]. It fits. As $y$ gets more positive [looking in the differential equation] the slope gets more negative [looks at the slope field]. Fits. Bottom half. Let's say we go to $y = 0$, slope is one. Ok that fits. let's go down a little more $y = -1$, slope -1 squared, let's see. It would be one so one minus one is zero, that’s starting to not fit because, because this is a $y$ squared [points to equation 2] it would have another equilibrium solution.

193 G: Ok.

194 J: Which means it’s not this [crosses out equation 2]. And we know it's not either of those because this [points to equation 3] depends on both, this [points to slope field A] depends on $y$. So this one is out [crosses out equation 3]. So on to the next one that does not depend on $t$ [points to equation 4]. One minus $y$ [pointing to equation 4], ok same equilibrium solution, as $y$ increases above one the slope gets more negative. As $y$ decreases below one, so like at zero the slope would be one [points to slope field] and then at -1 the slope would

be two, this one increases more, so this one is a possibility for A. As of what we have looked at so far, the best possibility. None of these others solely depend on $y$, so this is A.
195 G: Ok.
196 G: Before you move on to the next one, you said equilibrium solution, two or three times now, I am just curious what does that mean to you?
197 J: At, no matter what time or whatever it is, if $y$ ever hits that equilibrium solution, it will never change. Which it won't hit unless it starts there, but if we, whatever $y$ is, if $y$ starts at one on this equation $\frac{dy}{dt}$ will be zero, which means there will be no change in $y$, which means it will still be $y$ one second later, it will still be one. Which means there is still no change. So it'll just, no matter how much time passes there will never be any change in $y$.
198 G: So an equilibrium solution is what?
199 J: When there is no change in $y$ no matter how much $t$ or whatever the independent variable is, is changed. It is this line [draws $y = 1$ on slope field A].
200 G: Ok.
201 J: So.
202 G: With you.
203 J: That one is a, equilibrium solution and it has the right relation before like above and below and based on time.
204 G: Ok.
205 J: So it has all the right relationships…

In turn 176 Jordan noted that the slopes were "independent of time." To make this inference he utilized *functional variation* to determine which variables determined the value of the slope. There is a clear example of this in turn 180, when he said "This one [points to slope field B] if you go across the $t$ axis there is change, if you go up the $y$-axis there is change so this one depends on both." I asked him to elaborate on this further, and in turn 186 he said "at a specific time, the $\frac{dy}{dt}$ is dependent on $y$. And it goes the same way like if I am here [point on positive $x$ axis], it's slightly positive but if I go back to zero [points to origin], it's about zero." In this way Jordan varied the $t$ and $y$ values of the points he was considering on the slope field to see how the slope changed as the values of $t$ and $y$ changed (separately). In doing so he was able to determine the variables that $\frac{dy}{dt}$ depended on. In turn 190 Jordan utilized *functional dependence*, to "look for ones that don't have a $t$", and his *definition of equilibrium solutions* that "fit" the slope field to narrow his choices. In turn 192 he identified equation 2 as a potential match and began using *functional mapping* ("if $y$ is one, [$\frac{dy}{dt}$] is zero. So, lets see what it [$\frac{dy}{dt}$] is at $y\ = 2$, at $y = 2$ it [$\frac{dy}{dt}$] would be a negative 4"), and *functional variation* ("as $y$ gets more positive [looking in the differential equation] the slope gets more negative [looks at the slope field], fits.") to check if equation 4 matched the slope field. Specifically he used *functional mapping* to see if the correlation between the value of $y$ and $\frac{dy}{dt}$ present in the algebraic form of the differential equation were also present in the slope field representation. Additionally he used functional variation to see if the values of the slope changed in the same way the value of $\frac{dy}{dt}$ changed in the equation, given the same change in the value of $y$. At the end of turn 192 Jordan used his *definition of equilibrium solution*, to infer that equation 2 did not fit with slope field A; "that's starting to not fit because, because this is a $y$ squared [points to equation 2] it would have another equilibrium solution." He then moved to equation 4. Namely, Jordan noted that Slope field A only had one $y$ value where the slope was zero but the differential equation had two.

In turn 194 Jordan used his *definition of equilibrium solution* to infer that that equation four had the same equilibrium solution as slope field A, and *functional variation* to infer that the slope in slope field A had the same behavior of $\frac{dy}{dt}$ as $y$ decreased below the value of 1. Further he noted that none of the other equations only depended on $y$. From this he concluded that equation 4 represented slope field A in turn 195. I then asked Jordan in turn 196 what he meant by equilibrium solution, as part of his reply in turn 197 he said, "if $y$ ever hits that equilibrium solution, it will never change." In turn 198 looking for a clearer answer, I asked "so an equilibrium solution is what" to which he replied, "when there is no change in $y$ no matter how much $t$ or whatever the independent variable is, is changed. It is this line [draws $y = 1$ on slope field A]" (turn 199). Based on these two statements, Jordan looked for places where the value of $y$ did not change “no matter how much the independent variable is changed” to find equilibrium solutions in the slope field.

In summary, to complete Task 3 Jordan used the resources *functional variation, functional dependence, functional mapping,* and his *definition of equilibrium solutions* to check if the way the equations mapped values of $y$ (and $t$) to values of $\frac{dy}{dt}$ matched the way the slope field mapped $t, y$ points to slopes. To do so Jordan related $\frac{dy}{dt}$ in the differential equation with the “rates of change, these vectors, the tangent lines” in the slope field, and the $t, y$ points at which the vectors were located with values for $t$ and $y$ in the differential equation. This reasoning indicates that Jordan interpreted the differential equation as a relationship between the value of quantities and the value of their respective rates of change. This was something Jordan himself suggested later in the interview when asked how he was matching the slope field to the equations; “I am looking at the specific trends in $\frac{dy}{dt}$ relating to each variable. So, in this equation the trend would be as $t$ increases, $\frac{dy}{dt}$ increases, does this graph do that no” (turn 211).

**4.2.2.4 Interview 2 task 4.** Task 4 (see Figure 4.30) provided Jordan with a first order differential equation said to model a fish population, and an initial condition for the population. Jordan was tasked with determining the number of fish in the pond at a given time $t$. Jordan’s approach to this task was rather straight-forward; he found a general solution to the differential equation and used with the given initial condition to find the particular solution. This process took Jordan a little more than 2 minutes and comprised of reasoning which utilized the resources *the method of integrating factors, opposite operations reverse, the definition of general solution, function slots,* and *dependence.* His work can be seen in Figure 4.31, following the transcript.

TASK 4: Suppose the equation $y' + 4ty = 16t$ can be used to model the fish population $y$ in the campus Duck Pond. Determine the number of fish in the pond at a given time $t$, if the initial population is 5.

*Figure 4.30: Task 4 from Interview 2.*

234 J: Ok, suppose the equation ...[in audible], yay, more population.

235 J: Ok so last time I did this problem you asked me why I underlined what I underlined. So, I guess if I was solving this on a test I would re-write it [writes the differential equation on paper]. Ok, and then at a given time, so we are trying to find, the number of fish at time $t$. So we are trying to find $y(t)$ [writes $y(t) =?$].

236 G: Ok.

237 J: That's our goal. And we know $y_0$, why I underlined it, the initial population. [Writing] $y_0$ equals five, thousand.

238 G: Ok.

239 J: So here's the problem and then now I should solve it.

240 G: Ok.

241 J: And I'm trying to think of if there is an easier way that I can just logic my way through it, but don't see any so what I would do, I would just go through so integrating factor so that would be what? $e$ to the integral $4t$, so it would be like $2t^2$, so that's the integrating factor, and then $2t^2y$ equals integral of $16t\ e^{2t^2}$. I don't know its been a few days since I have actually solved a differential equation and I am skipping steps cause I don't like writing in sharpie.

242 G: Sorry.

243 J: So if you want me to keep going I can keep going.

244 G: If you could.

245 J: Ok, all right let's see if I can still integrate. So the integral of $16te^{2t^2}$ ah let's see, [inaudible] $4e^{2t^2}$, let's see. $y$ equals, [writing]. So that would be, If I actually did my math correctly that would be the general solution for the population of fish $y$ at any time $t$. And then the initial value is 5, so 5 equals 4 plus $C$. So $C$ equals 1, so $y$ equals $4e^{-2t^2}$ and that doesn't give me a $t$ it wants me to find the population at, but this is the population of fish at any point.

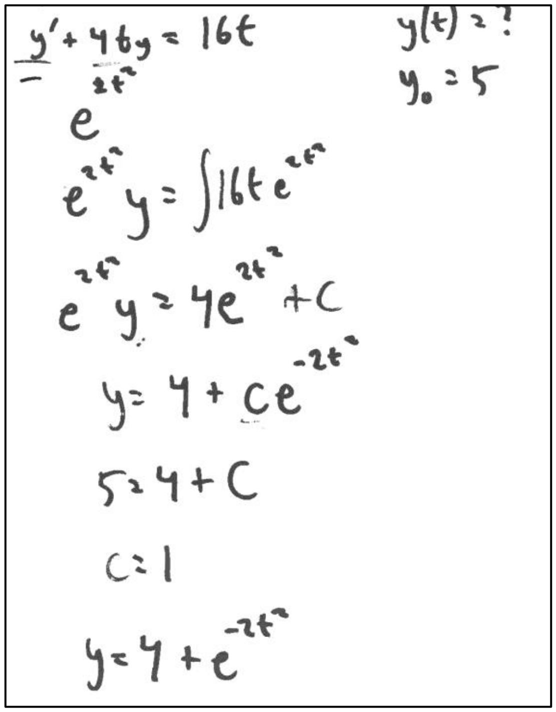


*Figure 4.31: Jordan's work utilizing the method of integrating factors.*

In turn 235 Jordan noted that the task was asking him to find $y(t)$, and in turn 241 he decided to utilize *the method of integrating factors;* he found the integrating factor $e^{2t^2}$, multiplied each side of the differential equation by the integrating factor and discussed integrating both sides of the equation (Jordan actually skipped the mechanics of this step for the left hand side, claiming he did not like to write with a sharpie marker). In turn 245 he found the general solution $y(t) = 4 + Ce^{-2t}$, and lastly used the initial condition to find the value of $C$, which yielded the function $y = 4 + e^{-2t^2}$. Jordan noted that this function represented "the population of fish at any point" (turn 245). In the process of completing the task Jordan also utilized the resource *opposite operations reverse* to transform $y'$ into $y$ by integrating, *the definition of general solution* to infer the solution of interest was of a certain form, *dependence* to infer the particular function of interest depended on the value of $C$, and *function slots* to appropriately place the initial conditions into the general solution to find the value of $C$. The utilization of these resources is supported by Jordan's responses to planned follow up questions.

After Jordan utilized the *method of integrating factors* to find the $P(t)$, I began asking him planned follow up questions. His responses to these questions both provide support of my claim that Jordan utilized the aforementioned resources, and illuminate their specific role in supporting his reasoning about the task. The first question regarded the purpose of integration.

246 G: Ok, um so why is integration appropriate when we solve for this differential equation?
247 J: Ok, so if I can try to repeat back the explanation for these steps when we did it in class, it was basically when you multiply both sides by this integrating factor it happens to be, like um, like, let's see if I can actually do this. $e^{2t^2}$ times $y'$ plus $4ty$ equals, actually let's just do. Basically there is a way to solve this out if I was thinking clearly that basically turns it into like $e^{2t}$ times the integral of $y'$ so, you can like separate it out I guess. I m trying to like logic my way back like the way we did it in class.
248 G: So I guess, I guess what I am wondering is how do you make sense of this integration, of why we are doing it. Why do we integrate is what I am asking.
249 J: Basically the whole point of what we are doing up to this point is to find a way to integrate to get rid of this $y'$, to put things all in terms of this $y$. So finding this integrating factor, multiplying by the integrating factor, whether this is done right or not is all to the point of just putting this in a form that we can integrate both sides to turn this $y'$ into just a $y$.
250 G: So in other words we are going to get, it sounds like on the right side, and you said you had skipped some steps, and are you thinking or are those steps indicative of integrating a $y'$. In other words…
251 J: Yeah.
252 G: So what you are saying is that?
253 J: I skipped this whole process.
254 G: Where we would actually integrate $y'$?
255 J: Yeah.
256 G: Ok, gotcha. And what do we get by integrating?
257 J: Just the equation entirely in terms of $y$ and $t$.
258 G: Um,
259 J: With an initial $C$. Whatever this plus $C$ is.
260 G: This guy or this guy? What is?
261 J: We are getting rid of $y'$. That's really all this is for is to get rid of that $y'$.

In turn 247 Jordan discussed his utilization of *the method of integrating factors*; "Basically there is a way to solve this out if I was thinking clearly that basically turns it into like $e^{2t}$ times the integral of $y'$ so, you can like separate it out I guess." Based on the work in Figure 4.31. I take this statement to mean that the integrating factor allows for "separating out the $y$," or in other words, getting just a $y$ on the left hand side (he talks about this further in turn 249). I then rephrased the question by asking him why we integrate to solve the differential equation, and in turn 249 he replied:

> "The whole point of what we are doing up to this point is to find a way to integrate to get rid of this $y'$, to put things all in terms of this $y$... to turn this $y'$ into just a $y$."

Further, in turn 257 he said integrating yields "just the equation entirely in terms of $y$ and $t$." For Jordan the purpose of the integration is not to remove the $y'$, but to turn it into "just a $y$", or in other words, to cancel the derivative of $y$ and just make it a $y$. This supports my claim that while reasoning with the task in turn 241, Jordan utilized *opposite cancel* to infer the integration would turn $y'$ into just a $y$. In turn 259, Jordan said that the equation resulting from the integration would also have a $C$. I asked him about this in turn 262.

262 G: Ok. Um, and so here we have this c thing that we had ended up solving for. What is $C$?
263 J: Um, so if we are looking at this whole slope field thing, c is determined by whatever initial condition you are given, which basically each path that could be drawn on this represents a specific $C$ which is a specific solution. The $C$ in the general solution it could represent any possible solution depending on what the initial value is which determines this $C$.
264 G: So, ok. And so that was another question. I think you called this [$y = 4 + Ce^{-2t^2}$] a general solution.
265 J: Yeah.
266 G: How do you think about or what do you mean by a general solution?
267 J: Um just the combination of solutions that could be put in this form [points to general solution from turn 245]. So $C$ can be anything based on whatever initial condition, or if you are given two separate conditions like the difference between them, but like whatever, like we did drag in class like a week or two ago. Like that interests me, and so like yes there is an equation that basically can say ok these are all the different possibilities of what can happen relating to drag. So that is the general equation, it's just hey this is how drag works. The general solution is, I mean the specific solution is ok, calculate the velocity of a ball that was dropped from 10 meters, at time equals half a second using the equation for drag.

In turn 263 Jordan utilized the resource *dependence* when explaining that $C$ is determined by the initial conditions; "$C$ is determined by whatever initial condition you are given, which basically each path that could be drawn on this represents a specific $C$ which is a specific solution." Further he noted that each $C$ value represents a different solution, and that the $C$ in the general solution "could represent any possible solution depending on what the initial value is." In turn 266 I asked Jordan to describe what he means by general solution. In turn 267 he said that a general solution is "just the combination of solutions that could be put in this form [points to general solution from turn 245]." Based on this, to Jordan, the general solution provided the form of the solution equation, and the $C$ within the general solution provided a solution specific to the initial condition. I take this as evidence that in turn 245 his *definition of general solution* was a resource, in that it supported him in finding the $C$ value which corresponded to the solution specific to the given initial condition. In other words, knowing that the solution needed to be of the form $y(t) = 4 + Ce^{-2t}$, he was able to coordinate *function slots* with the known form of the solution to place the initial conditions into the general solution to find the $C$ value which corresponded to the specific solution.

In summary Jordan utilized the *method of integrating factor, opposites operations cancel, general solution, dependence,* and *function slots* to find the general solution of the differential equation and then place the initial conditions into the general solution to find the particular solution that related to those initial conditions. The method of integrating factor consisted of "finding this integrating factor, multiplying by the integrating factor… all to the point of just putting this in a form that we can integrate both sides to turn this $y'$ into just a $y$" (turn 249). The resource *opposite operations reverse* was utilized in the form of integration, where integrating was the method of "turn[ing] this $y'$ into just a $y$." After integrating and getting "an equation for $y$ just in terms of $t$", Jordan then coordinated the *definition of general solution, dependence* and *function slots* to appropriately place the initial conditions into the known form of the solution to determine the value of $C$, and hence the solution specific to the initial conditions. While doing so, Jordan utilized $y'$ as the derivative of the function $y$ which represented the population of fish at a given time $t$. Lastly, this is indicative of Jordan interpreting the differential equation as providing a model of the quantity of interest.

**4.2.2.5 Discussion of interview 2.** While completing the tasks in Interview 2, collectively, Jordan expressed that the variables in the differential equations represented numbers, functions, points, rates of change and slopes. On an individual basis, Jordan only used certain combinations of these different representations of the components in the differential equation. Similarly, though the cumulative set of resources Jordan used across the tasks in Interview 2 is relatively large, the sets of resources he used to complete each task are comparatively smaller. Looking across the tasks the sets of resources he used to complete each task (see Table 4.8) and the patterns in which he used them, distinctly correspond to the ways in which he interpreted the entities from which the differential equation is composed.

For instance while Tasks 1 and 3 are very different from each other, Jordan expressed that the variables in the differential equations were values; $x$, $y$, and $t$ in the respective equations were values of quantities, and $\frac{dx}{dt}$ and $\frac{dy}{dt}$ represented the values of the rates of change of the respective quantities. In both tasks he utilized resources that supported him in characterizing the correlations between the values of the quantities and the values of their respective rates of change (*functional variation and functional mapping*). From these characterizations Jordan was able to then make inferences to complete the corresponding tasks. Namely, by correlating changes in the values of $x$ and $y$ to changes in the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ Jordan was able to determine which system represented competitive species. This same process was utilized By Jordan to determine which equations matched the slope fields; by comparing how the equations correlated values of $t$ and $y$ to values of $\frac{dy}{dt}$ to how the slope fields correlated $t, y$ points to the value of slopes. While the nature of the conclusions he drew were different (as required by the difference in the tasks), the resources he used to draw conclusions and the ways in which he interpreted the mathematical entities were the same.

While completing task 2 Jordan also expressed that the variables in the differential equation represented values, and worked with them as such in reference to the differential equation itself. Much like his reasoning in Task 1 and 3 Jordan used functional mapping variable substitution and functional dependence to characterize the nature of the correlations between the value of the quantity and the values of its rates of change ($P'$ and $P''$). There is however one important distinction in Jordan's reasoning with regard to Task 3, After determining how the values of the variables were correlated he then went on to use the values of the rates of change to determine general characteristics of the behavior of $P$. While doing so he mentioned that $P$ was dependent on the independent variable $t$. In this way while he utilized the variables $P$, $P'$ and $P''$ as values in the differential equation, he used these values to construct a graph that represented the behavior of the function $P(t)$. He did this with the utilization of rates indicate behavior; by knowing the values of $P'$ and $P''$ he could make inferences about the general behavior of $P$.

In this case interpreting $P$, $P'$ and $P''$ as values and $P$ as a function, his utilization of the resources functional dependence, functional mapping, variable substitution allowed him to correlate these values, and his utilization of the resource rates indicate behavior allowed him to determine the behavior of the function $P(t)$.

Lastly, while working with the initial value problem (Task 4) Jordan was not concerned with correlating values of $y$ with values of $y'$ or using the value of $y'$ to determine the behavior of $y$. In this case Jordan interpreted $y'$ as the derivative of $y$, and utilized resources (*the method of integrating factors* and *opposite operations reverse*) that allowed him get "just the equation entirely in terms of $y$ and $t$." In addition he used the *definition of general solution, dependence,* and *function slots* to find the specific equation that represented "the population of fish at any point." This set of resources is distinct from the other sets he used to complete the tasks, and in addition the result of his process was different from that of Tasks 1 and 3; Jordan found an equation that represented a specific correlation between the value of the fish population and the value of the independent variable ($t$).

*Table 4.8: Resources and interpretations Jordan expressed in Interview 2.*

| Task | Interpretation | Resource | Entities |
| --- | --- | --- | --- |
| I2-T1 | As a relationship between the value of quantities and the value of their respective rates of change. | *Definition of competing systems*<br>*Functional variation*<br>*Partial dependence*<br>*Parts contribute to the whole*<br>*Positives add – negatives take away* | $x$ and $y$ - Value of the respective species' populations<br>$\frac{dx}{dt}$ and $\frac{dy}{dt}$ - Values of the respective rates of change |
| I2-T2 | As a description of the behavior of the quantity of interest. | *Functional dependence*<br>*Parts contribute to the whole*<br>*Variable substitution*<br>*Functional mapping*<br>*Dependence*<br>*Rates indicate behavior*<br>*Larger magnitudes have larger impacts* | $P$ - Number<br>$P'$ – Rate of change of $P$<br>$P''$ – Change in the rate of change of $P$ |
| I2-T3 | As a relationship between the value of quantities and the value of their respective rates of change. | *Functional variation*<br>*Functional dependence*<br>*Functional mapping*<br>*Definition of equilibrium solutions* | $\frac{dy}{dt}$ –Slope<br>$t, y$- Point and value of $t$ and value of $y$ |
| I2-T4 | As providing a model of the quantity of interest. | *Method of integrating factor*<br>*Opposite operations reverse*<br>*Definition of general solution*<br>*Dependence*<br>*Function slots* | $y'$ - Derivative of $y$<br>$y(t)$ - Value of the population at a given $t$. |

### 4.2.3 Interview 3

Jordan completed a total of four tasks during Interview 3. In this section I only present the analysis of the first three tasks as Jordan completed the fourth task in the same manner as he did Task 4 from Interview 2. Recall that Task 4 from Interview 2 presented an initial value problem with the differential equation $y' + 4ty = 16t$, which was said to model the number of fish in the campus duck pond, and asked them to determine the fish population for some time $t$. The fourth task in Interview 3 was mathematically similar to that of Task 4 from Interview 2, but was situated around using the differential equation $B'(t) = 30 - 1.5B(t)$ to find the temperature of a cooling metal object for any value of $t$. While completing both tasks Jordan utilized the same reasoning pattern and set of resources, and in addition he interpreted $B'$ as a derivative, and $B(t)$ as the equation that represented the temperature of the metal object at a given time $t$.

**4.2.3.1 Interview 3 task 1.** Jordan was initially unsure of how to complete this task because, as he expressed, he did not know how to infer the size of the species, from their population values or their rates of change. After some prompting Jordan focused on discerning which species were predator and which were prey and determined that $y$ represented predators in both systems. Referring to the equation for $\frac{dy}{dt}$ in each system, he coordinated *positives add – negative subtract* with *functional variation* to reason that the positive sign in front of the $x$ terms meant $\frac{dy}{dt}$ would increase as $x$ increased, and that thusly $y$ must be prey. To determine which were large and which were small, Jordan coordinated the resources *partial dependence* and *rates indicate behavior* to determine the behavior of the prey populations, and then utilized *carrying capacity* to associate the behavior of $x$ in System A with large animals. While completing the task Jordan expressed that $x$ and $y$ in the differential equation represented the value of a quantity (population), and $\frac{dx}{dt}$ and $\frac{dy}{dt}$ represented the value of the quantities respective rates of change.

Lastly, Jordan interpreted the differential equations as descriptions of the behavior of the quantities of interest; he used the differential equations to determine the behavior of the species' populations from which he inferred the size of the species. I begin the presentation of the analysis that lead the identification of the resources and interpretation with transcript that starts just after Jordan finished reading the task.

11 J: That looks little bit more abstract than the last one.
12 G: Why do you say that?
13 J: Well in the last one it was just which ones were cooperating and which ones where in competition. This one what it is asking to separate them by seems like a lesser direct correlation. And like I guess, it doesn't really seem like just thinking about it that there would that much of a direct correlation. So it is hard to mathematically visualize something that you don't even know even think of as being a thing.
14 G: Ah, when you say a correlation, a correlation between what?
15 J: Um the number of predators and prey vs. the sizes that they are.
16 G: Ok.
17 J: I guess I would say that, I don't know how they are, I have zero clue just looking at the way that this question is phrased, what the rates of change should actually be. I don't know if, for any of these if there is more predators there is probably going to end up being a decrease in prey and it's probably going to be cyclical for all of these.
18 G: Ok.
19 J: Cause that's just typically how predator prey relations are.
20 G: Sure, so you don't know how to interpret $\frac{dx}{dt}$ and $\frac{dy}{dt}$, is that what you are saying?
21 J: I mean I guess, like I know what they are, I just have zero clue how to relate them to the size of the creatures.
22 G: Ok.
23 J: Like let's say, for the first example, large prey small predators. Ok, I would think of that as, ok if there is small predators you probably need more predators per prey, but that would be more of an initial condition issue than a rate of change issue.
24 G: So you are trying to figure out how you can relate the number in the population to the size of the population?
25 J: Yeah.

In this segment of transcript Jordan expressed not knowing how to utilize the relationship between the value of the population and the rates of change to infer the size of the populations. In turn 13 referring to the population values and their sizes he said it does not seem like "that much of a direct correlation", and in turn 21 he said he did not know how to relate their rates of change to their size. In turns 17 and 23 he discussed the characteristics he expected to see in the population values of a system with small predators and large prey, but attributed these characteristics to certain initial conditions (e.g., "if there is small predators you probably need more predators per prey, but that would be more of an initial condition issue than a rate of change issue"; turn 23.). At this point in the interview Jordan seemed stuck, so in turn 26 I prompted him to focus on determining which were predators and which were prey.

26 G: Well it seems like there are four things really that are being asked. Which one is predator, which one is prey, which one is large, which one is small.
27 J: Exactly, yeah.
28 G: And so if you, if you are totally lost on which one is large and which one is small, maybe you can look and tell me about the other ones.
29 J: Ok, so predator and prey one of them will have a positive correlation to where if there is more prey there will be more predators like a positive, like let's say $\frac{dy}{dt}$, if $y$ is predators, then if there is more prey there is more $x$ and $y$ will be increasing. So on this one [System A], $y$ looks like it would be a predator.
30 G: Can you say that last part again, I'm sorry. There were people making noise and I was just...
31 J: Ok.
32 G: You said something about if there is more $x$.
33 J: Yeah if there is more prey then there will be more food, which will lead to more predators.
34 G: Ok.
35 J: So if there is more $x$, then if we ignore this [the $y$ term], cause it's not dependent on $x$ if there is a positive $x$ if there is more $x$, then there will be a more positive change of $y$.

To identify which species were predators Jordan sought the equation with a "positive correlation"; "one of them will have a positive correlation to where if there is more prey there will be more predators" (turn 29). In other words, Jordan was looking for the equation that displayed an increase in the rate of change of one species, as the population of the other species increased. With this characteristic in mind, Jordan was able to quickly determine that in System A, $y$ represented the predator and $x$ represented the prey. Based on his statement in turn 35 he did this by coordinating the resources *positives add- negatives subtract ("*if there is a positive $x$*…)* and *functional variation* ("…if there is more $x$ then there will be a more positive change of $y$"). More specifically Jordan inferred that if there was a positive sign in front of the $x$ term in the equation for $\frac{dy}{dt}$, increases in the value of $x$ would relate to increases in the value of $\frac{dy}{dt}$, which meant $x$ was prey and $y$ was predator. In turn 39 his focus turned to System B, and he determined which was predator and which was prey much in the same way; "And so, looking at this one [System B] it looks like it is the same, which one is a predator, which one is a prey. You have a positive $xy$ and a negative $xy$." In this case, Jordan used the sign in front of the $xy$ term to indicate how increases in one species, related to change in the other species. A positive sign in front of the $xy$ term was a positive correlation (increase in one species would add to the rate of change of the other species), and a negative sign in front of the $xy$ term was a negative correlation (increase in one species would subtract from the rate of change of the other species).

While conducting the interview it was not clear how Jordan was making inferences about the nature of the "correlation" between the variables (turn 29). That is, it was not clear if he was characterizing the correlation based on the sign in front of the $xy$ term, the effect of larger and lager population values on the rate of change, or the rate of change being positive for certain population values. So I asked him a series of clarifying questions after turn 39. By and large these questions were not fruitful, mainly because I was asking him questions about reasoning that he was not employing. In turns 54 and 58 (transcript follows), however, Jordan was much more explicit about how he was reasoning and he verified that he was using the sign to indicate how the increase in the population of one species would affect the rate of change of the other species. I present this section of the transcript for two reasons, first it accounts for the otherwise large gap between talk turns, and his statements in turns 54 and 58 verify that he utilized *positives add-negatives subtract* to infer which species was predator and which was prey.

51 G: Ok, so if you could just mark that for me [which was predator and which was prey in each system]. And you said that you used that, one way you did it was knowing that if you had a large $x$, then $y$ would increase.

52 J: Mhm.
53 G: And then the same thing here. Another way was with the plus or minus signs on this last term.
54 J: That was the main way, the plus or minus signs, the positive or negative correlation.
55 G: Ok.
56 J: From the other variable.
57 G: When you say correlation you mean like between $\frac{dx}{dt}$ and $y$?
58 J: Yeah. It's, it's whatever if it is positive or negative relating to what the other variable is. So $\frac{dx}{dt}$ we're looking at, is it a plus $y$ or a minus $y$, it's a minus $y$ so it's the prey.

After clarifying how he determined which species were predators and which species were prey, I then prompted him to try to complete the remaining portion of the task, that is figuring out which were large and which were small.

61 G: Um, do you think you could try to figure out how to determine which one is large and which one is small?
62 J: Ok.
63 J: Um, so what, now what I am looking at is this first term in both this $\frac{dx}{dt}$ and this $\frac{dy}{dt}$. Because um, this is either, those are either going to be both large animals or small animals.
64 G: How do you know?
65 J: Because ah, I mean they are the opposite and the two equations are the opposite. So if they are both, so if $y$ is the predator in both, and one of them you are looking at say large predator small prey, and large predator small prey, either way they better match up.

It is worth noting that in turn 63 he said he was focusing on the first term in the pairs of differential equations on opposite corners ($\frac{dy}{dt}$ from System A and $\frac{dx}{dt}$ from System B, and $\frac{dy}{dt}$ from System B and $\frac{dx}{dt}$ from System A) because they had to be the same size; his reasoning here was correct. If $y$ in both systems represented the predators, then given the task, one had to be large and the other had to be small, thusly $y$ from System A had to be the same size as $x$ from System B, and vice versa.

73 J: So these crosses [the upper left and lower right, upper right and lower left equations], so now I'm looking at what makes them similar mathematically and maybe, how maybe how they interact as a single species without their predator slash prey counterpart. So if there is no predators to worry about for this system A $\frac{dx}{dt}$, I'm looking at this first term with $x$'s and

what I am doing over there is just trying to think through real quick some sort of behavioral…

74 G: Sure.

75 J: So like, for this one, it looks like given no $y$'s, $x$ will continually grow and increase and for either this term down here or this term up here, you have sort of that initial increase then decrease with this function.

76 G: Because of the formulas? Like this thing [point to the differential equation] tells you that it increases then decreases?

77 J: Yeah just, yeah that's just this function. So, and then yeah eventually it becomes negative and it starts, basically this, ignoring all of those outside terms that involve a relation of each other if you purely have $\frac{dx}{dt}$ in relation to this $x$, setting $y$ to zero, then it is going to have the $.3x$ which means it's gonna, I mean you could integrate it and get the actual function of $x$ which is quadratic. So this, this population is just going to take off with given no $y$, while this will eventually hit some limiting factor and just kind of possibly like be cyclical, grow, collapse, but its gonna hit some limiting point. So, at, I guess that's more popular in larger species, given that larger species, like they need more to survive, they need more space, they need more food. So, having some sort of limiting factor is more common in larger species, so I would say $x$ on this one is large prey, and $y$ predator is small. So like this would be the piranha-human relationship, but this would be ah, small prey.

To determine which species was large and which was small Jordan looked at "how they interact as a single species without their predator slash prey counterpart" (turn 73). More specifically used the differential equation to infer what the behavior of each prey species would be without the existence of the predator species. For instance in turn 77 when considering the differential equation $\frac{dx}{dt} = .3 - \frac{xy}{100}$, Jordan inferred that the population of species $x$ "is just going to take off with given no $y$." In doing so Jordan used the resources *partial dependence*, ("if you purely have $\frac{dx}{dt}$ in relation to this $x$,") and *rates indicate behavior* (*"*so this, this population is just going to take off*"*) to infer that species $x$ in System B would continually grow because the rate of change was always positive. Additionally, by coordinating these resources with *functional variation* Jordan determined that the prey (species $x$) in System A would have a limiting factor. In other words, Jordan inferred that as the value of $x$ increased, the value of the rate of change of $x$ decreased to zero, indicating that $x$ was approaching some maximum value. Jordan then used the resource *carrying capacity* to associate the limiting factor with larger species and thusly infer that System A represented small predators and large prey, and System B represented small prey and large predators. This is seen in the statement below.

> "So this, this population [$x$ in System B] is just going to take off with given no $y$. While this [$x$ in System A] will eventually hit some limiting factor and just kind of possibly like be cyclical, grow, collapse, but its gonna hit some limiting point. So, at, I guess that's more popular in larger species, given that larger species, like they need more to survive, they need more space, they need more food. so, having some sort of limiting factor is more common in larger species, so I would say $x$ on this one [System A] is large prey, and $y$ predator is small" (turn 77).

In summary Jordan completed this task by using the differential equation to infer which species were predators and which were prey, and then constructing a description of the behavior of the species from which he inferred the size of the species. To accomplish this Jordan utilized the resources *positives add-negatives subtract, functional variation, partial dependence*, *rates indicate behavior* and *carrying capacity*. Additionally he attended to $x$ and $y$ in the differential equation as the value of the populations and, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as the values of their respective rates of change. Jordan first made the determination that species $y$ represented the predators in both systems by identifying a positive correlation between $\frac{dy}{dt}$ and $x$; "if there is a positive $x$…" (*positives add - negatives subtract*) *"…* if there is more $x$, then there will be a more positive change of $y$"(*functional variation*). Lastly, to classify each species as large or small he used the differential equation to determine the behavior of each prey species without the existence of the predators. To do so he utilized *partial dependence*, to focus solely on how $x$ impacted $\frac{dx}{dt}$, *rates indicate behavior* to characterize the behavior of the populations of the prey ("so this, this population [$x$ in System B] is just going to take off with given no $y$, while this [$x$ in System A] will eventually hit some limiting factor and just kind of possibly like be cyclical, grow, collapse"; turn 77) and *carrying capacity* to associate large species with limiting factors. Based on this, Jordan used the relationship between the value of the quantities and the value of their respective rates of change to describe the behavior of the quantity of interest.

**4.2.3.2 Interview 3 task 2.** This was the second time Jordan interacted with a task asking Jordan to interpret a second order differential equation. This was a variant of T2-I2 (the coefficients in the equation were different), which was posed to Jordan in Interview 2, during which time interpreted the differential equation as a description of the behavior of the quantity of interest. His interaction with the task in Interview 3 was quite different from that of Interview 2. Here he interpreted the differential equation as an equation to be solved. During Interview 3 Jordan was quite frustrated by the differential equation because he did not know the situation in which the differential equation was embedded (this was seemingly a non-issue for Jordan during Interview 2). As will be outlined in the following paragraphs, Jordan suggested this had an impact on his ability to reason with the relationships present in the differential equation, and shifted the focus of Jordan's reasoning to the application of an analytical solution method. While completing this task Jordan utilized *the method of characteristics, the definition of general solution,* and *dependence,* to find the algebraic form of the solution to differential equation. In this case I argue Jordan interpreted the differential equation as an equation to be solved.

In this section, preserving temporal order, I present three segments of transcript from Jordan's interaction with Task 3. The first segment concerns his initial reaction to the differential equation, the second surrounds the ways in which he was interpreting $P$, $P'$ and $P''$, and the third segment is centered on his work while solving the differential equation. The following transcript begins with Jordan's first utterance after reading the task.

119 J: Hmm. Absolutely nothing.
120 G: Ok, tell me why, or tell me why not.

121 J: I can't think of anything directly to relate it to. I can think of ok, I can put this in standard form and get the characteristic equation and sit there and stumble my way through solving it.
122 G: Ok.
123 J: I need to study. But like its exactly what I was saying earlier, I can sit down and do the steps they have told me to do, I don't know what it means. I don't know where it comes from, I don't know why the characteristic equation works.
124 G: Ok.
125 J: I can go into ok $P''$ is the second derivative of, so its some sort of change in rate of change, so like I like physics, I see a double prime I immediately think of like an acceleration.
126 G: Ok.
127 J: So, let's look at it this way, $P'' + 6P' + 5P = 0$. Ok, what have we done that possibly looks similar to this? Ok, we did spring mass systems.
128 G:Ok so um I don't want to interrupt you, but what I want to make sure is that when, so when the question asks what it means to you, are you saying you are not sure of what this models?
129 J: Yeah, when I think differential equations I think model.
130 G: Ok.
131 J: Ok.
132 G: It seems like the things you are saying to me are that you're just not sure what this system, ah what this is representing.
133 J: Yeah, it means it's an equation for something, that's really all it means to me unless I have something that it could model or that it could relate to, its numbers that if I see it on a test I'll sit down and I'll solve it I, that's pretty much all it means to me and all I can relate it to. I can try and like think about it to understand it better. Ok, I've seen this in class where it's solving a spring system maybe where there is going to be some sort of oscillation where you have like the damping of a six and a $k$ of 5 on the spring like 1 mass.

Jordan starts out by saying the differential equation means "absolutely nothing" (turn 119) to him because he could not think of anything to relate it to, but that he could "stumble [his] way though solving it" (turn 121). Elaborating on this more, in turn 123 he said that he can perform the steps required to solve the differential equation, but he did not know what they mean or why they work. His statement in turn 133 helps illuminate why the differential equation meant "absolutely nothing" to him:

> It's an equation for something, that's really all it means to me unless I have something that it could model or that it could relate to, it's numbers that if I see it on a test I'll sit down and I'll solve it. I, that's pretty much all it means to me and all I can relate it to.

