CHANGE-OVER DESIGNS

by

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INTRODUCTION

In dairy husbandry, biological assay, agricultural crop rotation trials and various other fields, it is sometimes desirable or even necessary to apply different combinations of treatments in succession to the same subject or plot. At first experimenters, realizing that the effects of a particular treatment might be affecting the treatments applied after it, decided to leave an interval of time between two successively applied treatments. It was hoped that any lingering or "residual" effects would become negligible. For example, if the experiment consisted of testing animal feeds, a control feed was fed to the subjects during the interval lapse. This control interval necessarily increased the length of time necessary to complete the experiment. In some experiments the time factor is of critical importance. For example, in cow feeding experiments there is a necessity to complete the experiment during a single lactation, there being only so many months during the milking period.

A number of experimental designs have been constructed to eliminate the need for this "rest" interval, and in addition supply information about the residual or carry-over effects of a treatment from one time period to another. These designs are known by several names. Among them are change-over designs, carry-over designs, switch-over designs and cross-over designs.

At first these designs were not used to eliminate residual effects; the method which was commonly used " . . . was to base interpretations on the performances during only the latter portions of the experimental periods". [7]. Cochran, et. al. [3] in 1941, were the first to incorporate an analysis which permitted the elimination of these effects.

Since then other designs have been constructed. Advantages of these designs follow: (1) ease of analysis, (2) fewer observations, and (3) elimination of effects of treatments applied two or more periods before an observation.

As Williams [2] points out, there are two basic limiting factors on the feasibility of these designs: (1) the time element, and (2) the suitability of the subject or plot for repeated applications of different treatments.

Once the experimenter examines these factors, he must decide how much time can be allotted for each subject. Then he must divide this allotted time into "periods"; a different treatment will be applied to each period on each subject. The number of possible subjects must also be determined. Then he should estimate how many periods the residual effects of a treatment can be expected to last, as the number of residual effects eliminated affects the efficiency of the design. If it is the first time an experiment of this type is to be run, intuition will play a large part in this determination. If other experiments have been run before, the results may be of some aid. All of these factors contribute to choosing the proper design.

The purpose of this paper is to give the reader a rather broad look at change-over designs. No specific examples will be given as the literature already contains many, and usually the reference cited for a design contains at least one.

An extensive list of available designs will be presented in an appendix so the experimenter need not consult other sources when choosing his design. It will be necessary, however, to limit the number of treatments discussed to nine, most of these designs dealing with three, four, five or six treatments. This limitation is not seriously confining for most practical situations.

A detailed analysis of a number of major types of cross-over designs will be discussed. These designs come from the following sources. If no source is indicated, the designs were constructed by the author.

Type I Designs - Cochran, et. al. [3]
Type II Designs - Lucas [7]
Type III Designs - Patterson [4]
Type IV Designs - Berenblut [11]
Type V Designs - Williams [5]
Type VI Designs -
Type VII Designs - Berenblut [9] and [13]
Type VIII Designs -
Type IX Designs -

Normal equations, analysis of variance, variance estimates and efficiency comparisons will be presented for Designs I through V, also VII and VIII. No efficiencies are computed for Design IX. Also missing value formulae for one and two missing observations will be presented for Designs I through V. Designs I through IV are primarily for estimation of first-order residual effects. Designs V and VI are for second-order residual effects. Designs VII, VIII, and IX are orthogonal for the linear component of first-order residual effects, VII and VIII dealing with one treatment and IX dealing with more than one.

Lastly, some miscellaneous designs will be presented without analysis.

Unless otherwise stated, an assumption of additivity of all treatment effects will be assumed. A test will be given for this additivity under the assumption that the treatments correspond to equally spaced levels of a given treatment or treatments.

II

DESCRIPTION, CONSTRUCTION AND ANALYSIS

2.1 Notation

In the analysis of even one change-over design there is a necessity for a large amount of notation. In analyzing the designs, the following set of notations will hold unless otherwise stated:

n - number of treatments

m - number of squares

p - number of periods

T_v - total for treatment "v"

$T_j(k)$ - total for period "j" in square "k"

TC_i - total for subject "i"

$T_{k\cdot\cdot}$ - total for square "k"

G - over-all total

A_s - total of all observations immediately following treatment "s"

B_t - total of all observations following treatment "t" by two periods

T_{ab} - total of all observations which receive the direct effect of treatment "a" and the first order residual effect of treatment "b"

F'_v - sum of all subject totals receiving treatment "v" last

F''_v - sum of all subject totals receiving treatment "v" in the next to the last period

P_r - $\sum_{(r)} T_r(k)$; the sum of all the r^{th} period observations

M' - last treatment in the subject to which it refers

M" - next to the last treatment in the subject to which it refers

(TC_v) - sum of all subject totals which include treatment "v"

$\{TC_v\}$ - sum of all subject totals which include treatment "v", and treatment "v" is not in the last period; or

$\{TC_v\} = (TC_v) - F'_v$

The absence of a variable on a summation sign indicates summation over the entire range of each subscript.

The symbol $\sum_{(k)}$ shall indicate summation over all subscripts, subscript k held constant. Usually k will not be a subscript of the effect, but will be used rather to indicate summation of the effect over all its subscripts which exist for a given k.

2.2 Type I Designs

These designs were first discussed by Cochran, et. al. [3]; refer to Designs 1, 5, 9, 10, 13, 14, 17, 22, 23, 24, 25, 26, 29, 30, 31 and 32 for examples.

The basis for these designs is a "balanced" set of m Latin squares. Williams [2] states the following two conditions for balance:

(1) each treatment shall be preceded by each other treatment equally often and (2) each treatment shall occur equally often at each position, in order of application to the sites (so that the treatment effects shall be unaffected by possible effects of order of application).

Williams lists two advantages of a balanced design over an unbalanced design, these being (1) increased efficiency (more accurate estimates of effects) and (2) simplification of analysis.

He also proves that the above-mentioned balance can be achieved by using Latin squares, any number of squares for an even number of treatments and an even number of squares for an odd number of treatments.

The direct and first order residual effects from these designs can easily be seen to be non-orthogonal (the orthogonal case will be presented as Design II). From this non-orthogonality one finds there are two separate ways to compute the sums of squares for treatment effects. Both of these methods will be presented and will be as follows: (1) Direct (adjusted for residual) + Residual (unadjusted) and (2) Direct (unadjusted) + Residual (adjusted for direct).

The model for this design is

$$Y_{ijkvs} = \mu + C_i + \rho_j(k) + \tau_v + \theta_s + \varepsilon_{ijkvs}$$

$i = 1, 2, \ldots, mn$

$j = 1, 2, \ldots, n$

$k = 1, 2, \ldots, m$

$v = 1, 2, \ldots, n$

$s = 1, 2, \ldots, n$

where

C_i is the effect of the i^{th} subject,

$\rho_j(k)$ is the effect of the j^{th} period in the k^{th} square,

τ_v is the effect of the v^{th} treatment,

Θ_s is the effect of the s^{th} treatment on the observation which immediately follows it, and $\epsilon_{ijkvs} \sim N(0,\sigma_\epsilon^2)$.

The analysis comes from least square theory. The equation

$$\Sigma(\epsilon_{ijkvs})^2 = \Sigma(Y_{ijkvs} - \mu - C_i - \rho_j(k) - \tau_v - \Theta_s)^2$$

is minimized with respect to μ, C_i, $\rho_j(k)$, τ_v and Θ_s, and the following normal equations are obtained:

$$n^2m\hat{\mu} + n\Sigma\hat{C}_i + n\Sigma\hat{\rho}_j(k) + nm\Sigma\hat{\tau}_v + m(n-1)\Sigma\hat{\Theta}_s = G$$

$$n\hat{\mu} + n\hat{C}_i + n\Sigma\hat{\rho}_j(k) + \Sigma\hat{\tau}_v + \Sigma^i\hat{\Theta}_s = TC_i \qquad i = 1, 2, \ldots, nm$$

$$n\hat{\mu} + \sum_{(k)}\hat{C}_i + n\hat{\rho}_1(k) + \Sigma\hat{\tau}_v = T_1(k) \qquad k = 1, 2, \ldots, m$$

$$n\hat{\mu} + \sum_{(k)}\hat{C}_i + n\hat{\rho}_j(k) + \Sigma\hat{\tau}_v + \Sigma\hat{\Theta}_s = T_j(k) \qquad \begin{matrix} j = 2, 3, \ldots, n \\ k = 1, 2, \ldots, m \end{matrix}$$

$$nm\hat{\mu} + \Sigma\hat{C}_i + \Sigma\hat{\rho}_j(k) + nm\hat{\tau}_v + m\Sigma^{ii}\hat{\Theta}_s = T_v \qquad v = 1, 2, \ldots, n$$

$$m(n-1)\hat{\mu} + \Sigma^{iii}\hat{C}_i + \sum_{j=2}^{k}\hat{\rho}_j(k) + m\Sigma^{iv}\hat{\tau}_v + m(n-1)\hat{\Theta}_s = A_s$$

$$s = 1, 2, \ldots, n$$

where

$\Sigma^{i}\hat{\Theta}_s$ = the sum over all residual effects except when s is the last treatment in subject i,

$\Sigma^{ii}\hat{\Theta}_s$ = the sum over all residual effects except where s is the same as treatment v,

$\Sigma^{iii}\hat{C}_i$ = the sum over all subjects where s is not the final treatment,

and $\Sigma^{iv}\hat{\tau}_v$ = the sum over all v such that v is not the same as s.

One now applies the following constraints:

$$\Sigma\hat{C}_i = \sum_{(k)}\hat{\rho}_j(k) = \Sigma\hat{\tau}_v = \Sigma\hat{\Theta}_s = 0 .$$

Then the equations become

$$n^2m\hat{\mu} = G$$

$$n\hat{\mu} + n\hat{C}_i - \hat{\Theta}_{M'} = TC_i \qquad i = 1, 2, \ldots, nm$$

$$n\hat{\mu} + \sum_{(k)}\hat{C}_i + n\hat{\rho}_j(k) = T_j(k) \qquad \begin{array}{l} j = 1, 2, \ldots, n \\ k = 1, 2, \ldots, m \end{array}$$

$$nm\hat{\mu} + nm\hat{\tau}_v - m\hat{\Theta}_v = T_v \qquad v = 1, 2, \ldots, n$$

$$m(n-1)\hat{\mu} - \Sigma^{v}\hat{C}_i + \sum_{j=2}^{n}\hat{\rho}_j(k) - m\hat{\tau}_s + m(n-1)\hat{\Theta}_s = A_s \qquad s = 1, 2, \ldots, n$$

where $\Sigma^v \hat{C}_i$ = the sum over all those subjects where treatment s is in the final period.

Expressing $\hat{C}_i$ and $\hat{\rho}_j(k)$ in terms of $\hat{\tau}_v$ and $\hat{\theta}_s$ one arrives at the following set of reduced normal equations for $\hat{\tau}_v$ and $\hat{\theta}_v$.

$$\begin{bmatrix} nm & -m \\ -m & \frac{m}{n}(n^2-n-1) \end{bmatrix} \begin{bmatrix} \hat{\tau}_v \\ \hat{\theta}_v \end{bmatrix} = \begin{bmatrix} T'_v \\ A'_v \end{bmatrix} \qquad v = 1, 2, \ldots, n$$

where $$T'_v = T_v - \frac{1}{n} G$$

$$A'_v = A_v + \frac{1}{n^2}[nP_1 + nF'_v - (n+1)G] \quad .$$

Then
$$\begin{bmatrix} \hat{\tau}_v \\ \hat{\theta}_v \end{bmatrix} = \frac{1}{m^2(n^2-n-2)} \begin{bmatrix} \frac{m}{n}(n^2-n-1) & m \\ m & nm \end{bmatrix} \begin{bmatrix} T'_v \\ A'_v \end{bmatrix}$$

$$v = 1, 2, \ldots, n \; .$$

The analysis of variance can be seen in Table I.

TABLE I

ANALYSIS OF VARIANCE

Source	Degrees of Freedom	Sums of Squares
Subjects	nm-1	$\frac{1}{n}\Sigma TC^{2}_{i} - \frac{1}{n^{2}m}G^{2}$
Periods/squares	m(n-1)	$\frac{1}{n}\Sigma T^{2}_{j}(k) - \frac{1}{n^{2}}\Sigma T^{2}_{k\cdot\cdot}$
Direct (unadjusted)	n-1	$\frac{1}{nm}\Sigma T^{2}_{v} - \frac{1}{n^{2}m}G^{2}$
Residual (adjusted)	n-1	$\frac{m}{n}(n^{2}-n-2)\Sigma\hat{\theta}^{2}_{s}$
Residual (unadjusted)	n-1	- see below -
Direct (adjusted)	n-1	$\frac{(n^{2}-n-1)}{nm(n^{2}-n-2)}\Sigma\hat{\tau}^{2}_{v}$
Error	(n-1)(nm-m-2)	subtraction
Total	$n^{2}m-1$	$\Sigma Y^{2}_{ijkvs} - \frac{1}{n^{2}m}G^{2}$

Before explaining how to find Residual Sums of Squares (unadjusted) it will be necessary to explain how to find Direct (adjusted) and Residual (adjusted). Taking Direct (adjusted) as an example, first note that in the simplified normal equation for $\hat{\tau}_v$, T_v is called the unadjusted total for the direct effect of treatment v. By solving the remaining equations for $\hat{\theta}_v$ in terms of $\hat{\tau}_v$, one can substitute the result into the equation for $\hat{\tau}_v$, along with the estimate of $\hat{\mu}$, and get an equation only in terms of $\hat{\tau}_v$. This was shown to be

$$\hat{\tau}_v = \frac{1}{m^2(n^2-n-2)} \begin{bmatrix} \frac{m}{n}(n^2-n-1) & m \end{bmatrix} \begin{bmatrix} T'_v \\ A'_v \end{bmatrix} \qquad v = 1, 2, \ldots, n$$

or $$m^2(n^2-n-2)\hat{\tau}_v = \frac{m}{n}(n^2-n-1)T'_v + mA'_v \qquad v = 1, 2, \ldots, n .$$

Recalling what T'_v is, and substituting its value into the above equation, one gets

$$m^2(n^2-n-2)\hat{\tau}_v = \frac{m}{n}(n^2-n-1) \left[T_v - \frac{1}{n}G\right] + mA'_v \qquad v = 1, 2, \ldots, n .$$

Now, in this reduced normal equation for $\hat{\tau}_v$, if one makes the coefficient of the unadjusted total for $\hat{\tau}_v$ equal to unity, one gets

$$(1) \qquad \frac{nm(n^2-n-2)\hat{\tau}_v}{(n^2-n-1)} = T_v - \frac{1}{n}G + \frac{n}{(n^2-n-1)} A'_v \qquad v = 1, 2, \ldots, n .$$

The unadjusted direct total now can be said to be adjusted for $\hat{\mu}$ and $\hat{\theta}_v$, and the entire right-hand side of (1) is called the adjusted direct total. If one now takes the sum of the products of the estimators times their respective adjusted totals, one obtains the adjusted sum of squares for direct effects. The same procedure can be applied to the normal equation for $\hat{\theta}_v$, solving the remaining equations for $\hat{\tau}_v$ in terms of $\hat{\theta}_v$ and substituting in the same way.

It must now be explained how to find the unadjusted residual sum of squares. It should first be noted that by "unadjusted" one means that it is unadjusted only for direct effects. It actually will need to be adjusted for the remaining effects with which it is not mutually orthogonal.

Both the normal for $\hat{\tau}_v$ and the normal for $\hat{\theta}_v$ have been solved in terms of each other already. By taking the reduced normal equation for $\hat{\theta}_v$ in terms of $\hat{\tau}_v$, we get

$$\begin{bmatrix} -m & \frac{m}{n}(n^2-n-1) \end{bmatrix} \begin{bmatrix} \hat{\tau}_v \\ \hat{\theta}_v \end{bmatrix} = A'_v \qquad v = 1, 2, \ldots, n .$$

Ignoring the term for $\hat{\tau}_v$ yields

$$(2) \quad \frac{m}{n}(n^2-n-1)\hat{\theta}'_v = A_v + \frac{1}{n^2}[nP_1 + nF'_v - (n+1)G] \qquad v = 1, 2, \ldots, n .$$

Note that $\hat{\theta}'_v$ is not the same as $\hat{\theta}_v$. Now, A_v can be described as the unadjusted total for residual effects, and therefore the right-hand side of (2) will be the residual total adjusted for all effects except direct effects. One can now obtain the residual sum of squares (unadjusted for direct effects, adjusted for all other effects), which is denoted by residual (unadjusted), by taking these new estimates and multiplying them by their corresponding partially adjusted totals, and adding them together. This can be shown to be as follows

$$\text{Residual (unadjusted)} = \Sigma\hat{\Theta}'_v A'_v \qquad v = 1, 2, \ldots, n$$

$$= \frac{m}{n}(n^2-n-1)\Sigma\hat{\Theta}'^2_v \qquad v = 1, 2, \ldots, n .$$

The expected mean squares for the adjusted terms will be of some interest.

$$E[\text{MS Direct (adjusted)}] = \sigma^2_\varepsilon + \frac{nm(n^2-n-2)}{(n-1)(n^2-n-1)}\Sigma(\tau_v-\bar{\tau})^2$$

$$E[\text{MS Residual (adjusted)}] = \sigma^2_\varepsilon + \frac{m(n^2-n-2)}{n(n-1)}\Sigma(\Theta_s-\bar{\Theta})^2 .$$

Also, the variance of a difference between two adjusted direct effects is

$$\frac{2(n^2-n-1)}{nm(n^2-n-2)}\sigma^2_\varepsilon .$$

The variance of a difference between two unadjusted direct effects is

$$\frac{2}{nm}\sigma^2_\varepsilon .$$

The variance of a difference between two adjusted residual effects is

$$\frac{2n}{m(n^2-n-2)}\sigma^2_\varepsilon$$

and the variance of a difference between two unadjusted residual effects is

$$\frac{2(n^2-n-1)}{m}\sigma_\varepsilon^2 \quad .$$

2.3 Type II Designs

These designs were first presented in a paper by Lucas [7]. They are formed from the same designs given as Type I designs, by merely repeating the final period, forming an $(n+1)^{st}$ period identical to the n^{th} period. He refers to these designs as "extra-period designs".

In discussing change-over designs before his addition of the extra period, Lucas says, "In all of the published series of designs, the precision with which residual effects are estimated is considerably less than that with which the direct effects of treatments are estimated. This is in part because the residual effects are replicated fewer times than are the direct effects, but also in large share because the residual effects are non-orthogonal both to sequences and to direct effects." [7].

In his paper Extra Period Latin Square Change-Over Designs one can see that each treatment is now preceded by itself the same number of times that it is preceded by each other treatment, a condition which renders the direct and residual effects orthogonal to each other, and also renders residual effects orthogonal to subjects.

Example:	a b c	a b c	
	b c a	c a b	The letters
From Design 5	c a b	b c a	denote treatments.
	c a b	b c a	

Example:

From Design 9

a	b	c	d
b	d	a	c
c	a	d	b
d	c	b	a
d	c	b	a

While the replication of the final period does make residual effects orthogonal to subjects, it also makes direct effects non-orthogonal to subjects. However, as Lucas states, " . . . the degree of non-orthogonality is not great." [7]. The direct sum of squares must therefore be adjusted for subjects, while the residual sum of squares can be computed directly from the 2nd, 3rd, ..., (n+1)st periods. Since residual effects are orthogonal to all other effects, and since they do not, of course, occur in the first period, this sum of squares will be easily computed.

The model for this design is

$$Y_{ijkvs} = \mu + C_i + \rho_j(k) + \tau_v + \theta_s + \epsilon_{ijkvs}$$

$i = 1, 2, \ldots, nm$

$j = 1, 2, \ldots, (n+1)$

$k = 1, 2, \ldots, m$

$v = 1, 2, \ldots, n$

$s = 1, 2, \ldots, n$.

The normal equations before applying constraints are

$$nm(n+1)\hat{\mu} + (n+1)\Sigma\hat{C}_i + n\Sigma\hat{\rho}_j(k) + m(n+1)\Sigma\hat{\tau}_v + nm\Sigma\hat{\theta}_s = G$$

$$(n+1)\hat{\mu} + (n+1)\hat{C}_i + \sum_{(k)} \hat{\rho}_j(k) + \Sigma\hat{\tau}_v + \hat{\tau}_{M'} + \Sigma\hat{\Theta}_s = TC_i \qquad i = 1, 2, \ldots, nm$$

$$n\hat{\mu} + \sum_{(k)} \hat{C}_i + n\hat{\rho}_1(k) + \Sigma\hat{\tau}_v = T_1(k) \qquad k = 1, 2, \ldots, m$$

$$n\hat{\mu} + \sum_{(k)} \hat{C}_i + n\hat{\rho}_j(k) + \Sigma\hat{\tau}_v + \Sigma\hat{\Theta}_s = T_j(k) \qquad \begin{array}{l} j = 1, 2, \ldots, (n+1) \\ k = 1, 2, \ldots, m \end{array}$$

$$m(n+1)\hat{\mu} + \Sigma\hat{C}_i + \Sigma^i\hat{C}_i + \Sigma\hat{\rho}_j(k) + m(n+1)\hat{\tau}_v + m\Sigma\hat{\Theta}_s = T_v \qquad v = 1, 2, \ldots, n$$

$$nm\hat{\mu} + \Sigma\hat{C}_i + \Sigma\hat{\rho}_j(k) - \Sigma\hat{\rho}_1(k) + m\Sigma\hat{\tau}_v + nm\hat{\Theta}_s = A_s \qquad s = 1, 2, \ldots, n$$

which become, after applying the necessary constraints

$$nm(n+1)\hat{\mu} = G$$

$$(n+1)\hat{\mu} + (n+1)\hat{C}_i + \hat{\tau}_{M'} = TC_i \qquad k = 1, 2, \ldots, nm$$

$$n\hat{\mu} + \sum_{(k)} \hat{C}_i + n\hat{\rho}_j(k) = T_j(k) \qquad \begin{array}{l} j = 1, 2, \ldots, (n+1) \\ k = 1, 2, \ldots, m \end{array}$$

$$m(n+1)\hat{\mu} + \Sigma^i\hat{C}_i + m(n+1)\hat{\tau}_v = T_v \qquad v = 1, 2, \ldots, n$$

$$nm\hat{\mu} + nm\hat{\Theta}_s - \Sigma\hat{\rho}_1(k) = A_s \qquad s = 1, 2, \ldots, n$$

where $\Sigma^i\hat{C}_i$ = the sum of all subject effects in which treatment v appears last.

