Examining the Relationship Between Students' Measurement Schemes for Fractions and Their Quantifications of Angularity

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Dissertation submitted to the faculty of the Virginia Polytechnic Institute and State University in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
In
Curriculum and Instruction

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May 15, 2020
Blacksburg, VA

Keywords: Measurement Schemes, Fractions, Quantifications of Angularity

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ABSTRACT

In the basic understanding of measurement, students are expected to be able to subdivide a given whole into a unit and then change the position of that unit along the entire length of the whole. These basic operations of subdivision and change of position are related to the more formal operations of partitioning and iterating. In the context of fractions, partitioning and iterating play a fundamental role in understanding fractions as measures, where students are expected to partition a whole into an iterable unit. In the context of angle measurement, students are expected to measure angles as a fractional amount of a full rotation or a circle, by partitioning the circle into a unit angle and then iterating that unit angle to find the measure of the given angle. Despite this link between measurement, fractions, and angles, research suggests that there is a disconnect between students' concepts of measurement and geometry concepts, including angle and angle measurement. Therefore, one area of study that might help us understand this disconnection would be to investigate the relationship between students' concepts of measurement and their concepts of angle measurement.

This current study documents sixth, seventh, and eighth grade students' measurement schemes for fractions and their quantifications of angularity, and then investigates the relationship between them. This research is guided by the following question: What is the relationship between middle school students' measurement schemes for fractions and their quantifications of angularity? Results indicate that the majority of students involved in this study do not possess a measurement concept of fractions nor a measurement concept of angularity. However, these results demonstrate that there is a relationship between students' measurement

schemes for fractions and their quantifications of angularity. It is concluded that students who construct more sophisticated fraction schemes tend to construct more sophisticated quantifications of angularity.

Examining the Relationship Between Students' Measurement Schemes for Fractions and Their Quantifications of Angularity

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GENERAL AUDIENCE ABSTRACT

Although the concepts of measurement, fractions, and angle measurement are related, research suggests that there is a disconnect between students' concepts of measurement and geometry concepts, including angle and angle measurement. Therefore, one area of study that might help us understand this disconnection would be to investigate the relationship between students' concepts of measurement and their concepts of angle measurement. This current study documents sixth, seventh, and eighth grade students' understandings of measurement, as indicated by their fraction schemes, and angle measurement, as indicated by how they quantify angularity or the openness of an angle. This study then investigates the relationship between them. This research is guided by the following question: What is the relationship between middle school students' measurement schemes for fractions and their quantifications of angularity? Results indicate that the majority of students involved in this study do not possess a measurement concept of fractions nor a measurement concept of angularity. However, these results demonstrate that there is a relationship between students' measurement schemes for fractions and their quantifications of angularity. It is concluded that students who construct more sophisticated fraction schemes tend to construct more sophisticated quantifications of angularity.

ACKNOWLEDGEMENTS

This work would not have been possible without God, and the love, support, and encouragement from the many people who have impacted my life. I want to thank my committee for all their help and support. As a first-generation college student, this was not an easy task. But with their patience and help, I was able to succeed. Thanks to my advisor, Jay Wilkins, for the numerous days, hours, and minutes you spent mentoring me, reading papers, providing feedback, and meeting to discuss ideas. Thanks Andy Norton, Katy Ulrich, and Betti Kreye for all your feedback and support. Thanks David Hicks for being on my underground committee. Thanks Estrella Johnson for your support and mentorship. Thanks Jim Garrison for your philosophical insights, support, and mentorship. Thanks to all the other professors at Virginia Tech who further supported and encouraged me, whether it was during class, working on projects together, or simply chatting in the hallway. In your own unique way, each of you helped me to see how I could become successful; and hopefully one day I can impact and inspire students of my own the way you all have impacted and inspired me.

I would like to thank my husband, Ricky Mullins, for being my biggest supporter and encourager. Having watched you navigate this process as a first-generation college student provided more motivation than you will ever know. Thanks for being an editor, a reviewer, a critic, a “committee member”, an advisor, a colleague, and most of all a friend. I could not have made it to where I am today without your love and support. I know that you care about my work and want me to be successful, resulting in endless love, support, and encouragement. I cannot put into words how much I appreciate you and how thankful I am to have had you by my side through this process.

I would like to thank my parents, Jeff and Kim Stidham, for all the love, care, and support you have given me throughout the years, and for always supporting my educational adventures. I always said I wanted to be a lifelong student, and well, here I am! Thanks for pushing me to work hard in elementary, middle, and high school, to find my love for education. Also, thanks for encouraging me to go to college, even though you did not understand the process. You all always told J Scott and me that we had two choices once we graduated high school: (1) we could get a job and support ourselves, or (2) we could go to college and you all would support us. While J Scott chose to get a job, I wanted to go to college. This encouragement to attend undergraduate started my lifelong career in education. Thank you for trying your best to help me become a successful college student. Thanks for always asking about my homework, papers, and how work is going, and for listening to me talk about my research ideas even though they probably sounded like gibberish. You have helped shape me into the person I am today, and I could not ever thank you enough.

I would like to thank my grandparents, Mamaw Faye and Frank "Paps" Mullins and Mamaw Laura and the late Papaw Windy Stidham, who also provided endless love and support throughout my life. From birth, each one of you was always there for me, whether it was attending all my softball games, driving to Saba's to buy me some oatmeal cakes, or bringing ice cream over at 10 at night. You all also showed me the value of hard work, persistence, and patience that has helped me to become successful. Thanks for always telling me how proud you are of me, and for always showing your love.

I would like to thank my adopted grandparents, Vernon and Linda Boggs, for supporting me during my time in Blacksburg, three hours away from home. Thanks for helping me navigate

these waters so I could become successful. Thanks for the many dinners, chats, football games, and laughs you provided. Your love and support were instrumental in this process.

I would like to thank my brother and sister-in-law, Scott and Brittany Stidham, for their love and support. Thanks to my in-laws, Ricky and Terry Mullins, for their encouragement and support. Thanks to Christy and Allen Hamilton for your support and love as well. Even though you all have no idea what I am doing, or what Ricky is doing, I know you all are proud.

I would like to thank my many colleagues at Virginia Tech who helped me along my journey. Thanks Karen Zwanch for being an example and the predecessor among the mathematics education doctoral students. Also, thanks for all the chats, rants, mentoring sessions, and laughs. Thanks Suzanne Shelbourne for being my accountability partner, my writing buddy, and my friend. Thanks Tiffany LaCroix for being my partner for the past four years, not to mention a life saver after eye surgery. Thanks Jim Hill for being a critic, a reviewer, and a great friend and colleague. I appreciate the time you spent reading about math topics, which I am sure you did not find that interesting, and the numerous times you spent listening to me talk about those topics. Thanks Amanada Biviano, Erika Bass, Rachel Rupnow, and Pamela Lindstrom for offering your insights as more seasoned graduate students during my first few years. And thanks to Caryn Caruso, Jameson Jones, Lezly Jones, Kregg Quarles, Brigitte Sanchez, Kaitlyn Serbin, and Vlad Kokushkin, just to name a few, for providing me with an excellent network of, not only colleagues, but also friends.

I would like to thank all my friends at Open Door Baptist Church who supported me during my time in Blacksburg. Thanks Pastor Mike and Miss Crystal for always praying for me and for your love. You all will never know how much you mean to me. Special thanks to Becky

and Chip Shrader who became my adopted parents and provided endless support to help me make it through my final year.

Finally, I would like to thank all my teachers at J.W. Adams and Pound High School, and my professors at UVA-Wise, for helping me find an interest in math and a love for education.

TABLE OF CONTENTS

Chapter 1: Introduction 1

Measurement in Geometry 2

Measurement Schemes for Fractions 5

Quantifications of Angularity 7

Rationale for This Study 9

Purpose Statement and Research Question 12

Chapter 2: Literature Review and Theoretical Framework 13

Radical Constructivism 13

Piaget's Cognitive Development 14

Assimilation, Accommodation, and Equilibration 16

Piagetian Measurement 18

Development of Operational Measurement 19

Conservation of Length 20

Development of Units of Measurement for Length 23

Relating Measurement to Fractions 26

Fraction Operations and Schemes 30

Operations Involved with Fractions 31

Unitizing 31

Partitioning 32

Disembedding 33

Iterating 34

Splitting 35

Coordination of Units 36

Summary of Operations 38

Fraction Schemes 39

Part-Whole Scheme 40

Measurement Scheme for Unit Fractions 41

Measurement Scheme for Proper Fractions 42

Generalized Measurement Scheme for Fractions 44

Summary of Fraction Schemes 45

Relating Measurement to Angles 46

What is an Angle? 47

Different Conceptualizations of Angles 48

Angle Measurement 49

What Curriculum Standards Emphasizes 53

Quantifications of Angularity Framework 58

Gross Quantification 59

Intensive Quantification 61

Extensive Quantification 64

Ratio Quantification 68
Rate Quantification 70
Summary of Quantifications of Angularity 72
A New Reorganization Hypothesis Concerning Angles 73
Chapter 3: Methodology 85
Overview of the Study 86
Pilot Study 87
Development of Tasks 88
First Iteration of Tasks 88
First Round of Interviews 99
Second Iteration of Tasks 99
Second Round of Interviews 100
Final Iteration of Tasks 100
Mixed Methods Study 101
Phase One: Quantitative Survey Study 103
Participants 104
Measurement Schemes for Fractions Instrument (MSFI) Description 108
Fraction Scheme Coding 109
Quantifications of Angularity Instrument (QAI) Description 112
Quantifications of Angularity Coding 122

Interrater Reliability 125

Quantitative Data Analysis 125

Phase Two: Validation Study 127

Participants 128

Qualitative Data Collection 129

Interview Coding 131

Validation Analysis 133

Chapter 4: Results 134

Pilot Study Results 134

First Round of Interviews 134

First Round of Revisions 137

Second Round of Interviews 139

Final Iteration of Tasks 142

Phase One Results 145

Measurement Schemes for Fractions 147

Quantifications of Angularity Results 150

Relationship between Fraction Schemes and Quantifications of Angularity 154

Transition from Additive to Multiplicative Reasoning in the Context of Angles 162

Phase Two Results 166

Chapter 5: Conclusions 171

Students' Measurement Schemes for Fractions 171

Students' Quantifications of Angularity 175

Examining the Relationship between Fraction Schemes and Quantifications of Angularity 180

Transition from Additive to Multiplicative Reasoning in the Context of Angles 183

Validation of Students' Quantifications of Angularity 187

Test Content 187

Response Process 188

Internal Structure 189

Relations to Other Variables 190

Consequences of Testing 191

Insights, Contributions, and Implications for the Field 192

A New Conceptual Framework 192

A Revised Reorganization Hypothesis 193

A New Instrument 196

Curriculum and Teaching Implications 196

Future Work 199

Appendix A: Quantifications of Angularity Instrument- Original Set of Tasks 201

Appendix B: Quantifications of Angularity Instrument- Revised Set of Tasks 214

Appendix C: Quantifications of Angularity Instrument- Final Version 227

Appendix D: Table of Revisions between First Iterations and Final Iteration 240

Appendix E: Clinical Interview Questions 242

References.. 244

LIST OF FIGURES

Figure 1.1: Angle Measures in Concentric Circles 5

Figure 2.1: Scheme Theory Model (Modified from von Glasersfeld, 2000, 2002) 17

Figure 2.2: Curvy Length Compared to Straight Length Task 21

Figure 2.3: Example of Unitizing Unit Fractions 32

Figure 2.4: Example Task Designed to Elicit an Iterating Operation 34

Figure 2.5: Example Task Designed to Elicit a Splitting Operation 35

Figure 2.6: Example Task Designed to Elicit Three Levels of Units Coordination 37

Figure 2.7: Learning Progression for Measurement Schemes for Fractions (From Wilkins & Norton, 2018, p. 32) 40

Figure 2.8: Example Task Designed to Elicit a MSUF 42

Figure 2.9: Example Task Designed to Elicit a RPFS 43

Figure 2.10: Example Task Designed to Elicit a GMSF 45

Figure 2.11: Angles with the Same Measure but Different Size Circles (from Thompson et al., 2014, p. 2) 51

Figure 2.12: Comparing Angle Measurements Using Equivalent Arcs 57

Figure 2.13: Two 45º Angles with Different Ray/Side Lengths 61

Figure 2.14: Motions Used by Students as Described by Hardison (2018) 63

Figure 2.15: Using 90º Benchmark Angle for Comparisons 64

Figure 2.16: Using Angle Iteration and Partition for Comparing Angles 66

Figure 2.17: Using Angle Imposition for Comparing Angles 67

Figure 2.18: Example of Ratio Quantification 70

Figure 2.19: Example of Rate Quantification 71

Figure 2.20: Example of Arc Sweep 72

Figure 2.21: Example of Student who Focuses on Absolute Openness 75

Figure 2.22: Example of Student who Incorrectly Partitions Openness 76

Figure 2.23: Example of Student who Cannot Split in the Context of Angles 77

Figure 2.24: Example of Student who can Split in the Context of Angles 77

Figure 2.25: A New Reorganization Hypothesis 79

Figure 3.1: Task Designed to Assess Gross Quantification of Angularity 91
Figure 3.2: Task Designed to Assess Intensive Quantification of Angularity 92
Figure 3.3: Task Designed to Assess Extensive Quantification of Angularity 93
Figure 3.4: Task Designed to Assess Splitting Operation 94
Figure 3.5: Task Designed to Assess Ratio Quantification of Angularity 96
Figure 3.6: Task Designed to Assess Rate Quantification of Angularity 97
Figure 3.7: Task Designed to Assess Rate Quantification of Angularity 98
Figure 3.8: Task Designed to Assess an Intensive Quantification of Angularity 115
Figure 3.9: Task Designed to Assess Students' Comparisons 116
Figure 3.10: Task Designed to Assess an Extensive Quantification of Angularity 117
Figure 3.11: Task Designed to Elicit Splitting 118
Figure 3.12: Task Designed to Assess Multiplicative Reasoning 119
Figure 3.13: Task Designed to Assess a Ratio Quantification of Angularity 120
Figure 3.14: Task Designed to Assess a Rate Quantification of Angularity 121
Figure 5.1: A Revised Reorganization Hypothesis 195

LIST OF TABLES

Table 3.1: *Overview of Study Investigations* 87

Table 3.2: *QAI Item Blueprint with Example Solutions* 89

Table 3.3: *Total Numbers of Teachers and Classes Recruited for Phase One* 106

Table 3.4: *Total Numbers of Students who Participated in Phase One* 107

Table 3.5: *Student Demographics* 108

Table 3.6: *Fraction Operation/Scheme Item Breakdown for the MSFI* 109

Table 3.7: *Sample Coding Rubric for Overall Fraction Scheme Score* 111

Table 3.8: *Example Matrix Coding Scheme for Measurement Schemes for Fractions* 111

Table 3.9: *Ordinal Coding Scheme for Each Measurement Scheme for Fractions* 112

Table 3.10: *QAI Item Blueprint with Example Solutions* 113

Table 3.11: *Coding Guidelines for Quantifications of Angularity* 123

Table 3.12: *Example Matrix Coding Scheme for Quantifications of Angularity* 124

Table 3.13: *Ordinal Coding Scheme for Quantifications of Angularity* 125

Table 3.14: *Total Numbers of Students who Participated in Phase Two* 129

Table 4.1: *Description of Students who Participated in the Pilot Study* 135

Table 4.2: *Kappa Scores for Fraction and Angle Coding* 147

Table 4.3: *Students' Overall Fraction Schemes* 148

Table 4.4: *Overall Fraction Schemes by Grade* 149

Table 4.5: *Overall Fraction Schemes by Course* 150

Table 4.6: *Students' Overall Quantifications of Angularity* 151

Table 4.7: *Overall Quantifications of Angularity by Grade* 153

Table 4.8: *Overall Quantifications of Angularity by Course* 153

Table 4.9: *Correlation Between Each Fraction Scheme* 155

Table 4.10: *Correlation Between Each Quantification of Angularity* 155

Table 4.11: *Frequencies and Associated Percentages by Overall Fraction Scheme and Quantification* 157

Table 4.12: *Frequencies and Associated Percentages by Each Fraction Scheme and Overall Quantification* 160

Table 4.13: *Correlation Between Each Fraction Scheme and Each Quantification of Angularity* 161

Table 4.14: *Frequencies and Associated Percentages by Fractional Splitting and Angular Splitting* 163

Table 4.15: *Frequencies and Associated Percentages by Angular Splitting and Overall Quantification of Angularity* 164

Table 4.16: *Frequencies and Associated Percentages by Extensive and Angular Splitting* 165

Table 4.17: *Frequencies and Associated Percentages by Angular Splitting and Ratio* 166

Table 4.18: *Frequencies and Associated Percentages by Angular Splitting and Rate* 166

Table 4.19: *Weighted Kappa Scores for Interview Coding* 167

Table 4.20: *Interview Quantification of Angularity* 168

Table 4.21: *Difference Between Interview and QAI Quantification* 169

Table 4.22: *Frequencies Between Interview Quantification and QAI Quantification* 170

LIST OF ABBREVIATIONS

GMSF: Generalized Measurement Scheme for Fractions

IFS: Iterative Fraction Scheme

MSFI: Measurement Scheme for Fractions Instrument

MSPF: Measurement Scheme for Proper Fractions

MSUF: Measurement Scheme for Unit Fractions

PFS: Partitive Fraction Scheme

PrePWS: Pre Part-Whole Scheme

PUFS: Partitive Unit Fraction Scheme

PWS: Part-Whole Scheme

QAI: Quantifications of Angularity Instrument

RPFS: Reversible Partitive Fraction Scheme

Chapter 1: Introduction

It is through geometry that one purifies the eye of the soul.

–Plato

Geometry is a subject that is often misunderstood as being contained in its own mathematical silo, disconnected from other subjects and the world. However, Johannes Kepler once said, “Where there is matter, that is geometry” (Maclean, 2007, p. 188). This emphasizes the idea that geometry can be seen everywhere in the world. Furthermore, Pythagoras once said, “There is geometry in the humming of strings, there is music in the spacing of spheres” (Young, 1965, p. 113). From this, geometry is further described as being ever present and is directly related to many other things and has infinite applications. Despite this, geometry can be a subject that students either love or hate. Most often, sadly, students hate it (e.g., Melo & Martins, 2015; Phillips, 1953). In their investigation of this love-hate relationship between students and geometry, Melo and Martins (2015) found that students typically dislike geometry for the following reasons: students do not understand the content, because it is too difficult; students do not see the importance of it; students find it disconnected from other subjects; or students do not find it fun or interactive. In the 2015 Trends in International Mathematics and Science Study (TIMSS), it has also been documented that students in the United States have lower achievement scores in the geometry content domain than other mathematics content domains (Mullis et al., 2016). This demonstrates that geometry is a difficult subject for students to understand.

To highlight the difficult nature of geometry, consider the following example. In Virginia, as part of a high school Geometry course, students are expected to measure angles and solve problems involving angles (VDOE, 2016d). However, these curriculum standards focus on using degrees as a unit of measurement. Although curriculum standards in earlier grades and courses focus on measuring angles in the context of circles, recognizing that a one degree angle would be 1/360 of a circle (VDOE, 2016a), standard units of measurement (i.e., degrees) are emphasized. This lies in contrast to researchers' approach of encouraging students to use nonstandard units of measurement and then build towards standard units (Clements et al., 1996; Clements & Battista, 1986; Mitchelmore, 1997). Not allowing students to develop their own units of measurement could prevent students from fully understanding angle measurement. In fact, Moore (2013) states that this approach "might facilitate relating angle measures through calculations, but it fails to address the quantitative structure behind the *process* of determining an angle's measure" (p. 227, emphasis in original). In other words, students may be able to calculate angle measure by reading degrees from a protractor, but may not fully understand the process of obtaining an angle measurement. As a result, students may develop "shallow and fragmented angle measure understandings" (Moore, 2013, p. 226). This current study then seeks to investigates students' understandings of geometry concepts, and how they are and should be connected to other mathematical concepts.

Measurement in Geometry

Measurement forms the basis of students' understanding of counting, fractions, rational number reasoning, as well as proportional reasoning (Lamon, 2007; Steffe, 1992; Thompson et al., 2014). Measurement, in the conceptual sense, involves the coordination of partitioning a quantity into a unit and then iterating that unit to recreate the whole (Piaget et al., 1981/1960). Kieren (1980) summarizes it best:

> First, a unit must be partitioned off and then displaced without gaps or overlaps. This corresponds to a seriation. Second, the continuous units form inclusions—one piece

> included in two, and so on. Therefore, measurement is constructed from a synthesis of displacement and partitioning of an additive nature. (p. 101)

With this process, students are able to develop their own measurement unit, rather than simply reading standard units of inches from a ruler, thus allowing them to develop a more conceptual understanding of measurement.

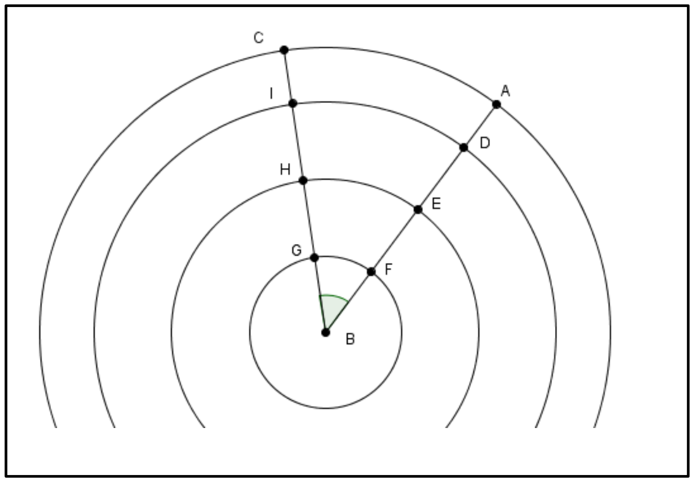
As noted in the Common Core State Standards Initiative (CCSSI, 2010), students begin learning measurement by indirectly comparing measurable attributes, and then formally by iterating a unit of length to obtain the number of times the unit spans the object's length. Other state standards support this notion of taking a unit of measure and laying it end-to-end, with no gaps or overlaps, to obtain the object's size or measurement (Indiana Department of Education [IDOE], 2014; Nebraska Department of Education [NDOE], 2015; Oklahoma State Department of Education [OSDOE], 2016; South Carolina Department of Education [SCDOE], 2018; Texas Education Agency [TEA], 2012). Thus, there is agreement that measurement should focus on iterating a unit to obtain an object's size, which supports the understanding of the aforementioned concept of measurement.

By extending this concept of measurement to angles, it should follow that angle measurement is obtained by taking some unit and iterating that unit a certain number of times. This emphasizes the notion that angle measurement can be obtained by "determining the fractional amount of the circle's circumference subtended by an angle, provided that the circle is centered at the vertex of the angle" (Moore, 2013, p. 227), and is supported in several curriculum standards (CCSSI, 2010; SCDOE, 2018; TEA, 2012; VDOE, 2016a, 2016b). Students who have a conceptual understanding of measurement should be able to determine an angle measurement by simply partitioning a circle into some unit angle, iterating that unit angle to obtain the circle, and identifying the number of iterations made. For example, given an arbitrary angle, with no measurement of degrees or radians, a student should be able to iterate that angle until they have created a circle, then count the number of iterations. If they could make 8 iterations, they would say that angle was 1/8 of the circle, or if they could make five iterations, they would say the angle was 1/5 of the circle. By focusing on angle measurement in relation to a fractional amount of circles (Moore, 2013), students may begin to understand angle measures, for example, as 1/4 of a circle or 1/7 of a circle, and then extend this concept of measurement to degrees and radians, thus providing a more meaningful understanding of the units of measurement (Clements & Battista, 1986).

However, students must also understand the multiplicative relationship between the length of the radius and the amount of rotation associated with the angle, denoted by the arc length, to help them understand angle measure in situations involving multiple circles, a single varying circle, or a group of concentric circles (Hardison, 2018; Moore, 2013; Thompson, 2008). For example, Moore (2013) explained that if the radius is extended, the arc length of the new angle will maintain the same proportional relationship to the circles' circumference. Consider the diagram in Figure 1.1. If a student were to create ∠GBF and then extend the length of $\overline{BG}$ to $\overline{BH}$ and length $\overline{BF}$ to $\overline{BE}$, the student would obtain the same angle, but the side lengths, the corresponding arc, and the circle's circumference would also increase in size. However, the relationship between the arc GF to the circumference of the corresponding circle and the relationship between arc HE to its corresponding circle would be the same proportion; that is, the arcs would be the same fractional part of the circumferences, and therefore the angles would have the same measurement. By having a conceptual understanding of measurement, students can better understand angle measurement in terms of this multiplicative relationship.

Figure 1.1

Angle Measures in Concentric Circles



Measurement Schemes for Fractions

One way to think about how students develop a concept of measurement is by examining their schemes. Schemes identify and describe the different actions and processes a student may use while solving tasks (Steffe, 2002, 2003; Steffe & Olive, 2010; von Glasersfeld, 1995b). Building upon Steffe and Olive's fraction schemes, Wilkins and Norton (2018) describe a progression of schemes for fractions to help explain the ways students think about fractions as measures. In their measurement schemes for fractions, Wilkins and Norton (2018) describe the different operations used, such as partitioning and iterating, as students think about fractions as measurements or sizes.

As a general description of these measurement schemes, students first begin with the operation of partitioning. Once they can partition a whole, they can begin to see fractions as parts of wholes, indicative of a *part-whole scheme* (PWS; Steffe, 2003; Steffe & Olive, 2010; Wilkins & Norton, 2018). In this case, they have no iterable unit, and simply rely on the number of parts taken out of the whole. Once students develop an iterating operation, they may then begin to recreate a whole by iterating a given part (Steffe, 2002, 2003; Steffe & Olive, 2010). This iterable unit of 1 is indicative of a *measurement scheme for unit fractions* (MSUF; Wilkins & Norton, 2018). Here, students recognize the relationship between the one unit and the number of iterations it takes to recreate the whole. Once students are able to simultaneously partition and iterate, known as splitting, they construct the *measurement scheme for proper fractions* (MSPF; Wilkins & Norton, 2018). Students understand the proper fraction *m/n* as simply *m* units of 1 from the *n* iterations needed to create the whole. Because students may not understand the partitions or units as 1/*n* but rather 1's, they are limited in their understanding of improper fractions. For example, they view 4/5 as four 1's used to make 5. They do not see 4/5 as four 1/5's. As a result, when working with improper fractions, they view 7/5 as seven 1's used to make 5, which does not make sense to them because seven 1's is bigger than five 1's. When students are able to establish an iterable unit of 1/*n*, they have constructed the *generalized measurement scheme for fractions* (GMSF; Wilkins & Norton, 2018). With this iterable unit, students are able to work with all fractions, proper and improper. They view 4/5 as four 1/5's and 7/5 as seven 1/5's.

These measurement schemes for fractions are directly related to the concept of measurement described above. They incorporate the operations of partitioning and iterating to help describe how students think about fractions. Furthermore, these descriptions of students' ways of thinking present fractions as a size or measure (Wilkins & Norton, 2018). Due to this direct connection to the concept of measurement, these measurement schemes for fractions can be used to categorize students' concepts of measurement in a developmental way.

Quantifications of Angularity

One way to examine students' concepts of angle measurement is through their quantifications of angularity. Quantification is the process of identifying an attribute of an object, assigning a value to that attribute, and defining a measurement where the unit of measure holds a proportional relationship to the attribute (Moore, 2013; Thompson, 2011). When thinking about angles, the attribute that is quantified is the openness of the angle (Hardison, 2018; Moore, 2013). Piaget (1965/1952) and Steffe (1991) explained several different types of quantifications, and Hardison (2018) examined students' different quantifications of angularity and the mental actions that were included in those quantifications. From these bodies of work, there are five different quantifications of angularity: gross, intensive, extensive, ratio, and rate. These quantifications form a developmental progression for categorizing students' concepts of angle and angle measurement.

With a *gross quantification*, students rely on perceptual material and make visual judgments (Piaget, 1965/1952; Steffe, 1991). Applying this to angularity, students visually compare the openness of the angle, or incorrectly compare the length of the rays, arcs, or size of the wedge created by the angle (Bütüner & Filiz, 2016). Moving away from their reliance on perceptual material, students with an *intensive quantification* compare angles by some established "measure." This measure is nonadditive and is not a formal measurement; it is some unit of comparison the student has constructed. For example, a student may compare two angles by rotating a ray through the interior of each angle and "timing" those durations. Whichever angle took longer to "sweep," the student may say that angle is larger (Hardison, 2018). In this case, the student is not actually using a stopwatch to time the durations, but is using some constructed timing system—not a formal timing—to "measure" the durations.

When students are able to establish an additive relationship between the attributes of each object being compared, in this case the openness of each angle, students are said to have an *extensive quantification* (Hardison, 2018; Kieren, 1980; Piaget, 1965/1952). For example, a student with an extensive quantification may partition a larger angle into parts the size of the smaller angle, or iterate the smaller angle to recreate the larger angle. From this, they are able to see how many times the smaller angle can be added to recreate the larger angle. In both of these cases, the relationship between the two angles' attributes are additive (e.g., if you put together four copies of the smaller angle, you get the larger angle).

Once students are able to establish a multiplicative relationship between the attributes, students are said to have a *ratio quantification* (Hardison, 2018; Thompson, 1994). Given the previous example, students can partition and iterate angles to recognize that the smaller angle is 1/4 the size of the larger angle and the larger angle is 4 times the size of the smaller angle. There is a clear multiplicative relationship between the openness of each angle. Finally, when students are able to maintain this multiplicative relationship within the context of different circles, especially concentric circles, they are said to have *a rate quantification* (Hardison, 2018). With a rate quantification, students conceive angularity as the fractional amount of the circle's circumference established by the minor arc of the angle (Moore, 2013). This enables students to maintain the multiplicative relationship between the length of the ray of the angle (radius), the minor arc, and the corresponding circle's circumference across all circles centered at the vertex of the angle (Hardison, 2018; Moore, 2013). These quantifications of angularity specifically describe the mental actions and operations students use in the context of angles, essentially providing a progression of schemes that can be used to describe student's concepts of angle measurement.

Rationale for This Study

The work of Piaget et al. (1981/1960) suggests that young children should be able to develop a concept of measurement, involving the processes of subdivision and change of position. This concept of measurement is emphasized in mathematics curriculum standards (CCSSI, 2010; SCDOE, 2018; TEA, 2012; VDOE, 2016a, 2016b). Because the concept of measurement focuses on subdivision and change of position, this has a direct link to students' understanding of fractions through the processes of partitioning and iterating (Lamon, 2007; Steffe, 2002; Steffe & Olive, 2010). Furthermore, research suggests that when fractions are taught from a measurement perspective, students are better equipped with the skills necessary to construct more sophisticated rational number constructs and fraction schemes (Lamon, 2007; Wilkins & Norton, 2018). As such, focusing on this concept of measurement can have an impact on students' understanding of fractions, which can also then impact their concepts of angle measurement.

After students have this foundational understanding of operational measurement, they can begin to understand fractions as measures, and then further extend this concept to angle measurement. Students typically learn about angles as formed by the intersection of two rays or lines (CCSSI, 2010; VDOE, 2016b, 2016c). This concept is then expanded in fourth and fifth grade, where students are introduced to identifying angle measurement as a fractional amount of a full rotation or a circle (CCSSI, 2010; VDOE, 2016a). Angle measurement is said to be measured in degrees. A degree is then defined as 1/360 of a circle, where an *n* degree angle represents *n* one-degree angles (CCSSI, 2010). This concept of angle measurement also involves the operations of partitioning and iterating, again highlighting the connection between measurement, fractions, and angle measurement. Research shows that when students understand angles as dynamic turns, they are able to gain a deeper understanding of angle measurement, which allows better connections between concepts (Clements & Burns, 2000). This also enables students to develop a more abstract conceptualization of angles (Mitchelmore & White, 2000), leading to higher performance and greater learning gains (Smith et al., 2014). However, research also shows that when students are limited to conceptualizing angles as static figures, they typically understand angle measurement by focusing on the length of rays, size of the arc drawn to represent the angle (not the subtended arc), or a linear distance between the sides of the angle (Barabash, 2017; Bütüner & Filiz, 2016; Clements & Battista, 1989, 1990; Clements, 2003; Clements et al., 1996; Piaget et al., 1981/1960). These limited understandings may cause students to continue to lag behind as they progress into Geometry and Trigonometry (Moore, 2013; Yigit, 2014). As Moore (2013) noted, when students have limited understanding of angle measure, "their ability to construct flexible trigonometric function understandings" (p. 226) is inhibited.

Furthermore, although this described concept of measurement and angle measurement is emphasized in the mathematics curriculum standards (CCSSI, 2010; IDOE, 2014; NDOE, 2015; OSDOE, 2016; SCDOE, 2018; TEA, 2012; VDOE, 2016a, 2016b), standard units of measurement are often emphasized (e.g., inches, feet, and degrees). Students typically are not provided opportunities to explore with non-standard units of measurement early on, and therefore may not fully understand measurement and angle measurement. It has also been documented that students often perform poorly on geometry and measurement tasks and assessments. Results from the 2015 TIMSS report demonstrated that the content domain of geometry was a problematic area for students, as indicated by their low scores (Mullis et al., 2016). For example, in fourth grade the average score for the Geometric Shapes and Measures domain was 525, compared to 546 for the Number domain and 540 for the Data Display domain[1]. In eighth grade, the average score for the Geometry domain was 500, compared to 520 for Number, 525 for Algebra, and 522 for Data and Chance. In addition, reports from the 2017 National Assessment of Educational Progress for fourth and eighth grade (National Center for Education Statistics [NCES], 2011) demonstrate that students' average scores for the categories of Measurement and Geometry were often lower than the other categories.

From these results, it is evident that students struggle with the domain of geometry and measurement. As discussed earlier, understanding angles and angle measurement is a major component of Geometry; students are expected to measure angles and solve problems involving angles (VDOE, 2016d). Due to these low scores for the geometry domain, it appears that students have limited understanding of concepts of measurement and geometry concepts, including angle and angle measurement. It is unclear whether students do not have the aforementioned concept of measurement involving partitioning and iterating, or if they simply cannot relate it to the context of angle measurement. Therefore, one area of study that might help us understand this would be an investigation of the relationship between students' concepts of measurement and their concepts of angle measurement. This will help gain insight into whether students (a) do not have a concept of measurement which then limits their concept of angle measurement, (b) do indeed have a concept of measurement, but are unable to apply it to the context of angles, or (c) do have a concept of measurement they can apply to the context of angles.

[1] Scores range from 0 to 1,000. However, student scores usually range from 300 to 700 (Mullins et al., 2016).

Purpose Statement and Research Question

The purpose of this study is to examine the relationship between students' concepts of measurement and their concepts of angle measurement. Specifically, this current study will document sixth, seventh, and eighth grade students' measurement schemes for fractions and their quantifications of angularity, and then investigate the relationship between them. As such, this research is guided by the following question: What is the relationship between middle school students' measurement schemes for fractions and their quantifications of angularity?

Chapter 2: Literature Review and Theoretical Framework

This chapter is organized into five main parts, each containing several sections. The first part describes the theoretical framework that guides this study and its methods. Because this study investigates students' understanding of fractions and angle measurement, it is necessary to first describe how understanding and knowledge are constructed through the lens of radical constructivism, then examine the possible ways students construct concepts of measurement, fractions, and angles. The second part of this chapter focuses on the concept of operational measurement and how students construct this concept. The third part of this chapter focuses on fractions. This part examines the relationship between the concept of measurement and fractions. Fraction operations and schemes are also discussed to better understand how students conceptualize fractions as measures. In addition, the measurement schemes for fractions framework, concerning students' measurement concepts of fractions, is described. The fourth part of this chapter focuses on angles and angle measurement. This part offers a literature review of students' conceptualizations of angles and the process for quantifying angularity. The framework used to explain students' quantifications of angularity is also described. The fifth and final part of this chapter ties together the two frameworks into one coherent framework that can be used to examine the relationship between students' measurement schemes for fractions and their quantifications of angularity.

Radical Constructivism

Radical constructivism is one branch of constructivism that places the individual at the center of the construction of knowledge. Von Glasersfeld (1995b) defined radical constructivism as an approach for understanding how knowledge is created, and how we come to know what we know. Within radical constructivism, knowledge is thought to reside in the "heads of persons," which then leads the person to “construct what he or she knows on the basis of his or her own experience” (von Glasersfeld, 1995b, p. 1). This comes out of Piaget’s (Garvin, 1977) argument that knowledge is not a copy of the objects we interact with, but rather is our interpretation of those objects and our actions performed on those objects.

There are two underlying principles of radical constructivism that comes from Piaget’s work: “The first is that knowledge is actively built by a cognizing subject” and the second is “that the function of cognition is to organize one's experiential world, not to discover an ontological reality” (Fleury, 1998, p. 158). Von Glasersfeld (1995b) noted that knowledge does not represent the world or even a picture of it, but rather knowledge contains actions, schemes, concepts, and thoughts. He went on to say that knowledge “pertains to the ways and means the cognizing subject has conceptually evolved in order to fit into the world as he or she experiences it” (von Glasersfeld, 1995b, p. 114). This means that as people experience the world, they begin to develop viable models and adapt their operations to fit within that model (Pepin, 1998; Piaget, 1980; Ulrich, et al., 2014), indicating that knowledge is viable, purposeful, and experiential (Larochelle & Bednarz, 1998). Piaget (1970, 1971, 1977) further clarified this process of knowledge development in terms of cognitive structures.

Piaget’s Cognitive Development

Piaget (1970, 1971, 1977) argued that knowledge can be explained through notions, operations, and structures. His explanation of knowledge, called genetic epistemology, focuses on “formalization– in particular, logical formalizations applied to equilibrated thought structures and in certain cases to transformations from one level to another in the development of thought” (Piaget, 1970, p. 1). Specifically, he states: “The fundamental hypothesis of genetic epistemology is that there is a parallelism between the progress made in the logical and rational organization of knowledge and the corresponding formative psychological process" (Piaget, 1970, p. 13). With this is mind, his explanation comprises both psychological and historical components, and focuses on knowledge as a continual process of construction and reorganization (Piaget, 1970). Therefore, knowledge is not static and is made up of actions and coordinations of actions.

For Piaget, we as humans are biological and psychological creatures who develop our knowledge through adaptation (Piaget, 1980; von Glasersfeld, 1995b, 2001). As Piaget saw it, cognition was "an instrument of adaptation, as a tool for fitting ourselves into the world of our experience" (von Glasersfeld, 1995b, p. 14). However, this type of adaptation for learning is much different from the biological adaptation seen in animals; adaptation in the biological sense "refers to the biological make-up, the genetically determined potential with which we are born; and learning is the process that allows us to build up skills in acting and thinking as a result of our own experience" (von Glasersfeld, 2002, p. 20). Although we are born with some innate actions or intuitions, we build off of these to develop more sophisticated knowledge structures (Piaget, 1970, 1971, 1977). Adaptation essentially is how the organism learns to survive the tests of the environment.

Instead of developing physical characteristics, like animals do when they evolve, children develop conceptual functions; and whichever ones are feasible or can be adapted for survival or achieving a goal are retained. This means that functions that were not adapted, did not fit with our experiential world, or did not work, were eliminated (von Glasersfeld, 2001). Therefore, knowledge becomes a viable model for our reality (von Glasersfeld, 1995a, 1982, 2002) and the construction of this knowledge can be explained through assimilation, accommodation, and equilibration.