From this statement we can see Jordan did not know what situation to relate the differential equation to and as a result thought of the differential equation as something to be solved; "I see it on a test I'll sit down and I'll solve it. I, that's pretty much all it means to me." The fact that Jordan did not know what situation to relate the differential equation to is not surprising; the presentation of the task is free from context. His statement, however, clearly indicates that the lack of context impacted the resources he could utilize and thusly his approach to the task as well. This is supported by his comment in turn 127 where he suggested that he was searching for situations in which the differential equation would be useful. In turn 136 I returned the focus of the conversation to how Jordan was thinking about $P''$, and as a result we began discussing how he thought about $P$ and $P'$ as well. The transcript that follows provides insight into why Jordan was searching for a situation to relate the differential equation to, and how this affected his ability to attend to the relationship between the value of the quantity and the value of its rates of change present in the differential equation.

136 G: Ok, so I interrupted you when you were saying um, that $p''$ was the something and then I…
137 J: It's some sort of rate of change of a rate of change, or an acceleration. I haven't used in any of the classes or any of the math or science classes I have done; I've never used a second derivative for anything other than acceleration.
138 G: Ok.
139 J: So I really don't know like how to explain what a second derivative is other than acceleration. I know it is a rate of change of a rate of change.
140 G: Ok.
141 J: I could say that it is a rate of change of a rate of change of population. That doesn't, like I haven't been told any words for that. It's math. And the only thing I know how to relate it to is acceleration, and then the only actual example of that we have done of any second order differential equations is springs. So I can say ok, maybe it's a spring.
142 G: Ok.
143 J: I haven't been taught anything past that, so I can't explain anything past that.
144 G: Let me just ask you some ah general stuff. How do you think about $P$ in this differential equation?
145 J: Because I am thinking about this as a motion of something maybe bobbing on a wave, or a spring, actually I don't even know if it is a cyclical function it could be over damped to the

point where it is probably just gonna pier out and stop. By what ever it is, what ever the movement is, I'm thinking $p$ is a position.

146 G: Ok, how about $P'$.

147 J: Velocity, rate of change is that if we are looking at on an $x$ axis, whatever we are looking at this particle this spring, is it going forwards or is it going backwards at any point in time.

148 G: Ok.

In turn 137 Jordan expressed that $P''$ was "some sort of rate of change of a rate of change, or an acceleration." Noting that he really had no other way of thinking about it other than an as acceleration, in turn 141 he said, "I could say that it is a rate of change of a rate of change of population. That doesn't, like I haven't been told any words for that. It's math," suggesting that if this differential equation was pertaining to population he would not know how to talk about it, or what the role of $P''$ would be (coincidently, in Interview 2, Jordan used population to contextualize T2-I2). Supporting this, in turn 143 he said that he couldn't explain the differential equation past it possibly representing the motion of a spring. Considering the statements in turns 141 and 143, it seems that while Jordan thought of $P''$ as the rate of change of a rate of change, this interpretation was somewhat meaningless for him because he didn't know what $t$ meant outside of the acceleration of an object. After discussing how he thought about $P''$, I asked him how he thought about $P$ and $P'$. In turn 145 Jordan expressed that he thought of $P$ as representing the position of an object in motion, and in turn 147 he discussed thinking of $P'$ as the velocity; "whatever we are looking at, this particle, this spring, is it going forwards or is it going backwards at any point in time" (turn 147). In this case, while Jordan expressed that $P'$ was the rate of change of $P$, and $P''$ was the rate of change of the rate of change of $P$ (much as he did in Interview 2), the fact that Jordan did not know what situation to relate the differential equation to seemed to deflect Jordan's ability to focus on the relationship between the value of the quantity and the value of its rates of change.

Based on the two previously discussed segments of transcript, Jordan was attending to $P$, $P'$, and $P''$ as three separate mathematical entities. That is, while Jordan referred to $P'$ as a rate of change, and $P''$ as a rate of change of a rate of change, at no point in the interaction did he express using the value of $P'$ or $P''$ to determine the behavior of $P$, nor did he discuss the differential equation representing a relationship between their values. I make this point because it played a large role in understanding how Jordan interpreted the differential equation; he interpreted it as an equation to be solved. That is, Jordan did not attend to the relationship present in the differential equation, rather, he noted that the differential equation meant “absolutely nothing” and immediately discussed the existence of a solution process. I now present the analysis identifying the resources Jordan used while solving the differential equation.

After the discussion concerning $P$, $P'$ and $P''$, I prompted Jordan to solve the differential equation, mainly because he mentioned being able to do so at the very beginning of the interaction. The following segment of transcript begins with me prompting Jordan to solve the differential equation and ends with Jordan discussing how he thinks of initial conditions. The work Jordan performed accompanying this segment of transcript can be seen in Figure 4.32.

150 G: Ok, um you had mentioned that you could stumble through and solve it. Could I trouble you stumble through it and solve it?
151 J: All right, we'll see if I can remember anything. All right, $r^2$ plus 6 $r$, plus 5 equals 0 [begins looking for roots].
152 J: I mean this I guess would be the general solution; I don’t have an initial condition so I can't really find out what the c's are, if I did it correctly, that is the general solution.
153 G: Ok, when you say general solution what does it mean to be a solution?
154 J: Ah wait is that a 5? Ah yeah I think I miss-read my own handwriting.
155 G: Ah that’s all right I do that all the time.
156 J: Yeah those are all fives. Um, any possible solution of this will fit into this form with a $C_1$ and a $C_2$. So if it is say spring and a mass, you can take this spring and this mass and stretch it to anywhere, put it anywhere, throw it, do whatever, give it any of these initial conditions and then you can find an exact solution to represent it.
157 G: You pointed to these and said initial conditions. What do you, what did you mean by that?

158 J: So if we are just gonna take this spring and say we are gonna start this by pulling it down two inches. And then we are gonna push it down to give it some sort of initial velocity, negative, and we are gonna let go. And then we can solve a way to, we can solve this general solution to give us a way of saying ok if you look at this solution this shows the specific motion of whatever it is that we are observing.
159 G: So when you point to these and said this initial condition, you meant like I pulled it down two inches and its moving such and such speed at that time.
160 J: Yeah position and velocity.

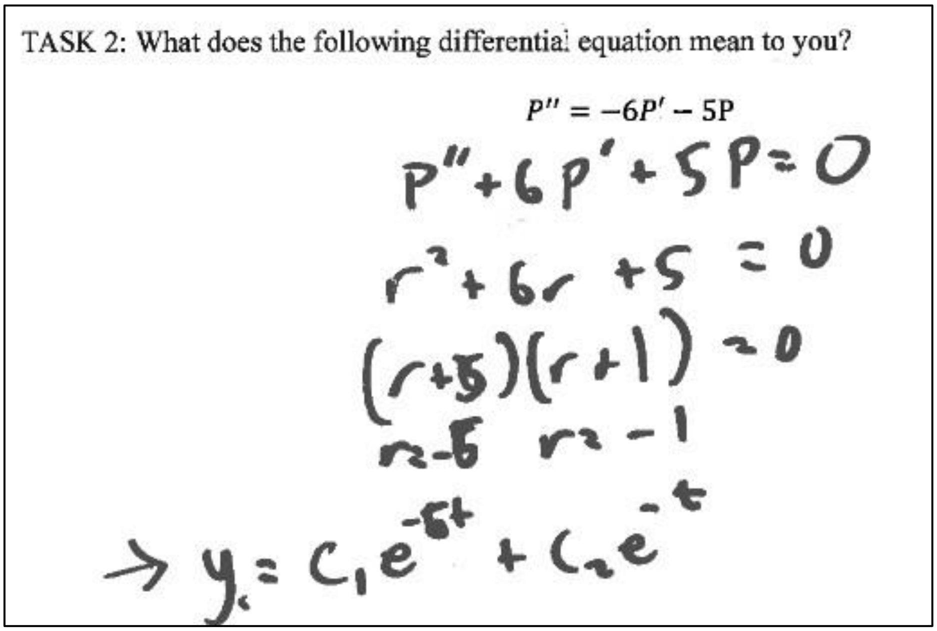


*Figure 4.32: Jordan attained an equation relating $y$ to the independent variable $t$.*

In turn 151 Jordan began solving for the roots of the characteristic equation, which he then used in turn 153 to get the general solution, $y = C_1 e^{-5t} + C_2 e^{-t}$. In this case, Jordan utilized the *method of characteristics* to set up the characteristic equation, solve for the roots of the equation (in this case, $r_1 = -5$ and $r_2 = -1$), and substitute the roots into the known form of the solution, $y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$. In turn 156 he coordinated *the definition of a general solution* and *dependence* to discuss that result of the solution process was an equation that provided the form of any of the exact solutions that were determined by a specific set of initial conditions. He elaborated on this in turn 158, noting that one can solve the general solution using the initial conditions to find the “specific motion of whatever it is we are observing.” In short, Jordan recognized that he could apply *the method of characteristics* to solve the differential equation for a general solution. Using *the definition of general* solution he inferred the process resulted in an equation that provided the form of any solution to the differential equation, each of which were determined by a specific set of initial conditions (*dependence*).

In summary, Jordan expressed that $P$ represented the value of a quantity, $P'$ represented its rate of change, and $P''$ was the rate of change of the rate of change. In addition he noted that other than knowing it was an equation for something, and without having a situation to relate it to, it's just something "I'll sit down and I'll solve" (turn 133). Based on how persistently Jordan expressed an absence of meaning in the differential equation, how quick he was to point out that he could solve it, and the fact that he did not discuss or utilize a relationship between the variables, he interpreted the differential equation as something to be solved. While engaged in the solution process, he utilized the *method of characteristics* to find the roots of the characteristic equation, and then put those roots into a known form of the solution. He then coordinated the *definition of general solution* and *dependence* to reason that any solution to the differential equation would have the form $y = C_1e^{-5t} + C_2e^{-t}$, and that one could find a particular solution by using the initial conditions to determine the exact values of $C_1$ and $C_2$.

**4.2.3.3 Interview 3 task 3.** This task presented Jordan with a non-linear differential equation, which was said to model the population of a species of owls. The task provided the students with two initial conditions and asked the students to determine the long-term behavior of the owl population for each initial condition. To complete this task Jordan first made an attempt to analytically solve the differential equation, though after quickly reaching a point where he did not know how to proceed I prompted him to reason about the long-term behavior directly from the differential equation. He then began to use *Euler's method, variable* *substitution,* and *parts contribute to the whole* to determine the future values of the population. While this was prompted by my question, I argue my question did not change the way he was interpreting the differential equation or the components from which it was composed, only the method (and some of the resources) he used to solve it. Then, unprompted, Jordan switched reasoning patterns and began to utilize *functional variation, functional mapping* and *rates indicate behavior* to determine the general behavior of the populations. Lastly he used *the definition of general solution* to (incorrectly) infer that behavior of the owl populations would be same for both initial conditions. Jordan's reasoning was indicative of interpreting the differential equation ion two ways, as providing a model of the quantity of interest, and as a description of the behavior of the quantity of interest.

After reading the task Jordan said, "Ok, so it looks like it just wants me to solve it" (turn 183). I asked him what it meant to "solve it" in turn 184, and as part of his response, in turn 187 he said (the graph corresponding to this segment of transcript can be seen in figure 4.33):

> "So it is asking ok, we are not really sure which one is right, um find what the equation would look like if there is actually 5 right now, maybe it looks something like that [draws hypothetical graph through 5 on the *P* -axis]. And then also find what it would be at 3, so maybe it's just kind of a more volatile system [draws hypothetical graph through 3 on the *P* -axis]. So it is just asking to find the *P* equals some function of time."

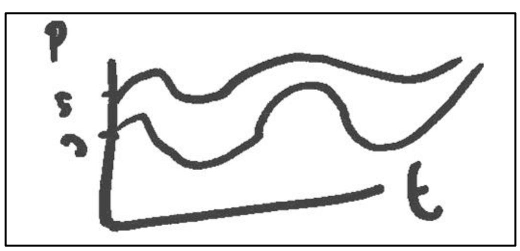



*Figure 4.33: Graphs illustrating the goal of the task.*

In this statement (turn 187) Jordan expressed that *P* as a function of time, and it was this function that the task was asking him to find. In turn 193 (transcript follows) Jordan mentioned there was a problem with regard to finding an equation for *P* as a function of time, namely that the differential equation had a non-linear term that he was unsure how to handle (Jordan's writing in turns 198 can be seen in Figure 4.34

190 G: Ok.
191 J: The problem is…
192 G: What's the problem?
193 J: We have a $P$ squared. I don't believe we have been taught how to deal with. hmm I am hoping this is not on our test. Because I don't know if I just haven't seen something like this in a while but I am pretty sure, this would be a non-linear equation right?
194 G: That's right.
195 G: Um just to give you piece of mind, I don't think you would have seen this before.
196 J: Ok. Yeah I don't think I have seen anything
197 G: I don't need you to have like a mini heart attack while I am interviewing you or something.
198 J: Oh yeah, I have a test Monday and I'm like this could be a rough test. Yeah no. So the answer, what it is asking me to do is solve this [writes $P' = \frac{P}{2} - \frac{P^2}{10}$], find some $P$ general solution [writes $P_{goal} = C_1 e^{kt}$] and then ah, find the $C_1$. I mean its first order so it might just have a c1, I don't know what the, what form an equation like this would have because I have never seen something like it.
199 G: Sure.

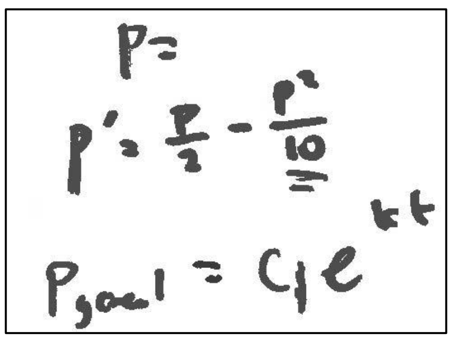



*Figure 4.34: Jordan's goal was to find the equation for P in terms of $t$.*

Based on turn 198, Jordan wanted to employ an analytical method to solve the differential equation by finding the general solution and then using the initial conditions to find any unknown constants. However, Jordan was unsure of how to approach the problem because it didn't fit a form he was familiar with; "I mean it's first order so it might just have a $C_1$, I don't know what the, what form an equation like this would have because I have never seen something like it." At this point it seemed as though Jordan reached stalemate; he knew the task was asking for an equation or graph, but he did not know what to do with the non-linear $y^2$ term which prohibited him from finding an equation that allow him construct such a graph. I then prompted Jordan to find an alternative solution method; this can be seen in the following segment of transcript. The work accompanying this portion of the transcript can be seen in Figure 4.35.

201 G: Is there a way that you could use the initial conditions and the differential equation itself?
202 J: I mean I could sit here and Euler's method it maybe. That might work.
203 G: Um, I am trying to think if that would be; just talk to me about what Euler's method would buy you. What would it get you?
204 J: Ok, so like let's say I am going to start with the $P$ equals 3 up here so. I have the $\frac{dP}{dt}$ equals 3 halves one minus.
205 J: Ok so if we plug it in, we have at our initial time ah, $\frac{dP}{dt}$ is $\frac{3}{5}$ maybe if I can read this. So, at time equals zero, we are going to start at 5, that's our initial condition or this one we started at 3. So we are starting at 3 and we have a $\frac{3}{5}$ slope, right there. So we are going to take that $\frac{3}{5}$ and we are going to go up and over a little bit, maybe we are going to move time ahead, we'll go with one, cause I don't know.
206 G: It's years.
207 J: Yeah its years, so we'll go with years because they are looking for long term prediction so if in that year it's a $\frac{3}{5}$ population increase and we started at 3, so now we are going to have, $\frac{18}{5}$ as our population after a year.
208 G: Ok.
209 J: So $P$ of 1 will be $\frac{18}{5}$, and then basically you jut plug that in again to $\frac{dP}{dt}$.

Before discussing turns 201-209, I first address the role my question in turn 201 had in changing the direction of Jordan's interaction with the task. While it is possible my prompt in turn 201 may have suggested the utilization of Euler's method to Jordan, I had no particular solution method in mind when asking the question. My question made mention of initial conditions because while working on other tasks during Interview 3, Jordan mentioned relying on initial conditions to find the value of c, and he expressed being asked to find a $C$ value in this task. In other words, my prompt was made in effort to shift Jordan's focus away from using an analytical method (because he admitted not knowing how to), toward focusing on the possibility of using the relationships in the differential equation itself. While my prompt was successful in doing so, I argue that I did not change the way Jordan was interpreting the differential equation, only the technique Jordan was using to solve it. That is, Jordan was still using this method to solve the differential equation. In addition, this was a technique Jordan already had available to him, my question simply prompted Jordan's utilization of it.

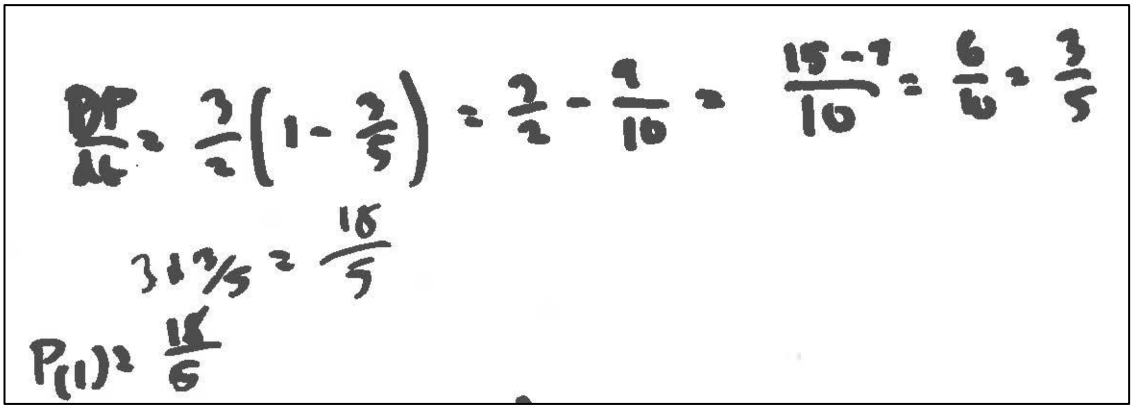


*Figure 4.35: Jordan's work while utilizing Euler's Method to complete the task.*

In segment of transcript (turns 201 - 209) Jordan completed one iteration of Euler's method. In turn 204 he substituted an initial population value of 3 into the differential equation to determine that $\frac{dP}{dt} = \frac{3}{5}$ when $P = 3$. He then multiplied the value of $\frac{dP}{dt}$ by the time step of 1 year and in turn 207 added the result to the value of *P* at the current time step to get $\frac{18}{5}$ for the value of the population after 1 year. While employing *Euler's Method* Jordan utilized the resource *variable substitution* to substitute the value of 3 into the differential equation for *P* to find the value of $P'$, and then coordinated *positives add- negatives take away* to add the increase in population to the initial value of $P$. In turn 209 he suggested that he would continue the process again to attain the value of *P* at the next time step. He then however, switched to reasoning with the relationship between the values of *P* and $P'$ in the differential equation itself to determine the behavior of the owl population.

210 G: And so essentially what it seems like you are doing is building this.
211 J: Yeah its one step at a time maybe its gonna just, I don't know what it's going to do. I'd probably just have to sit here and and do a couple more to see the trend. But, I guess it looks

like $P$ on the out, ahh, most likely its gonna do something like that [draws an increasing graph with horizontal asymptote].
212 G: Ok, why do you sat that, how do you know that?
213 J: Its population. They usually have some sort of max cap. ah if we start at say $p$ equals 3, when this gets multiplied out, actually I don't know cause these $P'$s are going to end up becoming this $P$ squared, so that will be the dominant term. Once its large enough to over come the fact that it is over ten so like if you start at the low value like one, $P'$ will be positive so it will initially be growing but once you get, like lets say we get to 10, you have 10 over here and then you have 5 over here so its a negative so its gonna have some sort of like up then down. But this is just a way of going step by step, and sitting here and saying ok at this point the slope is here and [draws slopes indicating the value of $\frac{dP}{dt}$ decreases as $y$ increases]
214 G: Gotcha ok. what happens at 5?
215 J: Same thing but starts a little higher up essentially.
216 G: Ok
217 J: It's the same relation, it's just going to start at a different point in the relation.

In turn 211 Jordan inferred that the graph corresponding to an initial condition of 3 would increase asymptotically to a maximum value by reasoning "it's population. They usually have some sort of max cap" (turn 213). He then checked this by considering how the value of $\frac{dP}{dt}$ changed as the value of $P$ increased, which can be seen in the following statement from turn 213:

> "So it will initially be growing but once you get, like let's say we get to 10, you have 10 over here [$\frac{P^2}{10}$] and then you have 5 over here [$\frac{P}{2}$] so it's [$\frac{dP}{dt}$] a negative so it's gonna have some sort of like up then down [draws graph which increases to a maximum value and then decreases; see Figure 4.36]."

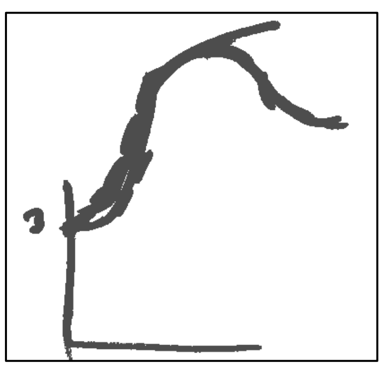



*Figure 4.36: Graph showing the behavior of $P$.*

Here, Jordan coordinated the resources *rates indicate behavior* and *functional variation*. More specifically, he determined how the value of $\frac{dP}{dt}$ would change as the value of $P$ increased (*functional variation*) and then used the values of $\frac{dP}{dt}$ to infer the behavior of the owl population (*rates indicate behavior*). At the end of turn 213, he utilized the resource *functional mapping* ("saying ok at this point the slope is here") to draw slopes on the graph of $P$ verifying that $P$ would increase and then decrease; these slopes can be seen in Figure 4.36. In this case Jordan used *functional mapping* to correlate values of $\frac{dP}{dt}$ to values on the function $P$. In this case Jordan began reasoning about $P$ and $P'$ as values in the differential equation that could be utilized to construct the behavior of the $P$. As such, I take this switch in reasoning and resources as an indication that Jordan was interpreting the differential equation as a description of the behavior of the quantity of interest.

In turn 214 I asked him what would happen in the case of the initial condition being 5. He replied that the behavior would be the same, but that it "starts a little higher up essentially" (turn 215). I take this inference as evidence of Jordan's utilization of the *definition of general solution.* His follow up in turn 216, "it's the same relation, it's just going to start at a different point in the relation," supports this. To clarify, Jordan used his *definition of general solution* to infer that all of the solution functions would have the same form, but just start from different places on the $P$-axis as dictated by the initial condition. More specifically, he used this resource to infer that the behavior of the owl population with an initial condition of 5 would be the same as that with an initial condition of 3, just higher up on the $P$-axis. In this case Jordan applied the notion of general solution to the form of the graphical representation of the solution functions, and as a result incorrectly inferred that the behavior of the owl population would be the same for two different initial conditions. I take this as evidence of the interaction between the two ways Jordan was interpreting the differential equation; that is, Jordan seemingly associated the behavior of the quantity of interest, with the equation relating the value quantity of interest to the value of the independent variable $P(t)$ (the model), to infer that the behavior for all solutions would be the same. More specifically, Jordan reasoned that all of the solution equations would have the same form (*the definition of general solution*) to infer that all of the graphs would have the same behavior. In this case the interaction between the interpretations supported Jordan in drawing an incorrect conclusion.

To complete this task Jordan initially attempted to solve the differential equation analytically to find the algebraic form of the differential equation, presumably to use it to graph the population with respect to time. However he was unsure of how to solve the differential equation analytically due to the non-linear term, but after some prompting, Jordan performed a single iteration of *Euler's Method.* His utilization of *Euler's method* consisted of the resources *variable substitution,* and *positives add- negatives subtract*. Specifically he used *variable substitution* to determine the value of $\frac{dP}{dt}$ for a specific population value, and *positives add-negatives subtract* to calculate an approximate value of the population after a certain amount of time. After this iteration he then focused on using the slope values to determine the behavior of the solution function, that is he then reasoned with the relationship between the value of $P$ and the value of $P'$ in the differential equation itself to infer the long term behavior of the owl population. He did this by coordinating *functional variation*, *functional mapping* and *rates indicate behavior*. More specifically, he considered how the value of $\frac{dy}{dt}$ changed as the value of $y$ increased, used this dynamic set of values for $\frac{dy}{dt}$ to infer the behavior of the solution function by correlating the value of $\frac{dy}{dt}$ to values of $P$ on the solution function. After constructing the graph of the solution function, Jordan used his *definition of general solution* to infer that if the initial population was 5, the graph would be the same as initially having a population of 3, because "it's the same relation" (turn 216). Lastly Jordan's reasoning is indicative of interpreting the differential equation as a description of the behavior of the quantity of interest.

**4.2.3.4 Discussion of interview 3.** Over the course of Interview 3, Jordan displayed four distinct sets of resources and ways of interpreting the differential equation and the components from which it was composed (see Table 4.9), each of which correspond to different ways of interpreting the differential equations; as a description of the behavior of the quantity of interest (I3-T1 and T3-T3), as an equation to be solved (I3-T2), and as providing a model of the quantity of interest (I3-T3). Each of these three tasks serve to demonstrate the interplay between the resources one has available to them and the ways in which they interpret the differential equations and its components. More specifically, there were instances in Jordan's interactions with each of the three tasks presented which exemplify how the interpretations supported the resources, and the reasoning patterns he utilized.

I attribute Jordan's struggle to complete Task 1 to a gap between what he interpreted the differential equation to provide and what he felt he needed to do to complete the task. Jordan was quite explicit about knowing he needed to find a way to use the correlation between the species' population values and the values of their rates of change to determine the size of the species, but that he was unaware of how to accomplish this. Ultimately he relied on using the differential equations to determine how the population of the prey would behave without the existence of the predators and then used this behavior to characterize the prey as large or small. That is to say, while Jordan wasn't sure of how to complete the task, Jordan's interpretation of the differential equation and its components supported his ability to determine the behavior of the prey without the predators, and thusly a way to reason about the size of the species. While his logic was flawed, the resources he utilized and the interpretations of the differential equation and its components nonetheless supported Jordan in completing the task. This interaction serves an as example of the importance of the students interpretations of the differential equation, its components and the resources they have available to them at the time. Namely, Jordan's interpretation of the differential equation and the resources he had available to him supported his ability to use the differential equation to determine the behavior of the species, from which he could determine the size of the population.

While completing Task 2 Jordan expressed frustration with knowing that $P$ represented a number, $P'$ was a rate of change, and $P''$ was the rate of change of the rate of change, but not knowing what situation to attribute them to. In this case, I argue that Jordan's interpretation of the differential equation and the components did not support his ability to reason with the relationship between the variables. Recall that though Jordan said $P''$ could be "the rate of change of the rate of change of population" he also expressed not knowing how to make sense of $P''$ in terms of population; this suggests that Jordan was not able to certain sets of resources because he did not know what "the rate of change of the rate of change of population" was. In other words, without a situation in which to embed the differential equation Jordan's interpretation of the differential equation and the components did not support the utilization of certain resources, but did support the utilization of others. Namely ones that corresponded to him being able to "sit down and… solve it."

In Task 3 Jordan was unsure of how to proceed in analytically solving the differential equation because of the non linearity of the $y^2$ term. As a result of my prompt, he began to use a numerical method (*Euler's method*) to find a graph that represented the solution. In this case, Hakeem's interpretation of the differential equation and its components combined with my prompt to focus on the relationship between the variables seemed to support Jordan in using the Euler's method to complete the task. Without prompting, however, Jordan then switched reasoning patterns and focused entirely on the relationship between $P$ and $P'$ in the differential equation to describe the behavior of the population. While the distinction between Euler's method and Jordan's second line of reasoning is subtle (he was no longer predicting values of $P$, just making inferences about general behavior), I argue this is indicative of an additional way of interpreting the differential equation. The identification of this second interpretation is significant because it interacted with first interpretation (as providing a model of the quantity of interest) to lead Jordan to incorrectly inferring that the behavior of the population with an initial condition of 3 would be the same as the behavior of the population with an initial condition of 5. That is Jordan associated the behavior of the solution, with the form of the equation and as a result misapplied the definition of general solution to the behavior of the solution (as opposed to the algebraic form of the equation). In short the interactions with Tasks 1, 2 and 3 illuminate how the interpretations and the resources supported Jordan in completing the various tasks.

*Table 4.9: Resources and interpretations Jordan expressed in Interview 3.*

| Task | Interpretation | Resources | Entity |
| --- | --- | --- | --- |
| I3-T1 | As descriptions of the behavior of the quantities of interest | *Positives add-negatives subtract*<br>*Functional variation*<br>*Partial dependence*<br>*Rates indicate behavior*<br>*Carrying capacity* | $x$ and $y$ - Values of the populations<br>$\frac{dx}{dt}$ and $\frac{dy}{dt}$ - Value of the populations rates of change |
| I3-T2 | As an equation to be solved. | *The method of characteristics*<br>*Definition of general solution*<br>*Dependence* | $P(t)$ - Equation that results from the solution process |
| I3-T3 | As a description of the behavior of the quantity of interest. | *Rates indicate behavior*<br>*Functional variation*<br>*Functional mapping*<br>*Definition of general solution* | $P$ - Function of time, represented the population of owls at a certain time $t$.<br>$P'$- Value of the rate of change of the population |
| | As providing a model of the quantity of interest | *Euler's method*<br>*Variable substitution*<br>*Positives add – negatives subtract* | $P'$ –Derivative of $P$<br>$P(t)$ - Value of the population at a given $t$. |

### 4.2.4 Interview 4

During Interview 4 Jordan completed a total of four tasks, each of which is presented in the analysis that follows. Over the course of the interview Jordan expressed three distinct sets of resources and ways of interpreting the entities from which the differential equations were composed. In addition Jordan's interaction with Tasks 2 and 3 further support the interaction between the interaction between the ways in which he interpreted the differential equation and the mathematical entities from which it was composed with regard to the application of certain sets of resources.

**4.2.4.1 Interview 4 task 1.** Task 1 consisted of the competing and cooperative species task Jordan had completed in Interviews 1 and 2. Jordan's reasoning with this task during Interview 4, though much more systematic and concise, was homologous with his completion of this task in Interview 2. That is, Jordan used the differential equation to relate changes in the value of the population of one species to changes in the value of the rate of change of the other species. From this he inferred which system represented the cooperative species and which system represented the competitive species. He did this by utilizing the *definition of cooperative systems* and the *definition of competing systems, functional variation, partial dependence, parts contribute to the whole, and positives add – negatives take away.* In addition his reasoning was indicative of interpreting the differential equation as a relationship between the value of quantities and the value of their respective rates of change.

The similarity between Jordan's reasoning in Interview 2 and Interview 4 was determined based on the significant overlap in the resources Jordan used, how he interpreted the differential equation, and how he expressed utilizing the entities from which the differential equation was composed. In addition, early in the interaction Jordan noted, "basically, I mean the same thing I said last time," (turn 8) and proceeded to outline his reasoning in a way that suggested he was recalling recognized aspects of the task. Due to this similarity, the presentation of Jordan's interaction with this task from Interview 4 is abbreviated; I provide the transcript related to Jordan's completion of the task (follows) and then concisely identify the resources Jordan used and his interpretation of the differential equations.

6 J: It's the competitive and cooperative one again.
7 G: Ok.
8 J: So this set [system A] is the cooperative set, this set [System B] is the competitive set. Basically, I mean the same thing I said last time, so this is the change in species $x$ with respect to time this is the change in species $y$ with respect to time [$\frac{dx}{dt}$ and $\frac{dy}{dt}$].
9 G: Ok.
10 J: So if we are looking at they're competitive or cooperative, we are looking at how they affect each other's rate of change so if we are looking for the affect of species $x$ on species $y$ and vice versa, so if we just ignore these parts, these are relative to the size of the actual species, cause we are just looking at the interaction between each other. With a greater amount of species, in this [upper equation in System A] scenario, with a greater amount of species $y$, um the rate of change of species $x$ will be greater so it'll grow faster or it will die slower either one, it'll be helped by $y$. It's [pointing to lower equation in System A] a positive relationship, so with more species $x$ there will be more of species $y$, a greater change in species $y$, I'm just assuming they are both growing right now.
11 G: Ok.
12 J: And vice versa for this, so ah species $x$ with respect to species $y$, if there is more of species $y$, species $x$ will grow slower or die faster. It'll have a more negative or a less positive rate, and vise versa for species $y$.
13 G: Ok, um here you had said, you pointed at this differential equation [$\frac{dx}{dt}$ in system A] and said you were looking at how $y$ affects $x$ and then you pointed at this one [$\frac{dy}{dt}$ in system A] and said I am looking at how $x$ affects $y$.
14 J: Yup.
15 G: Why is it that that is the case? In other words, why are you looking at how $y$ affects $x$ here and how $x$ affects $y$ here as opposed to how $x$ affects it self.
16 J: Just purely based off the question that is being asked. The question is about the relationship between the two species; are they cooperative or are they competitive. If it was asking oh if, will this species kill itself off as it grows, then we'd be looking at its rate of change relative to its actual size. So like for these, if there's a, if there is to many $x$ and like there is no $y$, it's eventually going to kill itself off. But we are not, that's the not question being asked right now, so we can kind of ignore the relative effects of the species size to its growth or death rate.
17 G: So in other words it's the interaction that you are talking about?
18 J: Yeah it's, we are just looking at the interaction for the two with this question.

In turn 8 Jordan declared that System A represented the cooperative species and System B represented the competing species, and then explained that $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are the rates of change with respect to time for species $x$ and species $y$ respectively. In turn 10, Jordan explained that to complete the task he was considering how each of the species affected the other's rate of change (*partial dependence*). He then used *functional variation* to do precisely that by noting, "with a greater amount of species $y$, um the rate of change of species $x$ will be greater" (turn 10). He justified this by using *parts contribute to the whole* to explain that increases in the rate of change of $x$ means $x$ will "grow faster or it will die slower either one, it'll be helped by $y$" (turn 10). More specifically, he reasoned that increases in $\frac{dx}{dt}$ would increase the birth, indicating $x$ would grow faster, or decrease the death rate, indicating $x$ would die slower. Jordan then repeated this process for the second equation in System A, noting "It's a positive relationship, so with more species $x$ there will be more of species $y$, a greater change in species $y$" (turn 10). Jordan utilized the resource *positives add- negatives subtract* when characterizing the interaction between species $x$ and species $y$ as a positive relationship. In turn 12 he then uses this same reasoning pattern in reference to System B to determine that "if there is more of species $y$, species $x$ will grow slower or die faster." Lastly, Jordan's statements concerning $x$ being "helped by $y$" (turn 10) in System A and $x$ being harmed by $y$ in System B (turn 12) are indicative of Jordan expressing the *definition of cooperative systems* and the *definition of competing systems.*

By his own admission, Jordan's reasoning was "the same thing [he] said last time" (turn 8). And after analysis, with the exception of the *definition of cooperative species,* Jordan expressed utilizing the same set of resources, and the same reasoning pattern that he did during his interaction with this task Interview 2. Additionally Jordan's language suggested that he was thinking about $x$ and $y$ as the value of a quantity (population), $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as the rates of change of those quantities, and the differential equation as a relationship between the values of quantities and the value of their respective rates of change. These are all consistent with the analysis of Jordan's interaction with this task during Interview 2. In short Jordan interpreted the system of differential equations in the same way he did during Interview 2 and used the same set of resources to accomplish the same goal.

**4.2.4.2 Interview 4 task 2.** Task 2 presented a graph of $\frac{dy}{dt} = 4 - y^2$, and asked Jordan to use the graph to determine the long-term behavior of possible solution functions (see Figure 4.37). To construct graphs of the potential solution functions Jordan utilized the resources *functional dependence, functional mapping, rates indicate behavior, definition of equilibrium solutions,* and *functional variation.* Jordan used these resources to characterize the behavior of the solution functions based on sets of $y$ values (e.g., at $y = 2$ $y(t)$ does not change and for $y$ values above 2, $y(t)$ is decreasing). In do so he expressed that $y$ represented the value of the function $y(t)$, and that $\frac{dy}{dt}$ was the value of the rate of change, or slope of the function. Lastly he interpreted the differential equation as a relationship between the value of a quantity and the value of the quantity's rate of change, or in Jordan's words, "how a number affects its rate" (turn 76). In the following paragraphs I describe how Jordan used these resources and interpretations to complete the task.

TASK 2: Below you are provided with a graph of a rate of change equation rather than the equation itself (Note that / depends only on $y$). Figure out the long-term behavior of possible solution functions, illustrate your conclusions with a suitable graph or graphs, and state your conclusions about the long-term behavior of these solutions.

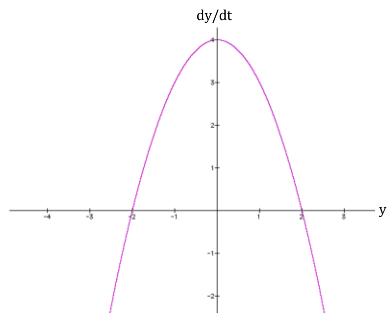



*Figure 4.37: Task 2 from Interview 4.*

The following portion of transcript, which is from the very beginning of Jordan's interaction with Task 2, demonstrates how he interpreted the differential equation and the entities from which it was composed.

76 J: [Begins drawing $y, t$ axis] Ok, so first thing I am going to say is, it gave me a $\frac{dy}{dt}$ and a $y$. And it's, it's so, this is kind of like the other problem, it's looking at how a population affects its own rate. How a number affects it's rate. So, and then it's asking about what an actual solution would look like with long-term behavior, so as $t$ approaches infinity. So I need to take this and put it into a $y$ with respect to $t$ instead of a $\frac{dy}{dt}$ with respect to $y$.