Solving for $\hat{\tau}_v$, one gets

$$nm(n+2)\hat{\tau}_v = (n+1)T_v - F'_v - G$$

and the analysis of variance Table II is formed as follows:

TABLE II

ANALYSIS OF VARIANCE

Source	Degrees of Freedom	Sums of Squares
Subjects (unadjusted for direct)	nm-1	$\frac{1}{(n+1)}\Sigma TC^2{}_i - \frac{1}{nm(n+1)}G^2$
Periods/squares	nm	$\frac{1}{n}\Sigma T^2{}_j(k) - \frac{1}{n(n+1)}\Sigma T^2{}_{k}..$
Direct (adjusted for subjects)	n-1	$\frac{nm(n+2)}{(n+1)}\Sigma\hat{\tau}^2{}_v$
Residual	n-1	- see below -
Error	(n-1)(nm-2)	subtraction
Total	nm(n+1)-1	$\Sigma Y^2{}_{ijkvs} - \frac{1}{nm(n+1)}G^2$

As was previously stated, by simply ignoring the first period one can obtain the residual sum of squares as follows:

$$\frac{1}{nm}\Sigma A^2{}_s - \frac{1}{n^2m}\left[\sum_{j=2}^{n+1} P_j\right]^2 \quad .$$

The expected mean square for adjusted direct effects is

$$E[\text{MS Direct (adjusted)}] = \sigma^2_\varepsilon + \frac{nm(n+2)}{(n^2-1)}\Sigma(\tau_v-\bar{\tau}\ ) \quad .$$

The variance of a difference between two adjusted direct effects is

$$\frac{2(n+1)}{nm(n+2)}\,\sigma^2_\varepsilon$$

and that for a difference between two residual effects is

$$\frac{2}{nm}\,\sigma^2_\varepsilon \quad .$$

2.4 Type III Designs

The third design to be discussed is a balanced incomplete Latin square design, first mentioned in 1950 in a paper by Patterson [4].

The basis for these designs is a set of completely orthogonal Latin squares. "This is a set of (n-1) squares such that when any two squares are superimposed, each letter of one square occurs (exactly) once with every letter of the other square." [1]. These squares are balanced for all orders of residual effects, however only first order effects will be considered here.

From this set of (n-1) orthogonal Latin squares, p corresponding rows (periods) are chosen, where $p \leq n$, such that balance is still maintained. This can always be done. Now one has a set of (n-1) $n \chi p$ Latin rectangles balanced for all orders of residual effects. Designs 5, 10, 17, 29, 31 and 32 are orthogonal sets for 3, 4, 5, 7, 8 and 9 treatments respectively. No orthogonal set of squares exists for $n=6$ treatments.

As an example, one could take the first three rows from each of the squares of Design 10.

$$\begin{array}{cccc} a & b & c & d \\ b & a & d & c \\ c & d & a & b \end{array} \qquad \begin{array}{cccc} a & b & c & d \\ d & c & b & a \\ b & a & d & c \end{array} \qquad \begin{array}{cccc} a & b & c & d \\ c & d & a & b \\ d & c & b & a \end{array}$$

Again, as in Design I, the combined sums of squares which lead to the total treatment effect must be computed in two different ways. Again, also remember that these two sets of sums of squares add to the same result. This fact can be used as a computational check. These two sets of sums of squares are

(1) Direct (adjusted for residual) + Residual (unadjusted),

and (2) Direct (unadjusted) + Residual (adjusted for direct).

The model for this design is

$$Y_{ijkvs} = \mu + C_i + \rho_j(k) + \tau_v + \Theta_s + \varepsilon_{ijkvs}$$

$i = 1, 2, \ldots, n(n-1)$
$j = 1, 2, \ldots, p$
$k = 1, 2, \ldots, (n-1)$
$v = 1, 2, \ldots, n$
$s = 1, 2, \ldots, n.$

The normal equations are

$$np(n-1)\hat{\mu} + p\Sigma\hat{C}_i + n\Sigma\hat{\rho}_j(k) + p(n-1)\Sigma\hat{\tau}_v + (p-1)(n-1)\Sigma\hat{\theta}_s = G$$

$$p\hat{\mu} + p\hat{C}_i + \sum_{(k)}\hat{\rho}_j(k) + \Sigma^{i}\hat{\tau}_v + \Sigma^{ii}\hat{\theta}_s = TC_i \qquad i = 1, 2, \ldots, n(n-1)$$

$$n\hat{\mu} + \sum_{(k)}\hat{C}_i + n\hat{\rho}_1(k) + \Sigma\hat{\tau}_v = T_1(k) \qquad k = 1, 2, \ldots, (n-1)$$

$$n\hat{\mu} + \sum_{(k)}\hat{C}_i + n\hat{\rho}_j(k) + \Sigma\hat{\tau}_v + \Sigma\hat{\theta}_s = T_j(k) \qquad \begin{matrix} j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, (n-1) \end{matrix}$$

$$p(n-1)\hat{\mu} + \Sigma^{iii}\hat{C}_i + \Sigma\hat{\rho}_j(k) + p(n-1)\hat{\tau}_v + (p-1)\Sigma^{iv}\hat{\theta}_s = T_v$$

$$v = 1, 2, \ldots, n$$

$$(p-1)(n-1)\hat{\mu} + \Sigma^{v}\hat{C}_i + \sum_{j=2}^{n+1}\hat{\rho}_j(k) + (p-1)\Sigma^{vi}\hat{\tau}_v + (p-1)(n-1)\hat{\theta}_s = A_s$$

$$s = 1, 2, \ldots, n$$

where $\Sigma^{i}\hat{\tau}_v$ = the sum over all treatment effects which appear in subject i,

$\Sigma^{ii}\hat{\theta}_s$ = the sum over all residual effects which appear in subject i,

$\Sigma^{iii}\hat{C}_i$ = the sum over all subject effects which include the effect of treatment v,

$\Sigma^{iv}\hat{\theta}_s$ = the sum of all residual effects excluding the residual effect of treatment v,

$\Sigma^{v}\hat{C}_i$ = the sum over all subject effects which include the residual effect of treatment s,

and $\Sigma^{vi}\hat{\tau}_v$ = the sum over all treatment effects excluding treatment s.

After applying the usual constraints, the normal equations become

$$np(n-1)\hat{\mu} = G$$

$$p\hat{\mu} + p\hat{C}_i + \Sigma^{i}\hat{\tau}_v + \Sigma^{ii}\hat{\theta}_s = TC_i \qquad i = 1, 2, \ldots, n(n-1)$$

$$n\hat{\mu} + \sum_{(k)} \hat{C}_i + n\hat{\rho}_j(k) = T_j(k) \qquad \begin{array}{l} j = 1, 2, \ldots, p \\ k = 1, 2, \ldots, (n-1) \end{array}$$

$$p(n-1)\hat{\mu} + \Sigma^{iii}\hat{C}_i + p(n-1)\hat{\tau}_v - (p-1)\hat{\theta}_v = T_v \qquad v = 1, 2, \ldots, n$$

$$(p-1)(n-1)\hat{\mu} + \Sigma^{v}\hat{C}_i - \Sigma\hat{\rho}_1(k) - (p-1)\hat{\tau}_s + (p-1)(n-1)\hat{\theta}_s = A_s$$

$$s = 1, 2, \ldots, n.$$

Solving the above system of equations in terms of $\hat{\tau}_v$ and $\hat{\theta}_s$, one gets

$$\begin{bmatrix} n(p-1) & -\frac{n}{p}(p-1) \\ -\frac{n}{p}(p-1) & \frac{(p-1)}{p}(np-n-1) \end{bmatrix} \begin{bmatrix} \hat{\tau}_v \\ \hat{\theta}_v \end{bmatrix} = \begin{bmatrix} T'_v \\ A'_v \end{bmatrix} \qquad v = 1, 2, \ldots, n$$

where, in this design

$$T'_v = T_v - \frac{1}{p}(TC_v) \qquad v = 1, 2, \ldots, n$$

$$A'_v = A_v + \frac{1}{n}P_1 - \frac{1}{np}G - \frac{1}{p}\{TC_v\} \qquad v = 1, 2, \ldots, n.$$

By ignoring the terms for $\hat{\theta}_v$ and $\hat{\tau}_v$ in the above two equations respectively, one arrives at the following unadjusted estimates, denoted $\hat{\tau}'_v$ and $\hat{\theta}'_v$

$$n(p-1)\hat{\tau}'_v = T'_v \qquad v = 1, 2, \ldots, n$$

and $$\frac{(p-1)(np-n-1)}{p}\hat{\theta}'_v = A'_v \qquad v = 1, 2, \ldots, n \; .$$

The adjusted estimates are formed by solving the previous set of equations for $\hat{\tau}_v$ and $\hat{\theta}_v$ as follows

$$(p-1)(n^2p^2-n^2p-np-n^2)\hat{\tau}_v = (np^2-np-p)T'_v + npA'_v \quad v = 1, 2, \ldots, n$$

$$(p-1)(n^2p^2-n^2p-np-n^2)\hat{\theta}_v = np^2A'_v + npT'_v \; .$$

The analysis of variance is now constructed as in Table III. The expected mean squares for the adjusted effects are

$$E[\text{MS Direct (adjusted)}] = \sigma^2_\varepsilon + \frac{(p-1)(n^2p^2-n^2p-np-n^2)}{p(n^2p-np-1)(n-1)}\Sigma(\tau_v-\bar{\tau}\;)^2$$

$$E[\text{MS Residual (adjusted)}] = \sigma^2_\varepsilon + \frac{(p-1)(np^2-np-p-n)}{p^2}\Sigma(\theta_s-\bar{\theta}\;)^2 \; .$$

The variance of a difference between two adjusted direct effects is

$$\frac{2p(np-n-1)}{(p-1)(n^2p^2-n^2p-np-n^2)}\,\sigma^2_\varepsilon$$

TABLE III

ANALYSIS OF VARIANCE

Source	Degrees of Freedom	Sums of Squares
Subjects	n(n-1)-1	$\frac{1}{p}\Sigma TC^2{}_i - \frac{1}{np(n-1)}G^2$
Periods/squares	(p-1)(n-1)	$\frac{1}{n}\Sigma T^2{}_j(k) - \frac{1}{np}\Sigma T^2{}_{k\cdot\cdot}$
Direct (unadjusted)	n-1	$n(p-1)\Sigma\hat{\tau}'^2{}_v$
Residual (adjusted)	n-1	$\frac{1}{p2}(p-1)(np^2-np-p-n)\Sigma\hat{\theta}^2{}_s$
Residual (unadjusted)	n-1	$\frac{1}{p}(p-1)(np-n-1)\Sigma\hat{\theta}'^2{}_s$
Direct (adjusted)	n-1	$\frac{1}{(np2-np-p)}(p-1)(n^2p^2-n^2p-np-n^2)\Sigma\hat{\tau}^2{}_v$
Error	(n-1)[n(p-1)-(p+1)]	subtraction
Total	np(n-1)-1	$\Sigma Y^2{}_{ijkvs} - \frac{1}{np(n-1)}G^2$

that of a difference between two unadjusted direct effects is

$$\frac{2}{n(p-1)}\sigma_\varepsilon^2$$

that of a difference between two adjusted residual effects is

$$\frac{2p^2}{(p-1)(np^2-np-p-n)}\sigma_\varepsilon^2$$

and that of a difference between two unadjusted residual effects is

$$\frac{2p}{(p-1)(np-n-1)} \sigma_{\varepsilon}^{2} \quad .$$

2.5 Type IV Designs

These designs were introduced by Berenblut [11]. He describes the construction of these designs for n treatments as follows (where a, b, ... denote treatments).

"Let

$$\begin{array}{lcl} \alpha & = & a\ b\ c\ d\ \ldots\ u\ v \\ \beta & = & v\ a\ b\ c\ \ldots\ t\ u \\ \gamma & = & u\ v\ a\ b\ \ldots\ s\ t \\ . & = & .\ .\ .\ .\ \ldots\ .\ . \\ \psi & = & d\ e\ f\ g\ \ldots\ b\ c \\ \phi & = & c\ d\ e\ f\ \ldots\ a\ b \\ \omega & = & b\ c\ d\ e\ \ldots\ v\ a \end{array}$$

"If n is odd, the design for n treatments can be written symbolically as

Period	Subject (1 to n^2)			
1	α	α	...	α
2	β	γ	...	α
3	γ	γ	...	γ
4	δ	ε	...	γ
5	ε	ε	...	ε
.	.	.	...	.
.	.	.	...	.
.	.	.	...	.
n-1	ϕ	ω	...	ψ
n	ω	ω	...	ω
n+1	ω	α	...	ϕ
n+2	ϕ	ϕ	...	ϕ
.	.	.	...	.
.	.	.	...	.
. .	.	.	...	.
2n-1	β	β	...	β
2n	α	β	...	ω

"If n is even, the lines for periods n and n+1, for periods n-1 and n-2, etc., are interchanged." These directions are not very clear. However, the designs can be easily constructed if one follows these simple rules:

(1) Define α, β, ..., ω as stated above.

(2) Write them down a page in forward order, then backward order, which yields a column the same as Berenblut's column one.

(3) Starting with the first letter in period one, replicate it n times altogether in a row, and do the same for every odd period.

(4) For the even periods, write the elements in order, α following the final letter, and so on.

Refer to Designs 2, 6, 11 and 18 for examples.

In this group of designs, direct effects and residual effects are orthogonal. Also, subjects and direct effects are orthogonal, but subjects and first order residual effects are not. Therefore, it will be necessary to split up the total sum of squares for subjects and residuals as follows:

Subjects (unadjusted) + Residual (adjusted for subjects).

The design consists of one block (replicated if desired) of treatments which is $2n \chi n^2$, i. e. 2n periods and n^2 subjects.

<u>Example</u>

Design 2

a	b	a	b
b	a	a	b
b	a	b	a
a	b	b	a

Or, utilizing the above notation, letting α = a b and β = b a, one gets

$$\begin{array}{cc} \alpha & \alpha \\ \beta & \alpha \\ \beta & \beta \\ \alpha & \beta \end{array} .$$

The model for this design is

$$Y_{ijvs} = \mu + C_i + \rho_j + \tau_v + \theta_s + \varepsilon_{ijvs}$$

$$\begin{aligned} i &= 1, 2, \ldots, n^2 \\ j &= 1, 2, \ldots, 2n \\ v &= 1, 2, \ldots, n \\ s &= 1, 2, \ldots, n \end{aligned}$$

where ρ_j is the effect of the j^{th} period.

The normal equations are

$$2n^3\hat{\mu} + 2n\Sigma\hat{C}_i + n^2\Sigma\hat{\rho}_j + 2n^2\Sigma\hat{\tau}_v + n(2n-1)\Sigma\hat{\theta}_s = G$$

$$2n\hat{\mu} + 2n\hat{C}_i + \Sigma\hat{\rho}_j + 2\Sigma\hat{\tau}_v + 2\Sigma^i\hat{\theta}_s + \hat{\theta}_{M'} = TC_i \qquad i = 1, 2, \ldots, n^2$$

$$n^2\hat{\mu} + \Sigma\hat{C}_i + n^2\hat{\rho}_1 + n\Sigma\hat{\tau}_v = P_1$$

$$n^2\hat{\mu} + \Sigma\hat{C}_i + n^2\hat{\rho}_j + n\Sigma\hat{\tau}_v + n\Sigma\hat{\theta}_s = P_j \qquad j = 2, 3, \ldots, 2n$$

$$2n^2\hat{\mu} + 2\Sigma\hat{C}_i + n\Sigma\hat{\rho}_j + 2n^2\hat{\tau}_v + (2n-1)\Sigma\hat{\theta}_s = T_v \qquad v = 1, 2, \ldots, n$$

$$n(2n-1)\hat{\mu} + 2\Sigma^{ii}\hat{C}_i + \Sigma^{iii}\hat{C}_i + n\sum_{j=2}^{2n}\hat{\rho}_j + (2n-1)\Sigma\hat{\tau}_v + n(2n-1)\hat{\theta}_s = A_s$$

$$s = 1, 2, \ldots, n$$

where $\Sigma^{i}\hat{\Theta}_s$ = the sum over all treatments which do not occur last

$\Sigma^{ii}\hat{C}_i$ = the sum over all subjects in which treatment s is not last

$\Sigma^{iii}\hat{C}_i$ = the sum over all subjects in which treatment s occurs last .

Upon applying the usual constraints, one gets the following set of normal equations

$$2n^3\hat{\mu} = G$$

$$2n\hat{\mu} - \hat{\Theta}_{M'} + 2n\hat{C}_i = TC_i \qquad i = 1, 2, \ldots, n^2$$

$$n^2\hat{\mu} + n^2\hat{\rho}_j = P_j \qquad j = 1, 2, \ldots, 2n$$

$$2n^2\hat{\mu} + 2n^2\hat{\tau}_v = T_v \qquad v = 1, 2, \ldots, n$$

$$n(2n-1)\hat{\mu} + n(2n-1)\hat{\Theta}_s - \Sigma^{iii}\hat{C}_i - n\hat{\rho}_1 = A_s \qquad s = 1, 2, \ldots, n .$$

Solving for $\hat{\Theta}_s$ yields the following equation

$$n\,(4n^2-2n-1)\hat{\Theta}_s = 2n^2A_s + nF'_s + 2nP_1 - (2n+1)G$$

$$s = 1, 2, \ldots, n .$$

One can now obtain the following analysis of variance as shown in Table IV.

TABLE IV

ANALYSIS OF VARIANCE

Source	Degrees of Freedom	Sums of Squares
Subjects (ignoring residuals)	n^2-1	$\frac{1}{2n}\Sigma TC^2{}_i - \frac{1}{2n^3}G^2$
Periods	$2n-1$	$\frac{1}{n^2}\Sigma P^2{}_j - \frac{1}{2n^3}G^2$
Direct Effects	$n-1$	$\frac{1}{2n^2}\Sigma T^2{}_v - \frac{1}{2n^3}G^2$
Residual (adjusted for subjects)	$n-1$	$\frac{(4n^2-2n-1)}{2}\Sigma\Theta^2{}_s$
Error	$2n^3-n^2-4n+3$	subtraction
Total	$2n^3-1$	$\Sigma Y^2{}_{ijvs} - \frac{1}{2n^3}G^2$

The expected mean square for adjusted residuals is

$$\sigma^2_\varepsilon + \frac{(4n^2-2n-1)}{2(n-1)}\Sigma(\Theta_s - \bar{\Theta}_s)^2 \quad .$$

The variance of a difference between two direct effects is

$$\frac{1}{n^2}\ \sigma^2_\varepsilon$$

and that for a difference between two adjusted residual effects is

$$\frac{4}{(4n^2-2n-1)}\ \sigma^2_\varepsilon \quad .$$

III

EFFICIENCIES

The method of determining efficiencies will be that of Patterson and Lucas [8] described as follows by Berenblut, " . . . the efficiency factor of design X compared with design Y is the ratio of the product of the number of observations and the variance of a contrast in Design Y to the corresponding quantity in Design X." [11].

Efficiencies will be presented for estimation of differences in direct and first order residual effects, for all comparisons for designs Type I through IV.

3.1 Differences in Direct Effects

It will first be necessary to determine the total number of observations χ variance of a difference between two direct effects for all designs.

Design	Number of Observations χ Variance $(\hat{\tau}_v - \hat{\tau}_{v'})$
I	$\frac{2n(n^2-n-1)}{(n^2-n-2)} \sigma_\varepsilon^2$
II	$\frac{2(n+1)^2}{(n+2)} \sigma_\varepsilon^2$
III	$\frac{2p^2(n-1)(np-n-1)}{(p-1)(np^2-np-p-n)} \sigma_\varepsilon^2$
IV	$2n\sigma_\varepsilon^2$

Efficiencies will be formed for n=3 and n=4. Also, since the variance for Type III depends on the number of periods chosen, for n=3, p=3 and for n=4, p=3. Note, when p=n Type III is equivalent to Type II.

The numbers in these tables refer to <u>efficiencies of Design X to Design Y</u>.

TABLE V

Y

		I	II	III	IV
	I	--	$\frac{64}{75}$	1	$\frac{4}{5}$
X	II	$\frac{75}{64}$	--	$\frac{75}{64}$	$\frac{15}{16}$
	III	1	$\frac{64}{75}$	--	$\frac{4}{5}$
	IV	$\frac{5}{4}$	$\frac{16}{15}$	$\frac{5}{4}$	--

n=3

Y

		I	II	III	IV
	I	--	$\frac{125}{132}$	$\frac{945}{748}$	$\frac{10}{11}$
X	II	$\frac{132}{125}$	--	$\frac{567}{425}$	$\frac{24}{25}$
	III	$\frac{748}{945}$	$\frac{425}{567}$	--	$\frac{136}{189}$
	IV	$\frac{11}{10}$	$\frac{25}{24}$	$\frac{189}{136}$	--

n=4

For "small" values of n Design IV is noticeably better than Designs I, II or III. As n increases, the efficiency of Design IV to the others approaches 1. Even for n=10 and p=5 the efficiencies of Design IV to I, II and III are 1.01, 1.01 and 1.19 respectively.