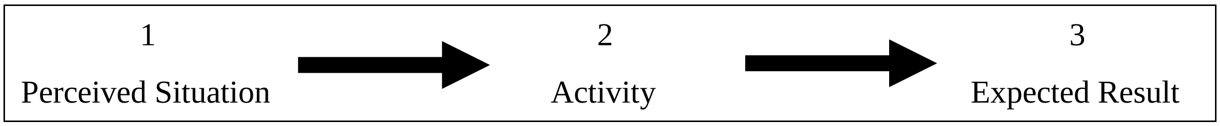
Assimilation, Accommodation, and Equilibration

Within radical constructivism, the idea is that the individual learner creates viable models of reality based on their own experiences (von Glasersfeld, 1995a, 1995b, 2000; Ulrich et al., 2014). In general, we construct our knowledge through assimilating new situations into prior knowledge and making accommodations in order to maintain equilibrium. Therefore, knowledge essentially is a continual process of assimilation, perturbation, accommodation, and equilibration (Piaget 1970, 1971, 1977). As we experience new situations every day, we continually construct new knowledge.

To more specifically describe this process, from Piaget's concepts of schemes, von Glasersfeld (1995b) developed a three-part model for scheme theory, consisting of three main components (Figure 2.1). First, the individual is presented with a situation, known as the "experiential situation" (von Glasersfeld, 1982, p. 625) or "perceptual situation" (von Glasersfeld, 2000, p. 8). When placed in this situation, a person will activate some mental action(s) or cognitive structure(s) related to some aspect of that situation based on their prior knowledge, through the process of assimilation. At this moment, the actor must recognize the situation, know which actions are related to it, and know what structure to apply (von Glasersfeld, 1990, 2000). Although not every situation or experience will be the same, there are similarities between situations which should trigger certain actions (von Glasersfeld, 1990, 1995b). When this new situation does not fit their current knowledge model, or the expected result is not produced, they must reorganize their knowledge in a different manner to accommodate this new experience. Finally, this will hopefully produce an expected result (von Glasersfeld, 1982, 1995b, 2000, 2002). However, if this process does not produce an acceptable outcome, or the person is presented with an unrecognizable situation, there is a perturbation which results in disequilibrium (Piaget 1980; von Glasersfeld, 1990, 1995b). At this moment, accommodations must be made in order to achieve the goal of equilibrium.

Figure 2.1

Scheme Theory Model (Modified from von Glasersfeld, 2000, 2002)



Equilibrium is the state at which there is a cohesive connection between schemes, structures, and systems. Thus, disequilibrium is the state at which some of these links and connections have become broken. When this occurs, the person must self-regulate and make modifications, known as accommodation, to their current cognitive structures to help resolve this imbalance caused by the perturbation (Piaget, 1970, 1971, 1977; von Glasersfeld, 1990, 2002). This new information is then fed back into the loop, where new information is assimilated, new actions are accommodated, and the result is examined. From this, we can see the continual construction Piaget was describing.

Ultimately, our main goal in life is to reach a state of equilibrium, to establish a fully functioning, connected structure. However, Piaget (1980) argues that there is no perfect state of equilibrium, but that we are always working towards a “better” equilibrium. As we develop and modify our existing structures, our means and goals change. Again, when there is a disconnect between the means and goal, we must self-regulate and modify our schemes. This means that our current schemes are always a result “from a modification or reorganization of one’s prior system of schemes” (Ulrich et al., 2014, p. 329). Therefore, schemes build off one another, and develop into a system of schemes. Von Glasersfeld (1995b) summarizes Piaget’s theory in the following manner: “cognitive change and learning in a specific direction take place when a scheme, instead of producing the expected result, leads to perturbation, and perturbation, in turn, to an accommodation that maintains or re-establishes equilibrium" (p. 68). Therefore, our knowledge is a viable model based on our experiences, meaning that whatever works for us in that situation becomes our knowledge. Due to this reliance on knowledge being individually constructed, it is then necessary to investigate individual students to examine their schemes and cognitive structures.

Piagetian Measurement

Before students can develop an understanding of fractions and angle measurement, they must first construct an initial understanding of measurement. To help elucidate the basic operations of measurement, consider the example of constructing length measurement. Piaget and colleagues (1981/1960) generally describe measurement as the ability to "take out of a whole one element, taken as a unit, and to transpose this unit on the remainder of a whole" (p. 3). They specifically define this as operational measurement; they further explain operational measurement as a coordination of subdivision and change of position[2]. Subdivision refers to the action of taking one element as a unit out of a whole. In the early stages of developing an understanding of measurement, students' subdivisions may not result in equal-sized divisions. However, in the more advanced stages, students' subdivisions result in equal-sized partitions. Change of position describes the action of taking that unit and transposing it the entire length of the whole. Piaget et al. note that even though it may not be possible to transpose the unit an integer number of times in the whole, students can still use that unit to find the length. For example, Piaget et al. discuss that in these instances, students may transpose the unit, find that it

[2] This coordination is not clearly defined by Piaget et al. (1981/1960). However, based on their descriptions, it appears that in earlier stages, this could be a sequential coordination. Then, in the later stages representing a more sophisticated understanding of measurement, this is a simultaneous coordination.

goes five and a half times, but then they will round their answer and claim that the unit goes five or six times. Therefore, the unit does not have to go into the whole an integer number of times, but it can still be used to exhaust or reestablish the whole. Once children can coordinate their actions of taking out a unit of the whole (subdividing) and then moving that unit the entire length of the whole (change of position), recognizing that the length of the objects remains the same, these children have developed operational measurement.

Development of Operational Measurement

Operational measurement involves the coordination of subdividing a whole into a unit, establishing that unit as a unit of measurement, and then transposing, or changing the position of that unit the entire length of the whole (Piaget et al., 1981/1960). Students then transfer the units of measurement between objects, to compare their measurements. Piaget et al. (1981/1960) defined three stages children, ages four to beyond seven, often progress through as they begin to develop operational measurement. In Stage I, children focus on visual and perceptual estimates for comparing objects. They often focus on one endpoint to determine which object is longer or taller, without regard to the other endpoint or the space between. This is what Piaget et al. refer to as visual transfer. Children in Stage II then begin using manual transfer to move objects side by side for comparisons. Instead of focusing on one endpoint, these children align one set of endpoints together and then evaluate the location of the other set of endpoints to determine which object is longer or taller. More sophisticated children in this stage may also use body transfer, that is, using their body to measure, such as comparing an arm's length to an object's length or height.

Between Stage II and III children measure more intuitively by using tools to compare lengths. However, these children can only compare the tool to each object, and not between both objects. For example, they can determine that the tool is longer than Object A and the tool is shorter than Object B, but cannot compare Object A to Object B. This is what Piaget et al. call intuitive transfer. Finally, in Stage III children no longer rely on visual judgements, and can now use a tool to compare Object A to Object B . They establish a unit of measurement, using a tool, and recreate the length of one object, understanding conservation of length. Once they find the measurement of Object A, they remember that measurement as they transfer the same unit of measurement or tool to Object B. They then recreate the length of Object B and compare the two measurements. At this point of coordinating the subdivisions of changes of positions, and then transferring the unit of measurement between objects, these children are said to have developed operational measurement.

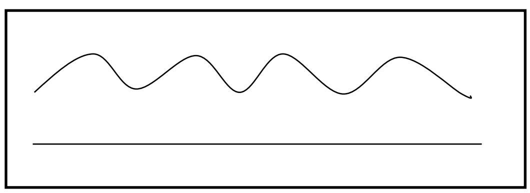
Conservation of Length

The key concept that impacts the development of operational measurement is conservation of length. Piaget et al. (1981/1960) claim that “underlying all measurement is the notion that an object remains constant in size throughout any change in position” (p. 90), which they define as conservation of length. However, this is a difficult concept for children to understand. As Piaget et al. argued, children must have a system of reference, referring to a coordinate system of space, in order to understand change of position and conservation of length. They simply cannot transpose or iterate the unit haphazardly; they must be able to keep track of the location of the changes of position, and know when the whole is exhausted, thus indicating when to stop iterating. Piaget et al. (1981/1960) note that “an understanding of measurement demands that the several reference points are linked in a systematic whole, which implies ‘coordinate axes’” (pp. 3–4). This spatial ability enables children to see the relationship between the subdivision and change of position, as well as conservation of length. Without it, children's understanding of conservation of length will be hindered.

Piaget et al. (1981/1960) offered several examples of tasks and children's responses for each stage to help describe how children develop conservation of length, leading to operational measurement. For example, children were given the task in Figure 2.2 and were asked to determine if the two strings have the same length. As part of the interview, after the children respond, the researcher straightened out the curvy string and repeated the question. The idea was to determine if the children recognize that no matter the shape of the curvy string, it will always be longer.

Figure 2.2

Curvy Length Compared to Straight Length Task



Children in Stage I often state that the two strings are the same length, even after tracing or stretching the curvy one out. In Stage IIA, children state the two strings are the same length at first. However, after the curved one is stretched out, they say it is longer. Once that string is curved again, they state that the two strings are the same. These children are focused on the furthest endpoint of each string in relation to one another. In Stage IIB, children focus on both sets of endpoints of the strings, but realize the curved string will be longer due to the curves. Finally, in Stage III, children logically know that the curved string is longer, and do not have to rely on endpoints.

In another task, children were presented with two identical sticks "with their extremities facing one another" (Piaget et al., 1981/1960, p. 95). Then, one stick is moved forward, and children were asked to compare lengths. Children in Stage I first state the sticks are the same. However, after moving one stick, the children say that the moved stick is now longer. Piaget et al. (1981/1960) argue that these children are "thinking only in terms of the further extremities and ignoring the nearer extremities" (p. 95). These children do not account for both ends simultaneously and do not think about the intervals between the endpoints remaining the same. Children in Stage IIA similarly state the sticks are equal before movement and that the moved one is longer after movement. These children also focus on the farthest endpoint to judge lengths and do not account for the intervals between endpoints. Although these children recognize that the change in length was due to the movement, Piaget et al. (1981/1960) note "the movements which they invoke are not Euclidean changes of position" (p. 97) because they are not accounting for the intervals between endpoints. Thus, there is no conservation of length. In the next stage, Stage IIB, children determine the sticks are equal before movement. After movement, the children realize that although one stick looks longer, it is not; it is simply due to the movement. They also recognize that one endpoint is longer on one stick, but the other endpoint is longer on the other stick; each stick is longer on one opposite end. They note:

> In the end they guess at conservation, without basing this notion on an exact composition of the spaces left empty by the change in position of the test objects and the corresponding spaces which are occupied: they do not realize that in every change of position these two factors are mutually compensating. Their thought does not yet embrace a system of fixed sites and deals only with the transformation of objects. That limitation precludes the operational conservation of length. It does however, admit of an intuitive conservation of relations of equality, which anticipates operation and may even come near to it. (Piaget et al., 1981/1960, p. 101)

Therefore, these children have an intuitive conservation of length, which is close to becoming fully operational, but is not there yet.

Finally, children in Stage III have a logical understanding of conservation of length, and it is no longer a guess or hypothesis. They recognize changes in position and realize that the length of the stick does not change when it is moved. Piaget et al. (1981/1960) argue that these more sophisticated children also recognize that "when an object undergoes a change of position the empty sites which have previously been occupied are equivalent to sites which were previously empty and are now filled, so that the overall length of the object remains constant" (p. 103). Thus, these children recognize length as an interval between two endpoints. From this collection of evidence, Piaget et al. (1981/1960) argue that "there can be no conservation of length, any more than of distance, unless there is a reference system which provides a common medium for all objects, whether moving or stationary" (p. 103). They go on to note that children must be able to coordinate the objects, their parts such as endpoints, and the intervals or empty space between them. Lastly, they note that it is necessary to consider both the filled and empty space because "such sites, whether empty of filled, form the essential framework of all metrics" (p. 103). Therefore, in order for children to understand conservation of length, they must understand changes of position and recognize length as the interval of filled space between endpoints.

Development of Units of Measurement for Length

In their analysis of how young children develop operational measurement, Piaget et al. (1981/1960) define the terms "length" and "distance" to make a clear distinction between them. They define *length* as the occupied or filled space between two points or objects, and *distance* as the empty space or linear separation between two points or objects. They go on to note that:

“Psychologically, the problems of distance are different from those of length, though logically, they are interdependent, since distance is the length of an interval and length is the distance taken up by an object” (p. 69). Concerning length, it is described as a filled space whereas distance is an empty space. Nonetheless, they both have a measurement. Coxford (1963) states that “*measurement* is the process whereby a number is given to a segment” (p. 424, emphasis in original). This number, or quantity, is created by laying a unit of measurement end to end to create a segment the same length as the object (Coxford, 1963), whether it is a filled or empty space.

From this, it is clear that the measurement of an object depends upon the tool the child uses to measure, or the unit the child subdivides the whole into. For example, if given a piece of string, a child may use a paperclip as their unit of measurement, another child may use an eraser, and another may use a pencil. Depending upon each child’s unit of measurement, they may obtain different numbers or quantities for the string’s length (i.e., 6 paperclips, 3 erasers, 1 pencil). However, if each child lays their number of units (i.e., 6 paperclips, 3 erasers, 1 pencil) end to end to recreate the piece of string, they should find that all their lengths are the same. The important take away is that measurement of length relates to number or quantity; whether it is a filled or empty space, there is a number associated with it.

However, for young children, this process of creating a unit of measurement is not an easy task. For example, Piaget et al. (1981/1960) argue that in order to understand a standard one foot ruler, an individual must actively construct the multiple copies of inches in order to make the ruler; the inches simply did not appear. Piaget et al. (1981/1960) further describe that it is necessary for students to understand that the inches and feet on a ruler are the “end-product of operations carried out in the past” (p. 27). Therefore, they go on to argue that there is a need to investigate how children construct initial units of measurement, not standard ones, to see how they are then able to transfer those initial units to other objects, ultimately leading to the development of standard measuring tools. From this argument, there is a distinction between the concept of measurement and the process of measuring. Concerning the former, when a child can coordinate subdivision and change of position, Piaget et al. believe that the child has developed a concept of operational measurement. However, concerning the latter, children simply read a measuring tool, made up of ready-made units, without any concern for subdivision and change of position.

Overall, there are many components of the development of operational measurement. In general, measurement is dependent upon awareness of and conservation of units (Coxford, 1963). Piaget et al. (1981/1960) determined that a child's understanding of length is "an outcome, not of a complementary relation between subdivision and change of position, but of their operational fusion" (128). This operational fusion emphasizes the coordination of subdivision and change of position. Children must first be able to realize that length is the amount of filled space between two points. Then, they must be able to subdivide that whole into a unit and iterate that unit the entire length of the whole (change of position). Next, children must be able to coordinate those subdivisions and changes of position to understand conservation of length. Once this coordination happens, these children have developed operational measurement.

Relating Measurement to Fractions

This part of the chapter focuses on the concept of fractions. These sections examine the relationship between the concept of measurement and fractions. Fraction operations and schemes are also discussed to better understand how students conceptualize fractions as measures. Finally, the measurement schemes for fractions framework concerning students' measurement concepts of fractions is described.

From the conceptual understanding of measurement as subdivision and change of position, there is an explicit connection to fractions. As discussed above, subdividing is related to partitioning a whole into a unit of measurement and change of position is related to iterating that unit the length of the object. These operations of partitioning and iterating are foundational mental operations required for understanding fractions (Kieren, 1980; Lamon, 2007, 2012; Steffe, 1992, 2002; Steffe & Olive, 2010). Other mental operations required for fractions include: (a) unitizing—treating an object, entity, or a collection of items as a unit or a whole (McCloskey & Norton, 2009; Norton & McCloskey, 2008; Steffe, 1992); (b) disembedding—taking one of those partitions or units out of the whole while maintaining the structure of the whole (Norton, 2008; Norton & McCloskey, 2008; Steffe, 1992); and (c) the ability to coordinate units—building and working with units at different levels (Hackenberg, 2007; Hackenberg et al., 2016; Steffe, 1992, 2002; Steffe & Olive, 2010). In the initial understanding of fractions, students need to be able to construct (unitize) a whole, separate (partition) that whole into a unit, pull out (disembed) one of those units from the whole, and repeat (iterate) that unit the length of the whole, while maintaining the relationship between the whole and the iterable unit (coordination of units; Norton & McCloskey, 2008; Steffe, 2002; Wilkins & Norton, 2011, 2018). For example, given a rod, a student needs to be able to consider that rod as the whole, partition it into a unit, such as 1/4, disembed one of those fourths, and iterate the 1/4 unit four times to recreate the whole, while realizing that four 1/4 units makes the whole and that the whole is four times as large as that unit. As demonstrated, these operations are similar to those that define the concept of measurement described by Piaget et al. (1981/1960), subdivision and change of position.

In understanding fractions and rational numbers, in general there are different concepts of fractions that students may develop. Kieren (1980) described five different subconstructs of rational numbers: (a) part-whole, (b) quotients, (c) measures, (d) ratios, and (e) operators. Each of these different subconstructs describe a different understanding of fractions. With a *part-whole* understanding, students understand 4/5 as 4 equal parts out of 5; they understand the relationship between the whole and number of parts. They recognize that a whole is broken into five equal parts, and they are only taking four of those parts out of the whole (Kieren, 1980; Lamon, 2012). Understanding fractions as parts of wholes indicates a purely additive relationship, where the parts add up to make the whole (Kieren, 1980). In the *quotient* subconstruct, students understand 4/5 as dividing 4 into 5 equal parts. This is similar to a part-whole understanding, but is different in that students are actually dividing the whole into parts (Kieren, 1980; Wilkins & Norton, 2018). The *measurement* subconstruct involves an understanding of 4/5 as 4 iterations of 1/5. This is directly related to Piaget et al.'s (1981/1960) concept of measurement. Here the fraction represents a size relative to the whole (Lamon, 2012; Wilkins & Norton, 2018). With a *ratio* concept, students understand 4/5 as a relationship or comparison of 4 to 5. This can either be a part-whole comparison or a part-part comparison (Lamon, 2012; Wilkins & Norton, 2018). With a part-whole comparison, the ratio represents a comparison of a part of the set to the whole set; in a part-part comparison, the ratio represents a comparison of a part of the set to another part of the set (Lamon, 2012). For example, a part-whole ratio comparison for 4/5 could be four girls to five students. A part-part comparison for 4/5 could be four girls to five boys in a class of nine students. Finally, an *operator* concept represent students understanding 4/5 as a rule for multiplying by 4/5 of something. This notion focuses explicitly on the multiplicative relationship between two quantities (Kieren, 1980; Lamon, 2007; Wilkins & Norton, 2018).

Despite these different subconstructs, there are similarities between each of them, and thus they are related to one another. Kieren (1980) notes that partitioning plays a major role in the development of each of these subconstructs, and iteration was especially useful for the measures subconstruct. Furthermore, the quotient and ratio subconstructs are related to the part-whole subconstruct, and the operator is essentially a more sophisticated measurement concept (Wilkins & Norton, 2018). In fact, Lamon (2007) found that the operator subconstruct is naturally developed from the measurement subconstruct. Although these subconstructs are related, certain ones promote a deeper understanding of fractions and rational numbers.

When students are taught fractions as measures, they are better prepared to solve problems involving fractions and also develop a deeper understanding of fractions and rational numbers (Lamon, 2007; Norton & Wilkins, 2009, 2012; Wilkins & Norton, 2018). Lamon (2007) found that by being introduced to fractions as measures, students were better able to develop other subconstructs of rational number, helping foster a better understanding of rational numbers. She claimed that the group of students who understood fractions as measures were the strongest "because it connected most naturally with the other subconstructs" (Lamon, 2007, p. 659) and therefore helped foster more meaningful understanding. In another study, Norton and Wilkins (2009) highlighted the importance of obtaining a measurement construct of fractions, as it is central to the progression of fractions schemes (Norton & Wilkins, 2012). Therefore, these findings emphasize the importance of helping students develop the concept of measurement, as it underlies other mathematical concepts such as fractions.

However, the manner in which students are often taught fractions focuses on fractions as a quantity and not a size (Lamon, 2007). This type of understanding of a fraction as a quantity is related to part-whole concepts (Steffe, 2002; Steffe & Olive, 2010; Wilkins & Norton, 2018). In the United States, most textbooks emphasize the part-whole concept of fractions and typically do not move beyond that concept (Watanabe, 2007). In their study, Boyce and Norton (2016) found that about 54% of the sixth graders they interviewed had not constructed a measurement meaning of fractions, and were classified as pre-fractional. Although a part-whole concept of fractions is important, there are limitations to this understanding (Hackenberg et al., 2016; Thompson & Saldanha, 2003; Wilkins & Norton, 2018). For example, when students attempt to work with improper fractions such as 8/5, they become confused when they try to understand 8 parts out of 5. They do not understand having more parts than what is in the whole. As a result, students often convert improper fractions into mixed numbers. This allows them to think in terms of part-whole. They are then able to understand 8/5 as one whole (1) and 3 out of 5 parts. Although it is an important foundational understanding for fractions, it is necessary to move beyond part-whole understandings and focus on fractions as sizes or measurements. Furthermore, because all the subconstructs of fractions involve the underlying operations of partitioning and iterating, fractions inherently relate to the concept of measurement. Therefore, it is necessary to examine the different ways in which students may develop this measurement concept of fractions.

Fraction Operations and Schemes

Schemes are “constructs used by teachers and researchers to model students’ cognitive structures” (McCloskey & Norton, 2009, p. 46). They identify and describe the different actions and processes a student may use while solving tasks. Von Glasersfeld (1995b) developed a three-part model for scheme theory, consisting of three main components (see Figure 2.1). This scheme theory model consists of a recognition template, mental actions, and a result. The mental actions employed in that situation that are used to carry out a specific function are known as operations. These operations are developed from abstractions of personal experience (von Glasersfeld, 1995b). Piaget (1970) defined an operation as “an action that can be internalized; that is, it can be carried out in thought as well as executed mentally” (p. 21). To further describe operations, von Glasersfeld notes that they reside in the mind, are unobservable, but “can only be *inferred* from observation” (p. 70, emphasis in original). In essence, operations do not have to be physically acted out (von Glasersfeld, 1995b), hence mental actions, and can only be inferred from observing student activity.

In contrast to these mental actions known as operations, a scheme describes or models a student’s cognitive structures (McCloskey & Norton, 2009). Piaget (1970) defined a scheme as “whatever is repeatable and generalizable” (p. 42). His definition focuses on a student’s general ways of operating. As discussed earlier, when a student’s assimilated mental actions produce an expected result, they reach a state of equilibrium. From here, the student is able to coordinate and repeat those notions and operations, leading to the development of structures known as schemes. Therefore, the difference in an operation and a scheme is that an operation is a certain mental action a student employs whereas a scheme is a three-part structure a student employs, including those individual mental actions. Essentially, operations are one key component of the entire scheme structure. When thinking about fractions within the realm of Steffe's (2002) reorganization hypothesis, there are certain operations and schemes that have been identified that are key for understanding fractions. These basic operations and schemes help provide a general framework for understanding and analyzing students' understanding of fractions.

Operations Involved with Fractions

When thinking about the concept of fractions, specifically grounded in Steffe's (2002) reorganization hypothesis, there are many different mental operations required for fractional understanding. These include unitizing, partitioning, disembedding, iterating, splitting, and the ability to coordinate different levels of units (Hackenberg, 2007; Hackenberg et al., 2016; Lamon, 1996, 2012; McCloskey & Norton, 2009; Norton, 2008; Norton & McCloskey, 2008; Norton & Wilkins, 2009, 2012; Olive, 1999; Steffe, 2002, 2003; Steffe & Olive, 2010; von Glasersfeld, 1981; Wilkins & Norton, 2011, 2018). Without these mental actions, students are limited in their understanding of fractions. Therefore, it is necessary to understand each of these operations and how they impact fractional understanding.

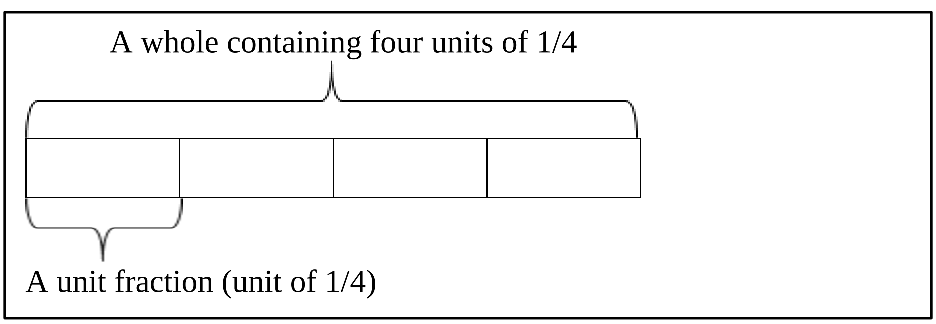
Unitizing

Unitizing can be described as "the cognitive assignment of a unit of measurement to a given entity; it refers to the size chunk one constructs in terms of which to think about a given commodity" (Lamon, 1996, p. 170). This means that anything can become a unit of measurement as long as the student views it as a separate entity from the given entity (Lamon, 1996, 2012; Norton, 2008). Consider a general example posed by Lamon (2012) in which you are given 24 cans of soda. Some possible ways of unitizing these individual cans is by treating this group as one 24-pack, two 12-packs, four 6-packs, or as 24 individual cans. In this case, unitizing is unique to each individual and is a “subjective process” (Lamon, 2012, p. 104). Nonetheless, it is also a process that occurs naturally as a student begins to separate and chunk items.

This process is necessary for students to understand fractions as they need to be able to unitize different objects, recognizing that wholes are made of smaller units, and thus leading to the development of composite units (Hackenberg et al., 2016; Steffe & Olive, 2010). Students need to be able to break a whole into smaller parts and then unitize the parts, thus establishing a composite unit with respect to the initial whole. For example, suppose a student broke the whole bar given in Figure 2.3 into four parts, took one of those parts, and then copied it three more times. A student may be able to then unitize each one of the parts into a new unit, known as a unit fraction (1/4), and furthermore unitize all four unit fractions into a composite unit, one whole containing four units of 1/4 (Hackenberg et al., 2016). From this example, it can be seen that unitizing is an important operation that is foundational for fractional understanding. It also plays a key role in the partitioning operation (Hackenberg et al., 2016).

Figure 2.3

Example of Unitizing Unit Fractions



Partitioning

Subdivision, as described by Piaget et al. (1981/1960), is similar to the fragmenting process described by Steffe and Olive’s (2010). This is simply the “act of breaking or fracturing a whole” (Lamon, 2012, p. 172) into different parts (Lamon, 1996). There are different levels of fragmenting, where partitioning is a special case that produces equal sized parts (Hackenberg et al., 2016; Steffe & Olive, 2010). For example, students who use a lower level of fragmenting may break a whole into two unequal parts, or may not use the entire whole. Students who use a higher level of fragmenting may make equal parts and "use up all of the material" (Hackenberg et al., 2016, p. 42). This level of fragmenting is known as partitioning, where the main focus is to produce equal sized parts (Steffe & Olive, 2010).

More clearly defined, partitioning is "the process of dividing an object or objects into a number of disjoint and exhaustive parts" (Lamon, 2012, p. 172), where the parts are "nonoverlapping and nonempty" (Lamon, 2012, p. 49). This indicates that the divided parts do not overlap and everything in the whole is contained in the parts. Further explaining the process of partitioning, Norton (2008) notes that partitioning enables "students to create equal groups from a discrete collection or continuous whole. The operation involves positing a specified number of slots in which to distribute the set or break up the whole into a continuous but partitioned whole" (p. 406). When students are able to partition a whole into equal parts and recognize that any one of those parts could be iterated a necessary number of times to recreate the whole, it is said that these students are using a more sophisticated process called equi-partitioning (Hackenberg et al., 2016; Steffe, 2002; Steffe & Olive, 2010).

Disembedding

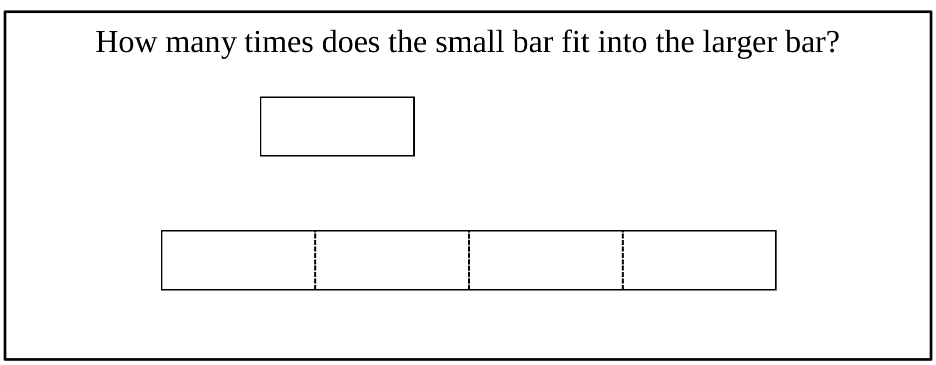
Once a student has partitioned a whole, if they have constructed a disembedding operation, they are then able to take one of those partitions out of the whole while maintaining the structure of the whole (Norton, 2008; Norton & McCloskey, 2008). "Disembedding is the fundamental mental operation on which part-whole comparisons are based" (Steffe & Olive, 1996, p. 118). For example, suppose a student partitions a whole into five parts. If they want to obtain 3/5, they understand to take 3 parts out of the 5. While doing so, they also maintain the whole consisting of five parts, or 5/5. They recognize that taking three of those parts out does not affect the whole, because those three are simply parts of the whole.

Iterating

Iteration is the process of copying a piece or part of a whole to recreate the whole (Hackenberg et al., 2016; Steffe, 2002; Steffe & Olive, 2010). This is similar to the process of change of position as described by Piaget et al. (1981/1960). Students ultimately make "connected copies of a part by repeating it" (Hackenberg et al., 2016, p. 186). These copies are identical copies, meaning they all are the same size and represent the same unit (McCloskey & Norton, 2009; Norton & McCloskey, 2008). For example, given the task in Figure 2.4, students may copy the small piece four times to recreate the whole.

Figure 2.4

Example Task Designed to Elicit an Iterating Operation



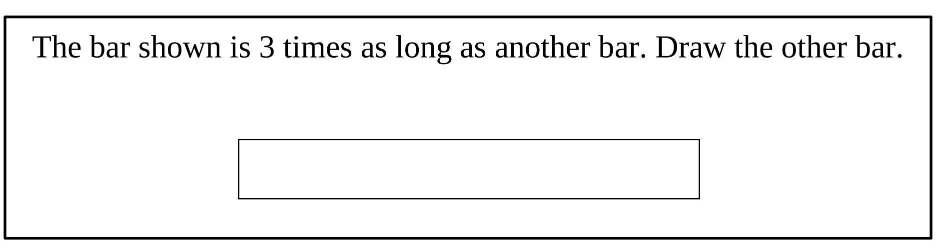
Here, the student is able to establish a relationship between the unit and the whole; that is a 1-to-4 relationship between the 1 unit being iterated and the whole quantity of 4 formed from the iterations (Hackenberg et al., 2016). Extending this iterating operation, students will eventually realize that only iterating the unit once will produce 1/4 of the whole, iterating it twice will produce 1/2 of the whole, and iterating it three times will produce 3/4 of the whole (Hackenberg et al., 2016; McCloskey & Norton, 2009; Norton & McCloskey, 2008; Steffe & Olive, 2010). This then leads students to recognize the multiplicative relationship between the two quantities: one unit is 1/4 the size of the whole, and the whole is four times as large as the one unit.

Splitting

The operations of partitioning and iterating can be viewed as inverses of each other, and when coordinated result in another operation called splitting (Hackenberg et al., 2016; McCloskey & Norton, 2009; Norton & Wilkins, 2009, 2012; Steffe 2002, 2003; Steffe & Olive, 2010; Wilkins & Norton, 2011, 2018). Steffe (2002) explicitly defined splitting as the simultaneous "*composition* of partitioning and iterating" (p. 19, emphasis in original).With splitting, students are able to reverse their thinking (Hackenberg, 2010; Hackenberg et al., 2016; Steffe, 2002). This allows students to think about partitioning and iterating as one operation, instead of two separate ones (Hackenberg et al., 2016; Norton, 2008; Steffe, 2002; Wilkins & Norton, 2011). Consider the example task given in Figure 2.5. The student is given a bar and told that it is three times as long as another bar. Norton and Wilkins (2009) explains that to solve tasks like this, that are "iterative in nature" (p. 151), a student must anticipate using partitions that could be iterated to recreate the whole. In this example, the student needs to understand to partition the given bar into three pieces while also realizing that iterating one of those pieces three times will recreate the whole.

Figure 2.5

Example Task Designed to Elicit a Splitting Operation



Oftentimes, however, students may interpret a splitting task as an iterating task (Hackenberg et al., 2016). For example, students may simply iterate the bar three times. In this case, students are not simultaneously composing their partitioning and iterating operations, and fail to employ the splitting operation. Once students have developed the splitting operation, they are better equipped to solve more advanced fractional tasks, thus developing more sophisticated fractional schemes (McCloskey & Norton, 2009; Norton & Wilkins, 2012; Steffe, 2002, 2003; Wilkins & Norton, 2011, 2018). Although not directly stated, the splitting operation appears to be an extension of Piaget et al.'s (1981/1960) concept of measurement as the coordination of subdivision and change of position (see [2] above).

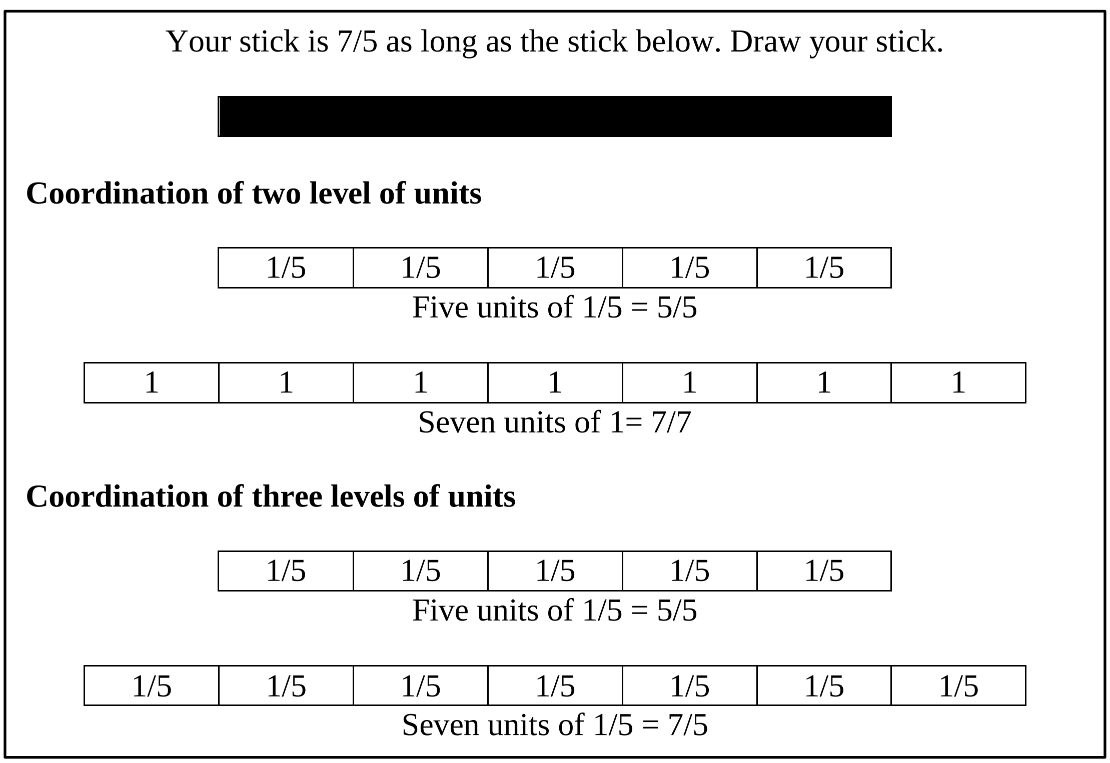
Coordination of Units

The coordination of units involves the process of building and working with units (Hackenberg, 2007; Hackenberg et al., 2016; Steffe, 1992, 2002; Steffe & Olive, 2010). When working with fractions, students create units through partitions and iterations; they then need to establish multiplicative relationships between those units and the whole, requiring coordination of units. In order to construct more sophisticated fraction schemes, students need to be able to coordinate different levels of units. For example, students typically begin by working with one level of units and then proceed to coordinate two levels of units (Hackenberg et al., 2016). From here, they are able to coordinate three levels of units in activity, meaning they can coordinate two levels of units and then, in activity, coordinate those two levels with a third level of units (Hackenberg, 2007; Steffe, 1992).

Once students interiorize their coordination of two levels of units with a third level of units, they can assimilate with three levels of units coordination, coordinating the three levels of units at once or as one whole multiplicative structure (Hackenberg, 2007). For example given the task in Figure 2.6, students who can coordinate two levels of units may partition the whole into five parts, take one of the five parts, and call it 1/5. Then after iterating the part seven times, they may lose track of the original whole and say the result is 7/7. They no longer think about the relationship between the unit (1/5) and the original whole (5/5), but now relate the iterated part as a 1 to the new whole (7). Through this process of coordinating units students often lose track of the original whole and are unable to relate their final result to the original whole. However, a student who can coordinate three levels of units may partition the whole into five pieces, recognize that each of those partitions represents a 1/5, and can furthermore iterating any one of them seven times and maintain the 1-to-5 relationship and the 1-to-7 relationship, resulting in 7/5. Here, students can simultaneously consider the whole unit of 1, the unit fraction 1/5, and the seven units of 1/5.

Figure 2.6

Example Task Designed to Elicit Three Levels of Units Coordination



Summary of Operations

For a complete understanding of fractions, students need to construct the operations of unitizing, partitioning, disembedding, iterating, splitting, and be able to coordinate three levels of units (Hackenberg, 2007; Hackenberg et al., 2016; Lamon, 1996, 2012; McCloskey & Norton, 2009; Norton, 2008; Norton & McCloskey, 2008; Norton & Wilkins, 2009, 2012; Olive, 1999; Steffe, 2002, 2003; Steffe & Olive, 2010; von Glasersfeld, 1981; Wilkins & Norton, 2011, 2018). Partitioning is related to Piaget et al.'s (1981/1960) subdivision operation and involves dividing or separating a whole into parts of equal size with no gaps or overlaps. This then enables students to begin breaking a whole into fractional units. Once students have broken a whole into pieces, they may begin iterating one of the pieces to recreate the whole, or another fractional part of the whole. Iterating is an extension of Piaget et al.'s (1981/1960) notion of change of position; that is taking one unit and transposing it along the entire length of the whole to recreate the whole.

When students can simultaneously coordinate their partitioning and iterating operations, it is said they have developed the splitting operation. Splitting describes the simultaneous action of partitioning and iterating, and appears to be related to Piaget et al.'s (1981/1960) description of measurement as the coordination of subdivision and change of position. At this point, students do not have to partition a whole first and then iterate one of those units; they can partition and iterate simultaneously. Finally, the ability to coordinate units helps students maintain the multiplicative relationships between different levels of units created by partitioning and iterating, with respect to the whole.