77 G: Why?

78 J: Cause it's, this [points to graph of $\frac{dy}{dt}$ with respect to $y$] is a slope representation sort of, I don't really know what this is called. But it's $\frac{dy}{dt}$ with respect to $y$, but it's asking for a solution, which would be $y$ with respect to $t$.

79 G: Ok.

80 J: So I just need to translate this [points to graph of $\frac{dy}{dt}$ with respect to $y$] to this [points to $y, t$ axis he drew] somehow.

81 G: Cool.

In turn 76 Jordan noted that the graph represented a relationship between the value of a quantity and the quantity's rate of change; "it's looking at how a population affects its own rate. How a number affects its rate." In this case Jordan utilized *functional dependence* to infer that the value of $y$ determined the value of $\frac{dy}{dt}$. Further he interpreted the task as asking him to find the behavior of $y$ with respect to $t$, as $t$ approached infinity. In turn 78 he noted that the graph "is a slope representation", which I take to mean that he thought of the graph as providing the slope of the solution function with respect to the value of the solution function. In turn 80, Jordan noted that he needed to translate the provided graph, into a graph of $y$ with respect to $t$, and proceeded to reason with the graph to construct a slope field (transcript follows). This suggests that Jordan was interpreting the differential equation as a description of the behavior of the quantity of interest.

82 J: So, we'll start with this. If $y$ is greater than 2, so we'll go one two three so here is two. And here is negative 2, if $y$ is greater than 2 or less than negative two, its going to approach 2 or approach positive 2 and negative 2 respectively, ah no, its going to be negative. So if it's greater than 2 $\frac{dy}{dt}$ will be negative, so if I am anywhere up here [points to area above $y = 2$ on the $y, t$ axis] it will be negative. It'll be more and more negative depending on the farther past it is so I could do that [draws slope field with negative slopes which decrease as $y$ increases], and at 2, at 2 and negative 2 there will be no change. So if you, if you can find a point where $\frac{dy}{dt}$ is zero, then those are stable solutions, stable points, if your graph ever hits this two there is going to be no change with respect to time so as time, continues, $y$ is going to stay the same. So those equilibrium solutions are at 2 and -2. So we are gonna mark those down first [draws lines $y = 2$ and $y = -2$].

83 G: Let me ask you a quick question. I don't want to side track you because you are explaining stuff pretty well, but it looks like you drew, 2 and -2 these, these are solid lines all the way across, these [points to vectors in slope field], are dashed lines. Can you tell me the differences between the two if there is one.

84 J: Um, I could draw this [points to line $y = 2$] as a series of dashed, it's I guess because this is definite, this is correct, I am labeling it as a solid line because like this is what's gonna happen here, its, I could just be because this is what my professor did, he drew them as straight lines, mainly because I think his dashed lines ran into each other, he didn't explain that, so it's convention that he did that I am now doing.

85 G: So you called this an equilibrium solution.

86 J: I'm assuming, I am guessing it's an equilibrium solution. It's, it's a point where, if $y$ is at that population or $y$ is at that number, it's not gonna change so it's never gonna waver or change if it hits there. So it's gonna hit that equilibrium.

87 G: So I guess I am just wondering, um, I guess you answered it, so I just want to make sure that's the case. You had said that your professor had drawn it as a solid line so that's why you are doing it, um but then you also called it an equilibrium solution and so I am wondering if you drew this thinking that it was just, thinking that it was a solution versus thinking that it was these as just representing the rate of change.

88 J: Yeah, I guess so. These are more like vectors of behavior, while this [points to $y = 2$] is an actual solution.

89 G: Is that how you were thinking about it when you drew it or did you just come to that conclusion when I said it?

90 J: Um a little bit of both, cause like I draw these [slope vectors], cause these aren't precise, these aren't exact, these are just estimations of behavior, this [points to $y = 2$] is exact to my drawing abilities, like this is a possible solution, these [points to vectors] are the behaviors, these are representing similar characteristics.

91 G: Ok, cool.

92 J: So looking at above two, it gets more and more negative as it gets farther and farther above 2, so you have this kind of behavior as it'll slowly approach 2 [draws decreasing slopes as $y$ increases, and then used the slope fields to draw decreasing functions] but looking below two, it's ah, like if it is slightly below 2 then is it a slightly negative slope, if it is farther below negative two it'll have a more negative slope, just because of this graph [$\frac{dy}{dt}$ vs. $y$]. So this one appears like it will kind of flow out. So if it's, if your initial $y$ of $t$ is below negative two, as time approaches infinity, $y$ will approach negative infinity. If your initial $y$, $y$ naught of $t$, is above two it will eventually approach two. And so now we'll look between. Anywhere else between, the slope is positive. It's the most positive at zero, so at $y$ equals zero, we have a pretty aggressive slope, closer to $y = 2$ the slope is zero, and closer to $y = -2$ this the slope is almost zero so we have this sort of behavior going through here. So if it is below negative 2 it'll approach negative infinity, anywhere else it will approach 2. So.

In turn 82 Jordan used *functional mapping* to correlate sets of $y$ values to sets of values of $\frac{dy}{dt}$ and *functional variation* to infer how the slope values changed within those sets of $y$ values. More specifically, Jordan considered the graph of $\frac{dy}{dt}$ with respect to $y$ and used *functional mapping* to determine that "if it's [$y$] greater than 2, $\frac{dy}{dt}$ will be negative" (turn 82). Then he used *functional variation* to further characterize how the slopes behaved for $y$ values greater than 2; "it'll [$\frac{dy}{dt}$] be more and more negative depending on the farther past [$y = 2$] it is" (turn 82). From this inference he then used *functional mapping* to construct a slope field such that the slopes decreased as $y$, increased (see Figure 4.38). That is, he correlated values of $\frac{dy}{dt}$ (as slope vectors) with values of $y$ on the slope field. He then coordinated *rates indicate behavior* and *functional mapping* to determine that for $y$ values of 2 and -2, $y$ would be not change because $\frac{dy}{dt}$ was zero. Using his *definition of equilibrium solution*, he then drew the lines $y = 2$ and $y = -2$ on the $y, t$

axis, and he did so without first drawing slope vectors. When I asked him about this, he said that was what his professor did in class (turn 84), but in turn 86 he said he was guessing it was an equilibrium solution because "it's a point where, if $y$ is at that population or $y$ is at that number, it's not gonna change so it's never gonna waver or change if it hits there. So it's gonna hit that equilibrium." In other words, he inferred the lines $y = 2$ and $y = -2$ were equilibrium solutions because $\frac{dy}{dt}$ was 0 at those $y$ values, and therefore he could draw the "actual solutions" (turn 88) without needing to use the "vectors of behavior"(turn 88).

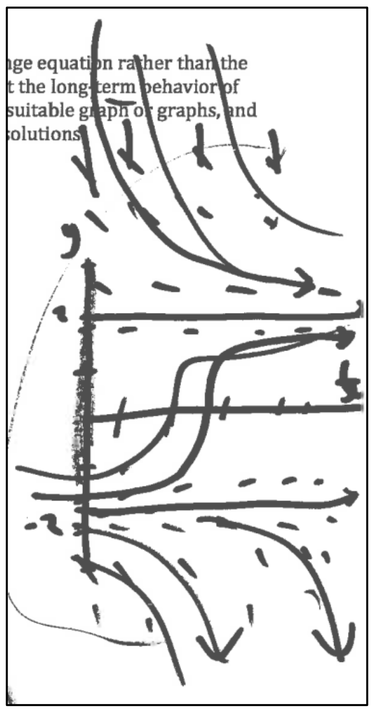



*Figure 4.38: Jordan's work while constructing possible solution functions.*

In turn 92 the conversation returned to determining the behavior of the possible solution functions, and Jordan discussed characterizing the behavior of $y$ for each of three sets of $y$-values: $y$-values above 2, $y$-values below -2, $y$-values between 2 and -2. To do this, he coordinated the resources *functional variation*, *functional mapping*, and *rates indicate behavior*. That is, Jordan determined how the value of the slopes would change as $y$ changed (*functional variation*), drew slopes representing this behavior on the $y, t$ axis (*functional mapping*), and then used the slopes to infer the behavior of $y$ (*rates indicate behavior*). Further he did this for each of the three sets of $y$-values. For instance, this reasoning pattern can be seen in the following statement from turn 92:

> "But looking below two, it's ah, like if it is slightly below 2 then it is a slightly negative slope, if it is farther below negative two it'll have a more negative slope [draws slopes which decrease as $y$ decreases], just because of this graph [$\frac{dy}{dt}$ vs. $y$]. So this one appears like it will kind of flow out. So if its, if your initial $y$ of $t$ is below negative two, as time approaches infinity, $y$ will approach negative infinity."

Using this set of resources and reasoning pattern, Jordan was able to correctly determine the long-term behavior of the different possible solutions for each of the sets of $y$ values. Specifically, he inferred that for initial values above two, the solution would approach two, for initial values between 2 and -2, the solution would also approach 2, and for initial values of $y$ below 2 the solution would approach negative infinity.

In summary Jordan used the resources *functional dependence* to infer that the graph of $\frac{dy}{dt} = 4 - y^2$ provided a characterization of how "a number affects its rate" (turn 76). He used *functional mapping* to correlate 4 sets of $y$ values to sets of values of $\frac{dy}{dt}$. Namely he used *functional mapping* to correlate:

- $y$-values of 2 and -2 corresponded to $\frac{dy}{dt}$ being 0,
- $y$-values greater than 2 corresponded to negative values of $\frac{dy}{dt}$,
- $y$-values less than -2 corresponded to negative values of $\frac{dy}{dt}$, and
- $y$-values between 2 and -2 corresponded to positive values of $\frac{dy}{dt}$.

For the $y$-values of 2 and -2, Jordan used his *definition of equilibrium solution* to infer that the lines $y = 2$ and $y = -2$ were solutions. For the remaining sets of $y$-values, Jordan used *functional variation* to determine how the slopes changed as $y$ changed, and then used *functional mapping* to create a slope field which graphically represented those changes. Finally, by considering the slopes and using *rates indicate behavior* he constructed graphs of the possible solutions, from which his was able to infer the long term behavior of the possible solution functions. Lastly, Jordan's reasoning suggests he interpreted the differential equation (in graphical form) as a description of the behavior of the quantity of interest.

**4.2.4.3 Interview 4 task 3.** This task presented Jordan with a system of differential equations, and asked him to discuss what it meant. Jordan's interaction with this task was quite different than that of Hakeem. Specifically, Jordan interpreted the differential equations as a set of equations to solve, and outlined how he would utilize the *method of eigenvalues* to do so, though Jordan expressed uncertainly with regard to his ability to carry out this solution process. Hakeem on the other hand, exclusively discussed the relationships between $y_1$, $y_2$, and $y_3$, and their derivatives, and never mentioned solving the system analytically or otherwise. Jordan did not voluntarily talk about the relationships between $y_1$, $y_2$, $y_3$, or their derivatives; when asked about such a relationship explicitly, he discussed not knowing how they were related because he was unsure of how the system of differential equations was generated, or what it represented.

Jordan's interaction with this task can be naturally separated into two parts: His discussion of the solution process he would use to solve the system of differential equations and his response to my follow up question regarding the relationships present in the system. I start the presentation with Jordan's opening statements.

153 J: Being completely honest with you, I can put this in a matrix, find an eigenvalue if there is one, probably, there probably is one because you have a $y$ times an a equals $y'$. So there is gonna be an eigenvalue or two $r$ three, and then I could solve the eigenpairs, but I have absolutely no idea what that means.
154 G: Ok bring me through what it is that you are thinking while you are doing that.
155 J: So, actually, I am going to save myself some time. So just [writing] $y' = Ay$, so this $y$ is just your $y_1$, $y_2$, $y_3$.
156 G: Right.
157 J: So and then this is essentially your coefficient matrix for this [points to the system of equations] cause this is just putting it in a different form. [Writing ] 3,3,-6, 5, -8, 1.

158 G: And what is this [points to the vector $y'$] in terms of, just because you have the different notations, you don't have to re-write it, just tell me what that represents.
159 J: Ah its $y_1'$, $y_2'$, $y_3'$. Just that vector which is the derivative of that [$y$] vector. So this [points to $y$] vector being put through this transformation [points to $A$] spits out its own derivative [points to $y$'].
160 G: Ok, hold on one second. Where did that language come from?
161 J: um, linear algebra. This is a transformation matrix and this is a vector, so this [$y'$] is the result of that vector [$y$] through that transformation matrix.

In turn 153 Jordan expressed being able to solve the differential equation but having "absolutely no idea what that means." After telling him to bring me through the process, he quickly re-wrote the differential equation into a matrix equation (see Figure 4.39), and discussed that $\vec{y}$' represented the derivative of $\vec{y}$. Additionally, he interpreted the matrix equation as representing a transformation of $\vec{y}$, or in his words "this [$y'$] is the result of that vector [$y$] through that transformation matrix" (turn 161). The conversation then shifted to which linear algebra class Jordan had taken, this portion of the transcript is omitted, and it picks up with my question about how Jordan knew there were eigenvalues and eigenvectors.

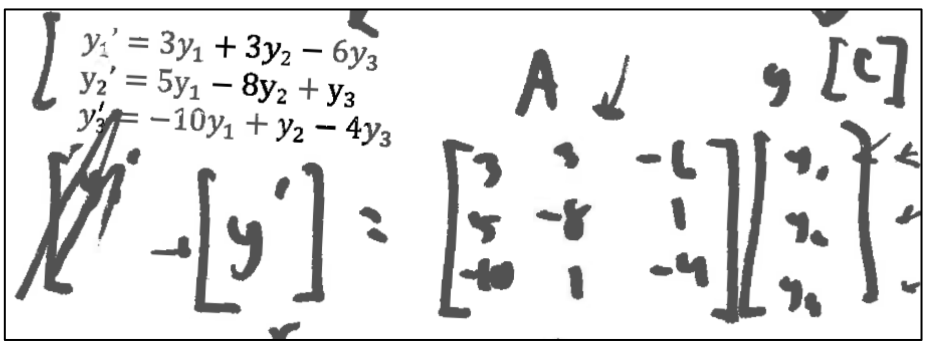



*Figure 4.39: Jordan rewrote the system of equations into a matrix equation.*

166 G: Can you just talk a little bit about how you know that has eigenvalues and eigenvectors? It just seemed like you were saying that because this [$A$] was multiplied by this [$y$], you know that it has eigenvalues and Eigen vectors. I am just curious what that means.
167 J: Ok let's see if I can…
168 G: Or I mean where it came from, how you know it, anything.
169 [long pause].
170 G: And if not, that's ok too.
171 J: I am trying to think, cause I know I did like 18 problems on this chapter like a week ago. So, normally like a lot of the problems we'd be given, we'd be given some specific $y_1$, $y_2$,

and $y_3$, and said ok, is this a solution. So you could plug in each of these, take the derivative of each of them individually, see if.
172 G: Sure.
173 J: And then do the math and see if $Ay$ actually equals $y'$. After that, I think it was something along the lines of, like going through and finding there is something, I think there is some way of like, if you find the eigenvalues of matrix $A$, and then you can get the eigenpair, and then once you get the Eigen pair, you can plug that into a formula, it's like $e^{it}$ or its like $e$ to the lambda $t$ times your actual eigenvector, plug all that stuff back in, put it into a new form, which would be more of like a general solution with a, with a like ay times $C$ vector, plug all that in and get your each individual $C$.
174 G: Gotcha.
175 J: I taught my roommate how to do this like 3 days ago, so I can't remember it all specifically, but…
176 G: Ok.
177 J: It is just a memorized process.
178 G: What you were talking about before.
179 J: To get out an answer in a form that we have been told.

From this portion of the transcript we can see that for Jordan putting the system into a matrix equation, finding the eigenvalues and eigenvectors, and then plugging those into the known form of the general solution is "a memorized process… to get out an answer, in a form that we have been told" (turns 177 and 179). Though Jordan admitted to not understanding the process, and was seemingly unable to complete it, this outlined method for solving systems of differential equations, *the method of eigenvalues*, was a resource for Jordan; he inferred that following this sequence of steps would lead to a solution to the system of differential equations. While Jordan did not solve the differential equation, it is evident from Jordan's statements and actions that he interpreted the system of differential equations as equations to be solved. The presentation now transitions to Jordan's discussion concerning the relationships present in the system of differential equations, which is included because it supports my claim regarding how Jordan interpreted the differential equation and explains (the rather unexpected) reason Jordan provided for not being able to reason with the system of differential equations.

After the discussion about how Jordan would solve the differential equation, I asked him how he thought about the relationship between $y_1$, $y_2$, and $y_3$. This was part of a planned follow up question that I asked all students that completed this task. I asked them this question hoping to elicit responses that would allow me to identify resources students use to reason about single differential equations that they also use to reason about systems of differential equations. In other words, the question was intended to support the collection of data that would allow for identifying the knowledge students were using to reason about single differential equations that was also useful for reasoning about systems of differential equations. In particular, this was useful for answering the third research question regarding how the resources supported the students understanding of differential equations. In the case of Jordan, he didn't see any relationship between them, and as the following transcript will show, this can be largely attributed to his uncertainty about how the system was generated.

183 G: I hear you. Two questions, how do you think about the relationship between $y_1$, $y_2$, and $y_3$?

184 J: See, that's I don't, well, when we set this up originally, like chapter 4.1 we got all of this [the system of equations] out of some initial equation. It was like [writing, see Figure 4.2.4.4] $y''' + y'' + y' = P(t)$, and somewhere along the road we said ok we are going to get rid of all these primes and just create a simple system. So we are going to say that $y$ equals $y_1$, so $y_1' = y'$, and we are going to set that equal to $y_2$, and so $y_1'' = y'' = y_2' = y_3$. So what we have basically done is taken some sort of second or third order I guess if we are doing this, $y_1'$,$y_2'$, $y_3'$, with a $y_1$,$y_2$,$y_3$, $y_3'$ is $y_4$, so I guess this would actually be third order. So then $y_3\ ' = y_4$ is actually $y'''$, so we are doing all of this to find the solution, the solution set to a third order differential equation.

185 G: So how is that related to how $y_1$,$y_2$, and $y_3$are related?

186 J: Well this is where this is a little bit of disconnect, where if we were given a problem like this [points to $y''' + y'' + y' = P(t)$] I could take that and set $y'''$ equal to this. Which would end up being $-y_3 + y_2 - y_1$ I mean I could do that, and I could say ok, so, and then basically create a matrix using $y_1$, $y_2$, and $y_3$ equals, so $y_1$ equals um, this would be $y_1'$,$y_2'$, $y_3'$ to end up getting this $y_4$ that we are looking for. So $y_4$ is this [points to $y_4 = y_3 + y_2 - y_1$] somehow. And I could say ok $y_1' = y_2$, so it is $0y_1 + 1y_2 + 0y_3$, $y_2'$ is $1y_3$, $y_3'$ is this, which would be -1, 1 -1. Now I have an A, now I have this $y_3'$ that I am looking for and I think you just, now we have this [matrix equation], now we solve it. So I have and then we went straight from this [nth order differential equations] to ok, if I am given a third or fourth order differential equation I can plug it into a matrix and go from there, but every single

equation you are pretty much ever going to do if you are taking it from one single differential equation, the way this is written is gonna give you something with this form: 0,1,0; 0,01, it its fourth order you would have a 0001, 5th order, 00001 next like you would have all of these basically just setting all of these $y_1{}' = y_2$. So I have zero connection between actually setting up one of these matrices versus being given something like this. I have no idea what on earth would create this sort of set up.

187 G: Oh because you don't have the same number patterns?

188 J: Yeah.

189 G: Ok.

In turn 184 Jordan made reference to the process of generating a system of equations from a single n$^{th}$ order differential equation (see Figure 4.40); this was the way in which systems of differential equations were introduced in Jordan's class. Based on his statements in turn 185, Jordan brought this up in reply to my question concerning how he thought about the relationship between $y_1$, $y_2$, and $y_3$, because as he explained, when generating a system from a 3$^{rd}$ order differential equation, $y_2$ is the derivative of $y_1$, and $y_3$ is the derivative if$y_2$. That is to say, in the case of generating a system from a single n$^{th}$ order differential equation, how $y_1$, $y_2$ … $y_n$ are related is somewhat clear to Jordan, $y_n + 1$ is the derivative of $y_n$. Jordan noted that the coefficients in the A matrix for the provided system of equations was not consistent with the pattern generated by transforming an n$^{th}$ order differential equation into a system of equations (see turn 186). As a result Jordan felt as though he could not characterize the relationship between $y_1$, $y_2$, and $y_3$ because he did not know what $y_1$, $y_2$, and $y_3$, represented. More specifically, he seemingly determined that $y_2$was not strictly the derivative of $y_1$, and that $y_3$ was not strictly the derivative of $y_2$, but did not know what they were comprised of. His last statement in turn 186 perhaps best summarizes this, "I have no idea what on earth would create this sort of set up."

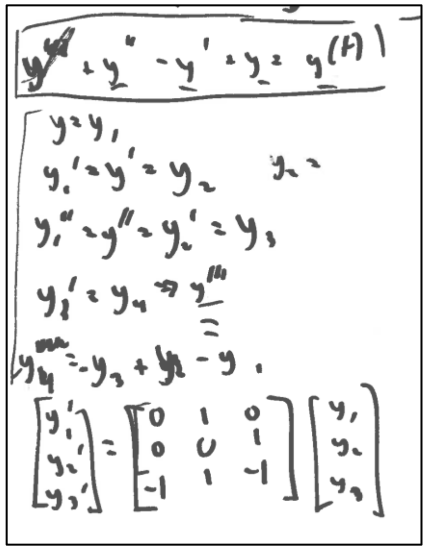


*Figure 4.40: Jordan generating a system of equations from an $n^{th}$ order equation.*

In summary, Jordan expressed utilizing the *method of eigenvalues* to solve the system of differential equations; re-writing the system into a matrix equation, finding the corresponding eigenpairs, and then plugging those eigenpairs into the known form of the solutions to the differential equation. Within the matrix form Jordan interpreted $\vec{y}'$ as the derivative of $\vec{y}$. In addition, he thought of the matrix equation as representing a transformation on $y$ that resulted in the derivative of $y$. Jordan, however, was unsure of the relationship between the entities from which the system was composed because he did not know "what on earth would create this sort of set up" (turn 186). In short, Jordan outlined the process he would use to solve the system of equations, and displayed a large degree of uncertainty with regard to understanding the relationships between the variables. Based on this Jordan interpreted the differential equations a set of equations to be solved.

**4.2.4.4 Interview 4 task 4.** Task 4 presented Jordan with a first order differential equation that was said to represent the amount of a list students had memorized and asked him to determine the amount of the list a student had memorized with respect to time, if the student initially had half of the list memorized (see Figure 4.41). As the analysis will show to complete this task Jordan utilized the resources *functional dependence, functional mapping, rates indicate behavior, definition of equilibrium solutions*, *dependence* and *functional variation* while reasoning with the relationship present in the differential equation. More specifically Jordan created a graph of $\frac{dL}{dt}$ with respect to $L$ and reasoned from it, much as he did to complete T2-I4, to determine the behavior of potential solution functions. This suggests he interpreted the differential equation as describing the behavior of the quantity of interest. Early in the interview Jordan also interjected that he could use the *method of integrating factors* to solve the differential equation; he outlined this process, but did not actually utilize it to complete the task. This is significant in terms of the analysis, however, as this was the first time Jordan completed a task of this manner without using an analytical solution technique. This suggests that the resources Jordan saw as applicable to tasks of this type may have expanded.

| |
|---|
| TASK 4: Suppose the amount of a list students memorize can be represented by the rate of change equation $L' = .5(1 - L)$, where $L$ represents the fraction of the list that is memorized at any time *t*. Determine the amount of the list memorized with respect to time *t*, if a student initially has half of the list memorized. |

*Figure 4.41: Task 4 from Interview 4.*

The following transcript begins with Jordan discussing his ability to solve the task.

218 J: Ok. This doesn't look that bad. So, it's giving me a rate, its asking for a solution in terms of a first order differential equation has a set of solutions that are, $L$ with respect to $t$. so, $L' = 0.5 - .5L$ I just distributed it out so I can reason with it a little better in my own head, and so, actually I will o back and draw that graph that I was given earlier, where it was kind of $\frac{dL}{dt}$ with respect to $L$.
219 G: Ok.
220 J: So, if $L$ is $0$ $\frac{dL}{dt}$ is going to be about .5. And cause you can't really know less than zero we can ignore everything over here.
221 G: in other words $L$ can't be less than zero.
222 J: Yeah, $L$ can't be less than zero and $L$ cant be greater than $L$. So you have zero to one, just because it is 0 to 100%.
223 G: Ok.
224 J: And then you have so at $L = 1$, you have $L' = 0$, and then its just a line.
225 G: Ok.

226 J: Ok, so I could actually sit here and solve this if I wanted to, I feel like it's been so long since I have actually just solved something that is a simple first order differential equation I would probably screw it up anyway.
227 G: Can you just tell me what it means to 'solve' it? You don't have to do it, just when you say solve it, what does that entail?
228 J: So basically just putting it in the form we'd been given at the beginning. So $L' + P(t) = G(t)$, take the integrating factor, just like this .5 and go through like divide it out, do it by the, just go through and solve it to the point that you get this, you end up getting $L = Ce^{.5t}$ or what ever it ends up being.
229 G: Sure I am with you.

In turn 218 Jordan suggested that the differential equation was "giving [him] a rate," and after distributing the .5 to both terms on the right hand side began reasoning with the differential equation to generate the behavior of the solution curves. Specifically, he determined an appropriate range of values for $L$, namely [0,1], and rates of change that corresponded to these boundary values. In turn 225 Jordan interjected that he could use the *method of integrating factors* to find the solutions, but suggested this was not his primary approach; "I could actually sit here and solve this if I wanted to." In response to me asking him to outline the process he would use (turn 226), he mentioned re-writing he equation so it would be in the form $L' + P(t) = G(t)$, finding the integrating factor, and then going "through and solving it to the point that get … $L = Ce^{.5t}$ or whatever it ends up being." (turn 228). Based on Jordan's previous interactions when utilizing the *method of integrating factors*, here Jordan was outlining the process of using the integrating factor method to find a *general solution* and then coordinating *dependence* and *function slots* to find the solution specific to the provided initial condition. After outlining this solution method, Jordan immediately returned to what appears to be his original approach to completing the task. This approach is seen in the following segment of transcript.

230 J: So, but without doing this [integrating factor method], um, we can just look at this [draws graph of $\frac{dL}{dt}$ with respect to $L$]. So at $L = 0$, so you can start. Well, we'll just do this. So at $L = 1$, your slope is zero, so at one [draws line $L = 1$], that's, that's your equilibrium solution kind of. And at $L = 0$ it's the most positive [draws positive slopes at $y = 0$], so, cause it's the highest. Then it is just gradually, it's linearly less positive, which when you

have linearly less positive slope, that will end up giving you [draws slope which decrease as $L$ increases to 1]. So if it is asking you at a specific point in time with however much you started, like let's say you start at time zero and you already know a quarter of the list, how well, how much of the list you are going to know with respect to time is gonna follow this trajectory. Where yeah, I guess you are going to start memorizing the list quickly, and you're going to memorize a lot of facts quickly and once you know more you are gonna learn it slower.

231 G: Ok.

232 J: That’s just essentially what this [the graph of $\frac{dL}{dt}$ with respect to $L$] is saying. The more you know the slower you learn more, and then this is a specific representation of that [points to slope field].

In turn 230 Jordan coordinated *functional dependence* and *functional mapping* to draw the graph of $\frac{dL}{dt}$ with respect to $L$ (see Figure 4.42). That is, he inferred that the value of $\frac{dL}{dt}$ depended on the value of $L$ (*functional dependence*) and then mapped values of $L$ to values of $\frac{dL}{dt}$ (functional mapping) to construct the graph. He then used the correlated values (e.g., $\frac{dL}{dt} = 0$ when $L = 1$) and *functional variation* (“it’s linearly less positive, which when you have linearly less positive slope”; turn 230) to draw a slope field. He then coordinated the resource *rates indicate behavior* and *dependence* to make inferences about the behavior of the graphs of the solution functions based on the value of the slopes and certain initial conditions. For instance in turn 230 Jordan said “where yeah, I guess you are going to start memorizing the list quickly, and you're going to memorize a lot of facts quickly and once you know more you are gonna learn it slower.” In addition Jordan utilized the *definition of equilibrium solutions* to infer that the line $y = 1$ is a also a solution to the differential equation.

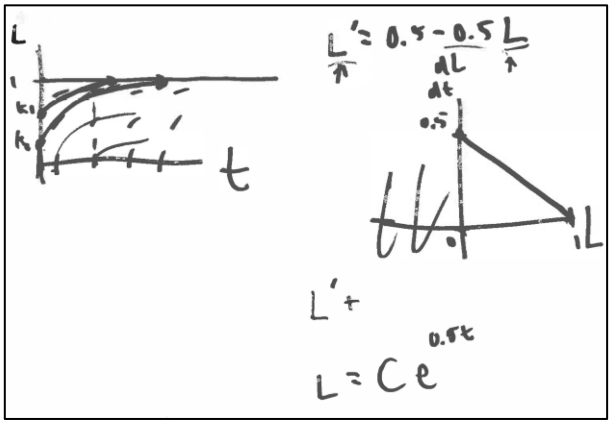



*Figure 4.42: Jordan constructed a graph of* $\frac{dL}{dt}$ *to construct solution functions.*

In summary, Jordan used the relationship between the value of a quantity and the quantity's rate of change to determine the behavior of the quantity. While doing so he interjected that he could utilize the *method of integrating factors* to complete the task, but wasn't confident in his ability to successfully utilize the method. This is significant because this interaction was the first time Jordan reasoning directly from the relationship in the differential equation to construct a graph of the solution function when he had an analytical solution method available to him. In all of the prior interactions with tasks such as this one, Jordan had utilized an analytical method to find the algebraic form of the solution. In this case however, Jordan constructed a graphical form of the differential equation (the graph of $\frac{dL}{dt}$ with respect to $L$) from which he constructed graphical representations of the solution functions. To generate the graphs of the solution functions, Jordan used *functional mapping* and *functional dependence* to construct a graph of $\frac{dL}{dt}$ with respect to $L$, then coordinated *functional mapping* and *functional variation* to construct a slope field, and then used *rates indicate behavior* and *dependence* to construct the graphs of the potential solution functions based on the slope values and different initial conditions. Lastly he utilized the *definition of equilibrium solutions* to infer that the line $y = 1$ is a solution to the differential equation, that is he realized the value of $\frac{dL}{dt}$ was zero at all $y$-value of 1, which indicated $y = 1$ was an equilibrium solution to the differential equation.

**4.2.4.5 Discussion of interview 4.** While completing the four tasks in Interview 4 Jordan expressed three ways of interpreting the differential equations, as a relationship between values of quantities and the value of the respective quantity's rate of change (T1-I4), as a description of the behavior of the quantity of interest (T2-I4, and T4-I4), and as equations to be solved (T3-I4). Table 4.10 shows the resources and interpretations of the components of the differential equations Jordan expressed for each task. In this discussion I only address tasks 2 and 3 as these tasks were somewhat novel to Jordan; Jordan had not encountered Task 2 prior to the interview and systems of differential equations were just beginning to be discussed in his differential equations course around the time of Interview 4.

Though Jordan had not seen a graphical representation of a differential equation (in the form a graph of $\frac{dy}{dt}$ with respect to $y$) prior to Interview 4, he immediately inferred that the graph related values of $y$ to values of $\frac{dy}{dt}$, and determined that to complete the task he needed to "put it into a $y$ with respect to $t$ instead of a $\frac{dy}{dt}$ with respect to $y$." To do this, Jordan graphically correlated the values of $y$ and $\frac{dy}{dt}$ in the form of a slope field having a $t, y$ axis. In other words, Jordan transformed the correlation provided by the graph of $\frac{dy}{dt}$ with respect to $y$, into a commensurate correlation, in the form of a slope field that provided a more explicit description the long-term behavior of $y$. In this case by interpreting the entities within the graphical representation of differential equation as slopes that corresponded to certain values of $y$, Jordan was able to construct the slope field and reason from it using resources that supported this.

Namely Jordan was supported in applying *functional mapping, functional variation* and *rates indicate behavior*.

While using these resources was fruitful for Jordan with regard to completing tasks 2 and 4, in the process of completing Task 3, Jordan suggested that he was unable to apply them. Namely Jordan noted that he had "no idea what on earth would create [that] sort of set up" in the system of equations. In his case Jordan was unsure of how the values of $y_1$, $y_2$, and $y_3$, were correlated to each other because the system of equations did not appear to be generated from an nth order differential equation. In other words, Jordan's statements suggested that for him, resources such as *functional mapping* and *functional variation* were not applicable in terms of correlating values of $y_1$, $y_2$, and $y_3$to the values of their rates of change. Therefore, Jordan made use of resources that would allow him to discuss aspects of the differential equation that he could relate to, namely solving it using the method of eigenvalues.

*Table 4.10: Resources and interpretations Jordan expressed while in Interview 4.*

| Task | Interpretation | Resources | Entities |
|---|---|---|---|
| I4-T1 | as a relationship between the value of quantities and the value of their respective rates of change. | *Definition of cooperative systems*<br>*Definition of competing systems,*<br>*Functional variation*<br>*Partial dependence*<br>*Parts contribute to the whole*<br>*Positives add – negatives take away.* | $x$ and $y$ - Value of the species populations<br>$\frac{dx}{dt}$ and $\frac{dy}{dt}$ - Values of the respective rates of change |
| I4-T2 | As a description of the behavior of the quantity of interest. | *Functional dependence*<br>*Functional mapping*<br>*Rates indicate behavior*<br>*Definition of equilibrium solutions*<br>*Functional variation* | $y$ - Number<br>$\frac{dy}{dt}$ – Value of the rate of change of $y$ |
| I4-T3 | As a set of equations to be solved | Method of eigenvalues | Jordan expressed not knowing what $y_1$, $y_2$, or $y_3$ were, but expressed that $\vec{y}$'was the derivative of $\vec{y}$ |
| I4-T4 | As a description of the behavior of the quantity of interest. | *Functional dependence*<br>*Functional mapping*<br>*Rates indicate behavior*<br>*Definition of equilibrium solutions*<br>*Dependence*<br>*Functional variation* | $L$ - Amount of a list a student had memorized<br>$L'$- Slope, the rate at which a student was memorizing the list. |

### 4.2.5 Interview 5

Jordan completed 5 tasks during Interview 5, only three of which are presented in this section. With the exception of the first task, Jordan had encountered all of the tasks in previous interviews. His interactions with Task 3 (similar to T2-I4) and 5 (similar to T4-I2) are omitted because Jordan completed them using the same resources, interpretations and reasoning patterns as he did when encountering them the first time. While, in terms of the analysis the fact that his approach did not change is important, for the purposes of presenting the analysis discussing Jordan's interaction with Task 3 and Task 5 would only serve to add redundant analysis to the chapter.

**4.2.5.1 Interview 5 task 1.** The first task of Interview 5 was given in the form of a verbal question asking Jordan to explain what it meant "for something to be a differential equation" (turn 5; transcript follows). In his response he used the resources *functional dependence*, *rates indicate behavior* and *equality condition* to describe differential equations as conditions concerning the behavior of certain solution functions. This indicated he was interpreting differential equations as descriptions of the behavior of the quantity of interest. While discussing what it meant to be a differential equation Jordan also described how they are different from other equations (which he equated with functions). Namely he indicated that: 1) functions are relationships between quantities, where as differential equations are relationships between a quantity and its behavior, 2) functions can be graphed, but only solutions to differential equations can be graphed, and 3) functions have independent and dependent variables but differential equations do not.

These differences between the way Jordan understood differential equations and the way he understood other equations (functions), is quite significant, and will be addressed in the discussion at the end of this chapter. Later in the interview Jordan found himself contradicting some of these statements. Additionally, some of these statements contradicted his earlier statements and actions from previous interviews (e.g. while completing T2-I4 Jordan graphed $\frac{dy}{dt}$ with respect to $y$, and in Interview 1 Jordan referred to $x$ and $y$ in the differential equation $\frac{dx}{dt} = 5x + 2xy$ as independent variables). I take such instances as evidence of the separation between the interpretations Jordan applied and the reasoning that corresponded with those interpretations. That is, the instances in which Jordan seemingly contradicted himself illuminate some of the distinctions between the various interpretations and the resources that correspond to the interpretations. In addition they provide insight into how the resources Jordan utilized in conjunction with the interpretations supported his different understandings of differential equations. In this subsection I present the analysis of the resources Jordan utilized while describing what differential equations are, and the distinctions he drew between them and other equations.

I start the discussion with a transcript from the very beginning of the task.

5 G: Alight so the first thing I have for you isn't actually a written task, it’s just something that I am gonna ask you. So what does it mean for something to be a differential equation?
6 J: Um it's an equation relating to how something is changing as opposed to various other factors of its current state.
7 G: Ok.
8 J: So its change depends on what its current state is and any number of other factors.
9 G: Could you give me an example?
10 J: Um, really anything where there is a $y$ a $y'$, any number of primes, anything where there is some sort of change or derivative relating to an initial $y$ or $y'$, some other version of itself. It's not a dependent variable and an independent variable the change is based off of where it’s, So like $y' = 2y + 6$.
11 G: Could you write that for me?
12 J: Ok [writing].
13 G: Um, so you said there is no independent or dependent variable. Can you tell me what you mean by that?
14 J: Well, in like earlier math classes calculus, all that, it’s $y$ equals whatever $x$, you can have $y = mx + b$ [writes this on paper], whatever it is, you are looking at the effect of $x$ on a $y$.