Efficiency alone should not be the criterion for selecting one design over another. If there is a cost per observation, certainly the experimenter may wish to consider the number of observations when making comparisons. In the above case of n=3, p=3 Designs I, II, III and IV have 18, 24, 18 and 54 observations respectively. If the cost per observation is great, one could find some justification for using Designs I, II or III instead of IV. Design III, although not as efficient as I, II or IV must be remembered to have only p<n periods, a distinct advantage over the other designs.

Design II also should not be overlooked. It is only slightly less efficient than Design IV, even for "small" n, and it is easy to construct and easy to analyze.

It can only be concluded that there is no set reason for choosing one design over another.

3.2 Differences in Residual Effects

The same four designs will now be compared for estimating differences in residual effects.

Again it will be necessary to find the total number of observations χ variance of a difference between two residual effects.

Design	Number of Observations χ Variance $(\hat{\theta}_s - \hat{\theta}_{s'})$
I	$\frac{2n^3}{(n^2-n-2)}\sigma_\varepsilon^2$
II	$2(n+1)\sigma_\varepsilon^2$
III	$\frac{2np^3(n-1)}{(p-1)(np^2-np-p-n)}\sigma_\varepsilon^2$
IV	$\frac{8n^3}{(4n^2-2n-1)}\sigma_\varepsilon^2$

Again, efficiencies will be found for the two cases, n=3, p=3 and n=4, p=3. The numbers in the tables refer to <u>efficiencies of Design X to Design Y</u>.

TABLE VI

		Y			
		I	II	III	IV
	I	--	$\frac{16}{27}$	1	$\frac{16}{29}$
X	II	$\frac{27}{16}$	--	$\frac{27}{16}$	$\frac{27}{29}$
	III	1	$\frac{16}{27}$	--	$\frac{16}{29}$
	IV	$\frac{29}{16}$	$\frac{29}{27}$	$\frac{29}{16}$	--

n=3

		Y			
		I	II	III	IV
	I	--	$\frac{50}{64}$	$\frac{405}{272}$	$\frac{8}{11}$
X	II	$\frac{64}{50}$	--	$\frac{162}{85}$	$\frac{256}{275}$
	III	$\frac{272}{405}$	$\frac{85}{162}$	--	$\frac{2176}{4455}$
	IV	$\frac{11}{8}$	$\frac{275}{256}$	$\frac{4455}{2176}$	--

n=4

One sees again that design IV is more efficient than Designs I, II, and III for small n. In fact, the same relationships hold here as in the case of direct effects. The same comments in regard to number of observations and advantages also apply.

3.3 Design Efficiency for Estimation of Residual Effects

Since residual effects never occur as many times in a design as direct effects, a design will always yield better estimate of direct effects. One might, however, be interested in estimating residual effects as the principle purpose of the design, or simply desire good estimates of residual effects. In this case he would be interested in how efficient a design is for estimating residual effects to direct effects. The same method utilized in the preceding section applies, but since the two variances come from the same design, no weighting factor will be necessary. Comparisons will be made for n=3, 5 and 10. P will be equal to 3 for Design III.

Numbers in Table VII represent the efficiency of a design for estimating residual effects as compared to direct effects.

TABLE VII

		n		
		3	5	10
Design	I	.556	.760	.890
	II	.800	.857	.917
	III	.556	.600	.633
	IV	.806	.890	.948

It can be seen that Design IV is better at estimation of residual effects in comparison to direct effects than the other designs in all cases. The total number of observations necessary to produce these results should not be overlooked, however. For n=5, Type I designs require a minimum of 50 observations, Type II require a minimum of 60, Type III a minimum of 60 (with $p=3$) and Type IV a minimum of 250. If cost is not relevant Design IV is certainly the best; however, if cost is a factor, Design II with less than one-fourth the number of observations could certainly be used.

IV

ESTIMATION OF MISSING VALUES

4.1 Notation

The below listed notation will hold for this chapter only, x and z will be considered to be missing observations. Before computation of missing value formulas, x and z must be set equal to zero.

$TC_i(x)$ = total for the subject containing x

$T_j(k)(x)$ = total for the period (in the square) containing x

$A_v(x)$ = total of all observations immediately following the treatment immediately preceding x

$T_k(x)$ = total for the square containing x

$$a = \begin{cases} -1 & \text{if x was to receive treatment v} \\ 0 & \text{otherwise} \end{cases}$$

$$a' = \begin{cases} -1 & \text{if x was to receive the first residual of treatment v} \\ 0 & \text{otherwise} \end{cases}$$

$$a'' = \begin{cases} -1 & \text{if x was to receive the second residual of treatment v} \\ 0 & \text{otherwise} \end{cases}$$

$$b = \begin{cases} 1 & \text{if x does not occur in the final period} \\ 0 & \text{otherwise} \end{cases}$$

$$b' = \begin{cases} -1 & \text{if treatment v occurs in the subject containing x} \\ 0 & \text{otherwise} \end{cases}$$

$$c = \begin{cases} -1 & \text{if treatment v is last on the subject containing x} \\ 0 & \text{otherwise} \end{cases}$$

$$c' = \begin{cases} -1 & \text{if treatment } v \text{ appears last or next to on the subject} \\ & \text{containing } x \\ 0 & \text{otherwise} \end{cases}$$

$$d = \begin{cases} -1 & \text{if } x \text{ is in the first period} \\ 0 & \text{otherwise} \end{cases}$$

$$d' = \begin{cases} -1 & \text{if } x \text{ is in the second period} \\ 0 & \text{otherwise} \end{cases}$$

$$d'' = \begin{cases} 0 & \text{is } x \text{ is in the first period} \\ 1 & \text{otherwise} \end{cases}$$

$$h = \begin{cases} 1 & \text{if } x \text{ and } z \text{ occur in the same period and block} \\ 0 & \text{otherwise} \end{cases}$$

$$h' = \begin{cases} 1 & \text{if } x \text{ and } z \text{ occur in the same subject} \\ 0 & \text{otherwise} \end{cases}$$

$$h'' = \begin{cases} 1 & \text{if } x \text{ and } z \text{ occur in the same block} \\ 0 & \text{otherwise} \end{cases}$$

$$k = \begin{cases} 1 & \text{if } x \text{ and } z \text{ receive the same treatment directly} \\ 0 & \text{otherwise} \end{cases}$$

$$k' = \begin{cases} 1 & \text{if } x \text{ and } z \text{ receive the same first residual treatment} \\ 0 & \text{otherwise} \end{cases}$$

$$k'' = \begin{cases} 1 & \text{if neither } x \text{ nor } z \text{ are in the first period} \\ 0 & \text{otherwise} \end{cases}$$

4.2 General Remarks

The missing value formulae which will be presented were obtained by the method suggested by Coons [16]. She states, "The purpose of this paper is to illustrate the full details of a method which can be used when one or more missing observations exist in an experiment of any statistical design. The advantages of this method are its generality of application and the ease with which exact tests of significance may be obtained. Here the word "exact" is used to mean exact when errors are normally and independently distributed."

She goes on to list six properties which she uses in order to justify her computational procedures. Of these six properties, two will be of interest here. These are

(1) "If an analysis of variance is made with symbols β_1, β_2, ..., β_q in the place of missing observations, then the best linear unbiased estimates of the missing observations are the quantities $\hat{\beta}_1$, $\hat{\beta}_2$, ..., $\hat{\beta}_q$ which minimize the error sum of squares."

(6) "The sum of squares for treatments obtained by analyzing the data augmented by the missing value estimates is always greater than or equal to the exact sum of squares for treatments."

The first property can best be seen from a simple example. Consider the following model:

$$Y_{ij} = \mu + \tau_i + \varepsilon_{ij} \qquad\qquad i = 1,2 \text{ and } j = 1,2$$

Now suppose observation Y_{12} is missing. Let β represent the missing observation. The full set of equations can now be rewritten as follows:

$$
\begin{aligned}
Y_{11} &= \mu + \tau_1 + \varepsilon_{11} \\
\beta &= \mu + \tau_1 + \varepsilon_{12} \\
Y_{21} &= \mu + \tau_2 + \varepsilon_{21} \\
Y_{22} &= \mu + \tau_2 \quad \varepsilon_{22}
\end{aligned}
$$

which can in turn be written as

$$
\begin{aligned}
Y_{11} &= \mu + \tau_1 + \beta x_{11} + \varepsilon_{11} \\
0 &= \mu + \tau_1 + \beta x_{12} + \varepsilon_{12} \\
Y_{21} &= \mu + \tau_2 + \beta x_{21} + \varepsilon_{21} \\
Y_{22} &= \mu + \tau_2 + \beta x_{22} + \varepsilon_{22}
\end{aligned}
$$

where the coefficients of β, that is x_{11}, x_{12}, x_{21} and x_{22} are equal to 0, -1, 0 and 0 respectively. These x_{ij}'s form the vector X.

Now, wherever a missing observation appears, one simply inserts a 0. These original observations, with the 0's, are considered to be the Y variable. The X variable takes on the value 0, except where it corresponds to a missing observation, in which case it is set equal to -1. This vector will be denoted as Z for a second missing value.

This paper will not present the entire analysis of convariance. Rather, the above method yields an estimate for β, which will be called either $\hat{x}$ or $\hat{z}$, (refer to Section 4.3) and which will be merely calculated and inserted into the original data. The usual analysis of variance is then performed.

From property 6 one sees that this test will not be exact. The F value that one calculates will be larger than that for an exact test. This problem will not be great, however. If the calculated F value is in the "acceptance" regions as compared to the tabulated value, the results are used without further concern. Only if the calculated F value is "slightly" larger than the tabulated F value will it be necessary to perform the exact test. For this test the reader is referred to Coons [16].

Note that in performing the analysis of variance one degree of freedom is lost from error and total <u>for each missing value</u>. This method yields the same results as the more familiar method of substituting an unknown quantity for the missing value and minimizing the error sum of squares.

This paper will merely present the missing value formulae. Type I designs will offer a more detailed description of how this was done.

The formulas are derived under the assumption that the residual effects of the missing value actually occurred. In actual experimentation it seems plausible that this assumption will generally be correct; for if it were not possible to apply a treatment during a particular period, it would be better to disregard the entire subject.

4.3 Missing Value Formulae

By following the method of Coons [16], the procedure reduces to the following. Where one missing value is to be estimated, it will be denoted by $\hat{x}$. If two values are to be estimated, they will be denoted by $\hat{x}$ and $\hat{z}$.

The general formulas will be as follows:

(1) For one missing value

$$\hat{x} = \frac{E_{xy}}{E_{xx}}$$

(2) For two missing values

$$\hat{x} = \frac{E_{xy}E_{zz}-E_{zy}E_{xz}}{E_{xx}E_{zz}-E^2_{xz}}$$

$$\hat{z} = \frac{E_{zy}E_{xx}-E_{xy}E_{zx}}{E_{xx}E_{zz}-E^2_{xz}} \quad .$$

The meanings of the various quantities will be explained under design Type I.

<u>Type I</u>: First it will be necessary to find the error sum of squares. Again, note that the data with the missing values set equal to zero are the Y values, and the X values are all zero except those corresponding to missing values, which are set equal to -1.

$$\text{Errors ss} = \Sigma Y^2{}_{ijkvs} - \frac{1}{n}\Sigma TC^2{}_{i} - \frac{1}{n}\Sigma T^2{}_{j}(k) + \frac{1}{n^2}\Sigma T^2{}_{k\cdot\cdot} - \frac{nm(n^2-n-2)}{(n^2-n-1)}\Sigma\hat{\tau}_{v}{}^2$$

$$- \frac{n}{m(n^2-n-1)}\Sigma\left[A_s + \frac{1}{n}P_1 + \frac{1}{n}F'_s - \frac{(n+1)}{n^2}G\right]^2$$

which simplifies to

$$\text{Error ss} = \Sigma Y^2_{ijkvs} - \frac{1}{n}\Sigma TC^2_{i} - \frac{1}{n}\Sigma T^2_{j}(k) + \frac{1}{n^2}\Sigma T^2_{k}..$$

$$- \frac{1}{nm(n^2-n-1)(n^2-n-2)}\Sigma[(n^2-n-1)T_v + nA_v + F'_v + P_1 - nG]^2$$

$$- \frac{1}{n^3m(n^2-n-1)}\Sigma[n^2A_v + nP_1 + nF'_v - (n+1)G]^2 \quad .$$

Now, for the case of one missing value one needs E_{xx} and E_{xy}. All the x values are known, so an exact value can be obtained for E_{xx} which differs depending on where in the design the missing value occurs. E_{xx} is simply the error sum of squares using only the x values.

The first bracketed quantity above reduces to

$$a(n^2-n-1) + bn + c + d + n \quad , \qquad v = 1, 2, \ldots, n$$

which will be called X_{vx}. The second quantity reduces to

$$bn^2 + dn + cn + n+1 \quad , \qquad v = 1, 2, \ldots, n$$

which will be called X'_{vx}. Note that $\Sigma X_{vx} = \Sigma X'_{vx} = 0$. This result holds true for all values of X_{vx} and X'_{vx} which will be computed because they come from values of $\hat{\tau}_v$ and $\hat{\theta}_v$ which also sum to zero by constraint.

E_{xx} can now be written as

$$\frac{nm(n-1)^2(n^2-n-1)(n^2-n-2)-n^2\Sigma X^2_{vx} - (n^2-n-2)\Sigma X'^2_{vx}}{n^3m(n^2-n-1)(n^2-n-2)} \quad .$$

E_{xy} is known as the error sum of products of the X and Y values.

Each squared quantity in the Error ss is replaced by the product of two quantities, the total for the x's and the total for the y's. For example, ΣTC^2_i will be written as $\Sigma(TC_i)x(TC_i)y$, where $(TC_i)x$ is the total of the x's for subject i, and $(TC_i)y$ is the total of the y's for the same subject.

The two quantities in brackets can again be reduced as follows

$$(n^2-n-1)T_v + nA_v + F'_v + P_1 - nG \qquad v = 1, 2, \ldots, n$$

for the first quantity, which shall be denoted by W_v, and

$$n^2A_v + nP_1 + nF'_v - (n+1)G \qquad v = 1, 2, \ldots, n$$

for the second quantity, which shall be denoted by W'_v. Note again that $\Sigma W_v = \Sigma W'_v = 0$. E_{xy} can now be written as

$$\frac{nm(n^2-n-1)(n^2-n-2)[nTC_i(x)+nT_j(k)(x)-T_k(x)\cdot\ ]-n^2\Sigma W_v X_{vx}-(n^2-n-2)\Sigma W'_v X'_{vx}}{n^3m(n^2-n-1)(n^2-n-2)} \quad .$$

For the case of two missing values it is necessary to find three more quantities. The quantities E_{zz} and E_{zy} are found in a similar manner to those for E_{xx} and E_{xy}. The fact that z occurs in a different location will change the values of the coefficients in the reduction of the bracketed quantities, and the same formulas are used to find

$$X_{vz} = a(n^2-n-1) + bn + c + d + n \qquad v = 1, 2, \ldots, n$$

and $$X'_{vz} = bn^2 + dn + cn + n + 1 \qquad v = 1, 2, \ldots, n \quad .$$

Note that these values do change for different missing values. The fact that the W_v and W'_v values remain unchanged yields the desired results for E_{zz} and E_{zy}, using the formulas for E_{xx} and E_{xy} respectively. This method will hold true for all remaining designs also. Only E_{xz} remains to be found. It is found from the Error ss also, by considering the X values as before, and the Z values also as from before. Note that the X values and Z values are all zero, except for one -1, which is in a different location for both sets of data. Since all the data is known, E_{xz} can be simply found as

$$\frac{-nm(n^2-n-1)(n^2-n-2)[n(h+h')-h'']-n^2\Sigma X_{vx}X_{vz}-(n^2-n-2)\Sigma X'_{vx}X'_{vz}}{n^3m(n^2-n-1)(n^2-n-2)} \quad .$$

The fact that the variables change for the different locations of the missing values eliminates the necessity for separate equations, this one case taking care of all combinations.

Again note that $\Sigma X_{vz} = \Sigma X'_{vz} = 0$, and will hold true for all designs. Once all the quantities are found, they can be substituted in the formulae at the beginning of the section. It will now only be necessary to give the various quantities represented above for the remaining designs, the method remains the same.

<u>Type II</u>: For one missing value

$$X_{vx} = a(n+1) - c + 1 \qquad v = 1, 2, \ldots, n$$

$$W_v = (n+1)T_v - F'_v - G \qquad v = 1, 2, \ldots, n$$

$$E_{xx} = \frac{nm(n+2)(n-1) - \Sigma X^2_{vx} - (n+1)(n+2)d''(1-\frac{1}{n})}{nm(n+1)(n+2)}$$

$$E_{xy} =$$

$$\frac{nm(n+2)TC_i(x) + m(n+1)(n+2)T_j(k)(x) - m(n+2)T_k(x) - \Sigma X_{vx}W_v}{nm(n+1)(n+2)}$$

$$+ \frac{(n+1)(n+2)d'' \left[A_v(x) - \sum_{j=2}^{n+1} P_j\right]}{nm(n+1)(n+2)} .$$

For two missing values: E_{zz} and E_{zy} are computed from the above formulas, the coefficients of X_{vx} changing for different missing values. One need again only find E_{xz}, which is

$$\frac{-m(n+2)[nh'+(n+1)h-h''] - \Sigma X_{vx}X_{vz} - (n+1)(n+2)[k'-\frac{k''}{n}]}{nm(n+1)(n+2)}$$

<u>Type III</u>: For one missing value

$$X_{vx} = \frac{(p-1)(np-n-1)}{p^2}[pa-b'] + \frac{(p-1)}{p^2}[npa'+pd+1-nb'b] \qquad v = 1, 2, \ldots, n$$

$$X'_{vx} = \frac{1}{np}[npa'+pd+1-nb'b] \qquad v = 1, 2, \ldots, n$$

$$W_v = \frac{(p-1)(np-n-1)}{p^2}[pT_v-(TC_v)] + \frac{(p-1)}{p^2}[npA_v+pP_1-G-n\{TC_v\}]$$

$$v = 1, 2, \ldots, n$$

$$W'_v = \frac{1}{np}[npA_v+pP_1-G-n\{TC_v\}] \qquad v = 1, 2, \ldots, n \ .$$

Some further notation will be required in order to reduce the complexity of the quantities to be found for this design.

Let $Q = \frac{(p-1)(np-n-1)}{p}$

and $Q' = \frac{(p-1)^2(n^2p^2-n^2p-np-n^2)}{p^2}$

$$E_{xx} = \frac{Q'Q(n^2p-np-n^2+2n-p-1)-np(n-1)\Sigma X^2{}_{vx}-np(n-1)Q'\Sigma X'^2{}_{vx}}{Q'Qnp(n-1)}$$

E_{xy} =

$$\frac{Q'Q[n(n-1)TC_i(x)+pT_j(k)(x)-(n-1)T_k(x)\cdot]-np(n-1)\Sigma X_{vx}W_v-Q'np(n-1)\Sigma X'_{vx}W'_v}{Q'Qnp(n-1)} \ .$$

For two missing values, E_{zz} and E_{zy} are again computed in a similar fashion as E_{xx} and E_{xy}. One need only find E_{xz}, which is

$$\frac{-Q'Q[n(n-1)h'+ph-(n-1)h'']-np(n-1)\Sigma X_{vx}X_{vz}-np(n-1)Q'\Sigma X'_{vx}X'vz}{Q'Qnp(n-1)} \ .$$

<u>Type IV</u>: For one missing value

$$X_{vx} = a' + \frac{(2n+1)}{2n^2} + \frac{c}{2n} + \frac{d}{n} \qquad v = 1, 2, \ldots, n$$

$$W_v = A_v - \frac{(2n+1)}{2n^2}G + \frac{F'v}{2n} + \frac{1}{n}P_1 \qquad v = 1, 2, \ldots, n$$

$$E_{xx} = \frac{(2n^3-n^2-3n+1)(4n^2-2n-1) - 4n^3\Sigma X^2_{vx}}{2n^3(4n^2-2n-1)}$$

$$E_{xy} = \frac{(4n^2-2n-1)[n^2TC_i(x)+2nT_j(k)(x)+nT_{vx}-T_k(x)-G]-4n^3\Sigma W_vX_{vx}}{2n^3(4n^2-2n-1)} .$$

For two missing values E_{zz} and E_{zy} are again computed in a similar fashion as E_{xx} and E_{xy}. One need only find E_{xz}, which is

$$\frac{-(4n^2-2n-1)(n^2h'+2nh+nk-h''-1) - 4n^3\Sigma X_{vx}X_{vz}}{2n^3(4n^2-2n-1)} .$$

V

DESIGNS FOR SECOND ORDER RESIDUAL EFFECTS

5.1 Type V Designs

General Remarks: When speaking about sets of (n-1) orthogonal Latin squares, it was mentioned that these squares are balanced for all orders of residual effects up to the $(n-1)^{st}$ order. In this section an analysis of these squares will be presented when direct, first order and second order residual effects are believed to be present. Refer to Designs 5, 10, 17, 29, 31 and 32. Additivity of all treatment effects will be assumed. These designs were first described by Williams [5].