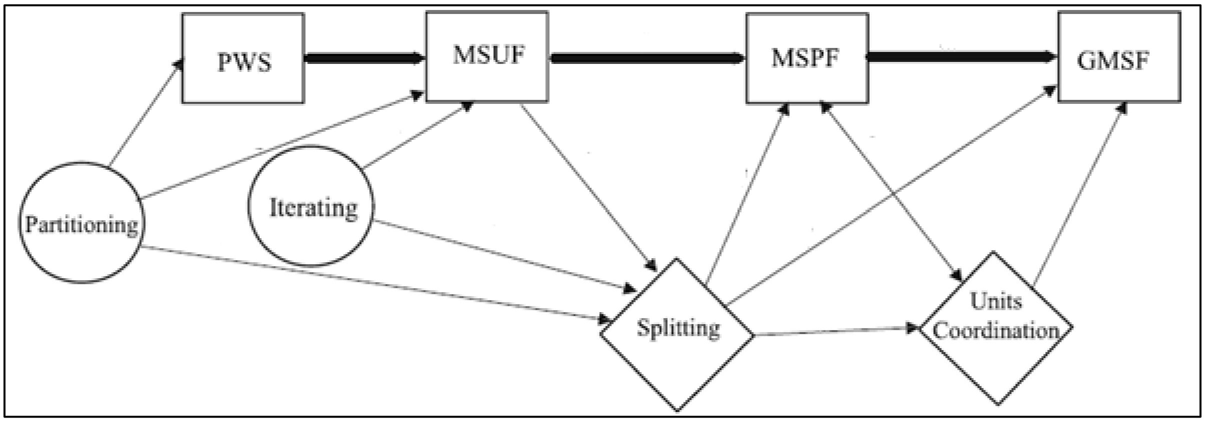
Fraction Schemes

Schemes describe the ways in which students think about certain concepts, and in this case fractions (Steffe, 2002, 2003; Steffe & Olive, 2010). Schemes are formed by the actions and operations students use and then how they interpret the results of those actions or operations. Schemes also describe a progression for how students may move from less sophisticated ways of thinking to more sophisticated ways of thinking (Steffe & Olive, 2010). In their work, Steffe and Olive (Steffe, 2002, 2003; Steffe & Olive, 2010) outlined the following schemes to help describe the ways in which students think about fractions: part-whole scheme (PWS), partitive unit fraction scheme (PUFS), partitive fraction scheme (PFS), reversible partitive fraction scheme (RPFS), and the iterative fraction scheme (IFS).

Building on Steffe and Olive's work, Wilkins and Norton (2018) reorganized the previous fraction schemes to describe them in terms of a measurement concept of fractions. In doing so, they combined some of the previous schemes into new ones, thus creating the following progression of schemes to describe the development of measurement concepts of fractions: part-whole scheme (PWS), the measurement scheme for unit fractions (MSUF), the measurement scheme for proper fractions (MSPF), and the generalized measurement scheme for fractions (GMSF). Figure 2.7 represents this progression as well as the relationship to the different operations that impact these schemes.

Figure 2.7

Learning Progression for Measurement Schemes for Fractions (From Wilkins & Norton, 2018, p. 32)



Part-Whole Scheme

With a PWS, students are able to partition a whole into parts and then disembed the given part(s) out of the whole (Steffe, 2003; Steffe & Olive, 2010; Wilkins & Norton, 2018). For example, suppose students are given a whole stick with a specific length. When asked to find 4/5 of the stick, students need to be able to partition the stick into five equal parts and then identify or disembed four of those parts. They are then able to recognize that fraction as 4/5, meaning 4 parts out of the 5 parts in the whole. As demonstrated, this requires the operation of partitioning and disembedding; students must be able to break a whole into parts before they can recognize how many parts of the whole. As a result, partitioning is a precursor for the development of a PWS, as presented in Figure 2.7. Although having a PWS is important for basic understanding of fractions as it provides a foundation, only thinking of fractions as parts of wholes limits students' fractional understanding (Hackenberg et al., 2016). With only a PWS, students do not recognize the partitions as iterable units; they are simply parts that make up the whole. As presented in Figure 2.7, iterating is not necessary for the construction of a PWS and may not develop until after a PWS. Therefore, students do not need to iterate the parts they made, because they are simply parts of the whole.

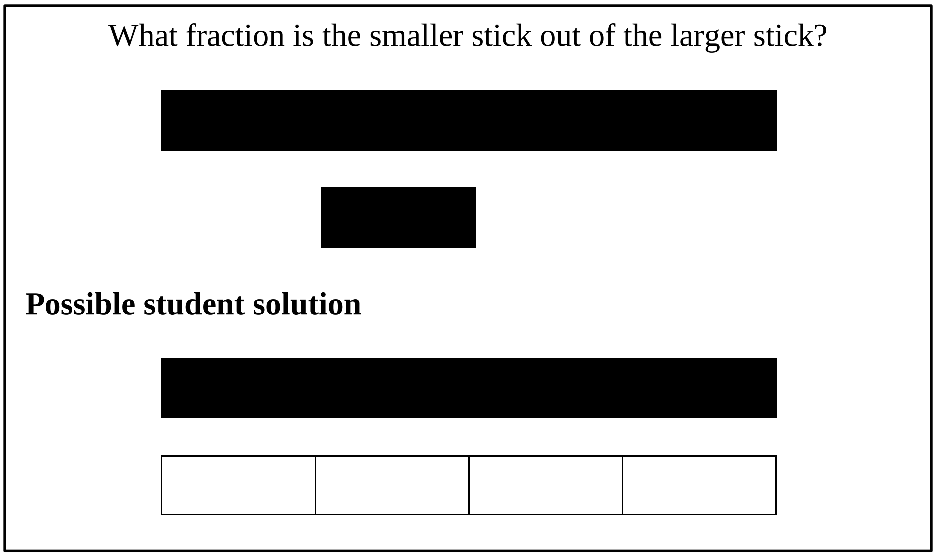
Measurement Scheme for Unit Fractions

Once students construct an iterating operation, and after students construct a PWS, they construct a MSUF (Figure 2.7; Wilkins & Norton, 2018). This scheme is synonymous to Steffe and Olive's PUFS where students are given a part of the whole, as well as the unpartitioned whole, and can figure out the size of the part by iterating it to recreate the whole (Steffe, 2002, 2003; Steffe & Olive, 2010). After students iterate the part to make the whole, they are then able to determine the size of the fractional part from the number of iterations. The MSUF is different from the PWS in that students now have established an iterable unit of 1, and recognize the relationship between that unit and the whole, *n*. They understand that the unit can be iterated *n* times to recreate the whole, establishing the notion of a unit fraction. They no longer rely on parts of the whole, but rely on the *n* iterations of the unit to name the fraction. This iterable unit of 1 is the distinguishing factor between the PWS and MSUF.

To describe the actions and operations involved in a MSUF, suppose a student is given the task in Figure 2.8. Students with a MSUF would iterate the smaller bar and determine that four iterations make up the larger bar. This would lead them to determine that the smaller bar is 1/4 of the larger bar. Similar to operating with a PWS, the student recognizes that the bar is one of those four iterations. However, they understand the reciprocal relationship between the iterable unit and the *n* iterations. Having this iterable unit is important for a measurement concept of fractions. This notion of a measurement concept of fractions extends Kieren's notion of fractions as measures. Kieren (1980) argued that being able to understand fractions as measures requires iteration, which is further consistent with Piaget et al.'s (1981/1960) concept of measurement as the coordination of subdivision and change of position. This notion of a fraction as a measure focuses on iterating a unit, but the MSUF focuses on obtaining an iterable unit, and is therefore more advanced than Kieren's (1980) notion.

Figure 2.8

Example Task Designed to Elicit a MSUF



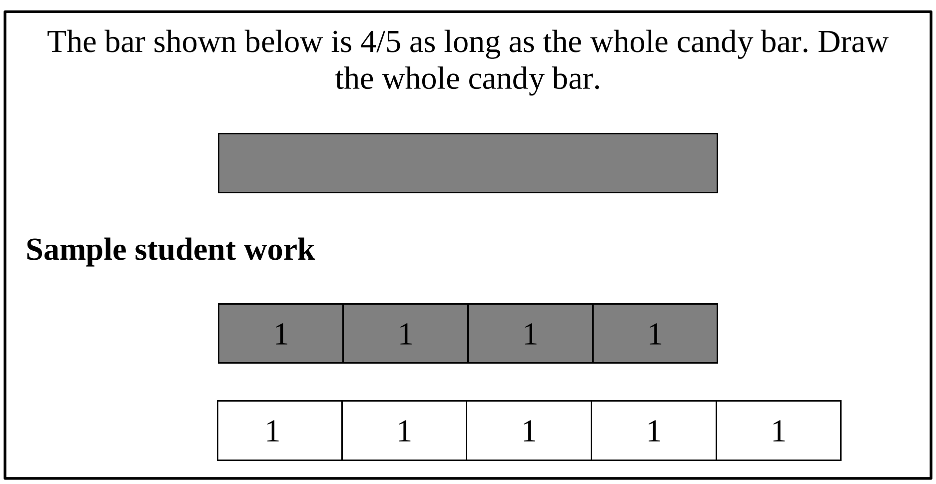
Measurement Scheme for Proper Fractions

After developing a MSUF, some students are able to generalize their scheme from unit fractions to proper fractions, thus developing a MSPF (Wilkins & Norton, 2018). The MSPF combines Steffe and Olive's PFS and the RPFS (Steffe 2002, 2003; Steffe & Olive, 2010). With a PFS, students are able to take an unpartitioned piece and compare it to an unpartitioned whole to determine the size of the piece. Here, they take the proper fraction, partition it into a unit fraction, establishing an iterable unit of 1, and then iterate that unit *n* times to recreate the whole. During these actions, students coordinate two levels of units sequentially: the unit fraction to the proper fraction, and the unit fraction to the whole. It is also important to note that in this situation students sequentially partition and then iterate the unit fraction. (McCloskey & Norton, 2009; Norton, 2008; Norton & McCloskey, 2008; Norton & Wilkins, 2009, 2012; Olive, 1999; Steffe, 2002, 2003; Steffe & Olive, 2010; Wilkins & Norton, 2011, 2018). Once a student can simultaneously partition and iterate, they have developed the splitting operation. This then enables them to reverse their thinking and construct a RPFS.

With a RPFS, students are able to take an unpartitioned piece and partition it into a unit fraction and iterate it *n* times to create the whole to determine the size of the piece. Although this sounds similar to the PFS, here students simultaneously partition and iterate. They are able to coordinate the partitions and iterations to visualize the whole, knowing when to stop iterating. For example, suppose a student is given the task in Figure 2.9. Students are given an unpartitioned bar and asked to find the whole. In order to do so, with a RPFS, a students will partition the bar into four pieces, each one representing a 1, and then iterate one piece five times to create the whole.

Figure 2.9

Example Task Designed to Elicit a RPFS



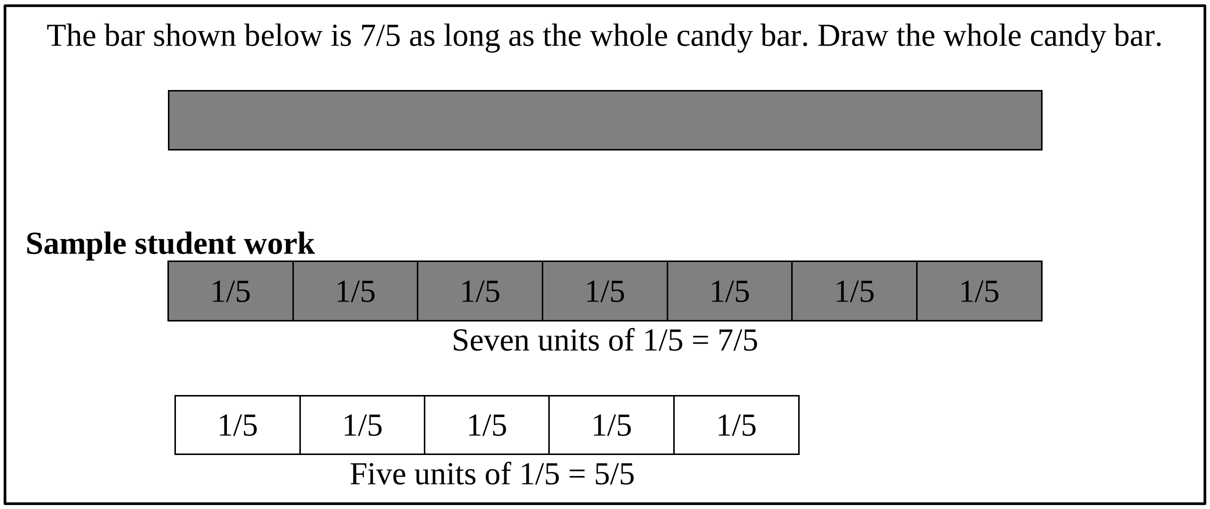
Since a MSPF combines a PFS and a RPFS, students with a MSPF are able to split. They are also able to reverse their partitioning of the fractional whole and iterating the unit fraction as an iterable unit of 1 to understand the proper fraction *m/n*; that is they maintain the relationship between the *n* iterations needed to create the whole and the *m* iterations taken. They understand the proper fraction *m/n* as simply *m* units of 1 from the *n* iterations needed to create the whole. They are able to work with proper fractions by recognizing the relationship between the unit fraction and the whole, and then by iterating the unit fraction to create the proper fraction (Wilkins & Norton, 2018). This requires an inverse relationship between the unit fraction and whole, as well as the unit fraction and proper fraction (Wilkins & Norton, 2018). Within this, they are coordinating three levels of units in activity: the 1 to *n* relationship, as well as the 1 to *m* relationship. This coordination happens sequentially, representing two two-level coordinations (Wilkins & Norton, 2018).

Generalized Measurement Scheme for Fractions

The final scheme is the GMSF (cf. Steffe and Olive's IFS, 2010). With a GMSF, students are able to use their splitting operation to work with improper fractions and smaller wholes (McCloskey & Norton, 2009; Norton, 2008; Norton & Wilkins, 2009, 2012; Olive, 1999; Steffe, 2002, 2003; Steffe & Olive, 2010; Wilkins & Norton, 2011, 2018). Here, students are able to establish an iterable unit fraction as 1/*n* instead of as an iterable unit of 1, as described in the MSPF. This allows them to create a whole using any fraction, thus requiring the coordination of units (Figure 2.7). With a GMSF, students are able to simultaneously coordinate three levels of units—the improper fraction, the unit fraction, and the whole. For example, given the task in Figure 2.10, a student with a GMSF can simultaneously partition the bar into seven pieces and iterate one of those pieces five times to create the whole. In doing so, they must recognize that each partition represents a 1/5 of the whole, and not a 1/7.

Figure 2.10

Example Task Designed to Elicit a GMSF



Summary of Fraction Schemes

Fraction schemes describe the different ways students think about fractions. Building on Steffe and Olive's work, Wilkins and Norton (2018) reorganized the fraction schemes to describe them in terms of a measurement concept of fractions (see Figure 2.7). Before the development of a PWS, partitioning is required (Wilkins & Norton, 2018). This enables students to partition a whole into parts and then determine how many parts of the whole is given. Here, students have no iterable unit; they simply rely on how many parts out of the whole. Once the iterating operation is developed, students can iterate a part to recreate the whole and then determine the size of the unit fraction, leading to the construction of a MSUF. With a MSUF, students have an iterable unit of 1. Students then rely on the *n* number of iterations of that unit to determine the size of the whole. Upon developing the splitting operation, students can simultaneously partition and iterate. They are able to reverse their partitions and iterations to maintain the relationship between iterating the unit fraction as an iterable unit of 1 and the number of iterations, *n*, to understand the proper fraction *m/n*. They understand the proper fraction *m/n* as simply *m* units of

1 from the *n* iterations needed to create the whole. At this point, students are attributed with a MSPF. Finally, after students are able to coordinate three levels of units, they can coordinate the improper fraction, the unit fraction, and the whole simultaneously. This indicates that students are able to establish an iterable unit fraction of 1/*n*, recognizing that 5/4 is five units of 1/4. At this point, students are said to have constructed a GMSF.

Relating Measurement to Angles

This part of the chapter focuses on angles and angle measurement. The following sections offer an overview of the relationship between the concept of measurement and angles. It also provides a literature review of students' conceptualizations of angles and the process for quantifying angularity. Finally, the framework used to explain students' quantifications of angularity is described.

The foundational concept of measurement as subdivision and change of position takes on a different meaning once students begin studying angle measurement. Instead of looking at linear measurement, measurement is now placed in the context on circles. The understanding and development of angle measurement requires students to use a different reference system for coordinating subdivision and changes of position (Piaget et al., 1981/1960). They must be able to recognize angle measurement "as a special kind of coordination between the length of the arms and the distance which separates them" (p. 182). That is, children need to recognize that angle measurement is the separation between two rays that make an angle, where the separation is made by rotating one ray about a point, thus emphasizing a dynamic construction (Piaget et al., 1981/1960). These dynamic actions, such as rotating, helps students realize that an angle is created by iterating a unit, whether that be a set rotation or a specified sweep (Clements & Burns, 2000; Clements et al., 1996; Smith et al., 2014). Nonetheless, the foundation concepts of subdivision and change of position still apply.

In the most conceptual understanding, angles are defined as fractional amounts of a full rotation or a circle (Moore, 2013; Thompson, 2008). This then allows students to recognize angle measurement as the angle’s openness, or separation. In order to understand the angle’s openness, Thompson (2008) argues that students must first “say what about an angle we [they] are measuring and the method by which we [they] derive a measure of it” (p. 35). Following Thompson's (2008) argument, Moore (2013) further argues that angle measurement should be developed “as representative of the same quantitative relationship” which can be achieved “by conceiving angle measure as the process of determining the fractional amount of the circle’s circumference subtended by an angle, provided that the circle is centered at the vertex of the angle” (p. 227). This directly relates to partitioning a circle into a piece, and then iterating that piece to determine the fractional amount of the circle that the angle comprises. However, this is not the initial understanding of angle and angle measurement that students develop.

What is an Angle?

There are many different definitions and concepts of angle that have been used to teach students about angles. For example, in their textbook, Henderson and Taimina (2005) define an angle as “the union of two rays (or segments) with a common endpoint” (p. 38). Charles (2011) also defines an angle as two rays with the same endpoint. This indicates that students must understand rays, line segments, and endpoints before they can understand angles. In another textbook, Greenberg (2008) uses the following definition for angles: “An angle with vertex A is a point A together with two distinct non-opposite rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (called the sides of the angle) emanating from A” (p. 18). From this definition, again students must understand what a ray, point, and vertex are to understand angles. However, this definition goes even farther than the previous one in that the rays cannot be opposite. Based on this definition, a straight line is not an angle. Similarly, Long (2009) defines an angle as "two rays with the same endpoint that do not lie on the same line" (p. 23), emphasizing that straight lines are not angles. Each of these definitions does not capture an angle as being created by rotating one ray about a point. This then may limit students' understanding of 0^{o}, 180^{o}, and 360^{o} angles.

Different Conceptualizations of Angles

Depending on how angles are defined and presented to students, students can develop different conceptualizations of angle. Researchers have documented students' conceptualizations of angles and categorized them into two categories: static and dynamic (Browning et al., 2008; Bütüner & Filiz, 2016; Clements & Burns, 2000; Kontorovich & Zazkis, 2016; Smith et al., 2014). With a static angle conception, angles are viewed as a geometric shape or figure, or as a pictorial/figurative representation (Kontorovich & Zazkis, 2016; Smith et al., 2014) in which the student focuses on what the angle looks like in relation to the location or position of the sides (Bütüner & Filiz, 2016). A common example of a static angle conception is the notion of an angle as a *union*. An angle as a *union* is seen as the intersection of two rays or line segments at a common vertex (Mitchelmore & White, 1998; Keiser, 2004). With this concept, students focus on the position of the angle, where both sides of the angle are visible, and not on the movement that was involved in the creation of the angle (Mitchelmore & White, 2000).

With a static angle conceptualization, students can have a difficult time identifying angles in different positions, orientations, and contexts. For example, students may struggle with identifying 0^{o}, 180^{o}, and 360^{o} angles (Smith et al., 2014) because they cannot see the two distinct rays and related vertex or endpoint. Students also confuse right angles in different orientations, often calling them left angles (Outhred, 1987), because they are focused on *how* the angle looks.

In contrast, with a dynamic angle conception, students understand that angles are created by movement or actions, in which students sweep, drag, or rotate a line or ray to make an angle (Clements & Burns, 2000). Additionally, students might use body movements or gestures to represent angles (Clements & Burns, 2000; Smith et al., 2014). A common example of a dynamic angle conception is the notion of an angle as a *turn.* When students think of angles as *turns*, they can recognize and identify the rotation used to make the angle; they understand that angles are created by rotating one line or ray, or by sweeping a line or ray (Browning et al., 2008; Bütüner & Filiz, 2016; Clements & Battista, 1989, 1990; Clements & Burns, 2000; Kontorovich & Zazkis, 2016; Smith et al., 2014; Yigit, 2014).

By conceptualizing angles as dynamic angles, students are able to understand a greater variety of angles in different contexts and orientations. They no longer focus on *how* the angle looks, but focus on the sweeping or rotation motion. For example, students are no longer limited to focusing on the two distinct rays and related vertex or endpoint and can now understand straight angles and semi-circles as a rotation of 180^o, and circles as 360^o. Students also no longer focus on the orientation of the angle, and are better prepared to understand right angles as 90^o no matter the orientation. With a dynamic conception of angles, students are better prepared to gain a more abstract understanding of angles.

Angle Measurement

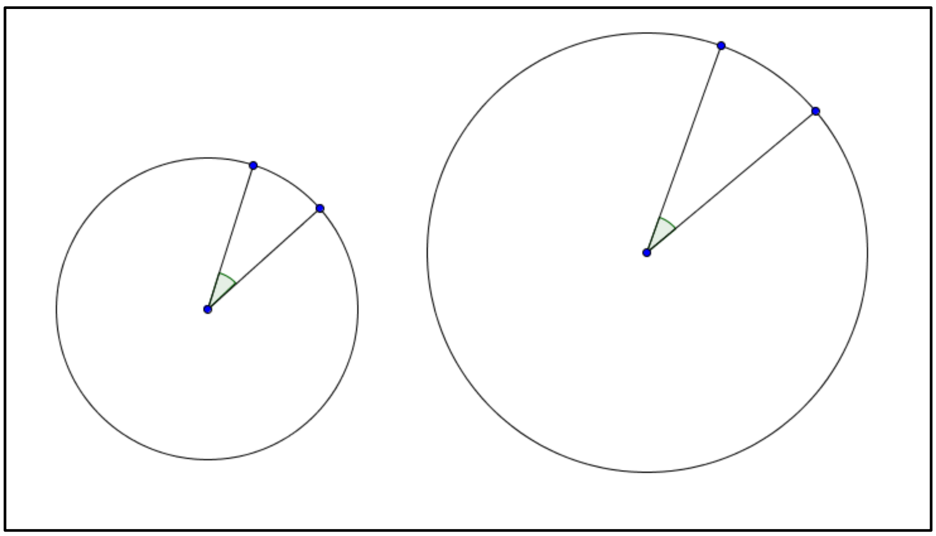
Piaget et al. (1981/1960) discuss the different stages students progress through as they learn about angles and angle measurement. In Stage I, students draw and “measure” angles from visual estimates. As they progress to Stage II, they begin using tools but still only view the angle as a perceptual figure (Piaget et al., 1981/1960). In Stage III, students develop more complex skills and try to use slope to "measure" angles. However, they still fail at recognizing angle measurement as the openness or separation between the two lines. It is only in Stage IV that students can begin to coordinate many actions within different dimensions to understand angle measurement as the separation between two lines. These students no longer see angles as solely figural, but as the result of operations and actions (Piaget et al., 1981/1960).

Because students can develop different conceptualizations of angle depending on how angles are defined and presented to students, this will also affect the different ways in which students understand angle measurement. If students do not understand what an angle is or how it is constructed, students are limited in their understanding of angle measurement (Barabash, 2017; Clements et al., 1996; Thompson et al., 2014). Thompson et al. (2014) claimed that even understanding angle as a size, from a magnitude perspective, can be misleading. They note that "any reference to an angle's magnitude would have little meaning because even when thinking of size, an angle would not have a unique size" (p. 2). For example, given the two angles in Figure 2.11, the two angles have the same measurement, but are presented in two different circles of different size; in this case, students may think the angles have different measures because one appears larger than the other, and they have different sizes.

This focus on the size of angles may cause some students to think that angle measurement can be determined by measuring the side lengths of the angles (Barabash, 2017; Bütüner & Filiz, 2016; Clements & Battista, 1989, 1990; Clements, 2003; Piaget et al., 1981/1960). Students may also think about angle measurements as being related to the size of the arc drawn to represent the angle (not the subtended arc) or the distance between the sides of the angle (Barabash, 2017; Bütüner & Filiz, 2016; Clements & Battista, 1989, 1990; Clements et al., 1996; Piaget et al., 1981/1960). This relates back to Piaget et al.'s (1981/1960) distinction between length and distance, where they define *length* as the occupied or filled space between two points or objects, and *distance* as the empty space or linear separation between two points or objects. When dealing with angle measurement, it is important to help students understand that it is not the linear distance between the two endpoints, but rather the separation between them.

Figure 2.11

Angles with the Same Measure but Different Size Circles (from Thompson et al., 2014, p. 2)



When angles are thought of as turns, they are understood as being constructed by sweeping, dragging, or rotating a line or ray (Andrews, 2002; Browning et al., 2008; Clements & Burns, 2000; Clements & Battista, 1989, 1990; Clements et al., 1996; Cullen et al., 2018; Mitchelmore, 1997; Smith et al., 2014). This process of a turn is developed by students using their informal knowledge of rotating objects using nonstandard units, then abstracting those ideas to develop more formal knowledge of rotating objects using standard units (Mitchelmore, 1997). For example, students may begin using nonstandard units of turns, such as rotating a bottle a half turn, or what they perceive as a half turn. Then, they may relate this to a clock hand rotating. From here, students can develop standard units of turns, such as minutes and hours. In their work, Clements et al. (1996) focused on angle construction and measurement in terms of units of turns. They focused on getting students to:

> (a) build up images of turn as physical rotation, a change in heading or orientation; (b) distinguish between smaller and larger turns (gross comparison); (c) construct and iterate units of turn; (d) estimate turn measures using certain units as benchmarks; and (e) recognize that different physical rotations can yield the same geometric effect. (p. 315)

This is similar to Mitchelmore's (1997) approach and is directly related to Clements and Battista's (1986) argument for focusing on nonstandard units of measurement and then build towards standard units of measurement.

By referencing a sweeping motion, students will be able to relate this to a circle, recognizing the angle measurement "by comparing the fraction of the circular arc and the circle's circumference" (Cullen et al., 2018, p. 146). With these sweeping or turning motions, students can relate this to their general measurement concept in terms of partitions and iterations. For example, in order to understand the standard units of measurement for angles, students would have to recognize that a turn represents a specific partition of a circle (Clements et al., 1996). A semi-circle could be created by partitioning a circle in half, which could then be conceived as a half turn. A student could also relate this to partitioning a circle into twelfths and then iterating that partition six times. Then building from this concept, they would be able to see that the partition of 1/12 would be 30^o, where they recognize a circle is 360^o and $\frac{360}{12} = 30$.

From this process, students are better prepared to understand tools for measuring angles, such as protractors. Instead of being provided a protractor to measure an angle, in which no concept of measurement is reinforced (Andrews, 2002), students would be able to see how a protractor was developed by iterating a unit of one degree 180 times (Barabash, 2017). The iterations of degrees would be implicitly seen in the tool, as suggested by Battista (2006). Therefore, there is a direct link between the basic concepts of measurement and the understanding of angle measurement. Students need to be able to understand partitions, iterations, and fractions to understand where standard units of angle measurements are derived.

With these conceptions of an angle as a dynamic construction, students are able to develop a more abstract conceptualization of angles, thus leading to a better and deeper understanding of angle measurement, as well as higher performance and greater learning gains (Clements & Burns, 2000; Cullen et al., 2018; Mitchelmore & White, 1998, 2000; Smith et al., 2014). For example, Clements and Burns (2000) discovered that when students use these dynamic actions for angles, they are able to develop schemes that can then be utilized in later situations. As a result, students gain a deeper understanding of angle measurement and clearer connections between concepts (Clements & Burns, 2000). In contrast, in their study, Bütüner and Filiz (2016) discovered that many high achieving sixth-grade students were limited to a static conception of angles, leading to many of these aforementioned misconceptions related to angle measurement. As a result, only 36% of students were able to correctly identify angles and only 39% were able to correctly compare the size of angles. By focusing on a dynamic conception of angles and with the concept of measurement, students are better able to see the whole, the partitions into a unit, the iterations of the unit, and the coordination between the units of measurement and the whole (Barrett et al., 2006). Therefore, students may be more apt to relate and expand the foundational principles of measurement to angles.

What Curriculum Standards Emphasizes

It is important to look at what curriculum standards emphasize to better understand how students might think about angle and angle measurement. The first place the Virginia Standards of Learning (SOLs) mention angles is in explaining the Kindergarten standards: “A vertex is the point at which two or more lines, line segments, or rays meet to form an angle” (VDOE, 2016c, p. 16). This definition supports the union concept of an angle. Then in first grade, this definition is expanded to include the idea that “An angle is formed by two rays that share a common endpoint called the vertex. Angles are found wherever lines or line segments intersect” (VDOE, 2016b, p. 20). This again supports the union concept of angle. This definition is used throughout second, third, and fourth grade as well. It is not until fifth grade that students are introduced to a more formal concept of angle and angle measurement, by placing angles within the context of a circle:

> - Angles are measured in degrees. A degree is 1/360 of a complete rotation of a full circle. There are 360 degrees in a circle.
> - To measure the number of degrees in an angle, use a protractor or an angle ruler. Before measuring an angle, students should first compare it to a right angle to determine whether the measure of the angle is less than or greater than 90 degrees. Students should understand how to work with a protractor or angle ruler as well as available computer software to measure and draw angles and triangles. (VDOE, 2016a, p. 25)

With this progression, students jump from understanding an angle as the intersection of two rays, line segments, or lines to thinking of angle measurement as 1/360 of a circle. If students begin by focusing on intersections of rays, this can be difficult to then expand this definition in the context of a circle. Furthermore, notice the emphasis placed on the use of protractors. Instead of allowing students to develop their own units, they move straight to using protractors and standard units of measurement (i.e., degrees).

Although the fifth grade standard states that one degree is 1/360 of a circle, they claim students should use protractors to measure angles. Despite the focus on the relationship between the angle and the circle, without a deep understanding of partitioning and iterating, this definition simply defines an angle in terms of degrees. This approach is what researchers have argued against in support of helping students use nonstandard units of measurement and then build towards standard units (Clements et al., 1996; Clements & Battista, 1986; Mitchelmore, 1997).

A similar progression is seen in the Common Core State Standards for Mathematics (CCSSI, 2010). Students begin learning about angles as the intersection of two rays or lines. This concept is then extended in fourth grade, where students are introduced to angle measurement as fractional parts of a full rotation or a circle:

> Recognize angles as geometric shapes that are formed wherever two rays share a common endpoint, and understand concepts of angle measurement:
>
> a. An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the two rays intersect the circle. An angle that turns through 1/360 of a circle is called a "one-degree angle," and can be used to measure angles.
> b. An angle that turns through *n* one-degree angles is said to have an angle measure of *n* degrees. (CCSSI, 2010, p. 31)

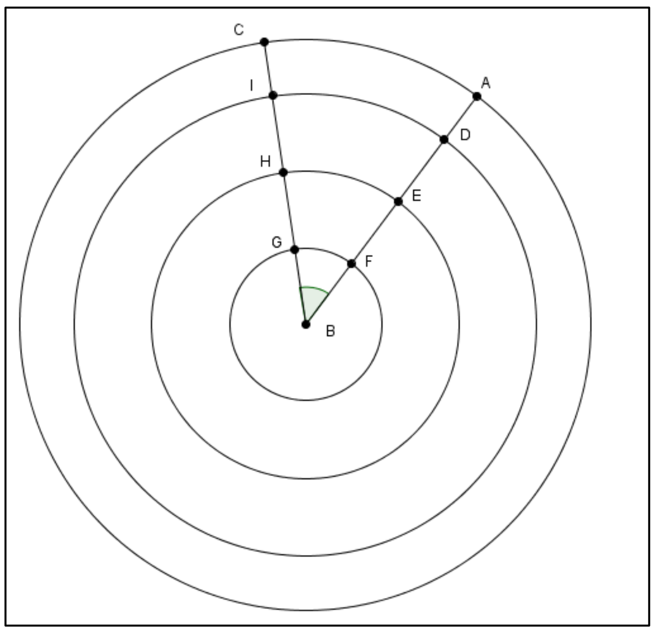
However, different from the Virginia standard, this definition not only focuses on angle measurement as a fractional amount of a circle, but also emphasizes understanding how to use non-standard units of measure to measure angles in terms of turns. There is no mention of a protractor. Students are expected to understand how angles are created which then allows them to develop an understanding of angles as turns. However, again this can be a drastic jump from thinking about angles as static figures to dynamic turns.

In both of these examples, the idea is that students recognize angles in terms of a circle, and that angle measurement is derived from that relationship to the circle. Thus, it is imperative to investigate this relationship between an angle, circle, and angle measurement. Moore (2013) noted that through his review of thirty elementary and secondary textbooks, most define angle measurement as "a number obtained by using a protractor, and a measurement of an arc length" (p. 227). He goes on to argue that these common concepts of angles and angle measurement as determined by degrees often lead students, and teachers, to develop "shallow and fragmented" (Moore, 2013, p. 226) understandings of angle measurement.

Although these concepts are important for helping students understand degrees as the standard unit of measurement for angle, and for relating right angles, complimentary angles, and supplementary angles through calculations, they "fail to address the quantitative structure behind the *process* of determining an angle's measure (Moore, 2013, p. 227, emphasis in original). Therefore, Thompson (2008) and Moore (2013) argue that angle measurement should be established as a quantitative relationship between the circle's circumference, the radius, and the arc length—that is the arc subtended by the angle. For example, a one degree angle would subtend 1/360 of the circumference of any circle at the vertex of the angle (Moore, 2013; Thompson, 2008). With this concept of angle measure, students maintain a relationship between the length of the arc and the circumference; this helps them recognize that if the length of the rays get longer or smaller, the angle measurement remains the same. For example, given the angles in Figure 2.12, some students might think ∠GBF is smaller than ∠CBA due to the length of the rays being shorter. However, with Moore's (2013) concept of angle measurement as the fractional amount of a circle's circumference, students would be able to recognize that ∠GBF and ∠CBA are in fact the same angle because the two arcs created by both angles, arc GF and arc CA, subtend the same fractional amount of each of the circle's circumference. As the length of the rays extend, a larger circle is created, resulting in a larger circumference. Even though the arcs and circles are getting larger, so is the circumference, and therefore the fractional amount or proportion remains the same.

Figure 2.12

Comparing Angle Measurements Using Equivalent Arcs



To summarize this concept of angle measurement, students need to understand that two rays intersect to form "an object...that has a measurable attribute of openness" (Moore, 2013, p. 228). This attribute can then be assigned a quantity to better understand that measurement. However, in order to quantify the openness of the angle, students must recognize the quantitative relationship between the circle's circumference, the radius, and the arc length created by the angle (Moore, 2013). This involves "coming to understand a unit in terms of a multiplicative relationship between a class of subtended arcs and the corresponding circle's circumference" (Moore, 2013, p. 228). What this means is that students ultimately develop a group of equivalent circles, where they recognize that although the circles get bigger, the proportional relationship between each of the subtended arcs and the circle's circumference remains the same (e.g., see Figure 2.12). Although some may argue that this is the most sophisticated concept of angle measure, and most difficult for students to understand, it is important to help students build towards this concept.

Quantifications of Angularity Framework

As discussed above, Moore's (2013) quantification of openness relies on the ability to maintain the relationship between the angle, subtended arc, and circle's circumference. However, students will not always begin with this quantification. Thompson (2011) defined quantification as "the process of conceptualizing an object and an attribute of it so that the attribute has a unit of measure, and the attribute's measure entails a proportional relationship...with its unit" (p. 37). He goes on to note that:

> Quantification is a process of settling what it means to measure a quantity, what one measures to do so, and what a measure means after getting one. Measurement comes into play *after* the quantification process, although measurement, from the start, is the motive for quantifying a quantity and trial or imagined measurements are involved while amidst the quantification process. (p. 38, emphasis in original)

Therefore, quantification is not the process of simply assigning a value or number to an object's attribute, nor is it the process of measuring. Instead, quantification is the process of identifying an attribute of an object, and then defining a unit of measurement that holds a multiplicative relationship to the attribute in order to assign a value to that attribute.

When looking at angle measurement, or quantifying angularity, students must identify the object, determine the attribute to be quantified, in this case the angle's openness or angularity, and then define a unit of measurement for that attribute. Since there are different conceptualizations of angles, students may identify different attributes to be measured (e.g., the linear length between the two rays' endpoints, or the linear separation of the rays). Therefore, there are different quantifications students may develop as they learn about angle and angle measurement. In his work, Hardison (2018) developed a framework for assessing students' quantifications of angularity—the openness of an angle. In his framework, he examined the different quantities students construct in relation to angle measure, being defined by the openness of the angle, and how students quantified those quantities. By combining Hardison's (2018) quantifications with Piaget's (1965/1952) and Steffe's (1991) explanation of quantifications, there are five different quantifications of angularity: gross, intensive, extensive, ratio, and rate. These quantifications represent the different mental actions and operations used to conceive angle measurement, and therefore represent a progression of schemes for quantifications of angularity.

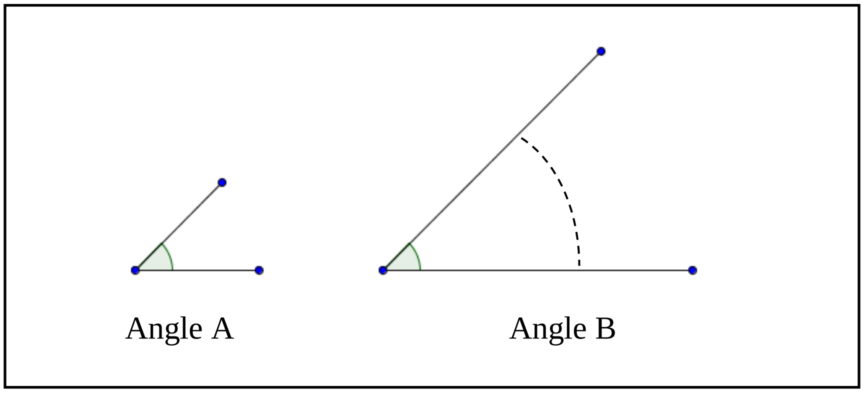
Gross Quantification

A gross quantification is similar to constructing a gross quantity, as described by Piaget (1965/1952). With a gross quantity, "quantification is restricted to the immediate perceptual relationships" (Piaget, 1965/1952, p. 11). This limits students to only being able to look at "relationships expressing 'more', 'equal to', or 'less' immediately perceived in the given qualities" (p. 76). Steffe (1991) further explains Piaget's notion of gross quantity by explaining: "a gross quantitative comparison might be isolated by a child in his or her perceptual field and might exist only temporarily in the immediate here and now" (p. 62). This implies a reliance on sensory and perceptual data. There is also no additive nor multiplicative relationship between the quantities (Piaget, 1965/1952). For example, given the task of the curvy versus straight string in Figure 2.2 as presented by Piaget et al. (1981/1960), children in Stage I often state that the two strings are the same length, even after tracing or stretching the curvy one out. This is an example of a gross quantity since the students simply use visual comparisons of the endpoints. The same is true in the context of angles.