> Whereas with differential equation it seems to be much more used for how something changes rather than what something is. Based off of, instead of this $x$ this independent, you can have an $x$ in there, like the population problems where it would be $2yx$. But it's, it's what makes a differential equation is more based off the fact that the, ah, rate of change is variable based on the actual value of $y$, or other derivatives of $y$ [writes $y'' = y' + y$].

In turn 8, indicative of the resource *functional dependence,* Jordan noted that a differential equation is an equation where a quantity's "change depends on what its current state is and any number of other factors." In turn 9 I asked him to provide an example and in turn 10, commensurate with his characterization, he said differential equations could be any equation where "there is a $y$ a $y'$, any number of primes, anything where there is some sort of change or derivative relating to an initial $y$ or $y'$, some other version of itself." In this case, "the change" depends on the value of what is changing or some other related factor. In Jordan's next statement, "it's not a dependent variable and an independent variable the change is based off of" (turn 10), he seemingly rejected this dependence, but his statement in turn 14 suggests that Jordan is applying a distinct definition of independent and dependent variables. Namely, Jordan suggested that independent and dependent variables are related to "what something is" (turn 14), where as differential equations relate to "how something changes" (turn 14). This will be explicated further in subsequent paragraphs.

To better understand this difference between differential equations and regular equations I asked Jordan to explain the distinction between them in turn 15, and he replied, "I don't really know what separates a differential equation, I can kind of recognize one" (turn 16). I then asked Jordan to compare $y' = 2y + 6$, to $y = mx + b$, and his replies provide great insight into his previous statements; the transcript follows.

19 G: Ok, so if this paper, if this [the two equations he wrote] wasn't here and I wrote this equation [$y = mx + b$], I just wrote this down on a piece of paper and wrote that [$y' = 2y + 6$] down on a piece of paper, um and asked you which one was a differential equation. It seems like you would be able to pick this [$y' = 2y + 6$] one, tell me why. What would it,

why is it, what is it that you see in this one that makes this a differential equation versus this [$y = mx + b$], not one.

20 J: A $\frac{dy}{dt}$ with a $y$ on the other side. It cannot all be condensed into $\frac{dy}{dx} = m$. Like this [$y = mx + b$] you can take the derivative of this with respect to $x$, and it is separate, they are two separate, this [$\frac{dy}{dx} = m$] is related to this [$y = mx + b$], but the lines I guess are kind of blurred with differential equations because it's more of a, you are looking for something that satisfies this [$y' = 2y + 6$], rather than a, you can graph this [points to $y = mx + b$], this is a function. A differential equation is a relationship that can be satisfied by functions.

21 G: So when you say that this [points to $y = mx + b$] is a function and that this [points to $y' = 2y + 6$] is a relationship. What's the difference between the function and the relationship?

22 J: Um this is a relationship of other functions, so like $y$ is a function I am assuming $y$ is a function, and the $y'$ is a characteristic of that function itself. This being its slope. It's what functions can satisfy these constraints.

23 G: Can you graph this [$y' = 2y + 6$]? It seemed like you said that that was a difference between the two. Maybe that's, do you see that as a difference that's probably a better question.

24 J: Um, I could graph, like for this [points to $y'' = y' + y$], if I just wanted to find a general solution [writes $y = e^t$ and $y = e^{2t}$ ], like these are solutions, these are functions these are solutions. One of these is a function, this is a function [$y = e^t$], this is a function [$y = mx + b$], and both of these can be graphed. Whereas this differential equation [$y'' = y' + y$], you could graph a phase plane on $y_1$ and $y_2$, and you could graph a general solution like a slope field type thing. But in terms of if we are gonna want to graph just a line or if it's a three dimensional space just any whatever you are trying to graph, this differential equation just describes, a set of solutions. Whereas this [$y = e^t$] is a function it is a solution, this [$y'' = y' + y$] is a relationship.

25 G: Ok cool. You had said that this was a solution to a differential equation.

26 J: Ok, yeah.

27 G: How do you think about solutions to differential equations as being different to solutions to equations like this [$y = mx + b$]?

28 J: Ok, well, if we are looking at a solution to a differential equation, we're usually looking at some sort of function that satisfies this. Like I'll go back to populations cause that's really the only context I have ever had for differential equations, is a couple populations follow this general guideline of behavior, we can find a solution that is, that would represent the specific behavior of a population over time. Whereas looking at just an equation, you plug in an $x$, you get a $y$ out. At time equals 2 seconds there are whatever, the ball is 10 feet in the air.

In turn 20 Jordan noted that differential equations correlate a function and its derivative, whereas regular equations are related, yet separate from their derivatives. Additionally he suggested that differential equations were not functions noting, "a differential equation is a relationship that can be satisfied by functions." I asked him to elaborate on this distinction and in turn 22 he expressed the resources *rates indicate behavior* and *equality condition* when he said:

> "Um this [points to $y' = 2y + 6$] is a relationship of other functions, so like $y$ is a function I am assuming $y$ is a function, and the $y'$ is a characteristic of that function itself. This [points to $y'$] being its slope. It's what functions can satisfy these constraints."

Here Jordan used *rates indicate behavior* to explain that "$y'$ is a characteristic of [the] function itself", and *equality condition* to describe differential equations as constraint that only certain functions satisfy. I then asked him if differential equations could be graphed (this seemed to be an additional distinction he made in turn 20), and in his reply in turn 24 suggested that it was possible to graph solutions to differential equations but not the differential equation itself. It is important to note that in his reply Jordan equated solutions to differential equations with functions, and the equation $y = mx + b$ with being a function as well; this is important for understanding what Jordan meant in turn 14 concerning independent and dependent variables, which becomes clearer in light of turn 28. In turn 28 when he specifically addressed how he thought about solutions to differential equations as opposed to standard equations. More specifically he indicated that differential equations provided a "general guideline of behavior", and that their solution functions represent specific behavior. In light of turn 28 independent and dependent variables detail the value of specific quantities (e.g., "At time equals 2 seconds there are whatever, the ball is 10 feet in the air"; turn 28), where as differential equations are "general guideline[s] of behavior." This helps illuminate the meaning behind Jordan's statement in turn 14 when he was discussing the lack of dependent and independent variables in a differential equation. Namely, Jordan is expressing that differential equations correlate a set of values (what a function's "current state is and any number of other factors"; turn 8) to a behavior, whereas a regular equation or a function is a correlation between two values (“At time equals 2 seconds there are whatever, the ball is 10 feet in the air”; turn 28).

In summary, Jordan described differential equations as conditions between a function’s value and its behavior. He used the resources *functional dependence, rates indicate behavior* and *equality condition* to provide this description and described $y$ as the value of a function that determined $y'$, the behavior of $y$. In addition he expressed three distinctions between functions and differential equations: 1) Functions are relationships between the value of two quantities, where as differential equations are relationships between a quantity and its behavior, 2) functions can be graphed, but only solutions to differential equations can be graphed, and 3) functions have independent and dependent variables but differential equations do not. Based on these distinctions, it seems that for Jordan differential equations are not functions because of the of the differences between what the mathematical relationships correlate, not how they correlate them; functions correlate values of quantities, where as differential equations correlate the value of a quantity to its behavior.

**4.2.5.2 Interview 5 task 2.** This task asked students to find a differential equation that had the solution $y = 6$. While completing this task in Interview 5 Jordan did not seem to recall his first interaction with this task (T4-I1). While completing this task Jordan provided two differential equations each having the solution $y = 6$. Each of the differential equations seemed to be the result of two separate interpretations. To generate the first Jordan interpreted differential equations as a description of the behavior of the quantity of interest, and to generate the second, he seemingly interpreted differential equations as a relationship between the value of a quantity and the value of the quantities rate of change. To create the first differential equation $y' = 0$, Jordan utilized the resources *functional mapping*, *rates indicate behavior* and *equality* *condition.* Jordan coordinated *functional dependence, the definition of equilibrium solution,* and *equality condition* to generate the second differential equation $y' = y - 6$. The following segment of transcript shows Jordan constructing both differential equations.

112 J: All righ [writes $y' = 0$].
113 G: Ok, can you...
114 J: That's what I am gonna go with.
115 G: Ok, tell me how you came to that conclusion.
116 J: Ok. So the differential equation, back to what we started with $y'$ and $y$. Base, first order basic differential equation. solution, is a solution to where the derivative of $y$ with the behavior of $y$ fits into the description of the differential equation, the bounds. So, we have $y = 6$, ok $y = 6$, I'll put that on a $t, y$ plane just to visualize it. $y = 6$, ok, well, the slope doesn't change. So, that's nice. So, if I solve this out, I don't know, I've never solved $y' = 0$, I'm sure you'd solve it the same way you would solve a normal one. But, it [$y' = 0$] fits the behavior no matter, I could make it a zero $y$ just to make it look more like a differential equation even though it is the same thing, its ah, the slope is gonna be zero. I could make it a little bit more legitimate looking like if I did more like a $\frac{dy}{dt}$ equal let's see, $6 - y$. I think that would work. Cause that would be a solution to this, I mean that was just the same thing that was on here with this question. But it's anything that this line would fit the bounds. so any differential equation that has an equilibrium of $y = 6$.
117 G: so you pointed to this one [$y' = y - 6$] and said that this was the same as what's on here [Slope field A T4-I2].
118 J: Yeah.
119 G: Could you tell me a little bit about what you meant or how they are the same.
120 J: Ok, so let's look at $A$, $A$, I mean these all represent different solutions, that's what the phase plane [sic, pointed to slope field] does. Well if you look at the equilibrium solution that's just the line $y = 1$, yes all of these other lines work and satisfy $1 - y$, the easiest one and the equilibrium solution is this $y = 1$ cause at $y = 1$, $\frac{dy}{dt}$ equals zero according to this, always. And on the line $y = 1$, $\frac{dy}{dt}$ equals zero always. So.
121 G: how is that the same as what's over here?
122 J: so with this, this would give you a very similar phase plane [sic] to that. You'd have kind of an equilibrium solution and it would kinda [draws] do this kinda thing. And this would be solution at $y = 6$. Cause at $y = 6$, $\frac{dy}{dt}$ is gonna be 0, which means as time increases your never going to change your $y$ which means your gonna end up with this line $y = 6$. So, its kind of, you can go from here to there and there back to here and...

In turn 112 Jordan (correctly) declared the differential equation $y' = 0$ has the solution $y = 6$. In response to my clarifying question, in turn 116 he graphed the line $y = 6$ "on a $t, y$ plane just to visualize it," and noted that the slope did change. He then said he could solve the differential equation $y' = 0$ to show the line $y = 6$ was a solution, but that he did not need to because "it fits the behavior, no matter" (turn 116). Here he utilized *rates indicate behavior* when explaining the differential equation matched the behavior of $y = 6$, presumably meaning that the equation $y' = 0$ corresponds to the line $y = 6$ because the slopes do not change. To construct the differential equation he utilized *functional mapping* to correlate points on the function with the value of its slope, from which he inferred the slope did not change. Noting that the slope was always zero, he constructed a differential equation that reflected this. In doing so he utilized *equality condition*, in this case however, *equality condition* was not applied to the differential equation itself, but to the qualitative characteristics $y' = 0$ provided concerning the behavior its solution function. More specifically Jordan used *equality condition* when justifying the differential equation fit the behavior of $y = 6$; "it [$y' = 0$] fits the behavior no matter." In this case Jordan interpreted the differential equation as describing the behavior of the quantity of interest.

After determining that $y' = 0$ was a solution, and seemingly realizing this equation did not fit with the characterization he provided in Task 1, (needing both a $y'$ and a $y$), he said that he could also make it $y' = 0y$ "just to make it look more like a differential equation even though it is the same thing, its ah, the slope is gonna be zero." He then provided the "more legitimate" (turn 116) differential equation $y' = y - 6$, noting that any differential equation with the equilibrium solution $y = 6$ would work. After discussing how this equation was similar to one from another task, in turn 122 he drew the slope field for $y' = y - 6$ (see Figure 4.43). He then said this was the case because "at $y = 6$, $\frac{dy}{dt}$ is gonna be 0," suggesting that he determined $y' = y - 6$ because it satisfied the condition of $y'$ being 0 when $y$ was 6. This reasoning process suggests that Jordan used *functional dependence* to get an equation where the value of $y'$ was determined by the value of $y$, *the definition of equilibrium solutions* to infer that $y'$ is always zero at $y = 6$ (he used this exact language in turn 120 when noting $y = 1$ was an equilibrium solution for the equation $y' = 1 - y$; "the equilibrium solution is this $y = 1$ cause at $y = 1$, $\frac{dy}{dt}$ equals zero… always"), and then coordinated these resources with *equality condition* to generate an equation where the value of $y'$ depended on the value of $y$, such that $y' = 0$ when $y = 6$. In this way, the focus was not on generating a differential equation that described the behavior of the solution functions, but rather on generating a differential equation that satisfied the condition of having an equilibrium solution at $y = 6$. In this way he interpreted the differential equation as a relationship between the value of a quantity and the value of the quantity's rate of change.

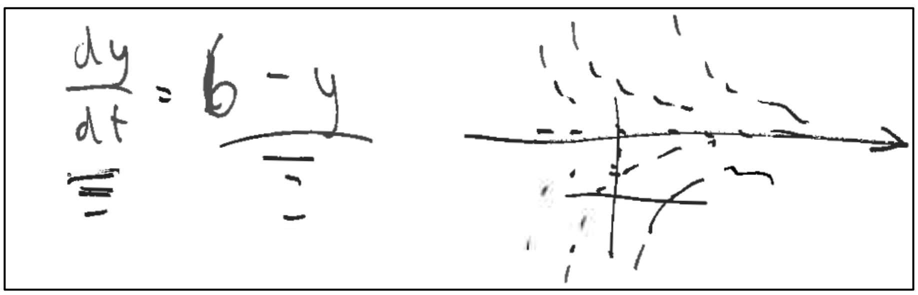



*Figure 4.43: Jordan justified $y = 6$ is a solution to $\frac{dy}{dt} = 6 - y$ because $\frac{dy}{dt} = 0$ when $y = 6$.*

In short, Jordan used *equality condition* and functional mapping to build a differential equation that provided a description of a function whose behavior matched that of $y = 6$. The resource functional mapping was used to infer that the slope was always zero on the line $y = 6$, and *equality condition* was utilized in the process of generating a differential equation that correctly described the behavior of the line $y = 6$. In which case Jordan interpreted the differential equation as a description of the quantity of interest. In addition to this, Jordan volunteered an alternative differential equation which he constructed using functional dependence, the definition of equilibrium solutions and *equality condition*. By coordinating these three resources Jordan was able to construct a differential equation where the value of $y'$

depended on the value of $y$, such that the value of $y'$ was 0 when the value of $y$ was 6. In doing so Jordan interpreted the differential equation as a relationship between the value of the quantity and the value of the quantity's rate of change.

**4.2.5.3 Interview 5 task 3.** Task 3 comprised of an initial value problem in the form of a system of two first order differential equations (see figure 4.44). Jordan quickly solved the system by re-writing it into a matrix equation and then utilizing the *method of eigenvalues.* While completing this task Jordan expressed that $y_1$ and $y_2$ were functions that satisfied the system of differential equations and that $y_1{}'$ and $y_2{}'$ indicated how the functions behaved. In addition Jordan utilized *the method of eigenvalues* and *equality condition,* to find the solution and then discuss what it meant to be a solution to the differential equation. In doing so he interpreted the differential equation as something to be solved.

Solve the following system of differential equations:

$$y_1{}' = y_1 - 3y_2$$
$$y_2{}' = y_1 + 5y_2 \quad \text{where } y_1(0) = 1 \text{ and } y_2(0) = 1.$$

*Figure 4.44: Task 2 from Interview 5.*

Jordan read the task noted, "ok, so it wants me to solve it" (turn 33), and went to work solving the system. He re-wrote the system as a matrix equation and then started talking through the solution process (transcript follows). His work can be seen in Figure 4.45

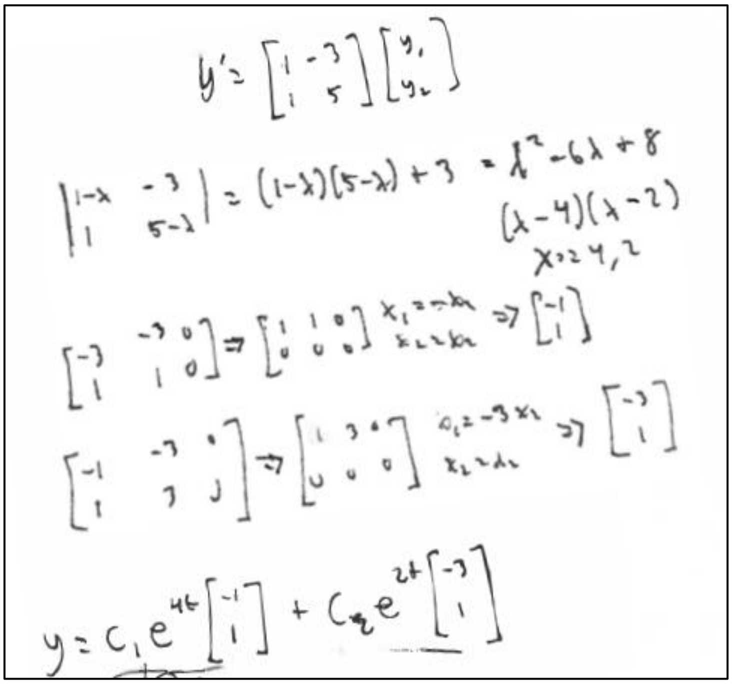


*Figure 4.45: Jordan's work solving the initial value problem.*

37 J: Um I guess I should be talking you through what I am doing. So putting it in a matrix, finding the eigenvalues...

38 J: So, I got the eigenvalues, go back through it again get the eigenvectors [solves for eigenvectors, then uses eigenpair to write the general solution of the system].

39 G: Ok, um. So what does it mean to be solution to this system of differential equations?

40 J: Um, the behavior of $y_1$ [underlined first term in the solution] and $y_2$ [underlined second term in the solution] um, have the relationship defined by this differential equation.

41 G: Ok, what is that relationship?

42 J: Oh actually they gave us initial values, I can do that later if you want but, it's I don't really know what this relationship is, but it is this is the system that describes how these two variables, how these two functions, interact with each other and change as their current state changes.

43 G: Ok, how do you think about $y_1$ and $y_2$?

44 J: Um, $y_1$ and $y_2$ are both functions that satisfy like, this together. It's, it's the set of two functions that will work.

45 G: Ok, how about $y_1{}'$ and $y_2{}'$, how do you think of those?

46 J: It's um, so this is more where we started getting into the phase plane stuff, I guess is how they showed it that I don't really understand.

47 G: Ok

48 J: But its ah, it's basically how the ah, the different functions behave, as, usually we are throwing it in with time, so its how these two functions are going to behave and like what they are going to trend to.

49 G: Ok. When you are talking about, I know you said that you don't really understand the phase plane, could you tell me how those things are related to the phase plane though.

50 J: Um the phase plan is kind of like a projection of how these ah, this $y_1$ and this $y_2$. I guess these aren't really $y_1$ and $y_2$, these both... but if I did separate this out into $y_1$'s and $y_2$'s, how those $y_1$'s and $y_2$'s are gonna interact with each other.

Jordan quickly and systematically completed this task using the *method of eigenvalues* as can be seen in both the transcript (turns 37 and 38), and his work. This is the same process Jordan outlined while working on Task 3 from Interview 4, but was unable to complete. In turn 39 I asked him what it mean to be a solution to the system, Jordan's reply was indicative of the resource *equality condition;* "the behavior of $y_1$and $y_2$ um, have the relationship defined by this differential equation" (turn 40). Namely, Jordan expressed that solutions have to satisfy the relationship present in the differential equation. When asked what that relationship was, Jordan was unable to explain it, but he knew that the system described how the "two variables [points to $y_1$ and $y_2$], how these two functions, interact with each other and change as their current state changes" (turn 42) Further, when I asked him how he thought about $y_1{}'$ and $y_2{}'$, he said they both had to deal with the phase plane, in that they showed the behavior of $y_1$ and $y_2$as they interacted with one another (turns 48 and 50).

In this case Jordan interpreted the differential equation as an equation to be solved. This is not solely based on the fact that Jordan solved the initial value problem; the task prompted him to do just that. My inference is based on two main factors. First while Jordan stated that the differential equation contained a relationship between how the two functions interacted he did not know what this relationship was. Additionally he attributed this relationship to being depicted in the phase plane, and referred to $y_1$ and $y_2$ as "functions that will work." Second, he didn't use this relationship to complete the task but solved the differential equation and used the existence of a relationship in the differential equation to discuss the significance of $y_1$ and $y_2$.

In short, Jordan used *the method of eigenvalues* to complete the task, and *equality condition* to discuss how the solution he found satisfies the condition set by the differential equation. He expressed $y_1$ and $y_2$ represented the solution “functions that will work” and that $y_1{}'$ and $y_2{}'$ were used in the phase plane which represented how the two solutions behaved as they interacted with one another. In doing so he Jordan interpreted the system of differential equations as a system to be solved.

**4.2.5.4 Discussion of interview 5.** Over the course of Interview 5 Jordan utilized three distinct sets of resources each corresponding to a different way of interpreting the differential equations and its components (see Table 4.11). Here I discuss Jordan’s interaction with Tasks 1 and 2, starting with Task 2.

Jordan reasoned about Task 2 in two ways. Initially Jordan utilized a set of resources that enabled him to reason about the behavior of the function $y = 6$ to construct a differential equation which fit the behavior of $y = 6$, namely $y = 0$. By interpreting the differential equation as a description of the behavior of the quantity of interest, and $y'$ as the behavior of the quantity, he was able to use *rates indicate behavior* to infer $y' = 0$ and coordinate this with *equality condition* to verify the differential equation reflected the behavior of the solution function. He then switched reasoning to determine a “more legitimate” differential equation. This was seemingly the result of considering a differential equation as a relationship between the value of a quantity and the value of the quantity’s rate of change. Namely he realized the differential equation $y' = 0$ did not reflect such a relationship, and began to use resources which supported his ability to construct such an equation. Namely he used functional mapping and the *definition of equilibrium solution* to correlate each point on the line $y = 6$ with $y' = 0$. Then, using *equality condition* he constructed a differential equation which reflected this correlation. In this case his interpretations of the differential equation and the resources that supported each interpretation afforded him the ability to determine two differential equations. In addition the interpretations influenced the form of the differential equations he solved. That is $y' = 0$ reflected the behavior of the solution $y = 6$, and the differential equation $y' = y - 6$ reflected the relationship between the values of the function and the corresponding slope values.

The statements Jordan made while completing Task 1 concerning the differences between functions and differential equations were rather intriguing. Mainly because over the course of the five interviews, there were points where Jordan contradicted these statements (including two times in Interview 5). Here I attempt to provide an explanation for this in terms of the way he expressed interpreting differential equations in Task 1; as a description of the behavior of the quantity of interest. Recall that Jordan made the following distinctions between differential equations and functions:

1) Functions are relationships between the value of two quantities, where as differential equations are relationships between a quantity and its behavior
2) Functions can be graphed, but only solutions to differential equations can be graphed,
3) Functions have independent and dependent variables but differential equations do not.

Given that he was interpreting the differential equation as a description of the behavior of the quantity of interest, and that he kept referring to $y'$ as the behavior of $y$, it seems plausible that Jordan was not interpreting $y'$ in the differential equation he provided, $y' = 2y + 6$, as a value. In fact Jordan did not use the word value in reference to $y'$ while completing the task. With this in mind, it seems viable that the differential equation did not relate two quantities, didn't have a dependent variable ($y'$ was a behavior), and thusly couldn't be graphed. While this explanation is somewhat unverifiable, it is commensurate with the reasoning and interpretations he expressed while completing Task 1.

*Table 4.11: Resources and interpretations Jordan expressed in Interview 5.*

| Task | Interpretation of the differential equation | Resources | Entities |
|---|---|---|---|
| I5-T1 | As descriptions of the behavior of the quantities of interest | *Functional dependence*<br>*Rates indicate behavior*<br>*Equality condition* | $y$ – Function of $t$<br>$y'$ – Behavior of $y$ |
| I5-T2 | As descriptions of the behavior of the quantities of interest | *Functional mapping,*<br>*Rates indicate behavior*<br>*Equality condition* | $y$ – Function of $t$<br>$y'$ – Behavior of $y$ |
| | As a relationship between the value of the quantity and the value of the quantity's rate of change. | *Functional dependence*<br>*Definition of equilibrium solution*<br>*Equality condition* | $y$ – Value of the function<br>$y'$ – Value of the rate of change |
| I5-T3 | As an equation to solved | *Method of eigenvalues*<br>*Equality condition* | $y_1$, $y_2$, $y_3$ – "functions that will work" |

## 4.3 Conclusions

I conclude this chapter by discussing the various roles of the individual resources in the students' completion of the tasks and drawing connections between the resources and current literature on student understanding of function, rate of change and differential equations. In doing so I provide further validation of the resources I identified, and I construct an argument for which resources relate to student understanding of function and rate of change. In addition by relating the resources to findings from the literature I outline how taking a Knowledge in Pieces perspective can both support and illuminate these findings.

This discussion culminates in answering the first research question by identifying the resources the students used relating to function and rate of change. I reserve the presentation of the analysis regarding changes in the resources and how they interacted to support the students' understanding of differential equations for Chapter 5. It is noteworthy that although the discussion that follows characterizes the various ways in which the students used the individual resources, the characterization is not all-inclusive; rather, my goal in this section is to generalize the students' usage of each resource within the different interpretations of the differential equations. It should also be noted that what follows artificially separates the resources from the reasoning patterns in which they were applied. Although this sacrifices a description of the ways each of the students used specific sets of resources to complete the tasks, it explicates the different support each resource provided regarding the different ways the students interpreted the differential equations.

### 4.3.1 Comparison of Resource Utilization Between Hakeem and Jordan

In this section I describe some of the similarities and differences with regard to the combinations of resources and interpretations Hakeem and Jordan expressed while completing the same tasks. There were three main ways the students approaches differed across the tasks with regard to the interpretations and resources: sometimes they expressed the same interpretations but different sets of resources, sometimes they expressed different interpretations but utilized many of the same resources, and at other times they expressed different interpretations and different resources. To explicate this I compare their utilization of resources across I2-T1, I3-T2, and I3-T3. These tasks were chosen for the discussion in this section because they highlight the various combinations of resources and interpretations Hakeem and Jordan each expressed while completing the different tasks. More specifically, to complete I2-T1, Jordan and Hakeem expressed different interpretations yet somewhat overlapping resources. With regard to I3-T2, both the interpretations and resources utilized by each student were vastly different, and to complete I3-T3 Jordan and Hakeem each expressed the same interpretations of the differential equation but did not use the same sets of related resources.

**4.3.1.1 Different interpretations - overlapping resources.** Hakeem interpreted the differential equations as a description of the behavior of the quantities of interest. In addition he thought of the variables $x$ and $y$ as the population values of the respective species, which were inputs in the differential equation, and he thought of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as the value of the rates of change of the species, which were outputs in the differential equation. Additionally he utilized the following resources to complete the task*: larger magnitudes have larger impacts, parts contribute to the whole, functional mapping, rates indicate behavior, unrestricted change the definition of cooperative systems, positives add- negatives take away, functional mapping,* and *the definition of competitive systems.*

Hakeem used the resources *larger magnitudes have larger impacts, parts contribute to the whole* and *functional mapping* when reasoning with System A to relate the values of the populations to their respective rates of change. He then used *rates indicate behavior* and *functional mapping* to determine the how the size of the populations related to their behavior. He then used *unrestricted change* to infer that populations would have oscillatory behavior indicating the growth of one species is beneficial to the other species, which was consistent with *the definition of cooperative systems*. When reasoning with System B Hakeem used the resources *partial dependence, positives add – negatives take away,* and *functional mapping* to determine that the growth of one species was harmful to the other species. He then used *the definition of competitive systems* to characterize the relationship between the two populations as competitive.

Jordan used *partial dependence* to determine which variables from each of the differential equations to consider. Specifically, to detremine how the change in one population impacted the rate of change of the other, he inferred that he only needed to consider the $y$ variables in the equations for $\frac{dx}{dt}$ and only the $x$ variables in the $\frac{dy}{dt}$ equations. He then coordinated the resources *functional variation* and *positives add–negatives take away* to determine how increases in one population related to changes in the other. For example, knowing that coeffeceient in front of the $y$ term in the equation $\frac{dx}{dt} = 2y$ is positive, Jordan was able to infer that increases in $y$ would relate to an increase the rate of change of $x$. Lastly Jordan used his *definition of competing systems* to infer that System B was competing because "as $y$ increases, $\frac{dx}{dt}$, so the change in $x$ decreases, so that would be a competitive nature." In addition he utilized $x$ and $y$ as population values, and $\frac{dx}{dt}$ and $\frac{dy}{dt}$ as the rate of change of the population, which were themselves composed of the sum of a birthrate and death rate. Based on the resources Jordan utilized and the ways in which he utilized the various entities in the differential equation, he interpreted the differential equation as a relationship between the value of quantities and the values of their respective rates of change. The resources utilized by both students can be seen in Figure 5.3.

| Hakeem: I2-T1 | | Jordan: I2-T1 | |
|---|---|---|---|
| Interpretation | Resources | Interpretation | Resources |
| Description of the behavior of the quantity of interest | Definition of competitive systems<br>Definition of cooperative systems<br>Functional mapping<br>Functional variation<br>Larger magnitudes have larger impacts<br>Parts contribute to the whole<br>Positives add- negatives take away<br>Rates indicate behavior<br>Unrestricted change | Relationship between the value of quantities and the value of their respective rates of change. | Definition of competing systems<br>Functional variation<br>Partial dependence<br>Parts contribute to the whole<br>Positives add – negatives take away |

*Figure 4.46: Resources and interpretations expressed while completing I2-T1.*

Comparing students', Jordan and Hakeem expressed different interpretations of the differential equation but utilized many of the same resources to attain different goals. Jordan used resources that supported him in constructing relationships between the values of the populations and their rates of change, from which he directly made inferences about their competing or cooperative nature. Hakeem, on the other hand, constructed relationships between the values, to then make inferences about the interrelated changes in the population values. From the behavior of the populations Hakeem determined which system represented cooperative species and which represented competing species. Additionally, the resources *functional variation*, *parts contribute to the whole* and *positives add - negatives subtract* supported both students abilities to relate the value of quantities to the value of their rates of change.

**4.3.1.2 Different interpretations - different resources.** At the beginning of his interaction with I3- T2 Hakeem reasoned with the relationships between $P$, $P'$ and $P''$ to construct the behavior of $P$. This was indicative of interpreting the differential equation as a description of the behavior of the quantity of interest. To determine the behavior of $P$, he used the resources *functional variation, parts contribute to the whole,* and *positives add - negative take away* to determine how changes in the values of $P$ and $P'$ related to changes in the values of $P''$ and then made use of *rates indicate behavior* and *unrestricted change* to relate the values of $P''$ and $P'$ to the behavior of $P$. As a result, he determined that $P$ would oscillate between positive and negative values. In this case he utilized $P$, $P'$, and $P''$ as dynamic values in the differential equation values to infer the behavior of $P$, which represented, as he said in the beginning of the interview, "anything, and it's changing with respect to I guess some other variable" (turn 60).

Immediately after determining the behavior of $P$, Hakeem switched interpretations and made use of the *method of characteristics* to find the roots of the characteristic equation $\lambda^2 + 6\lambda + 5 = 0$ and used them to determine the solution function $P(t) = C_1e^{-t} + C_2e^{-5t}$. He then used *functional variation* to make inferences about the behavior of $P(t)$ as $t$ approached infinity. In this case he did not explicitly utilize the relationships between the variables; rather he utilized their coefficients to construct the characteristic equation. In addition, $P(t)$ was a result of the solution method, and his attention was not focused on $P'$ or $P''$. In this case Hakeem interpreted the differential equation as a providing a model of the quantity of interest.

Jordan expressed that $P$ represented the value of a quantity, $P'$ represented its rate of change, and $P''$ was the rate of change of the rate of change. In addition he noted that other than knowing it was an equation for something, and without having a situation to relate it to, it's just something "I'll sit down and I'll solve" (turn 133). Based on how persistently Jordan expressed an absence of meaning in the differential equation, how quick he was to point out that he could solve it, and the fact that he did not discuss or utilize a relationship between the variables in the differential equation or the solution he found he interpreted the differential equation as something to be solved. While engaged in the solution process, he utilized the *method of characteristics* to find the roots of the characteristic equation, and then put those roots into a known form of the solution. He then coordinated the *definition of general solution* and *dependence* to reason that any solution to the differential equation would have the form $y = C_1e^{-5t} + C_2e^{-t}$, and that one could find a particular solution by using the initial conditions to determine the exact values of $C_1$ and $C_2$.

With regard to I3-T2, the interpretations and resources expressed by the each of the two students were vastly different (see Figure 5.4). Hakeem expressed two interpretations: one while reasoning about the nature of the value of $P$ and one while reasoning about algebraic form of the solution to the differential equation. In addition he used information gained from both sets of resources to determine the long-term behavior of the value of $P$. Jordan, on the other hand, expressed resources that supported that interpretation in that they provided him with the ability to analytically solve the differential equation. The main difference in the approaches between the students was that Jordan did not determine the long term behavior of the solution he found.

| Hakeem: I3-T2 | | Jordan: I3-T2 | |
|---|---|---|---|
| Interpretation | Resources | Interpretation | Resources |
| Description of the behavior of the quantity of interest | Functional variation<br>Parts contribute to the whole<br>Positives add - negative take away<br>Rates indicate behavior<br>Unrestricted change | As an equation to be solved. | The method of characteristics<br>Definition of general solution<br>Dependence |
| Equation providing a model of the quantity of interest | Method of characteristics<br>Functional variation | | |

*Figure 4.47: Resources and interpretations expressed while completing I3-T2.*

**4.3.1.3 Same interpretations – different resources.** While completing task I3-T3, Hakeem interpreted the differential equation in two ways: as a description of the behavior of the quantity of interest and as providing a model of the quantity of interest. These two interpretations are signified by the two approaches Hakeem took to predict the long-term behavior of the owl populations and the distinct set of resources associated with the approaches. While interpreting the differential equation as a description of the quantity of interest Hakeem utilized *variable substitution, rates indicate behavior, functional mapping, parts contribute to the whole,* and *larger magnitudes have larger impacts* to determine the behavior of the quantity of interest. Namely he used *variable substitution, functional mapping, parts contribute to the whole,* and *larger magnitudes have larger impacts* to determine how the values of $P$ related to $P'$, and then *rates indicate behavior* to determine that $P$ would generally decrease, unless $P$ was small. To determine the exact behavior of $P$, Hakeem used an analytical solution method to find the general solution. Once Hakeem attained the general solution, he used *dependence* and *function slots* to place the initial conditions into the general solution and find the solution pertaining to the specific initial conditions. He then applied *functional variation* to the solution function to determine the behavior of $P(t)$ as $t$ approached infinity. Thus, his first approach provided him with general characteristics of the behavior of $P$, (it would decrease except for small values of $P$), and his second approach allowed him to determine the exact nature of the populations for each initial condition.

While in the process of completing this task Jordan's reasoning is indicative of interpreting the differential equation as a description of the behavior of the quantity of interest. Jordan initially attempted to solve the differential equation analytically to find the algebraic form of the differential equation, presumably to use it to graph the population with respect to time. However he was unsure of how to solve the differential equation analytically due to the non-linear term, but after some prompting, Jordan performed a single iteration of *Euler's Method.* His utilization of *Euler's method* consisted of the resources *variable substitution* and *positives add-negatives subtract*. Specifically he used *variable substitution* to determine the value of $\frac{dP}{dt}$ for a specific population value, and *positives add- negatives subtract* to calculate an approximate value of the population after a certain amount of time. After this iteration he then focused on using the slope values to determine the behavior of the solution function; that is, he then reasoned with the relationship between the value of $P$ and the value of $P'$ in the differential equation itself to infer the long term behavior of the owl population. He did this by coordinating *functional variation*, *functional mapping* and *rates indicate behavior*. More specifically, he considered how the value of $\frac{dy}{dt}$ changed as the value of $y$ increased, used this dynamic set of values for $\frac{dy}{dt}$ to infer the behavior of the solution function by correlating the value of $\frac{dy}{dt}$ to values of $P$ on the solution function. After constructing the graph of the solution function, Jordan used his *definition of general solution* to infer that if the initial population was 5, the graph would be the same as initially having a population of 3, because "it's the same relation" (turn 216).

Both Jordan and Hakeem expressed the same two interpretations while completing this task; however, the resources used in support of each interpretation were quite different (see Figure 5.5). The students each used different sets of resources to attain the same goals; both Jordan and Hakeem sought to relate values of time to the value of the owl population, but each needed to use the relationship in the differential equation to determine the long term behavior of population. To complete each of these goals, the students used vastly different knowledge resources.

| Hakeem: I3-T3 | | Jordan: I3-T3 | |
|---|---|---|---|
| Interpretation | Resources | Interpretation | Resources |
| Description of the behavior of the quantities of interest | Functional mapping<br>Larger magnitudes have larger impacts<br>Parts contribute to the whole<br>Rates indicate behavior<br>Variable substitution | Description of the behavior of the quantities of interest | Definition of general solution<br>Functional mapping<br>Functional variation<br>Rates indicate behavior |
| Providing a model of the quantity of interest | Definition of general solution<br>Dependence<br>Function slots<br>Functional variation | Providing a model of the quantity of interest | Euler's method<br>Positives add – negatives take away<br>Variable substitution |

*Figure 4.48: Resources and interpretations expressed while completing I3-T3.*

**4.3.1.4 Discussion of the differences.** These examples are representative of the ways the students' utilization of the resources differed. In some instances the students' approaches to the tasks were different because the objectives they associated with the tasks differed; for example in the case of I3-T2 Hakeem was trying to determine the long term behavior of the value of $P$, whereas Jordan was trying to find the function that satisfied the differential equation. At other times the students used different combinations of resources to complete similar objectives. For example in the case of I2-T1 both students sought to understand the nature of the relationships between the species represented in both systems. To do this Jordan was able to reason directly from the relations between the values of the populations and the values of their respective rates of change, where as Hakeem used these same relations to determine the behavior of the population values. From this behavior he was able to make inferences about the nature of the relationships between the species. It is important to note that despite the differences in the combinations of resources, overall the solutions the students attained were commensurate. In addition, their interactions with the tasks were composed of the same over arching set of resources; namely, those located in the Table 4.1.