Care must be taken in construction of the analysis of variance, for direct, first and second order effects are nonorthogonal.

The combined treatment sum of squares for direct, first order residual and second order residual effects will be broken down in the following four ways:

(1) Direct (adjusted 1^{st} and 2^{nd} order) + 1^{st}(adjusted 2^{nd} order) + 2^{nd} (unadjusted)

(2) Direct (unadjusted) + 1^{st} (adjusted direct and 2^{nd} order) + 2^{nd} (adjusted 1^{st} order)

(3) Direct (adjusted 1^{st} order) + 1^{st} (unadjusted) + 2^{nd} (adjusted direct and 1^{st} order)

(4) Direct (unadjusted) + 1^{st} (adjusted direct) + 2^{nd} (adjusted direct and 1^{st} order).

There are reasons for picking these specific sets of sums of squares, these being [5],

(1) Method 1 yields a test on direct effects,

(2) Method 2 yields a test on first order effects,

(3) Methods 3 and 4 both yield a test for second order effects,

and (4) Methods 3 and 4 together yield tests on direct and first order effects if for some reason second order effects are to be ignored.

The model for this design is

$$Y_{ijkvst} = \mu + C_i + \rho_j(k) + \tau_v + \Theta_s + \Phi_t + \varepsilon_{ijkvst}$$

$$\begin{aligned} i &= 1, 2, \ldots, n(n-1) \\ j &= 1, 2, \ldots, n \\ k &= 1, 2, \ldots, (n-1) \\ v &= 1, 2, \ldots, n \\ s &= 1, 2, \ldots, n \\ t &= 1, 2, \ldots, n \end{aligned}$$

where Φ_t represents the effect of the t^{th} treatment on the observation in the second period following it.

The normal equations are

$$n^2(n-1)\hat{\mu}+n\Sigma\hat{C}_i+n\Sigma\hat{\rho}_j+n(n-1)\Sigma\hat{\tau}_v+(n-1)^2\Sigma\hat{\Theta}_s+(n-1)(n-2)\Sigma\hat{\Phi}_t = G$$

$$n\hat{\mu}+n\hat{C}_i+\sum_{(k)}\hat{\rho}_j(k)+\Sigma\hat{\tau}_v+\Sigma^{i}\hat{\Theta}_s+\Sigma^{ii}\hat{\Phi}_t = TC_i \qquad i = 1, 2, \ldots, n(n-1)$$

$$n\hat{\mu}+\sum_{(k)}\hat{C}_i+n\hat{\rho}_1(k)+\Sigma\hat{\tau}_v = T_1(k) \qquad k = 1, 2, \ldots, (n-1)$$

$$n\hat{\mu}+\sum_{(k)}\hat{C}_i+n\hat{\rho}_2(k)+\Sigma\hat{\tau}_v+\Sigma\hat{\Theta}_s = T_2(k) \qquad k = 1, 2, \ldots, (n-1)$$

$$n\hat{\mu}+\sum_{(k)}\hat{C}_i+n\hat{\rho}_j(k)+\Sigma\hat{\tau}_v+\Sigma\hat{\Theta}_s+\Sigma\hat{\Phi}_t = T_j(k) \qquad \begin{matrix} j = 3, 4, \ldots, n \\ k = 1, 2, \ldots, (n-1) \end{matrix}$$

$$n(n-1)\hat{\mu}+\Sigma\hat{C}_i+\Sigma\hat{\rho}_j(k)+n(n-1)\hat{\tau}_v+(n-1)\Sigma^{iii}\hat{\Theta}_s+(n-2)\Sigma^{iv}\hat{\Phi}_t = T_v$$

$$v = 1, 2, \ldots, n$$

$$(n-1)^2\hat{\mu}+\Sigma^{v}\hat{C}_i+\sum_{j=2}^{n}\hat{\rho}_j(k)+(n-1)\Sigma^{vi}\hat{\tau}_v+(n-1)^2\hat{\Theta}_s+(n-2)\Sigma^{vii}\hat{\Phi}_t = A_s$$

$$s = 1, 2, \ldots, n$$

$$(n-1)(n-2)\hat{\mu}+\Sigma^{viii}\hat{C}_i+\sum_{j=3}^{n}\hat{\rho}_j(k)+(n-2)\Sigma^{ix}\hat{\tau}_v+(n-2)\Sigma^{x}\hat{\Theta}_s+(n-1)(n-2)\hat{\Phi}_t = B_t$$

$$t = 1, 2, \ldots, n$$

where $\Sigma^{i}\hat{\Theta}_s$ = the sum over all treatments in subject i except the final one,

$\Sigma^{ii}\hat{\Phi}_t$ = the sum over all treatments in subject i except those in the two final periods,

$\Sigma^{iii}\hat{\Theta}_s$ = the sum over all s where s is not equal to v,

$\Sigma^{iv}\hat{\Phi}_t$ = the sum over all t where t is not equal to v,

$\Sigma^{v}\hat{C}_i$ = the sum over all subjects for which treatment s is not in the final period,

$\Sigma^{vi}\hat{\tau}_v$ = the sum over all v where v is not equal to s,

$\Sigma^{vii}\hat{\Phi}_t$ = the sum over all t where t is not equal to s,

$\Sigma^{viii}\hat{C}_i$ = the sum over all subjects where treatment t is not in either of the final two periods,

$\Sigma^{ix}\hat{\tau}_v$ = the sum over all v where v is not equal to t,

and, $\Sigma^{x}\hat{\Theta}_s$ = the sum over all s where s is not equal to t.

After applying the usual constraints, the normal equations become

$$n^2(n-1)\hat{\mu} = G$$

$$n\hat{\mu}+n\hat{C}_i-\hat{\Theta}_{M'}-\hat{\Phi}_{M'}-\hat{\Phi}_{M''} = TC_i \qquad i = 1, 2, \ldots, n(n-1)$$

$$n\hat{\mu}+\sum_{(k)}\hat{C}_i+n\hat{\rho}_j(k) = T_j(k) \qquad \begin{array}{l} j = 1, 2, \ldots, n \\ k = 1, 2, \ldots, (n-1) \end{array}$$

$$n(n-1)\hat{\mu}+n(n-1)\hat{\tau}_v-(n-1)\hat{\Theta}_v-(n-2)\hat{\Phi}_v = T_v \qquad v = 1, 2, \ldots, n$$

$$(n-2)^2\hat{\mu}-\Sigma^{xi}\hat{C}_i-\Sigma\hat{\rho}_1(k)-(n-1)\hat{\tau}_s+(n-1)^2\hat{\Theta}_s-(n-2)\hat{\Phi}_s = A_s$$

$$s = 1, 2, \ldots, n$$

$$(n-1)(n-2)\hat{\mu}-\Sigma^{xii}\hat{C}_i-\Sigma\hat{\rho}_1(k)-\Sigma\hat{\rho}_2(k)-(n-2)\hat{\tau}_t-(n-2)\hat{\Theta}_t+(n-1)(n-2)\hat{\Phi}_t = B_t$$

$$t = 1, 2, \ldots, n$$

where $\Sigma^{xi}\hat{C}_i$ = the sum over all subject effects where treatment s occurs last in subject i,

$\Sigma^{xii}\hat{C}_i$ = the sum over all subject effects where treatment t is either in the last or next to last period in subject i.

By solving the normal equations in terms of $\hat{\tau}_v$, $\hat{\Theta}_s$ and $\hat{\Phi}_t$ only, one arrives at the following set of reduced normal equations:

$$\begin{bmatrix} n(n-1) & -(n-1) & -(n-2) \\ -(n-1) & \frac{(n-1)(n^2-n-1)}{n} & -\frac{(n+1)(n-2)}{n} \\ -(n-2) & -\frac{(n+1)(n-2)}{n} & \frac{(n+1)(n-2)^2}{n} \end{bmatrix} \begin{bmatrix} \hat{\tau}_v \\ \hat{\Theta}_v \\ \hat{\Phi}_v \end{bmatrix} = \begin{bmatrix} T'_v \\ A'_v \\ B'_v \end{bmatrix}$$

$$v = 1, 2, \ldots, n$$

where $T'_v = T_v - \frac{1}{n}G$

$$A'_v = A_v + \frac{1}{n^2}[nF'_v + nP_1 - (n+1)G]$$

and $$B'_v = B_v + \frac{1}{n^2}[nF'_v + nF''_v + nP_1 + nP_2 - (n+2)G].$$

The various adjusted and unadjusted effects which will be necessary to find the different treatment sums of squares will be found by modifying this set of equations.

At this point it will be necessary to change the notation somewhat, in order to incorporate the different adjusted and unadjusted estimates. The following notation will hold throughout the remainder of this section.

$\hat{\tau}_v$ - estimate of unadjusted direct effect

$\hat{\tau}_a$ - estimate of direct effect adjusted for 1st order residual effect

$\hat{\tau}_{ab}$ - estimate of direct effect adjusted for both 1st and 2nd order residual effects

$\hat{\theta}_v$ - estimate of unadjusted 1st order residual effect

$\hat{\theta}_b$ - estimate of 1st order residual effect adjusted for 2nd order residual effect

$\hat{\theta}_t$ - estimate of 1st order residual effect adjusted for direct effect

$\hat{\theta}_{tb}$ - estimate of 1st order residual effect adjusted for both direct and 2nd order residual effects

$\hat{\Phi}_v$ - estimate of unadjusted 2nd order residual effect

$\hat{\phi}_t$ - estimate of 2^{nd} order residual effect adjusted for direct effect

$\hat{\phi}_{at}$ - estimate of 2^{nd} order residual effect adjusted for both direct and 2^{nd} order residual effects

The following equations yield the required estimates:

$$n^2(n-1)\hat{\tau}_v = nT_v-G \qquad v = 1, 2, \ldots, n$$

$$n(n-1)(n^2-n-1)\hat{\theta}_v = n^2A_v+nF'_v+nP_1-(n+1)G \qquad v = 1, 2, \ldots, n$$

$$n(n+1)(n-2)^2\hat{\phi}_v = n^2B_v+n[F'_v+F''_v+P_1+P_2]-(n+2)G \qquad v = 1, 2, \ldots, n$$

$$n(n-2)(n^2-2n-1)\hat{\theta}_b = n(n-2)A_v+(n-1)F'_v+nB_v+F''_v+(n-1)P_1+P_2-nG \qquad v = 1, 2, \ldots, n$$

$$n(n-2)^2(n^2-2)\hat{\phi}_t = n^2(n-1)B_v+n(n-1)[F'_v+F''_v]+n(n-1)P_1+n(n-2)T_v-(n^2+2n-4)G \qquad v = 1, 2, \ldots, n$$

$$n(n+1)(n-2)(n-1)\hat{\theta}_t = n^2A_v+nF'_v+nP_1+nT_v-(n+2)G \qquad v = 1, 2, \ldots, n$$

$$n(n+1)(n-2)(n-1)\hat{\tau}_a = (n^2-n-1)T_v+nA_v+F'_v+P_1-nG \qquad v = 1, 2, \ldots, n$$

$$\begin{aligned} n(n-1)(n-2)(n^3-n^2-5n-2)\hat{\tau}_{ab} = {} & (n+1)(n-2)(n^2-2n-1)T_v+n(n+1)(n-2)A_v \\ & +(2n^2-2n-2)F'_v+n^2(n-1)B_v+n(n-1)F''_v \\ & +(2n^2-2n-2)P_1+n(n-1)P_2-n^2(n-1)G \end{aligned} \qquad v = 1, 2, \ldots, n$$

$$\begin{aligned} n(n-1)(n-2)(n^3-n^2-5n-2)\hat{\Theta}_{tb} = {} & n(n-2)(n^2-2)A_v+n(n-1)(n+2)B_v \\ & +(n^3-n^2-n+2)F'_v+(n-1)(n+2)F''_v \\ & +(n^3-n^2-n+2)P_1+(n-1)(n+2)P_2 \\ & +n(n+1)(n-2)T_v-n(n+2)(n-1)G \end{aligned}$$

$$v = 1, 2, \ldots, n$$

$$\begin{aligned} n(n-2)(n^3-n^2-5n-2)\hat{\Phi}_{at} = {} & n(n^2-1)B_v+n(n+2)A_v+n(n+1)F'_v+(n^2-1)F''_v+(n^2+n+1)P_1 \\ & +(n^2-1)P_2+n^2T_v-(n^2+4n+2)G \end{aligned}$$

$$v = 1, 2, \ldots, n \, .$$

The analysis of variance can now be written as shown in Table VIII.

TABLE VIII

ANALYSIS OF VARIANCE

Source	Degrees of Freedom	Sums of Squares
Subjects	$n(n-1)-1$	$\frac{1}{n}\Sigma TC^2{}_i - \frac{1}{n^2(n-1)}G$
Periods/squares	$(n-1)^2$	$\frac{1}{n}\Sigma T^2{}_j(k) - \frac{1}{n^2}\Sigma T^2{}_{k\cdot\cdot}$
Treatment effects	$3(n-1)$	- see subanalysis -
Error	n^3-3n^2+2	subtraction
Total	$n^2(n-1)-1$	$\Sigma Y^2{}_{ijkvst} - \frac{1}{n^2(n-1)}G^2$

The sums of squares for the treatment effects will be displayed in a subanalysis (see Table IX). Remember only one of the four sets of sums of squares can be used to calculate the error sum of squares.

The variance of a difference between any two estimates of the same kind is as follows:

$$\text{Variance } (\hat{\tau}_{ab} - \hat{\tau}_{ab'}) = \frac{2(n+1)(n^2-2n-1)}{n(n-1)(n^3-n^2-5n-2)} \sigma_\varepsilon^2$$

$$\text{Variance } (\hat{\tau}_v - \hat{\tau}_{v'}) = \frac{2}{n(n-1)} \sigma_\varepsilon^2$$

$$\text{Variance } (\hat{\tau}_a - \hat{\tau}_{a'}) = \frac{2(n^2-n-1)}{n(n^2-1)(n-2)} \sigma_\varepsilon^2$$

$$\text{Variance } (\hat{\theta}_{tb} - \hat{\theta}_{tb'}) = \frac{2(n^2-2)}{(n-1)(n^3-n^2-5n-2)} \sigma_\varepsilon^2$$

$$\text{Variance } (\hat{\theta}_b - \hat{\theta}_{b'}) = \frac{2}{(n^2-2n-1)} \sigma_\varepsilon^2$$

$$\text{Variance } (\hat{\theta}_t - \hat{\theta}_{t'}) = \frac{2n}{(n^2-1)(n-2)} \sigma_\varepsilon^2$$

$$\text{Variance } (\hat{\theta}_v - \hat{\theta}_{v'}) = \frac{2n}{(n-1)(n^2-n-1)} \sigma_\varepsilon^2$$

$$\text{Variance } (\hat{\Phi}_{at} - \hat{\Phi}_{at'}) = \frac{2(n^2-1)}{(n-2)(n^3-n^2-5n-2)} \sigma_\varepsilon^2$$

$$\text{Variance } (\hat{\Phi}_t - \hat{\Phi}_{t'}) = \frac{2n(n-1)}{(n-2)^2(n^2-2)} \sigma_\varepsilon^2$$

TABLE IX

SUBANALYSIS OF TREATMENT SUM OF SQUARES

Source	Degrees of Freedom	Sums of Squares
Direct (adjusted for 1st and 2nd)	n-1	$\frac{n(n-1)(n^3-n^2-5n-2)}{(n+1)(n^2-2n-1)}\Sigma\hat{\tau}^2_{ab}$
(1) 1st (adjusted for 2nd)	n-1	$(n^2-2n-1)\Sigma\hat{\theta}^2_b$
2nd (unadjusted)	n-1	$\frac{(n+1)(n-2)^2}{n}\Sigma\hat{\phi}^2_v$
Direct (unadjusted)	n-1	$n(n-1)\Sigma\hat{\tau}^2_v$
(2) 1st (adjusted for direct and 2nd)	n-1	$\frac{(n-1)(n^3-n^2-5n-2)}{(n^2-2)}\Sigma\hat{\theta}^2_{tb}$
2nd (adjusted for direct)	n-1	$\frac{(n-2)^2(n^2-2)}{n(n-1)}\Sigma\hat{\phi}^2_t$
Direct (adjusted for 1st)	n-1	$\frac{n(n-1)(n-2)(n+1)}{(n^2-n-1)}\Sigma\hat{\tau}^2_a$
(3) 1st (unadjusted)	n-1	$\frac{(n-1)(n^2-n-1)}{n}\Sigma\hat{\theta}^2_v$
2nd (adjusted for direct and 1st)	n-1	$\frac{(n-2)(n^3-n^2-5n-2)}{(n^2-1)}\Sigma\hat{\phi}^2_{at}$
Direct (unadjusted)	n-1	$n(n-1)\Sigma\hat{\tau}^2_v$
(4) 1st (adjusted for direct)	n-1	$\frac{(n+1)(n-1)(n-2)}{n}\Sigma\hat{\theta}^2_t$
2nd (adjusted for direct and 1st)	n-1	$\frac{(n-2)(n^3-n^2-5n-2)}{(n^2-1)}\Sigma\hat{\phi}^2_{at}$

$$\text{Variance } (\hat{\Phi}_v - \hat{\Phi}_{v'}) = \frac{2n}{(n+1)(n-2)^2} \sigma_\varepsilon^2$$

Missing Values: In finding missing value formulae it will be necessary to assume that the second order residual effect of the missing observation is actually present. The procedure is identical to that in Section 4.3. It will again only be necessary to compute E_{xx} and E_{xy}, E_{zz} and E_{zy} being computed in a like manner. It will also be necessary to compute E_{xz}, after which the formulas from Section 4.3 can be applied directly.

It will be necessary to introduce some simplifying notation.

Let $$D = (n+1)(n-2)^2(n^2-2n-1) \quad ,$$

$$D' = \frac{n(n-1)(n^3-n^2-5n-2)}{(n+1)(n^2-2n-1)} \quad ,$$

and $$D'' = \frac{(n^3-3n^2+4)}{n} \quad .$$

The remaining notation is the same as that from Section 4.3.

Let $$X_{vx} = a + \frac{1}{(n^2-2n-1)}[a'n+c] + \frac{1}{(n+1)(n-2)(n^2-2n-1)}[a''n^2(n-1)$$
$$+c'n(n-1)+2d(n^2-n-1)+d'n(n-1)+n^2(n-1)] \qquad v = 1, 2, \ldots, n$$

$$X'_{vx} = D''[a'n^2+n+cn+dn-1] + \frac{(n+1)(n-2)}{n}[a''n^2+c'n+dn+d'n+n+2]$$
$$v = 1, 2, \ldots, n$$

$$X''_{vx} = [a''n^2+c'n+dn+n+2] \qquad v = 1, 2, \ldots, n$$

$$W_v = T_v + \frac{1}{(n^2-2n-1)}[nA_v+F'_v] + \frac{1}{(n+1)(n-2)(n^2-2n-1)}\{n^2(n-1)B_v+n(n-1)[F'_v+F''_v]$$

$$+2(n^2-n-1)P_1+n(n-1)P_2-n^2(n-1)G\} \qquad v = 1, 2, \ldots, n$$

$$W'_v = D''[n^2A_v+nF'_v+nP_1-(n+1)G]+\frac{(n+1)(n-2)}{n}\{n^2B_v+n[F'_v+F''_v+P_1+P_2]-(n+2)G\}$$

$$v = 1, 2, \ldots, n$$

$$W''_v = n[nB_v+F'_v+F''_v+P_1+P_2]-(n+2)G \qquad v = 1, 2, \ldots, n\ .$$

Now,

$$E_{xx} = \frac{n^2DD'D''(n-1)^2-n^4DD''\Sigma X^2_{vx}-D'\Sigma X'^2_{vx}-DD'\Sigma X''^2_{vx}}{n^4DD'D''}$$

$$E_{xy} =$$

$$\frac{n^2DD'D''[nTC_i(x)+nT_j(k)(x)-T_k(x)\cdot]-n^4DD''\Sigma X_{vx}W_v-D'\Sigma X'_{vx}W'_v-DD'\Sigma X''_{vx}W''_v}{n^4DD'D''}\ .$$

As before, the variables in the above formulas for X_{vx}, X'_{vx} and X''_{vx} change for each different location in the design the missing value assumes and X_{vz}, X'_{vz} and X''_{vz} are found from the same formulae.

$$E_{xz} =$$

$$\frac{-n^2DD'D''[nh'+nh-h'']-n^4DD''\Sigma X_{vx}X_{vz}-D'\Sigma X'_{vx}X'_{vz}-DD'\Sigma X''_{vx}X''_{vz}}{n^4DD'D''}\ .$$

Efficiency: Type V designs will be compared to Designs I through

IV for the same cases as in Section III. Comparisons will be made first for differences in direct effects. The numbers in the tables represent efficiencies of Design V to Designs I through IV. The cases n=3, p=3 and n=4, p=3 are presented.

TABLE X

Design	I	II	III	IV
V	$\frac{5}{32}$	$\frac{2}{15}$	$\frac{5}{32}$	$\frac{1}{8}$

n=3

Design	I	II	III	IV
V	$\frac{143}{175}$	$\frac{65}{84}$	$\frac{351}{340}$	$\frac{26}{35}$

n=4

The method of analysis in Design V is primarily for second order residual effects, and if none were assumed to exist, the design could be analyzed by the method of Design I. This is the reason for the low efficiencies. Also note, however, that this design is not as efficient as Design III if p=n, but it is more efficient than Design III if p<n.

The next two sections of Table XI contain comparisons for estimating differences in first order residual effect. The numbers represent efficiencies of Design V to Designs I through IV for the same two cases as previously illustrated.