When examining students' understanding of angles, Piaget et al. (1981/1960) found that students who are limited in measuring linear length are also limited in their measuring of angularity. Just like comparing the two strings, a student who has developed a gross quantification for angularity may compare the two angles in terms of how they appear. One way this could be done it to visually compare the two objects to see which one is more open (Piaget et al., 1981/1960). Other students may compare the side lengths of the angles, that is, the length of the rays (Bütüner & Filiz, 2016). For example, given the two 45° angles in Figure 2.13, some students may say that Angle B is larger because its rays extend further than the rays of Angle A. Some students may also compare the size of the arc representing each angle (Bütüner & Filiz, 2016). For example, imagine that the arc denoting Angle B was placed at the dotted arc; students may think Angle B is larger due to the size of that arc being larger. Others may compare the angles by the amount of space between the rays or the size of the wedge (Bütüner & Filiz, 2016). For example, some students may say that the wedge made by Angle B is larger than the one made by Angle A.

Figure 2.13

Two 45° Angles with Different Ray/Side Lengths



Intensive Quantification

Once students no longer rely solely on the figurative material, students can construct an intensive quantity, thus developing an intensive quantification. Steffe (1991) explains that intensive quantities occur once students coordinate two gross quantities. However, these quantities are purely nonadditive (Kieren, 1980). An intensive quantity is “the name given to any magnitude which is not susceptible of actual addition, as for example temperature” (Piaget, 1965/1952, p. 244). Piaget (1965/1952) explains that if a student has a quantity of water with a temperature of 15° and another with a temperature of 25°, the student cannot add these two temperatures to create a mixture that is 40°. The quantity simply offers some measure of comparison. Kieren (1980) further explains that these intensive quantities are “‘measurable’ in the sense that they may be arranged in a series showing differences in degrees of the quantity under consideration” (p. 93). As another example, consider the task of the curvy versus straight string in Figure 2.2 as presented by Piaget et al. (1981/1960). Some children in Stage II focus on both sets of endpoints of the strings, but realize the curved string will be longer due to the curves. Steffe (1991) argues that these children have abstracted the perceptual relationships, and as a result these students can compose one length with the other (Piaget, 1965/1952). However, these quantities from the curvy string and straight string are nonadditive because there is no specific measure for a curve versus a straight segment. Students just know that a curve will have more length because it will take longer to walk a curved path than if walking straight (Piaget et al., 1981/1960).

Although Hardison (2018) does not describe an intensive quantifications of angularity, insight into these quantifications can be obtained by relating the description of intensive quantities to Hardison's work with angles. In the context of angles, Hardison (2018) suggested that students may use a radial sweep to compare the angles (Figure 2.14). A radial sweep "involves the rotation a [sic] single ray (or segment), whose endpoint is fixed at the vertex of the angle, through an angle's interior" (Hardison, 2018, p. 301). After making these sweeps, students compare the duration of those sweeps to see which angle is larger (Hardison, 2018). With this method, students simply determine how long it took for the sweeps to reach the other edge of the angle. This is related to the concept of an angle as a turn and the resulting actions performed by students (Clements et al., 1996; Clements & Burns, 2000; Smith et al., 2014). Although Hardison (2018) provides this radial sweep as an example of a gross quantification, since he did not include an intensive quantification category, I argue that this mental action and motion used by students qualifies as an intensive quantification because students are moving past perceptual comparisons and are relying on the duration of the sweeps. They are "timing" those durations to determine which angle is larger, but these durations are nonadditive because they have no specific measure.

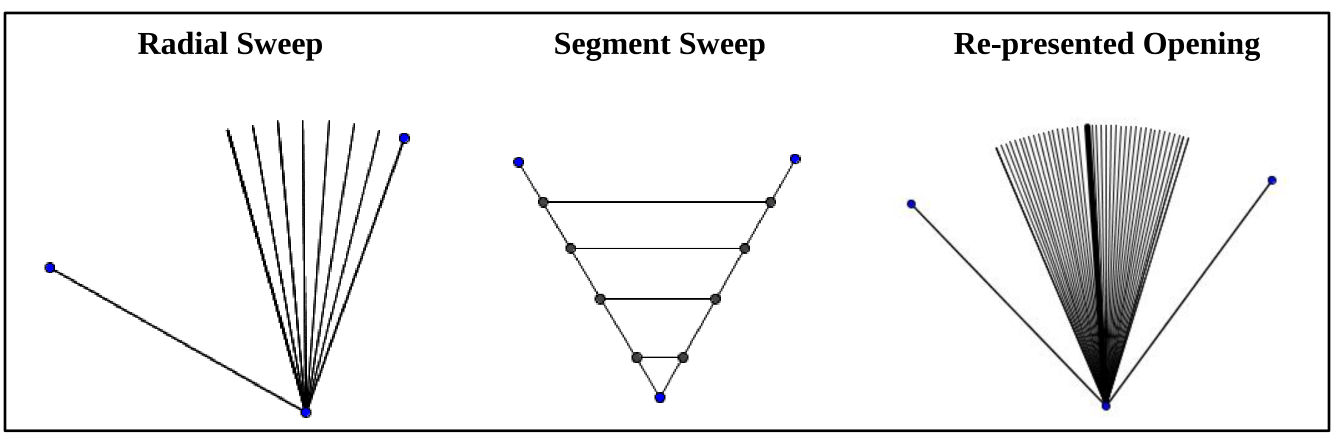
Another method suggested by Hardison (2018) is that students might compare two angles by performing a segment sweep. That is, given two angles, students might imagine drawing segments within an angle, moving out towards the end, and compare which one gets larger faster

(Figure 2.14). Students compare two angles “by comparing the growths of the sweeping segments without any noticeable simultaneous sensorimotor activity” (Hardison, 2018, p. 95). Students are therefore able to establish a relationship between the growth rates of the segments within each angle. Hardison (2018) argues that these students do not necessarily rely on the figurative material or physical acts of drawing out the segments in one angle. Some students might draw the segment sweeps in one angle, and then sequentially imagine drawing them in the other angle (Hardison, 2019). This use of a segment sweep would also be considered indication of an intensive quantification of angularity.

Another motion that students may use in developing an intensive quantification is by re-presenting an opening (Figure 2.14). Re-presentation of an opening is the ability to bring forth a mental image of an angle previously viewed once the perceptual material is absent (Hardison, 2018). This “involves imagining two distinct rays (or segments) opening from a closed position” (Hardison, 2018, p. 301). In these cases, students will recreate the angle’s openness from a closed position using hand motions or physical material.

Figure 2.14

Motions Used by Students as Described by Hardison (2018)

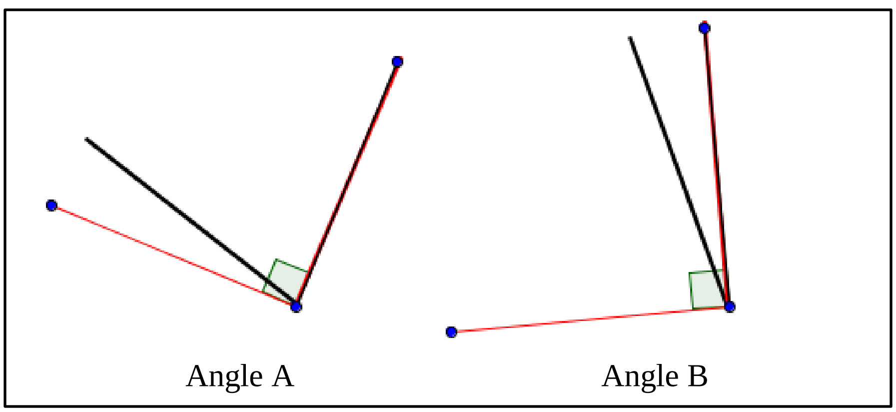


Extending the visual comparisons used in gross quantification, another way that students may compare angles is by the action of superimposition (Hardison, 2018). With superimposition, students place on angle atop the other to determine which one is more open. For example, given Angle A and Angle B in Figure 2.13, a student may place Angle A on top of Angle B and realize that they create the same size angle. However, some students may still claim that the length of the rays for Angle B is longer and the dotted arc is larger.

Finally, students may compare angles to benchmark angles, such as a 90° angle (Hardison, 2018; Hardison & Lee, 2019; Piaget et al., 1981/1960). Given two angles, a student may create a 90° angle on top of both angles. Then the student is able to relate each angle to the 90° angle to determine which angle is larger. For example, given the angles in Figure 2.15, a student may begin by drawing a 90° angle on both angles. Upon completion, a student may say that Angle A is larger because it is closer to making a 90° angle than Angle B.

Figure 2.15

Using 90° Benchmark Angle for Comparisons



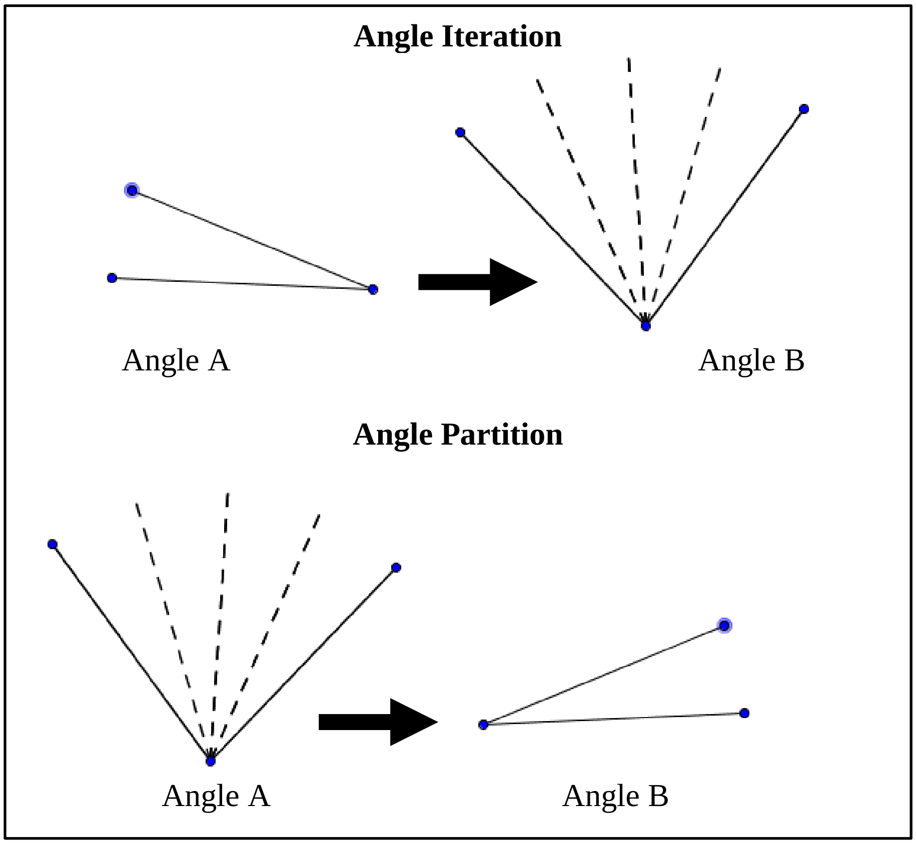
Extensive Quantification

When students are able to think about the relationship between two quantities additively, it is said that students have established an extensive quantification (Kieren, 1980; Piaget, 1965/1952). Because the relationship between the attributes are additive (Kieren, 1980), it is at this moment students can begin introducing units into their quantities (Steffe, 1991). For example, suppose students are given two rows of eight toothpicks, one row that is zig-zagged and made to look shorter, such as the task presented by Piaget et al. (1981/1960). Children in Stage II may move the toothpicks to straighten the zig-zagged one to determine it is indeed longer. Children in Stage III logically know that the length of the two rows of toothpicks are equal because they both have eight toothpicks. They recognize the number of toothpicks as units, and compare eight toothpicks to eight toothpicks to determine the lengths are equal.

Students use similar actions to compare angles. Piaget et al. (1981/1960) suggested that some students may still rely on visual estimates but by intuitively visualizing the comparison through the use of other mental operations. Students are able to develop extensive quantifications through the use of operations such as partitioning and iterating (Hardison, 2018; Piaget et al., 1981/1960). For example, a student may iterate a smaller angle a certain number of times to determine a larger angle is four copies of the smaller angle (Figure 2.16). They understand that by adding four copies of the smaller angle they can create the larger angle. However, they do not recognize the entire multiplicative relationship that Angle B is 4 times the size of Angle A and A is 1/4 units of B.

Figure 2.16

Using Angle Iteration and Partition for Comparing Angles

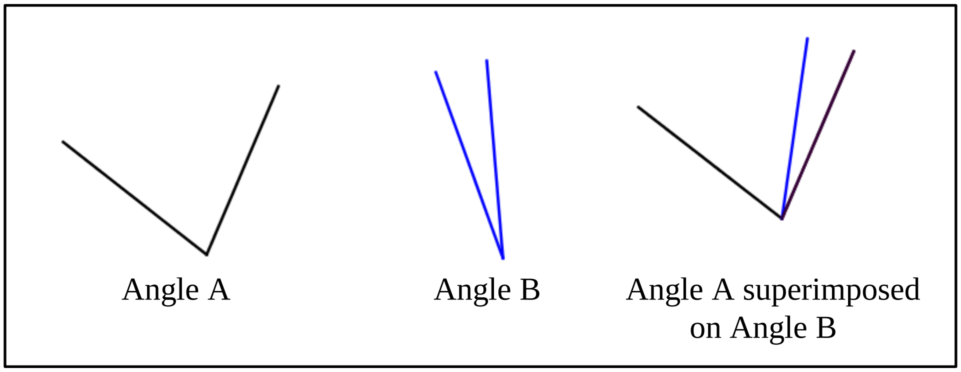


Another student may partition a larger angle into units the same size as the smaller angle to determine the size of the smaller angle. For example given the angles in Figure 2.16, a student may partition Angle A into parts the size of Angle B, thus producing four partitions. From this they may say that it takes four Angle Bs to make Angle A. However, these explanations are based solely on the additive relationship between the angles; students do not understand the multiplicative relationship between Angles A and B.

Other students may extend their superimposition action of placing an angle on top of another to compare their units of openness (Piaget et al., 1981/1960). Instead of visually comparing to determine which angle is larger, students are able to make claims concerning units, such as Angle A is four copies of Angle B (Figure 2.17). Again, students do not recognize the multiplicative relationship between Angles A and B.

Figure 2.17

Using Angle Imposition for Comparing Angles



Another action that students may use is to compare angles to benchmark angles, such as a 90° angle (Piaget et al., 1981/1960). This is different than the visual comparison for intensive quantification in that students make claims involving units. For example, given the angles in Figure 2.15, some students may say that Angle A is about 75° because it is close to 90°, and that Angle B is about 20° because it is close to 0°. In all extensive quantifications, students rely on operations to produce additive units that enable them to compare the openness of the given angles, but do not recognize the multiplicative relationship between the units.

In all of the previous quantifications—gross, intensive, and extensive—students are comparing objects based on attributes either visually, non-additively, or additively (Kieren, 1980). Kieren (1980) argues that with these quantifications, students have not reached measurement yet, but that these types of quantifications are necessary for measurement to take place. However, because measurement is defined by the coordination of subdivision and change of position (Piaget et al., 1980/1961), these students have an initial understanding of angle measurement. Since some of the motions and actions involve partitioning and iterating, it seems they appear to have these foundational operations necessary for measurement.

Ratio Quantification

A ratio quantification comes about once a student is able to recognize a multiplicative relationship between the two quantities being compared (Hardison, 2018; Thompson, 1994). Different from the extensive quantification where students may develop additive relationships between quantities, here students only rely on the multiplicative relationship and understand it in its totality. Thompson (2014) describes this difference as “comparing two quantities additively creates a difference; comparing two quantities multiplicatively creates a ratio” (p. 185). Whereas with an extensive quantification students are producing units that are then applied to both quantities, with a ratio students are able to measure both quantities in terms of one another. For example, given two angles A and B, students can measure Angle A in units of Angle B and can also measure Angle B in units of Angle A. Thompson (1994) also notes that “even though the result is expressed in the same way as what is often called a ‘unit rate,’ the comparison described is between two specific, non-varying quantities, and hence is a ratio comparison” (p. 191).

In the context of angles, a student who demonstrates a ratio quantification may iterate a smaller angle a certain number of times to determine the smaller angle is 1/4 unit of the larger angle, and the larger is 4 units of the smaller (Figure 2.16). They fully understand that the two quantities are multiplicatively related. Another student may partition a larger angle into units the same size as the smaller angle to determine the size of the smaller angle. For example, given the angles in Figure 2.16, a student may partition Angle A into parts the size of Angle B, thus producing four partitions. From this they may say that Angle A is 4 units of Angle B, and Angle B is 1/4 units of Angle A. Other students may extend their superimposition action of placing one angle on top of another to compare their units of openness (Piaget et al., 1981/1960); From this students recognize that Angle B is 1/4 units of Angle A and Angle A is 4 units of Angle B (Figure 2.17). These actions are similar to those described as extensive quantifications, however, these results are multiplicative and not additive.

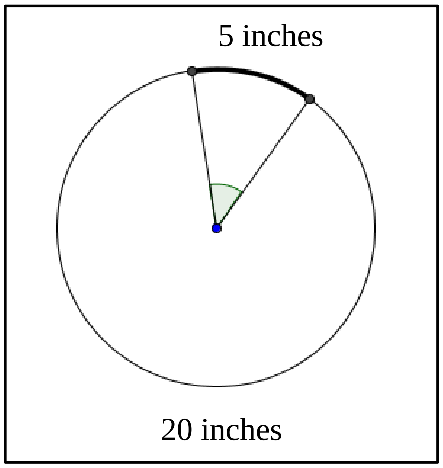
Hardison (2018) further describes that when a student makes a ratio quantification, they make "multiplicative comparisons of circular quantities (e.g., arc lengths)" (p. 45). In this case, students relate angularity to the context of circles. Hardison (2018) gives the following example to help explain this process:

> If an angle is conceived in terms of a radial sweep, he might insert a circular arc into the angular context by imagining the trace of a single point along the rotating ray. If the individual could enact extensive quantitative operations on both an angle and a single circle centered at the angle's vertex, then he might quantify angularity as a ratio. (p. 45)

He goes on to explain that, for example, given the tasks in Figure 2.18, students may recognize that four iterations of the arc will recreate the circumference of the circle. From this, the student can understand that the angle given is 1/4 the full circle. He argues that "such reasoning entailing a multiplicative comparison of an arc length and a circumference would constitute a ratio quantification of angularity" (p. 45). In both of these cases, the student is relying on a multiplicative relationship between the arc length and circumference of the circle. This is similar to Moore's (2013) quantification of openness, where students must have the ability to maintain the relationship between the angle, subtended arc, and circle's circumference. However, it is important to note that this is within the context of one given circle.

Figure 2.18

Example of Ratio Quantification



Rate Quantification

A rate quantification occurs when students are able to extend their ratio quantification into the context of multiple circles, a single varying circle, or a group of concentric circles. Thompson (1994) distinguishes a ratio and rate in the following way: whereas a ratio may only appear as a single quantity, a rate represents the entire structure (Thompson, 1994). The entire structure in this case being the proportional relationship between a circle's radius length, arc length, and circumference (Moore, 2013). Once a rate quantification is established, students can conceive of angularity as "the fractional amount of a circle's circumference subtended by an angle, provided that the circle is centered at the vertex of the angle" (Moore, 2013, p. 227).

Hardison (2018) further describes the rate quantification as being able to understand "the multiplicative comparisons of circular lengths as being invariant across all possible circles" (p. 46). With this notion, students can recognize that no matter the length of the rays, as long as the arcs created by the angles are the same fractional amount of each circle's circumference, then the two angles are congruent. For example, given the angles in Figure 2.12, with a rate quantification, students would be able to recognize that ∠GBF and ∠CBA are in fact congruent angles because the subtended arcs have the same quantitative relationship with respect to each circles' circumference. Students would recognize that as the ray length gets longer, the circle gets bigger, meaning it has a larger circumference; despite this, the relationship between the length of the ray, the length of the arc, and the circumference remains the same proportional relationship. As another example of a rate quantification, suppose students were given the example in Figure 2.19. If students can recognize that each and *every* circle centered at the angle's vertex, no matter what size, would create an arc 1/4 the length of that circle's circumference, then Hardison (2018) argues that these students would demonstrate a rate quantification. Again, students must be able to recognize the multiplicative relationship between the subtended arc length and the circle's circumference across all possible circles.

Once students have a rate quantification, they may use an arc sweep to compare angles. An arc sweep is when students imagine "the interior of an angle being swept out by a circular arc bounded by the sides of an angle and where the circle containing the arc is centered at the vertex of the angle" (Hardison, 2018, p. 329). For example, in Figure 2.20, a student may sweep out an arc, creating different arcs made by concentric circles, and recognize the multiplicative relationship between the radius, length, and circumference.

Figure 2.19

Example of Rate Quantification

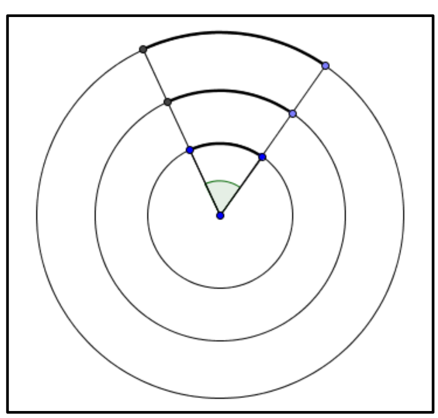
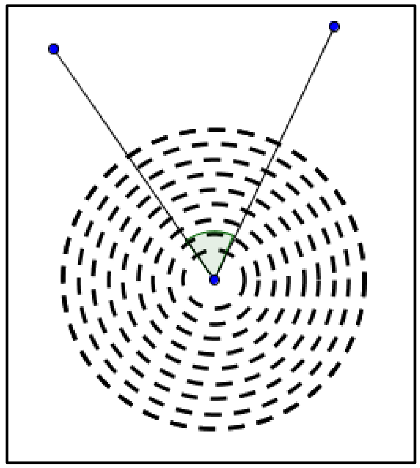


Figure 2.20

Example of Arc Sweep



Summary of Quantifications of Angularity

There are five different quantifications of angularity that can be derived from Piaget's (1965/1952) and Steffe's (1991) explanation of quantifications and Hardison's (2018) quantifications of angularity: gross, intensive, extensive, ratio, and rate. These quantifications help describe students' ways of thinking about angle measurement, and ultimately represent a progression of schemes. A gross quantification represents actions wherein students simply compare angles by visual judgements. An intensive quantification is characterized students moving away from visual judgements and introducing some type of measure for comparing the angles. These measures are not yet units, but are some comparison quantity such as duration of a sweep. An extensive quantification arises once students introduce units into their measures of comparisons. Instead of timing durations, students may use partitions and iterations to count how many times larger Angle A is than Angle B. However, these quantities are not multiplicative. Once students recognize the multiplicative relationship between their quantities of measures, then they have reached a ratio quantification (Hardison, 2018). Students are now able to recognize that Angle B is 1/4 units of Angle A and Angle A is 4 units of Angle B (Figure 2.17).

Finally, once students are able to extend these multiplicative relationships into other contexts, specifically cases involving multiple circles, a single varying circle, or a group of concentric circles, a rate quantification is obtained.

These quantifications can be categorized into two different groups. A gross, intensive, and extensive quantification represents a non-circular quantification (Hardison, 2018, 2019; Hardison & Lee, 2019). With these quantifications, students are making visual judgments or comparisons based on units with the given angle(s). Students do not relate the angles to the context of circles. A ratio and rate quantification represent circular quantifications (Hardison, 2018, 2019; Hardison & Lee, 2019) because students are able to place the angle(s) in the context of circles. Especially with a rate quantification, students must recognize that the relationship between the length of the ray, subtended arc, and circumference is invariant across all possible circles when the circles are centered at the angle's vertex (Hardison, 2018). They must be able to recognize that within multiple circles, a single varying circle, or a group of concentric circles (e.g., Figure 2.12), the angle measurement remains that same, regardless of the size of the circle, as long as the circles remain centered at the angle's vertex.

A New Reorganization Hypothesis Concerning Angles

Considering the two frameworks for measurement schemes for fractions (Wilkins & Norton, 2018) and quantifications of angularity (Hardison, 2018), a connection can be made by examining the underlying operations involved in each scheme and quantification. According to Steffe's (2002) reorganization hypothesis, partitioning and iterating, as well as the ability to coordinate units (Steffe & Olive, 2010), are important and necessary operations for understanding whole numbers, which are then reorganized to understand fractions. Hackenberg extended this reorganization hypothesis from fractions to Algebra (see Hackenberg, 2013;

Hackenberg & Lee, 2015; Lee & Hackenberg, 2014). Similarly, I extend this reorganization hypothesis from fractions to geometry concepts, specifically angle measurement. I argue that the operations used for fractions are reorganized to understand angle measurement. Therefore, I hypothesize that students must first develop a measurement concept, as indicated by their fraction schemes, before they are able to construct more sophisticated quantifications of angularity.

Recall from the measurement schemes for fractions framework, that the schemes form a hierarchical structure from PWS to MSUF to MSPF to GMSF. Furthermore, the partitioning operation is a precursor to the construction of a PWS, iterating is a developmental precursor to MSUF, splitting is a precursor to MSPF, and the coordination of units is a precursor to GMSF. Similarly, the quantifications of angularity framework follows a hierarchical developmental structure: gross, intensive, extensive, ratio, and rate. This developmental structure also builds on the operations of partitioning, iterating, and the ability to coordinate different levels of units. Gross and intensive quantifications rely on visual judgments and do not depend on these operations. It is not until students are able to operate on angles through partitioning and iterating that they construct an extensive quantification of angularity. In other words, students need to be able to partition and iterate in the context of angles, implying those two operations are developmental precursors.

Furthermore, students must reorganize these operations in the context of angles. In particular, students must reorganize what it means to partition or iterate the “openness” of an angle, and not focus only on particular parts of the angle. For example, students who focus solely on the absolute openness, created by the endpoints of the rays, without relating it to the overall angle, have not reorganized these operations to the context of angles. Consider the example of student work in Figure 2.21. This student is focusing on the absolute openness made by the endpoints of the rays. They have taken that openness and iterated that linear length to create their new angle. Now consider the example of student work in Figure 2.22. This student partitions the openness of the angle by partitioning the ray length. Once they have partitioned the length of the angle five times, they draw their new angle the same length as that fifth partition. Notice how this new angle is actually larger than the original angle. In both these cases, there is indication that students cannot transfer the partitioning and iterating operations to the context of angles.

Figure 2.21

Example of Student who Focuses on Absolute Openness

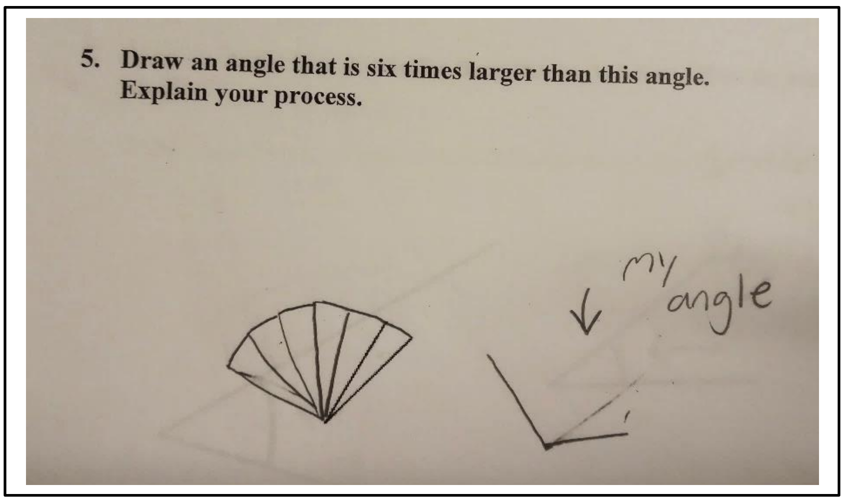
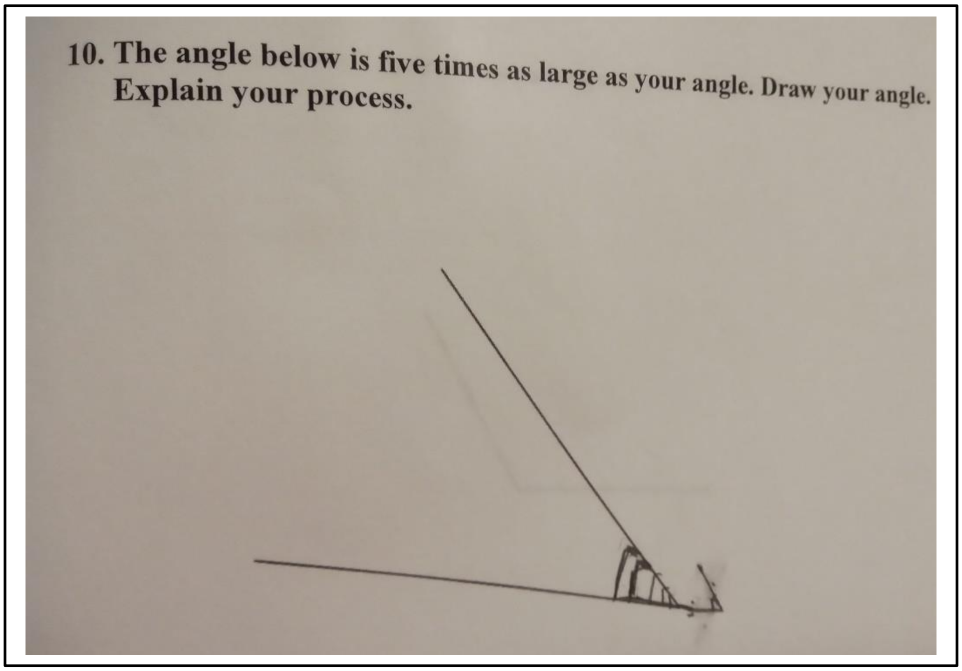


Figure 2.22

Example of Student who Incorrectly Partitions Openness



For a ratio quantification, students rely on the multiplicative relationship between those partitions/iterations and the given angles, therefore splitting needs to be a precursor. Students must first be able to “split” in the context of angles, recognizing what it means to split the “openness” of an angle. Otherwise, they have not reorganized their splitting operations to the context of angles. For example, consider the example of student work in Figure 2.23. This student is focusing on the absolute openness, made by the endpoints of the rays. They have taken that linear openness and partitioned that linear length into five sections. Then, they have taken one of those sections, and drawn their new angle to have the same linear openness. Notice how if we were to extend the length of the rays for this new angle to become the same length as the original angle, it would be about one half the original angle. Now consider the example of student work in Figure 2.24. They have correctly split the openness into five parts, and then drew one of those angles as their new angle. In addition to being able to split, students need to be able to coordinate at least two levels of units to maintain the proportional relationship between the angle, arc, and the circle's circumference.

Figure 2.23

Example of Student who Cannot Split in the Context of Angles

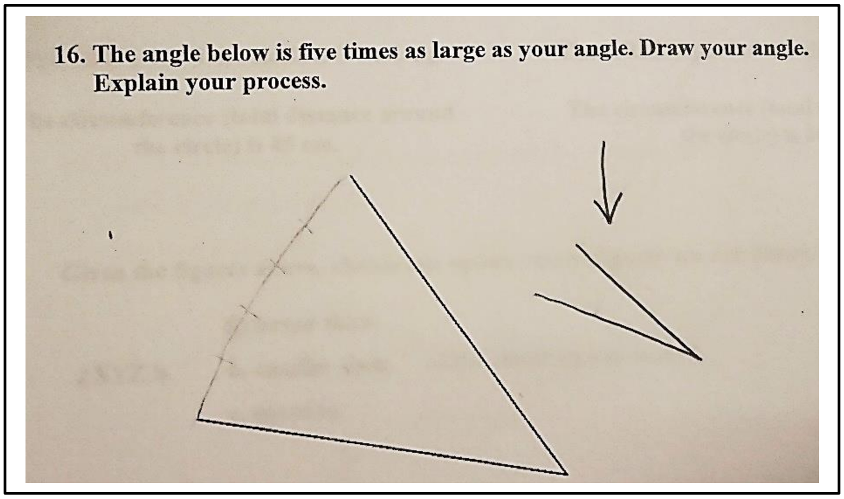
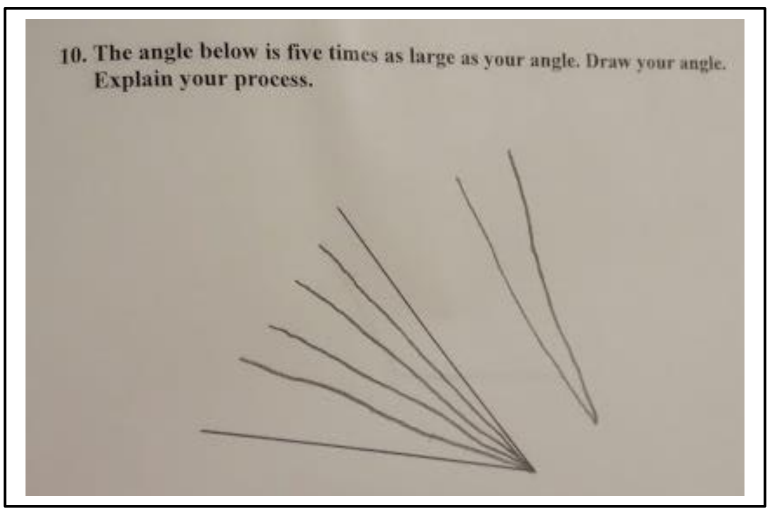


Figure 2.24

Example of Student who can Split in the Context of Angles

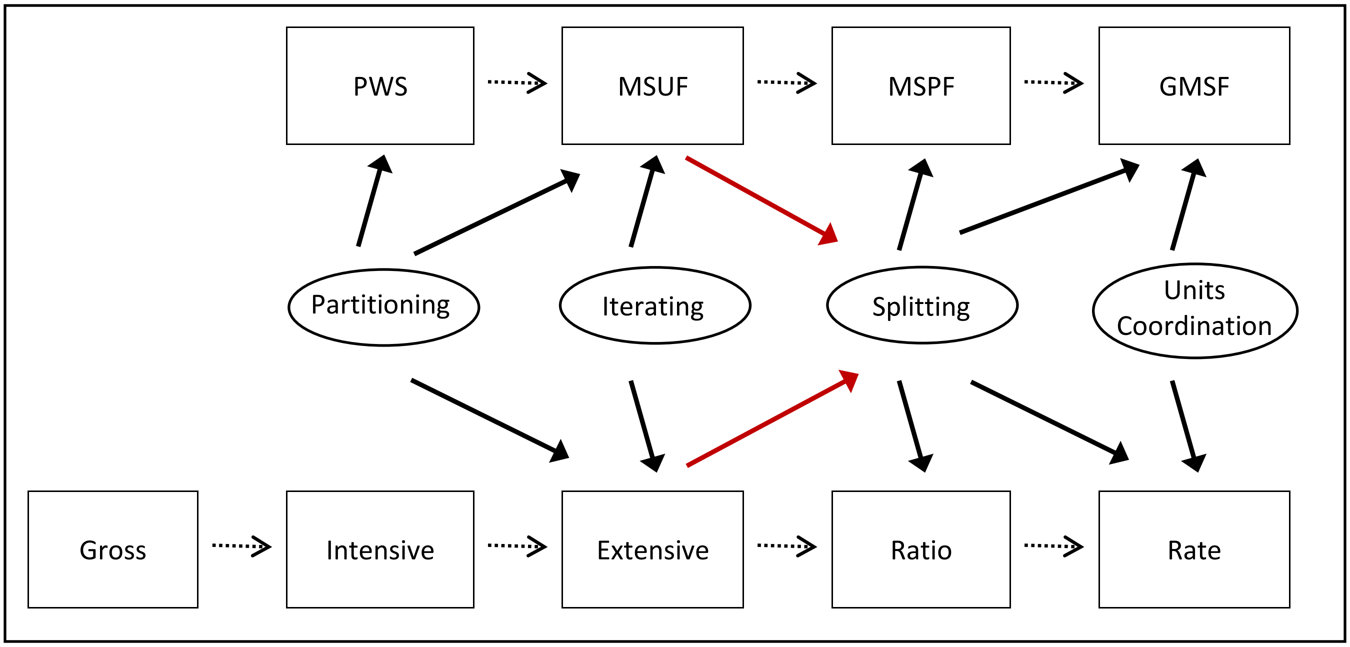


Finally, for a rate quantification, students extend the previous actions and operations used for a ratio quantification (i.e., splitting and two levels of units coordination) into the context of multiple circles. Students also need to maintain the proportional relationship between the arcs, radii, and circumferences of multiple circles, necessitating three levels of units coordination.

In the quantifications of angularity framework, students are reorganizing their fraction operations to use in the context of angle measurement. This connection is visualized in Figure 2.25. The two frameworks (i.e., measurement schemes for fractions and quantifications of angularity), as indicated by the solid boxes, and the underlying operations (i.e., partitioning, iterating, splitting, and units coordination), as indicated by the circles, that tie them together are aligned in a new progression. For example, it is not until after students have developed the splitting operation that they construct a MSPF. Similarly, it is not until after students understand what it means to split the openness of an angle that they construct a ratio quantification. Therefore, it may be possible that a MSPF is necessary for the construction of a ratio quantification. It is also important to note that the coordination of units plays a role in the lower fraction schemes. For example, coordinating two levels of units is required for MSPF. However, to maintain consistency between the two frameworks, the units coordination in the diagram refers to three levels of units coordination. It is also important to note that the dotted arrows indicate a linear progression, that is all stages to the left precede the stages to the right. For example, a PWS precedes a MSUF, and a ratio quantification precedes a rate quantification.

Figure 2.25

A New Reorganization Hypothesis



Relating the underlying operations between the fraction schemes and quantifications of angularity, I hypothesize that there is a connection between these two frameworks, and thus a relationship between the two developmental hierarchies exists. A gross quantification is purely visual and involves no additive nor multiplicative relationship between the quantities (Piaget, 1965/1952). In addition, there are no units involved or related to the quantities being compared. Therefore, this quantification does not require the construction of the aforementioned mental operations. An intensive quantification moves beyond figurative material and involves some nonadditive quantity (Kieren, 1980; Steffe, 1991). However, similar to a gross quantification, there are no units involved, and therefore again does not require the construction of the aforementioned operations. From this description, gross and intensive quantifications do not involve the mental operations of partitioning and iterating, and therefore may be developed without a measurement concept or the construction of more sophisticated measurement schemes for fractions.

The partitioning operation involves the process of breaking a whole into some number of parts to create a quantity (Lamon, 1996, 2012). Once a student has developed the partitioning operation, they are then able to construct a PWS. With a PWS, students are able to partition a whole into parts and then disembed the given part(s) out of the whole (Steffe, 2003; Steffe & Olive, 2010; Wilkins & Norton, 2018). For an extensive quantification of angularity, students begin using additive units for comparing quantities (Kieren, 1980; Piaget, 1965/1952; Steffe, 1991). In describing an extensive quantification, Hardison, (2018) noted that students need to be able to break an angle into some unit they can then use to compare it to another angle, thus requiring the partitioning operation. From these descriptions, the partitioning operation is a precursor to a PWS and an extensive quantification, indicating that students need to construct at least a PWS before developing an extensive quantification. However, iterating is also needed for an extensive quantification.