In addition, though there were instances in which the students both expressed many of the same resources to complete a certain task, there was not a single instance in which they utilized exactly the same set of resources. The existence of differences in the ways students interpret and complete certain tasks is expected within the Knowledge in Pieces perspective. Being rooted in Radical Constructivism, the construction of knowledge is considered to be idiosyncratic; a process that occurs based on ones own experiences and the meaning they make of those experiences. As a learner encounters different situations and comes to associate certain knowledge as productive in those situations they begin to organize that knowledge in their own unique ways; no one shares exactly the same experiences. Along these lines, Wagner (2010) noted, “any assumption that we as individuals or as a community all come to see the things we classify as ‘the same’ by use of the same interpretive mechanisms seems dubious” (p. 450). Though I make no claims about both Hakeem and Jordan classifying the tasks as “the same,” the fact that the students, even while interpreting the differential equations in the same way, did not use the same resources to complete a single task supports his claim.

### 4.3.2 Discussion of Hakeem’s and Jordan’s resources

To help provide a cleaner organizational structure for their presentation, I start the discussion of the resources with those that were utilized in fewer instances and transition to those that were identified more frequently. As such, I begin the discussion with *definition of competing systems, definition of competing systems, continuity, carrying capacity,* and *partial dependence,* as these resources were identified in a small subset of the tasks.

Over the course of the five interviews Hakeem and Jordan utilized the *definition of competing systems* and the *definition of cooperative systems*, rather unsurprisingly, strictly for the competing and cooperative species task (T1-I1, T1-I2 and T1-I3). It is worthy to note, however, that their applications of these resources differed. While both Jordan and Hakeem used these resources to characterize the interaction of two species represented in two different systems of equations, Jordan consistently applied this resource in consideration of how the value of the population of one species related the value of the rate of change of the other species (e.g., how values of $x$ related to values of $\frac{dy}{dt}$). Hakeem, on the other hand, strictly applied this resource to the nature of the relationship between the respective population values (e.g., how changes in sets of values of $y$ related to changes in sets of values of $x$). In this way the resources were only applied to the behavior of the quantities for Hakeem, and the relationship between the value of the quantity and the value of the quantity's rate of change for Jordan.

In addition, it is worthy to note that the resource *definition of competing systems* developed as a result of the students' problem solving activity with regard to the competing and cooperative species task. Both students upon their first interaction with the task expressed that they expected both populations to have negatives rates of change if they were competing species; Jordan noted their rates of change could be small or negative, and Hakeem expressed their rates of change would both be negative, but this was not fruitful for either student. They then transitioned to considering changes in the growth (or decline) of both of the species' populations as a result of their interaction. Jordan used the values of the respective populations' rates of change to consider the changes in the growth (or decline) of the two species, whereas Hakeem used changes in the population values. That is, they transitioned from separately considering the populations' changes to considering how the population value of one species related to the other.

The resource *continuity* was utilized solely by Hakeem in the tasks that involved the graph of $\frac{dy}{dt}$ versus $y$ (T2-I4 and T3-I5). In both cases he used this resource in reference to drawing solutions on a vector field to explain that he could find slope values at any point, and that those slope values were changing in a smooth fashion. In addition, Jordan utilized the resource *carrying capacity* only once, in task T1-I3 (small predator - large prey, and large predator - small prey) to infer that the behavior of the prey he was considering represented a large species because the value of their population was limited by some factor. In this case Jordan asserted that the behavior indicated by the differential equation represented that of a large species, because large species tend to have factors that limit the size of their populations. Lastly, both Jordan and Hakeem used the resource *partial dependence* exclusively in the competing and cooperative species task. This resource supported the students in completing the task by allowing them to look at how the two species interacted. More specifically, by only considering certain relationships between variables in the differential equation they could focus on how the two species interacted, as opposed to, in Jordan's words, considering "how a population affects its own rate" (turn 76; T1-I4). In the case of Jordan and Hakeem, these resources (*definition of competing systems*, the *definition of competing systems, continuity, partial dependence* and *carrying capacity*) were utilized very specifically to accomplish very specific goals.

Both Jordan and Hakeem utilized the resource *equality condition* while reasoning about differential equations and their solutions throughout the interviews. In general this resource supported the students in determining and justifying why proposed solutions were actually solutions to a differential equation. Hakeem consistently applied this to a relationship between a function and its derivative function, such as in T4-I5, where Hakeem noted that $y = 6$ is a solution to the differential equation $y' = 0$ because the derivative of $y = 6$ is $y' = 0$. Jordan also used *equality condition* in this manner but displayed more variation regarding the interpretations of the components to which he applied the resource. Jordan used *equality condition* to: relate the behavior of graphed solution functions to the behavior described by the differential equation (e.g., $y = 6$ does not change value, and $y' = 0$ means the solution's value doesn't change; T2-I5), discuss solution functions retaining the equality in the differential equation when substituted into the differential equation (being "functions that will work," turn 44; T3-I5), and correlate the value of the quantity and the rate of change of the quantity (e.g., $y' = y - 6$ requires solutions to have a slope of zero when $y = 6$; T2-I5).

These cases of the students' utilization of *equality condition* can be linked to two categories within Raychaudhuri's (2008) framework concerning student thinking about solutions to differential equations: solutions as either an entity with a certain form or as a object that satisfies the differential equation. The latter characterization consists of three images: the differential equation as an object that is generated by the process of taking derivative of the solution function, the solution function as an object that is generated by the process of integrating the differential equation, and the solution as a object that satisfies the differential equation as a result of the defining process. Raychaudhuri's framework would characterize Hakeem's response as indicating he saw the differential equation as an object that is generated by taking the derivative of the solution function. In the case of Jordan using *equality condition* to relate solutions to being "functions that work," he would be categorized as using the defining process of solutions as objects that satisfy the differential equation. In both of these cases the identification of the resource *equality condition* and the students' interpretations of the components that were supported by *equality condition* parallel the Raychaudhuri's framework.

*Equality condition* was not the only resource utilized by Jordan and Hakeem in the process of justifying solutions or creating correspondences between values. Jordan used the *definition of equilibrium solutions* to make inferences about the relation between the value of a quantity and the value of its rate of change in ways that supported him in productively reasoning about solutions. While matching the vector field in T3-I2 and creating a vector field from which to reason in T2-I4, Jordan used the *definition of equilibrium solutions* to match the appropriate differential equation to the slope field and graph the behavior of solutions, respectively. More specifically, knowing that the value of $y'$ was identically 0 for certain values of the quantity (due to the existence of equilibrium solutions) supported Jordan in appropriately selecting matching equations for the provided slope fields in T3-I2, and in determining the behavior of the quantity of interest in T2-I4. In both cases it was the correlation between the value of the quantity and the value of its rate of change that allowed Jordan to productively reason about the task.

In this paragraph I address the five resources concerning the various methods students utilized while completing the tasks (*method of characteristic equations, method of integrating factor, method of eigenvalues, Euler's method*, and *method of separation of variables*). While each of these resources is distinct in that they represent a different algorithm the students used while reasoning about the various tasks, I collectively characterize them here as having one of two goals; solving the (systems of) differential equations, or finding equations that relate the quantity of interest to an independent variable (the model the differential equation provided; I will elaborate on this difference in Chapter 5). Jordan's and Hakeem's application of these resources was seemingly related to the form of the differential equation, "cause then there is a method that always works …then I can just use a go to method like integrating factor" (Hakeem, in reference to why it was important for him to note that he was working with a linear first order differential equation, Interview 1, turn 123). Jordan made similar comments, for instance when completing Task 2 from Interview 1, he said he would solve the differential equation by "put[ing] it in the standard thing [form] then, integrating factor" (turn 98). This is not surprising because each of the methods (with the exception of Euler's method) is generally presented as being associated with differential equations of certain forms. For instance, the method of characteristic equations is applicable to equations of the form $ay''(t) + by'(t) + cy(t) = 0$, and the method of separation of variables is applicable to differential equations of the form $\frac{dy}{dt} = g(y)h(t)$.

Hakeem and Jordan each utilized *opposite operations reverse* to infer that integration techniques were appropriate for solving differential equations (Jordan, T4-I2; Hakeem, T3-I1) because integrating reverses differentiating, and leaves just the original function. For instance Jordan noted "Basically the whole point of what we are doing up to this point is to find a way to integrate to get rid of this $y'$, to put things all in terms of this $y$" (turn 247, Interview 2), and Hakeem said "the integral is just the anti-derivative and in a differential equation you have a derivative the $\frac{dp}{dt}$, so to get rid of that you would take the integral of it, and it becomes a function of just $t$" (turn 125, Interview 1). The utilization of *opposite operations reverse* with regard to anti-differentiation and differential equations is linked to Raychaudhuri's (2008) finding that students think of solutions to differential equations as being generated by the process of integration. From this, the students' utilization of the resource *opposite operations reverse* with respect to anti-differentiating the derivative is commensurate with the findings of previous researchers of student learning in differential equations.

Hakeem and Jordan typically made use of the *definition of general solution* and *dependence* when interpreting the differential equation as an equation to be solved or as providing a model of the quantity of interest. In general the *definition of general solution* provided the students with a way, after applying one of the algebraic solution methods, to infer the algebraic form of all of the solutions to a differential equation. Additionally, *dependence* provided the students with a way to use the equation resulting from the solution process (the general solution) to find the particular equation that related to a certain set of initial conditions. Collectively, by inferring the form of the solutions and their dependence on a set of initial conditions, the students could find solutions specific to a provided set of initial conditions (e.g., T4-I1, T3-I3). While one may argue this sequence in the students' problem solving activity was the result of following a memorized sequence of steps, I argue this is not the case; "the $C$ in the general solution, it [the general solution] could represent any possible solution depending on what the initial value is which determines this $C$" (Jordan, turn 263, Interview 2). Here Jordan is using the *definition of general solution* to justify the process of using the general solution.

These resources were also used in conjunction with other interpretations of the differential equation. For instance Jordan used the *definition of general solution* to (incorrectly) infer that the behavior of the owl population (T3-I3) represented in the graph he constructed for the initial condition $P(0) = 3$,was indicative of the behavior of the population with an initial condition of $P(0) = 5$. His reasoning was that all of the solutions would have the same form. In addition, when interpreting he differential equation as a relationship (T4-I1) between a function and its derivative function Hakeem used *dependence* to infer that the derivative of a function was determined by the function itself; in other words, he reasoned that different functions have different derivatives.

Over the course of the five interviews, Jordan and Hakeem each used the resource *rates indicate behavior* in roughly half of the tasks they completed. In the vast majority of the instances in which this resource was utilized, it was done in conjunction with interpreting the differential equation as a description of the behavior of the quantity of interest. *Rates indicate behavior* had two main roles with regard to supporting Hakeem and Jordan in describing the behavior of the quantity of interest. First, and foremost, the students used this resource to make inferences about the direction and magnitude of a quantity's change. For instance, by considering the value of the rate of change of a quantity (either in the form of the slope of a tangent vector or derivative (function) value), the students could make inferences about whether the quantity was increasing or decreasing, and how fast. Second, by associating the value of the rate of change with tangent vectors, the students were supported in building graphical representations of the behavior of the solutions. This was particularly useful for them while working with slope fields because they would utilize the tangent vectors as indicating the direction in which a solution was moving at a particular instance and could thusly use the tangent vectors to draw the solution functions.

In addition, though there were relatively few of these instances, Jordan utilized the resource *rates indicate behavior* while interpreting the differential equation as a relationship between the value of a function and the value of the function's rate of change. By looking at the behavior of a graph (e.g., not changing) at certain points Jordan could correlated the values represented by those points with values of the rate of change. For instance while completing T4-I1 (finding a differential equation with the solution $y = 6$), Jordan graphed the line $y = 6$, noted that the value of the function didn't change, and from that determined that $y' = 0$ when $y = 6$. In this case *rates indicate behavior* was used to infer the value of $y'$ based on the behavior of the solution function, which in turn was used to construct a differential equation which had the condition $y' = 0$ when $y = 6$ ($y' = y - 6$).

These ideas are commensurate with multiple findings from previous research concerning students' utilization of rate of change in differential equations. For instance, Rasmussen and Whitehead (2003) noted that students used rates as quantities that determine the behavior of a function. This is precisely how Hakeem and Jordan utilized the value of the rate of change when interpreting the differential equation as a description of the behavior of the quantity of interest; For instance when discussing Task 1 during Interview 4 Jordan noted, "um the rate of change of species $x$ will be greater so it'll grow faster or it will die slower either one", and Hakeem while completing Task 2 during interview one noted, " so, $\frac{dp}{dt}$ dictates whether $P$ is just increasing or decreasing, or how much its increasing how much its decreasing." These statements were paradigmatic of their reasoning with the resource *rates indicate behavior* while interpreting the differential equations as a description of the behavior of the quantity of interest.

A finding that resonates with the reasoning Jordan and Hakeem expressed while using *rates indicate behavior* to complete various tasks during the series of interviews is that of Keene (2008) who noted that students used the sign of the rate of change to determine if solutions were increasing or decreasing. Hakeem was explicit about while completing the competing and cooperative species task during Interview 4 "what's gonna happen is that $\frac{dx}{dt}$ will be positive and $\frac{dy}{dt}$ will be negative so $x$ will start increasing and $y$ will start decreasing"(turn 10). In addition Jordan expressed a similar idea while completing Task 2 from Interview 2 when he noted that if " $P''$ is positive and $P'$ is positive and if $P$ initial is positive, there is nothing bringing it [makes motion tracing over graph of $P$] back it will just continually grow" (turn 105). That is he used the positives rates of change to justify that the solution would "continually grow" and not decrease.

I now discuss the following seven resources: *functional dependence, functional mapping, functional variation, larger magnitudes have larger impacts, parts contribute to the whole,* *positives add-negatives subtract,* and *variable substitution*. Over the course of the five interviews, both Jordan and Hakeem utilized some combination of these seven resources in the process of completing nearly every task. In some cases the students used these resources to reason about the relationship between values in the differential equation, and in other cases they used the resources to reason about the relationship between values in the solution functions. Based on the nature of the usage of these resources, I organize them into two sets: (a) resources that supported the students in perceiving (or recognizing) and utilizing the relations between the components in the differential equation (i.e., relationships between the value of quantities and the values of the rate of change of the quantity of interest and a function and its derivative function), and (b) resources that supported the establishment of a specific characterization of that relation (e.g., as the value of quantity increases the value of the quantity's rate of change decreases or when the quantity takes on a certain value its rate of change is a certain value). I explicate this in the following paragraphs where I discuss how certain combinations of these resources are directly related with the principles underlying various findings regarding students' understanding of function. Before drawing connections between the function literature and the resources, however, I first briefly outline some of the ways students understand functions and reason with functions.

Summarizing the work of multiple researchers (e.g., Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson, 1998; Dubinsky & Harel, 1992), Carlson, Oehrtman, and Engelke (2010) describe two ways in which student understanding of function can be categorized; as having an *action view of function* and as having a *process view of function*. Having an *action view of function* is characterized as an operational notion where, generally speaking, symbolic procedures are used to evaluate a function at a specific value. Students with this view of function tend to place emphasis on determining evaluated values of functions as opposed to interpreting a function as a mapping between values in a domain set and values in a range set. Students with a *process view of function* can envision how a continuum of values being input into the function result in a continuum of values as outputs. In addition Carlson et.al (2010) discuss *covariational reasoning,* a type of reasoning required while considering the dynamic nature of the input and output values of a function. More specifically, the authors note that *covariational reasoning* regards considering " how one variable changes while imagining successive amounts of change in the other" (p. 117).

In this paragraph I provide explanatory examples of the students utilization of the resources in (a) that supported the students in perceiving and utilizing the relations between the components in the differential equation: *functional dependence, functional variation,* and *functional mapping*. Generally speaking the students used these three resources to relate values of a quantity to the value of its rate of change. For instance in T2-I2 Jordan used *functional dependence* to determine that in the equation $P'' = 3P' + P$, the value of $P''$ depended on the value of $P'$ and $P$ and that $P$ depended on the value $t$. In this same task, with regard to the same equation, Hakeem used *functional variation* to infer that changes in $P$ and $P'$ related to changes in $P''$. Further, both students used *functional mapping* to infer that the values of $P$, $P'$ and $P''$ all correlated to the same point in the $P, t$ plane on the solution $P(t)$; each student drew a set of slope vectors representing $P'$ at specific values of $P$ and discussed the role of the value of $P''$ (at each point) in changing the value of $P'$ (see Figures 4.8 and 4.29). In short, these resources supported the students in perceiving and utilizing the defined relationship between the components in the differential equation in different ways. Namely by using *functional dependence*, the students could infer that the value of one quantity determined another. By using *functional variation* the students could infer that changes in the value of one quantity related to changes in the value of another quantity, and similarly *functional mapping* supported the students in inferring that sets of values of one quantity correlated to sets of values of another.

I now address the resources in category (b), resources that supported the establishment of a specific characterization of the relations they perceived between the components in the differential equation; l*arger magnitudes have larger impacts, parts contribute to the whole, positives add-negatives subtract*, and *variable substitution.* The students' utilization of these resources was not always explicit, especially towards the end of the series of interviews. That is to say, the students' observable actions and statements did not always plainly indicate their use of these resources. I believe this was because the students were able to more easily recognize the nature of the relationships defined by the differential equation as their experience grew, and as a result they did not need to verbalize or otherwise express the utilization of the resources. Nevertheless, the students used these resources to determine exactly how certain values depended on others (e.g., specific values of $\frac{dx}{dt}$ given specific values of $x$ and $y$), how changes in in one value related to changes in other values (e.g., how the direction and magnitude of changes in one value related to the direction and magnitude of change in another value), and how certain sets of values were correlated to other values (e.g., how $t, y$ pairs were mapped to values of $\frac{dy}{dt}$.) In the following paragraph I furnish a description of the support these resources provided in the form of paradigmatic examples. These examples explicate how the resources in category (b) were utilized in support of the resources in category (a). It should be noted however, that individual resources in category (b) are not restricted to supporting specific resources in category (a). That is to say, while the examples demonstrate how the students used different combinations of resources from each category to attain certain goals in certain tasks, other combinations were used as well. The examples I provide also demonstrate the connections to having an action view of function, process view of function, and using covariational reasoning.

The resource *variable substitution* was often utilized in conjunction with *functional dependence* to characterize exactly how a specific value of one variable (or set of variables) related to the specific value of another. For instance while working on the competing and cooperative species task during Interview 1, Jordan, aware that the values of $x$ and $y$ determined the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ (*functional dependence*), substituted the value of 10 into the differential equations for both $x$ and $y$ to determine specific values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$. In this case, Jordan utilized *variable substitution* to find specific values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ for specific values of $x$ and $y$. This is commensurate with how Carlson, Oehrtman, and Engelke (2010) describe an *action view of function;* as being dominated by an image of procedures used to evaluate a function at a specific value. In this example Jordan did exactly this, he substituted 10 into the differential equation for $x$ and $y$, and calculated the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$. In this case, as a result of utilizing the resources *functional dependence* and *variable substitution* Jordan displayed an action view of the differential equation by evaluating the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ by using 10 as an input value for both $x$ and $y$. Here I expand beyond an action view of *function* to an action view of the *differential equation* because the differential equations (at least those the students encountered during the interviews) can be thought of as functions that relate values or as equations that relate functions. This raises some interesting questions that warrant further investigation. Namely, since differential equations relate functions, and also relate evaluated values of functions, it may be the case that having an action view of differential equations exists in two layers, actions on functions and actions on their evaluated values.

The students also displayed reasoning that was commensurate with having a process view of function. For instance while completing Task 2 from Interview 2, recall that Jordan reasoned with the relationship between values of $P$, $P'$ and $P''$ as defined by the differential equation $P'' = 3P' + P$ to determine the behavior of the solution functions. Specifically he used *functional mapping* to establish the correlation between the values of $P$, $P'$ and $P''$ (as triplets) and *positives add – negatives subtract* to further define the nature of correlation by inferring that positive values of $P$ and $P'$ related to positive values of $P''$ (he then used *rates indicate behavior* to infer that all of these values would be increasing, for a positive $P$ and $P'$). While completing the task he related a continuum of increasing values of $P$ and $P'$ to a continuum of values of $P''$. Further he then correlated the each of the sets of related values of $P$, $P'$ and $P''$ to individual points on the solution curve over a continuum of $t$ values. This can be seen in the graph he constructed (see Figure 4.48), and his associated transcript from Interview 2, turn 129:

129: …This number change [$P'$] and concavity [$P''$] you can put at every point relating to $t$ and you would just get, you would get this graph [marks points with slopes and concavities, then draws a new graph, see Figure 4.48]

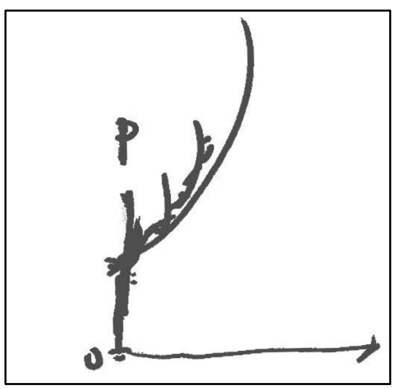



*Figure 4.49: Jordan Coordinating values of $P$, $P'$ and $P''$*

It is important to note that there are two aspects of his reasoning that are similar to having a process view of function. First he envisioned how $P''$ changed over a continuum of input values for $P$ and $P'$. Second he envisioned the dynamic triplets of values $(P, P', P'')$ as each being related to a specific value of $t$, but over a continuum of $t$ values. In this case the utilization of *functional mapping* and *positives add – negatives take away* is commensurate with having a process view of the differential equation. Again, I expand beyond a process view of *function* to a process view of a *differential equation* because Jordan used the differential equation to relate the values $P, P'$, and $P''$ over a continuum of values, but also used the notion that these were functions to relate the triplets over a continuum of $t$ values.

While completing Task 2 in Interview 3, Hakeem used *functional variation* and *positives add - negatives take way,* and in doing so, displayed level 2 covariational reasoning ("coordinating the direction of change of one variable with changes in the other variable" p. 118, Carlson et al., 2010) while working with the differential equation $P'' = -6P' - 5P$.

72 H: So here, $P$ represents anything and it's changing with respect to I guess some other variable, so it could be like a population changing over time. So, ah, and I guess what this equation represents is that, so P" is I guess decreasing as I guess P' and P are increasing.
73 G: Ok
74 H: And I guess what that means is that if $P''$ is decreasing as $P'$ and $P$ are increasing, by $P''$ decreasing, would cause I guess $P'$ and $P$ to both decrease. And when both of these are negative $P''$ would be positive. So it's a, it cycles back and forth.
75 G: Ok, what is the 'it' that cycles back and forth?

In turn 60, Hakeem displayed covariational reasoning when he coordinated the changes in both $P$ and $P'$ with the changes in $P''$. He reasoned that increases in both $P$ and $P'$ would relate to decreases in $P''$, and from this he was able to then determine the behavior of $P'$ and $P$. Inferring values of $P''$ changed with the values of $P$ an $P'$ (*functional variation*) and using *positives add negative subtract* to make inferences about how changes in $P$ an $P'$ related to changes in $P''$ is commensurate with mental action 2 within covariational reasoning. This mental action is defined as "Coordinating the direction of change of one variable with changes in the other variable" (p. 118, 2010).

Lastly the students' utilization of the resources in categories (a) and (b) can be linked to the work of Donovan (year) who used Sfard's reification theory to discuss student understanding of differential equations as an object, namely as functions. Under Sfard's reification theory there are three requirements for an having an object conception: "seeing the same object under different disguises" (Sfard, 1992, p. 76), seeing that the various representations are linked by a unifying abstract construct (Sfard, 1991), and being able to recognize the abstract quality present in the various representations quickly, without performing calculations. Donovan noted the first requirement was satisfied when the students attend to the relationship between the function's value and its rate of change across multiple representations. The second requirement was met if students could easily transition between the symbolic representation of a differential equation and its graphical representation. Lastly students that could quickly identify the same differential equation across multiple representations they were said to meet the last requirement.

The resources in categories (a) and (b) are related to Donovan's characterization of understanding the differential equation as an object (function). The resources *functional dependence*, *functional variation,* and *functional mapping* were utilized by Hakeem and Jordan across multiple representations of the differential equation including vector fields, graphs of $\frac{dy}{dt}$ with respect to $y$, and algebraic representations. These three resources supported the students in perceiving and utilizing the relationships between the value of the quantities and the values of their rates of change. I take the students' utilization of these three resources across the multiple representations to characterize the relationship between the values of the quantities and their rates of change, as Donovan did, as evidence of meeting he first requirement.

With regard to the second requirement, transitioning between representations, both Hakeem and Jordan did this frequently when constructing vector fields from either a graph of $\frac{dy}{dt}$

with respect to $y$ (such as Hakeem did while completing T2-I5) or an algebraic equation (such as Jordan did while completing T4-I4). By utilizing combinations of resources from category (a) and category (b), the students could characterize the relationship present in one representation and reconstruct it in a new representation; often this new representation afforded some information that the previous one did not. Lastly I take the fact that the students began constructing these different representations while making their utilization of the resources less obvious as evidence of the students being able to recognize the same differential equation in different representations without calculations. In this way, the Knowledge in Pieces perspective provides more mechanisms through which one can characterize conceptualizing a differential equation as an object. And as such it provides a finer level of detail at which researchers can describe and explain how such an understanding can develop. In the case of Jordan and Hakeem, by coordinating the resources in categories (a) (resources that supported the students in perceiving (or recognizing) and utilizing the relations between the components in the differential equation), and (b) (resources that supported the establishment of a specific characterization of that relation across different representations) they were able to work with the relationship between the quantities and their rates of change in ways that were commensurate with having object understanding of the differential equation.

Based on the connections to the literature as outlined above, I conclude that the resource *rates indicate behavior* concerns student understanding of rate of change, and *functional dependence, functional mapping, functional variation, larger magnitudes have larger impacts, parts contribute to the whole, positives add-negatives subtract,* and *variable substitution* concern student understanding of function. It should be noted that the resources *equality condition,* the *definition of general solution,* and *opposites cancel* were also tied to the literature related to function because these resources are connected with student understanding of solutions, and in differential equations solutions are functions. As a result I leave open the possibility for other resources to be tied to function (and rate of change) within the topic of differential equations.

In conclusion, in this section I discussed the different ways the students used the individual resources. In addition I tied these resources to students understanding of function, rate of change and differential equations. While doing so I provided additional support for the identified resources and used them to briefly outline how documented understandings related to these topics could manifest themselves though the utilization of the resources. The next chapter in the dissertation consists of describing how the resources changed over time and how they interacted to support the students understanding of differential equations (Chapter 5).

# CHAPTER 5: CHANGES IN THE UTILIZATION OF RESOURCES AND THE INTERACTION BETWEEN FUNCTION AND RATE OF CHANGE

In this chapter I address the second and third research questions concerning how the students' utilization of the resources changed as they progressed though their differential equations course, and how the students' resources concerning function and rate of change influenced one another during the development of their understanding of differential equations. I answer these questions by making use of and building on the analysis presented in Chapter 4. Whereas Chapter 4 discussed the students' responses to individual tasks as single instances, the analysis presented in this chapter concerns identifying differences and similarities in the interpretations and resources the students expressed while completing the various tasks over the series of interviews. To do this I use both the sets of resources the students used to complete the tasks and the interpretations of the differential equations they expressed while doing so.

To answer the second research question, I present analysis concerning similarities and differences in the interpretations expressed by the students and the resources they utilized in conjunction with the interpretations over the series of interviews. More specifically I discuss changes in the ways the students interpreted the tasks and changes in the sets of resources the students used within the interpretations. The interpretations provide insight into the students' perceived utility of the resources for a certain task, the goals the students aimed to accomplish, and the mathematical entities (e.g., values and functions) they reasoned with to accomplish those goals. The resources represent the knowledge the students coordinated with the information from the task to accomplish these goals. By analyzing for similarities and differences in the sets of resources used in conjunction with the same interpretations, I was able to determine how the resources developed over time to provide the students with different ways of accomplishing the same goals. In addition, by examining how each of the student's interpretations within related tasks changed over the series of interviews, I was able to discuss how the sets of resources were utilized over time with regard to certain tasks. Note that I do not explicitly compare across interpretations in this chapter because the analysis presented in concluding section of Chapter 4 describes how the different sets of resources were utilized across interpretations.

To answer the third research question, which regards how the resources for function and rate of change interact, I present analysis concerning the ways in which the resources for function and rate of change came together while the students expressed interpreting the differential equations in the same way. That is, I discuss how the resources for function and rate of change supported the students' abilities to complete various tasks centered around the different interpretations. I do this by making use of the analysis presented in Chapter 4 and identifying regularities in the ways the resources for function and rate of change came together to support the students' ability to complete the tasks in which they expressed the same interpretations. As presented in Chapter 4, the different interpretations account for the different sets of resources utilized by the students when completing the tasks. In this sense they provide a student-centered way of explaining the various and sometimes contradictory ways in which the students reasoned to complete tasks. In addition, they highlight some of the different aspects of the students' understanding of the rather expansive notion of a differential equation, and illuminate the specific aspects the students utilized while completing certain tasks.

## 5.1 Changes in the Students' Utilization of the Resources

Over the series of interviews Hakeem expressed five different ways of interpreting the differential equations, and Jordan expressed a total of four of those same five interpretations. Though it is not surprising that the different interpretations were related to different sets of resources that supported the students in completing different goals, there was a significant amount of variation in the sets of resources the students used to complete tasks while expressing the same interpretations. Specifically, in the instances in which the students interpreted the differential equations in the same way across tasks, the sets of resources they utilized to complete those tasks, even across instances in the same interview, were not exactly the same. In this section I compare across these tasks and discuss the similarities and differences in the sets of resources the students utilized within these interpretations, and I discuss instances in which the students expressed different interpretations for the same or similar tasks between subsequent interviews. In doing so I show how the students' utilization of the resources changed over time and thusly explicate their systematization and reorganization. Further, in the process I delineate the differences between the interpretations and outline the salient features of each of the interpretations with regard to their relation to certain resources.

### 5.1.1 Hakeem

In this section I present analysis comparing and contrasting the resources Hakeem utilized across different tasks. As the analysis that follows will show, generally speaking Hakeem's utilization of the resources changed in two ways: (1) as the series of interviews progressed, sets of resources were able to be used across a wider array of tasks; and (2) sets of resources related to different interpretations became associated with one another in ways which provided support or justification of the attainment of certain solutions. For instance, to complete some of the tasks Hakeem leveraged different sets of resources and interpretations to glean different information from the same differential equation to construct a solution or provide an alternative explanation for his reasoning; over time, Hakeem no longer needed to leverage two interpretations to glean different information to complete the task because one interpretation and set of resources provided him with the ability to do so.

Using the tables in Chapter 4 that summarized the resources and interpretations Hakeem expressed while completing each of the tasks in the series of interviews (namely, Tables 4.2, 4.3, 4.4, 4.5 and 4.6), I constructed Figure 5.1. Recall from Chapter 4 the five interpretations are: Description of the behavior of the quantity of interest, Relationship between the value of the quantity and the value of the quantity's rate of change, and Relationship between the a function and the function's derivative, Equation to be solved, and Equation that provides a model of the quantity of interest. These are referred to respectively as Description of Behavior, Relation Between Values, Relation Between Functions, Equation, and Model in Figure 5.1. The figure shows the interpretations Hakeem expressed while completing each of the tasks. The cells on the main diagonal represent expressing a single interpretation and the cells on the off diagonal represent expressing multiple interpretations. The tasks listed in each cell indicate the interview and task for which those interpretations were expressed. For instance, the "I3-T1" in the "Description of Behavior" row and "Description of Behavior" column indicates during Interview 3 while completing Task 1 Hakeem interpreted the differential equation as a Description of the behavior of the quantity of interest. The "I5-T1" in the "Relation Between Values" row and "Description of Behavior" column indicates that Hakeem expressed two while completing Task 1 from Interview 5: (1) Relationship between the value of the quantity and the value of the quantity's rate of change and (2) a Description of the behavior of the quantity of interest.

It is important to note that even though Hakeem expressed the same interpretation(s) while completing tasks within the same cells, the sets of resources utilized for those individual tasks were not necessarily identical (this can be seen by examining across Tables 4.2 – 4.6 in Chapter 4). Therefore, though Figure 5.1 allows for comparing and contrasting interpretations expressed across the tasks and interviews, it does not provide exact details concerning the changes in Hakeem's utilization of certain resources within the interpretations. To determine exactly how Hakeem's utilization of resources changed over time, I compare the sets of resources utilized to complete the tasks in the same cells over the interviews, and identify instances in which Hakeem expressed different interpretations for the same or similar tasks from one interview to the next (for instance while completing I2-T2 Hakeem interpreted the differential equation $P' = 3P' + P$ as a description of the behavior of the quantity of interest, but while completing I3-T2, a similar task, Hakeem expressed the additional interpretation the differential equation $P'' = -6P' - 5P$ as an equation which provided a model of the quantity of interest). In the following paragraphs I discuss the changes in the utilization of the resources by analyzing the tasks across subsequent interviews.

| Hakeem | | | | | |
|---|---|---|---|---|---|
| Interpretation | Description of Behavior | Relation Between Values | Relation Between Functions | Model | Equation |
| Description of Behavior | I1-T1<br>I2-T1, I2-T2<br>I3-T1<br>I4-T1<br>I5-T3 | | | | |
| Relation Between Values | I5-T1 | I2-T3<br>I5-T5 | | | |
| Relation Between Functions | | I4-T3<br>I5-T4 | I1-T4 | | |
| Model | I3-T2, I3-T3<br>I4-T2<br>I5-T2, I5-T1 | | | I1-T2 | |
| Equation | | | | | I1-T3 |

*Figure 5.1: Interpretations Hakeem expressed for each task.*

Before analyzing across interviews, I briefly remind the reader of the results of the analysis of the tasks from Interview 1. Recall from the analysis presented in Chapter 4 that Hakeem encountered four tasks in Interview 1 and interpreted the differential equations in each of those tasks differently. To complete task (I1-T1) he utilized resources that enabled him to reason with the relationship between variables in the differential equation to construct a description of the behavior of the two species in each of the systems of differential equations. While discussing the differential equation P'=3P (I1-T2), Hakeem used resources which enabled him determine the solution function that related values of $P$ to specific values of an independent variable (in this case $t$), and to discuss how the value of $P$ would change as $t$ changed by using that same function. In Task 3, when he encountered a first order differential equation said to model a fish population and was asked how one could use the differential equation to find the fish population at a given time, Hakeem made use of resources that enabled him to analytically solve the differential equation for a function that satisfied the differential equation. Lastly, in Task 4 (I1-T4), Hakeem interpreted the differential equation as relationship between a function and its derivative function and used resources that enabled him to find a differential equation that represented the relationship between the solution $y = 6$ and its derivative function $y' = 0$. The cumulative list of resources supporting each interpretation expressed by Jordan during interview 1 can be seen in Table 5.1. In Chapter 4 I discussed similarities and differences in the Hakeem's utilization of the resources across these four tasks; I now compare his utilization of the resources to those he used while completing the tasks in Interview 2.

In Interview 2 Hakeem encountered three tasks. The first (I2-T1) was identical to the competing and cooperative species task from Interview 1 (I1-T2). The second task (I2-T2) was a variation of I1-T2 that involved a second order differential equation as opposed to the first order differential equation he encountered in the first interview. Lastly, the third task was novel in that it asked Hakeem to match a set of vector fields to corresponding differential equations.

*Table 5.1: Resources supporting the interpretations Hakeem expressed in Interview 1.*

| Interpretation of the differential equation | Resources supporting the interpretation | |
|---|---|---|
| Description of the behavior of the quantity of interest | Functional dependence<br>Functional variation<br>Functional mapping<br>Rates indicate behavior<br>Parts contribute to the whole | Positives add- negatives take away<br>Larger magnitudes have larger impacts<br>Definition of cooperative systems<br>Definition of competing systems |
| Equation providing the model of the quantity of interest | Rule for differential equations of the form $P' = kP$<br>Rates indicate behavior | Larger magnitudes have larger impacts<br>Function slots<br>Functional dependence |
| Equation to be solved | Functional dependence<br>Dependence | Method of integrating factors<br>Opposite operations reverse |
| Relationship between a function and the function's derivative. | Functional dependence<br>Dependence | Equality condition<br>Opposite operations reverse |

During his second encounter with the competing and cooperative species task Hakeem interpreted the differential equations in the same way he did during his first encounter, as a description of the behavior of the quantity of interest, and utilized many of the same resources. The resources *functional mapping, functional variation, larger magnitudes have larger impacts, parts contribute to the whole, positives add-negatives take away, rates indicate behavior,* the *definition of cooperative systems* and the *definition of competing systems* were all utilized in the first and second encounter; additionally each was utilized in the same manner they were during the first encounter. Specifically he used *functional mapping* and *functional variation,* in conjunction with *larger magnitudes have larger impacts, parts contribute to the whole and positives add-negatives take away* to relate population values of species $x$ and species $y$ to values of their respective rate of change, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ based on the relationship defined by the differential equation, and then *rates indicate behavior* to determine how the values of the populations were changing for specific sets of values of $x$ and $y$. Based on the behavior of the species, he was then able to characterize the two species as competing or cooperative based on the *definition of* *competing systems* and *the definition of cooperative systems*.