TABLE XI

Design	I	II	III	IV
V	$\frac{3}{28}$	$\frac{4}{63}$	$\frac{3}{28}$	$\frac{12}{203}$

n=3

Design	I	II	III	IV
V	$\frac{52}{70}$	$\frac{65}{112}$	$\frac{1053}{952}$	$\frac{208}{385}$

n=4

This case also illustrates the low efficiencies of Design V to Designs I through IV. Also, the efficiencies are lower than for direct effects. Type V is still better than Type III for $p<n$.

5.2 Type VI Designs

If an experimenter wanted as much information about second order residual effects as possible, as might be the case in testing medicines, and if he could afford to have more periods than was necessary for Design V, he could choose a design of this type. The reader is referred to Designs 3, 7 and 19. It can easily be shown that all treatment effects, direct, first and second order, are orthogonal. The only non-orthogonality in the designs is found between residual effects and subject effects.

For the most part these designs have no practical value, and are merely included for the sake of interest. Some attempt was made toward generalization of construction, but as practicality is small, it was abandoned.

The sources for the analysis of variance for these designs will be as follows:

Source
Subjects (unadjusted)
Periods
Direct
1st order (adjusted for subjects)
2nd order (adjusted for subjects)
Error
Total

VI

DESIGNS BALANCED FOR THE LINEAR COMPONENT OF RESIDUAL EFFECTS

6.1 Type VII Designs

General Information: In the following designs only one treatment will be tested. However, it will be tested at a number of equally spaced levels, so that the treatment must be quantitative. Equal spacing permits easier analysis for linear and curvature components. A set of Latin squares is constructed in such a manner that the linear component of residual effects and the linear, quadratic, ... components of direct effects are orthogonal. This type of change-over design was first discussed by Berenblut [13] where he deals with a specific example for four levels. In a second paper by Berenblut [9] he extends these designs to five levels, and includes a test for linear direct χ linear residual interaction. This is essentially a test for additivity of direct and residual effects, assuming that direct and residual effects are predominantly linear. He gives no designs for n>5.

In his paper he assumes a model proposed by Finney [17] in which " . . . errors are uncorrelated but first residual effects are multiples of corresponding direct effects" [9].

He then gives the following reasoning and conditions for assuming linearity of residual effects:

"It is . . . general for the constant of proportionality between residual and direct effects to be less than unity; if in fact, the residual effects are very small by comparison with direct effects, even the presence of some curvature in the direct effects will not seriously affect the linearity of residual effects, so long as the linear component in the direct effects is predominant. We extend this idea to quantitative treatments in general, and take as the conditions for assuming linearity of residual effects (i) direct effects to have a predominant linear component, (ii) residual effects to be small by comparison with direct effects and proportional to them." [9].

For four levels he gives one design, that being Design 12. For five levels he gives 12 designs, those being Designs 15a through 15l. Each design has a different degree of non-orthogonality, and must have a unique analysis of variance. All 13 designs will be analyzed entirely.

Note that Designs 12 and 15a through 15l utilize the same notation as the rest of the designs, but that in this case the letters represent the different levels of a given quantitative treatment in either ascending or descending order. For example, a could represent the low level, b the next level, etc. Also, in this analysis v will represent one level of the given treatment, and T_v will represent the sum over all the observations at the v^{th} level.

Without Interaction: Only the design for four levels will be analyzed without interaction. It will also be analyzed in the following section under an interaction model.

Model I $Y_{ij} = \mu+C_i+\rho_j+\tau_L\xi_1+\tau_Q\xi_2+\tau_C\xi_3+\theta_L\eta_1+\theta_Q\eta_2+\theta_C\eta_3+\epsilon_{ij}$

$$i = 1, 2, \ldots, 8$$
$$j = 1, 2, 3, 4$$

where τ_L, τ_Q and τ_C are the linear, quadratic and cubic components of direct effects; θ_L, θ_Q and θ_C are the linear, quadratic and cubic effects of first order residual effects, and ξ_1, ξ_2, ξ_3, η_1, η_2, and η_3 are the orthogonal polynomials for four levels defined as follows:

TABLE XII

Treatment Applied	a	b	c	d	
ξ_1	-3	-1	1	3	η_1
ξ_2	1	-1	-1	1	η_2
ξ_3	-1	3	-3	1	η_3
	a	b	c	d	Previous Treatment

Applying the constraints $\Sigma\hat{C}_i = \Sigma\hat{\rho}_j = 0$, the normal equations become

$$32\hat{\mu} = G$$

$$4\hat{\mu}+4\hat{C}_1-\hat{\theta}_L+\hat{\theta}_Q+3\hat{\theta}_C = TC_1$$

$$4\hat{\mu}+4\hat{C}_2-3\hat{\theta}_L-\hat{\theta}_Q-\hat{\theta}_C = TC_2$$

$$4\hat{\mu}+4\hat{C}_3+3\hat{\theta}_L-\hat{\theta}_Q+\hat{\theta}_C = TC_3$$

$$4\hat{\mu}+4\hat{C}_4+\hat{\theta}_L+\hat{\theta}_Q-3\hat{\theta}_C = TC_4$$

$$4\hat{\mu}+4\hat{C}_5+\hat{\theta}_L+\hat{\theta}_Q-3\hat{\theta}_C = TC_5$$

$$4\hat{\mu}+4\hat{C}_6+3\hat{\Theta}_L-\hat{\Theta}_Q+\hat{\Theta}_C = TC_6$$

$$4\hat{\mu}+4\hat{C}_7-3\hat{\Theta}_L-\hat{\Theta}_Q-\hat{\Theta}_C = TC_7$$

$$4\hat{\mu}+4\hat{C}_8-\hat{\Theta}_L+\hat{\Theta}_Q+3\hat{\Theta}_C = TC_8$$

$$8\hat{\mu}+8\hat{\rho}_j = P_j \qquad j = 1, 2, 3, 4$$

$$160\hat{\tau}_L = -3T_a-T_b+T_c+3T_d$$

$$32\hat{\tau}_Q-24\hat{\Theta}_Q = T_a-T_b-T_c+T_d$$

$$160\hat{\tau}_C = -T_a+3T_b-3T_c+T_d$$

$$120\hat{\Theta}_L-\hat{C}_1-3\hat{C}_2+3\hat{C}_3+\hat{C}_4+\hat{C}_5+3\hat{C}_6-3\hat{C}_7-\hat{C}_8 = -3A_a-A_b+A_c+3A_d$$

$$24\hat{\Theta}_Q-24\hat{\tau}_Q+\hat{C}_1-\hat{C}_2-\hat{C}_3+\hat{C}_4+\hat{C}_5-\hat{C}_6-\hat{C}_7+\hat{C}_8 = A_a-A_b-A_c+A_d$$

$$120\hat{\Theta}_C+3\hat{C}_1-\hat{C}_2+\hat{C}_3-3\hat{C}_4-3\hat{C}_5+\hat{C}_6-\hat{C}_7+3\hat{C}_8 = -A_a+3A_b-3A_c+A_d \;.$$

Solving the above equations for the residual linear, quadratic and cubic effects, one gets

$$110\hat{\Theta}_L = -3A_a-A_b+A_c+3A_d+\frac{1}{4}[TC_1+3TC_2-3TC_3-TC_4-TC_5-3TC_6+3TC_7+TC_8]$$

$$4\hat{\Theta}_Q = A_a-A_b-A_c+A_d-\frac{1}{4}[TC_1-TC_2-TC_3+TC_4+TC_5-TC_6-TC_7+TC_8]+\frac{3}{4}[T_a-T_b-T_c+T_d]$$

$$110\hat{\Theta}_C = -A_a+3A_b-3A_c+A_d-\frac{1}{4}[3TC_1-TC_2+TC_3-3TC_4-3TC_5+TC_6-TC_7+TC_8] \text{ .}$$

$\hat{\tau}_Q$ is also needed

$$64\hat{\tau}_Q = 11[T_a-T_b-T_c+T_d]+12[A_a-A_b-A_c+A_d]-3[TC_1-TC_2-TC_3+TC_4+TC_5-TC_6-TC_7+TC_8] \text{ .}$$

Two methods of analysis will be presented. The first will be for testing residual effects, the second for testing direct effects assuming residual effects are predominantly linear.

The first analysis of variance follows in Table XIII. Note that v represents the different levels of the given treatment. A test for significance of curvature of residual effects is

$$\frac{(4\hat{\Theta}^2{}_Q + 110\hat{\Theta}^2{}_C)/2}{MSE} \sim F_{2,15} \text{ .}$$

Also, the test for linear residual effects is

$$\frac{110\hat{\Theta}^2{}_L}{MSE} \sim F_{1,15} \text{ .}$$

However, this analysis does not yield a test for unadjusted direct effects.

If one could assume that the curvature components of residual

TABLE XIII

ANALYSIS OF VARIANCE

Source	Degrees of Freedom	Sums of Squares
Subjects	7	$\frac{1}{4}\Sigma TC^2{}_i - \frac{1}{32}G^2$
Periods	3	$\frac{1}{8}\Sigma P^2{}_j - \frac{1}{32}G^2$
Direct (unadjusted)	3	$\frac{1}{8}\Sigma T^2{}_v - \frac{1}{32}G^2$
Residual (adjusted)	3	$110\hat{\theta}^2{}_L + 4\hat{\theta}^2{}_Q + 110\hat{\theta}^2{}_C$
Linear	1	$110\hat{\theta}^2{}_L$
Quadratic	1	$4\hat{\theta}^2{}_Q$
Cubic	1	$110\hat{\theta}^2{}_C$
Error	15	subtraction
Total	31	$\Sigma Y^2{}_{ij} - \frac{1}{32}G^2$

effects are negligible it is possible to obtain a test for unadjusted direct effects and also for linear residual effects. This is because direct effects are orthogonal to linear residual effects, but not to quadratic residual effects. One merely pools the quadratic residual sum of squares unadjusted for direct effects with error. Also, the cubic residual sum of squares should be pooled with error. This yields the analysis of variance in Table XIV. The test for linearity of residual effects now becomes

$$\frac{110\hat{\theta}^2{}_L}{MSE} \sim F_{1,17}$$

in addition to the usual tests for direct effects.

TABLE XIV

ANALYSIS OF VARIANCE

Source	Degrees of Freedom	Sums of Squares
Subjects (unadjusted)	7	$\frac{1}{4}\Sigma TC^2{}_i - \frac{1}{32}G^2$
Periods	3	$\frac{1}{8}\Sigma P^2{}_j - \frac{1}{32}G^2$
Direct (adjusted for residual quadratic)	3	$160\hat{\tau}^2{}_L + \frac{64}{11}\hat{\tau}^2{}_Q + 160\hat{\tau}^2{}_C$
Linear	1	$160\hat{\tau}^2{}_L$
Quadratic (adjusted)	1	$\frac{64}{11}\hat{\tau}^2{}_Q$
Cubic	1	$160\hat{\tau}^2{}_C$
Residual linear (adjusted subjects)	1	$110\hat{\theta}^2_L$
Error	17	subtraction
Total	31	$\Sigma Y^2{}_{ij} - \frac{1}{32}G^2$

The variances of the different effects are:

$$\text{Variance } (\hat{\tau}_L) = \text{Variance } (\hat{\tau}_C) = \frac{1}{160}\sigma_\varepsilon^2$$

$$\text{Variance } (\hat{\tau}_Q) = \frac{11}{64}\sigma_\varepsilon^2$$

$$\text{Variance } (\hat{\theta}_L) = \text{Variance } (\hat{\theta}_C) = \frac{1}{110}\sigma_\varepsilon^2$$

$$\text{Variance } (\hat{\theta}_Q) = \frac{1}{4}\sigma_\varepsilon^2$$

With Interaction: The example just discussed will be analyzed first to illustrate the procedure. Then the 12 designs for five levels will be analysed.

Four Levels

Model II

$$Y_{ij} = \mu+C_i+\rho_j+\tau_L\xi_1+\tau_Q\xi_2+\tau_C\xi_3+\theta_L\eta_1+(\theta\tau)\zeta+\varepsilon_{ij} \quad \begin{matrix} i = 1, 2, \ldots, 8 \\ j = 1, 2, 3, 4 \end{matrix}$$

where everything is defined as in Model I, and $(\theta\tau)$ represents the effect of the interaction of linear direct χ linear residual.

The coefficients ζ are found by multiplying together the two orthogonal polynomials ξ_1 and η_1, both from linear terms. Upon doing this, one gets the following values of ζ as shown in Table XV. An asterisk is used to represent a treatment sequence which does not occur.

The normal equations after constraints are applied become

TABLE XV

		Treatment Applied			
		a	b	c	d
Previous Treatment	a	*	3	-3	*
	b	3	*	*	-3
	c	-3	*	*	3
	d	*	-3	3	*

$$32\hat{\mu} = G$$

$$4\hat{\mu}+4\hat{C}_1-\hat{\theta}_L+3(\widehat{\theta\tau}) = TC_1$$

$$4\hat{\mu}+4\hat{C}_2-3\hat{\theta}_L+3(\widehat{\theta\tau}) = TC_2$$

$$4\hat{\mu}+4\hat{C}_3+3\hat{\theta}_L+3(\widehat{\theta\tau}) = TC_3$$

$$4\hat{\mu}+4\hat{C}_4+\hat{\theta}_L+3(\widehat{\theta\tau}) = TC_4$$

$$4\hat{\mu}+4\hat{C}_5+\hat{\theta}_L-3(\widehat{\theta\tau}) = TC_5$$

$$4\hat{\mu}+4\hat{C}_6+3\hat{\theta}_L-3(\widehat{\theta\tau}) = TC_6$$

$$4\hat{\mu}+4\hat{C}_7-3\hat{\theta}_L-3(\widehat{\theta\tau}) = TC_7$$

$$4\hat{\mu}+4\hat{C}_8-\hat{\Theta}_L-3(\widehat{\Theta\tau}) = TC_8$$

$$8\hat{\mu}+8\hat{\rho}_j = P_j \qquad j = 1, 2, 3, 4$$

$$160\hat{\tau}_L = -3T_a-T_b+T_c+3T_d$$

$$32\hat{\tau}_Q = T_a-T_b-T_c+T_d$$

$$160\hat{\tau}_C = -T_a+3T_b-3T_c+T_d$$

$$120\hat{\Theta}_L-\hat{C}_1-3\hat{C}_2+3\hat{C}_3+\hat{C}_4+\hat{C}_5+3\hat{C}_6-3\hat{C}_7-\hat{C}_8 = -3A_a-A_b+A_c+3A_d$$

$$216(\widehat{\Theta\tau})+3[\hat{C}_1+\hat{C}_2+\hat{C}_3+\hat{C}_4-\hat{C}_5-\hat{C}_6-\hat{C}_7-\hat{C}_8] = 3[T_{ab}+T_{ba}+T_{cd}+T_{dc}-T_{bd}-T_{db}-T_{ac}-T_{ca}]$$

Solving for $\hat{\Theta}_L$ and $(\widehat{\Theta\tau})$,

$$110\hat{\Theta}_L = -3A_a-A_b+A_c+3A_d+\frac{1}{4}[TC_1+3TC_2-3TC_3-TC_4-TC_5-3TC_6+3TC_7+TC_8]$$

$$\begin{aligned}198(\widehat{\Theta\tau}) = {} & 3[T_{ab}+T_{ba}+T_{cd}+T_{dc}-T_{bd}-T_{db}-T_{ac}-T_{ca}] \\ & -\frac{3}{4}[TC_1+TC_2+TC_3+TC_4-TC_5-TC_6-TC_7-TC_8] \ .\end{aligned}$$

Define $f(T) = [T_{ab}+T_{ba}+T_{cd}+T_{dc}-T_{bd}-T_{db}-T_{ac}-T_{ca}]$

$g(B) = \frac{1}{4}[TC_1+TC_2+TC_3+TC_4-TC_5-TC_6-TC_7-TC_8]$

Then one gets,

$$198(\widehat{\Theta\tau}) = 3f(T) - 3g(B).$$

The analysis of variance can now be presented as in Table XVI.

TABLE XVI

ANALYSIS OF VARIANCE

Source	Degrees of Freedom		Sums of Squares
Subjects (unadjusted)	7		$\frac{1}{4}\Sigma TC^2{}_i - \frac{1}{32}G^2$
Periods	3		$\frac{1}{8}\Sigma P^2{}_j - \frac{1}{32}G^2$
Direct	3		$\frac{1}{8}\Sigma T^2{}_v - \frac{1}{32}G^2$
Linear		1	$160\hat{\tau}^2{}_L$
Deviations		2	$32\hat{\tau}^2{}_Q + 160\hat{\tau}^2{}_C$
Linear Residual (adjusted for subjects)	1		$110\hat{\theta}^2{}_L$
Linear Direct χ Linear Residual (adjusted)	1		- see below -
Error	16		subtraction
Total	31		$\Sigma Y^2{}_{ij} - \frac{1}{32}G^2$

Note that v represents different levels of the same treatment.

Linear Direct χ Linear Residual Sum of Squares $= 198(\widehat{\theta\tau})^2$

$$= \frac{1}{198}[3f(T)-3g(B)]^2$$

$$= \frac{1}{22}[f(T)-g(B)]^2$$

$$\text{Variance } (\hat{\theta}_L) = \frac{1}{110}\sigma^2_\varepsilon$$

$$\text{Variance } (\widehat{\theta\tau}) = \frac{1}{198}\sigma^2_\varepsilon \ .$$

Since direct effects are orthogonal to all other effects, a test statistic for testing direct effects is

$$\frac{\text{Direct MS}}{\text{EMS}} \sim F_{3,16} \ .$$

Five Levels: As previously noted there are no general formulas for analyzing Designs 15a through 151.

All the designs are orthogonal for direct, linear residual and linear χ linear effects, but neither linear residual nor linear χ linear effects are orthogonal to subject effects. Each design contains a different degree of entanglement so as to render the sums of squares different in each instance.

The normal equations for μ, ρ_j, τ_L, τ_Q, τ_C and τ_q will be the same for all 12 designs, but those for C_i, θ_L and $(\theta\tau)$ will change. The normal equations will be similar to those for the previous case of four levels, and will not be presented. Only the method of analysis will be presented here.

Certain properties and sums of squares will be the same for all the designs and will be given first. Thereafter, only four quantities will be needed to complete the analysis.

Model

$$Y_{ij} = \mu+C_i+\rho_j+\tau_L\xi_1+\tau_Q\xi_2+\tau_C\xi_3+\tau_q\xi_4+\theta_L\eta+(\theta\tau)\zeta+\varepsilon_{ij} \qquad \begin{array}{l} i = 1, 2, \ldots, 10 \\ j = 1, 2, 3, 4, 5 \end{array}$$

where τ_L, τ_Q, τ_C, θ_L, and $(\theta\tau)$ are defined as in the case for four levels, τ_q is the quartic component of direct effects, and ξ_1, ξ_2, ξ_3, ξ_4 and η are the orthogonal polynomials for five levels and for the linear level in five levels respectively. They are defined as follows:

TABLE XVII

Treatment Applied	a	b	c	d	e	
ξ_1	-2	-1	0	1	2	η
ξ_2	2	-1	-2	-1	2	
ξ_3	-1	2	0	-2	1	
ξ_4	1	-4	6	-4	1	
	a	b	c	d	e	Previous Treatment

ζ is the product of ξ_1 and η defined as shown in Table XVIII.

Where an * indicates the combination does not occur. It should be noted that some of the above combinations without an asterisk do not occur for some designs. For example, consider Design 15a. Treatment e never immediately follows treatment a, or vice versa. Therefore, if either of the totals T_{ae} or T_{ea} were required for a general formula, they would simply be zero for this design.

The general analysis of variance is seen in Table XIX. Note that v represents the different levels.

TABLE XVIII

		Treatment Applied				
		a	b	c	d	e
Previous Treatment	a	*	2	0	-2	-4
	b	2	*	0	-1	-2
	c	0	0	*	0	0
	d	-2	-1	0	*	2
	e	-4	-2	0	2	*

For all designs let $h(\Theta) = -2A_a - A_b + A_d + 2A_e$,

and let

$f(T) = 2[T_{ab}+T_{ba}+T_{de}+T_{ed}-T_{ad}-T_{da}-T_{be}-T_{eb}]-4[T_{ae}+T_{ea}]-[T_{bd}+T_{db}]$.

Now, the sum of squares for Linear Residual (unadjusted for interaction) can be shown to be

$$\frac{1}{76}(A')^2$$

and the sum of squares for Linear χ Linear Interaction (adjusted) is

$$C^*[AT'+C^{**}A']^2,$$

where $A' = h(\Theta) + \frac{1}{5}[g_1(B)]$,

$AT' = f(T) + \frac{1}{5}[g_2(B)]$,

and C^*, C^{**}, $g_1(B)$ and $g_2(B)$ must be defined separately for each of the twelve designs.