The iterating operation involves the process of copying a piece or part of a whole to recreate the whole (Hackenberg et al., 2016; Steffe, 2002; Steffe & Olive, 2010). Once a student has developed the iterating operation, they are able to construct a MSUF. With a MSUF, students have established an iterable unit of 1, and recognize the relationship between that unit and the whole, *n*. They understand that the unit can be iterated *n* times to recreate the whole, establishing the notion of a unit fraction. They no longer rely on parts of the whole, but rely on the *n* iterations of the unit to name the fraction. As mentioned earlier, with an extensive quantification, students need to be able to break an angle into some unit, and then iterate the unit to compare it to another angle (Hardison, 2018), thus requiring the iterating operation. Therefore, an extensive quantification requires both the partitioning and iterating operations, which then links it to a

PWS and a MSUF. Therefore, it seems that after a student has constructed a PWS and MSUF they are then able to construct an extensive quantification.

The splitting operation is the simultaneous action of partitioning and iterating. With splitting, students do not have to partition a whole first and then iterate one of those units; they can partition and iterate simultaneously. Once this operation is developed, students are able to reverse their thinking and thus construct a MSPF. With a MSPF, students are able to work with proper fractions by recognizing the relationship between the unit fraction and the whole, and then by iterating the unit fraction to create a proper fraction (Wilkins & Norton, 2018). This requires an inverse, or multiplicative, relationship between the unit fraction and whole, as well as the unit fraction and proper fraction (Wilkins & Norton, 2018). Also, students are able to coordinate three levels of units in activity. With a ratio quantification, students are able to recognize a multiplicative relationship between two quantities being compared (Hardison, 2018; Thompson, 1994). When comparing angles, a student may partition a larger angle into units the same size as the smaller angle and then iterate that unit to determine the relationship between the large and small angle, which represent the underlying actions of the splitting operation. In other instances, students need to maintain the relationship between the angle and subtended arc, as well as the multiplicative relationship between the arc length and circumference of the circle, requiring the coordination of three levels of units in activity. From these descriptions, splitting precedes the construction of a MSPF, and also precedes the construction of a ratio quantification. Furthermore, once students have developed the splitting operation and can coordinate three levels of units in activity within the context of fractions, they can then reorganize these operations in the context of angles. This suggests that the construction of a MSPF is related to the construction of a ratio quantification. However, it is not clear that it would precede the other but may co-develop.

The coordination of units involves building and working with units. With a GMSF, students are able to establish an iterable unit of $1/n$, create a whole using any fraction, and simultaneously coordinate three levels of units. With a rate quantification, students understand the multiplicative structure between angles and circles, that being the proportional relationship between a circle's radius length, arc length, and circumference (Moore, 2013). Therefore, assimilating with three levels of units coordination is a precursor to both a GMSF and a rate quantification, thus providing a link between those two constructs. This suggests that the construction of a GMSF is related to the construction of a rate quantification

Overall, this new reorganization hypothesis links the measurement schemes for fractions to the quantifications of angularity through the underlying operations. To summarize, a gross and intensive quantification of angularity does not involve any of the mental actions and operations associated with measurement, in particular, partitioning and iterating. Thus, these quantifications may develop without a concept of measurement, as indicated by the measurement schemes for fractions. A PWS and MSUF are related to an extensive quantification by the partitioning and iterating operations. The construction of the splitting operation makes it possible for a student to construct a MSPF; similarly, the underlying splitting operation, once reorganized to the context of angles, is required to construct a ratio quantification of angularity. They both would also require students to maintain three levels of units in activity. Furthermore, to move beyond an extensive quantification of angularity students would need to develop multiplicative reasoning with angles. A GMSF is related to a rate quantification by the coordination of units. Although these measurement schemes for fractions are related to the quantifications of angularity, it is unclear if the fractions schemes precede or co-develop with the quantifications of angularity. However, it does appear that students need to develop a concept of measurement, as indicated by the measurement schemes for fractions, before they can develop more sophisticated quantifications of angularity.

From this new reorganization hypothesis, I hypothesize that a concept of measurement precedes the ability to quantify angularity using additive units, and therefore a measurement concept is necessary for the construction of more sophisticated quantifications of angularity. In other words, students need to construct more sophisticated measurement schemes of fractions, indicating a measurement concept, in order to construct more sophisticated quantifications of angularity. For example, a PWS does not represent a measurement concept of fractions. Therefore students with only a PWS would lack the necessary operations and concepts to form more sophisticated quantifications of angularity. However, the other fraction schemes represent a measurement concept of fractions and therefore could be related to more sophisticated quantifications of angularity. Concerning the quantifications of angularity, a gross and intensive quantification do not require a measurement concept of angle. These quantifications are based on visual comparisons and do not involve units, therefore eliminating a requirement for measurement. As a result, they could be constructed by a student with any fraction scheme. However, extensive, ratio, and rate involve the use of units and are therefore more sophisticated; thus students would need a measurement scheme for fractions in order to construct these quantifications. It is important to note that this is not a one-to-one correspondence (i.e., PWS to gross, MSUF to extensive), but rather a conditional relationship. Students need a measurement concept before they can quantify angularity using additive units. This indicates that students' measurement schemes for fractions are positively related to their quantifications of angularity.

Further describing this development, gross and intensive quantifications precede a measurement concept, but a measurement concept precedes the construction of more sophisticated quantifications of angularity (i.e., extensive, ratio, and rate). Therefore, a measurement concept is necessary for the construction of an extensive quantification, implying that students need to construct at least a MSUF, a measurement concept of fractions, to construct an extensive quantification. However, again it is unclear if a MSUF precedes an extensive quantification or if they co-develop. Once students have a measurement concept, indicated by at least a MSUF, they are then able to construct more sophisticated quantifications of angularity. This implies that a MSUF precedes the construction of a ratio and rate quantification. In order to construct a ratio or rate quantification, students need to be able to reason multiplicatively. Therefore, multiplicative reasoning precedes the construction of a ratio quantification, implying that students need to be able to split before moving to a ratio quantification.

In summary, a gross and intensive quantification precede a measurement concept. A measurement concept then precedes more sophisticated quantifications of angularity. Once students have a MSUF, they are able to construct an extensive; once students have multiplicative reasoning, indicated by splitting (note the red arrows in Figure 2.25), they are able to construct a ratio and rate quantification. Again, this is not a one-to-one correspondence, but a hypothesis that a measurement concept is required for more sophisticated quantifications of angularity. Once students have that measurement concept, the fraction schemes and quantifications may co-develop. A student may not have to construct a GMSF to construct a rate quantification, but they do have to have at least the splitting operation. Again, this new reorganization hypothesis links the measurement schemes for fractions to the quantifications of angularity through the underlying operations.

Chapter 3: Methodology

The purpose of this study was to examine the relationship between students' concepts of measurement and their concepts of angle measurement. Specifically, this research was guided by the following question: What is the relationship between middle school students' measurement schemes for fractions and their quantifications of angularity? This study first involved a pilot study to aid in the design of the angularity tasks, and to help assess the validity of students' quantifications of angularity as measured by the tasks. Then, this study used quantitative survey data to assess students' measurement schemes for fractions and their quantifications of angularity and further test for a relationship between these two constructs. Finally, clinical interviews were used to further assess the validity of students' quantifications of angularity based on the devised instrument.

For the validation of students' quantifications of angularity, this study employed a mixed methods approach to better understand students' quantifications of angularity. This mixed methods research design took a sequential explanatory design (Creamer, 2017; Creswell & Plano Clark, 2007), with a quantitative priority. This study was conducted in two phases: Phase One included the quantitative strand, and Phase Two included the qualitative strand. The quantitative data collected was used to examine the relationship between students' measurement schemes for fractions and their quantifications of angularity. The qualitative data was used to confirm and validate inferences from the quantitative data. This chapter will be presented in three sections: an overview of the study, a description of the pilot study, and a detailed explanation of the mixed methods study.

Overview of the Study

This study involved two different investigations, one focused on the validity of students' quantifications of angularity and the other focused on examining the relationship between students' measurement schemes for fractions and quantifications of angularity. These investigations were conducted at different times, and are described in Table 3.1. The first investigation involved a pilot study to initially assess the validity of students' quantifications of angularity based on a new instrument. During this pilot study, students were interviewed and qualitative data was collected. The next component of this study involved the mixed methods study. This was done in two phases: (1) the quantitative strand and (2) the qualitative strand. During this time, students were given both the Measurement Schemes for Fractions Instrument (MSFI) and the Quantifications of Angularity Instrument (QAI). This provided quantitative data concerning students' understanding of fractions and angle measurement. Next, using this quantitative data from both the MSFI and the QAI, data analysis was conducted to answer the research question. Finally, students who participated in these surveys were recruited for interviews focused on their quantifications of angularity. This qualitative data was used to confirm and validate the inferences made from the quantitative data.

Table 3.1

Overview of Study Investigations

	Pilot Study	Mixed Methods	
		Phase One: Quantitative Strand	Phase Two: Qualitative Strand
Purpose	- Assess validity of quantifications of angularity	- Assess students' concepts of fractions and angle measurement - Examine relationship between student's measurement schemes for fractions and their quantifications of angularity	- Assess validity of quantifications of angularity
Data Collection Tool	- Interviews	- Measurement Schemes for Fractions Instrument (MSFI) - Quantifications of Angularity Instrument (QAI)	- Interviews
Type of Data	- Qualitative	- Quantitative	- Qualitative - Quantitative

Pilot Study

The quantifications of angularity tasks were created based on a reviewing tasks in previous literature designed to elicit student thinking concerning angles. This resulted in a total of 24 original angularity tasks (see the "Quantifications of Angularity Instrument (QAI) Description" section for more details). Since there is limited research concerning students quantifications of angularity, and there is no holistic instrument that has been designed and validated to assess students' quantifications of angularity, it was important to determine if these angularity tasks would be useful for assessing student's quantifications of angularity. Therefore, a pilot study was conducted to provide an initial validation of the QAI to ensure that the tasks measured what they were intended to measure (Krupa et al., 2019). Kane (2013) clarifies that "validity is not a property of the test. Rather, it is a property of the proposed interpretations and uses of the test scores" (p. 3). Krupa et al. (2019) further argue that without "a rigorous validation process...claims made based on the measure are worthless, unsubstantiated, and potentially misinforming research" (p. 1). Therefore, it was necessary to test the QAI to ensure that the scores are valid, and in this case they measure students' quantifications of angularity, to ensure more credible claims. The five main components for assessing and ensuring validity presented in the *Standards* (American Educational Research Association [AERA] et al., 2014) are: test content, response processes, internal structure, relations to other variables, and consequences of testing (Krupa et al., 2019). However, because the main purpose of this study was to examine the relationship between student's measurement schemes for fractions and their quantifications of angularity, a small scale validation was conducted during the pilot of these tasks that focus primarily on test content, response processes, and consequences of testing. The detailed results of the pilot study are provided in Chapter 4.

Development of Tasks

First Iteration of Tasks

The QAI was created by reviewing tasks in previous literature designed to elicit student thinking concerning angles. Twenty-four tasks were created to potentially provoke particular ways of thinking that are consistent with the five different levels associated with quantifications of angularity (Appendix A). A test blueprint with examples of student thinking is provided in Table 3.2. These preliminary tasks were designed to elicit the actions associated with each quantification of angularity. These tasks were designed to assess if students possess a certain quantification or not. It is also important to understand that even though students may possess a more sophisticated quantification, it can only be inferred from the tasks if a student does or does not possess a certain quantification. Therefore, tasks were developed to specifically assess each type of quantification to help distinguish which quantification the student possessed.

Table 3.2

QAI Item Blueprint with Example Solutions

Tasks	Purpose	Quantifications of Angularity	Examples of Solutions
1-4	Assess gross quantification	Gross	- Claims angle with shorter rays is smaller -Claims angle with shorter arc is smaller
		Not Gross	- Correctly compares the angles
5-8	Assess intensive quantification	Gross	- Incorrectly draws angle based on size of rays or arc - Incorrectly orders angles
		Intensive	- Correctly draws angle with more, less, or same openness - Uses radial sweep, segment sweep, superimposition, or re-presents opening to prove - Discusses amount of rotation or openness to prove -Correctly orders angles - Uses benchmark angles for comparison
9-12	Assess extensive quantification	Not Extensive	- Incorrectly iterates angle
		Extensive	- Correctly partitions larger angle into smaller angles - Correctly iterates smaller angle to determine how many times it will take to make the larger angle - Superimpose smaller angle to determine how many times it will take to make the larger angle - *Discusses additive relationship between angles*
13-16	Assess splitting operation	Splitting	- Simultaneously partitions and iterates angle to draw their angle
17-20	Assess ratio quantification	Not Ratio	- Incorrectly compares angles - No multiplicative relationship
		Ratio	- Partitions or iterates angle the correct number of times to obtain smaller/larger angle -*Multiplicative explanation*

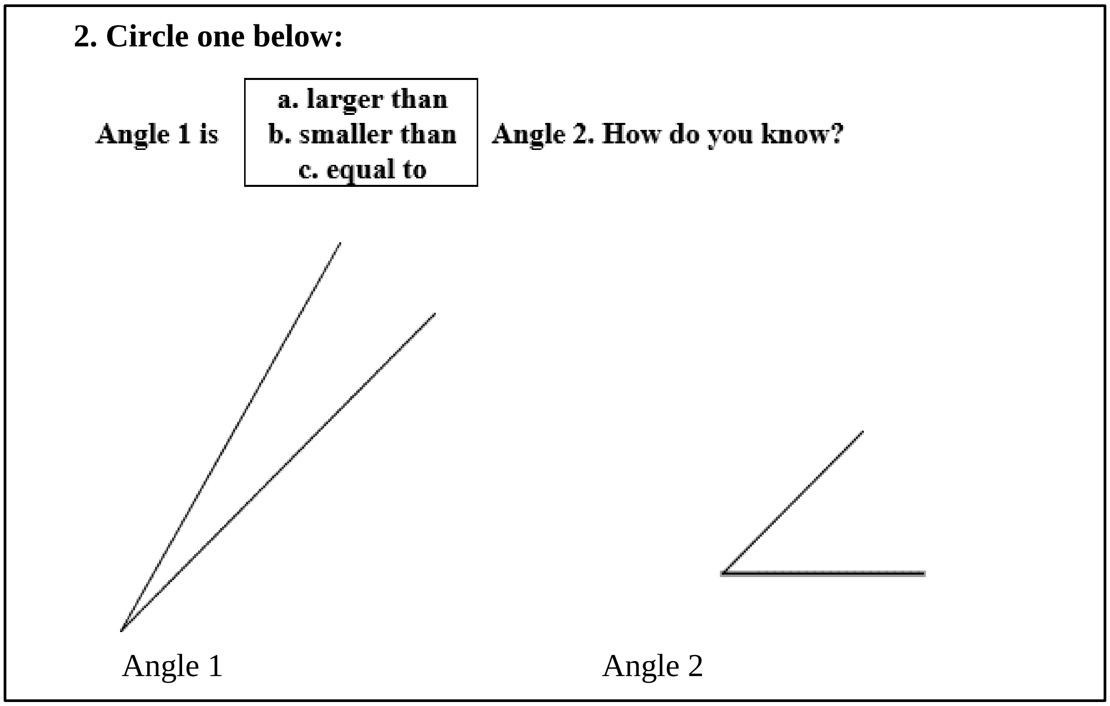
Table 3.2 Continued

Tasks	Purpose	Quantifications of Angularity	Examples of Solutions
21-24	Assess rate quantification	Not Rate	- Incorrectly compare angles - States ∠GBF is smaller than ∠IBD etc. based on length of rays, arcs, sector, or different size circles - States ∠GBF is smaller than ∠IBD based on how fast it grows (segment sweep) - States angles are equal based on superimposition visual comparison - States angles are equal based on same units found during superimposition
		Rate	- States all angles are equal based on concentric circles - States all angles are equal based on proportional relationship between ray, arc, and corresponding circle

To generally describe the QAI, the tasks were grouped based on the type of quantification it was designed to assess. Each task was designed to elicit particular ways of thinking and actions relative to the different types of quantifications. For example, the first four tasks assessed whether a student possessed a gross quantification or not. Consider Task 2 in Figure 3.1. This task purposefully presented the smaller angle with longer rays. A student with a gross quantification would compare angles based purely on visual comparisons, and therefore would claim that Angle 1 is larger. Other tasks within this category presented similar situations, including examples where a larger angle was represented with a smaller arc and two congruent angles were presented with different length rays. If the student answered the questions correctly, it was inferred that the student had constructed a more sophisticated quantification than gross.

Figure 3.1

Task Designed to Assess Gross Quantification of Angularity

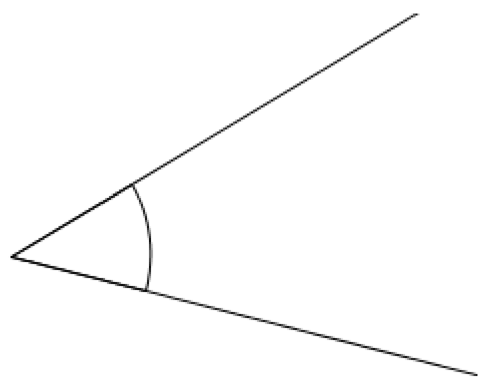


In the next set of tasks (5-8), these four tasks assessed whether a student had a gross or at least an intensive quantification. These tasks focused on how students understand openness and the specific actions used to compare angles. Consider, for example, Task 7 in Figure 3.2. This task asked students to draw an angle that was less open and then prove how it was less open. A student with at least an intensive quantification would correctly draw an angle that was less open, and might use actions such as radial sweep, segment sweep, or superimposition to prove the new angle was less open. Other tasks within this category presented similar situations where students must draw an angle that was more open or had the same openness. The goal behind these tasks was to elicit specific actions for proving the angles had different or the same amount of openness. These tasks also helped assess if a student had a gross quantification, in which they drew angles with shorter/longer ray lengths or smaller/larger arcs, or could not correctly order the angles. Overall, if a student was able to draw an angle that was more open, less open, or the same openness and prove it using a specific method other than visual judgments, it was determined that the student had at least an intensive quantification. If the student was unable to correctly solve the tasks, it was inferred that the student had a gross quantification, provided they demonstrated indications of such quantification in the previous set of tasks. The final task within this group asked students to order a group of angles from smallest to largest. The goal behind this task was to assess if students could use benchmark angles for comparisons. The angles within this task purposefully had short rays for obtuse angles and long rays for acute angles, to help assess whether the student had constructed a gross or at least an intensive quantification.

Figure 3.2

Task Designed to Assess Intensive Quantification of Angularity

7. Draw an angle that is smaller than the angle below. How do you know your angle is smaller?

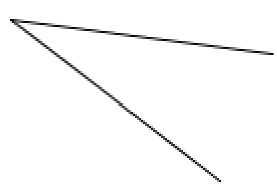


The next four tasks (9-12) helped determine whether a student possessed an extensive quantification or not. These tasks focused on how students compared and related angles to one another. They were also designed to assess whether students used additive thinking. For example, Task 9 in Figure 3.3 asked students to make an angle that was three times bigger than the given angle. Even though this was written in a multiplicative manner, students with an extensive quantification could interpret this in an additive manner. The goal was to determine whether students could correctly partition and iterate angles to create smaller/larger angles. If students created smaller/larger angles by drawing shorter/longer ray lengths or smaller/bigger arcs, then it was inferred that students did not have an extensive quantification. However, if students could properly partition and iterate to create the desired angle, it was inferred that they possessed at least an extensive quantification. Students' explanations were also used to infer their understandings. For example, if a student noted that it took three of the smaller angle to make the larger angle, or something that conveyed an additive nature, it was inferred that they had at least an extensive quantification. However, if students discussed the angles multiplicatively, it was inferred that these students had at least an extensive quantification. Although they may possess a more sophisticated quantification, from these tasks it could only be inferred that they had at least an extensive quantification.

Figure 3.3

Task Designed to Assess Extensive Quantification of Angularity

9. Draw an angle that is three times larger than this angle:
Explain your process:

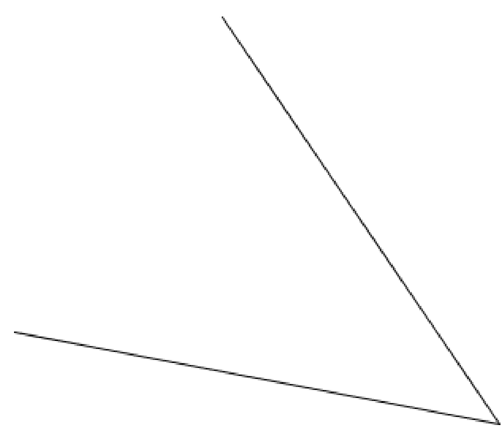


Tasks 13 through 16 assessed students' splitting operation. Although this was not part of the quantifications of angularity framework, these tasks provided a threshold between an extensive and ratio quantification, moving from additive thinking to multiplicative thinking. Also, since splitting is relevant to the measurement schemes for fractions, it was important to include such tasks in the angularity instrument for comparisons. These splitting tasks helped assess if a student was using additive or multiplicative reasoning for their quantifications. For example, given Task 13 (Figure 3.4), students who could split would partition the angle into five parts, and take one of those partitions as their angle, recognizing that iterating it five times would recreate the given angle. If a student could not split, it was inferred that they had not constructed a ratio quantification. If a student could split, it was inferred that they had constructed at least an extensive quantification.

Figure 3.4

Task Designed to Assess Splitting Operation

13. The angle below is five times as large as your angle. Draw your angle: Explain your process:



To assess a ratio quantification, Tasks 17 through 20 were designed to elicit multiplicative reasoning. These tasks asked students to compare the angles in terms of one another and to relate angles to the context of circles. For example, Task 17 (Figure 3.5) asked students to measure Angle 1 in terms of Angle 2, and vice versa. The goal behind this task was for students to partition and iterate the angles for comparison, noting a multiplicative relationship. If students noted that Angle 2 was 1/4 Angle 1, then it was inferred that the student possessed a ratio quantification. Another task within this group, Task 19 (Figure 3.5) gave students a circle with a central angle. They were told the circumference was 48cm and the arc made by the angle was 8cm, and were asked how they could find the measure or size of the angle. It was not intended that students would respond with an answer in degree units, but that students with a ratio quantification would recognize the multiplicative relationship. For example, students may note that since it would take six iterations to recreate the circle, the angle was 1/6 of the circle. If students responded with such explanations, it was inferred that they possessed a ratio quantification. If students did not respond with a multiplicative explanation, it was inferred that they did not possess a ratio quantification.

Finally, four tasks (21-24) were designed to assess a rate quantification. These tasks helped determine if a student recognized that the relationship between the length of the ray, subtended arc, and circumference was invariant across all possible circles when the circles were centered at the angle's vertex. For example, consider Task 21 in Figure 3.6. Students who responded that all the angles were congruent was attributed a rate quantification. Students who responded that they were not equal would not be attributed a rate quantification. For Task 23 in Figure 3.7, students with a rate quantification would recognize that although the circle changed size, the angle remained the same. It was inferred that students who responded that the angle remained the same possessed a rate quantification. It was inferred that students who responded the angle would change did not possess a rate quantification.

Figure 3.5

Task Designed to Assess Ratio Quantification of Angularity

17. Measure Angle 1 in terms of Angle 2. Write your measurement of Angle 1 below. Measure Angle 2 in terms of Angle 1. Write your measurement below of angle 2 below.

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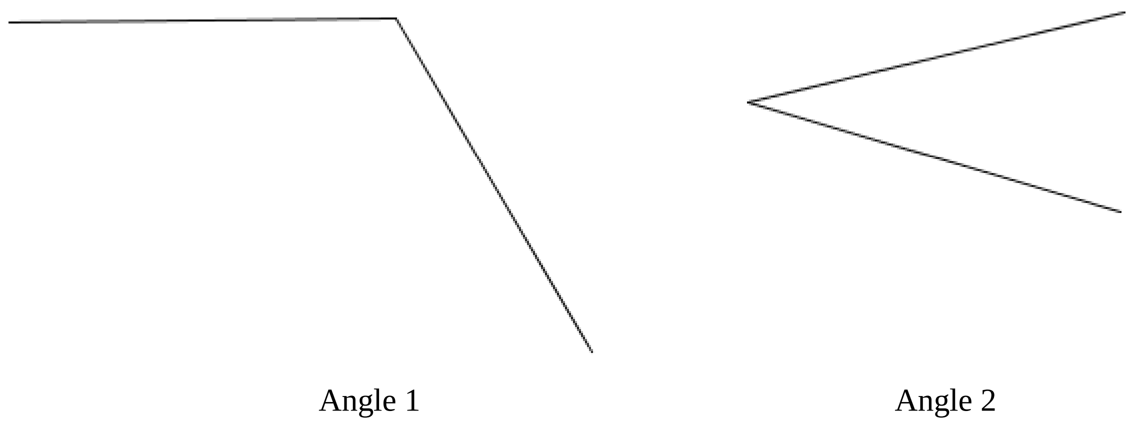


19. The circumference or total distance around the circle is 48cm. The thick arc is 8cm long. Determine the measure/size of the angle in relation to the circle.

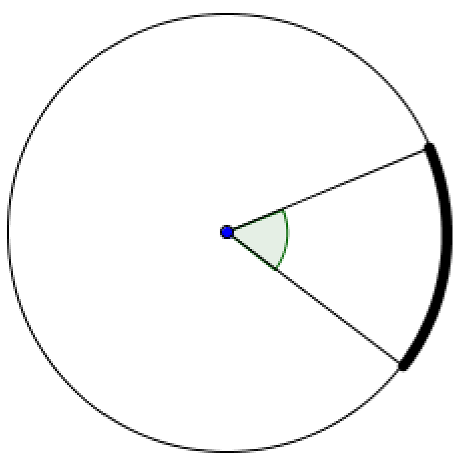
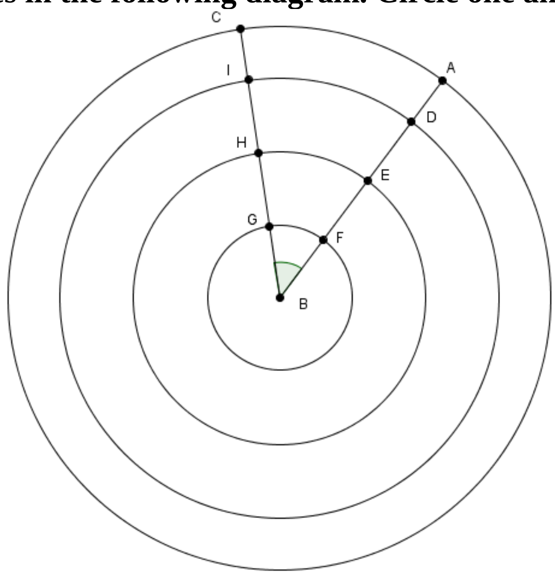


Figure 3.6

Task Designed to Assess Rate Quantification of Angularity

21. Compare the angles in the following diagram. Circle one answer.



∠GBF is
a. larger than
b. smaller than
c. equal to
∠IBD. How do you know?

∠ IBD is
a. larger than
b. smaller than
c. equal to
∠CBA. How do you know?

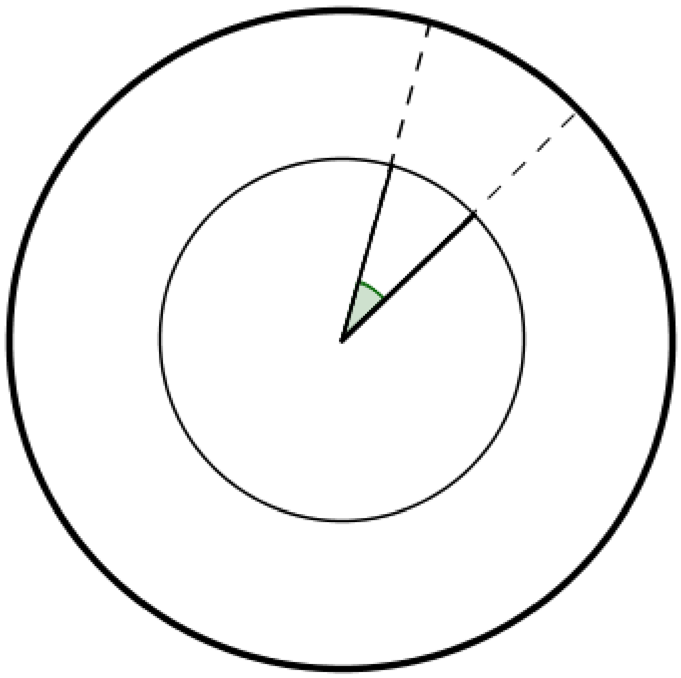
∠HBE is
a. larger than
b. smaller than
c. equal to
∠GBF. How do you know?

∠CBA is
a. larger than
b. smaller than
c. equal to
∠HBE. How do you know?

Figure 3.7

Task Designed to Assess Rate Quantification of Angularity

23. Suppose you are given the angle below. If the smaller circle 1 was enlarged to the dark circle 2, what would happen to the size of the angle?



Once these tasks were developed, they were evaluated by one expert researcher in the mathematics education field, in addition to my own evaluation, to help assess test content, face validity, and the quality of each task (AERA et al., 2014; Krupa et al., 2019). We also determined if the items were balanced, meaning there was an even distribution of the types of tasks concerning each type of quantification. For example, we investigated whether there was an equal amount of extensive quantification items as ratio quantification items. Upon examination, we felt the tasks would provide valid assessments of students' quantifications of angularity, and we deemed them appropriate to use for pilot testing.

First Round of Interviews

After we evaluated the tasks for test content, face validity, and quality of tasks, the tasks were then presented to three students. These students were recruited using a purposeful sampling method (Creswell & Plano Clark, 2007). Students were purposefully recruited from fifth, sixth, seventh, eighth, ninth, and tenth grade. Three students agreed to participate: a seventh grade girl, an eighth grade girl, and a tenth grade boy. After receiving parental consent and student assent, I interviewed each student to ask questions about their methods for solving each task. This helped evaluate response processes, to determine how students actually respond to test questions compared to the hypothesized response (AERA et al., 2014; Krupa et al., 2019). During this time, I investigated whether students thought about the question in the manner that I intended. I looked to see if students interpreted the question in a different way than it is written. Next, I evaluated the consequences of testing (AERA et al., 2014; Krupa et al., 2019) by examining how students responded to the items during the interview. I examined to see if there were any unintended consequences of test bias or failure to represent the intended construct. Finally, these interviews were used to see if the tasks could be used to identify students' quantifications of angularity.

Second Iteration of Tasks

After these initial interviews, the previous expert researcher and myself discussed results of the pilot testing and made changes to the tasks. For example, two of the students stated that the Tasks 17 and 18 did not make sense and suggested rewording it to be more specific. Furthermore, students were also able to easily solve Tasks 21 through 24, even though they did not have a ratio/rate quantification. This suggested that those items were too easy and were not valid for assessing a ratio or rate quantification. A more detailed description of this iteration is provided in the Results section associated with the Pilot Study. These revised tasks are presented in Appendix B.

Second Round of Interviews

Once we addressed all the concerns brought forth from the three initial interviews, I conducted a second round of pilot testing with two more students. These students were again recruited using a purposeful sampling method (Creswell & Plano Clark, 2007). Students were purposefully recruited from fifth, sixth, and seventh grade to provide variability. The two students who agreed to participate were a fifth grade boy and a seventh grade girl. After receiving parental consent and student assent, I interviewed each student to ask questions about their methods for solving each task, including the revised tasks. This helped evaluate response processes, to determine how students actually respond to test questions compared to the hypothesized response (AERA et al., 2014; Krupa et al., 2019). During this time, I investigated whether students thought about the new questions in the manner that I intended. I looked to see if students interpreted the question in a different way than it is written. Next, I evaluated the consequences of testing (AERA et al., 2014; Krupa et al., 2019) by examining how students responded to the items during the interview. I examined to see if there were any unintended consequences of test bias or failure to represent the intended construct. Finally, these interviews were used to see if the new tasks could be used to identify students' quantifications of angularity.

Final Iteration of Tasks

After the second round of pilot testing, revisions to the tasks were again made based on students' responses. For example, students still noted that Tasks 17 and 18 were confusing. Again, a more detailed description of this iteration is provided in the Results section associated with the Pilot Study. Once the tasks were revised, they were given to the original expert researcher in the mathematics education field, along with two more experts in the field of mathematics education. This again helped assess test content and face validity (AERA et al., 2014; Krupa et al., 2019). The three experts evaluated the quality of each task, determined if the items were balanced, and evaluated the tasks for test content. Based on feedback from the experts, additional modifications were made to the overall instrument resulting in 23 total tasks for the QAI. These 23 final tasks were then used for the main study and are included in Appendix C. A table of these revisions between the first and the final iteration is also provided in Appendix D. These tasks and the final instrument are also presented and discussed in the Results section associated with the Pilot Study.

Mixed Methods Study

Radical constructivism supports the notion that knowledge is an individual construction. Thus, it is necessary to gain insight into how students think about and solve mathematical tasks. A mixed methods approach, including both quantitative and qualitative data, provided an opportunity to better examine students' concepts of fractions and angle measurement. Quantitative data was used to help describe students' measurement schemes for fractions and their quantifications of angularity; qualitative data further provided insight into how students think about angularity, and also helped confirm and validate their quantifications of angularity as inferred from the QAI. Oftentimes, it is difficult to tell from survey data exactly how students are thinking about a particular task. There may be evidence that they are operating at a certain level or with a specific scheme; however, upon inquiring into their thinking, they demonstrate different behaviors, which provides different insight into their knowledge. By considering both quantitative and qualitative data, the researcher can better understand the students' ways of thinking and operating.

With a mixed methods approach, the goal is to "enhance validity through *triangulation*" (Creamer, 2017, p. 3, emphasis in original). This triangulation involves "*verification through multiple data points or multiple types of data about the same phenomenon*" (Creamer, 2017, p. 3, emphasis in original). In this case, both quantitative and qualitative data helped with triangulation, to provide a more detailed description of the students' knowledge. Creswell and Plano Clark (2007) argue that "the use of quantitative and qualitative approaches in combination provides a better understanding of research problems than either approach alone" (p. 5). Again, through the lens of radical constructivism, it is imperative that the researcher enter the mind of the learner (Zazkis & Hazzan, 1998) to better understand their ways of thinking, and not to solely rely on quantitative survey data. A mixed methods approach integrates both quantitative and qualitative data to allow "the analyst to develop a richer, more analytically dense, more complete, and confidently argued response to their research question(s) (Bryman, 2006; Fetters, Curry, & Creswell, 2013; Fielding, 2012)" (Bazeley, 2018, p. 12). This integration occurs before the researcher makes final conclusions (Bazeley, 2018; Creswell & Plano Clark, 2007). This linkage of quantitative and qualitative data is known as mixing (Creamer, 2017; Creswell & Plano Clark, 2007). Mixing can occur at any phase of the study (i.e., design, data collection, sampling, analysis, or inferences; Creamer, 2017), but needs to occur before the final conclusions are made to offer better insight into the problem (Bazeley, 2018; Creswell & Plano Clark, 2007).

In the present study, mixing occurred at the data collection, sampling, analysis, and inferences phases. This study also took a sequential explanatory design (Creamer, 2017; Creswell & Plano Clark, 2007), where the quantitative data was given priority. First, quantitative survey data was collected, and then qualitative data was collected through student interviews, to help explain the survey data and validate inferences from the survey data. Also, during data collection, nested sampling (Creamer, 2017) was used for qualitative interviews. Students who participated in the interviews were a subsequent subgroup of those who participated in the quantitative survey. Mixing was achieved in the analysis phase through the sequential analysis of the quantitative and qualitative data. These analyses were then compared, and relationships, patterns, and trends between them were identified (Creamer, 2017; Creswell & Plano Clark, 2007). This allowed the researcher to weave the two data strands together to make one cohesive argument, and in this case, an argument about the validity concerning students' quantifications of angularity. Thus, mixing in the inference phase occurred through the development of a meta-inference related to validity. Meta-inferences are "inferences that link, compare, contrast, or modify inferences generated by the qualitative and quantitative strand" (Teddlie & Tashakkori, 2009, p. 300; as cited by Creamer, 2017, p. 15). By mixing the quantitative and qualitative data, meta-inferences lead to more credible conclusions (Creamer, 2017; Creswell & Plano Clark, 2007), since the conclusions are better supported by both strands of data. In this case, by weaving together the quantitative and qualitative data, a meta-inference about the validity of students' quantifications of angularity could be gained. In summary, this study was conducted in two phases. The first phase focused on gathering survey data from students. Phase Two focused on obtaining qualitative data through student interviews to enhance understanding of students' thinking and to validate inferences from survey data.

Phase One: Quantitative Survey Study

During Phase One, quantitative survey data was collected through two different instruments. One instrument was used to assess students' fractions schemes, the MSFI, and another instrument was used to assess their quantifications of angularity, the QAI. The purpose of using the MSFI was to assign a measurement scheme for fractions to each student (see Norton & Wilkins, 2009, 2012; Wilkins & Norton, 2011; Wilkins et al., 2013). The purpose of using the QAI was to assign a quantification of angularity to each student. These assignments were then used for statistical analysis to determine if there is a relationship between them. The data from the QAI was later used in conjunction with student interviews to assess validity.

Participants

Participant selection was couched in a purposeful sampling method (Creswell & Plano Clark, 2007). Participants included sixth, seventh, and eighth grade students at one middle school. The purpose of recruiting sixth, seventh, and eighth grade students was that some of them have not had a formal Geometry course at this point in middle school, but most have had some type of instruction centered on angles. For example, students begin learning about angles as the intersection of two rays or line in first grade, which is then carried through fourth grade (VDOE, 2016b). In fifth grade students learn about angle measurement as a fractional amount of a full rotation or a circle and focus on angle measurements in degrees (VDOE, 2016a). I contacted a local school district and obtained permission from the superintendent to conduct data collection within that specific district. According to Virginia Tech Institution Review Board (IRB) protocol, superintendent approval was necessary before obtaining IRB approval. After obtaining superintendent permission and IRB approval, I then contacted the principal of one middle school and requested permission to conduct data collection.

This middle school is a small school, located in the rural Southeast. It houses sixth through eighth grade students. The highest level of math offered at the middle school was Algebra; three students enrolled in Geometry were bused to the high school to take their class. The middle school also operated on a block schedule, where students took four classes each day, each class lasting 90 minutes. This school had 100% pass rate the past three years for the Algebra I test (VDOE, 2019). However, the school had an average pass rate of 76% for sixth grade mathematics, 71.6% for seventh grade mathematics, and 74.3% for eighth grade mathematics (VDOE, 2019). About 50% of these students in this middle school were economically disadvantaged (VDOE, 2020). In terms of student demographics, 85% of students were White non-Hispanic, about 4% were Black, 3% were Hispanic, and less than 1% were Asian (VDOE, 2020). During the 2019-2020 school year that data was collected, there were a total of 501 students enrolled in the school. Of these 501 students, 176 were sixth graders, 158 were seventh graders, and 167 were eighth graders (VDOE, 2020).