There were slight differences in the resources he used across the two interviews however; namely, Hakeem did not use *functional dependence* in the second encounter (or at least make the utilization observable), and he made use of *unrestricted change*, a resource that he did not express during his first encounter. These changes in the resources he utilized afforded Hakeem the ability to focus on the behavior of the populations over a longer period of time. Whereas in Interview 1 Hakeem made inferences concerning the behavior of the populations for specific conditions with regard to their population values (e.g., large or small), by incorporating *unrestricted change* into the set of resources he utilized in Interview 2 Hakeem was able to make inferences about how the populations would transition from being large to being small. For example, in Interview 2 Hakeem noted that the populations would "keep going back between $x$ being big and $y$ being small and then $y$ being big and $x$ being small" (turn 8), which was an inference he made by using *unrestricted change* – one he did not make during Interview 1. More specifically in Interview 1, Hakeem only used *rates indicate behavior* to discuss the behavior of the populations at instances in which the populations were large or small, but in Interview 2 he used *unrestricted change* to build on these inferences to discuss the behavior of the population values transitioning between being large and small.

This difference between the sets of resources may seem insignificant, however, this is indicative of the systematization of the knowledge resources Hakeem was applying to complete the task; he no longer needed to use *functional dependence* to relate values of $x$ and $y$ to values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$, rather he only expressed applying *functional variation* to consider the respective changes in the values. In addition, he incorporated *unrestricted change*, which supported him in discussing how the values of the population transitioned from large to small and small to large.

Lastly, the other resources and their roles in the solution process remained the same, indicating they still had the same utility for Hakeem, for this particular task.

While completing task I2-T2, which presented a homogeneous second order differential equation, Hakeem interpreted the differential equation as a description of the behavior of the quantity of interest. I consider this task to be a variation of I1-T2, which presented a first order homogeneous differential equation, but Hakeem did not express the same interpretation across the two tasks. Rather, he interpreted the differential equation in I2-T2 the same way he did the competing and cooperative species task from Interviews 1 and 2. To complete I2-T2 Hakeem made use the resources *functional mapping, functional variation, parts contribute to the whole, rates indicate behavior, unrestricted change* and *variable substitution.* With the exception of *variable substitution*, this collection of resources is a subset of those he used while completing the competing and cooperative species task during Interview 2. Further, the resources shared across the two tasks (I2-T1 and I2-T2) were utilized in the same ways. In particular, Hakeem used *functional mapping, functional variation,* and *parts contribute to the whole* to use the relationship defined by the differential equation to determine how values of $P$, $P'$ and $P''$ were related; and *rates indicate behavior* and *unrestricted change* to discuss how the value of $P'$ changed based on the value of $P''$ and how the value of $P$ changed based on the value of $P'$. In this case Hakeem incorporated the resource *variable substitution* which supported his ability to consider the different cases of the values of $P$ and $P'$:

> "But if I started plugging in… so you have $P$ is positive and $\frac{dP}{dt}$ is positive, $P''$ has to be positive. …But say both were negative… also this [$P''$] would have to be negative… say this [$P'$] was positive and this [$P$] was negative but the total equation was such that $P''$ was positive…" (turn 111, Interview 2).

Based on this the second order differential equation $P'' = 3P' + P$ presented in the second interview was interpreted differently than the first order differential equation $P' = 3P$

presented in the first interview. It was, however, interpreted the same way as the differential equations presented in the competing and cooperative species task. This indicates that despite the significant differences in the tasks (I2-T1 and I2-T2), the circumstances Hakeem perceived in the two tasks were associated with similar sets of resources, though each task required slight variations in the exact set of resources utilized in order to accommodate for slight perceived differences in the tasks. For instance, I2-T1 required the use of the *definition of competing systems* and the *definition of cooperative systems*, and I2-T2 required the use of *variable substitution* to reason about the different combinations of values of $P$ and $P'$.

While completing the last task he encountered in Interview 2 (I2-T3), which asked Hakeem to match vector fields to their respective differential equations, Hakeem expressed a new interpretation of differential equations; as relationships between the value of quantities and the value of the rate of change of the quantity of interest. Though this was a different interpretation from those he expressed before, the resources he utilized overlapped with those in I2-T1 and I2-T2. More specifically, the common resources were *functional mapping*, *functional variation*, and *parts contribute to the whole*, which were utilized to determine relationships between pairs of values of $t$ and $y$ and the value of $\frac{dy}{dt}$. In the case of I2-T3 the use of the common resources demonstrates these resources were able to support different interpretations. The cumulative list of resources supporting each interpretation expressed by Jordan during interview 1 can be seen in Table 5.2. Having discussed changes in Hakeem's utilization of the resources between Interview 1 and Interview 2, I now discuss the interactions from Interview 3.

*Table 5.2: Resources supporting the interpretations Hakeem expressed in Interview 2.*

| Interpretation of the differential equation | Resources supporting the interpretation | |
|---|---|---|
| Description of the behavior of the quantity of interest | Functional dependence<br>Functional variation<br>Functional mapping<br>Rates indicate behavior<br>Parts contribute to the whole | Positives add- negatives take away<br>Larger magnitudes have larger impacts<br>Definition of cooperative systems<br>Definition of competing systems<br>Unrestricted change<br>Variable substitution |
| Relationship between the value of a quantity and the value of the quantity's rate of change | Functional dependence<br>Functional variation | Functional mapping<br>Parts contribute to the whole |

Hakeem completed 3 tasks in Interview 3: The first task (I3-T1) was novel and was a variation of the competing and cooperative species task, which involved determining the size of the predators and prey in two different systems of equations. The second task was similar to I2-T2 in that it asked Hakeem to discuss what $P'' = -6P' - 5P$ meant to him. The last task presented Hakeem with a nonlinear differential equation with two different initial conditions and asked Hakeem to determine the long-term behavior of the owl population modeled by the differential equation for each initial condition.

While completing the first task Hakeem interpreted the differential equation as a description of the behavior of the quantity of interest, utilized many of the resources he used while completing the tasks from Interview 2 that shared this interpretation, and used them in much of the same way. Specifically, Hakeem made use of *functional variation* and *functional mapping* in conjunction with *parts contribute to the whole, positives add- negatives take away*, and *larger magnitudes have larger impacts* to relate (changes in the) values of $x$ and $y$ to (changes in the) values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$. He then used *rates indicate behavior* and *unrestricted change* to determine how the population values of $x$ and $y$ changed based on the values of $\frac{dx}{dt}$ and

$\frac{dy}{dt}$. From this he was able to determine which species was predator and which was prey (in this case, $x$ represented prey and y represented predators in both systems); however, this set of resources did not enable Hakeem to determine the size of each species. He utilized *partial dependence* to consider how values of $x$ impacted values of $\frac{dy}{dt}$ across both systems: if $x$ had a large impact on the value of $\frac{dy}{dt}$ he inferred $x$ was large, and if $x$ had a small impact on the value of $\frac{dy}{dt}$ he inferred $x$ was small. Hakeem had not expressed *partial dependence* in any of the previous interviews, and he seemingly incorporated it into this particular task for the specific purpose of determining the size of the species.

When in the process of completing the second and third tasks Hakeem expressed the same combination of interpretations of the differential equations: as a description of the behavior of the quantity of interest, and as an equation that provides a model of the quantity of interest. This was the first time Hakeem expressed multiple interpretations of a differential equation within a single task. This is significant in terms of changes in the resources as Hakeem associated the two interpretations and their respective sets of resources; both were perceived as being productive within the demands of the individual tasks. Further, to adequately describe the solution, Hakeem leveraged the sets of resources related to each interpretation to attain different information, which he then combined to complete the task. To explore the utilization of the two distinct interpretations and sets of resources, consider the following statement Hakeem made while completing the second task:

> "So, ah well I guess this equation [points to the solution $P(t) = C_1e^{-t} + C_2e^{-5t}$] and the differential equation in general are both just ways to represent the way that $P$ is I guess changing over I guess time. So, what this equation, this [points to $P(t) = C_1e^{-t} + C_2e^{-5t}$] equation I guess is a little more specific in which in what is happening to $P$, but it's also kind of incomplete right because there is two constants here, that you have to figure out, and you can't do that without a initial

values" (turn 82, Interview 3).

Based on this statement, Hakeem viewed both the differential equation and the solution function as ways to describe changes in $P$ with regard to changes in $t$, showing he in fact associated both interpretations with the single task. Further, the fact that Hakeem viewed the solution $P(t) = C_1 e^{-t} + C_2 e^{-5t}$ as both "a little more specific" and "incomplete" helps illuminate why he needed the two interpretations and sets of resources and perceived them as being productive in this task. He couldn't attain all of the necessary information from either set of resources alone; rather, the two sets of resources were utilized separately but towards a common goal. It is important to note, that while Hakeem associated these two interpretations and sets of resources with this single situation (task), the interpretations did not merge in that the information they afforded Hakeem was supplementary. I now discuss the resources associated with interpreting the differential equation as a description of the behavior of the quantity of interest for Task 2 and Task 3.

In terms of changes in Hakeem's utilization of resources, I compare the two sets within each of the tasks to those utilized in previous tasks; I do this because of the separate nature in the way Hakeem utilized two sets of resources to complete each of the tasks. I first compare the resources related with interpreting the differential equation as a description of the behavior of the quantity of interest, and then the resources related with interpreting the differential equation as an equation that provides a model of the quantity of interest.

The set of resources Hakeem utilized to complete I2-T2 (related with interpreting the differential equation as a description of the behavior of the quantity of interest) was a subset of those utilized while completing I3-T1, I2-T1 and I2-T2. In addition the resources were utilized in the same ways. Specifically he used *functional variation* in conjunction with *parts contribute to* *the whole* and *positives add-negatives take away* to relate the changes in the values of $P'$ and $P$ with changes in the value of $P''$ based on the relationship defined by the differential equation. He then coordinated *rates indicate behavior* and *unrestricted change* to discuss how the value of $P''$ related to changes in the value of $P'$, and how the value of $P'$ related to changes in $P$.

The set of resources for I3-T3 (related with interpreting the differential equation as a description of the behavior of the quantity of interest) was a distinct combination of those used in previous tasks. Hakeem used *variable substitution* and *functional mapping* in conjunction with *parts contribute to the whole* and *larger magnitudes have larger impacts* to relate the values of $P$ and $P'$ as per the relationship defined by the differential equation, and then used *rates indicate behavior* to infer how $P$ was changing based on values of $P'$. With the exception of *variable substitution*, the set of resources formed a subset of those used in I1-T1, I2-T1, and I3-T1. In addition, with the exception of *larger magnitudes have larger impacts*, the resources were a subset of those utilized in I2-T2. In this case Hakeem associated sets of resources from these four tasks with being useful for I3-T2.

I take the fact that the resources Hakeem utilized to complete I3-T2 and I3-T3 were also used in previous tasks as evidence of the assimilation of each of the tasks as similar to those that he encountered previously, be it the result of recognizing or actively structuring them as so. In both of these tasks Hakeem made use of resources that had already proven productive in other tasks and situations. In the case of I3-T2, Hakeem used a combination of resources he had used in multiple other instances. In the case of I3-T3, Hakeem associated the task with resources utilized in different tasks that had proven useful in accomplishing similar goals. I now discuss the sets of resources related with interpreting the differential equation as an equation that provided a model of the quantity of interest for both Task 2 and 3.

The last instance in which Hakeem expressed interpreting the differential equation as an equation that provided a model of the quantity of interest was during Interview 1 (I1-T2). In comparing I1-T2 to both I3-T2 and I3-T3, *function slots* is shared between I1-T2 and I3-T3, however, comparing I3-T2 to I3-T3, the sets of resources shared *functional variation*. In terms of the differences in the resources utilized between the tasks, Hakeem analytically solved the differential equation $P'' = -6P' - 5P$ (from I3-T2) with the *method of characteristic equations* to find the solution function $P(t) = C_1 e^{-t} + C_2 e^{-5t}$, and then used *functional variation* to make inferences about how $P$ changed as $t$ changed.

With regard to I3-T3, Hakeem analytically solved the differential equation for the general solution $P(t) = \left(\frac{1}{5} + Ce^{\frac{-t}{2}}\right)^{-1}$, and then coordinated the initial conditions (such conditions were not provided in I3-T2) with *function slots, dependence,* and the *definition of general solution* to find a particular solution for each of the two initial condition provided – $P(t) = \frac{1}{5} + \frac{2}{5}e^{-\frac{t}{2}}$ for $P(0) = 3$ and $P(t) = 5$ for $P(0) = 5$. He then used *functional variation* to discuss how the owl population changed as the value of $t$ increased for each of the particular solutions. The incorporation (as compared to I3-T2) of the three resources *function slots, dependence,* and the *definition of general solution* can be explained by the existence of the initial conditions in I3-T3; these resources were each utilized in coordination with the initial conditions in the process of determining the particular solutions. The cumulative list of resources supporting each interpretation expressed by Jordan during interview 3 can be seen in Table 5.3. Having compared across tasks in which these interpretations and resources were utilized for Interview 3, the presentation now transitions to Interview 4.

*Table 5.3: Resources supporting the interpretations Hakeem expressed in Interview 3.*

| Interpretation of the differential equation | Resources supporting the interpretation | |
|---|---|---|
| Description of the behavior of the quantity of interest | Functional dependence<br>Functional variation<br>Functional mapping<br>Rates indicate behavior<br>Parts contribute to the whole | Positives add- negatives take away<br>Larger magnitudes have larger impacts<br>Definition of cooperative systems<br>Definition of competing systems<br>Partial Dependence |
| Equation providing the model of the quantity of interest | Rule for differential equations of the form $P' = kP$<br>Rates indicate behavior<br>Function slots | Method of characteristic equations<br>Larger magnitudes have larger impacts<br>Functional dependence<br>Definition of general solution<br>Functional variation |

There were a total of three tasks Hakeem completed in Interview 4. The first (I4-T1) was the exact same competing and cooperative species task Hakeem encountered during Interviews 1 and 2. The second task (I4-T2) was novel in that it presented a graphical representation of a non-linear differential equation in the form of a graph of $\frac{dy}{dt}$ with respect to $y$, and asked Hakeem to use the graph to determine the long term behavior of $y$. The last task (I4-T3) asked Hakeem what a system of three differential equations meant to him. This last task was considered to be a variation of I2-T2 and I3-T2 in that they were identical except for the type of differential equations involved. Whereas I2-T2 presents Hakeem with a single second order differential equation, I3-T2 presented a system of three differential equations.

During this encounter with the competing and cooperative species task (I4-T1) Hakeem interpreted the differential equations as a description of the behavior of the quantities of interest and expressed the utilization of fewer resources to complete the task. Hakeem did not incorporate new resources to complete the task, which I take as evidence of the systematization of the resources. In other words, Hakeem's approach was more streamlined than his previous approaches; Hakeem expressed the utilization of fewer resources (compared to his interaction with the task during Interviews 1 and 2) that enabled him to attain the same goals in a more systematic fashion. It should also be noted that while Hakeem seemingly utilized fewer resources, it may be that encountering similar tasks numerous times allowed him to recognize aspects of it from previous encounters, thus decreasing the need to in engage problem solving activity which would have otherwise made the utilization of certain resources observable.

When determining the long-term behavior of $y$ from the graph of $\frac{dy}{dt}$ with respect to $y$ (I4-T2), Hakeem interpreted the differential equation as a description of the behavior of the quantity of interest and as an equation that provides a model of the quantity of interest. In this case, Hakeem perceived the task as similar to I3-T2 and I3-T3 as indicated by expressing the same combination of interpretations (and similar sets of resources) across the three tasks. Again, Hakeem utilized two sets of resources separately to acquire different information from the task, which he then combined in the process of determining the long term behavior of the value of $y$. In this case however, the individual sets of resources related with the respective interpretations were slightly different than those utilized in I3-T2 and I3-T2.

While interpreting the differential equation as a description of the behavior of the quantity of interest, Hakeem used the resources *functional mapping*, *functional variation* and *variable substitution* to relate values of $y$ to value of $\frac{dy}{dt}$ graphically in the form of a vector field. He used the vectors, which represented the values of $\frac{dy}{dt}$ to infer the behavior of $y$, using the resource *rates indicate behavior*. To do this however, Hakeem incorporated the resource *continuity*, which supported him in constructing the solution curves from discrete vectors. Though constructing and reasoning about the solution curves from a vector field was not new to Hakeem, this was the first time Hakeem expressed the resource *continuity.* In this case *continuity* was used to accommodate the values of $\frac{dy}{dt}$ between the discrete values represented on the vector field. While completing I4-T2 Hakeem attended to the existence of a continuum of values between those represented discretely on the vector field, and used *continuity* to accommodate for these values; values he did not seemingly attend to during his interaction with I3-T2.

While interpreting the differential equation as an equation that provided a model of the quantity of interest, Hakeem expressed the utilization of *separation of variables* and *functional variation.* These two resources supported Hakeem, similar to his encounter with I3-T2 and I3-T3, in analytically solving for the solution function that he used to relate changes in the value of $y$ to changes in the values of $t$. In this case, the resource *separation of variables* was used in place of the *method of characteristic equations* (when compared to I3-T2), plausibly because Hakeem perceived the *separation of variables* as more appropriate for solving the differential equation $\frac{dy}{dt} = -y^2 + 4$ than the method of characteristic equations.

Hakeem's interaction with I4-T3 was significant in terms of changes in the utilization of resources in that he expressed a new combination of interpretations (and sets of resources) while completing the task. In this case Hakeem not only associated the task with those he encountered previously, but he associated the respective sets of resources with this task as well. Specifically, Hakeem used the resources *functional variation, variable substitution, rates indicate behavior,* and *functional dependence* to discuss relationships between the values of $y_1$,$y_2$, and $y_3$, and $y_1$', $y_2$', and $y_3$' based on the differential equations. He used these resources while interpreting the differential equation as a relationship between the values of quantities and the value of the quantity of interest. In addition he used *functional variation, variable substitution* and *equality condition* while discussing characteristics of solutions to the system of differential equations. These resources were utilized while interpreting the differential equation as a relationship between a function and its derivative function.

In terms of changes in the sets of resources, this was the first time *rates indicate behavior* was utilized to support interpreting the differential equation as a relationship between values. Hakeem had not previously utilized this resource while expressing this particular interpretation. Here Hakeem used *rates indicate behavior* to infer that changes in the value of $y_1$' would affect the values of not only $y_1$, but $y_2$ and $y_3$ as well the values of $y_2$' and $y_3$' (which related to changes in the value of $y_2$ and $y_3$) were impacted by changes in the value of $y_1$ (based on the relationship defined by the differential equations). Regarding to resources related to interpreting the differential equation as a relationship between a function and the function's derivative, Hakeem made use *variable substitution* for the first time within this particular interpretation; he used *functional dependence* and *equality condition* the previous time he expressed this interpretation (I1-T4). Up to this point *variable substitution* was utilized by Hakeem to replace variables in the differential equation with values. In this case, however, Hakeem used *variable substitution* when discussing replacing the variables with functions. The cumulative list of resources supporting each interpretation expressed by Hakeem during interview 4 can be seen in Table 5.4 (follows) The presentation now transitions to the changes observed in Interview 5.

With regard to Interview 5, the analysis of five tasks was presented in Chapter 4. Recall that the first task (I5-T1) was novel and was in the form of a verbal question asking, "what does it mean to be a differential equation?" The second task (I5-T2 provided Hakeem with a system of two differential equations and prompted him to provide a graph of the solutions. Task 3 from Interview 5 (I5-T3) was similar to I4-T2, in that it provided a graph of $\frac{dy}{dt}$ with respect to $y$ and asked Hakeem to determine the long term behavior of $y$. The fourth task Hakeem encountered was the same as I1-T4, which asked Hakeem to find a differential equation which had a solution of $y = 6$. Lastly, Task 5 (I5-T5) was the same as I2-T3, which asked Hakeem to match the vector fields to their respective differential equations. Changes in Hakeem's utilization of resources can be seen in Tasks 2, 3 and 4; His responses to Tasks 1 and 5 did not indicate a difference in the utilization of resources.

*Table 5.4: Resources supporting the interpretations Hakeem expressed in Interview 4.*

| Interpretation of the differential equation | Resources supporting the interpretation | |
|---|---|---|
| Description of the behavior of the quantity of interest | Functional dependence<br>Functional variation<br>Functional mapping<br>Rates indicate behavior<br>Parts contribute to the whole | Positives add- negatives take away<br>Larger magnitudes have larger impacts<br>Definition of cooperative systems<br>Definition of competing systems<br>Partial Dependence<br>Continuity |
| Equation providing the model of the quantity of interest | Rule for differential equations of the form $P' = kP$<br>Rates indicate behavior<br>Function slots<br>Functional dependence | Method of characteristic equations<br>Larger magnitudes have larger impacts<br>Definition of general solution<br>Functional variation<br>Separation of variables |
| Relationship between the value of a quantity and the value of the quantity's rate of change | Functional dependence<br>Functional variation<br>Functional mapping | Parts contribute to the whole<br>Variable substitution<br>Rates indicate behavior |
| Relationship between a function and the function's derivative. | Functional dependence<br>Dependence<br>Equality condition | Opposite operations reverse<br>Variable substitution |

Hakeem expressed two ways of interpreting the differential equations while completing I5-T2: the differential equation as a description of the behavior of the quantity of interest, and also as an equation that provides a model of the quantity of interest. Though this combination of interpretations was not new (Hakeem expressed this combination of interpretations while completing I3-T2, I3-T3, I4-T2 and I5-T1), this is the first time he expressed this combination of interpretations (and related sets of resources) while completing a task involving a system of differential equations. In this case Hakeem associated the task with interpretations and resources that were previously productive in tasks involving single differential equations. That is, Hakeem structured I5-T2 as similar to those encountered previously (specifically I3-T2, I3-T3, I4-T2 and I5-T1), and used many of the same resources to complete the task: *functional dependence*, *functional variation, functional mapping, variable substitution,* and *rates indicate behavior*. These resources were considered by Hakeem to be productive in completing the task. Further this indicates that these resources were productive pieces of prior knowledge for Hakeem in that he was now successfully utilizing knowledge once associated with single differential equations to reason similarly with systems of equations.

The set of resources he used while expressing the differential equation as equation that provides a model of the quantity of interest, however, reflect an accommodation for the demands of a system of differential equations (as compared single differential equations). Namely, Hakeem made use of *the method of eigenvalues* to analytically solve the differential equation; this was the first time he expressed this resource.

The resources and interpretations Hakeem expressed while completing I5-T3 reflect a rather significant change in the organizational structure of Hakeem's knowledge. While completing this task, Hakeem interpreted the differential equation as a description of the behavior of the quantity of interest. Recall that during his previous interaction with this task (I4-T3), Hakeem expressed two interpretations and used different sets of resources to glean different pieces of information that combined to determine the long-term behavior. During his interaction in Interview 5, Hakeem seemingly no longer needed both sets of resources (and interpretations) and used just one to complete the task. Specifically Hakeem only expressed interpreting the differential equation as a description of the behavior of the quantity of interest and used *functional dependence, functional mapping variable substitution, rates indicate behavior,* *functional variation* and *continuity* to complete the task; he did not express interpreting the differential equation as an equation that provided a model of the quantity of interest nor did he use an analytical solution method to determine the specific solution function. I take this change as indicative of the reorganization of the sets of resources. Though it may be the case that the resources related to interpreting the differential equation as providing a model of the quantity of interest were still applicable to this task, they were no longer necessary to complete the task. In this case an entire set of resources once used to complete the task were not cued.

The resources and interpretations Hakeem expressed while completing I5-T4 indicate a significant change as well. The first time Hakeem encountered this task (I1-T4) he utilized resources that enabled him to relate a function ($y = 6$) to its derivative function ($\frac{dy}{dt} = 0$). During interview 5, Hakeem utilized a subset of these same resources (*equality condition* and *opposite operations reverse*) to accomplish the same goals; however, he also used *equality condition, functional mapping* and *functional variation* to infer that the value of "the derivative is zero along it [$y = 6$]." In this case, Hakeem was able to justify the solution he attained while interpreting the differential equation as a relationship between a function and the function's derivative by using resources that allowed him to relate the values of the function's derivative to points along the function.

Lastly, Hakeem's utilization of resources such as *parts contribute to the whole, positives add-negatives take way,* and *larger magnitudes have larger impacts* were not identified in Interview 5 at all. One explanation for this is that the utilization of these resources did not simply cease; rather, as Hakeem's familiarity with the tasks increased his utilization of resources became more systematized. Thusly the instances in which he expressed the utilization of these resources decreased. In other words, as the interviews progressed Hakeem restructured his knowledge so that the utilization of these resources was no longer at the forefront of the knowledge he was expressing. The cumulative list of resources supporting each interpretation expressed by Hakeem during interview 5 can be seen in Table 5.5. The discussion now turns to summarizing the changes in Hakeem's utilization of the resources.

*Table 5.5: Resources supporting the interpretations Hakeem expressed in Interview 5.*

| Interpretation of the differential equation | Resources supporting the interpretation | |
|---|---|---|
| Description of the behavior of the quantity of interest | Functional dependence<br>Functional variation<br>Functional mapping<br>Rates indicate behavior<br>Parts contribute to the whole | Positives add- negatives take away<br>Larger magnitudes have larger impacts<br>Definition of cooperative systems<br>Definition of competing systems<br>Partial Dependence<br>Continuity |
| Equation providing the model of the quantity of interest | Rule for differential equations of the form $P' = kP$<br>Rates indicate behavior<br>Function slots<br>Functional dependence<br>Functional variation | Method of characteristic equations<br>Method of eigenvalues<br>Larger magnitudes have larger impacts<br>Definition of general solution<br>Separation of variables<br>Functional mapping |
| Relationship between the value of a quantity and the value of the quantity's rate of change | Functional dependence<br>Functional variation<br>Functional mapping<br>Variable substitution | Parts contribute to the whole<br>Rates indicate behavior<br>Equality condition |
| Relationship between a function and the function's derivative. | Functional dependence<br>Dependence<br>Equality condition | Opposite operations reverse<br>Variable substitution |

Over the series of interviews, Hakeem's utilization of resources changed in a few important ways. Perhaps the most evident change was that Hakeem began to use different sets of resources (and interpretations) within single tasks as the interviews progressed. Initially Hakeem used a single set of resources associated with one interpretation to complete each task. As the interviews progressed, however, Hakeem began to associate different combinations of these sets of resources with individual tasks. By associating the different sets of resources with each task

Hakeem was supported in both justifying why certain solutions were correct and in leveraging the different information provided by the task. In some cases the different interpretations afforded Hakeem two different lines of reasoning that provided him with different information, which could be leveraged to construct a single image of the solutions. In other cases, the utilization of two interpretations afforded Hakeem with alternative explanations or justifications for the solutions he attained.

In addition to the sets of resources becoming applicable across a wider array of tasks, the resources also expanded in terms of the supporting different interpretations. These changes occurred in two ways: first, resources initially related to one interpretation could be incorporated in support of another interpretation. Such as the case with *equality condition,* which was originally used by Hakeem with regard to the differential equation as a relationship between functions in Interview 1, but was later used in Interview 5 when discussing the differential equation as a relationship between values. In this way, resources could be applied in different ways to accomplish different goals. Second, the sets of resources within interpretations changed as the demands of the tasks changed. For instance, in the tasks that Hakeem perceived it necessary to determine particular solutions, he made of use *function slots.* Hakeem did not use this resource in other instances.

Lastly, Hakeem's utilization of resources became systematized over time as indicted by the fact that fewer resources were identified within his completing of each task as the interviews progressed. As discussed previously, expressions of the utilization of resources such as *positives add –negatives take away* and *larger magnitudes have larger impacts* were never identified in Hakeem's problem solving activities in Interview 5. In other words, over the series of interviews the utilization of certain resources seemed to become systematized. Furthermore, entire sets of resources (and interpretations) seemed to be no longer perceived as necessary to complete certain tasks. In the case of the Interview 5 Hakeem no longer needed to use the solution function to reason about the long term behavior of the solution, because as he noted the differential equation itself "is telling you how $y$ is changing with $t$" (turn 76).

### 5.1.2 Jordan

In general, changes in Jordan's utilization of the resources occurred across the sets related to certain interpretations; over the series of interviews Jordan incorporated resources previously related with certain interpretations into sets of resources related with other interpretations. In this case, the applicability and utility of the resource expanded. Further, there were few instances in which Jordan expressed different interpretations (and sets of resources) during the completion of a single task. In the limited number of cases when this occurred, it was seemingly due to the original interpretation and related set of resources not adequately supporting his completion of the task. Along these lines, though resources transitioned across interpretations, the interpretations themselves did not seem to become associated with one another; this can generally be seen in Figure 5.2. which was constructed using the tables in Chapter 4 that summarized the resources and interpretations Jordan expressed while completing each of the tasks in the series of interviews (namely, Tables 4.7, 4.8, 4.9, 4.10 and 4.11), I constructed Figure 5.2. Note that Jordan did not express interpreting the differential equation as a relationship between a function and the function's derivative. I start the presentation of the changes in Jordan's utilization of the resources with a brief discussion of the resources and interpretations he expressed during interview 1.

| Jordan | | | | | |
|---|---|---|---|---|---|
| Interpretation | Description | Relation between Values | Relation between function | Model | Equation |
| Description | I1-T3<br>I2-T2<br>I4-T2, I4-T4<br>I5-T1 | | | | |
| Relation between values | I5-T2 | I1-T1, I1-T4<br>I2-T1, I2-T3<br>I3-T1<br>I4-T1 | | | |
| Relation between functions | | | | | |
| Model | I3-T3 | | | I2-T4<br>I4-T5 | |
| Equation | I1-T2 | | | | I3-T2<br>I4-T2<br>I5-T3 |

*Figure 5.2: Interpretations Jordan expressed for each task.*

Jordan completed a total of 4 tasks during interview 1. The first of which (I1-T1) was the competing and cooperative species task. To complete I1-T1 Jordan used resources that supported him in relating the values of the population of species $x$ and species $y$ to values of their rates of change. Specifically these resources were *functional dependence, functional variation functional mapping, partial dependence* and *variable substitution*. To complete the second task (I1-T2), which asked Jordan to explain what $P' = 3P$ meant to him, Jordan used the resources *functional dependence*, *functional variation* and *rates indicate behavior*, to describe how the value of $P$ changed over time. In addition, Jordan expressed the alternative interpretation of the differential equation as an equation to be solved and used the *method of integrating factors* and *dependence* to find and then reason with $P(t) = ce^{3t}$, the solution to the differential equation. Jordan interpreted the differential equation in the third task (I1-T3), which asked him to describe how one could use a differential equation said to model a fish population to determine the fish population at a certain time $t$, as a description of the behavior of the quantity of interest. Much like he did while completing I1-T2, Jordan used *functional dependence, functional variation, rates indicate behavior,* and the additional resource (not expressed in the completion of I1-T2) *variable substitution*. In this case the resources supported Jordan in relating the values of the fish population to the value of the rate of change of the population and then using the value of the population to describe the nature of the population. While completing the last task, which asked Jordan to find a differential equation with the solution $y = 6$, Jordan expressed interpreting the differential equation as a relationship between the value of quantities and the value of the rate of change of the quantity of interest. The cumulative list of resources supporting each interpretation can be seen in Table 5.6. The discussion now transitions to Interview 2.

*Table 5.6: Resources supporting the interpretations Jordan expressed in Interview 1.*

| Interpretation of the differential equation | Resources supporting the interpretation | |
|---|---|---|
| Description of the behavior of the quantity of interest | Functional dependence<br>Functional variation | Rates indicate behavior<br>Variable substitution |
| Relationship between the value of quantities and the rate of change of the quantity of interest | Functional dependence<br>Functional variation<br>Functional mapping<br>Partial dependence | Rates indicate behavior<br>Variable substitution<br>Equality condition<br>Definition of competing systems |
| Equation to be solved | Dependence | Method of integrating factors |

Jordan encountered four tasks during Interview 2. The first task (I2-T1) was the same as I1-T1. The second task (I2-T2) was a variation of I1- T2 and asked Jordan to discuss what $P'' = 3P' + P$ meant to him. Task 3 (I2-T3) was novel in that it asked Jordan to match vector fields to their respective differential equations, and the last task Jordan encountered was a variation of I1-T3 in that it presented an initial value problem (with a different differential equation) regarding a differential equation said to model a fish population.

While completing the first task, Jordan interpreted the differential equations in the same way he did during the first encounter. Although Jordan interpreted the differential equation the same way, he expressed a different set of resources; he seemingly made use of only three resources from the first interview (*partial dependence, functional variation*, and the *definition of competing species*) and expressed two that he not previously expressed while completing any of the tasks (*parts contribute to the whole* and *positives add –negatives take away*). While Jordan's overall approach to this task was the same, his reasoning was more systematic; as discussed in Chapter 4, he did not need to revisit System B twice as he did during the first interaction, and he was much more confident of his answer. It is not clear if this was because of his utilization of the two additional resources (*parts contribute to the whole* and *positives add –negatives take away*), or if he was more confident in making inferences concerning the relationship between the values of $x$ and $\frac{dy}{dt}$ and between the values of $y$ and $\frac{dx}{dt}$.

With regard to the second task (I2-T2), Jordan interpreted the differential equation in the same way as he did I1-T2. While completing the task Jordan used four of the same resources he did to complete I1-T2 (*functional mapping, functional dependence, variable substitution,* and *rates indicate behavior*), and used them in the same way. He utilized three additional resources while completing the task, however: *parts contribute to the whole, dependence* and *larger magnitudes have larger impacts.* The inclusion of these three resources supported Jordan in inferring that the behavior of $P$ (which Jordan interpreted as representing a population value) depended on the initial values of $P$ and $P'$. Namely, by coordinating these three resources Jordan was able to infer the different ways the population would be behave for different combinations of initial values of $P$ and $P'$. This was something Jordan did discuss while completing I1-T2.

While matching the vector field to their respective equations (I2-T3), a novel task, Jordan interpreted the differential equations as a relationship between the value of quantities and the value of the rate of change of the quantity of interest. Jordan made use of three resources that had previously been utilized to relate the value of a quantity to the value of the quantity's rate of change, *functional variation*, *functional dependence,* and *functional mapping.* Here his utilization of *functional dependence* was different than that of the way he utilized the resource in previous tasks. In this case, Jordan did not use it to relate specific values; rather he used it to determine which equations potentially matched the vector fields by looking for questions where the value of $\frac{dy}{dt}$ depended on the same variables as the vectors in the vector field. In addition, Jordan made use of the *definition of equilibrium solutions,* a resource he had not expressed previously while completing any of the tasks. This resource was of particular use to Jordan in this task as it supported him in being able to relate pairs of values of $y$ and $t$ to values of $\frac{dy}{dt}$. Namely, by recognizing the existence of an equilibrium solution in the vector field or algebraic form of the differential equation Jordan was able relate certain y values with $\frac{dy}{dt} = 0$, which provided Jordan an addition means of comparing the equations to the vector fields.

While completing the last task in Interview 2, which was a variation of I1-T3 Jordan interpreted the differential equation as an equation that provides the model of the quantity of interest. This was the first time Jordan expressed this interpretation. The resources he utilized that supported Jordan in completing the task, and making this interpretation were *opposite operations reverse*, the *method of integrating factors,* the *definition of general solution, dependence,* and *function slots.* Both the *method of integrating factors* and *dependence* had been utilized by Jordan previously to complete I1-T2, when interpreting the differential equation as an equation to be solved. In both cases Jordan used the *method of integrating factors* to analytically solve the differential equation and *dependence* to infer that the exact solution was determined by initial conditions. The difference, however, is that in this case Jordan then used the additional resources *definition of general solution, dependence,* and *function slots* to discuss the solution as a representation of the value of the fish population for any given value of $t$. The cumulative list of resources Jordan used over the series of interview in support of each interpretation can be seen in Table 5.7. The presentation of changes now transitions to discussing the resources and interpretations Jordan expressed in Interview 3.

*Table 5.7: Resources supporting the interpretations Jordan expressed in Interview 2.*

| Interpretation of the differential equation | Resources supporting the interpretation | |
|---|---|---|
| Description of the behavior of the quantity of interest | Functional dependence<br>Functional variation<br>Rates indicate behavior<br>Variable substitution | Parts contribute to the whole<br>Positives add- negatives take away<br>Larger magnitudes have larger impacts<br>Dependence |
| Relationship between the value of quantities and the rate of change of the quantity of interest | Functional dependence<br>Functional variation<br>Functional mapping<br>Partial dependence<br>Rates indicate behavior<br>Variable substitution | Equality condition<br>Definition of competing systems<br>Parts contribute to the whole<br>Positives add- Negatives take away<br>Definition of equilibrium solution |
| Equation that provides a model of the quantity of interest | Dependence<br>Function slots | Method of integrating factors<br>Opposite operations reverse<br>Definition of general solution |

During Interview 3 Jordan completed three tasks. The first task (I3-T1) presented Jordan with two systems of differential equations each said to represent predators and prey, and asked him to determine the size of the predators and prey in each system. This task was a variation of I1-T1 and I2-T1. The second task (I3-T2) was similar to I2-T2 in that it asked Jordan to discuss what the differential equation $P'' = -6P' - 5P$ meant to him. The last somewhat novel task presented Jordan with a non-linear differential equation said to model an owl population and two initial conditions, and asked him to determine the long-term behavior of the owl population.