TABLE XIX

ANALYSIS OF VARIANCE

Source	Degrees of Freedom	Sums of Squares
Subjects (unadjusted)	9	$\frac{1}{5}\Sigma TC^2{}_i - \frac{1}{50} G^2$
Periods	4	$\frac{1}{10} \Sigma P^2{}_j - \frac{1}{50} G^2$
Direct	4	$\frac{1}{10} \Sigma T^2{}_v - \frac{1}{50} G^2$
Linear	1	$\frac{1}{100}[-2T_a-T_b+T_d+2T_e]^2$
Quadratic	1	$\frac{1}{140}[2T_a-T_b-2T_c-T_d+2T_e]^2$
Cubic	1	$\frac{1}{100}[-T_a+2T_b-2T_d+T_e]^2$
Quartic	1	$\frac{1}{700}[T_a-4T_b+6T_c-4T_d+T_e]^2$
Linear Residual (unadjusted)	1	- see text -
Linear Direct χ Linear Residual	1	- see text -
Error	30	subtraction
Total	49	$\Sigma Y^2{}_{ij} - \frac{1}{50} G^2$

Design

15a $g_1(B) = [2TC_1+TC_3-2TC_4-TC_5-TC_6+TC_8+2TC_9-2TC_{10}]$

$g_2(B) = 2[-2TC_1-TC_2+TC_3+TC_7-TC_8+2TC_{10}]$

$C^* = .0115$

$C^{**} = -.0421$

15b $g_1(B) = [TC_2+2TC_3-2TC_4-TC_5-2TC_7+2TC_8-TC_9+TC_{10}]$

$g_2(B) = 2[-TC_1-2TC_2+TC_3+TC_6-TC_8+2TC_9]$

$C^{*} = .0116$

$C^{**} = -.0210$

15c $g_1(B) = [2TC_1+TC_3-2TC_4-TC_5+2TC_6-2TC_7-TC_8+TC_9]$

$g_2(B) = 2[-2TC_1-TC_2+TC_3+2TC_6-TC_8+TC_9]$

$C^{*} = .01157$

$C^{**} = .0105$

15d $g_1(B) = [TC_2-TC_3+2TC_4-2TC_5+2TC_6+TC_7-TC_8-2TC_9]$

$g_2(B) = [-2TC_1-4TC_2-2TC_3-TC_5+3TC_6+6TC_7+2TC_8-4TC_9+2TC_{10}]$

$C^{*} = .0120$

$C^{**} = .0463$

15e $g_1(B) = [2TC_2+TC_3-TC_4-2TC_5+2TC_6+TC_7-TC_8-2TC_{10}]$

$g_2(B) = [-2TC_1+2TC_3+6TC_4-4TC_5-TC_6-4TC_7+3TC_{10}]$

$C^{*} = .0092$

$C^{**} = -.0210$

15f $g_1(B) = [2TC_2+TC_3-2TC_4-TC_5+2TC_6-TC_8-2TC_9+TC_{10}]$

$g_2(B) = [-3TC_1-4TC_2+2TC_3+3TC_6-2TC_8+5TC_9-TC_{10}]$

$C^{*} = .0106$

$C^{**} = -.0232$

15g $g_1(B) = [TC_2+2TC_3-2TC_4-TC_5-TC_6+TC_7-2TC_8+2TC_9]$

$g_2(B) = 2[-TC_1-2TC_2+TC_3+2TC_7-TC_8+TC_{10}]$

$C^{*} = .01157$

$C^{**} = .0210$

15h $g_1(B) = [2TC_2-TC_3-2TC_4+TC_5-TC_6+2TC_7+TC_8-2TC_9]$

$g_2(B) = [5TC_1-4TC_2-2TC_3+3TC_4-TC_5-4TC_7+3TC_8+3TC_9-3TC_{10}]$

$C^* = .01427$

$C^{**} = .0630$

15i $g_1(B) = [TC_1-2TC_2-2TC_3-TC_5+TC_6+2TC_8-2TC_9-TC_{10}]$

$g_2(B) = 2[-2TC_1+TC_3-TC_4+3TC_5+3TC_6-TC_7-TC_8-2TC_{10}]$

$C^* = .01162$

$C^{**} = 0$

15_j $g_1(B) = [2TC_1-2TC_2+TC_3-TC_4-TC_6+2TC_7+TC_8-2TC_9]$

$g_2(B) = [3TC_1-TC_2-4TC_4+3TC_5-4TC_7+2TC_8+3TC_9-2TC_{10}]$

$C^* = .0106$

$C^{**} = 0$

15k $g_1(B) = [TC_1-2TC_2-TC_3+2TC_4-TC_6+2TC_7+TC_8-2TC_9]$

$g_2(B) = 2[-2TC_1-TC_3-TC_5+2TC_6+TC_8+TC_{10}]$

$C^* = .01157$

$C^{**} = -.0105$

15l $g_1(B) = [-TC_1+2TC_3-2TC_4+TC_5+TC_6-2TC_8+2TC_9-TC_{10}]$

$g_2(B) = 2[-TC_2-TC_3-2TC_4+TC_7+TC_8+2TC_9]$

$C^* = .01158$

$C^{**} = .0210$

Variances: If interaction effects can be assumed negligible, and no test for it is being made, the variance of $\hat{\Theta}_L$ is the same for all 12 designs, being equal to $\frac{1}{76}\sigma^2_\varepsilon$. If, however, one is testing using the interaction model, the different variances of $\hat{\Theta}_L$ and $\widehat{(\Theta\tau)}$ can be found from Table XX.

TABLE XX

Design	Variance$(\hat{\theta}_L)/\sigma^2_\varepsilon$	Variance $(\widehat{\theta\tau})/\sigma^2_\varepsilon$
15a	.01318	.01159
15b	.01316	.01158
15c	.01316	.01157
15d	.01318	.01200
15e	.01316	.00920
15f	.01316	.01600
15g	.01316	.01158
15h	.01321	.01427
15i	.01316	.01162
15j	.01316	.01059
15k	.01316	.01157
15l	.01316	.01158

Designs e and j are particularly good for estimating $(\widehat{\theta\tau})$ while Designs d, f and h are not quite as good as the remainder of them. All designs are about equal with respect to estimation of $\hat{\theta}_L$, Design h having the largest variance.

6.2 Type VIII Designs

General Remarks: Unfortunately there have been no designs published of Type VII for greater than five levels, and Berenblut [9] states that none exist for two or three levels. The extra-period type II designs can, however, be analyzed under the same model, also yielding a test for interaction. These designs are orthogonal for linear, quadratic, ... direct effects and linear residual effects. The linear residual effect is now orthogonal to subjects, but the direct effects are no longer orthogonal to subjects. The linear χ linear interaction effect is still nonorthogonal to subjects, but is now also nonorthogonal to periods.

The efficiency of these designs to Type VII designs is quite low, and it would be advisable to use one of the latter designs if one is available. However, Type VIII designs are quite easy to find, exist for two and three levels, and are not much more complicated to analyze than those of Type VII. Again there is no general formula to follow for this analysis.

Two designs will be analyzed, one for three levels and one for four levels. The four level design will be compared to the four level Type VII design.

Since the linear residual effect is completely orthogonal to all other effects it need not be adjusted, and

$$\frac{(\text{Linear Residual MS})}{\text{EMS}}$$

will always be a proper test statistic for linear residual effects for these designs.

Three Levels: Design 5 will be used as an extra-period design. The model will be

$$Y_{ij} = \mu + C_i + \rho_j + \tau_L \xi_1 + \tau_Q \xi_2 + \theta_L \eta + (\theta\tau)\zeta + \varepsilon_{ij} \qquad \begin{array}{l} i = 1, 2, \ldots, 6 \\ j = 1, 2, 3, 4 \end{array}$$

where ξ_1 and ξ_2 represent the two orthogonal polynomials corresponding to the linear and quadratic components for three levels, η represents the linear orthogonal polynomial for three levels, and ζ is merely $\xi \chi \eta$ representing the linear direct χ linear residual interaction.

ξ_1, ξ_2 and η are defined as follows:

TABLE XXI

Treatment Applied	a	b	c	
ξ_1	-1	0	1	η
ξ_2	1	-2	1	
	a	b	c	Previous Treatment

and ζ is defined as :

TABLE XXII

Treatment Applied

		a	b	c
Previous Treatment	a	1	0	-1
	b	0	0	0
	c	-1	0	1

Note that in this analysis the letters a, b, c ... in the designs represent quantitative equally spaced levels of the same treatment in either ascending or descending order.

After applying the constraints $\Sigma\hat{C}_i = \Sigma\hat{\rho}_j = 0$, the normal equations become:

$$24\hat{\mu} = G$$

$$4\hat{\mu} + 4\hat{C}_1 + \hat{\tau}_L + \hat{\tau}_Q + (\widehat{\theta\tau}) = TC_1$$

$$4\hat{\mu} + 4\hat{C}_2 - \hat{\tau}_L + \hat{\tau}_Q = TC_2$$

$$4\hat{\mu} + 4\hat{C}_3 - 2\hat{\tau}_Q - (\widehat{\theta\tau}) = TC_3$$

$$4\hat{\mu} + 4\hat{C}_4 - 2\hat{\tau}_Q - (\widehat{\theta\tau}) = TC_4$$

$$4\hat{\mu} + 4\hat{C}_5 + \hat{\tau}_L + \hat{\tau}_Q = TC_5$$

$$4\hat{\mu} + 4\hat{C}_6 - \hat{\tau}_L + \hat{\tau}_Q + (\widehat{\theta\tau}) = TC_6$$

$$6\hat{\mu} + 6\hat{\rho}_1 = P_1$$

$$6\hat{\mu} + 6\hat{\rho}_2 - 2(\widehat{\theta\tau}) = P_2$$

$$6\hat{\mu} + 6\hat{\rho}_3 - 2(\widehat{\theta\tau}) = P_3$$

$$6\hat{\mu} + 6\hat{\rho}_4 + 4\widehat{(\theta\tau)} = P4$$

$$16\hat{\tau}_L + \hat{C}_1 - \hat{C}_2 + \hat{C}_5 - \hat{C}_6 = -T_a + T_c$$

$$48\hat{\tau}_Q + \hat{C}_1 + \hat{C}_2 - 2\hat{C}_3 - 2\hat{C}_4 + \hat{C}_5 + \hat{C}_6 = T_a - 2T_b + T_c$$

$$12\hat{\theta}_L = -A_a + A_c$$

$$8\widehat{(\theta\tau)} + \hat{C}_1 - \hat{C}_3 - \hat{C}_4 + \hat{C}_6 - 2\hat{\rho}_2 - 2\hat{\rho}_3 + 4\hat{\rho}_4 = T_{aa} + T_{cc} - T_{ac} - T_{ca}$$

Solving for $\hat{\tau}_L$ one gets

$$15\hat{\tau}_L = -T_a + T_c - \frac{1}{4}[TC_1 - TC_2 + TC_5 - TC_6]$$

It will also be necessary to solve for $\hat{\tau}_Q$ and $\widehat{(\theta\tau)}$. Two quantities in terms of both effects are obtained,

$$45\hat{\tau}_Q - \frac{3}{2}\widehat{(\theta\tau)} = Q'$$

$$-\frac{3}{2}\hat{\tau}_Q + 3\widehat{(\theta\tau)} = AT'$$

where

$$Q' = T_a - 2T_b + T_c - \frac{1}{4}[TC_2 + TC_2 - 2TC_3 - 2TC_4 + TC_5 + TC_6]$$

and

$$AT' = T_{aa} + T_{cc} - T_{ac} - T_{ca} - \frac{1}{4}[TC_1 - TC_3 - TC_4 + TC_5] + \frac{1}{3}[P_2 + P_3 - 2P_4] .$$

It will only be necessary to solve for $\widehat{(\theta\tau)}$

$$\frac{177}{60}\widehat{(\theta\tau)} = AT' + \frac{1}{30}Q'$$

The analysis of variance can now be constructed as in Table XXIII. The variances of $\hat{\theta}_L$ and $\widehat{(\theta\tau)}$ are

$$\text{Variance } (\hat{\theta}_L) = \frac{1}{12}\sigma^2_\varepsilon \qquad \text{Variance } \widehat{(\theta\tau)} = \frac{60}{177}\sigma^2_\varepsilon$$

The test statistic for additivity of direct and residual effects is

$$\frac{177}{60}\widehat{(\theta\tau)}^2/MSE \sim F_{1,11}.$$

Four Levels: Design 9 is a completely balanced Latin square for four treatments. It will be used as an extra-period design with the treatments a, b, c, and d representing four equally spaced levels of a given treatment.

The model is

$$Y_{ij} = \mu + C_i + \rho_j + \tau_L\xi_1 + \tau_Q\xi_2 + \tau_C\xi_3 + \theta_L\eta + (\theta\tau)\zeta + \varepsilon_{ij}$$

$$i = 1, 2, 3, 4$$
$$j = 1, 2, 3, 4, 5$$

TABLE XXIII

ANALYSIS OF VARIANCE

Source	Degrees of Freedom	Sums of Squares
Subjects (unadjusted)	5	$\frac{1}{4}\Sigma TC^2{}_i - \frac{1}{24}G^2$
Periods (unadjusted)	3	$\frac{1}{6}\Sigma P^2{}_j - \frac{1}{24}G^2$
Direct (adjusted only for subjects)	2	$15\hat{\tau}_L{}^2 + \frac{1}{45}Q'^2$
Linear (adjusted for subjects)	1	$15\hat{\tau}_L{}^2$
Quadratic (unadjusted for linear χ linear)	1	$\frac{1}{45}Q'^2$
Linear Residual	1	$12\hat{\theta}^2{}_L$
Linear Direct χ Linear Residual (adjusted)	1	$\frac{177}{60}(\widehat{\theta\tau})^2$
Error	11	subtraction
Total	23	$\Sigma Y^2{}_{ij} - \frac{1}{24}G^2$

where the notation is the same as that for four level Type VII designs with interaction, except that in this case ζ is defined as shown in Table XXIV because all possible combinations occur.

After applying the constraints $\Sigma\hat{C}_i = \Sigma\hat{\rho}_j = 0$, the normal equations become:

TABLE XXIV

		Treatment Applied			
		a	b	c	d
Previous Treatment	a	9	3	-3	-9
	b	3	1	-1	-3
	c	-3	-1	1	3
	d	-9	-3	3	9

$$20\hat{\mu} = G$$

$$5\hat{\mu}+5\hat{C}_1+3\hat{\tau}_L+\hat{\tau}_Q+\hat{\tau}_C+14(\widehat{\theta\tau}) = TC_1$$

$$5\hat{\mu}+5\hat{C}_2+\hat{\tau}_L-\hat{\tau}_Q-3\hat{\tau}_C-14(\widehat{\theta\tau}) = TC_2$$

$$5\hat{\mu}+5\hat{C}_3-\hat{\tau}_L-\hat{\tau}_Q+3\hat{\tau}_C-14(\widehat{\theta\tau}) = TC_3$$

$$5\hat{\mu}+5\hat{C}_4-3\hat{\tau}_L+\hat{\tau}_Q-\hat{\tau}_C+14(\widehat{\theta\tau}) = TC_4$$

$$4\hat{\mu}+4\hat{\rho}_1 = P_1$$

$$4\hat{\mu}+4\hat{\rho}_2 = P_2$$

$$4\hat{\mu}+4\hat{\rho}_3-20(\widehat{\theta\tau}) = P_3$$

$$4\hat{\mu}+\hat{\rho}_4 = P_4$$

$$4\hat{\mu}+\hat{\rho}_5+20(\widehat{\theta\tau}) = P_5$$

$$100\hat{\tau}_L+3\hat{C}_1+\hat{C}_2-\hat{C}_3-3\hat{C}_4 = -3T_a-T_b+T_c+3T_d$$

$$20\hat{\tau}_Q+\hat{C}_1-\hat{C}_2-\hat{C}_3+\hat{C}_4 = T_a-T_b-T_c+T_d$$

$$100\hat{\tau}_C+\hat{C}_1-3\hat{C}_2+3\hat{C}_3-\hat{C}_4 = -T_a+3T_b-3T_c+T_d$$

$$80\hat{\theta}_L = -3A_a-A_b+A_c+3A_d$$

$$\begin{aligned}400(\widehat{\theta\tau})+14\hat{C}_1-14\hat{C}_2-14\hat{C}_3+14\hat{C}_4-20\hat{\rho}_3+20\hat{\rho}_5 = {} & [T_{bb}+T_{cc}-T_{bc}-T_{cb}] \\ & +3[T_{ab}+T_{ba}+T_{cd}+T_{dc}-T_{ac}-T_{bd}-T_{ca}-T_{db}] \\ & +9[T_{aa}+T_{dd}-T_{ad}-T_{da}] \ .\end{aligned}$$

Solving for $\hat{\tau}_L$ and $\hat{\tau}_C$ one gets

$$96\hat{\tau}_L = L' \qquad \text{and} \qquad 96\hat{\tau}_C = C'$$

$$L' = -3T_a-T_b+T_c+3T_d-\frac{1}{5}[3TC_1+TC_2-TC_3-3TC_4]$$

and

$$C' = -T_a+3T_b-3T_c+T_d-\frac{1}{5}[TC_1-3TC_2+3TC_3-TC_4] \quad .$$

Let $\qquad 80\hat{\theta}_L = R'$

where $R' = -3A_a - A_b + A_c + 3A_d$.

Now, solving for $\hat{\tau}_Q$ and $\widehat{(\theta\tau)}$ one gets the following set of equations:

$$\frac{96}{5}\hat{\tau}_Q - \frac{56}{5}\widehat{(\theta\tau)} = Q'$$

$$-\frac{56}{5}\hat{\tau}_Q + \frac{216}{5}\widehat{(\theta\tau)} = AT'$$

where $Q' = T_a - T_b - T_c + T_d - \frac{1}{5}[TC_1 - TC_2 - TC_3 + TC_4]$

and $AT' = [T_{bb} + T_{cc} - T_{bc} - T_{cb}] + 3[T_{ab} + T_{ba} + T_{cd} + T_{dc} - T_{ac} - T_{bd} - T_{ca} - T_{db}]$
$+9[T_{aa} + T_{dd} - T_{ad} - T_{da}] - \frac{14}{5}[TC_1 - TC_2 - TC_3 + TC_4] + 5[P_3 - P_5]$.

It will only be necessary to find the adjusted interaction effect, which is

$$\frac{110}{3}\widehat{(\theta\tau)} = AT' + \frac{7}{12}Q' \quad .$$

The analysis of variance can now be constructed as in Table XXV. Note that in this design and in the previous design for three levels $\frac{\text{Linear Residual MS}}{\text{MSE}}$ is a valid test for linearity of residual effects because of orthogonality.

The sum of squares for direct effects can be split up as shown in Table XXVI.

The variance of $\widehat{(\theta\tau)}$ is $\frac{3}{110}\sigma_\varepsilon^2$

TABLE XXV

ANALYSIS OF VARIANCE

Source	Degrees of Freedom	Sums of Squares
Subjects (unadjusted)	3	$\frac{1}{5}\Sigma TC^2{}_i - \frac{1}{20}G^2$
Periods (unadjusted)	4	$\frac{1}{4}\Sigma P^2{}_j - \frac{1}{20}G^2$
Direct (adjusted for subjects)	3	- see text -
Linear Residual	1	$\frac{1}{80}R'^2$
Linear Direct χ Linear Residual (adjusted)	1	$\frac{110}{3}(\widehat{\theta_\tau})^2$
Error	7	subtraction
Total	19	$\Sigma Y^2{}_{ij} - \frac{1}{20}G^2$

TABLE XXVI

Source	Degrees of Freedom	Sums of Squares
Direct (adjusted for subjects)	3	$\frac{1}{96}[L'^2 + 5Q'^2 + C'^2]$
Linear (adjusted subjects)	1	$\frac{1}{96}L'^2$
Quadratic (adjusted subjects)	1	$\frac{5}{96}Q'^2$
Cubic (adjusted subjects)	1	$\frac{1}{96}C'^2$

and that for $\hat{\theta}_L$ is $\frac{1}{80}\sigma^2_\varepsilon$.

Efficiency: The design for four levels just discussed will be compared with design Type VII for four levels with interaction. It will be necessary to find the number of observations χ the variance of the effects to be compared. The effects to be compared are $\hat{\theta}_L$ and $(\widehat{\theta\tau})$.

First for $\hat{\theta}_L$:

Design	Number of Observations χ Variance $\hat{\theta}_L$
Type VII	$32(\frac{1}{110}) = .291$
Type VIII	$20(\frac{1}{80}) = .250$

The efficiency of Type VIII to Type VII designs is 1.164. Therefore, if one were mainly interested in estimating first order residual effects, one should use a Type VIII design which yields a slightly better estimate. This statement can only be made here for four levels. No attempt will be made in this paper to extend this result to five levels or more.

Now the two types of designs will be compared for estimation of interaction effects.

Design	Number of Observations χ Variance $(\widehat{\theta\tau})$
Type VII	$32(\frac{1}{198}) = .165$
Type VIII	$20(\frac{3}{110}) = .545$

The efficiency of Design VIII to Design VII is seen to be .303. One sees that Design VII, or Berenblut's design is far better at estimating the interaction effect.

It can only be concluded that the type of design utilized should be chosen by the amount of precision desired of the two effects. If more precision is desired for first order residual effects than interaction effects, design Type VIII should be used, and vice versa.

6.3 Type IX Designs

General Remarks: Type VII and VIII designs can be quite useful if the experimenter is only interested in testing one type of treatment at various equally spaced levels. Often this is not the case. What if he were interested in testing, say, two treatments, and still wanted some test for additivity? As long as his treatments are quantitative there may be another design he could use. The author has devised some designs which can be analyzed for more than one treatment, each at this same number of equally spaced levels.

Examples will be given for two treatments, each at two levels; two treatments, each at three levels; and three treatments, each at two levels.

Only the first example will be analyzed.