After obtaining permission from the principal, I recruited one teacher from each grade level, resulting in one sixth grade teacher, one seventh grade teacher, and one eighth grade teacher. This resulted in two sixth grade math classes, four seventh grade math classes, and three Algebra classes (Table 3.3). The Algebra classes include both seventh and eighth grade students, but were mostly eighth grade students. Participants were then recruited from these teachers' classrooms, giving a possible sample of 180 students. The three teachers agreed to incorporate the instruments as normal classroom activity and data was collected during students' regularly scheduled math class.

Table 3.3

Total Numbers of Teachers and Classes Recruited for Phase One

Teacher	Block	Class	Number of Students
1	1st	7th grade "gifted" math	15
	2nd	6th grade math	25
	3rd	6th grade math	18
2	1st	Algebra	15
	3rd	Algebra	20
	4th	Algebra	23
3	1st	7th grade math	22
	2nd	7th grade math	22
	4th	7th grade math	21
Total			**180**

A week before data was collected, parent information forms were sent home, to allow parents the option to opt students out of their data being included as part of the overall dataset. On the day of data collection, student assent was also obtained, giving students the option to opt out of having their data included in the dataset. Only students who agreed to participate, and whose parents did not opt out, were included in the study. Students were also given instructions to provide some indication that they attempted a task, even if they were unable to provide a solution. For example, some students drew stars beside tasks they did not know. This was then used to determine if students attempted to solve the tasks or did not complete any tasks. To determine which students were included in data analysis, only students who had parental permission, gave student assent, and did not skip more than two pages on the surveys were included. Students who did not solve any tasks or skipped more than two pages on the surveys were removed from the study. The total number of students included in this study are described in Table 3.4 and Table 3.5.

Overall, 155 students were included in this study. This included 152 students who completed the MSFI and 152 students who completed the QAI. It is important to note that the 152 students who completed the MSFI were not be the same 152 students who completed the QAI. In relation to the MSFI, 24.3% of students were enrolled in sixth grade, 50.7% were enrolled in seventh grade, and 25% were enrolled in eighth grade. Additionally, 24.3% of students were enrolled in Math 6, 45.4% were enrolled in Math 7, and 30.3% were enrolled in Algebra. For the QAI, 23.7% of students were enrolled in sixth grade, 49.3% were enrolled in seventh grade, and 27% were enrolled in eighth grade. Furthermore, 23.7% of students were enrolled in Math 6, 44.1% were enrolled in Math 7, and 32.2% were enrolled in Algebra. From the 152 students included in this study, 54.6% were boys and 45.4% were girls.

Table 3.4

Total Numbers of Students who Participated in Phase One

Course	Possible Number of Students	Instrument	Parent Opt Out	Student Opt Out	Total Surveyed*
Math 6	42	MSFI	2	3	37
		QAI	2	3	36
Math 7	80	MSFI	0	2	69
		QAI	0	2	67
Algebra	58	MSFI	7	2	46
		QAI	7	2	49
Total	**180**	MSFI			**152**
		QAI			**152**

* Some students were absent and did not complete a survey. Thus, the total number of students surveyed may be different.

Table 3.5

Student Demographics

Instrument	Grade	Course	Gender	Total Surveyed	Total
MSFI	6	Math 6	Boy Girl	18 19	37 (24.3%)
	7	Math 7	Boy Girl	34 35	77 (50.7%)
		Algebra	Boy Girl	4 4	
	8	Algebra	Boy Girl	13 25	38 (25%)
QAI	6	Math 6	Boy Girl	17 19	36 (23.7%)
	7	Math 7	Boy Girl	33 34	75 (49.3%)
		Algebra	Boy Girl	4 4	
	8	Algebra	Boy Girl	15 26	41 (27%)
Totals			**Boys** **Girls**	**69 (45.4%)** **83 (54.6%)**	**152**

Measurement Schemes for Fractions Instrument (MSFI) Description

Each student who was present on the first day of data collection was given the MSFI. The measurement schemes for fractions tasks were designed by Norton and Wilkins (see Norton & Wilkins, 2009, 2012; Norton et al., 2018; Wilkins & Norton, 2011; Wilkins et al. 2013) and partially validated for assessing students' ways of operating with fractions (Wilkins et al., 2013). These tasks included 28 items that are grouped into 4 tasks per operation or scheme. For example, there are four items for each operation: splitting and units coordination. There are also four items for each fraction scheme: four items for the PWS, four items for PUFS, four items for PFS, four items for RPFS, and four items for IFS. Table 3.6 presents the breakdown of the survey by fraction operation/scheme and the number of items used for the MSFI.

Table 3.6

Fraction Operation/Scheme Item Breakdown for the MSFI

Measurement Schemes for Fractions Tasks	
Fraction Scheme/Operation	Number of Items
PWS*	4
PUFS*	4
Splitting*	4
PFS	4
RPFS*	4
Units Coordination	4
IFS*	4
Total	**28**

*Indicates tasks included in the MSFI (Total of 20)

However, because the measurement schemes for fractions rename and also combine some of these schemes (Wilkins & Norton, 2018), only the PWS, PUFS, splitting, RPFS, and IFS tasks were used. This means that a total of 20 tasks were used in the MSFI. The PWS tasks assess students' construction of a PWS. The PUFS tasks assess students' construction of a MSUF. The RPFS tasks assess students' construction of a MSPF. The splitting tasks also help assess the possession of a MSPF, since splitting is a developmental precursor to that particular scheme. Finally, the IFS tasks assess students' construction of a GMSF.

In addition, two different forms of the instrument were created and used in data collection. The same items were used in both forms of the survey, just in a different order. This helped control for testing effects.

Fraction Scheme Coding

I followed the coding scheme outlined by Wilkins and Norton (2011). In this coding scheme, each measurement scheme for fractions task was scored with a 1–demonstrate strong evidence of operating in alignment with scheme or operation, or a 0–demonstrate no evidence of operating in alignment with scheme or operation. For tasks in which students show some indication of compatibility with scheme, but do not fall within either of these categories, scores of .4 and .6 were given to indicate a leaning towards more compatibility or less compatibility with the scheme (Wilkins & Norton, 2011). Two coders independently scored each student's survey, scoring each item individually. These scores were then summed to provide an overall score for each operation or scheme. By grouping similar tasks related to a particular fraction scheme or operation, the individual task scores were summed to calculate a total scheme or operation score (Wilkins & Norton, 2011). For each grouping containing four tasks, if a student had a score greater than or equal to 3 for each category, it was inferred that they possess that specific operation or scheme and was coded as 1 (Wilkins & Norton, 2011). If a student had a score less than or equal to 2, it was inferred that the student did not possess that particular operation or scheme and was coded as 0 (Wilkins & Norton, 2011). For students with scores between 2 and 3, their work was reviewed holistically for that operation or scheme, and each individual researcher made a final decision concerning whether they possessed the operation or scheme (Wilkins & Norton, 2011). In summary, each fraction scheme (i.e., PWS, MSUF, MSPF, GMSF) was coded 0 or 1 to indicate whether a student had constructed the scheme or not. A summary of the coding rubric for the operations/schemes is provided in Table 3.7.

Table 3.7

Sample Coding Rubric for Overall Fraction Scheme Score

	Initial Coding			
Each Fraction Task Score	0– indicates no action constructed in relation to scheme/operation	.4– leaning more towards no indication (less compatibility)	.6– leaning more towards indication (more compatibility)	1– indicates action constructed in relation to scheme/operation
Sum of 4 tasks	Sum less than or equal to 2 for each category, student does not possess operation/scheme	Sum between 2 and 3, will be reviewed and then coded as 0 or 1		Sum greater than or equal to 3, student possess operation/scheme
Overall Code	0– does not possess			1– does possess

This coding resulted in a matrix of binary codes for each student, regarding each measurement scheme for fractions (i.e., PWS, MSUF, MSPF, GMSF; Table 3.8). For example, suppose Student 1 had a sum greater than 3 for the PWS and MSUF tasks, but a score less than 2 for splitting, MSPF, and GMSF tasks. They would be given a binary code of 1 for PWS and MSUF, but a 0 for the others. Using this information each student received an ordinal code based on their overall scores for each scheme. This code was generated based on the highest scheme they possess. If a student received a code of 0 for all of the schemes, they were given a code of 0, and attributed PrePWS. This coding scheme is provided in Table 3.9.

Table 3.8

Example Matrix Coding Scheme for Measurement Schemes for Fractions

	Each Scheme Code				**Overall Scheme Code**
Student	PWS	MSUF	MSPF	GMSF	
1	1	1	0	0	**2**
2	1	0	0	0	**1**
3	1	1	1	0	**3**
4	0	0	0	0	**0**

Table 3.9

Ordinal Coding Scheme for Each Measurement Scheme for Fractions.

Overall Scheme Code				
0–PrePWS	1–PWS	2–MSUF	3–MSPF	4–GMS

Quantifications of Angularity Instrument (QAI) Description

Each student who was present on the second day of data collection was given the QAI. The QAI was created by reviewing tasks in previous literature designed to elicit student thinking concerning angles, and was revised based on the results of the pilot study (Chapter 4). A total of 23 tasks were used in the main study (Appendix C). A test blueprint of the final instrument is provided in Table 3.10.

These tasks were designed to assess students quantifications of angularity. We determined that the initial "gross" tasks did not validly provide indication of a gross quantification of angularity, since students would have to answer the questions incorrectly. This was inconsistent with the scoring of the other tasks. To maintain the consistency in the scoring the of tasks, the final version of the tasks did not include tasks designed to assess a gross quantification. A gross quantification of angularity was inferred if a student did not provide indication of a more sophisticated quantification. For example, if a student missed the intensive tasks we inferred they had a gross quantification.

In addition, an examination of the quantifications of angularity framework revealed a large jump between the extensive and ratio quantification. The idea that students jump directly from using additive units to using multiplicative units in relation to a circle seemed like a stretch. Therefore, we felt it was necessary to include two tasks to help determine if students could use multiplicative units, just not in relation to circles. Furthermore, to help align the measurement schemes for fractions with the quantifications of angularity, and to help understand students' transition from additive to multiplicative thinking about angles, we also included four splitting tasks in the context of angles. These 23 tasks were purposefully developed to specifically assess each stage of quantification and to help distinguish which quantification each student possessed. In addition, two different forms of the instrument were created and used in data collection. The same items were used in both forms of the survey, just in a different order. This helped control for testing effects.

Table 3.10

QAI Item Blueprint with Example Solutions

<table>
<tr><th>Tasks</th><th>Purpose</th><th>Quantifications of Angularity</th><th>Examples of Solutions</th></tr>
<tr><td rowspan="2">1-4</td><td rowspan="2">Assess intensive quantification</td><td>Gross</td><td>- Incorrectly draws/compares angle based on size of rays or arc</td></tr>
<tr><td>Intensive</td><td>- Correctly draws/compares angles with more, less, or same openness
- Uses radial sweep, segment sweep, superimposition, or re-presents opening to prove
- Discusses amount of rotation or openness to prove
- Uses benchmark angles for comparison</td></tr>
<tr><td>5</td><td>Determine how students compare angles</td><td>N/A</td><td>-Line is the largest vs. Circle is the largest
-First angle is smallest because it is skinny vs. First angle is largest because it is long</td></tr>
<tr><td rowspan="2">6-9</td><td rowspan="2">Assess extensive quantification</td><td>Not Extensive</td><td>- Cannot partition/iterate angle</td></tr>
<tr><td>Extensive</td><td>- Partitions larger angle into smaller angles
- Iterates smaller angle to determine how many times it will take to make the larger angle</td></tr>
</table>

Table 3.10 Continued

Tasks	Purpose	Quantifications of Angularity	Examples of Solutions
			- Superimpose smaller angle to determine how many times it will take to make the larger angle - *Discusses additive relationship between angles*
10-13	Assess splitting operation	Splitting	- Simultaneously partitions and iterates angle to draw their angle
14-15	Assess multiplicative thinking	Multiplicative	-Determine angles are fractions of one another (1/4 smaller and 4 times larger)
16-19	Assess ratio quantification	Not Ratio	- No multiplicative relationship
		Ratio	- Recognizes multiplicative relationship
20-23	Assess rate quantification	Not Rate	- Incorrectly compare angles - States angles are equal based on superimposition visual comparison - States angles are equal based on same units found during superimposition -No multiplicative relationship
		Rate	- Correctly compares angles based on proportional relationship between ray, arc, and corresponding circle -Recognizes multiplicative/proportional relationship

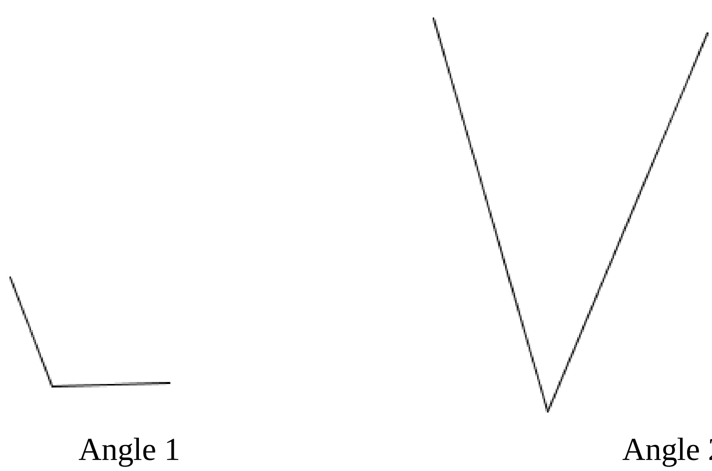
To generally describe the instrument, the tasks were grouped based on the type of quantification they were designed to assess. Each task was designed to elicit particular ways of thinking and actions relative to the construction of the different stages of quantifications. For example, the first four tasks assessed whether a student had constructed an intensive quantification or not. Consider Task 2 in Figure 3.8. This task purposefully presented the smaller angle with longer rays. A student with a gross quantification would compare angles based purely on visual comparisons, and therefore would claim that Angle 1 is larger. A student with an intensive quantification would compare the angles using some non-additive measure, and state that Angle 1 was larger than Angle 2. Other tasks within this category present similar situations, including examples where a larger angle was represented with a smaller arc and two congruent angles were presented with different length rays. If the student answered the questions correctly, it was inferred that the student had constructed at least an intensive quantification.

Figure 3.8

Task Designed to Assess an Intensive Quantification of Angularity

1. Circle one option below:

	a. larger than	
Angle 1 is	**b. smaller than**	**Angle 2. How do you know?**
	c. equal to	

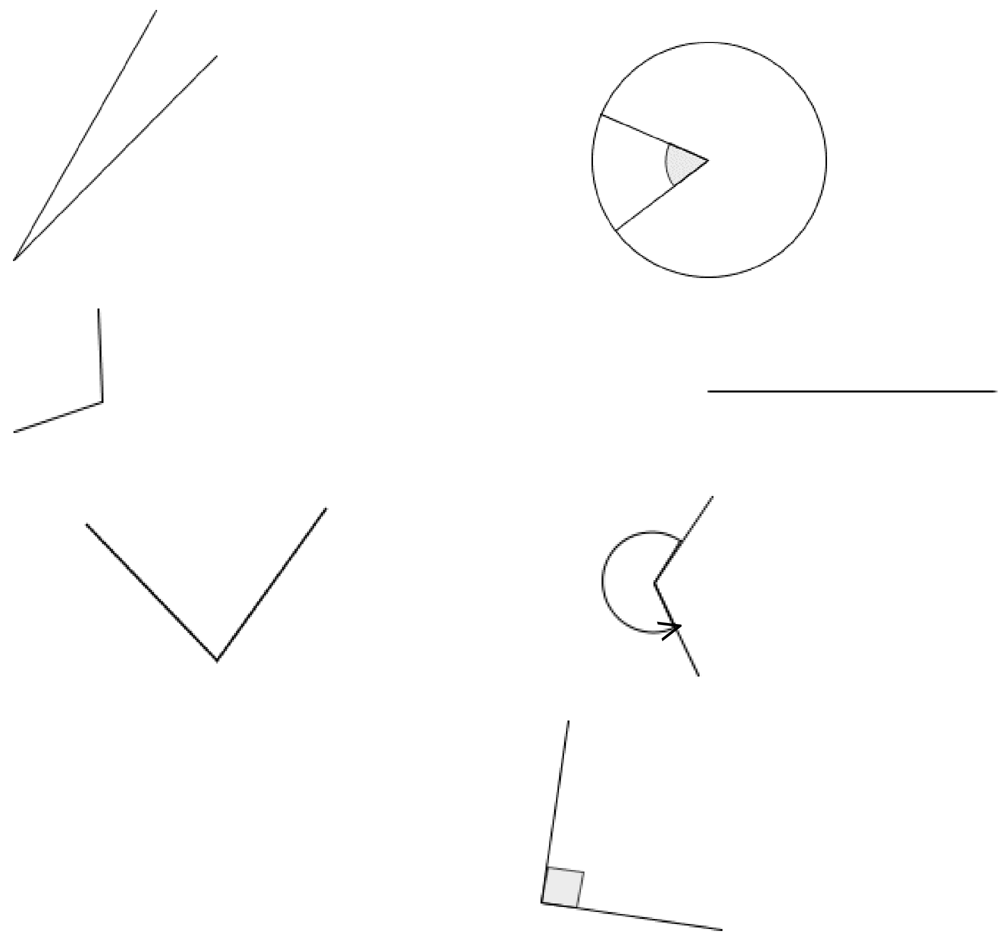


The next task, Task 5 asked students to order angles. This task helped determine how students were thinking about angles and how they were able to compare them. For example, given Task 5 in Figure 3.9 a student with a gross quantification would most likely say that the first angle was the largest because the rays are the longest. A student with an intensive quantification would likely be able to correctly order most of the angles using some measure for openness. This item was not included in the scoring to determine students' quantifications of angularity.

Figure 3.9

Task Designed to Assess Students' Comparisons

5. **Order the angles below from smallest to largest (1-7). Label the SMALLEST ANGLE with a 1 and label the LARGEST with a 7. Explain your process.**

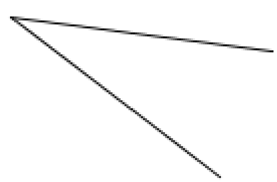


The next four tasks (6-9) helped determine whether a student possessed an extensive quantification or not. These tasks focused on how students compared and related angles to one another. They were also designed to assess whether students used additive thinking. For example, Task 6 in Figure 3.10 asked students to make an angle that was three times bigger than the given angle. Even though this was written in a multiplicative manner, students with an extensive quantification could interpret this in an additive manner. The goal was to determine whether students could correctly partition and iterate angles to create smaller/larger angles. If students created smaller/larger angles by drawing shorter/longer ray lengths or smaller/bigger arcs, then it was inferred that students did not have an extensive quantification. However, if students could properly partition and iterate to create the desired angle, it was inferred that they possessed at least an extensive quantification. Students' explanations were also used to infer their understandings. For example, if a student noted that it took three of the smaller angle to make the larger angle, or something that conveyed additive thinking, it was inferred they had constructed at least an intensive quantifications. However, if students discussed the angles multiplicatively, it was inferred that these students had constructed at least an extensive quantification. Although they may have constructed a more sophisticated quantification, from these tasks it could only be inferred whether they had at least an extensive quantification.

Figure 3.10

Task Designed to Assess an Extensive Quantification of Angularity

**6. Draw an angle that is four times larger than this angle.
Explain your process.**

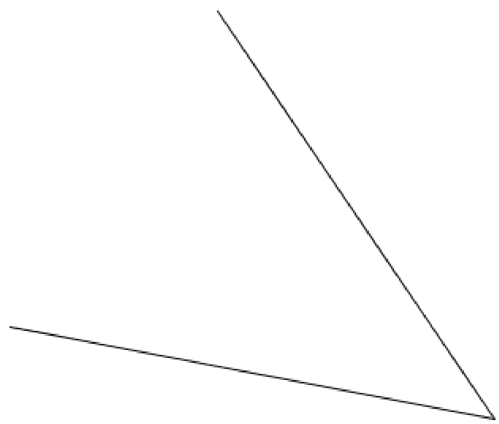


Tasks 10 through 13 assessed students' construction of a splitting operation in the context of angles (Figure 3.11). Although this was not part of the quantifications of angularity framework, these tasks provided a threshold between an extensive and ratio quantification, moving from additive thinking to multiplicative thinking. Also, since splitting is relevant to the measurement schemes for fractions, it was important to include such tasks in the angularity instrument for comparisons. These splitting tasks helped assess if a student was using additive or multiplicative reasoning for their quantifications.

Figure 3.11

Task Designed to Elicit Splitting

10. The angle below is five times as large as your angle. Draw your angle. Explain your process.

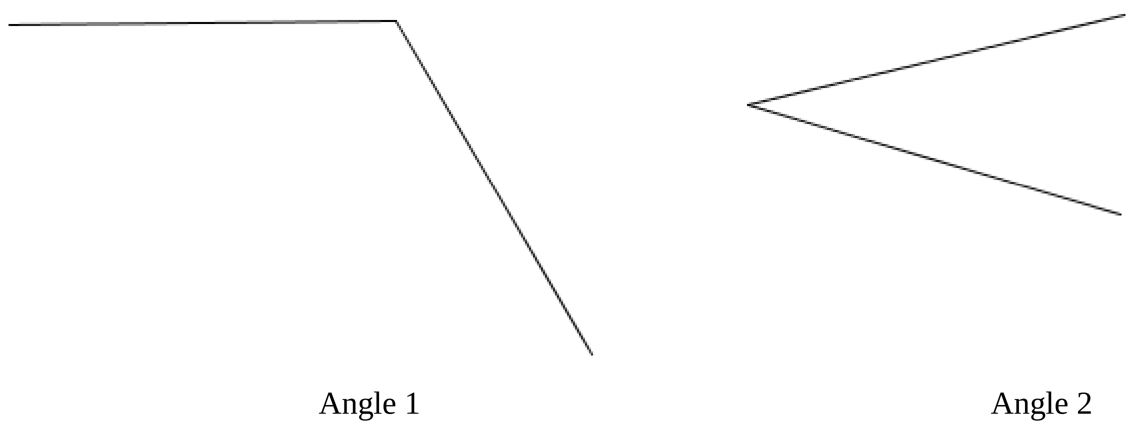


The next two tasks, Tasks 14 and 15, further helped assess if a student was using additive or multiplicative reasoning for their quantifications (Figure 3.12). If a student could solve these tasks and write their answers in fractions, it was inferred that they were using multiplicative thinking. If a student could not answer in terms of fractions, it was inferred that they were not using multiplicative thinking. These tasks were also used to aid in the inference of a ratio quantification.

Figure 3.12

Task Designed to Assess Multiplicative Reasoning

14. What fraction is Angle 2 of Angle 1?
What fraction is Angle 1 of Angle 2?



To assess a ratio quantification, Tasks 16 through 19 were designed to elicit multiplicative reasoning. These tasks represented multiplicative relationships between angles and circles. For example, Task 16 (Figure 3.13) gave students a circle with a central angle. They were told the circumference was 49cm and the arc made by the angle was 7cm, and were asked to find the measure of the angle. It was not intended that students would respond with an answer in degree units, but that students with a ratio quantification would recognize the multiplicative relationship. For example, students may note that since it would take seven iterations to recreate the circle, the angle was 1/7 of the circle. If students responded with such explanations, it was inferred that they had constructed a ratio quantification. If students did not respond with a multiplicative explanation, it was inferred that they had not constructed a ratio quantification. The size of the angles for these tasks were intentionally not drawn to scale in order to keep students from obtaining an answer through iteration. These tasks were also worded so students would have to work forwards and backwards, either finding the measure of the angle given the arc length and circumference, or finding the circumference given the angle measure and arc length.

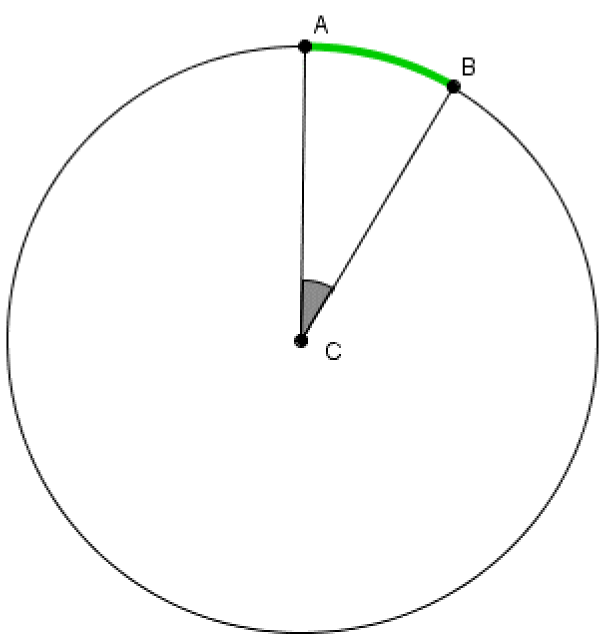
Figure 3.13

Task Designed to Assess a Ratio Quantification of Angularity

16. The circumference (total distance around the circle) is 49 cm. (Not drawn to scale)

The GREEN part of the circle is 7 cm long.

What is the measure of ∠ACB in relation to the circle?



Finally, four tasks (20-23) were designed to assess a rate quantification given situations involving concentric circles. Three of these tasks helped determine if a student recognized that the relationship between the length of the ray, subtended arc, and circumference was invariant across all possible circles when the circles were centered at the angle's vertex. For example, consider Task 23 in in Figure 3.14. Students with a rate quantification would recognize that although the circle changed size, the angle remained the same. They would also recognize that the multiplicative relationship between the arc of Circle 1 and its circumference is 1/8. Then, understanding the relationship between the arc, circumference, and angle, they would note that the angle would be 1/8 of Circle 2 as well, calculating the arc length of Circle 2 to be 12 cm. These tasks were intentionally not drawn to scale so that students could not obtain an answer through iteration. These tasks were also worded so students would have to work forwards and backwards, either finding the measure of the angle given the arc lengths and circumferences, or finding one of the circumferences given the angle measure and arc lengths. One task also involved two non-concentric circles, so that students would have to relate the proportional relationships of each circle without visually comparing them.

Figure 3.14

Task Designed to Assess a Rate Quantification of Angularity

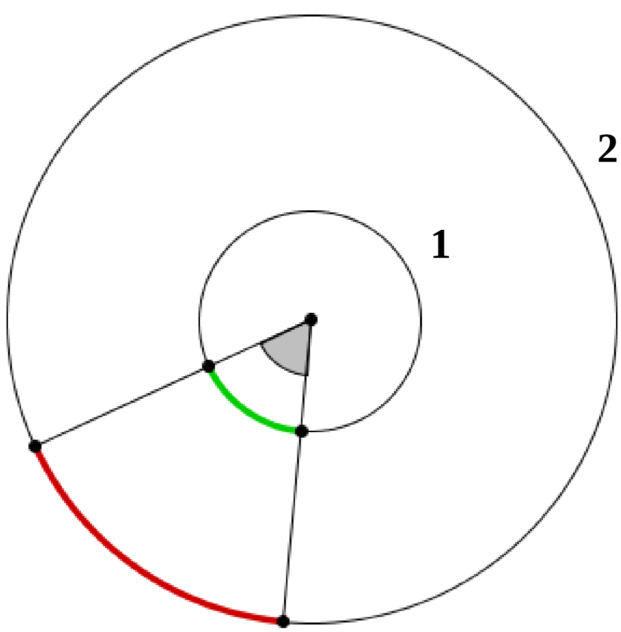
23. Circle 1 has a circumference (total distance around the circle) of 56 cm.

The GREEN part of Circle 1 is 7 cm long.

The circumference (total distance around the circle) of Circle 2 is 96 cm.

How long is the RED part of Circle 2?

(Figure is not drawn to scale)



Quantifications of Angularity Coding

Each angularity task used in the main study was evaluated by two coders, independently, using the coding guidelines (Table 3.11). Coders investigated responses for consistency with particular ways of thinking and actions used. For example, coders inferred students' actions based on written responses including: visual judgements, partitioning/iterating, or possibly multiplicative reasoning. Coders also examined responses to infer each action as described by Hardison (2018): segment sweep, re-presented opening, radial sweep, and arc sweep. A coding scheme similar to that described for the fraction scheme coding (Table 3.7) was used. Each task was given a 0, .4, .6, or 1 based on students' responses and the compatibility with the descriptions of the quantifications of angularity. After summing the scores for the grouping of tasks, if a student had a score greater than or equal to 3 for each category, it was inferred that they possessed that quantification and was coded as 1. If a student had a score less than or equal to 2, it was inferred that the student did not possess that particular quantification and was coded as 0. For students with scores between 2 and 3, their work was reviewed holistically for that quantification, and the researchers individually made a final decision concerning whether they possessed the quantification. Specifically for a ratio quantification, the two multiplicative tasks were used to help distinguish an extensive from a ratio quantification during the review.

Table 3.11

Coding Guidelines for Quantifications of Angularity

Description of Approaches	Description of Actions	Quantifications of Angularity
• Use only visual judgments	• Compare length of rays • Compare size of arcs • Compare size of wedges/sectors	Gross
• Measure attribute with non-additive units • Rely on visual perceptions	• Draw/use radial sweep to compare duration of sweeps • Draw/use segment sweep to compare openness (how fast angle grows) • Use re-presented opening to compare angles (hand motions for openness) • Use of superimposition to compare angles by visual comparisons • Visually compare angles to benchmark angles	Intensive
• Measure with additive units • Use partitioning/iterating operations in additive sense	• Use operations of partitioning and iterating to compare angles • Use of superimposition to compare angles by units • Compare angles to benchmark angles using units	Extensive
• Use multiplicative reasoning • Use partitioning/iterating operations in multiplicative sense • Measure angles in units of one another	• Use operations partitioning, iterating, and splitting to compare angles • Relate angles to circles for comparison (draw circles) • Draw radial sweep to compare arc length • Maintain multiplicative relationship between radius, circle, and arc	Ratio
• Use proportional reasoning to understand relationship of angles in all possible circles	• Draw/use arc sweep • Recognize angle relationship in concentric circles	Rate

This coding resulted in a matrix of binary codes for each student, regarding each quantification of angularity (i.e., intensive, extensive, ratio, and rate; Table 3.12). For example, suppose Student 1 had a sum greater than 3 for the intensive and extensive tasks, but a score less than 2 for splitting, ratio, and rate tasks. They would be given a binary code of 1 for intensive and extensive quantification, but a 0 for the others. It is also important to reiterate that the gross quantification was not specifically determined based on the tasks on the QAI. Instead, if a student received a code of 0 for all stages of quantifications, they were attributed a gross quantification. We determined that if a student attempted to solve tasks on the QAI, then they at least had some understanding of angle and therefore possessed a gross quantification.

After creating a matrix of binary codes, each student was assigned a quantifications of angularity code representing their highest stage of quantification (Table 3.13). These codes were assigned based on the most sophisticated quantification each researcher determined the student had constructed. For example, Student 1 would be assigned a code of 2 because the highest quantification they possess is extensive. Student 4 would be assigned a 0 because based on the logic of not having possessed any quantification, we inferred they had only a gross quantification.

Table 3.12

Example Matrix Coding Scheme for Quantifications of Angularity

	Each Quantification Code				**Overall Quantification Code**
Student	Intensive	Extensive	Ratio	Rate	
1	1	1	0	0	**2**
2	1	0	0	0	**1**
3	1	1	1	0	**3**
4	0	0	0	0	**0**

Table 3.13

Ordinal Coding Scheme for Quantifications of Angularity

Overall Quantification Code				
0–Gross	1–Intensive	2–Extensive	3–Ratio	4–Rate

Interrater Reliability

About 10% of the students were randomly selected from both the MSFI and QAI. These instruments were used for calibrating rater scoring. The two coders independently scored 16 MSFIs and 16 QAIs. The two coders then met to discuss and reconcile any disagreements concerning coding. This ensured that the scoring rubric was calibrated, as well as to promote accuracy and reliability in coding the surveys.

Next, the two coders independently coded the remaining 136 MSFIs and 136 QAIs. The number of agreements and disagreements were calculated between each scheme and each quantification code using the binary matrix for each construct. For students' measurement schemes for fractions scores and quantifications of angularity scores, Cohen's Kappa (κ; Cohen, 1960, 1968) was calculated to assess interrater reliability. κ (Cohen, 1960, 1968) is a statistic that assesses "the level of agreement between observers that is corrected for levels of agreement that would be expected by chance" (Warner, 2013, p. 906). Since κ treats disagreements equally (Cohen, 1968), it was an appropriate statistics to use (Siegel & Castellan, 1988) for these dichotomous codes.

Quantitative Data Analysis

To examine the relationship between students' measurement schemes for fractions and their quantifications of angularity, Goodman and Kruskal's Gamma (G) and Spearman Rank Correlation (r_s) was calculated. Spearman Rank Correlation is similar to a Pearson Correlation, but it used when variables represent ordinal data (Siegel & Castellan, 1988). However, because there was a possibility that there may be many ties in the codes between the two variables, Gamma was also used (Siegel & Castellan, 1988). Also, since the data was positively skewed, these non-parametric statistics were used.

Gamma and Spearman Rank Correlation assess the strength of association between the two variables (Berry et al., 1976; Hryniewicz, 2006; Somers, 1980; Siegel & Castellan, 1988), provided there is an association. In calculating G and r_s, there is the assumption that data must be ordinal (Siegel & Castellan, 1988). This assumption was met in this present study since students were given ordinal codes representing the different measurement schemes for fractions and quantifications of angularity, which progress from a least sophisticated to more sophisticated categorization (see Table 3.9 and Table 3.13). The second assumption is that the data has a weak monotonic association (Berry et al., 1976; Siegel & Castellan, 1988). This means that as one variable increases in value, the other variable should also increase in value.

Based on theory and literature, I hypothesized that a concept of measurement precedes the ability to quantify angularity using additive units, and therefore a measurement concept is necessary for the construction of more sophisticated quantifications of angularity. In other words, students who construct more sophisticated measurement schemes of fractions are able to construct more sophisticated quantifications of angularity. Since the most sophisticated concepts of angles involve understanding angle measurement as a fractional amount of a circle, students with a more sophisticated measurement scheme for fractions would potentially have a deeper understanding of angle and angle measurement, leading to a more sophisticated quantification of angularity. However, again, this is not a one-to-one correspondence, but rather a hypothesis that a concept of measurement precedes quantifications of angularity. Therefore, these two construct were correlated to measure their association.

After this association was assessed, I further investigated my hypothesis concerning the relationship between student's measurement schemes for fractions and their quantifications of angularity. For example, in Chapter 2 I hypothesized that a gross and intensive quantification of angularity does not involve any of the mental actions and operations associated with measurement (i.e., partitioning and iterating), and thus, may develop without a concept of measurement, as indicated by their measurement schemes for fractions. A PWS and MSUF are related to an extensive quantification by the partitioning and iterating operations. A MSPF is related to a ratio quantification by the splitting operation. A GMSF is related to a rate quantification by the units coordination operation. A developmental progression can be seen where a gross and intensive quantification precede a measurement concept. A measurement concept then precedes more sophisticated quantifications of angularity. Once students have a MSUF, they are able to construct an extensive quantification; once students have multiplicative reasoning, indicated by splitting, they are able to construct a ratio and rate quantification. However, it is unclear if the fractions schemes precede or co-develop with the quantifications of angularity.

To investigate these relationships, each individual scheme score, represented by nominal scores, were correlated with students' overall quantifications of angularity, using *G*. Then, each individual fraction scheme score was correlated with each quantification of angularity score, using Phi coefficients. The goal was to provide a more thorough examination of the relationships between students' measurement schemes for fractions and their quantifications of angularity.

Phase Two: Validation Study

During Phase Two, qualitative data was collected through clinical interviews to gain insight into students' ways of thinking and to further validate students' quantifications of angularity based on their classifications from the QAI. Oftentimes, it is difficult to tell from survey data exactly how students are thinking about a particular task. There may be evidence that they are operating at a certain level or with a specific scheme; however, upon inquiring into their thinking, they demonstrate different behaviors, which provides different insight into their knowledge. Therefore, the purpose of the clinical interviews was to investigate students' ways of thinking about tasks and their related actions and operations. Clinical interviews allow for follow-up questions and provide opportunities for students to think aloud and describe their process for solving tasks. This enables the researcher to make more trustworthy inferences concerning students' classifications.

The classifications based on the clinical interviews provide a criterion measure of students' quantifications by which to compare students' quantifications as determined by the written survey, the QAI, further evaluating the validity of the quantifications of angularity. The five main components for assessing and ensuring validity are: test content, response processes, internal structure, relations to other variables, and consequences of testing (AERA et al., 2014; Krupa et al., 2019). The purpose of the validation of students' quantifications of angularity during Phase Two was used to assess relations to other variables; in other words, students' classifications from the interview was treated as their "true" quantification of angularity and was compared to their QAI classification. This qualitative data was then combined with data from the pilot study and Phase One to provide a more informed description of students' quantifications of angularity.

Participants

Participants were selected by nested sampling (Creamer, 2017). A subsequent subgroup of students who participated in the QAI were recruited to participate in clinical interviews.

Participants were selected based on three criteria. First, students must have agreed to and participated in the QAI. Second, students must have agreed to participate in the clinical interviews and return a parent consent and student assent form. Third, students were selected from different grade levels, to hopefully provide variation in their quantifications of angularity.

On the day the QAI was administered, parental permission forms were sent home. All students were asked to return the signed parental permission form within the next week in order to participate in the clinical interviews. Only students who returned a parental permission form were asked to participate in the interviews. In total, 24 students returned signed parental permission forms granting permission for their child to be interviewed. However, on the days students were pulled to be interviewed, one student was absent, and one student did not want to participate. Therefore, a total of 22 interviews were conducted. This included 9 students from Math 6, 9 students from Math 7, and 4 students from Algebra (Table 3.14).

Table 3.14

Total Numbers of Students who Participated in Phase Two

Course	Possible Number of Students	Number of Returned Signed Forms	Number of Students Absent	Student Opt Out	Total Interviewed
Math 6	36	9	0	0	9
Math 7	67	9	0	0	9
Algebra	49	6	1	1	4
Total	**153**				**22**

Qualitative Data Collection

Qualitative data was collected through the use of semi-structured clinical interviews. Clinical interviews are a data-collection and observational tool that enables the researchers to investigate students' mathematical thinking (Cobb & Steffe, 1983, Zazkis & Hazzan, 1998), and helps uncover student's thinking, reasoning and development of mental structures (Clement, 2000). The general purpose of a clinical interview is to gain insight into the mental structures, constructions, and processes students use in relation to a particular mathematical concept (Clement, 2000; Cobb & Steffe, 1983). Clinical interviews can also be used to discover, identify, and evaluate the competence of a child's cognitive abilities (Ginsburg, 1981).

The clinical interview was influenced by Piaget's (1975) clinical method for investigating children's' cognitive development (Clement, 2000; Ginsburg, 1981, 1997; Posner & Gertzog, 1982). Piaget (1975) focused on asking various questions that would allow children to "talk freely" and encourage "spontaneous tendencies" (p. 4), thus capturing children's true interests and reactions. The clinical interview helps uncover children's natural mental inclinations, thought processes, as well as their mental context (Ginsburg, 1997). During the interview, the role of the researcher is not to teach the child how to solve the task or to play the game, but rather to observe the child's behaviors and actions to better understand their way of thinking. These student observations allow the researcher to understand how a student thinks about a particular concept and what mental actions they perform in order to solve a task (Clement, 2000). Overall, the idea is to gain insight into how students think about and solve tasks to paint a clearer picture of their understanding of particular concepts.