Jordan interpreted the differential equation in the first task (I3-T1) differently than he did those of I1-T2 and I2-T1; in this case he interpreted the differential equation as a description of the behavior of the quantity of interest. While doing so he utilized the resources *positives add – negatives take away, functional variation, rates indicate behavior, partial dependence* and *carrying capacity*. Jordan had utilized the first three resources in previous tasks and in similar ways while supporting the interpretation of the differential equation as a description of the behavior of the quantity of interest. Prior to Jordan's interaction with this task the resource *partial dependence* was exclusively utilized while completing I1-T1 and I2-T1. In I2-T1, it was utilized in much of the same way: to consider how the values of $x$ related to values of $\frac{dy}{dt}$ and how values of $y$ related to values of $\frac{dy}{dt}$. In addition this was the first (and only) instance in which Jordan expressed *carrying capacity*, which was used to relate the behavior of the species to their size. More specifically, Jordan had expressed that he was unsure of how to determine the size of the species based on their rates of change, and *carrying capacity* afforded Jordan a way to relating the existence of a limiting factor in the population's behavior to being large.

Jordan's interaction with Task 2, (I3-T2) represented a significant change in his utilization of the resources as he completely changed his approach to completing a specific task from one interview to the next, seemingly due to the way he perceived the variable $P''$ in the differential equation. In this case Jordan interpreted the differential equation $P'' = -6P' - 5P$ as an equation to be solved, because as he said in turn 141 "I could say that it is a rate of change of a rate of change of population. That doesn't, like I haven't been told any words for that. It's math." The fact that Jordan did not know exactly how to contextualize $P''$ seemed to deflect Jordan's ability to focus on the relationship between the value of the quantity and the value of its rates of change. This is somewhat intriguing because situating the task within the setting of population was precisely what Jordan did while interacting with the similar task from interview

2. In this case however, Jordan was left with only one way of approaching the problem: “I see it on a test I’ll sit down and I’ll solve it. I, that’s pretty much all it means to me” (turn 133).

To solve the differential equation Jordan made use of three resources: the *method of characteristic equations,* the *definition of general solution,* and *dependence*. The first resource was new, a resource Jordan had not previously expressed. Jordan had utilized the second two resources previously while interpreting the differential equation as an equation that provided the model of the quantity of interest. In this case Jordan used the *definition of general solution* and *dependence* in a way commensurate to that of his previous utilizations but the resources now supported a different interpretation.

In the process of completing the last task in Interview 3 (I3-T3) Jordan expressed two interpretations of the differential equations, and thusly, used two separate sets of resources. Initially, while interpreting the differential equation a providing a model of the quantity of interest, he was unsuccessful in finding a method that would allow him to analytically solve the differential equation. After some prompting Jordan then began to utilize *Euler’s method* in conjunction with *variable substitution* and *positives add - negatives take away* to construct an approximation of the solution curve for the initial condition $P(0) = 3$. This was the first time Jordan had expressed utilizing *Euler’s method* and, in addition, the first time Jordan had used *variable substitution* and *positives add - negatives take away* while interpreting the differential equation as providing a model of the quantity of interest. Jordan had previously used these resources to relate values in the differential equation, and in this case he was using these same resources to relate values of variables within the equation defining Euler’s method.

After completing one iteration of Euler’s method, however, he suddenly switched interpretations and began to use a set of resources that enabled him to construct a representation of the solution curve by reasoning directly from the relationship defined by the differential equation. In this case he used the resources *functional variation, functional mapping, rates indicate behavior* and the *definition of general solution* to construct solution curves for both initial conditions. The utilization of the *definition of general solution* indicated a shift in the utilization of the resource, one that resulted in incorrectly characterizing the behavior of the owl population for the initial condition $P(0) = 5$. In this case after Jordan constructed the solution for $P(0) = 3$, he used the *definition of general solution* to infer the behavior was the same for all initial conditions. The change in the utilization is in the representation of the solution of the differential equation to which Jordan applied the resource. In the previous instances when Jordan expressed utilizing the *definition of general solution* Jordan consistently applied the resource to the algebraic form of the solution; in the case of I3-T3, however, Jordan applied the resource to the graphical form of the solution. This lead him to incorrectly characterize the solution for the initial condition $P(0) = 5$ as having the same form as the one he constructed for $P(0) = 3$. The cumulative list of resources supporting each interpretation can be seen in Table 5.8. The discussion now transitions to Interview 4.

Table 5.8: Resources supporting the interpretations Jordan expressed in Interview 3.

| Interpretation of the differential equation | Resources supporting the interpretation | |
|---|---|---|
| Description of the behavior of the quantity of interest | Functional dependence<br>Functional variation<br>Rates indicate behavior<br>Variable substitution<br>Carrying capacity | Parts contribute to the whole<br>Positives add- negatives take away<br>Larger magnitudes have larger impacts<br>Dependence<br>Definition of general solution |
| Equation to be solved | Dependence<br>Definition of general solution | Method of integrating factors<br>Method of characteristic equations |
| Equation that provides a model of the quantity of interest | Dependence<br>Function slots<br>Euler's method<br>Variable substitution | Method of integrating factors<br>Opposite operations reverse<br>Definition of general solution<br>Positives add- negatives take away |

Recall that during Interview 4 Jordan completed 4 tasks. The first was the competing and cooperative species task from Interviews 1 and 2. Jordan completed this task in a way that was commensurate with his second interaction during interview 2. Task 2 (I4-T2) was novel in that it presented a graph of an autonomous differential equation in the form of $\frac{dy}{dt}$ with respect to $y$. The third task (I4-T3) was a variation of I2-T2 and I3-T2 in that it asked Jordan to discuss what a system of three differential equations meant to him. Lastly, the fourth task (I4-T4) was similar to I2-T4, as it presented an initial value problem concerning a differential equation that represented the amount of a list students memorized with respect to time.

There were a few notable changes in the ways Jordan utilized the resources evident in this interview. During the completion the second task Jordan utilized the *definition of equilibrium solutions* while interpreting the differential equation as a description of the behavior of the quantity of interest. Prior to this interaction his utilization of this resource was always in service to interpreting the differential equation as a relationship between the values of quantities and the value of the rate of change of the quantity of interest. In this case Jordan used the *definition of equilibrium solutions* to infer that the lines $y = 2$ and $y = -2$ were solutions. An additional change is reflected in the fact that while completing the third task from Interview 4 Jordan expressed a new resource, the *method of eigenvalues,* which supported him in interpreting the differential equations in I4-T3 as equations to be solved. More specifically Jordan used this resource to analytically solve the system of differential equations.

Perhaps the most significant change was reflected in the approach Jordan took while completing I4-T4. This task was a variation of I2-T4; both tasks presented Jordan with an initial value problem consisting of a first order differential equation. Whereas in the previous encounters with tasks of this type (I1-T3, I2-T4,) Jordan interpreted the differential equation as an equation which provided a model of the quantity of interest, in this case Jordan interpreted the differential equation as a description of the behavior of the quantity of interest. More specifically, as opposed to analytically solving the differential equation for a function that related the value of the quantity of interest with the value of its independent variable, Jordan used resources that provided him the ability to reason directly from the relationship defined by the differential equation to describe the behavior of the quantity of interest. Though the resources associated with interpreting the differential equation as a description of the behavior of interest did not change, they afforded him the ability of interpreting differential equations within initial value problems in a new way. The cumulative list of resources supporting each interpretation over the series of interviews can be seen in Table 5.9. I now address Interview 5.

*Table 5.9: Resources supporting the interpretations Jordan expressed in Interview 4.*

| Interpretation of the differential equation | Resources supporting the interpretation | |
|---|---|---|
| Description of the behavior of the quantity of interest | Functional dependence<br>Functional variation<br>Rates indicate behavior<br>Variable substitution<br>Carrying capacity<br>Dependence | Parts contribute to the whole<br>Positives add- negatives take away<br>Larger magnitudes have larger impacts<br>Definition of general solution<br>Definition of equilibrium solutions |
| Equation to be solved | Dependence<br>Definition of general solution | Method of integrating factors<br>Method of characteristic equations<br>Method of Eigenvalues |
| Relationship between the value of quantities and the rate of change of the quantity of interest | Functional dependence<br>Functional variation<br>Functional mapping<br>Partial dependence<br>Rates indicate behavior<br>Variable substitution | Equality condition<br>Definition of competing systems<br>Parts contribute to the whole<br>Positives add- Negatives take away<br>Definition of equilibrium solution |

In Chapter 4 Jordan's interaction with three tasks is presented: The first task, a question posed verbally was, "To you, what is a differential equation?" The second task involved finding a differential equation with the solution $y = 6$, a task Jordan encountered during the first Interview (I1-T4). Finally, the third task (I5-T3) presented an initial value problem concerning a system of differential equations; I considered this task to be a variation of I4-T3, which asked Jordan to explain what a certain system of differential equations meant to him. There were two instances in which the Jordan's utilization of the resources changed in interview 5.

The first change was reflected in the first task. While completing I5-T1 Jordan described differential equations as conditions between a function's value and its behavior. He used the resources *functional dependence, rates indicate behavior* and *equality condition* to provide this description and described $y$ as the value of a function that determined $y'$, the behavior of $y$. In doing so, *equality condition* was used to support interpreting the differential equation as a description of the behavior of the quantity of interest. This was the first time Jordan used the resource *equality condition* while interpreting the differential equation as a description of the behavior of the quantity of interest. The resource e*quality condition* was used by Jordan only one time prior to interview 5 (I1-T4); namely he used it to justify why a function (or equation) was a solution to a differential equation by showing that for the function $y = 6$ the value of $y' = 0$ for all points on the line. In the case of I5-T2, Jordan did not coordinate the resource with the value of $y'$ and $y$, rather he applied *equality condition* to the behavior of $y$, noting that "at $y = 6\ \frac{dy}{dt}$ is gonna be zero which means that you are never going to change your $y$, which means that you are gonna end up with this line $y = 6$."

The second and only other change noted in the utilization of the resources also concerned *equality condition.* In the case of the third task (I5-T3), Jordan interpreted the system of equations as equations to be solved (as he did during interview 4). To solve the system of equations Jordan used the *method of eigenvalues* and *equality condition*. This was the first time Jordan used *equality condition* to support interpreting the differential equation as an equation to be solved. In addition, during his completion of I5-T3 this resource supported Jordan in justifying the solution he found (as it did in all of the instances in which he used the resource), the difference however was that here Jordan applied the resource to a system of differential equations; Jordan had applied the *method of eigenvalues* to the system of equation in I4-T3. His extension of the resource *equality condition* to systems of differential equations indicates that for Jordan, it was a productive prior knowledge which he could use to reason about systems. The cumulative list of resources supporting each interpretation can be seen in Table 5.10. The discussion now transitions to summarizing the changes in Jordan's utilization of the resources.

*Table 5.10: Resources supporting the interpretations Jordan expressed in Interview 5.*

| Interpretation of the differential equation | Resources supporting the interpretation | |
|---|---|---|
| Description of the behavior of the quantity of interest | Functional dependence<br>Functional variation<br>Rates indicate behavior<br>Variable substitution<br>Carrying capacity<br>Dependence | Parts contribute to the whole<br>Positives add- negatives take away<br>Larger magnitudes have larger impacts<br>Definition of general solution<br>Definition of equilibrium solutions<br>Equality condition |
| Equation to be solved | Dependence<br>Definition of general solution<br>Equality condition | Method of integrating factors<br>Method of characteristic equations<br>Method of Eigenvalues |
| Relationship between the value of quantities and the rate of change of the quantity of interest | Functional dependence<br>Functional variation<br>Functional mapping<br>Partial dependence<br>Rates indicate behavior<br>Variable substitution | Equality condition<br>Definition of competing systems<br>Parts contribute to the whole<br>Positives add- Negatives take away<br>Definition of equilibrium solution |

Looking across the series of interviews, Jordan's utilization of the resources changed in that a greater number of resources became able to support each of the interpretations. In this way not only did the perceived situations in which the resources could be utilized expand, there became more ways in which individual resources could be utilized. For example, in the first interview (I1-T4), *equality condition* was used by Jordan to discuss differential equations as a condition between the value of a quantity and the value of its rate of change. In the last interview, however, Jordan utilized this resource in two a different ways: Jordan applied *equality condition* to the relationship between the behavior of a solution and that as expressed by the differential equation (I5-T2). This particular change in the utilization was documented while Jordan interacted with the same task; I1-T4 and I5-T2 both asked Jordan to find a differential equation with a solution of $y = 6$, and were presented in exactly the same way (though roughly 11 weeks apart). Therefore, this change is not strictly the result of the task intrinsically supporting one interpretation or another, but rather it is indicative of a change in the resources that supported Jordan in making a different interpretation of the differential equation. In addition, with regard to I5-T3 (a variation of I4-T3) though his interpretation of the differential equation during this task was not different from that of I4-T3, Jordan used *equality condition* to discuss solution functions as being "functions that will work" (turn 44), or in other words to explain that solutions are functions that satisfy the differential equation.

Though the individual resources began to support multiple interpretations and expanded in terms of the ways in which they could be utilized, there were only a few instances in which Jordan expressed multiple ways of interpreting the differential equations (and thusly different ways of using the applied resources) within individual tasks. Of the three instances in which Jordan expressed multiple interpretations (I1-T2, I3-T3, and I5-T2 see Figure 5.2), two of them contained inferences that were imprecise and sometimes left Jordan confused. Wagner (2010) suggests that some applications of knowledge that result in "awkward and imprecise conclusions" (p. 452) are indicative of students assimilating situations as similar to those they have encountered before, and applying knowledge that worked well in one situation in another situation where the knowledge may not fit particularly well. This description characterizes Jordan's interactions with I1-T2 and I3-T3.

Although these moments were productive in terms of learning, I posit that this is somewhat connected to the frustrations Jordan expressed concerning his understanding of the course material; he may have been aware of the lack of connections between knowledge he was applying to complete the tasks. More specifically, I do not believe that Jordan perceived the different interpretations he expressed as being related to one another.

## 5.2 Interaction Between Function and Rate of Change

In this section I address the third research question by discussing how the resources for function and rate of change interacted to support the students' completion of various tasks. I do this by building on the analysis presented in Chapter 4 by using the analysis of the resources and interpretations as data. More specifically I analyze across the tasks completed by the students, seeking to identify regularities in their approaches specifically involving resources previously identified as relating to function and rate of change. These resources can be seen in the Table 5.11. This is different than the analysis presented in section 5.1 as here I specifically looked for ways in which *rates indicate behavior* interacted with *functional dependence, functional variation*, *functional mapping, variable substitution, parts contribute to the whole, larger magnitudes have larger impacts*, and *positives add- negatives subtract.* In addition I discuss how the interaction supported the students understanding of differential equations by drawing connections with the relevant literature regarding student understanding of differential equations.

Due to the large number of different combinations of resources the students utilized while completing the tasks and the fact that the students often used different sets of resources to accomplish similar goals, analyzing the interactions between function and rate of change at the grain size of individual resources on a task by task basis would not be practical. Rather, to answer the third research question, I zoom out to the level of interpretations. More specifically, in this section I discuss regularities in the students' utilization of resources related to function and rate of change with regard to approaches students took while expressing the same interpretation of the differential equations across tasks.

*Table 5.11: Resources for function and resources for rate of change*

| Resources concerning function | Resources concerning rate of change |
|---|---|
| Functional dependence | Rates indicate behavior |
| Functional variation | |
| Functional mapping | |
| Larger magnitudes have larger impacts | |
| Parts contribute to the whole | |
| Positives add – negatives subtract | |
| Variable substitution | |

**5.2.1 The Role of Function and Rate of Change Within the Interpretations.** Over the series of the five interviews Hakeem and Jordan collectively expressed five ways of interpreting the differential equations. Some of these interpretations were expressed over a broad range of tasks, whereas others were expressed while working on a small collection of specific tasks. In addition, there were several instances when two or more interpretations were utilized while completing a single task. While both Hakeem and Jordan often used different resources over the series of tasks presented in the interviews, there were significant regularities in their problem activity within each of the interpretations. The regularities help inform the students' understanding of differential equations. Specifically they highlight key aspects of the students understanding that were productive for completing the various tasks. Here I specifically focus on the regularities that involved resources related to function and resources related to rate of change.

Over the series of tasks, resources for both function and rate of change were expressed in only three of the interpretations of the differential equations: As a description of the behavior of the quantity of interest, as a relationship between values of quantities and the value of the rate of change of the quantity of interest, and as an equation that provides a model of the quantity of interest. In the following subsections, I discuss the ways in which resources for function and rate of change interacted within each of the three interpretations.

The other two interpretations of the differential equations, an equation to be solved and a relationship between functions the function's derivative did not involve structuring solutions to differential equations in ways that required the utilization of resources related to both function and rate of change. In these cases the students were focused, respectively, on constructing algebraic representations of the solutions based on analytical methods and statements of equality in the differential equation after substituting functions into the differential equation.

**5.2.1.1 As a description of the behavior of the quantity of interest.** Jordan and Hakeem each expressed interpreting the differential equation as a description of the behavior of the quantity of interest in over half of the tasks they completed. When interpreting the differential equation in this way their focus was on using the relationship between the variables in the differential equation to glean the information necessary for characterizing the behavior of the quantity of interest (be it the shape of the solution curve, long term behavior, or an understanding of how changes in one quantity relate to changes in another). To acquire the needed information the students correlated values of the quantity with values of the quantity's rate of change through the relationship present in the differential equation, and then used properties of rate of change to infer qualitative characteristics of the quantity. This was accomplished in various ways by the students: for instance when completing I4-T1 Hakeem verbally noted how changes in species $x$ related to changes in species $y$, both students structured solution curves from vector fields they constructed while completing I5-T3, and in the case of Jordan's interaction with Task 2 from Interview 2, he related the different sets of values for $P$, $P'$ and $P''$ to different points on a single solution curve to construct a graphical representation of $P$.

In instances such as these the students' goal was not to determine an exact equation or graphical representation of the relation between the value of the quantity and its independent variable, but rather to characterize the nature of the quantity of interest. While there were variations from task to task with regard to the exact way the students constructed the behavior of the quantity of interest, this general process can be seen, as discussed in detail in Chapter 4, in Hakeem's approach to tasks such as I1-T1, I2-T2, I3-T3, I4-T2, and I5-T2. This can also bee seen in Jordan's approach to tasks such as I1-T1, I1-T2, I2-T2, I3-T3, and I5-T2.

In each of these cases, the exact set of resources the students used were different. However, as can be seen Chapter 4, to relate the values of quantities to the value of their rate of change, Hakeem and Jordan used (combinations) of *functional dependence, functional variation,* and *functional mapping.* The exact combination of resources they utilized was dependent on whether they were focusing on aspects such as relating changes in the values, constructing vector fields, or relating single values of a quantity to its rate of change. In addition to using these three resources they would use different combinations of the resources p*arts contribute to the whole, positives add – negatives take away, larger magnitudes have larger impacts,* and *variable substitution* to characterize the exact nature of the relationships. The students' utilization of different combinations of these resources is discussed explicitly in section 4.4.

In terms of using the relationship between values to determine the behavior of the quantity of interest, the students used *rates indicate behavior* to accomplish this. In fact, when looking across each of the tasks, *rates indicate behavior* was evident in every instance in which the students interpreted the differential equation as a description of the behavior of the quantity of interest. Generally speaking this resource allowed both Jordan and Hakeem to infer whether the value of a quantity was increasing and decreasing and how fast. Further they were able to apply this resource to different representations of $\frac{dy}{dt}$ (e.g., values, vectors and slopes).

In terms of the interaction between the resources related to function and rate of change, both Jordan's and Hakeem's interaction with I2-T2 serves as a paradigmatic example (though this can be seen in analysis presented in Chapter 4 for each of the previously listed tasks). Here I discuss only Jordan's interaction. Recall that Jordan used *functional dependence* to infer the existence of an independent variable, which he coordinated with *functional mapping* to infer the values of $P$, $P'$, and $P''$ all corresponded to the same value of $t$. Jordan used *dependence* to infer the solution functions were determined by the initial value of $P$ and $P'$, and *parts contribute to the whole* to determine how the values of $P$ and $P'$ related to the value of $P''$. In addition, he used *rates indicate behavior* to determine new values of $P$ based on the values of $P'$ and $P''$. Lastly he coordinated *larger magnitudes have larger impacts, variable substitution, and parts contribute to the whole* to infer that larger values of $P$ and $P'$ would lead to larger values of $P''$. In short, Jordan used these resources to discuss the process of using values of $P$, $P'$ and $P''$ to build specific solution functions. While doing se he utilized the differential equation to relate values of $P$, $P'$ and $P''$ to construct the solution function $P$ (see Figure 5.6).

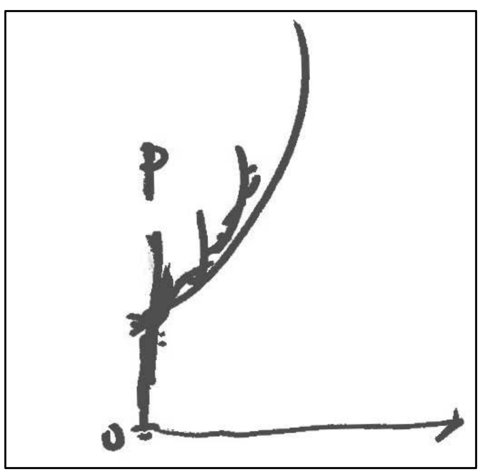



*Figure 5.3: Jordan used resources related to function and rate of change.*

**5.2.1.2 As a relationship between values.** When interpreting the differential equation as a relationship between values of quantities and the value of the rate of change of the quantity of interest, there were many aspects of the students' problem solving activity common to those observed in instances in which they were identified as interpreting the differential equation as a description of the behavior of the quantity of interest. The main difference between the two approaches was that while expressing the former interpretation the students determined relationships between the variables, but did not build on those relationships to structure solutions as they did with the latter interpretation. A suitable paradigmatic example of the students' approach while expressing this interpretation could be represented by either of the students' interactions with the vector matching tasks (I2-T3 or I5-T5), though this can be seen in analysis of Hakeem's interactions with I1-T4, I4-T3, and I5-T4, and Jordan's interactions with tasks such as I1-T1, I2-T1, and I5-T2.

For instance, while completing I2-T1 Jordan used *partial dependence* to determine which variabels from each of the differential equations to consider. Specifically, to determine how the change in one population impacted the rate of change of the other, he inferred that he only needed to consider the $y$ variables in the equations for $\frac{dx}{dt}$ and only the $x$ variables in the $\frac{dy}{dt}$ equations. To do so he coordinated the resources *functional variation* and *positives add–negatives take away* to determine how increases in one population related to changes in the other. For example, by knowing that coeffeceient in front of the $y$ term in the equation $\frac{dx}{dt} = 2y$ is positive, Jordan was able to infer that increases in $y$ would relate to an increase the rate of change of $x$. Lastly Jordan used his *definition of competing systems* to infer that System B was competing because "as $y$ increases, $\frac{dx}{dt}$, so the change in $x$ decreases, so that would be a competitive nature." In this case Jordan worked exclusively with the relationship between $x$, $y$ $\frac{dx}{dt}$ and $\frac{dy}{dt}$, and did not build a description of the populations of species $x$ and species $y$; he did so by simply reasoning with the nature of the relationship between the variables.

Though it was not generally the case that resources related to function interacted with resources for rate of change in tasks in which I identified the students interpreting the differential equation as a relationship between values that is not to say such interactions did not occur. For instance when completing I1-T4 Jordan coordinated *functional mapping*, *equality condition* and *rates indicate behavior* to construct the differential equation $\frac{dy}{dt} = y - 6$. By using *rates indicate behavio*r he was able infer that the value of the slope of the line $y = 6$ was zero, and with *functional mapping* and *equality condition* he correlated $\frac{dy}{dt}$ being 0 with y being 6 (*functional mapping*) to create the differential equation $\frac{dy}{dt} = y - 6$. In this case, Jordan was able to use *rates indicate behavior* to determine which values of $y$ correlated to which values of $\frac{dy}{dt}$, and *functional mapping* and *equality condition* to construct the equation which represented the correlation. In this case, resources for function and rate of change interacted to support his ability to determine the algebraic form of the differential equation based on the given solution.

After looking across the Tables in Chapter 4, with the exception of the previously mentioned case (Jordan, I1-T4), when Jordan and Hakeem expressed interpreting the differential equations as a relationship between values of quantities and the value of the rate of change of the quantity of interest, there was no other interaction between the resources for function and resources for rate of change identified in this study. I do however, leave open the possibility of the existence of other resources for function and rate of change that may have been utilized by the students. The fact that there were able to interpret symbols such as $y'$ and $\frac{dy}{dt}$ as rates of change supports the existence of such resources. There are two important implications here: The first is that more research is needed concerning the identification of resources for rate of change. The second implication is that it may be the case that not all differential equations tasks that require relating values of a quantity and its rate of change require resources for both function and rate of change; perhaps in some cases the students can utilize $\frac{dy}{dt}$ as a value without attending to it as representing a rate of change.

**5.2.1.3 An equation that provides a model.** Over the course of the interviews this interpretation was expressed 7 times by the students: two times by Jordan and five times by Hakeem. Only Hakeem expressed both resources related to function and resources related to rate of change while expressing this interpretation, and further, he only did so once (I1-T2). In general, in instances when the students expressed this interpretation they would use analytical solution methods to solve the differential equation, and if provided initial conditions, use resources such as *the definition of general solution, dependence* and *function slots* to determine a particular solution function that represented a relationship between the value of the quantity and the value of the quantity's independent variable. In such cases the students were structuring the solutions based on the algebraic form of the functions attained as a result of analytically solving the differential equations. In these instances it was not the differential equation itself that provided a characterization of the quantity of interest, but rather the solution function provided by solving the differential equation.

To explicate this consider that when completing I3-T2 and I4-T2 Hakeem used *functional variation* to discuss the values the solution function would attain as the value of the independent variable increased toward infinity. Here Hakeem determined a function that related the independent variable $t$ to the dependent (quantity of interest) $P$, and reasoned with that function to characterize how the value of $P$ behaved as $t$ approached infinity. In the case of Hakeem's interaction with I1-T2, Hakeem used his rule for solutions of the form $P' = kP$ to determine the equation for $P(t)$ which represented a model of a population. He then used *function slots* and *functional dependence* to situate this model within certain initial conditions, the existence of which he supposed. Lastly, he reasoned about the coefficient in front of $P$ to make inferences about the behavior of the values of the population that the differential equation provided a model for. More specifically in terms of the interaction between function and rate of change, he considered the coefficient to be a "rate constant" and, utilizing the resources *larger magnitudes have larger impacts* and *rates indicate behavior,* he inferred how fast or slow the population would grow was based on this rate constant; larger rate constants related to faster growth, smaller rate constants related to slower growth. In this case, the resources for function and rate of change supported Hakeem's understanding of differential equations in that they enabled him to infer that the population values, which were changing as $t$ changed, would increase faster for larger values of the rate constant.

**5.2.1.4 Discussion of the interaction between function and rate of change.** The interconnected nature of the resources for function and the resources for rate of change tended to emerge while the students were completing tasks in which they interpreted the differential equation as a description of the behavior of the quantity of interest. In these cases the students were actively structuring the solution functions based on the relationship present in the differential equation between the quantity and its rate of change. Across the three interpretations however, the interaction between the resources related to the respective two concepts allowed the students to determine relationships between values, construct representations of the solution functions, determine long-term behavior of the quantities based on the differential equation itself and discuss relationships between the differential equations and their solution functions. More specifically the resources related to function supported the students in determining how the values of the quantity related to the value of the rate of change, and these relationships were then used in conjunction with *rates indicate behavior* to construct characterizations of the solutions.

Interplay between rate of change and function of this nature is not novel; Whitehead and Rasmussen (2003) noted that students use notions of rate to build images of function while structuring solutions for single differential equations. In addition Whitehead and Rasmussen noted that students were able to reason about solutions to differential equations by almost effortlessly transitioning between substituting values for $y$ and treating $y$ as a continuously changing function by using the value of the rate of change to glean information from the differential equation. In terms of the research presented here, the analysis illuminates finer aspects of the knowledge students utilize to structure solutions in these ways based on relationships in the differential equations. More specifically, the students structured solutions to the differential equations in two ways; by reasoning with the relationship between the variables in the differential equation, and reasoning with the relationship between the variables in the solution functions. These two ways of structuring the solutions were always accomplished using a subset of the resource *functional dependence, functional variation, functional mapping, variable substitution, parts contribute to the whole, larger magnitudes have larger impacts,* and *positives add- negatives subtract* - to with the relationships in the equations. In addition to structure the solutions by reasoning directly from the relationship in the differential equation the students always utilized *rates indicate behavior.*

Along these lines, this also helps illuminate Donovan's (2007) findings that students with the ability to conceptualize ordinary differential equations as functions were able to more easily see the differential equations as connected to and informative of their solutions, treat $y$ as both a function and a variable in the differential equation, interpret $\frac{dy}{dt}$ as a variable, slope or derivative, and understand $t$ as an independent variable in both the differential equations and their solutions. With respect to the students in this study, the interaction between resources related to function and resources related to rate of change supported these abilities. For instance, by utilizing functional dependence or functional mapping the students were able to reason with $t$ as an independent variable in both the differential equations and their solutions, which then supported their ability to utilize rates indicate behavior to determine the exact nature of the solutions. In addition, by mapping values of $y$ and $t$ to different values of $\frac{dy}{dt}$, the students were treating $\frac{dy}{dt}$ as a variable, explicitly using the relationship in the differential equation to structure solutions and build connections between the differential equations and their solutions. Here, resources for function played a key role in the students' problem solving process with regard to reasoning with the differential equations as functions; resources for function provided the students with an ability to determine the relationships between the variables across different representations of the differential equations and supported the students in reasoning with different representations of solution functions. In addition these different representations provided the students with different information, and different ways of reasoning with $\frac{dy}{dt}$ and how it relates to $y$ as both a function and a variable.

The resources provided similar support with regard to connections between differential equations and their solutions when interpreting the differential equation as a relation between the value of a quantity and the value of the quantity's rate of change. In particular when completing I5-T4, Hakeem was able to discuss, what an expert might consider to reflect a pointwise comparison between the value of a function and the value of the function's derivative; "So I guess another way to look at it is if you graph, $y$ and $t$ and you graph the line $y = 6$, the derivative is zero along it" (turn 101). In doing so, Hakeem was able to establish another connection between differential equations and their solutions.

Lastly, though resources related to both function and rate of change were expressed individually, the connection between resources related to these concepts did not seem prevalent when the students were interpreting the differential equation as a an equation to be solved or as a relation between functions and their derivatives, at least not with regard to the resources identified in this study. I posit this is due to the students' focus on the algebraic form of the solutions as opposed to the relation between the values in the solutions. In other words, the students seemed to be focused on finding "functions that will work" (Jordan, Interview 5 turn 44), or functions that satisfy the conditions set by the differential equations when interpreting the differential equations in these two ways. In these cases the students' did not appear to be connecting the relations present in the differential equation with those present in the solution function, rather their focus was on finding expressions that retained the equality in the differential equation without attending to what the components in the differential equation represented. It would be interesting to see how this generalizes to students enrolled in courses where the focus was not on analytical solution methods.

There are two important implications here in terms of student learning of differential equations. The first is that the interaction between function and rate of change had a significant role in the students' reasoning with regard to structuring solutions. This suggests that if one wishes to foster students' abilities to reason about solutions to differential equations and the connections between differential equations and their solutions, developing resources for function and resources for rate of change are imperative. Second, the fact that not every task elicited resources related to function and resources related to rate of change indicates that not only do different tasks elicit or require different resources, but that certain tasks might provide more opportunities for students to develop certain understandings than others. For instance, tasks that prompt students to graphically structure solution functions seemed to elicit interactions between function and rate of change in ways that tasks prompting the students to match vector fields to equations did not. In this way, this analysis can be used to inform curriculum materials with regard to identifying tasks that may be useful for providing opportunities for students to develop certain mathematical ideas.

The remainder of the dissertation, Chapter 6, is devoted to drawing conclusions, discussing implications, and addressing future directions. I also discuss some of the limitations of the study.

# CHAPTER 6: DISCUSSION OF FINDINGS, IMPLICATIONS, AND FUTURE DIRECTIONS

In this chapter I connect the findings discussed in Chapters 4 and 5 and situate them within the broader setting of student learning of differential equations. I then discuss the contributions made to the larger mathematics community, practical implications, constraints of the study, and future directions.

## 6.1 Summary of Findings

Before drawing connections I quickly synthesize the findings regarding each of the research questions. To remind the reader the research questions were:

1. What resources concerning function and rate of change do students utilize to complete various differential equations tasks?
2. How do these resources change as the students progress through a differential equations course?
3. How do students' resources concerning rate of change and function influence one another during the development of their understanding of differential equations?

### 6.1.1 Resources

In Chapter 4 I presented analysis that identified a multitude of resources two students used to complete various tasks over a series of five interviews. In addition I identified the ways the students interpreted both the differential equations and their components while completing each of the tasks. The result of this analysis is a collection of fine-grained knowledge resources that were productive for the students with regard to completing various tasks involving differential equations, as well as a set of characterizations informing the ways students structured the tasks. In addition the analysis explicates how each of two students used these resources while completing individual tasks, which support and build on the findings of others by illuminating the fine-grained knowledge students use in instances such as characterizing solutions to differential equations, constructing solution curves from vector fields, analytically finding solutions, and understanding the phenomenon being represented by the differential equation.

The analysis reveals that the students used a multitude of different resources while completing the tasks (for the glossary of knowledge resources see Table 4.1) for a variety of different purposes. Some of the resources supported the students in perceiving (or recognizing) and utilizing the relations between the components in the differential equation (i.e., relationships between the value of quantities and the values of the rate of change of the quantity of interest and a function and its derivative function), and others supported the establishment of a specific characterization of those relations. Further, various resources were used while analytically solving the differential equations (e.g., *separation of variables, method of integrating factors*, and *method of characteristic equations*) and reasoning about solutions (e.g., *rates indicate behavior, definition of general solution, dependence* and *equality condition*). Lastly, depending on the mathematical entity (e.g., function, value, slope) the student was coordinating a resource with, the utility of certain the resources was slightly different. For instance, *functional dependence* provided support for relating specific values in a differential equation, but it also supported the students in determining which equations potentially matched a vector field based on which variables the value of $\frac{dy}{dt}$ depended on. As another example, *rates indicate behavior* was often used to determine if a solution curve was increasing or decreasing and how quickly, but it could also be used to characterize the value of slope at a point on a given solution curve.

The analysis also demonstrates that the students used different sets of these resources to complete different tasks. The students' expressed different interpretations of the differential equations accounted for the students' utilization of different sets of resources at different times. Thusly the interpretations provide insight into the students' perceived utility of certain sets of resources for a given task at a given time. Broadly speaking the collection of resources provided the students with the ability to interpret the differential equations as relationships between values, characterizations of the behavior of solution curves, equations to be solved, equations that provided models, and as conditions between functions. The different interpretations can be viewed as different ways the students structured the tasks based on the coordination between the attributes they perceived in the situation and the resources they associated with being productive in that situation. In this sense the interpretations illuminated the goals the students associated with each of the tasks and thusly situated the resources within the perceived goals of the tasks.

Lastly, though it was not a goal of this dissertation (nor typically the goal of research using Knowledge in Pieces) the resources I identified as being utilized by the students illuminate potential aspects of knowledge used by students when displaying an understanding of the differential equations in a way that is compatible with having an action view of function and a process view of function. It is important to note the two perspectives from which these constructs (resources, and action and process views) emerged – Knowledge in Pieces and APOS – are very different. Due to the fine-grained characterization of knowledge provided by Knowledge in Pieces, however, it provides a powerful way of illuminating, further characterizing, and building on constructs established from other perspectives on individual student learning. In chapter 4, I used the identified resources to pose implications for building on the notion of having action and process views of function to having action and process views of differential equations. These implications stemmed from drawing connections between the resources I identified and published research literature concerning student understanding of function. As such, the characterizations of action and process views of differential equations require further exploration and research. In the following paragraphs I summarize these characterizations.

The resource *variable substitution* was often utilized in conjunction with *functional dependence* to characterize exactly how a specific value of one variable (or set of variables) related to the specific value of another. While working on the competing and cooperative species task during Interview 1, Jordan, aware that the values of $x$ and $y$ determined the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ (*functional dependence*), substituted the value of 10 into the differential equations for both $x$ and $y$ to determine specific values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$. In this case, Jordan utilized *variable substitution* to find specific values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ for specific values of $x$ and $y$. This is commensurate with how Carlson, Oehrtman, and Engelke (2010) describe an *action view of function;* as being dominated by an image of procedures used to evaluate a function at a specific value. In this example Jordan did exactly that: he substituted 10 into the differential equation for $x$ and $y$ and calculated the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$. In this case, as a result of utilizing the resources *functional dependence* and *variable substitution,* Jordan displayed an action view of the differential equation by evaluating the values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$ using 10 as an input value for both $x$ and $y$.

In addition, the students displayed reasoning that was similar to what researchers have described as having a process view of function. For instance while completing Task 2 from Interview 2, recall that Jordan reasoned with the relationship between values of $P$, *P'* and *P''* as defined by the differential equation $P'' = 3P' + P$ to determine the behavior of the solution functions. Specifically he used *functional mapping* to establish the correlation between the values of $P$, *P'* and *P''* (as triplets), *positives add – negatives subtract* to further define the nature of correlation by inferring that positive values of $P$ and *P'* related to positive values of *P''*, and *rates indicate behavior* to infer that all of these values would be increasing for a positive $P$ and *P'*. While doing so he related a continuum of increasing values of $P$ and *P'* to a continuum of values of $P''$. Further he then correlated each of the sets of related values of $P$, $P'$ and $P''$ to individual points on the solution curve over a continuum of $t$ values. It is important to note that there are two aspects of his reasoning that are similar to having a process view of function. First he envisioned how $P''$ changed over a continuum of input values for $P$ and $P'$. Second he envisioned the dynamic triplets of values $(P, P', P'')$ as each being related to a specific value of $t$ but over a continuum of $t$ values. In this case the result of his utilization of *functional mapping* and *positives add – negatives take away* is commensurate with having a process view of function.