Two Treatments at Two Levels: Construction of the design is as follows. Take Design 12. Let treatment a be one level of treatment one, denoted by a_1. Let treatment d be the second level of treatment one, denoted by a_2. Let treatments b and c be the two levels of treatment two, denoted by b_1 and b_2 respectively. The following design is now obtained:

$$\begin{array}{cccccccc}
a_1 & a_1 & a_2 & a_2 & b_1 & b_1 & b_2 & b_2 \\
b_1 & b_2 & b_1 & b_2 & a_1 & a_2 & a_1 & a_2 \\
a_2 & a_2 & a_1 & a_1 & b_2 & b_2 & b_1 & b_1 \\
b_2 & b_1 & b_2 & b_1 & a_2 & a_1 & a_2 & a_1
\end{array}$$

The four levels of the two treatments can actually be thought of as four separate treatments. Therefore, each level (or treatment) is seen to appear once in each subject and twice in each period, rendering treatments orthogonal to both subjects and periods.

Other orthogonal properties will be discussed after presentation of the normal equations.

The model for this design is

$$Y_{ij} = \mu+C_i+\rho_j+\tau_1\xi_1+\Theta_1\xi_2+\tau_2\eta_1+\Theta_2\eta_2+(\tau_1\tau_2)\lambda_1+(\Theta_1\Theta_2)\lambda_2+(\tau_1\Theta_2)\zeta_1+(\tau_2\Theta_1)\zeta_2+\varepsilon_{ij}$$

$$i = 1, 2, \ldots, 8$$
$$j = 1, 2, 3, 4$$

where τ_1 represents the (linear) direct effect of treatment 1,

τ_2 represents the (linear) direct effect of treatment 2,

Θ_1 represents the (linear) residual effect of treatment 1,

Θ_2 represents the (linear) residual effect of treatment 2,

$(\tau_1\tau_2)$ represents a linear contrast in the observations orthogonal to the direct effects of treatments 1 and 2; or 1 versus 2,

$(\Theta_1\Theta_2)$ represents a linear contrast in the observations orthogonal to the residual effects of treatments 1 and 2; or 1 versus 2,

$(\tau_1\theta_2)$ represents the interaction effect treatment 1 direct χ treatment 2 residual,

and $(\tau_2\theta_1)$ represents the interaction effect treatment 2 direct χ treatment 1 residual.

ξ_1, ξ_2, η_1 and η_2 can be defined as follows:

TABLE XXVII

Treatment Applied	a_1	a_2	b_1	b_2	
ξ_1	-1	1	0	0	ξ_2
η_1	0	0	-1	1	η_2
	a_1	a_2	b_1	b_2	Previous Treatment

ζ_1 and ζ_2 can be defined as follows:

TABLE XXVIII

		Treatment Applied			
		a_1	a_2	b_1	b_2
Previous Treatment	a_1	*	*	1	-1
	a_2	*	*	-1	1
	b_1	1	-1	*	*
	b_2	-1	1	*	*
		ζ_1		ζ_2	

and λ_1 and λ_2 can be defined as follows:

TABLE XXIX

Treatment Applied

	a_1	a_2	b_1	b_2
λ_1	1	1	-1	-1
λ_2	-1	-1	1	1

The normal equations under the constraints $\Sigma\hat{C}_i = \Sigma\hat{\rho}_j = 0$ are:

$$32\hat{\mu} = G$$

$$4\hat{\mu} + 4\hat{C}_1 - \hat{\theta}_2 - (\widehat{\tau_1\theta_2}) + 2(\widehat{\tau_2\theta_1}) + (\widehat{\theta_1\theta_2}) = TC_1$$

$$4\hat{\mu} + 4\hat{C}_2 + \hat{\theta}_2 + (\widehat{\tau_1\theta_2}) - 2(\widehat{\tau_2\theta_1}) + (\widehat{\theta_1\theta_2}) = TC_2$$

$$4\hat{\mu} + 4\hat{C}_3 - \hat{\theta}_2 + (\widehat{\tau_1\theta_2}) - 2(\widehat{\tau_2\theta_1}) + (\widehat{\theta_1\theta_2}) = TC_3$$

$$4\hat{\mu} + 4\hat{C}_4 + \hat{\theta}_2 - (\widehat{\tau_1\theta_2}) + 2(\widehat{\tau_2\theta_1}) + (\widehat{\theta_1\theta_2}) = TC_4$$

$$4\hat{\mu} + 4\hat{C}_5 - \hat{\theta}_1 + 2(\widehat{\tau_1\theta_2}) - (\widehat{\tau_2\theta_1}) - (\widehat{\theta_1\theta_2}) = TC_5$$

$$4\hat{\mu} + 4\hat{C}_6 + \hat{\theta}_1 - 2(\widehat{\tau_1\theta_2}) + (\widehat{\tau_2\theta_1}) - (\widehat{\theta_1\theta_2}) = TC_6$$

$$4\hat{\mu} + 4\hat{C}_7 - \hat{\theta}_1 - 2(\widehat{\tau_1\theta_2}) + (\widehat{\tau_2\theta_1}) - (\widehat{\theta_1\theta_2}) = TC_7$$

$$4\hat{\mu} + 4\hat{C}_8 + \hat{\theta}_1 + 2(\widehat{\tau_1\theta_2}) - (\widehat{\tau_2\theta_1}) - (\widehat{\theta_1\theta_2}) = TC_8$$

$$8\hat{\mu} + 8\hat{\rho}_j = P_j \qquad\qquad j = 1, 2, 3, 4$$

$$16\hat{\tau}_1 = T_{a_2} - T_{a_1}$$

$$12\hat{\theta}_1 - \hat{C}_5 + \hat{C}_6 - \hat{C}_7 + \hat{C}_8 = A_{a_2} - A_{a_1}$$

$$16\hat{\tau}_2 = T_{b_2} - T_{b_1}$$

$$12\hat{\theta}_2 - \hat{C}_1 + \hat{C}_2 - \hat{C}_3 + \hat{C}_4 = A_{b_2} - A_{b_1}$$

$$32(\widehat{\tau_1\tau_2}) = T_{a_1} + T_{a_2} - T_{b_1} - T_{b_2}$$

$$24(\widehat{\theta_1\theta_2}) + \hat{C}_1 + \hat{C}_2 + \hat{C}_3 + \hat{C}_4 - \hat{C}_5 - \hat{C}_6 - \hat{C}_7 - \hat{C}_8 = A_{a_1} + A_{a_2} - A_{b_1} - A_{b_2}$$

$$12(\widehat{\tau_1\theta_2}) \quad \hat{C}_1 + \hat{C}_2 + \hat{C}_3 - \hat{C}_4 + 2(\hat{C}_5 - \hat{C}_6 - \hat{C}_7 + \hat{C}_8) = T_{a_2b_2} + T_{a_1b_1}$$
$$- T_{a_2b_1} - T_{a_1b_2}$$

$$12(\widehat{\tau_2\theta_1}) + 2(\hat{C}_1 - \hat{C}_2 - \hat{C}_3 + \hat{C}_4) - \hat{C}_5 + \hat{C}_6 + \hat{C}_7 - \hat{C}_8 = T_{b_1a_1} + T_{b_2a_2}$$
$$- T_{b_2a_1} - T_{b_1a_2} .$$

It can be seen that subject effects are nonorthogonal to residual effects, direct χ residual interaction effects and the linear contrast among the residual effects.

One first needs to obtain $\hat{\Theta}_1$ and $\hat{\Theta}_2$ as follows:

$$11\hat{\Theta}_1 = A'_1$$

$$11\hat{\Theta}_2 = A'_2$$

where

$$A'_1 = A_{a_2} - A_{a_1} + \frac{1}{4}[TC_5 - TC_6 + TC_7 - TC_8]$$

and
$$A'_2 = A_{b_2} - A_{b_1} + \frac{1}{4}[TC_1 - TC_2 + TC_3 - TC_4] \quad .$$

One needs next to solve for $(\widehat{\Theta_1\Theta_2})$:

$$22(\widehat{\Theta_1\Theta_2}) = (A_1A_2)'$$

where

$$(A_1A_2)' = A_{a_1} + A_{a_2} - A_{b_1} - A_{b_2} + \frac{1}{4}[TC_5+TC_6+TC_7+TC_8-TC_1-TC_2-TC_3-TC_4] \quad .$$

Lastly, one needs to solve for the interaction effects; doing so yields a set of simultaneous equations in both estimates as follows:

$$7(\widehat{\tau_1\theta_2}) + 4(\widehat{\tau_2\theta_1}) = (T_1A_2)'$$

$$4(\widehat{\tau_1\theta_2}) + 7(\widehat{\tau_2\theta_1}) = (T_2A_1)'$$

where

$$(T_1A_2)' = T_{a_1b_1}+T_{a_2b_2}-T_{a_2b_1}-T_{a_1b_2}+\frac{1}{4}[TC_1-TC_2-TC_3+TC_4-2TC_5+2TC_6+2TC_7-2TC_8] \ ,$$

and

$$(T_2A_1)' = T_{b_1a_1}+T_{b_2a_2}-T_{b_1a_2}-T_{b_2a_1}+\frac{1}{4}[-2TC_1+2TC_2+2TC_3-2TC_4+TC_5-TC_6-TC_7+TC_8] \ .$$

The adjusted estimates now become

$$33(\widehat{\tau_1\theta_2}) = 7(T_1A_2)' - 4(T_2A_1)'$$

$$33(\widehat{\tau_2\theta_1}) = 7(T_2A_1)' - 4(T_1A_2)' \quad .$$

The analysis of variance is shown in Table XXX. Only one pair of interaction sums of squares should be used when finding the error sum of squares. As a check, note that

1 Direct χ 2 Residual (adjusted) Sum of Squares + 1 Residual χ 2 Direct (unadjusted) Sum of Squares = 1 Residual χ 2 Direct (adjusted) Sum of squares + 1 Direct χ 2 Residual (unadjusted) Sum of Squares.

TABLE XXX

Source	Degrees of Freedom	Sums of Squares
Subjects (unadjusted)	7	$\frac{1}{4}\Sigma TC^2{}_i - \frac{1}{32}G^2$
Periods	3	$\frac{1}{8}\Sigma P^2{}_j - \frac{1}{32}G^2$
Direct Effects	3	$\frac{1}{8}[T^2{}_{a_1}+T^2{}_{a_2}+T^2{}_{b_1}+T^2{}_{b_2}]-\frac{1}{32}G^2$
Treatment 1 Direct	1	$\frac{1}{16}[T_{a_1}-T_{a_2}]^2$
Treatment 2 Direct	1	$\frac{1}{16}[T_{b_1}-T_{b_2}]^2$
Treatment 1 vs Treatment 2	1	$\frac{1}{32}[T_{a_1}+T_{a_1}-T_{b_1}-T_{b_1}]^2$
Residual Effects	3	$11[\hat{\theta}^2{}_1+\hat{\theta}^2{}_2+2(\widehat{\theta_1\theta_2})^2]$
Treatment 1 Residual	1	$11\hat{\theta}^2{}_1$
Treatment 2 Residual	1	$11\hat{\theta}^2{}_2$
Treatment 1 vs Treatment 2	1	$22(\widehat{\theta_1\theta_2})^2$
1 Dir χ 2 Res (adj 1 Res χ 2 Dir)	1	$\frac{33}{7}(\widehat{\tau_1\theta_2})^2$
1 Res χ 2 Dir (unadjusted)	1	$\frac{1}{7}(T_2A_1)'^2$
1 Res χ 2 Dir (adj 1 Dir χ 2 Res)	1	$\frac{33}{7}(\widehat{\tau_2\theta_1})^2$
1 Dir χ 2 Res (unadjusted)	1	$\frac{1}{7}(T_1A_2)'^2$
Error	13	subtraction
Total	31	$\Sigma Y_{ij} - \frac{1}{32}G^2$

Since in this design a treatment never directly follows itself, the interaction between a treatment direct effect and its own residual effect is of no concern. Also, the interactions between the two effects $(\widehat{\tau_1\tau_2})$ and $(\widehat{\Theta_1\Theta_2})$ and the two interaction effects are considered to be negligible.

The variances of the different effects are as follows:

$$\text{Variance } (\hat{\tau}_1) = \text{Variance } (\hat{\tau}_2) = \frac{1}{16}\sigma^2_\varepsilon \quad ,$$

$$\text{Variance } (\hat{\Theta}_1) = \text{Variance } (\hat{\Theta}_2) = \frac{1}{16}\sigma^2_\varepsilon \quad ,$$

and

$$\text{Variance } (\widehat{\tau_1\Theta_2}) = \text{Variance } (\widehat{\tau_2\Theta_1}) = \frac{7}{33}\sigma^2_\varepsilon \quad ,$$

Two Treatments at Three Levels: A design for two treatments each at three equally spaced levels is:

$$\begin{array}{cccccccccccccccccc}
a_1 & a_1 & a_1 & a_2 & a_2 & a_2 & a_3 & a_3 & a_3 & b_1 & b_1 & b_1 & b_2 & b_2 & b_2 & b_3 & b_3 & b_3 \\
b_1 & b_2 & b_3 & b_2 & b_3 & b_1 & b_3 & b_1 & b_2 & a_1 & a_2 & a_3 & a_2 & a_3 & a_1 & a_3 & a_1 & a_2 \\
a_3 & a_3 & a_3 & a_1 & a_1 & a_1 & a_2 & a_2 & a_2 & b_2 & b_2 & b_2 & b_3 & b_3 & b_3 & b_1 & b_1 & b_1 \\
b_2 & b_3 & b_1 & b_3 & b_1 & b_2 & b_1 & b_2 & b_3 & a_2 & a_3 & a_1 & a_3 & a_1 & a_2 & a_1 & a_2 & a_3 \\
a_2 & a_2 & a_2 & a_3 & a_3 & a_3 & a_1 & a_1 & a_1 & b_3 & b_3 & b_3 & b_1 & b_1 & b_1 & b_2 & b_2 & b_2 \\
b_3 & b_1 & b_2 & b_1 & b_2 & b_3 & b_2 & b_3 & b_1 & a_3 & a_1 & a_2 & a_1 & a_2 & a_3 & a_2 & a_3 & a_1
\end{array}$$

The analysis of this design will not be considered here.

Three Treatments at Two Levels: A design for three treatments each at two equally spaced levels is:

a_1	a_1	a_1	a_1	a_2	a_2	a_2	a_2	b_1	b_1	b_1	b_1	b_2	b_2	b_2	b_2	c_1	c_1	c_1	c_1	c_2	c_2	c_2	c_2
b_1	b_2	c_1	c_2	b_1	b_2	c_1	c_2	a_1	a_2	c_1	c_2	a_1	a_2	c_1	c_2	a_1	a_2	b_1	b_2	a_1	a_2	b_1	b_2
c_1	c_2	b_1	b_2	c_2	c_1	b_2	b_1	c_1	c_2	a_1	a_2	c_2	c_1	a_2	a_1	b_1	b_2	a_1	a_2	b_2	b_1	a_2	a_1
b_2	b_1	c_2	c_1	b_2	b_1	c_2	c_1	a_2	a_1	c_2	c_1	a_2	a_1	c_2	c_1	a_2	a_1	b_2	b_1	a_2	a_1	b_2	b_1
c_2	c_1	b_2	b_1	c_1	c_2	b_1	b_2	c_2	c_1	a_2	a_1	c_1	c_2	a_1	a_2	b_2	b_1	a_2	a_1	b_1	b_2	a_1	a_2
a_2	a_2	a_2	a_2	a_1	a_1	a_1	a_1	b_2	b_2	b_2	b_2	b_1	b_1	b_1	b_1	c_2	c_2	c_2	c_2	c_1	c_1	c_1	c_1

The analysis of this design will not be considered here.

VII

MISCELLANEOUS DESIGNS

Designs 16, 27 and 28 are examples of balanced designs which do not fit any of the types already discussed. They might be classified as designs for n treatments, p periods ($p<n$) and m squares. No attempt will be made to analyze these designs.

Patterson, in regard to these designs, states, "When rectangles are considered it is found that there are not many balanced designs requiring fewer than n(n-1) units." [14]. Of course a design based on Latin rectangles for n(n-1) units can always be formed from orthogonal Latin squares, which have been denoted as Type III designs.

Another group of designs stems from what have been called Type VI designs. As was noted, the three Designs 3, 7 and 19 are completely orthogonal for all treatment effects up to second order residual effects. This orthogonality has been extended in Designs 4, 8 and 20 up to third order residual effects and in Design 21 up to fourth order residual effects. These designs are presented to give the reader an idea of what can be constructed. Other designs could be constructed for other numbers of treatments, but practicality does not seem to warrant the effort. Unfortunately the author is not aware of any other designs with the same orthogonality properties, but of smaller dimensions. As the presented designs were constructed by trial and error, none for smaller dimensions were found, and it does not seem that any might exist.

VIII

SUMMARY

A number of different types of change-over designs are analyzed. The analysis of variance for each type is given explicitly, along with variances and expected mean squares. In some cases different designs are compared and efficiencies are obtained.

While efficiency is certainly a criterion for choosing a design, it has been shown that another major factor of consideration is the number of observations. In some cases a more efficient design leads to more periods than would necessarily be required. In these cases the subjects may not be able to handle this increase in number of periods. It also may be the case that there are simply not enough subjects available.

In any case, no specific design can be recommended for all purposes. Each different problem requires its own solution, and the necessity to choose the best design available, for whatever reasons the problem dictates.

APPENDIX

DESIGNS

Designs will be listed by number of treatments. Rows represent periods and columns represent subjects.

2 Treatments

Design 1

a	b
b	a

Design 2

a	b	a	b
b	a	a	b
b	a	b	a
a	b	b	a

Design 3

a	a	b	b
a	b	b	a
a	b	a	b
b	b	a	a
b	a	a	b
b	a	b	a

Design 4

a	a	a	a	b	b	b	b
a	a	b	b	a	a	b	b
a	b	a	b	a	b	a	b
a	a	b	b	b	b	a	a
b	a	b	a	a	b	a	b
b	b	b	b	a	a	a	a
b	a	b	a	b	a	b	a
b	b	a	a	a	a	b	b

3 Treatments

Design 5

a	b	c
b	c	a
c	a	b

a	b	c
c	a	b
b	c	a

Design 6

a	b	c	a	b	c	a	b	c
c	a	b	b	c	a	a	b	c
b	c	a	b	c	a	b	c	a
b	c	a	a	b	c	c	a	b
c	a	b	c	a	b	c	a	b
a	b	c	c	a	b	b	c	a

Design 7

a	a	a	b	b	b	c	c	c
a	b	c	b	c	a	c	a	b
a	b	c	a	b	c	a	b	c
c	c	c	a	a	a	b	b	b
c	a	b	a	b	c	b	c	a
c	a	b	c	a	b	c	a	b
b	b	b	c	c	c	a	a	a
b	c	a	c	a	b	a	b	c
b	c	a	b	c	a	b	c	a

Design 8

a	a	a	b	b	b	c	c	c
a	b	c	b	c	a	c	a	b
a	b	c	a	b	c	a	b	c
a	b	c	c	a	b	b	c	a
c	c	c	a	a	a	b	b	b
c	a	b	a	b	c	b	c	a
c	a	b	c	a	b	c	a	b
c	a	b	b	c	a	a	b	c
b	b	b	c	c	c	a	a	a
b	c	a	c	a	b	a	b	c
b	c	a	b	c	a	b	c	a
b	c	a	a	b	c	c	a	b

4 Treatments

Design 9

a	b	c	d
b	d	a	c
c	a	d	b
d	c	b	a

Design 10

a	b	c	d
b	a	d	c
c	d	a	b
d	c	b	a

a	b	c	d
d	c	b	a
b	a	d	c
c	d	a	b

a	b	c	d
c	d	a	b
d	c	b	a
b	a	d	c

Design 11

a	b	c	d	a	b	c	d	a	b	c	d	a	b	c	d
d	a	b	c	c	d	b	a	b	c	d	a	a	b	c	d
c	d	a	b	c	d	a	b	c	d	a	b	c	d	a	b
b	c	d	a	a	b	c	d	d	a	b	c	c	d	a	b
b	c	d	a	b	c	d	a	b	c	d	a	b	c	d	a
c	d	a	b	b	c	d	a	a	b	c	d	d	a	b	c
d	a	b	c	d	a	b	c	d	a	b	c	d	a	b	c
a	b	c	d	d	a	b	c	c	d	a	b	b	c	d	a

Design 12

a	b	c	d
b	a	d	c
d	c	b	a
c	d	a	b

a	b	c	d
c	d	a	b
d	c	b	a
b	a	d	c

5 Treatments

Design 13

a	b	c	d	e
b	c	d	e	a
d	e	a	b	c
e	a	b	c	d
c	d	e	a	b

a	b	c	d	e
c	d	e	a	b
b	c	d	e	a
e	a	b	c	d
d	e	a	b	c

Design 14	a b c d e	a b c d e
	b c d e a	e a b c d
	e a b c d	b c d e a
	c d e a b	d e a b c
	d e a b c	c d e a b

Design 15a	a b c d e	a b c d e
	b a e c d	d e a c b
	c d b e a	e d b a c
	d e a b c	c a e b d
	e c d a b	b c d e a

Design 15b	a b c d e	a b c d e
	b a d e c	d e b a c
	e c a b d	e d a c b
	d e b c a	b c d e a
	c d e a b	c a e b d

Design 15c	a b c d e	a b c d e
	b a e c d	d c e a b
	c d b e a	c e d b a
	d e a b c	b d a e c
	e c d a b	e a b c d

Design 15d	a b c d e	a b c d e
	b a e c d	d c a e b
	e c d a b	b e d c a
	d e a b c	c a e b d
	c d b e a	e d b a c

Design 15e	a b c d e	a b c d e
	b a e c d	c a d e b
	e d b a c	b c e a d
	d c a e b	d e a b c
	c e d b a	e d b c a

Design 15f	a b c d e	a b c d e
	b a e c d	d a e b c
	d c b e a	b e d c a
	e d a b c	c d a e b
	c e d a b	e c b a d