The clinical interviews took place in one sitting, lasting approximately 30 minutes. During the semi-structured clinical interviews, I presented students with tasks from the QAI, as well as several new tasks. Some of these tasks and questions were derived from the interview questions used by Hardison (2018). I used a systematic, logical approach to determine which questions to ask during the interviews (Appendix E). For example, after some preliminary questions and sorting activity, students were presented with the splitting tasks. These tasks were chosen to present first because they represented the transition between additive and multiplicative thinking. If students successfully solved these tasks, I moved to the ratio tasks. However, if they could not solve the splitting tasks, I moved to the extensive tasks. In this way I was able to potentially reduce the amount of time needed to determine students' stage of quantification while at the same time maximize the time to differentiate among stages.

During the interview, I inquired into how they solved certain tasks and what they did to solve them. This provided better insight into their quantifications of angularity, helping to confirm their quantifications of angularity scores. These interviews were recorded for retrospective analysis, and written work was digitized. Field notes were also taken to document students' behaviors or actions in the moment during the interview process. The main focus during the interview was to validate students' quantifications of angularity scores, ultimately validating the instrument. These interviews were used to classify students' quantifications of angularity. These classifications then served as the criterion measure for comparison to the results from the QAI, for evaluating the relation to other measures component of validity (Krupa et al., 2019). That is, the classifications from the interviews were correlated with classifications from the QAI to determine the magnitude of the association, which provided a measure of criterion validity for the classifications based on the QAI. Further details of the validation study are discussed later in the Validation Study section.

Interview Coding

Clement (2000) discusses two different purposes of clinical interview studies: generative and convergent. Each of these purposes will lead to different types of analysis. A generative purpose leads to an interpretive analysis, in which the researcher open-codes the data. Here, there usually is not predetermined codes, but rather categories and themes develop from observation as the researcher analyzes the data. Clement (2000) further explains that this interpretive analysis allows the researcher to develop “grounded theoretical models for learning processes” (p. 558). Different from the generative purpose, the convergent purpose of the clinical interview is to “provide reliable, comparable, empirical findings that can be used to determine frequencies, sample means, and sometimes experimental comparisons for testing a hypothesis” (p. 558). This purpose typically leads to coded analysis, in which predetermined codes or categories are used. Clement (2000) describes this as using criteria specific to a phenomenon and then coding all occurrences of that phenomenon in the transcript. This analysis is similar to using a framework or established model to categorize students’ behaviors as particular levels or stages.

Based on these two purposes, the purpose of the clinical interviews in this study was convergent. Therefore, pre-determined codes were used to analyze students’ responses and their quantifications of angularity. The pre-determined codes (i.e., gross, intensive, extensive, ratio, and rate) were derived from the quantifications of angularity framework described above. Each participants’ responses to the tasks was analyzed and compared to the descriptions of the quantifications of angularity. Every instance of a specific quantification (i.e., gross, intensive, extensive, ratio, and rate) was coded based on the criteria provided in Tables 3.10 and 3.11. Students’ responses were analyzed to investigate their approaches and actions relative to the different quantifications of angularity. During the interview, I recorded students’ responses and made notes about their preliminary quantifications based on their responses. After the interviews were over, two coders independently watched the interviews and evaluated student’s responses. Each coder assigned an overall quantification of angularity classification. The raters then met to reconcile these differences and reach consensus on a single classification.

To evaluate interrater reliability, weighted Kappa (κ_ω; Cohen, 1968) was calculated for the independent ratings. κ_ω considers both the amount of agreement and disagreement that could occur by chance (Cohen, 1968). In this case, disagreements are not treated equally, which allows for a more precise assessment (Cohen 1968). Since the quantification codes are ordinal (e.g., 0-4) κ_ω is an appropriate statistic to use. The Kappa scores for the two coders' agreements and disagreements are presented in Table 4.19.

Validation Analysis

These classifications based on the interviews were also used to further assist with the assessment of validity. As discussed earlier, the five main components for assessing and ensuring validity presented in the *Standards* (AERA et al., 2014) are: test content, response processes, internal structure, relations to other variables, and consequences of testing (Krupa et al., 2019). The goal here was to examine the relation to other variables. This qualitative data was used to classify each student into a particular quantification of angularity. This classification served as a criterion to compare with each students' quantifications based on the QAI. A validity coefficient, Gamma (*G*), was calculated between students' classifications from the interview and the QAI, which provided a measure of criterion validity. A Spearman Rank Correlation was also calculated to help assess criterion validity.

Chapter 4: Results

This chapter presents the results from the different phases of the study and will be divided into three parts. The first part will describe the results of the pilot study. The second part will describe the results of the quantitative survey data. The third part will focus on the results of the validation of students' quantifications of angularity, by combining the results of the pilot study and the final interviews.

Pilot Study Results

The purpose of the pilot study was to conduct a small-scale validation of the QAI that focused primarily on test content, response processes, and consequences of testing. During pilot testing, I, along with one expert researcher in the mathematics education field, evaluated the tasks to help assess test content, face validity, and the quality of each task (AERA et al., 2014; Krupa et al., 2019). Interviews were then conducted with fifth through tenth grade students to provide an initial validation of the QAI to ensure that the tasks measured what they were intended to measure (Krupa et al., 2019). These interviews were also used to classify students' quantifications of angularity.

First Round of Interviews

For the first round of interviews during the pilot study, three students agreed to participate: a seventh grade girl, an eighth grade girl, and a tenth grade boy. These students, and the students from the second round of interviews, are described in Table 4.1 along with their quantifications of angularity. Overall, test content was not initially found to be balanced as some tasks were extremely easy for students and did not represent the intended range of quantifications. However, students did respond in the intended manner to the majority of the tasks. There were some issues with a few tasks, which are described below. Also, there were no apparent negative consequences associated with testing.

Table 4.1

Description of Students who Participated in the Pilot Study

Round of Interviews	Student	Gender	Grade	Quantification of Angularity
1st	Chloe	Girl	7	Intensive
	Mia	Girl	8	Extensive
	Max	Boy	10	Rate
2nd	Nate	Boy	5	Ratio
	Sara	Girl	7	Rate

Looking at specific tasks, all three students said that Task 8 (Appendix A) was somewhat confusing. Two students asked if the line was supposed to be included in the ordering, because they did not know if it was supposed to be an angle or a divider on the page. Another problem with this task was that students did not understand the notation for the angle greater than 180^{o}. They suggested using an arrow to denote the angle's rotation above 180^{o}. In addition, two students did not understand the angle within the circle. However, this was purposefully included to determine if students could recognize angles within circles.

All three students in the first round of interviews had issues with Tasks 17 and 18 (Appendix A). Chloe and Mia stated that Tasks 17 and 18 did not make sense, and it was only after I explained the intent of the task that they were able to solve them. Even though Max could solve these two tasks, he suggested that younger students would not understand the intent of the task. All three suggested rewording these tasks to be more specific. For example, Max suggested asking "How many times will Angle 2 fit into Angle 1?" Chloe and Mia suggested asking something similar to "How many times smaller is Angle 2 than Angle 1?"

Other tasks that the students thought were problematic were the tasks associated with the rate quantification of angularity. Chloe noted that she had been taught that when the circles get bigger, the angles remain the same, and relied on that knowledge to solve those tasks. For example, for Task 21, Chloe said the length of the rays did not matter because they are all the same angle. When I asked her how she knew that the ray length did not matter, she said that was just something she was taught. Chloe was unable to relate the proportional relationship between the circles and angles, but relied on something she had memorized. Similarly, Max noted that these were things he was taught early on when learning about angles and thought these tasks may be easy for students.

Furthermore, looking across the ratio and rate tasks it appeared that the tasks did not require the necessary conceptual understanding to successfully solve them. For example, Chloe did not think about angles multiplicatively, and only sometimes demonstrated additive thinking. Based on her thinking, I felt that she had constructed only an intensive quantification. Chloe struggled with partitions and iterations and relied on visual estimates and judgments. She did not draw any lines to represent her partitions or iterations. However, Chloe was able to obtain correct answers to the ratio and rate problems. She simply divided the numbers to obtain the answer. In contrast, Mia had constructed an extensive quantification and the splitting operation, but could not solve the ratio or rate problems. She drew partitions and iterations to figure out the relationship between angles and could easily solve the splitting tasks. However, she incorrectly solved the ratio and rate problems, stating that as circles get larger, angles get larger. Max was different in that he could relate angles to circles, providing evidence of having constructed a rate quantification. Max began the interview by defining an angle as “a measurement out of a circle.” He easily solved all tasks, drawing partitions and iterations, as well as recognizing the multiplicative relationship between angles. He also recognized that the change in circle size does not change angle measure, representing a true rate quantification. When looking at the correctness of the tasks, Chloe and Max correctly solved the ratio and rate tasks but demonstrated completely different thinking. Furthermore, Mia demonstrated more sophisticated thinking than Chloe but could not solve the ratio and rate tasks. Taken in tandem, these examples provide evidence that these tasks did not provide information to make valid assessments of a rate or ratio quantification. These tasks violated the test content of validity, since they did not measure the construct they were intended to measure (Krupa et al., 2019). They also violated the response process component of validity because students did not respond to the tasks in the manner that I predicted (Krupa et al, 2019).

First Round of Revisions

After the results of the first round of interviews, I and the expert researcher developed new tasks, hoping to address the students' concerns and to be able to make more valid assessments of students' quantifications of angularity. These revised tasks are presented in Appendix B. We revised six of the tasks to make them more visually pleasing and easier to read and understand. For example, we removed the box on Tasks 1 through 4 to make the questions less distracting. We also added an arrow to the item in Task 8 to help denote that it was an angle greater than 180^o.

We struggled with revising old Tasks 17 and 18 because the goal of the tasks was to assess whether students could measure angles in terms of one another. However, based on the students' responses, we changed the wording from "Use Angle 1 to measure Angle 2" to "Measure Angle 1 in terms of Angle 2." Furthermore, after hearing students' concerns about notation and not understanding what questions we were trying to ask, we thought it may be better to change the wording in old Tasks 19 and 20 from the vagueness of “the thick arc” and “the measure of the angle” to “Arc AB” and “the measure of ∠ACB” and include labels on the pictures. Even though the students made no mention of a concern for these tasks, we felt this would help prevent students from incorrectly solving these tasks because they did not understand which part of the circle we were referencing. We also changed these tasks so that the pictures were not drawn to scale, to prevent students from visually comparing what fraction of the circle each angle was. We hoped that this change would prevent students from iterating the angle to figure out how many times it would fit into the circle, so we could assess if they truly recognized the multiplicative relationship.

Another revision we made is that we combined old Tasks 21 and 22 into one new task to eliminate redundancy. In this new Task 21 (Appendix B), we also removed the answer choice box, like in Tasks 1 through 4, to reduce distractions in the question. We also made the length of the ray on each side of an angle unequal in some cases. This was to help assess if students relied on the length of the rays or if they truly understood the relationship for angles within concentric circles.

Finally, we created three new tasks, Tasks 22 through 24 (Appendix B), to assess a rate quantification. Based on the theoretical description of rate quantification (see Figure 2.19) this quantification is truly an extension of ratio into the context of concentric circles, meaning that students apply the proportional relationship between the length of the ray, arc length, and circumference to multiple circles. Therefore, these new tasks needed to be related to the ratio tasks. These new tasks presented similar situations as new Tasks 19 and 20 (Appendix B), but involved multiple circles. For example, new Task 23 (Appendix B) presented two circles where students were given the circumference and arc length in Circle 2. They then had to apply this multiplicative relationship to Circle 1 in order to figure out the relation of the angle to Circle 1. We felt that this was a more valid task for assessing rate than the earlier tasks because it required students to maintain this proportional relationship between the angle, arc, and circumference for two different circles.

Second Round of Interviews

For the second round of interviews during the pilot study, students were purposefully recruited from fifth, sixth, and seventh grade to provide variability in students' age, grade, and possibly depth of understanding. The two students who agreed to participate were a fifth-grade boy and a seventh-grade girl. During these interviews, I presented the original tasks (Appendix A) as well as the revised tasks (Appendix B) to see which ones students thought were easier to read and understand, helping to address test content and response process. I also used the new tasks to determine if they were useful for making more valid assessments of students' quantifications of angularity. Evaluating the instrument, overall, test content was balanced and represented the intended range of quantifications. Students' responded in the intended manner to the new tasks. Although there were no major issues with any of the tasks, students did make some suggestions for improvements. Also, there were no apparent negative consequences of testing.

Both Nate and Sara agreed that for Tasks 1 through 4 that the box made the questions easier to read, but felt the revised tasks were also fine. For Task 8, they both also stated that they were unsure if the line was supposed to be part of the ordering or if it was a divider. Sara stated that the arrow did not help her understand that the angle was greater than 180^{o}, but Nate said he understood the arrow notation.

For Tasks 17 and 18, both Sara and Nate stated that the question did not make sense. When compared to the original question, they claimed that neither question was appropriate. They did not understand what the task was asking them to do by saying “Use Angle 2 to measure Angle 1.” After I explained what I was trying to assess, they suggested rewording the question to ask, “What fraction of Angle 2 is Angle 1?” or “How much larger is Angle 2 than Angle 1?”

For Tasks 19 and 20, Nate and Sara were able to solve those tasks. They understood the notation used and said it was clear what the task was asking. However, Sara stated that using “size” and “measure” in different questions was a bit confusing. She knew that measure and size were meant to mean the same thing, but thought other students might confuse size with how much larger the angle was, and not relate it to angle measure. Therefore, she suggested using measure in both questions to help younger students understand. Nate also said that he knew measure and size meant the same thing, but said that using both words was not confusing. Sara also suggested asking “What fraction of the circle?” to help eliminate the confusion between measure and size. Nate also suggested including a clarifying statement to say the pictures were not drawn to scale to prevent confusion. These suggestions were considered in the next iteration of tasks.

For new Task 21, Nate and Sara said it was clearly written and easy to understand. They also stated that having different length rays was not confusing, because they knew the angles would remain the same even if the circle gets bigger. However, Nate thought that maybe this question may be too easy since this was something they had learned in fourth grade.

Finally, Nate and Sara thought that Tasks 22 through 24 were clearly written, and that they understood the notations used for arcs and angles. Nate again mentioned clarifying that the pictures were not drawn to scale to prevent potential confusion. When he solved the problem, he noted that his answer was not consistent with the picture, and therefore he must be wrong. When I explained that the picture was purposefully not drawn to scale, he said that would be an important thing to note for students. Sara made another suggestion and claimed that using color may help some students. She thought the color would help students understand which arc and circle was being referenced in the questions, and may help with confusion about notation.

Based on the students' responses, I felt that these tasks enabled me to make more valid assessments of students' quantifications of angularity. For example, the ratio and rate tasks assessed students' ratio and rate quantifications. Throughout the interview, Nate relied heavily on degrees as his standard unit of measure. When asked to draw angles that were five times smaller, or for the splitting tasks, he stated that he could not do so precisely because he did not know the degrees. However, when asked to estimate, he properly used his partitioning, iterating, and splitting operations.

On the ratio and rate tasks, he related angle measurement to a fractional amount of a circle. When working with the rate tasks, he became a bit confused. For example, on Task 23 he said the smaller Circle 1 looked to be about one half of the larger Circle 2, and therefore given the proportion of 10/60, he would divide that by two and get 5/30. He then reduced his fraction to 1/6 and claimed that the angle would be 1/6. However, he said that if Circle 1 was 3 times smaller, he would have divided the proportion by three. He went on to say that if the question simply asked about Circle 2, then the answer would just be 10/60 or 1/6. He did not recognize these were the same proportional relationship.

From his responses, I inferred that Nate had constructed a ratio quantification. He could relate angle measurement to a fractional amount of a circle, but struggled when relating to multiple circles. From Sara's responses, I inferred that she had constructed a rate quantification.

She, like Nate, could use her iterating, partitioning, and splitting operations to solve the tasks. She also understood angle measurement as a fractional amount of a circle and could maintain the proportional relationship between multiple circles. Overall, I felt the new tasks provided more valid assessments of students' quantifications of angularity. These new tasks were more balanced in terms of assessing the construct they were intended to measure. The new ratio and rate tasks provided information to make valid assessments of a rate or ratio quantification, also helping provide a more consistent internal structure. Students also responded in the manner I predicted they would, and there were no apparent negative consequences of testing.

Final Iteration of Tasks

Based on the second round of interviews, the tasks were further revised. After thinking about how the tasks were evaluated, I and the expert researcher determined that the first four tasks intended to assess a gross quantification were not consistent with the assessment of the other quantifications. For example, in order to be categorized as a gross quantification, a student would have to respond incorrectly. However, if a student solved the tasks correctly, they would be coded as not having a gross quantification. Again, this was inconsistent with how the other tasks were designed for assessment. Therefore, we felt it would be better to combine the first seven tasks into four tasks that would elicit students thinking associated with an intensive quantification. Then based on logic, if a student attempted to solve any tasks, but did not score high enough to be categorized as intensive, extensive, ratio, or rate, they would by default be assigned a gross quantification.

Another revision I and the expert researcher made was that we revised the ordering task. We rearranged the order of the items within the new Task 5 to help students understand that the line was part of the ordering (Appendix C). We also kept the arrow to denote the angle greater than 180°. We also changed new Tasks 14 and 15 (Appendix C) to make them clearer as to what we were asking students to do, because no student was able to understand the question. Based on students' suggestions, we rewrote it to explicitly ask students to find what fraction is Angle 1 of Angle 2.

After we addressed the major issues brought up from students, I sent the tasks to the original expert researcher in the mathematics education field, along with two more experts. This was done again to assess test content and face validity (AERA et al., 2014; Krupa et al., 2019). The three experts evaluated the quality of each task, determined if the items were balanced, and evaluated the tasks for test content. Overall, they determined that the tasks were not balanced, some tasks did not provide a valid assessment of students' quantifications of angularity, and therefore raised several concerns.

One thing that the expert reviewers brought up was that the partitioning tasks, Tasks 10 and 12 (Appendix B), were too difficult; they felt that these tasks required the splitting operation and therefore assessed multiplicative thinking instead of additive for extensive. One expert reviewer also noted that asking students to draw an angle that is 1/7 of the given angle requires a multiplicative understanding, and that students who are additive would not be able to solve the problem. Therefore, we revised the extensive tasks to truly assess additive thinking and to also remove the issue of using multiplicative language. For example, we created two new tasks, Tasks 7 and 9 (Appendix C) that asked students "How many times will Angle 1 fit into Angle 2?" These tasks required students to partition into an odd number (i.e. five and three times). We purposefully did this so students could not partition the angle in half and then half again to create four, as some students in the pilot study did. We then revised the new iterating tasks, Tasks 6 and 8 (Appendix C) to require students to iterate an even number of times (i.e., four and six times) to provide variability and balance. We also revised two splitting tasks, Task 12 and 13 (Appendix C) so students were not asked to split into thirds in two tasks and to provide variability with the number of splits required.

Another concern the experts brought up was that they did not think that Task 21 (Appendix B) was valid for assessing rate. They felt this task was too simple for students to solve and did not emphasize the multiplicative relationship between the angle, arc, and circumference. They argued that some students would be able to correctly solve this task by using visual comparisons, as some did in the pilot study. They suggested creating a new task to maintain consistency between the four rate tasks. A new task, Task 20 (Appendix C) was created that involved nonconcentric circles to assess if students could maintain and transfer the proportional relationship between the angle, arc, and circumference between circles.

One final concern that one expert reviewer brought up was that the tasks involving formal mathematical language of “arc” would be too difficult for students. The term arc is not introduced into the middle school curriculum standards until Geometry. Therefore, it could be possible that some students would miss the problem because they did not understand the language. We felt that this situation might potentially lead to an underestimation of students’ quantification of angularity. It is possible that students might have the knowledge to solve the problem, but if they did not understand the language, they may get the answer wrong for the wrong reasons. Therefore, taking this suggestion, as well as Sara’s suggestion, we revised the ratio and rate questions to include color and to eliminate formal mathematical language and notations. We also included a statement in all of the tasks involving pictures of circles to note that the pictures were not drawn to scale.

We also addressed the issue of item balance across the instrument. We felt that Tasks 14 and 15 (Appendix C) were not valid for assessing ratio, since ratio emphasized the relationship between the angle, arc, and circumference However, we wanted to keep Tasks 14 and 15 since they assessed whether students were thinking multiplicative or not. We felt these would be good for helping discriminate between extensive and ratio. Then, to be consistent, with having four tasks per quantification, we created two new ratio tasks and two new rate tasks. These revisions resulted in a total of 23 tasks we then used in the QAI for the main study (Appendix C). A table of these revisions between the first iteration of tasks and the final version of tasks is provided in Appendix D.

Phase One Results

During Phase One, quantitative survey data was collected through two different instruments. One instrument, the MSFI, was used to infer students' construction of fractions schemes and categorize them by stage associated with the fractions schemes; and another instrument, the QAI, was used to infer students' quantifications of angularity and categorize them by stage. These assignments were then used to attest for an association between students' level of measurement scheme for fractions and their quantification of angularity. Results from Phase One will be presented in three sections. The first section will describe results from the data collected with the MSFI. The second section will describe results based on the data collected with the QAI. The third section will examine the relationship between students' measurement schemes for fractions and their quantifications of angularity.

I first present evidence for the interrater reliability of the two instruments based on the remaining surveys after rater calibration was conducted using approximately 10% of the surveys. From the remaining 136 MSFIs and 136 QAIs, the number of agreements and disagreements were calculated between each scheme and each quantification code using the binary matrix for each construct (Table 4.2). κ scores for PWS, MSUF, splitting, MSPF, and GMSF were .949, .821, .709, .606, and .428 respectively. Landis and Koch (1977) state that κ scores between .81 and 1.00 are “almost perfect”, between .61 and .80 are “substantial,” and between .41 and .60 are “moderate” (p. 165). Accordingly, the κ scores for PWS and MSPF are almost perfect, Splitting and MSPF are substantial, and GMSF is moderate. It is important to note that in this sample of students, it was rare for students to have constructed a MSPF or a GMSF, with only five students classified as having constructed a MSPF and only six students classified as having a GMSF. Due to these low numbers of students in each category, Viera and Garrett (2005) claim that “Kappa may not be reliable for rare observations” (p. 362). Therefore, overall percent agreement was also calculated to display coder agreement (Table 4.2), and demonstrates almost perfect agreement (Landis & Kock, 1977) between coders for all fraction scheme classifications. Together these statistics provide evidence of high reliability between the raters.

κ scores for intensive, extensive, splitting, ratio, and rate scores were .917, .684, .651, .891, and 1.00 respectively. The κ scores for intensive, ratio, and rate are almost perfect, and extensive and splitting are substantial (Landis & Koch, 1977). Overall percent agreement was calculated to further display coder agreement, and demonstrates a remarkably high percent agreement between coders for all quantifications of angularity classifications. Again, this evidence suggests high reliability between raters.

Table 4.2

Kappa Scores for Fraction and Angle Coding

Instrument	Scheme/ Quantification	Number of Disagreements	Number of Agreements	Kappa Score	Percent Agreement
MSFI	PWS	3	133	.949	.978
	MSUF	8	128	.821	.941
	Splitting	15	121	.709	.890
	MSPF	7	129	.606	.949
	GMSF	5	131	.428	.963
QAI	Intensive	5	131	.917	.963
	Extensive	20	116	.684	.853
	Splitting	19	117	.651	.860
	Ratio	3	133	.891	.978
	Rate	0	136	1.00	1.00

Measurement Schemes for Fractions

This part of the quantitative phase of the study involved 152 students in sixth, seventh, and eighth grade. The purpose of using the MSFI was to assign a fraction scheme to each student: PWS, MSUF, MSPF, or GMSF. In order to determine a student's overall categorization, students were assigned to the highest scheme for which it was inferred they had constructed. For example, students who did not construct any measurement schemes for fractions, indicating that they were pre-fractional, were categorized as Pre Part-Whole Scheme (PrePWS). Based on the categorizations from the MSFI, the percentage of students for each scheme are presented in Table 4.3. Of the 152 students, 27.6% were PrePWS, 50.7% had constructed a PWS, 14.5% had constructed a MSUF, 3.3% had constructed a MSPF, and 3.6% had constructed a GMSF. These results show that the majority (78.3%) of these middle school students had constructed at most a PWS. This also indicates that less than one fourth of the students (21.8%) had constructed a more sophisticated fraction scheme, that is, a measurement scheme for fractions, moving beyond PWS understanding.

Table 4.3

Students' Overall Fraction Schemes

	Frequency	Percent
PrePWS	42	27.6%
PWS	77	50.7%
MSUF	22	14.5%
MSPF	5	3.3%
GMSF	6	3.9%
Total	152	100%

Looking at these results by grade level (Table 4.4), of the 37 sixth grade students, 37.8% were PrePWS, 54.1% had constructed at most a PWS, 8.1% had constructed at most a MSUF, and 0% had constructed a MSPF or a GMSF. Of the 77 seventh grade students, 27.3% were PrePWS, 58.4% had constructed at most a PWS, 10.4% had constructed at most a MSUF, 2.6% had constructed at most a MSPF, and 1.3% had constructed a GMSF. Of the 38 eighth grade students, 27.6% were PrePWS, 50.7% had constructed at most a PWS, 14.5% had constructed at most a MSUF, 3.3% had constructed at most a MSPF, and 3.9% had constructed a GMSF. Looking at the patterns within the data, it appears that as students progress in grade level, their fraction schemes advance developmentally. As students progress in grade level, the percentage of students who had constructed a more sophisticated fraction scheme increases, and the percentage of students who had constructed a less sophisticated fraction scheme decreases.

Although all sixth graders were enrolled in Math 6, it is important to note that some seventh graders were enrolled in Algebra, and therefore it may be helpful to also look at the distribution of fraction schemes by course. Percentages by course are provided in Table 4.5. Results show that of the 69 students enrolled in Math 7, 29% were PrePWS, 60.9% had constructed at most a PWS, 10.1% had constructed at most a MSUF, and 0% had constructed a MSPF or a GMSF. Of the 46 students enrolled in Algebra, 17.4% were PrePWS, 32.6% had constructed at most a PWS, 26.1% had constructed at most a MSUF, 10.9% had constructed at most a MSPF, and 13% had constructed a GMSF. Again, looking at the patterns within the data, it appears that as students progress in course level, their fraction schemes advance developmentally. As students progress in course level, the percentage of students who had constructed a more sophisticated fraction scheme increases, and the percentage of students who had constructed a less sophisticated fraction scheme decreases.

Table 4.4

Overall Fraction Schemes by Grade

	6th		7th		8th	
	Frequency	Percent	Frequency	Percent	Frequency	Percent
PrePWS	14	37.8%	21	27.3%	7	27.6%
PWS	20	54.1%	45	58.4%	12	50.7%
MSUF	3	8.1%	8	10.4%	11	14.5%
MSPF	0	0%	2	2.6%	3	3.3%
GMSF	0	0%	1	1.3%	5	3.9%
Total	37	100%	77	100%	38	100%

Table 4.5

Overall Fraction Schemes by Course

	Math 6		Math 7		Algebra	
	Frequency	Percent	Frequency	Percent	Frequency	Percent
PrePWS	14	37.8%	20	29%	8	17.4%
PWS	20	54.1%	42	60.9%	15	32.6%
MSUF	3	8.1%	7	10.1%	12	26.1%
MSPF	0	0%	0	0%	5	10.9%
GMSF	0	0%	0	0%	6	13%
Total	37	100%	69	100%	46	100%

Quantifications of Angularity Results

This part of the quantitative phase of the study also involved 152 students in sixth, seventh, and eighth grade. The purpose of using the QAI was to assign a quantification of angularity to each student: gross, intensive, extensive, ratio, or rate. In order to determine a student's overall categorization, students were assigned to the highest quantifications for which it was inferred they had constructed. For example, students who did not construct any quantifications were categorized as having a gross quantification. The percentage of students in each quantification are presented in Table 4.6. Results show that of the 152 students, 32.2% had constructed at most a gross quantification, 31.6% had constructed at most an intensive quantification, 24.3% had constructed at most an extensive quantification, 6.6% had constructed at most a ratio quantification, and 5.3% had constructed a rate quantification. These results indicate that the majority (88.1%) of these middle school students had constructed at most an extensive quantification. This also indicates that only 11.9% of the students were able to understand the multiplicative relationship between angles and circles.

Table 4.6

Students' Overall Quantifications of Angularity

	Frequency	Percent
Gross	49	32.2%
Intensive	48	31.6%
Extensive	37	24.3%
Ratio	10	6.6%
Rate	8	5.3%
Total	152	100%

Looking at these results in terms of grade level (Table 4.7), of the 36 sixth grade students, 50% had constructed at most a gross quantification, 30.6% had constructed at most an intensive quantification, 19.4% had constructed at most an extensive quantification, and 0% had constructed a ratio or rate quantification. Of the 75 seventh grade students, 37.3% had constructed at most a gross quantification, 32% had constructed at most an intensive quantification, 20% had constructed at most an extensive quantification, 5.3% had constructed at most a ratio quantification, and 5.3% had constructed a rate quantification. Of the 41 eighth grade students, 7.3% had constructed at most a gross quantification, 31.7% had constructed at most an intensive quantification, 36.6% had constructed at most an extensive quantification, 14.6% had constructed at most a ratio quantification, and 9.8% had constructed a rate quantification. Examining the patterns within the distribution of quantifications of angularity across courses, it appears that as students progress in grade level, their quantifications advance developmentally. As students progress in grade level, the percentage of students who had constructed a more sophisticated quantification increases, and the percentage of students who had constructed a less sophisticated quantification decreases.

Again, it is important to note that some seventh graders were enrolled in Algebra, and therefore it may be helpful to also look at the distribution of quantifications of angularity across courses. Percentages for each quantification by course are provided in Table 4.8. Results show that of the 67 students enrolled in Math 7, 41.8% had constructed at most a gross quantification, 32.8% had constructed at most an intensive quantification, 19.4% had constructed at most an extensive quantification, 3% had constructed at most a ratio quantification, and 3% had constructed a rate quantification. Of the 49 students enrolled in Algebra, 6.2% had constructed at most a gross quantification, 30.6% had constructed at most an intensive quantification, 34.7% had constructed at most an extensive quantification, 16.3% had constructed at most a ratio quantification, and 12.2% had constructed a rate quantification. Again, the patterns within this data indicate that as students progress in course level, their quantifications advance developmentally. As students progress in course level, the percentage of students who had constructed a more sophisticated quantification increases, and the percentage of students who had constructed a less sophisticated quantification decreases.

Table 4.7

Overall Quantifications of Angularity by Grade

	6th		7th		8th	
	Frequency	Percent	Frequency	Percent	Frequency	Percent
Gross	18	50%	28	37.3%	3	7.3%
Intensive	11	30.6%	24	32%	13	31.7%
Extensive	7	19.4%	15	20%	15	36.6%
Ratio	0	0%	4	5.3%	6	14.6%
Rate	0	0%	4	5.3%	4	9.8%
Total	36	100%	75	100%	41	100%

Table 4.8

Overall Quantifications of Angularity by Course

	Math 6		Math 7		Algebra	
	Frequency	Percent	Frequency	Percent	Frequency	Percent
Gross	18	50%	28	41.8%	3	6.2%
Intensive	11	30.6%	22	32.8%	15	30.6%
Extensive	7	19.4%	13	19.4%	17	34.7%
Ratio	0	0%	2	3%	8	16.3%
Rate	0	0%	2	3%	6	12.2%
Total	36	100%	67	100%	49	100%

Relationship between Fraction Schemes and Quantifications of Angularity

Before I investigated the relationship between students' measurement schemes for fractions and their quantifications of angularity, it was necessary to investigate the hypothesized hierarchy underlying both frameworks (see Figure 2.25). Although the results from the quantitative surveys provided preliminary evidence that there was a developmental progression for both the fraction schemes and quantifications of angularity, more investigation was needed. Therefore, students' scores for each fraction scheme were correlated with one another. Students' scores for each quantification of angularity were also correlated with one another. Results of the fraction schemes correlations are presented in Table 4.9. Results of the quantifications of angularity correlations are presented in Table 4.10.

Examining the correlations for both frameworks, the largest correlations occur closest to the main diagonal. Moving further away from the main diagonal, the correlations become smaller, indicating a simplex correlation structure (Davison et al., 1978; Guttman, 1955). This structure pattern provides evidence to indicate that there is an underlying order for these fraction schemes and quantifications of angularity, further indicating that these frameworks are hierarchical. Therefore, it was feasible to use students' highest attributed fraction scheme and highest attributed quantifications of angularity in future analyses.

Table 4.9

Correlation Between Each Fraction Scheme

	PWS	MSUF	MSPF	GMSF
PWS	1	.171*	.168*	.136
MSUF	.171*	1	.486**	.310**
MSPF	.168*	.486**	1	.522**
GMSF	.136	.310**	.522**	1
Frequency	105	32	9	6
Percentage	67.7%	20.6%	5.8%	3.9%

**Indicates significance at 0.01 level
*Indicates significance at 0.05 level

Table 4.10

Correlation Between Each Quantification of Angularity

	Gross	Intensive	Extensive	Ratio	Rate
Gross	1	-.971**	-.490**	-.237**	-.163*
Intensive	-.971**	1	.446**	.244**	.167*
Extensive	-.490**	.446**	1	.301**	.145
Ratio	-.237**	.244**	.301**	1	.495**
Rate	-.163*	.167*	.145	.495**	1
Frequency	49	101	51	16	8
Percentage	31.6%	65.2%	32.9%	10.3%	5.2%

**Indicates significance at 0.01 level
*Indicates significance at 0.05 level

To examine the relationship between students' measurement schemes for fractions and their quantifications of angularity, student data from each instrument was paired to provide two data points for each student. As described above 152 students completed the MSFI and the QAI. However, these were not the same 152 students. A total of 155 unique students completed either the MSFI or the QAI. Of these 155, three students did not complete the MSFI and three different students did not complete the QAI. For this part of data analysis, students with missing data were removed. There were a total of 149 students with matching student data from both the MSFI and QAI. Earlier in Chapter 2, I hypothesized that a concept of measurement precedes the ability to quantify angularity using additive units, and therefore a measurement concept is necessary for the construction of more sophisticated quantifications of angularity. I also claimed that these measurement schemes for fractions could be used to categorize students' concepts of measurement. Therefore, the fraction schemes are treated theoretically as the independent variable and quantifications of angularity are treated as the dependent variable.

The association between fraction schemes and quantifications of angularity was tested using the Goodman and Kruskal's Gamma (*G*) statistic. This association was based on students' highest attributed fraction scheme and quantification of angularity. It is important to note that students who had not constructed any fraction schemes were categorized as PrePWS and students who had not constructed any quantifications of angularity were attributed a gross quantification. Results indicate that there is a strong, positive and statistically significant association ($G=.643, p<.001$) between students' fraction schemes and their quantifications of angularity. In addition, a Spearman Rank Correlation was also calculated. Results indicate, again, that there is a positive and statistically significant correlation ($r_s=.526, p<.001$) between students' fraction schemes and their quantifications of angularity. Based on these results, students' construction of more sophisticated fraction schemes relate to the construction of more sophisticated quantifications of angularity.

A contingency table, along with frequencies, is presented in Table 4.11 to show the distribution of students' highest attributed fraction scheme and highest attributed quantification of angularity. Examining the distribution, consistent with the correlation coefficients, lower fraction schemes tend to be associated with less sophisticated quantifications of angularity, and the higher fractions schemes tend to be associated with more sophisticated quantifications of angularity. Looking at the patterns of data, no students who were PrePWS had constructed a ratio or rate quantification, and no students who had constructed a MSPF or GMSF had constructed a gross or intensive quantification. Furthermore, students who had constructed a MSUF or higher tend to construct more sophisticated quantifications of angularity.

Table 4.11

Frequencies and Associated Percentages by Overall Fraction Scheme and Quantification

	Gross	Intensive	Extensive	Ratio	Rate	Total
PrePWS	21 52.5%	15 37.5%	4 10%	0 0%	0 0%	40
PWS	26 34.2%	28 36.8%	16 21.1%	4 5.3%	2 2.6%	76
MSUF	2 9.1%	3 13.6%	11 50%	3 13.6%	3 13.6%	22
MSPF	0 0%	0 0%	3 60%	2 40%	0 0%	5
GMSF	0 0%	0 0%	2 33.3%	1 16.7%	3 50%	6
Total	49	46	36	10	8	149

It is important to note that the finding that four students were PrePWS but had also constructed an extensive quantifications suggested a developmental inconsistency. How could students without a MUSF construct an extensive quantification of angularity? The surveys of these four students were reexamined to investigate potential misclassification of these students. One student was able to correctly solve two PWS tasks, one MSUF, and one GMSF task. Although he completed all tasks on the survey, his work demonstrated inconsistency in his thinking and responses. Another student was able to correctly solve two PWS tasks, one MSUF, two splitting, and one MSPF task. Again, even though this student would solve the more complex tasks, he did not answer some of the tasks, and only drew pictures. The third student only correctly solve one MUSF and one splitting task. This student skipped many tasks on the survey; for example, he would solve two tasks then skip two more, but never skipped more than two pages. Since he did not answer many tasks, it is unclear that if he had completed the tasks this would have provided evidence that he had indeed constructed a more sophisticated fraction scheme. The fourth student was able to correctly solve one PWS task, three MSUF tasks, and two splitting tasks. She was similar to the third student and skipped several tasks. Again, it is unclear that if she had completed those tasks she would have provided evidence that she had constructed a more sophisticated fraction scheme. For these students, it appears that they were able to solve more complex tasks, but their thinking and effort was not consistent across the survey. Therefore, the fact that these students were classified as PrePWS might be explained as measurement error associated with the survey.

Based on my new reorganization hypothesis, students need to have constructed at least a MSUF, a measurement concept of fractions, to construct of an extensive quantification. Once students have a measurement concept, indicated by at least a MSUF, they are then able to construct more sophisticated quantifications of angularity, implying that a MSUF precedes the construction of a ratio and rate quantification. In order to further examine the relationship between students' fractions schemes and quantifications of angularity, each of the individual fraction scheme scores were correlated with students' highest quantifications of angularity. This was done to potentially better pinpoint the developmental progression associated with the fraction schemes and the necessary precursors for the construction of a more sophisticated quantification of angularity. Frequencies were entered into 2x5 contingency tables, and *G* was calculated to examine the magnitude of the relationship between each fraction scheme and students' highest quantification of angularity. A contingency table that shows the distribution of each fraction scheme and quantifications of angularity is provided in Table 4.12.

Results indicate that there is a positive and statistically significant correlation ($G=.488, p<.001$) between PWS and the quantifications of angularity. There is a strong, positive and statistically significant correlation ($G=.778, p<.001$) between MSUF and the quantifications of angularity. There is a strong, positive and statistically significant correlation ($G=.702, p<.001$) between splitting and the quantifications of angularity. There is a strong, positive and statistically significant correlation ($G=.831, p<.005$) between MSPF and the quantifications of angularity. There is a strong, positive and statistically significant correlation ($G=.914, p<.05$) between GMSF and the quantifications of angularity. These positive correlations indicate that students who construct more sophisticated fraction schemes tend to construct more sophisticated quantifications of angularity. Examining the distribution, results show that lower fraction schemes are associated with less sophisticated quantifications of angularity, and the higher fractions schemes are associated with more sophisticated quantifications of angularity. Moreover, no students who had constructed a PWS had constructed a ratio or rate quantification.