This expansion beyond an action view of *function* to an action view of the *differential equation* is possible because the differential equations (at least those the students encountered during the interviews) can be thought of as functions that relate values or as equations that relate functions. This raises some interesting questions that warrant further investigation. Namely, because differential equations relate functions and also relate evaluated values of functions, it may be the case that having an action view of differential equations exists in two layers: actions on functions and actions on their evaluated values. In addition the students used the differential equation to relate the values of the variables over a continuum, but also used the notion that the differential equations related functions, which in and of themselves relate the value of a quantity to a continuum of values of an independent variable. In the case of Jordan, he used the differential equation to relate the values $P''$ over a continuum of values of $P$ and $P'$, but also used the notion that these were functions to relate the triplets over a continuum of $t$ values.

### 6.1.2 Changes in the Utilization of the Resources

The analysis in Section 5.1 discusses the various ways the students' utilization of the resources changed over time by examining differences in the sets of resources and interpretations the students expressed across tasks and interviews. In doing so Section 5.1 builds on the analysis presented in Chapter 4. In general the students' utilization of the resources changed in the following ways: (1) resources became applicable in new tasks, (2) resources began to support different interpretations, and (3) multiple sets of resources and interpretations became associated with individual tasks (most notable in the case of Hakeem ). The first change, resources becoming applicable in new tasks, reflects a tenet of Constructivist epistemologies, namely that students utilize prior knowledge while constructing new knowledge. In this case, resources that were productive in a certain situation have become associated with being productive in a new situation. This is exemplified by the students' utilization of *equality condition*, *functional dependence, functional variation* and *functional mapping* early in the series of interviews to reason about the relationship between single differential equations and their solutions, and later utilizing these same resources to discuss the relationship between systems of differential equations and their solutions. Here, the situations differed but the resources were productive for accomplishing the same goal: relating differential equations and their solutions. In this case the resources served as valuable pieces of prior knowledge useful for completing tasks involving more mathematically advanced topics.

In the case of the second change in the utilization of resources, providing support to an increasing number of interpretations, this change furnished students with new ways of perceiving or structuring tasks. As the resources became transcendent of different interpretations, the students began to utilize the same resources to attain different goals. For example, in the first interview (I1-T4), *equality condition* was used by Jordan to discuss differential equations as a condition between the value of a quantity and the value of its rate of change. In the last interview, however, Jordan utilized this resource in two a different ways: Jordan applied *equality* *condition* to the relationship between the behavior of a solution and that as expressed by the differential equation (I5-T2). In this way *equality condition* was used in support of a new interpretation and provided him with a means of relating the behavior of the solutions to the behavior indicated by the differential equation.

This did not just impact the how the students perceived and structured new tasks; comparing Jordan's approach to I4-T2 and I5-T3, Jordan's interpretations of the differential equations across from one interaction to the next changed, as did the resources he used to solve the task. In other words, as the utility of certain resources changed the students could use these resources to accomplish more goals and as a result did not need to rely on as many resources for completing the tasks. For instance, recall that during his interaction in Interview 5, Hakeem seemingly no longer needed two sets of resources (and interpretations) and used just one to complete the task. Specifically Hakeem only expressed interpreting the differential equation as a description of the behavior of the quantity of interest and used *functional dependence*, *functional mapping variable substitution, rates indicate behavior*, *functional variation* and *continuity* to solve the task; he did not express interpreting the differential equation as an equation that provided a model of the quantity of interest, nor did he use an analytical solution method to determine the specific solution function. I take this change as indicative of the reorganization of the sets of resources. Though it may be that the resources related to interpreting the differential equation as providing a model of the quantity of interest were still applicable to this task, they were no longer necessary to complete it. In this case an entire set of resources once used to complete the task were not cued, presumably because he could use the above mentioned resources to complete the goals he once needed two sets of resources and interpretations to complete.

In terms of different sets of resources and interpretations becoming viable for the completion of single tasks, by associating the different sets of resources (related with different interpretations) with each task students were supported in both justifying why certain solutions were correct and in leveraging the different information provided by the task. In other words, different interpretations could be utilized productively while in the process of completing a single task. In this way the resources began to support different aspects of the students' understanding of differential equations. This was most notable in the analysis of Hakeem who, towards the end of the series of interviews, expressed two or more interpretations in the majority of the tasks he encountered. Jordan on the other hand did not express as many interpretations. One take away from this is that while for each student the resources themselves supported multiple ways of interpreting and working with the differential equations, this does not mean that the students saw these interpretations or sets of resources as related.

### 6.1.3 Interaction Between Function and Rate of Change

Lastly, Section 5.2 addresses the third research question and discusses how the resources for function and rate of change interacted to support the students' understanding of differential equations. Connections between resources related to function and resources related rate of change were most prevalent during instances when the students interpreted the differential equation as a description of the behavior of the quantity of interest. In these cases the students were actively structuring the solution functions based on the relationship present in the differential equation. The interaction between the resources for the two concepts allowed the students to construct and reason with relationships in the differential equations to build representations of the solution functions, determine long-term behavior of the quantities, structure solution functions, and work with the terms in the differential equations as both functions and values.

In addition, the connection between resources related to function and rate of change did not seem prevalent when the students were interpreting the differential equation as an equation to be solved or as a relation between functions and their derivatives (at least not with regard to the resources identified in this study). Within these two interpretations the students were not explicitly reasoning about the quantities represented by the variables. In these cases the students seemed to be focused, respectively, on constructing algebraic representations of the solutions based on analytical methods and on stating equality in the differential equation after substituting functions into the differential equation. That is not to say that such interpretations (or tasks that elicit them) are not valuable, rather that such interpretations and tasks do not seem, based on the analysis of Hakeem and Jordan, to require an interaction between function and rate of change.

### 6.1.4 Synthesis of Results

As stated in Chapter 1, these research questions were meant to come together to tell a single story about student learning in differential equations; namely to follow the students through their differential equations course and identify resources they used while completing various tasks, how the utilization of those resources changed, and how the resources came together to support the students' understanding of differential equations. In the analysis of the two students emerges a relatively unique story for each with regard to the development of their knowledge structures. While both of the students utilized many of the same resources, their application of them in the individual tasks was different more times than not. Similarly the interpretations of the differential equations and their components within tasks were often different. Though such idiosyncrasies in the knowledge students apply is not novel to Knowledge in Pieces, the differences in the ways the knowledge was (re)organized between the two students is intriguing.

Over the series of interviews Hakeem seemed to develop a multitude of connections between the knowledge resources, and on multiple levels; the same resources began to support Hakeem in making multiple interpretations, the same interpretations (and related sets of resources) became associated with increasingly diverse sets of tasks, and different interpretations (and related sets of resources) became associated with each other for certain tasks. In this way, not only was Hakeem utilizing prior knowledge to complete new tasks involving more “advanced” mathematical topics, but he was also creating and leveraging connections between the resources that constituted his prior knowledge. These connections allowed Hakeem to successfully utilize multiple reasoning strategies while completing a single task.

Jordan, who was also quite successful in completing the tasks, created many connections between the resources that were valuable and highly productive with regard to the mathematics he encountered as he progressed through the course. Over the series of interviews, much like for Hakeem, single resources began to support multiple interpretations, and Jordan became able to use those resources in different tasks to make different inferences as the demands of the different tasks required. Jordan, however, did not seem to establish associations between the interpretations (and their related sets of resources) within individual tasks the way Hakeem did. While there were instances in which Jordan expressed different interpretations and used different sets of resources related to these interpretations to complete tasks, the utilization of the second set of resources and interpretation often followed an unsuccessful attempt to complete the task. While these instances represent productive learning opportunities, why there were such differences in the connections established between the resources is unclear, and certainly warrants further investigation.

In any case, regardless of how the connections developed or the differences in how the resources were organized between the two students, an important take-away is that these connections and organizations involved the same set of resources identified in the analysis presented in Chapter 4. In other words, while different connections between the resources were created, and different sets of the resources were perceived as productive differently between the students, they used the same "core" set of resources to complete the various tasks, just in different combinations based on the students individual perceptions of the task. Practically speaking then, the first step in supporting student understanding of differential equations is fostering the development of these resources.

## 6.2 Contributions

This study makes numerous contributions to the field of undergraduate mathematics education. First, the study is unique in that it makes use of Knowledge in Pieces to examine student learning of differential equations longitudinally over the course of a semester. Knowledge in Pieces was originally developed within the context of physics education (diSessa, 1993), but has been slowly gaining traction in field of mathematics education. This study contributes to the relatively small number of studies utilizing Knowledge in Pieces to examine learners' knowledge in mathematics education (e.g., Adiredja, 2014; Campbell, 2011; Kapon, Ron, Hershkowitz, & Dreyfus, 2015; Pratt & Noss, 2002; Wagner, 2006, 2010). Along these lines, this dissertation begins the first step of the long journey that is determining how students' knowledge is organized around situations involving differential equations.

Secondly, I provide a detailed discussion of students' utilization of fine-grained knowledge over a broad range of tasks, and how that knowledge changes as the students' experience grows over the course of the semester. In doing so I support and build on the findings of other mathematics education researchers across a broad range of domains. For instance, the utilization of *opposite operations reverse* with regard to the role of anti-differentiation in reasoning about differential equations is linked to Raychaudhuri's (2008) finding that students think of solutions to differential equations as being generated by the process of integration. In addition, Whitehead and Rasmussen (2003) noted that students use notions of rate and function to structure solutions for differential equations. More specifically they noted that students were able to reason about solutions to differential equations by almost effortlessly transitioning between substituting values for $y$ and treating $y$ as a continuously changing function by using the value of the rate of change to glean information from the differential equation. Hakeem and Jordan, while interpreting the differential equation as a description of the behavior of the quantity of interest, used *functional dependence, functional variation, functional mapping* and *rates indicate behavior* to accomplish precisely what Rasmussen and Whitehead discussed in their 2003 study. By coordinating these resources to determine the nature of the relationships between the values in the differential equation, and then how those values change Jordan and Hakeem were able to construct solution functions from the differential equations.

As a last example, the resources identified herein provide a finer level of detail from which researchers can describe and explain how students can develop an understanding of ordinary differential equations as functions. In the case of Jordan and Hakeem, by coordinating the resources that supported the students in perceiving (or recognizing) and utilizing the relations between the components in the differential equation, and resources that supported the establishment of a specific characterization of that relation across different representations, they were able to work with the relationship between the quantities and their rates of change in ways that were commensurate with understanding of the differential equation as a function. As noted by Donovan (2007) such understanding is important with regard to their ability to structure solutions to differential equations and understand the relationship between differential equations and their solutions. The analysis presented in this dissertation explicates the resources students utilized and the various ways students utilize them while constructing solutions and relating them their respective differential equations.

Another contribution is made by the identification of the interpretations made by the students with regard to the differential equations and their components. The interpretations provide a characterization of the different ways students understand differential equations and the mathematical entities from which they are composed. Further they highlight general patterns in the ways in which students structure and reason with various tasks. For instance, when interpreting the differential equations as a description of the behavior of the quantity of interest, the students reasoned from relationships present in the differential equation to characterize the solution functions. Additionally, in terms of interpreting the differential equation as an equation that provided a model of the differential equation, the students solved the differential equation for the equation that related the value of the to some independent variable and then worked with that equation to characterize the quantity of interest. In this case the differential equation was not used to describe the quantity of interest but rather provided a way to get an equation used to model the quantity of interest.

These interpretations did not only provide insight into the students' general approaches to completing the task; differences in the individual students' applications of knowledge across the tasks were accounted for by the different interpretations. As their experience grew with the topic, the students were able to perceive the variables as representing different mathematical entities (values, slopes, functions, and in some cases even as operations; Hakeem expressed $y'$ as representing an operation on $y$), which was related to how they interpreted the differential equations.

Lastly, the fact that Hakeem's and Jordan's knowledge allowed them to perceive attributes differently in the same instances highlights an additional complexity within the students' organization of knowledge. Namely, the students began to associate resources that supported them in interpreting attributes differently in a particular situation. The notion that students must conceptualize the variables in the differential equations as representing multiple objects is not new; Stephan and Rasmussen (2002) discussed the emergence of this as a classroom mathematics practice when analyzing a community of students in an IO-DE classroom. As previously noted Whitehead and Rasmussen (2003) also suggested the importance of this while structuring solutions to differential equations. The ability to perceive a single attribute in a situation in multiple ways (in this case entities in the differential equation as both a function and a variable), however, is not something explicitly addressed by Knowledge in Pieces and reflects a potential need to account for this within the organizational structure of knowledge. Hakeem's utilization of multiple interpretations of the differential equations and their entities seems to support this; strictly speaking it wasn't the situation that changed, but rather his perception of the differential equations and their components that differed. This suggests that students are not limited to associating single knowledge elements with any one particular type of situation.

### 6.3 Practical implications

The students in this study were quite successful in completing the various tasks they encountered over the series of interviews. Their ability to be successful was supported by the large number of resources they had available to them, their flexibility with regard to being able to interpret the differential equations in different ways, and the connections they made between the resources and they tasks in which they perceive those resources as productive. Along these lines, when teaching students differential equations mathematics educators should strive for providing opportunities for students to construct these resources, interpretations, and connections. Based on the analysis of Hakeem and Jordan, such learning opportunities may be created by providing students with tasks that prompt them to focus on relationships between the variables (and functions) in the differential equations, the quantities those variables (and functions) represent, and the connections between differential equations and their solutions. Further, opportunities to promote student understanding can be created by supporting the utilization of productive knowledge across a wide array of situations; more specifically by using sets of instructional tasks that include different representations of differential equations (e.g., vector fields, algebraic equations, and graphical representations) across different situations. Such tasks provide students with opportunities to make use of prior knowledge in new situations, instances which can foster the students understanding of differential equations.

With regard to the resources themselves, there were quite a few resources which were identified across a wide array of tasks and interpretations, which seems to indicate their overarching importance with regard to student understanding of differential equations. Resources such as *functional mapping*, *functional variation*, *functional dependence*, *rates indicate behavior* and *equality condition* were utilized by the students when solving the differential equations analytically, working with the relationships in the differential equations, and discussing the connections between the differential equations and their solutions. Furthermore, these resources were productive across many topics within differential equations such as first and second order differential equations, systems of differential equations, as well as linear and non-linear differential equations. With this in mind, developing resources such as these is an important aspect of fostering a robust understanding of differential equations.

In addition, it may be the case that the different interpretations expressed by Hakeem and Jordan can provide entry points for students learning differential equations for the first time. More research is needed to see if these ways of interpreting the differential equations generalize across students in other classes and universities. Given that the interpretations played a significant role in Hakeem's and Jordan's problem solving activity, however, it seems promising that they would serve as valuable ways of teaching students different topics within differential equations. For instance, fostering the understanding that variables in the differential equations are also functions could potentially be accomplished by leveraging students' innate (yet separate) interpretations of differential equations as relationships between functions, and as relationships between values. Along these lines, it is important to note that neither Jordan nor Hakeem seemed to switch back and forth between interpretations in the process of completing a single task. Additionally, in Jordan's case there did not seem to be established connections between these interpretations. With regard to teaching, this implies using different ways of explaining differential equations to students also requires drawing connections between those different explanations.

In terms of student learning, this research suggests that understanding differential equations is composed of a combination of multiple ways of conceptualizing differential equations and the entities they are relating, and resources that support successfully applying those conceptualizations in different situations. For instance, it may not be necessary for students to always structure solutions based on the relationships in the differential equation; they may productively complete certain tasks (in a mathematically rigorous way) using alternative solution methods and resources that support the completion of different steps that lead to attaining the same goal. Thusly if one were to develop a framework designed to characterize student learning of differential equations it would be beneficial to incorporate these different conceptualizations.

Looking forward, based on the results it seems plausible that a student's understanding of differential equations may first develop around situations cueing resources centered on relationships between values and then transition to supporting the student in perceiving and structuring tasks in ways that cue resources productive for reasoning about relationships between functions. While more research is needed to determine how such an understanding develops, this is commensurate with Rasmussen's (2001) assertion that understanding solutions as functions is a "fundamental leap" for students and Whitehead and Rasmussen's (2003) claim that understanding terms in differential equations as both a variable and a function is one of the most difficult images for students to construct. This also aligns with the fact that Hakeem was able to reason about the differential equations as relationships between values and functions, but Jordan did not express reasoning with the differential equation as relationships between functions. Together with Whitehead and Rasmussen's claim, this seems to imply that building such an image requires the construction of resources that allow the students to perceive the terms as both functions and variables in a single situation. Developmentally speaking it seems plausible that this demands first associating situations in which the terms are perceived as functions with situations in which the terms are perceived as variables.

## 6.4 Constraints

There are a few constraining factors associated with this study. Although the participants in this study were students from an honors section and a regular section, the fact that the dissertation only concerns two students is limiting in terms of generalizability of the study.

Further study is warranted investigating (changes in) the knowledge utilized by other students both within these particular courses and at other universities. Additionally, the findings may be limited to students that have taken a course where the focus is on analytical methods. Despite the fact that the students were enrolled in two differential courses, with two different instructors, both of the courses heavily focused on material involving analytical solution methods. While this is of great value as a significant number of differential equations courses place emphasis on these topics, it is unknown how these results would compare to students enrolled in a course that focuses on other solution methods.

Further, I was the only researcher performing the analysis of the students' knowledge. Though there were instances in which I leveraged the insight and experience of other mathematics education researchers by seeking their validation of various portions of my analysis, ultimately, I was the only researcher responsible for identifying the students' utilization of resources. This impacted the scope of the resources identified, the characterizations of the resources, and my ability to consider multiple alternative explanations. Had there been additional researchers involved in the identification of the resources, their additional perspectives and experiences may have lead to the identification of additional resources.

Finally, although the tasks posed to the students over the course of the five interviews consisted of a variety of different types of differential equations and representations thereof, they did not encompass certain aspects of differential equations typically encountered in an introductory course. For instance, tasks did not involve existence and uniqueness theorems or the law of superposition. Thusly it may be the case that resources identified in this study serve as productive knowledge for tasks involving these topics as well. Or, alternatively that students use additional resources, not identified in this study, while completing such tasks. For these reasons, more research is warranted to identify resources the students use while completing tasks involving the various theorems students encounter in an introductory differential equations course.

### 6.5 Future research

One of the goals underlying this study was to take a step toward understanding how students come to utilize prior knowledge in a new situation while learning differential equations. From the perspective of Transfer in Pieces this requires first structuring the new situation in a way that is similar to that of a previously encountered situation. While I believe this dissertation did take a significant step in understanding the structures imposed by the students, the complexities involved with regard to the rather large number of resources students used while completing individual tasks, and the different ways those resources became associated over time raises more questions than answers. For instance, were there resources the students used to relate only specific mathematical entities, and what contextual attributes were these resources organized around? In other words, how can one account for the students' abilities to express the same interpretations over such a diverse set of tasks?

In addition there was a significant amount of variation in the sets of resources the students utilized even when expressing the same interpretations. This also raises important questions regarding how students perceive or structure situations as similar. Namely, if students structure vastly different tasks similarly, and use significantly different resources in the process of completing the tasks, what exactly is it about the tasks that the students perceive as similar, and what does it mean for two tasks to be similar?

Lastly, but perhaps most importantly, given the importance of the resources identified in this study, it would be beneficial to know in what courses these resources developed, how they developed, and how their development can be supported. Some resources, such as those related to analytical solution methods, were certainly developed with the course, but others such as *functional variation* and *rates indicate behavior* were likely developed in previous courses. This provides merit for studies identifying resources such as those discussed here in other mathematical domains.

# REFERENCES

Adiredja, A. (2014). *Leveraging Students' Intuitive Knowledge About the Formal Definition of a Limit* (Unpublished doctoral dissertation). University of California, Berkeley CA.

Arslan, S. (2010). Do students really understand what an ordinary differential equation is? *International Journal of Mathematical Education in Science and Technology*, *41*(7), 873-888.

Bajracharya, R., and Thompson, J. (2014) Student understanding of the fundamental theorm of calculus at the mathematics-physics interface. *Proceedings from Eighteenth Annual Conference on Research in Undergraduate Mathematics Education*

Black, K. E., & Wittmann, M. C. (2009). *Understanding the use of two integration methods on separable first order differential equations*. arXiv preprint arXiv:0902.0748.

Camacho-Machín, M., Perdomo-Díaz, J., & Santos-Trigo, M. (2012). An exploration of students' conceptual knowledge built in a first ordinary differential equations course (Part I). *The Teaching of Mathematics, XV*(1), 1–20.

Campbell, M. E. (2011). Modeling the co-development of strategic and conceptual knowledge in mathematical problem solving. (Doctoral dissertation). Retrieved from http://gradworks.umi.com/3498781.pdf

Carlson, M., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for Research in Mathematics Education*, 352-378.

Christensen, W. M., & Thompson, J. R. (2012). Investigating graphical representations of slope and derivative without a physics context. *Physical Review Special Topics-Physics Education Research*, *8*(2), 023101.

Clement, J. (2000). Analysis of clinical interviews: Foundations and model viability. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (547-589). Mahwah, NJ: Lawrence Erlbaum Associates.

Cui, L., Rebello, N. S., & Bennett, A. G. (2006, February). College students' transfer from calculus to physics. In *2005 Physics Education Research Conference* (Vol. 818, pp. 37-40).

Dana-Picard, T. & Kidron, I. (2008). Exploring the phase space of a system of differential equations: Different mathematical registers. *International Journal of Science and Mathematics Education*, *6*, 695–717.

DiSessa, A. A. (1993). Toward an epistemology of physics. *Cognition and instruction*, *10*(2-3), 105-225.

Disessa, A. A., & Sherin, B. L. (1998). What changes in conceptual change?. *International Journal of Science Education*, *20*(10), 1155-1191.\

DiSessa, A. A., & Wagner, J. F. (2005). What coordination has to say about transfer. *Transfer of learning from a modern multi-disciplinary perspective*, 121-154.

Donovan, J. E. I. (2002). Students' understanding of first-order differential equations. (Doctoral dissertation, State University of New York at Buffalo). Available from *ProQuest Dissertations and Theses database*. (UMI No. 3063116)

Donovan, J.E., II. (2007). The Importance of the Concept of Function for Developing Understanding of First-Order Differential Equations in Multiple Representations [Electronic Version]. *Thirteenth Annual Conference on Research in Undergraduate Mathematics Education* from http://cresmet.asu.edu/crume2007/eproc.html.

Dubinsky, E., & Harel, G. (1992). *The concept of function: Aspects of Epistemology and Pedagogy* (Vol. 25). Washington, DC: Mathematical Association of America.

Dubinsky, E., & McDonald, M. A. (2002). APOS: A constructivist theory of learning in undergraduate mathematics education research. In *The teaching and learning of mathematics at university level* (pp. 275-282). Springer Netherlands.

Glasersfeld, E. V. (1979). Cybernetics, experience, and the concept of self. *A cybernetic approach to the assessment of children: Toward a more humane use of human beings*, 67-113.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (517-545). Mahwah, NJ: Lawrence Erlbaum Associates.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A" proceptual" view of simple arithmetic. *Journal for research in Mathematics Education*, 116-140.

Habre, S. (2000). Exploring students' strategies to solve ordinary differential equations in a reformed setting. *Journal of Mathematical Behavior, 18*(4), 455–472.

Hammer, D. (2000). Student resources for learning introductory physics. *American Journal of Physics*, *68*(S1), S52-S59.

Hammer, D., & Elby, A. (2003). Tapping epistemological resources for learning physics. *The Journal of the Learning Sciences*, *12*(1), 53-90.

Ju, M. K., & Kwon, O. N. (2007). Ways of talking and ways of positioning: Students' beliefs in an inquiry-oriented differential equations class. *The Journal of Mathematical Behavior*, *26*(3), 267-280.

Keene, K. A. (2007). A characterization of dynamic reasoning: Reasoning with time as parameter. *The Journal of Mathematical Behavior*, *26*(3), 230-246.

Keene, K. A (2008). Ways of reasoning: Two case studies in an inquiry-oriented differential equations class. In *Proceedings of the Thirtieth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

Keene, K. A., Rasmussen, C., & Stephan, M. (2012). Gestures and a chain of signification: the case of equilibrium solutions. *Mathematics Education Research Journal*, *24*(3), 347-369.

Kohler, W. E., & Johnson, L. W. (2003). *Elementary differential equations*. ($2^{nd}$ ed.) Boston, MA: Addison-Wesley.

Kapon, S., Ron, G., Hershkowitz, R., & Dreyfus, T. (2014). Perceiving permutations as distinct outcomes: the accommodation of a complex knowledge system. *Educational Studies in Mathematics*, 1-22.

Lobato, J. E. (1996). Transfer reconceived: How sameness is produced in mathematical activity. Unpublished doctoral dissertation, University of California, Berkeley.

Lobato, J. (2003). How Design Experiments Can Inform a Rethinking of Transfer andViceVersa. *Educational Researcher, 32*(1), 17-20.

Lobato, J. (2006). Alternative perspectives on the transfer of learning: History, issues, and challenges for future research. *The Journal of the Learning Sciences*, *15*(4), 431-449.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research*, 60(1), 1-64.

Mallet, D. G., & McCue, S. W. (2009). Constructive development of the solutions of linear equations in introductory ordinary differential equations. *International Journal of Mathematical Education in Science and Technology*, *40*(5), 587-595.

Markovits, Z., Eylon, B. S., & Bruckheimer, M. (1986). Functions today and yesterday. *For the learning of mathematics*, 18-28.

Marnyanskii, I. A. (1975). Psychological characteristics of pupils' assimilation of the concept of a function. *Soviet studies in the psychology of learning and teaching mathematics*, *13*, 163-172.

Monk, S. (1992). Students' understanding of a function given by a physical model. *The concept of function: Aspects of epistemology and pedagogy*, *25*, 175-194.

Nguyen, D. H., & Rebello, N. S. (2011). Students' understanding and application of the area under the curve concept in physics problems. *Physical Review Special Topics-Physics Education Research*, *7*(1), 010112.

Pratt, D. & Noss, R. (2002). The Microevolution of Mathematical Knowledge: The Case of Randomness. *Journal of the Learning Sciences*, 11(4), 455–488.

President's Council of Advisors on Science and Technology. (2012). *Engage to excel*. Washington, DC: White House.

Rasmussen, C. (2001). New directions in differential equations: A framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior, 20*, 55-87.

Rasmussen, C., & Blumenfeld, H. (2007). Reinventing solutions to systems of linear differential equations: A case of emergent models involving analytic expressions. *The Journal of Mathematical Behavior*, *26*(3), 195-210.

Rasmussen, C. L., & King, K. D. (2000). Locating starting points in differential equations: A realistic mathematics education approach. *International Journal of Mathematical Education in Science and Technology*, *31*(2), 161-172.

Rasmussen, C., & Kwon, O. (2007). An inquiry oriented approach to undergraduate mathematics. *Journal of Mathematical Behavior, 26*, 189-194.

Rasmussen, C. & Stephan, M. (2001). Creating the space of solution functions to differential equations. In *Proceedings of the Twenty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

Rasmussen, C., & Rhodehamel, B. (2006). Students' proofs for the shape of graphs of solutions in the phase plane. In *Proceedings of the Twenty-Sixth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

Rasmussen, C., & Wawro, M. (in press). Post calculus research in undergraduate mathematics education. Invited chapter in J. Cai, (Ed.), *The compendium for research in mathematics education.* Reston VA: National Council of Teachers of Mathematics.

Rasmussen, K. W. C. (2003). Undergraduate Sstudents' mental operations in systsms of differential equations. In *Proceedings of the Twenty-Third Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*.

Raychaudhuri, D. (2007). A layer framework to investigate student understanding and application of the existence and uniqueness theorems of differential equations. *International Journal of Mathematical Education in Science and Technology*, *38*(3), 367-381.

Raychaudhuri, D. (2008). Dynamics of a definition: a framework to analyze student construction of the concept of solution to a differential equation. *International Journal of Mathematical Education in Science and Technology*, *39*(2), 161-177.

Raychaudhuri, D. (2014). Adaptation and extension of the framework of reducing abstraction in the case of differential equations. *International Journal of Mathematical Education in Science and Technology*, *45*(1), 35-57.

Roschelle, J. (2000). Choosing and using video equipment for data collection. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (709–729). Mahwah, NJ: Lawrence Erlbaum Associates.

Rossman, G. B., & Rallis, S. F. (2003). *Learning in the field: An introduction to qualitative research*. Sage.

Seidman, I. (2006). *Interviewing as Qualitative Research: A Guide for Researchers in Education and the Social Sciences*. Teachers College Press.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, 22, 1-36.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification-the case of function. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (25 ed., pp. 59-84). Washington DC: Mathematical Association of America.

Sfard, A. (1994). Reification as the birth of a metaphor. *For the learning of mathematics*, 14(1), 44 - 55.

Sfard, A., & Thompson, P. W. (1994). *Problems of reification: Representations and mathematical objects.* Paper presented at the Annual Meeting of the International Group for the Psychology of Mathematics Education– North American Chapter, Baton Rouge, LA.

Smith III, J. P., Disessa, A. A., & Roschelle, J. (1994). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *The journal of the learning sciences*, *3*(2), 115-163.

Steffe, L. P., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267–306). Mahwah, NJ: Lawrence Erlbaum Associates.

Stephan, M., & Rasmussen, C. (2002). Classroom mathematical practices in differential equations. *The Journal of Mathematical Behavior, 21*(4), 459–490.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational studies in mathematics*, *12*(2), 151-169.

Thompson, P. W. (1994). Images of rate and operational understanding of the Fundamental Theorem of Calculus. Educational Studies in Mathematics, 26(2-3), 229–274.

Tremblay, M. A. (1989). The key informant technique: A non-ethnographic application. In R. G. Burgess (Ed.)., *Field research: A sourcebook and manual* (pp. 151-161). New York: Routledge.

Trigueros, M. (2004). Understanding the meaning and representation of straight line solutions of systems of differential equations. In McDougall, D.E & Ross, J. A. (Eds.). (2004). *Proceedings of the twenty-sixth annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*

Vinner, S., & Dreyfus, T. (1989). Images and Definitions for the Concept of Function. *Journal for Research in Mathematics Education*, 20(4), 356-366.

Von Glasersfeld, E. (1995). *Radical Constructivism: A Way of Knowing and Learning. Studies in Mathematics Education Series: 6*. Falmer Press, Taylor & Francis Inc., Bristol, PA

Wagner, J. F. (2006). Transfer in pieces. *Cognition and Instruction*, *24*(1), 1-71.

Wagner, J. F. (2010). A transfer-in-pieces consideration of the perception of structure in the transfer of learning. *The Journal of the Learning Sciences*, *19*(4), 443-479.

Wittmann, M. C. (2006). Using resource graphs to represent conceptual change. *Physical Review Special Topics-Physics Education Research*, *2*(2), 020105.

Wittmann, M. C., & Black, K. E. (2015). Mathematical actions as procedural resources: An example from the separation of variables. *Physical Review Special Topics-Physics Education Research*, *11*(2), 020114.

Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. *CBMS Issues in Mathematics Education*, *8*, 103-122.

Zandieh, M., & McDonald, M. (1999). Student understanding of equilibrium solution in differential equations. In *Proceedings of the 21st annual meeting of the North American chapter of the international group for the psychology of mathematics education* Columbus: ERIC.

## APPENDIX

Interview 1:

TASK 1: In this task, we look at systems of rate of change equations designed to inform us about the future populations for two species that are either competitive (that is, both species are harmed by interaction), or cooperative (that is, both species benefit from interaction, for example bees and flowers). Which system of rate of change equations describes competing species and which system describes cooperative species? Explain your reasoning. (Rasmussen & Whitehead, 2003)

| **A** | **B** |
|---|---|
| $\frac{dx}{dt} = -5x + 2xy$ | $\frac{dx}{dt} = 3x - 2xy$ |
| $\frac{dy}{dt} = -4y + 3xy$ | $\frac{dy}{dt} = y - 4xy$ |

TASK 2: What does the following differential equation mean to you?

$$P' = 3P$$

TASK 3: Suppose the equation $P' = 2P + 2t$ can be used to model the fish population in the duck pond. How might it be used to determine the number of fish in the pond at a given time?

TASK 4: Provide a differential equation which has the equilibrium solution $y = 6$.

Interview 2:

TASK 1: In this task, we look at systems of rate of change equations designed to inform us about the future populations for two species that are either competitive (that is, both species are harmed by interaction), or cooperative (that is, both species benefit from interaction, for example bees and flowers). Which system of rate of change equations describes competing species and which system describes cooperative species? Explain your reasoning. (Rasmussen & Whitehead, 2003)

| **A** | **B** |
|---|---|
| $\frac{dx}{dt} = -5x + 2xy$ | $\frac{dx}{dt} = 3x - 2xy$ |
| $\frac{dy}{dt} = -4y + 3xy$ | $\frac{dy}{dt} = y - 4xy$ |

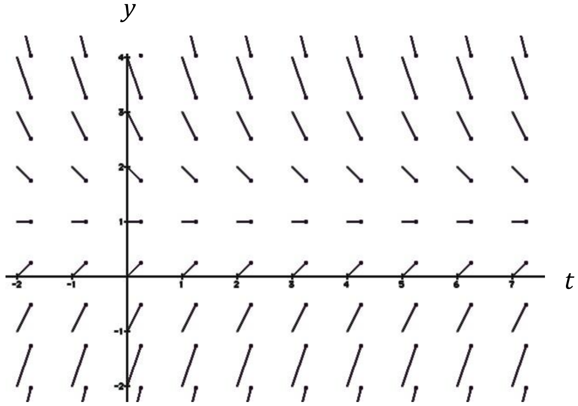
TASK 2: What does the following differential equation mean to you?

$$P'' = 3P' + P$$

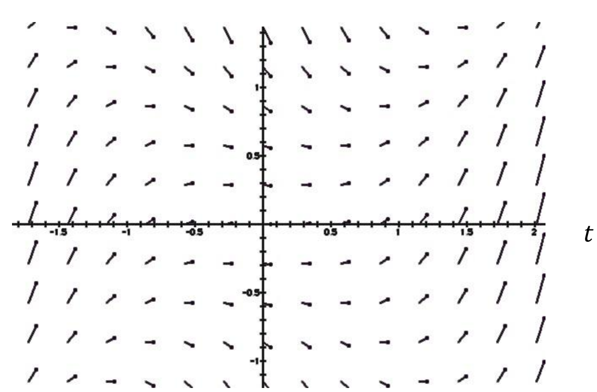
TASK 3: Suppose the equation $y' + 4ty = 16t$ can be used to model the fish population $y$ in the campus Duck Pond. Determine the number of fish in the pond at a given time $t$, if the initial population is 5.

TASK 4: Below are three different tangent vector fields and six rate of change equations. Without using technology, identify which differential equation is the best match for each tangent vector field (Thus you will have three rate of change equations left over). Explain your reasoning.

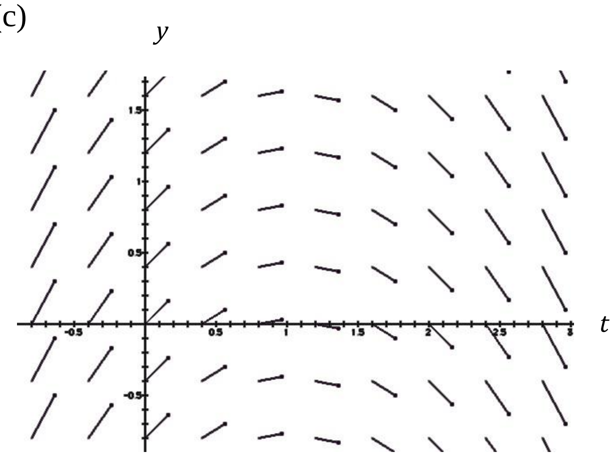
(a)




(b)




(c)




$\frac{dy}{dt} = t - 1$ $\quad \frac{dy}{dt} = 1 - y^2$ $\quad \frac{dy}{dt} = y^2 - t^2$ $\quad \frac{dy}{dt} = 1 - y$ $\quad \frac{dy}{dt} = t^2 - y^2$ $\quad \frac{dy}{dt} = 1 - t$

* Task from IO-DE Curriculum (Rasmussen & Kwon, 2007)

Interview 3:

TASK 1: Consider the following systems of rate of change equations:

System A

$$\frac{dx}{dt} = 3x\left(1 - \frac{x}{10}\right) - 20xy$$
$$\frac{dy}{dt} = -5y + \frac{xy}{20}$$

System B

$$\frac{dx}{dt} = 0.3x - \frac{xy}{100}$$
$$\frac{dy}{dt} = 15y\left(1 - \frac{y}{17}\right) + 25xy$$

In both of these systems, *x* and *y* refer to the number of two different species at time *t*. In particular, in one of these systems the prey are large animals and the predators are small animals, such as humans and piranhas. The other system has very large predators and very small prey.

Which system represents the small prey and large predator? Which system represents the large prey and small predator? Explain your reasoning. (IODE materials, Rasmussen & Kwon, 2007)

TASK 2: What does the following differential equation mean to you?

$$P'' = -6P' - 5\mathrm{P}$$

TASK 3: A group of biologists are making predictions about the spotted owl population in a forest in the Pacific Northwest. The differential equation the scientists use to model the spotted owl population is $\frac{dP}{dt} = \frac{P}{2}\left(1 - \frac{P}{5}\right)$, where *P* is in hundreds of owls and *t* is in years. The problem is that the current number of owls is only approximately known.

Because they can only get an estimate of the population, they make long-term predictions of the owl population for various initial conditions. Two of these initial conditions are *P* = 3.0 and *P* = 5.0. Determine the long-term predictions for these initial conditions based on the differential equation. (IODE materials, Rasmussen & Kwon, 2007)

TASK 4: Suppose the equation $B'(t) = 30 - 1.5B(t)$ can be used to model the temperature B, in degrees Fahrenheit of a cooling metal object. How might it be used to determine the temperature of the object at a given time *t*?