Design 15g

a b c d e	a b c d e
b a d e c	c e d a b
e c a b d	d c e b a
d e b c a	e a b c d
c d e a b	b d a e c

Design 15h

a b c d e	a b c d e
d a e b c	c a e b d
b c d e a	d c a e b
e d a c b	e d b c a
c e b a d	b e d a c

Design 15i

a b c d e	a b c d e
b c d e a	e a b c d
c e a b d	b d a e c
e d b a c	c e d b a
d a e c b	d c e a b

Design 15j

a b c d e	a b c d e
c d b e a	c a e b d
d e a c b	d c b e a
b c e a d	e d a c b
e a d b c	b e d a c

Design 15k

a b c d e	a b c d e
b c e a d	d a e c b
c e d b a	c d b e a
e d a c b	e c a b d
d a b e d	b e d a c

Design 15l

a b c d e	a b c d e
d a b e c	c e b a d
e d a c b	b d e c a
c e d b a	e a d b c
b c e a d	d c a e b

Design 16

a	b	c	d	e
b	c	d	e	a
d	e	a	b	c

a	b	c	d	e
d	e	a	b	c
c	d	e	a	b

Design 17

a	b	c	d	e
b	c	d	e	a
c	d	e	a	b
d	e	a	b	c
e	a	b	c	d

a	b	c	d	e
c	d	e	a	b
e	a	b	c	d
b	c	d	e	a
d	e	a	b	c

a	b	c	d	e
d	e	a	b	c
b	c	d	e	a
e	a	b	c	d
c	d	e	a	b

a	b	c	d	e
e	a	b	c	d
d	e	a	b	c
c	d	e	a	b
b	c	d	e	a

Design 18

a	b	c	d	e	a	b	c	d	e	a	b	c	d	e	a	b	c	d	e	a	b	c	d	e
e	a	b	c	d	d	e	a	b	c	c	d	e	a	b	b	c	d	e	a	a	b	c	d	e
d	e	a	b	c	d	e	a	b	c	d	e	a	b	c	d	e	a	b	c	d	e	a	b	c
c	d	e	a	b	b	c	d	e	a	a	b	c	d	e	e	a	b	c	d	d	e	a	b	c
b	c	d	e	a	b	c	d	e	a	b	c	d	e	a	b	c	d	e	a	b	c	d	e	a
b	c	d	e	a	a	b	c	d	e	e	a	b	c	d	d	e	a	b	c	c	d	e	a	b
c	d	e	a	b	c	d	e	a	b	c	d	e	a	b	c	d	e	a	b	c	d	e	a	b
d	e	a	b	c	c	d	e	a	b	b	c	d	e	a	a	b	c	d	e	e	a	b	c	d
e	a	b	c	d	e	a	b	c	d	e	a	b	c	d	e	a	b	c	d	e	a	b	c	d
a	b	c	d	e	e	a	b	c	d	d	e	a	b	c	c	d	e	a	b	b	c	d	e	a

Design 19

a	a	a	a	a	b	b	b	b	b	c	c	c	c	c	d	d	d	d	d	e	e	e	e	e
a	b	c	d	e	b	c	d	e	a	c	d	e	a	b	d	e	a	b	c	e	a	b	c	d
a	b	c	d	e	c	d	e	a	b	e	a	b	c	d	b	c	d	e	a	d	e	a	b	c
b	b	b	b	b	c	c	c	c	c	d	d	d	d	d	e	e	e	e	e	a	a	a	a	a
b	c	d	e	a	c	d	e	a	b	d	e	a	b	c	e	a	b	c	d	a	b	c	d	e
b	c	d	e	a	d	e	a	b	c	a	b	c	d	e	c	d	e	a	b	e	a	b	c	d
c	c	c	c	c	d	d	d	d	d	e	e	e	e	e	a	a	a	a	a	b	b	b	b	b
c	d	e	a	b	d	e	a	b	c	e	a	b	c	d	a	b	c	d	e	b	c	d	e	a
c	d	e	a	b	e	a	b	c	d	b	c	d	e	a	d	e	a	b	c	a	b	c	d	e
d	d	d	d	d	e	e	e	e	e	a	a	a	a	a	b	b	b	b	b	c	c	c	c	c
d	e	a	b	c	e	a	b	c	d	a	b	c	d	e	b	c	d	e	a	c	d	e	a	b
d	e	a	b	c	a	b	c	d	e	c	d	e	a	b	c	a	b	c	d	b	c	d	e	a
e	e	e	e	e	a	a	a	a	a	b	b	b	b	b	c	c	c	c	c	d	d	d	d	d
e	a	b	c	d	a	b	c	d	e	b	c	d	e	a	c	d	e	a	b	d	e	a	b	c
e	a	b	c	d	b	c	d	e	a	d	e	a	b	c	a	b	c	d	e	c	d	e	a	b

Design 20

a	a	a	a	a	b	b	b	b	b	c	c	c	c	c	d	d	d	d	d	e	e	e	e	e
a	b	c	d	e	b	c	d	e	a	c	d	e	a	b	d	e	a	b	c	e	a	b	c	d
a	b	c	d	e	c	d	e	a	b	e	a	b	c	d	b	c	d	e	a	d	e	a	b	c
a	b	c	d	e	d	e	a	b	c	b	c	d	e	a	e	a	b	c	d	c	d	e	a	b
b	b	b	b	b	c	c	c	c	c	d	d	d	d	d	e	e	e	e	e	a	a	a	a	a
b	c	d	e	a	c	d	e	a	b	d	e	a	b	c	e	a	b	c	d	a	b	c	d	e
b	c	d	e	a	d	e	a	b	c	a	b	c	d	e	c	d	e	a	b	e	a	b	c	d
b	c	d	e	a	e	a	b	c	d	c	d	e	a	b	a	b	c	d	e	d	e	a	b	c
c	c	c	c	c	d	d	d	d	d	e	e	e	e	e	a	a	a	a	a	b	b	b	b	b
c	d	e	a	b	d	e	a	b	c	e	a	b	c	d	a	b	c	d	e	b	c	d	e	a
c	d	e	a	b	e	a	b	c	d	b	c	d	e	a	d	e	a	b	c	a	b	c	d	e
c	d	e	a	b	a	b	c	d	e	d	e	a	b	c	b	c	d	e	a	e	a	b	c	d
d	d	d	d	d	e	e	e	e	e	a	a	a	a	a	b	b	b	b	b	c	c	c	c	c
d	e	a	b	c	e	a	b	c	d	a	b	c	d	e	b	c	d	e	a	c	d	e	a	b
d	e	a	b	c	a	b	c	d	e	c	d	e	a	b	c	a	b	c	d	b	c	d	e	a
d	e	a	b	c	b	c	d	e	a	e	a	b	c	d	c	d	e	a	b	a	b	c	d	e
e	e	e	e	e	a	a	a	a	a	b	b	b	b	b	c	c	c	c	c	d	d	d	d	d
e	a	b	c	d	a	b	c	d	e	b	c	d	e	a	c	d	e	a	b	d	e	a	b	c
e	a	b	c	d	b	c	d	e	a	d	e	a	b	c	a	b	c	d	e	c	d	e	a	b
e	a	b	c	d	c	d	e	a	b	a	b	c	d	e	d	e	a	b	c	b	c	d	e	a

Design 21

a	a	a	a	a	b	b	b	b	b	c	c	c	c	c	d	d	d	d	d	e	e	e	e	e
a	b	c	d	e	b	c	d	e	a	c	d	e	a	b	d	e	a	b	c	e	a	b	c	d
a	b	c	d	e	c	d	e	a	b	e	a	b	c	d	b	c	d	e	a	d	e	a	b	c
a	b	c	d	e	d	e	a	b	c	b	c	d	e	a	e	a	b	c	d	c	d	e	a	b
a	b	c	d	e	e	a	b	c	d	d	e	a	b	c	c	d	e	a	b	b	c	d	e	a
b	b	b	b	b	c	c	c	c	c	d	d	d	d	d	e	e	e	e	e	a	a	a	a	a
b	c	d	e	a	c	d	e	a	b	d	e	a	b	c	e	a	b	c	d	a	b	c	d	e
b	c	d	e	a	d	e	a	b	c	a	b	c	d	e	c	d	e	a	b	e	a	b	c	d
b	c	d	e	a	e	a	b	c	d	c	d	e	a	b	a	b	c	d	e	d	e	a	b	c
b	c	d	e	a	a	b	c	d	e	e	a	b	c	d	d	e	a	b	c	c	d	e	a	b
c	c	c	c	c	d	d	d	d	d	e	e	e	e	e	a	a	a	a	a	b	b	b	b	b
c	d	e	a	b	d	e	a	b	c	e	a	b	c	d	a	b	c	d	e	b	c	d	e	a
c	d	e	a	b	e	a	b	c	d	b	c	d	e	a	d	e	a	b	c	a	b	c	d	e
c	d	e	a	b	a	b	c	d	e	d	e	a	b	c	b	c	d	e	a	e	a	b	c	d
c	d	e	a	b	b	c	d	e	a	a	b	c	d	e	e	a	b	c	d	d	e	a	b	c
d	d	d	d	d	e	e	e	e	e	a	a	a	a	a	b	b	b	b	b	c	c	c	c	c
d	e	a	b	c	e	a	b	c	d	a	b	c	d	e	b	c	d	e	a	c	d	e	a	b
d	e	a	b	c	a	b	c	d	e	c	d	e	a	b	e	a	b	c	d	b	c	d	e	a
d	e	a	b	c	b	c	d	e	a	e	a	b	c	d	c	d	e	a	b	a	b	c	d	e
d	e	a	b	c	c	d	e	a	b	b	c	d	e	a	a	b	c	d	e	e	a	b	c	d
e	e	e	e	e	a	a	a	a	a	b	b	b	b	b	c	c	c	c	c	d	d	d	d	d
e	a	b	c	d	a	b	c	d	e	b	c	d	e	a	c	d	e	a	b	d	e	a	b	c
e	a	b	c	d	b	c	d	e	a	d	e	a	b	c	a	b	c	d	e	c	d	e	a	b
e	a	b	c	d	c	d	e	a	b	a	b	c	d	e	d	e	a	b	c	b	c	d	e	a
e	a	b	c	d	d	e	a	b	c	c	d	e	a	b	b	c	d	e	a	a	b	c	d	e

6 Treatments

Design 22

a	b	c	d	e	f
c	d	e	f	a	b
b	c	d	e	f	a
e	f	a	b	c	d
f	a	b	c	d	e
d	e	f	a	b	c

7 Treatments

Design 23

a	b	c	d	e	f	g
b	c	d	e	f	g	a
e	f	g	a	b	c	d
c	d	e	f	g	a	b
g	a	b	c	d	e	f
f	g	a	b	c	d	e
d	e	f	g	a	b	c

a	b	c	d	e	f	g
g	a	b	c	d	e	f
d	e	f	g	a	b	c
f	g	a	b	c	d	e
b	c	d	e	f	g	a
c	d	e	f	g	a	b
e	f	g	a	b	c	d

Design 24

a	b	c	d	e	f	g
b	c	d	e	f	g	a
e	f	g	a	b	c	d
d	e	f	g	a	b	c
c	d	e	f	g	a	b
f	g	a	b	c	d	e
g	a	b	c	d	e	f

a	b	c	d	e	f	g
f	g	a	b	c	d	e
c	d	e	f	g	a	b
e	f	g	a	b	c	d
g	a	b	c	d	e	a
d	e	f	g	a	b	c
b	c	d	e	f	g	a

Design 25

a	b	c	d	e	f	g
b	c	d	e	f	g	a
g	a	b	c	d	e	f
c	d	e	f	g	a	b
f	g	a	b	c	d	e
d	e	f	g	a	b	c
e	f	g	a	b	c	d

a	b	c	d	e	f	g
g	a	b	c	d	e	f
b	c	d	e	f	g	a
f	g	a	b	c	d	e
c	d	e	f	g	a	b
e	f	g	a	b	c	a
d	e	f	g	a	b	c

Design 26

a	b	c	d	e	f	g
b	c	d	e	f	g	a
d	e	f	g	a	b	c
g	a	b	c	d	e	f
c	d	e	f	g	a	b
e	f	g	a	b	c	d
f	g	a	b	c	d	e

a	b	c	d	e	f	g
g	a	b	c	d	e	f
e	f	g	a	b	c	d
b	c	d	e	f	g	a
f	g	a	b	c	d	e
d	e	f	g	a	b	c
c	d	e	f	g	a	b

Design 27

a	b	c	d	e	f	g
b	c	d	e	f	g	a
d	e	f	g	a	b	c
g	a	b	c	d	e	f

a	b	c	d	e	f	g
g	a	b	c	d	e	f
e	f	g	a	b	c	d
b	c	d	e	f	g	a

Design 28

a	b	c	d	e	f	g
f	g	a	b	c	d	e
g	a	b	c	d	e	f

a	b	c	d	e	f	g
d	e	f	g	a	b	c
f	g	a	b	c	d	e

a	b	c	d	e	f	g
g	a	b	c	d	e	f
d	e	f	g	a	b	c

Design 29

a	b	c	d	e	f	g
b	c	d	e	f	g	a
c	d	e	f	g	a	b
g	a	b	c	d	e	f
f	g	a	b	c	d	e
d	e	f	g	a	b	c
e	f	g	a	b	c	d

a	b	c	d	e	f	g
c	d	e	f	g	a	b
e	f	g	a	b	c	d
f	g	a	b	c	d	e
d	e	f	g	a	b	c
g	a	b	c	d	e	f
b	c	d	e	f	g	a

a	b	c	d	e	f	g
d	e	f	g	a	b	c
g	a	b	c	d	e	f
e	f	g	a	b	c	d
b	c	d	e	f	g	a
c	d	e	f	g	a	b
f	g	a	b	c	d	e

a	b	c	d	e	f	g
e	f	g	a	b	c	d
b	c	d	e	f	g	a
d	e	f	g	a	b	c
g	a	b	c	d	e	f
f	g	a	b	c	d	e
c	d	e	f	g	a	b

Design 29 -
Continued

a	b	c	d	e	f	g
f	g	a	b	c	d	e
d	e	f	g	a	b	c
c	d	e	f	g	a	b
e	f	g	a	b	c	d
b	c	d	e	f	g	a
g	a	b	c	d	e	f

a	b	c	d	e	f	g
g	a	b	c	d	e	f
f	g	a	b	c	d	e
b	c	d	e	f	g	a
c	d	e	f	g	a	b
e	f	g	a	b	c	d
d	e	f	g	a	b	c

Design 30

a	b	c	d	e	f	g	h
b	c	d	e	f	g	h	a
h	a	b	c	d	e	f	g
c	d	e	f	g	h	a	b
g	h	a	b	c	d	e	f
d	e	f	g	h	a	b	c
f	g	h	a	b	c	d	e
e	f	g	h	a	b	c	d

Design 31

a	b	c	d	e	f	g	h
b	a	d	c	f	e	h	g
c	d	a	b	g	h	e	f
d	c	b	a	h	g	f	e
e	f	g	h	a	b	c	d
f	e	h	g	b	a	d	c
g	h	e	f	c	d	a	b
h	g	f	e	d	c	b	a

a	b	c	d	e	f	g	h
e	f	g	h	a	b	c	d
b	a	d	c	f	e	h	g
f	e	h	g	b	a	d	c
g	h	e	f	c	d	a	b
c	d	a	b	g	h	e	f
h	g	f	e	d	c	b	a
d	c	b	a	h	g	f	e

a	b	c	d	e	f	g	h
g	h	e	f	c	d	a	b
e	f	g	h	a	b	c	d
c	d	a	b	g	h	e	f
h	g	f	e	d	c	b	a
b	a	d	c	f	e	h	g
d	c	b	a	h	g	f	e
f	e	h	g	b	a	d	c

a	b	c	d	e	f	g	h
h	g	f	e	d	c	b	a
g	h	e	f	c	d	a	b
b	a	d	c	f	e	h	g
d	c	b	a	h	g	f	e
e	f	g	h	a	b	c	d
f	e	h	g	b	a	d	c
c	d	a	b	g	h	e	f

Design 31 - Continued

a	b	c	d	e	f	g	h
d	c	b	a	h	g	f	e
h	g	f	e	d	c	b	a
e	f	g	h	a	b	c	d
f	e	h	g	b	a	d	c
g	h	e	f	c	d	a	b
c	d	a	b	g	h	e	f
b	a	d	c	f	e	h	g

a	b	c	d	e	f	g	h
f	e	h	g	b	a	d	c
d	c	b	a	h	g	f	e
g	h	e	f	c	d	a	b
c	d	a	b	g	h	e	f
h	g	f	e	d	c	b	a
b	a	d	c	f	e	h	g
e	f	g	h	a	b	c	d

a	b	c	d	e	f	g	h
c	d	a	b	g	h	e	f
f	e	h	g	b	a	d	c
h	g	f	e	d	c	b	a
b	a	d	c	f	e	h	g
d	c	b	a	h	g	f	e
e	f	g	h	a	b	c	d
g	h	e	f	c	d	a	b

Design 32

a	b	c	d	e	f	g	h	i
b	c	a	e	f	d	h	i	g
c	a	b	f	d	e	i	g	h
d	e	f	g	h	i	a	b	c
e	f	d	h	i	g	b	c	a
f	d	e	i	g	h	c	a	b
g	h	i	a	b	c	d	e	f
h	i	g	b	c	a	e	f	d
i	g	h	c	a	b	f	d	e

a	b	c	d	e	f	g	h	i
g	h	i	a	b	c	d	e	f
d	e	f	g	h	i	a	b	c
b	c	a	e	f	d	h	i	g
h	i	g	b	c	a	e	f	d
e	f	d	h	i	g	b	c	a
c	a	b	f	d	e	i	g	h
i	g	h	c	a	b	f	d	e
f	d	e	i	g	h	c	a	b

a	b	c	d	e	f	g	h	i
i	g	h	c	a	b	f	d	e
e	f	d	h	i	g	b	c	a
f	d	e	i	g	h	c	a	b
b	c	a	e	f	d	h	i	g
g	h	i	a	b	c	d	e	f
h	i	g	b	c	a	e	f	d
d	e	f	g	h	i	a	b	c
c	a	b	f	d	e	i	g	h

a	b	c	d	e	f	g	h	i
h	i	g	b	c	a	e	f	d
f	d	e	i	g	h	c	a	b
i	g	h	c	a	b	f	d	e
d	e	f	g	h	i	a	b	c
b	c	a	e	f	d	h	i	g
e	f	d	h	i	g	b	c	a
c	a	b	f	d	e	i	g	h
g	h	i	a	b	c	d	e	f

Design 32 - Continued

a	b	c	d	e	f	g	h	i
c	a	b	f	d	e	i	g	h
b	c	a	e	f	d	h	i	g
g	h	i	a	b	c	d	e	f
i	g	h	c	a	b	f	d	e
h	i	g	b	c	a	e	f	d
d	e	f	g	h	i	a	b	c
f	d	e	i	g	h	c	a	b
e	f	d	h	i	g	b	c	a

a	b	c	d	e	f	g	h	i
d	e	f	g	h	i	a	b	c
g	h	i	a	b	c	d	e	f
c	a	b	f	d	e	i	g	h
f	d	e	i	g	h	c	a	b
i	g	h	c	a	b	f	d	e
b	c	a	e	f	d	h	i	g
e	f	d	h	i	g	b	c	a
h	i	g	b	c	a	e	f	d

a	b	c	d	e	f	g	h	i
e	f	d	h	i	g	b	c	a
i	g	h	c	a	b	f	d	e
h	i	g	b	c	a	e	f	d
c	a	b	f	d	e	i	g	h
d	e	f	g	h	i	a	b	c
f	d	e	i	g	h	c	a	b
g	h	i	a	b	c	d	e	f
b	c	a	e	f	d	h	i	g

a	b	c	d	e	f	g	h	i
f	d	e	i	g	h	c	a	b
h	i	g	b	c	a	e	f	d
e	f	d	h	i	g	b	c	a
g	h	i	a	b	c	d	e	f
c	a	b	f	d	e	i	g	h
i	g	h	c	a	b	f	d	e
b	c	a	e	f	d	h	i	g
d	e	f	g	h	i	a	b	c

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CHANGE-OVER DESIGNS

James Mark Mason

Abstract

When it is necessary to apply several different treatments in succession to a given subject, the residual effect of one treatment on another must be taken into consideration. A number of various designs have been developed for this purpose. A number of them are presented in this paper and can be summarized as follows:

Type I: Balanced for first-order residual effects. For n, the number of treatments, even, any number of Latin squares can be used; for n odd, an even number of squares is necessary.

Type II: Formed by repeating the final period of Type I designs. Direct and residual effects are orthogonal.

Type III: Formed from $p<n$ corresponding rows of $n-1$ orthogonal $n \chi n$ Latin squares.

Type IV: Complete orthogonality except for subjects and residuals. Very efficient but large numbers of observations are necessary.

Type V: Designs balanced for first and second order effects. Also formed from orthogonal Latin squares.

Type VI: Designs orthogonal for direct, first and second order residuals. Designs presented for $n=2$, 3 and 5.

Type VII: Orthogonal for linear, quadratic, ... components of direct and linear component of residual effects. Analysis includes linear direct χ linear residual interaction. Designs given for n=4, 5.

Type VIII: Type II designs analyzed under model for Type VII designs. Less efficiency, but designs available for all n.

Type IX: Designs useful for testing more than one treatment and direct χ residual interactions.

Analysis for most designs includes normal equations, analysis of variance, variances of estimates, expected mean squares, efficiencies and missing value formulas.

A list of designs is presented in an appendix.