Of the students who had constructed a PWS, about 56% of them had not constructed an extensive quantification or higher. It can also be seen that MSUF and extensive are related; once students have constructed a MSUF, about 85% of them construct an extensive quantification of angularity or higher. Of the students who had not constructed a MSUF, 77% of them constructed an intensive quantification or below. This provides evidence that a measurement concept is necessary for the construction of at least an extensive quantification of angularity. Therefore, a MSUF is important for students to construct more sophisticated quantifications of angularity.

Table 4.12

Frequencies and Associated Percentages by Each Fraction Scheme and Overall Quantification

		Gross	Intensive	Extensive	Ratio	Rate	Total
PWS	Not Constructed	22 48.9%	15 33.3%	8 17.8%	0 0%	0 0%	45
	Constructed	27 26.0%	31 29.8%	28 26.9%	10 9.6%	8 7.7%	104
MSUF	Not Constructed	47 40.2%	43 36.8%	20 17.1%	4 3.4%	3 2.6%	117
	Constructed	2 6.3%	3 9.4%	16 50%	6 18.8%	5 15.6%	32
MSPF	Not Constructed	49 35.0%	46 32.9%	31 22.1%	7 5%	7 5%	140
	Constructed	0 0%	0 0%	5 55.6%	3 33.3%	1 11.1%	9
GMSF	Not Constructed	49 34.3%	46 32.2%	34 23.8%	9 6.3%	5 3.5%	143
	Constructed	0 0%	0 0%	2 33.3%	1 16.7%	3 50%	6

Based on these results indicating that a MSUF is necessary for the construction of at least an extensive quantification, each measurement scheme for fractions was correlated with each of the more sophisticated quantifications of angularity (i.e., extensive, ratio, and rate) to further examine this relationship. Results of these correlations are presented in Table 4.13. Since these were dichotomous scores for each fraction scheme and quantification, it was necessary to calculate Phi coefficients. Results indicate that there is a strong, positive correlation between MSUF and extensive (Φ=.494, p<.001) which was also statistically significant. Although all other correlations, except for MSPF and rate, were also statistically significant, this correlation between MSUF and extensive was the strongest. This correlation provides evidence that MSUF is an important stage in this developmental progression for the construction of more sophisticated quantifications of angularity. Students who have not constructed a MSUF tend to not construct an extensive quantification. However, students who have constructed a MSUF tend to construct an extensive quantification.

Table 4.13

Correlation Between Each Fraction Scheme and Each Quantification of Angularity

	Extensive	Ratio	Rate
MSUF	.494**	.346*	.238**
MSPF	.297**	.276**	.065
GMSF	.216**	.370**	.406**
Frequency	50	16	8
Percentage	33.6%	10.7%	5.4%

**Indicates significance at 0.01 level
*Indicates significance at 0.05 level

Transition from Additive to Multiplicative Reasoning in the Context of Angles

Based on the theoretical framework for the quantifications of angularity, an extensive quantification requires both the partitioning and iterating operations which enables students to reason additively when comparing angles. After students have developed an additive way to compare angles, they move to using multiplicative units for comparing of angles. However, the idea that students jump directly from using additive units to using multiplicative units seemed like a stretch. Therefore, I hypothesized that before students construct a ratio quantification, they need to construct the splitting operation in the context of angles (see Figure 2.25). For a ratio quantification, students rely on the multiplicative relationship between those partitions/iterations and the given angles, therefore splitting needs to be a precursor. Students must first be able to "split" in the context of angles, recognizing what it means to split the "openness" of an angle. In summary, an extensive quantification of angularity precedes the construction of the splitting operation in the context of angles. The splitting operation in the context of fractions precedes the splitting operation in the context of angles. Once students have this splitting operation concerning both fractions and angles, they are then able to think multiplicatively and construct a ratio quantification, which can also be extended to a rate quantification. However, angular splitting and a ratio quantification may co-develop, but angular splitting should precede a rate quantification. By examining the role that splitting in each context plays in this progression, we can better understand how students move from additive to multiplicative reasoning in the context of angles.

The first examination explored the relationship between splitting in the context of fractions, denoted Splitting F, and splitting in the context of angles, denoted Splitting A. Frequencies were entered into a 2x2 contingency table (Table 4.14), and *G* was calculated to examine the magnitude of the relationship between fractional and angular splitting. Results indicate that there is a strong, positive and statistically significant correlation (*G=.815, p<.001*) between fractional splitting and angular splitting. Looking at the distribution of frequencies, only 13.5% of students who had not constructed fractional splitting were able to construct angular splitting. However, 60.5% of students who had constructed fractional splitting were able to construct angular splitting. This indicates that students' construction of fractional splitting is related to the construction of angular splitting.

Table 4.14

Frequencies and Associated Percentages by Fractional Splitting and Angular Splitting

		Splitting A		
		Not Constructed	Constructed	Total
Splitting F	Not Constructed	96 86.5%	15 13.5%	111
	Constructed	15 39.5%	23 60.5%	38
Total		111	38	149

Note: G=.815, *p*<.001

The second examination explored the relationship between splitting in the context of angles, denoted Splitting A, and the quantifications of angularity. The goal was to determine if angular splitting was necessary for the construction of more sophisticated quantifications of angularity. Frequencies were entered into a 2x5 contingency table (Table 4.15), and *G* was calculated to examine the magnitude of the relationship. Results indicate that there is a strong, positive and statistically significant correlation (*G=.913, p<.001*) between angular splitting and the quantifications of angularity. Looking at the distribution of frequencies, 81% of students who had not constructed angular splitting had a gross or intensive quantification. No students who had not constructed angular splitting had a rate quantification, and only 3 had a ratio. Of the students who had constructed angular splitting, 86.8% were extensive or above. This indicates that students' construction of angular splitting is related to the construction of more sophisticated quantifications of angularity.

Table 4.15

Frequencies and Associated Percentages by Angular Splitting and Overall Quantification of Angularity

		Gross	Intensive	Extensive	Ratio	Rate	Total
Splitting A	Not Constructed	49 44.1%	41 36.9%	18 16.2%	3 2.7%	0 0%	111
	Constructed	0 0%	5 13.2%	18 47.4%	7 18.4%	8 21.1%	38
Total		49	46	36	10	8	149

Note: G=.913, p<.001

To help examine this developmental progression for students' ability to construct more sophisticated quantifications, the extensive quantification was correlated with students' angular splitting. The goal was to determine if an extensive quantification was necessary for the construction of angular splitting. Frequencies were entered into a 2x2 contingency table (Table 4.16), and G was calculated to examine the magnitude of the relationship. Results indicate that there is a strong, positive and statistically significant correlation ($G=.865$, $p<.001$) between extensive quantification and angular splitting. Looking at the distribution of frequencies, only 9% of students who had not constructed an extensive quantification had constructed angular splitting. Of the students who had constructed an extensive quantification, 42.0% had not constructed angular splitting. This indicates that students' tend to construct an extensive quantification before they are able to construct angular splitting.

Table 4.16

Frequencies and Associated Percentages by Extensive and Angular Splitting

		Splitting A		
		Not Constructed	Constructed	Total
Extensive	Not Constructed	90 90.9%	9 9.1%	99
	Constructed	21 42.0%	29 58.0%	50
Total		111	38	149

Note: G=.865, p<.001

The second examination explored the relationship between angular splitting and a ratio and rate quantification of angularity. The goal was to determine if angular splitting was necessary for the construction of a ratio or rate quantification. Frequencies were entered into a 2x2 contingency tables (Table 4.17 and Table 4.18), and *G* was calculated to examine the magnitude of the relationship. Results indicate that there is a strong, positive and statistically significant correlation between angular splitting and a ratio quantification ($G=.899$, $p<.001$) and between angular splitting and a ratio quantification ($G=1.00$, $p<.001$). Looking at the distribution of frequencies, only 2.7% of students who had not constructed angular splitting also had constructed a ratio quantification. Of the students who had constructed angular splitting, 65.8% had not constructed a ratio quantification. This indicates that students' construction of angular splitting precedes the construction of a ratio quantification of angularity. Exploring the relationship between angular splitting and a rate quantification, students who had not constructed angular splitting had not constructed a rate quantification. Of the students who had constructed angular splitting, 78.9% had not constructed a rate quantification. Again, this provides evidence that angular splitting is necessary for the construction of a ratio and rate quantification.

Table 4.17

Frequencies and Associated Percentages by Angular Splitting and Ratio

		Ratio		
		Not Constructed	Constructed	Total
Splitting A	Not Constructed	108 97.3%	3 2.7%	111
	Constructed	25 65.8%	13 34.2%	38
Total		133	16	149

Note: G=.899, *p*<.001

Table 4.18

Frequencies and Associated Percentages by Angular Splitting and Rate

		Rate		
		Not Constructed	Constructed	Total
Splitting A	Not Constructed	111 100%	0 0%	111
	Constructed	30 78.9%	8 21.1%	38
Total		141	8	149

Note: G=1.00, *p*<.005

Phase Two Results

Phase Two consisted of clinical interviews with students who participated in the QAI. The purpose of the clinical interviews was to investigate students' ways of thinking about tasks and their related actions and operations and to classify students into stages of quantification of angularity with increased accuracy. These classifications then served as the criterion measure for comparison to the results from the QAI, for evaluating the relationship to other measures (Krupa et al., 2019). That is, the classifications from the interviews were correlated with classifications from the QAI to determine the magnitude of the association, which provided a measure of criterion validity for the classifications based on the QAI. There were a total of 22 students who participated in these clinical interviews.

To evaluate interrater reliability, weighted Kappa (κ_ω; Cohen, 1968) was calculated for the independent ratings based on independently observing the interviews. κ scores for the overall quantification of angularity classification for each student interviewed was .809 (Table 4.19). Landis and Koch (1977) state that κ scores between .81 and 1.00 are "almost perfect", between .61 and .80 are "substantial", between .41 and .60 are "moderate" (p. 165). Accordingly, the κ_ω scores for the interview classifications were almost perfect. The two raters disagreed on only four students' classifications. The two raters then reconciled these four disagreements and decided on a single classification. Results of the interview coding are presented in Table 4.20. Of the 22 students, 36.4% students were attributed a gross quantification, 31.8% were attributed an intensive, 22.7% were attributed an extensive, 0% were attributed a ratio, and 9.1% were attributed a rate.

Table 4.19

Weighted Kappa Scores for Interview Coding

Instrument	Number of Disagreements	Number of Agreements	Weighted Kappa Score
QAI	4	18	.809

Table 4.20

Interview Quantification of Angularity

	Frequency	Percent
Gross	8	36.4%
Intensive	7	31.8%
Extensive	5	22.7%
Ratio	0	0%
Rate	2	9.1%
Total	22	100%

These classifications from the interviews were then compared to students' quantifications based on the QAI. A validity coefficient, Gamma, was calculated between students' classifications from the interview and the QAI, which provided a measure of criterion validity. This provided a comparison between students' QAI classifications and their interview classifications, the criterion (Kane, 2013). Results indicate that there is a statistically significant positive relationship between students' interview quantification and their QAI quantification (G=.623, p<.05). A Spearman Rank Correlation was also calculated, indicating that there is a positive correlation between students' interview quantification and their QAI quantification, which was statistically significant (r_s=.546, p<.01). These validity coefficients represent a strong association between students' classifications from the QAI and the interview (Cohen, 1992). This provides evidence that the scores from the QAI are valid for predicting students' quantifications of angularity.

Further assessing this association, the difference between students' interview quantifications and QAI quantifications was calculated to determine how the two classifications differed. These differences are presented in Table 4.21. Negative scores indicate that students' interview quantifications were lower than their QAI quantifications, and positive scores indicate that their interview quantifications are higher than their QAI quantifications. For example results show that for two students, students were attributed a quantification that was two stages higher than their attributed quantification from the interview. These two students were assigned a gross quantification from the interview, but their QAI indicated that they had constructed an extensive quantification. Overall, 59.1% of students' interview quantifications were the same as their QAI quantification. Furthermore, 86.4% of students' interview quantifications were equal or within one stage of their QAI quantification. Only 3 (13.6%) of the students' interview quantifications were two stages different.

Table 4.21

Difference Between Interview and QAI Quantification

	Frequency	Percent
-2	2	9.1%
-1	2	9.1%
0	13	59.1%
1	4	18.2%
2	1	4.5%
Total	22	100%

A contingency table is presented in Table 4.22 to show the alignment between students' interview quantification and their QAI quantification and to help further describe this association. Looking at the distribution, there were two students who were attributed ratio based on the QAI, but who were not attributed ratio from the interviews. As noted above, there were two students who were assigned a gross quantification from the interview, but their QAI indicated that they had constructed an extensive quantification. Another student who was attributed a gross quantification in the interview was given an intensive quantification on the QAI. Two students who were attributed an intensive from the interview but two were given gross on the QAI. There was more variability in the extensive quantification, as one student was given gross, one an intensive, and another ratio based on the QAI. For rate, there was one student who had rate from the interview but ratio on the QAI. The relative consistency of these classifications can also be seen in Table 4.22 by the fact that the ratings lie close to the diagonal consistent with the overall Kappa score. Overall, the criterion measure, students' interview classifications, had a strong association to the test scores, students' QAI classifications. This provides evidence that the scores from the QAI are valid for predicting students' quantifications of angularity (Kane, 2013; Krupa et al., 2019).

Table 4.22

Frequencies Between Interview Quantification and QAI Quantification

		QAI					
		Gross	Intensive	Extensive	Ratio	Rate	Total
Interview	Gross	5	1	2	0	0	8
	Intensive	2	5	0	0	0	7
	Extensive	1	1	2	1	0	5
	Ratio	0	0	0	0	0	0
	Rate	0	0	0	1	1	2
	Total	8	7	4	2	1	22

Chapter 5: Conclusions

The purpose of this study was to examine the relationship between students' concepts of measurement and their concepts of angle measurement. Specifically, the goal was to document sixth, seventh, and eighth grade students' measurement schemes for fractions and their quantifications of angularity and then investigate the relationship between them. This research was guided by the following question: What is the relationship between middle school students' measurement schemes for fractions and their quantifications of angularity? This final chapter will present a summary of the findings and a discussion of the results from the study to provide an answer to the research question. Insights, contributions, and implications for the field of mathematics education will be offered to better understand the results of this study. Finally, suggestions for future work concerning the teaching and learning of angles will be provided.

Students' Measurement Schemes for Fractions

From the quantitative survey of 152 sixth, seventh, and eighth grade students, results show that the majority of these middle school students had only constructed at most a PWS (see Table 4.2). This also indicates that less than one fourth of the students had constructed a more sophisticated fraction scheme, moving beyond part-whole understanding. This result supports prior research that shows that most students typically develop a part-whole understanding by the fifth grade (Norton, 2008; Olive, 1999; Wilkins & Norton, 2018). For example, Boyce and Norton (2016) found that about 54% of the sixth graders they interviewed had not constructed a measurement meaning of fractions.

Students begin learning about fractions as parts of wholes in third grade (CCSSI, 2010). This understanding is then related to a measurement concept of fractions in third grade, where a fraction a/b is understood as "a parts of size 1/b" (CCSSI, 2010, p. 24). Research suggests that a measurement concept of fractions is necessary for students' to understand improper fractions, more sophisticated concepts of fractions, as well as more complex mathematical concepts and topics (Hackenberg et al., 2016; Kieren, 1980; Lamon, 2007; Thompson & Saldanha, 2003; Wilkins & Norton, 2018). However, looking at Virginia's curriculum standards, students begin learning about fraction as parts of wholes in first grade (VDOE, 2016b). Although they note that fractions have different meanings, such as measurement (VDOE, 2016b), this measurement concept is not directly addressed in any of the standards and the notion of part-whole is emphasized. This could possibly be related to the results from this study, which indicate that most of these middle school students had not developed a measurement concept of fractions. Also, research shows that most textbooks in the United States emphasize the part-whole notion of fractions and typically do not move beyond that notion (Watanabe, 2007). This limited view of fractions as parts of wholes prevents students from understanding fractions as other constructs, such as measures (Hackenberg et al., 2016; Thompson & Saldanha, 2003; Wilkins & Norton, 2018). Taken together, this focus on a part-whole understanding of fractions may be related to students' lack of the development of a measurement concept of fractions.

Examining students' fraction schemes by grade level, results show that the majority of each grade level was comprised of students categorized as PrePWS or PWS (see Table 4.4). In fact, 91.9% of sixth grade, 85.7% of seventh grade, and 78.3% of eighth grade students were PrePWS or PWS. Again, the fact that Virginia's curriculum standards do not emphasize a measurement meaning of fractions may help explain why the majority of these middle school students were unable to construct a fraction scheme higher than PWS. However, what is concerning is that, 37.8% of sixth graders, 27.3% of seventh graders, and 28.6% of eighth graders were PrePWS. This indicates that almost a third of students in each grade level had no concept of fraction, and could not understand fractions as parts of wholes. Overall, less than 10% of sixth graders, 15% of seventh graders, and 22% of eighth graders had a measurement concept of fractions.

Further examining these results, less than 10% of sixth graders had constructed a MSUF and could work with an iterable unit of 1. This percentage improved for seventh and eighth grade, but still was less than 15%. None of the sixth graders, and less than 5% of seventh and eighth graders were able to generalize their scheme from unit fractions to proper fractions to develop a MSPF. Also, none of the sixth graders, less than 2% of seventh, and less than 4% of eighth graders had constructed a GMSF. Comparing these results to other studies, Wilkins and Norton found that within four different studies, over 60% of sixth graders had constructed a MSUF (Norton & Wilkins, 2009, 2013; Norton et al., 2018; Wilkins & Norton, 2011). In two other studies, they found that 61% of one group of seventh graders and 65% of another group had constructed a MSUF (Norton & Wilkins, 2010, 2013). From their overall results of these six studies, Wilkins and Norton (2013, 2018) determined that students who had constructed a MSUF by the end of sixth grade were 13 times the odds of students who had not constructed a MSUF to construct the splitting operation. After the construction of the splitting operation, students are enabled to construct more sophisticated fraction schemes. Since most of the students in this study did not have a measurement concept of fractions, and had not constructed a MSUF, they are less likely to develop a more sophisticated fraction scheme by the end of their next grade level.

Looking at students' fraction schemes by course, results also indicate that the majority of students enrolled in Math 6 and Math 7 were PrePWS or PWS, and 50% of students enrolled in Algebra were PrePWS or PWS (see Table 4.5). Again, these percentages indicate that the majority of students do not have a measurement concept of fractions, and have not moved beyond a part-whole understanding of fractions. What is also concerning is that 17.4% of Algebra students were PrePWS (see Table 4.5), meaning they had no concept of fraction, and did not understand fractions as parts of wholes. Due to this lack in understanding, these students are significantly less likely to develop a more sophisticated fraction scheme by the end of their next grade level, possibly limiting students' understanding of more complex mathematical concepts and topics (Hackenberg et al., 2016; Kieren, 1980; Lamon, 2007; Thompson & Saldanha, 2003; Wilkins & Norton, 2018).

Looking at the distribution of these results across grade level and course, it is evident that there is a developmental progression for these fraction schemes. Results indicate that as students progress in grade level and more advanced courses, their fraction schemes advance developmentally. Moving from sixth to seventh to eighth grade, the percentages of the lower fraction schemes (i.e., PrePWS and PWS) most often become smaller and the percentages of the higher fraction schemes (i.e., MSUF, MSPF, and GMSF) become larger. A similar pattern is seen when comparing Math 6 to Math 7 to Algebra. Further examining this progression, a hierarchy can be seen through the correlation table (see Table 4.9). Results indicate that the fraction schemes represent a simplex correlation structure, meaning there is an order for these fraction schemes (Davison, et al., 1978; Guttman, 1955), further indicating that they are hierarchical. This provides further support for Wilkins and Norton's (2018) claim that these fraction schemes represent a hierarchy and learning progression.

Overall, the results of this study show that the majority of these middle school students do not possess a measurement concept of fractions. Also, there were many students who do not possess any fraction concepts. Even though Virginia's curriculum standards do not emphasize a measurement meaning of fractions and focus mainly on part-whole understanding, it is alarming that such a large percentage of students either do not have a measurement concept of fractions or do not have a fraction concept at all. Moving forward, these students are less likely to develop more sophisticated fraction schemes and therefore continue to lag behind (Wilkins & Norton, 2013, 2018).

Students' Quantifications of Angularity

From the quantitative survey of 152 sixth, seventh, and eighth grade students, results show that the majority of these middle school students had constructed at most an intensive quantification (see Table 4.6). This indicates that about two thirds of the students did not use units when comparing angles; they used visual comparisons or nonadditive units. Less than one quarter of the students had constructed additive units for comparing angles, and about 12% of students had constructed multiplicative units and could relate angles to circles. This aligns with Bütüner and Filiz's (2016) finding indicating that many high achieving sixth grade students limited a static conception of angles, with only 36% of students being able to correctly identify angles and 39% able to correctly compare the size of angles. This also aligns with prior research that shows that some students think that angle measurement can be determined by measuring the side lengths of the angles, comparing the size of the arc representing the angle, or simply measuring the linear distance between the sides of the angle (Barabash, 2017; Bütüner & Filiz, 2016; Clements & Battista, 1989, 1990; Clements, 2003; Clements et al., 1996; Piaget et al., 1981/1960).

Although standards documents emphasize understanding angles within the context of circles in fourth (CCSSI, 2010) and fifth grade (VDOE, 2016a), results of this study indicate that students are not making that connection and are unsuccessful in quantifying angularity. However, research indicates that quantifying angularity is no easy task for adults, let alone children. For example, prior research shows that undergraduate and pre-service teachers struggle to connect angles to circles. Moore (2013) found that undergraduate precalculus students focused mainly on degrees for measuring and comparing angles, and struggled when no units were given. In another study, Hardison and Lee (2019) found that 59% of pre-service elementary teachers did not use units for comparing angles. Taken in tandem, these results indicate that although curriculum standards may emphasize angle measurement as a fractional amount of a circle (CCSSI, 2010; VDOE, 2016a), quantifying angularity remains a difficult concept for both adults and children.

Examining students' quantifications of angularity by grade level, results show that the majority of students in sixth and seventh grade were categorized as gross or intensive: over 80% of sixth grade and about 70% of seventh grade; only 39% of eighth grade were categorized as gross or intensive (see Table 4.7). Surprisingly, 7.3% of eighth graders had constructed only a gross quantification. Although these results support prior work that demonstrates students often view angles as static figures (Barabash, 2017; Bütüner & Filiz, 2016; Clements & Battista, 1989, 1990; Clements, 2003; Clements et al., 1996; Kontorovich & Zazkis, 2016; Piaget et al., 1981/1960; Smith et al., 2014), it is concerning that these students are not using units for comparison. Without these conceptions of an angle as a dynamic construction, students are unable to develop a more abstract conceptualization of angles, thus preventing them from developing a better and deeper understanding of angle, as well as limiting their future performance and learning gains (Clements & Burns, 2000; Cullen et al., 2018; Mitchelmore & White, 1998, 2000; Smith et al., 2014).

Looking at the percentage of students who did use units, less than 20% of sixth graders had constructed an extensive quantification and therefore could work with additive units. This percentage improved for seventh and eighth grade, but still was low (see Table 4.7). None of the sixth graders, and about 5% of seventh graders were able to develop multiplicative units for comparing angles, and construct a ratio quantification. This percentage improved for eighth grade, where about 15% of students had constructed a ratio quantification. Also, none of the sixth graders, about 5% of seventh graders, and less than 10% of eighth graders were able to extend their ratio quantification into the context of multiple circles. Taking these results together, all of the sixth graders, 89.3% of seventh graders, and 75.6% of eighth graders had not developed multiplicative units and could not relate angles to the context of circles.

Further examining the construction of additive and multiplicative units, it is beneficial to look at the distribution by course. Math 6 results were the same as the sixth grade results because all sixth graders were enrolled in Math 6. However, Math 7 and Algebra results were different than the seventh and eighth grade results. Over 40% of students in Math 7 were not using units to compare angles, but were simply comparing angles based on visual judgements (see Table 4.8). Of the students who could work with units, less than 20% of them were able to use additive units and only 6% were able to use multiplicative units. This distribution changed for Algebra, with only 6.2% of students who were not using units to compare angles. About 35% of Algebra students were able to use additive units and almost 30% were able to use multiplicative units.

These results provide evidence for the claim that students' understanding of angle measurement as a fractional amount of a circle is not being supported (Moore, 2013; Thompson, 2008). The majority of students had neither constructed multiplicative units nor were using multiplicative thinking to compare angles. This indicates that these students were unable to understand the multiplicative relationship between their quantities of measures, specifically maintaining the relationship between the angle, subtended arc, and circle's circumference

(Hardison, 2018; Moore, 2013). Based on prior work, these results make sense since most students most often have a static conception of angles (Barabash, 2017; Bütüner & Filiz, 2016; Clements & Battista, 1989, 1990; Clements, 2003; Clements et al., 1996; Kontorovich & Zazkis, 2016; Piaget et al., 1981/1960; Smith et al., 2014).

Although these results support prior work demonstrating that students are often limited to a static conception of angles, these results raise several concerns. First, these students have not constructed an understanding of angles within the context of circles despite curriculum standards emphasizing this notion in fourth (CCSSI, 2010) or fifth grade (VDOE, 2016a). From Piaget et al.'s work (1981/1960), students within this age range should have moved past visual comparisons and focus on angular separation. Secondly, it is also surprising that 94% of Math 7 and 71.5% of Algebra students were not thinking multiplicatively and could not relate angles to circles. If this trajectory continues, when these students move into Geometry and Trigonometry without multiplicative units, they will continue to lag behind. Having a conceptual understanding of angle measurement is a necessary component for pursuing further geometric topics such as right triangles, trigonometry functions, and radian measure (Moore, 2013; Yigit, 2014). Thirdly, these results also offer more evidence as to why students perform poorly on geometry and measurement tasks and assessments (Melo & Martins, 2015; Mullis et al., 2016; NCES, 2011), and further supports the claim that there is a disconnect between students' measurement and geometry concepts.

A highlight from these results is that as students progress in grade level, their quantifications of angularity advance developmentally. As students move from sixth to seventh to eighth grade, the percentage of students who had constructed a more sophisticated quantification (i.e., extensive, ratio, and rate) increases, and the percentage of students who had constructed a less sophisticated quantification decreases (i.e., gross and intensive). A similar pattern is seen when comparing Math 6 to Math 7 to Algebra. Further examining this progression, a hierarchy can be seen through the correlation table (Table 4.10). Results indicate that the quantifications of angularity represent a simplex correlation structure, meaning there is an order for these quantifications (Davison, et al., 1978; Guttman, 1955), further indicating that they are hierarchical. This supports my hypothesis that that these quantifications of angularity represent a hierarchy and learning progression. By combining Hardison's (2018) quantifications with Piaget's (1965/1952) and Steffe's (1991) explanation of quantifications, I was able to develop a progression of schemes for quantifications of angularity, including the different mental actions and operations used to conceive angle measurement. Earlier, I claimed that an extensive quantification represented a transition from visual concepts of angle, to an additive manipulation of units to describe angle, which was necessary for the construction of more sophisticated quantifications (i.e., ratio and rate). Students needed to be able to use additive units before they could use multiplicative units. The results of the study support this claim, that there is a progression to these quantifications.

Overall, the results of this study show that the majority of these middle school students do not possess a measurement concept of angle. Most students had not constructed a quantification above intensive and did not use units for comparing angles. In addition, most students could not relate angles within the content of circles. Although curriculum standards do focus on angle measurement as a fractional amount of a circle (CCSSI, 2010; VDOE, 2016a), students are not understanding this concept. Furthermore, it is concerning that students were not using additive nor multiplicative units. If this trajectory continues, these students will continue to lag behind as they progress into Geometry and Trigonometry, as this concept of angle measurement is a necessary component for pursuing further geometric topics (Moore, 2013; Yigit, 2014).

Examining the Relationship between Fraction Schemes and Quantifications of Angularity

Based on the theoretical framework, I hypothesized that students' development of a measurement concept, as indicated by their fraction schemes, is positively related to their development of quantifications of angularity. Therefore, to investigate this hypothesized relationship, students' overall fraction schemes were correlated with their overall quantifications of angularity. From the correlational analysis relating students' highest fraction scheme and quantification, there was a statistically significant and strong positive correlation ($G=.643, p<.001; r_s=.526, p<.001$) between students' fraction schemes and their quantifications of angularity. This means that students' construction of more sophisticated fraction schemes relates to the construction of more sophisticated quantifications of angularity.

Looking at the distribution of students' highest attributed fraction scheme and overall quantification of angularity (see Table 4.11), lower fraction schemes tend to be associated with less sophisticated quantifications of angularity, and the higher fractions schemes tend to be associated with more sophisticated quantifications of angularity. For example, from the students who were PrePWS, 90% of them were attributed a gross or intensive, and only 10% were attributed an extensive quantification (recall that the classification of these 4 students' were likely underestimated based on their inconsistent responses on the MSFI). For the students who had constructed a measurement concept of fractions, they tended to construct higher quantifications of angularity. For example, of the students who had constructed a MSUF, about 73% had constructed at least an extensive quantification, and about 27% had constructed at least a ratio quantification of angularity. Of the students who had constructed a MSPF or a GMSF, none of them were attributed a gross or intensive quantification and 40% and 67% of them, respectively, were attributed with at least a ratio quantification.

From the analysis of the overall distribution of fraction schemes, it was determined that the majority of these middle school students did not possess a measurement concept of fractions. Consistent with the hypothesis of the study and the positive correlation between schemes and quantification, it seems clear why the majority of these middle school students did not possess a measurement concept of angle (i.e., most did not possess a quantification above intensive). Because the majority of students were PrePWS or PWS; they had not constructed a concept of measurement and had not constructed an iterable unit of 1. Based on the theoretical framework, the construction of a concept of measurement precedes the ability to quantify angularity using at least additive units, and therefore a measurement concept is necessary for the construction of more sophisticated quantifications of angularity. However, due to the lack of a measurement concept of fractions, moreover an iterable unit of 1, they were unable to extend a measurement concept into the context of angles, further explaining why students did not use additive units in the context of angles. In other words, students did not possess a measurement concept and therefore could not quantify angularity using additive units.

In order to investigate the hypothesis that students need to construct at least a MSUF, corresponding to a measurement concept of fractions, to construct an extensive quantification of angularity (see Figure 2.25), each of the individual fraction scheme scores were correlated with students' highest quantifications of angularity. Results indicated that there is a strong, positive and statistically significant correlation ($G=.778, p<.001$) between MSUF and the quantifications of angularity. Looking at the distribution of each fraction scheme with students' highest quantification (see Table 4.12), the majority of students who had not constructed a MSUF had constructed at most an intensive quantification (77%). Of these students, 17.1% had constructed an extensive quantification. However, once students had constructed a MSUF, the majority had also constructed at least an extensive quantification (84.4%). Recall that Kieren's (1980) notion of fraction as a measure focuses on iterating a unit, resulting in an extensive quantity. The MSUF involves an iterable unit, a multiplicative quantity, and is therefore more advanced than Kieren's (1980) notion. Therefore, it seems reasonable that some students who construct fractions as measures (in the Kieren sense) are developing extensive quantities before they are able to construct multiplicative quantities.

To further investigate this relationship between extensive and MSUF, each measurement scheme for fractions was correlated with each of the more sophisticated quantifications of angularity (i.e., extensive, ratio, and rate). Results indicated that there was a strong, positive correlation between each fraction scheme (i.e., MSUF, MSPF, GMSF) and the quantifications (see Table 4.13). However, MSPF was not related to a rate quantification. The strongest correlation was between MSUF and extensive ($\Phi=.494, p<.01$), providing evidence that MSUF is an important stage in this developmental progression for the construction of more sophisticated quantifications of angularity. This provides evidence that a measurement concept, as indicated by a MSUF, is necessary for the construction of at least an extensive quantification. This further supports the hypothesis that once students have a measurement concept they are poised to construct more sophisticated quantifications. Once students have an iterable unit of 1, they are able to extend their measurement concept into the context of angles

Overall, these results indicate that the fraction schemes and quantifications of angularity frameworks are hierarchical and represent a developmental progression. The results of this study also support the notion that students' measurement schemes for fractions are related to their quantifications of angularity. As students construct more sophisticated fraction schemes, they are poised to construct more sophisticated quantifications of angularity. Since measurement precedes the development of more sophisticated quantifications, once students develop a measurement concept, as indicated by the measurement schemes for fractions, they are then able to construct more sophisticated quantifications of angularity. Once students have an iterable unit of 1, they are better enabled to work with units (Wilkins & Norton, 2018). This allows them to extend and transfer these operations into the context of angles. Therefore, once students develop a MSUF they tend to construct at least an extensive quantification.

Transition from Additive to Multiplicative Reasoning in the Context of Angles

In the quantifications of angularity framework, students jump from an extensive quantification where they use additive units to a ratio quantification where they use multiplicative units in the context of circles. I hypothesized that splitting may help explain this developmental progression. Before students construct a ratio quantification, they need to construct the splitting operation in the context of angles (see Figure 2.25). For a ratio quantification, students rely on the multiplicative relationship between those partitions/iterations and the given angles, therefore splitting needs to be a precursor. Wilkins and Norton (2018) describe the transition from the construction of a MSUF to the construction of a MSPF as being associated with children’s construction of splitting. Splitting makes it possible for children to reverse their thinking which is necessary for the construction of a MSPF. This reversible thinking also seems necessary for students’ construction of a ratio understanding of angles. To examine the role of splitting in the construction of the different stages of quantification of angularity, I examined several relationships associated with splitting in the context of angles.

First, I examined the relationship between fractional splitting and angular splitting. I thought students needed to be able to split in the context of fractions before they split in the context of angles. Results indicated that fractional splitting was strongly related to angular splitting ($G=.815$, $p<.001$). However, from an examination of the contingency table (see Table 4.14) it is unclear if fractional splitting precedes angular splitting, but instead they appear to co-develop. This is seen in the fact that the off-diagonal cases are evenly split, in some cases students seem to construct splitting in the context of angles first, and in other cases it was the opposite. However, when students construct angular splitting, they must recognize what it means to split the openness of an angle instead of merely splitting one of the rays, or the opening of the angle. Students must know what it means to split before they can reorganize their splitting operations to the context of angles. However, the transfer of the splitting operation from one context to the other may co-develop.

Next, I examined the relationship between angular splitting and all the quantifications of angularity. I thought that since splitting is necessary for the construction of more sophisticated fraction schemes (Wilkins & Norton, 2018), that angular splitting would be necessary for the construction of more sophisticated quantifications (see Figure 2.25). Results indicated that angular splitting was statistically significantly related to the quantifications of angularity ($G=.913$, $p<.001$). Looking at the contingency table (see Table 4.15) it appears that angular splitting precedes the construction of more sophisticated quantifications. For example, no students who had not constructed angular splitting were able to develop a rate quantification, and only three students had a ratio. Also looking at the contingency table, it appeared that extensive seemed be an important stage in this developmental progression. Of the students who did not construct angular splitting, 16.2% had an extensive quantification. On the flip side, of the students who had constructed angular splitting, 47.4% had an extensive quantification. It appears as if students construct an extensive quantification prior to the construct of splitting in the context of angles.

To further investigate this, I examined the relationship between an extensive quantitation and angular splitting. I thought that an extensive would precede angular splitting since students move from basic partitions and iterations to the simultaneous coordination of these two operations, which is defined as the splitting operation (see Figure 2.25). Results indicated that the construction of an extensive quantification was strongly related to angular splitting ($G=.865$, $p<.001$). Looking at the contingency table (see Table 4.16) it appears that an extensive quantification precedes the development of angular splitting based on the increased number of students in the lower left cell and fewer students in the upper right cell. Again, this supports the theoretical framework suggesting that students need to develop additive units, working with partitions and iterations, before they can coordinate those partitions and iterations.

Finally, I examined the relationship between an angular splitting and a ratio and rate quantification of angularity. This was to provide further insight into the role that angular splitting plays in the construction of more sophisticated fraction schemes. I hypothesized that students would need to be able to split in the context of angles before they were able to work with multiplicative units (see Figure 2.25). Results indicated that angular splitting was statistically significantly related to a ratio ($G=.899$, $p<.001$) and a rate quantification of angularity ($G=1.00$, $p<.001$). Looking at the contingency tables (see Table 4.17 and Table 4.18) it is clear that angular splitting precedes the construction of a ratio and rate quantification. For example, only three (2.7%) students who had not constructed angular splitting were able to develop a rate quantification, and no students had a ratio. This seems to suggest that angular splitting is necessary for the construction of a ratio and rate quantification. A ratio quantification comes about once a student is able to recognize a multiplicative relationship between the two quantities being compared (Hardison, 2018; Thompson, 1994). They also have to be able to maintain the multiplicative relationship between the arc length and circumference of the circle (Hardison, 2018; Moore, 20103). Therefore, students must not only be able to work with multiplicative units, but also relate this relationship in the context of circles. This then makes sense that angular splitting, enabling the ability to work with multiplicative units, would occur before students transfer these multiplicative units into the context of circles.

To summarize, in my new reorganization hypothesis, I argue that the transition from an extensive to a ratio quantification appears to be a huge developmental jump. Students move from using additive units to maintaining multiplicative relationships between arc lengths and circumferences of circles. I argued that the splitting operation could help explain this developmental jump, and provide further insight into this transition. Results from this exploration indicates that fractional splitting and angular splitting are related. Students need to be able to relate their splitting operation in the context of fractions to a splitting operation in the context of angles. However, it is important to note that although these two operations are related, it is unclear whether one precedes another or they co-develop. Just as students co-construct units coordination with whole numbers and units coordination with fractions (Boyce, & Norton, 2016), it may be that students co-construct splitting and angular splitting. Nonetheless, these results indicate that students need to have at least an extensive quantifications before they can construct angular splitting; that is, they need to be able to work with additive units before moving to working with multiplicative units. Once they have an extensive quantification, they can develop angular splitting, which enables them to construct a ratio and rate quantification.

Overall, angular splitting plays a major role in this developmental progression as students transition from an extensive to a ratio to a rate quantification.

Validation of Students' Quantifications of Angularity

The five main components for assessing the validity of an instrument are test content, response processes, internal structure, relations to other variables, and consequences of testing (AERA et al., 2014; Krupa et al., 2019). The validation of students' quantifications of angularity was conducted in multiple parts: (a) the pilot study was used to assess test content, response process, internal structure, and consequences of testing; (b) Phase One provided insight into internal structure and reliability; and (c) Phase Two was used to assess relations to other variables. By combining the results from these three parts, better insight can be gained into the validity of the QAI. Each component for assessing validity will be discussed.

Test Content

Test content was evaluated by combining results of the Pilot Study with results of the clinical interviews in Phase Two. When evaluating test content, one must examine the "relationship between the content of a test and the construct it is intended to measure" (AERA et al, 2014, p. 14). In this case, I evaluated the relationship between the content of the QAI to see if it could be used to assess students' quantifications of angularity. The development of the QAI involved several different iterations of testing and revising. During pilot testing, interviews were conducted with five students to evaluate the tasks, and to ensure they measured what they were intended to measure (Krupa et al., 2019). Three expert researchers were also involved in evaluating the tasks. Results from the first round of interviews raised concerns about the quality of the tasks and indicated that some tasks were not valid for assessing a ratio and rate quantification. After these concerns were addressed, results from the second round of interviews indicated that these revised tasks were more valid for assessing a ratio and rate quantifications. However, students still did not understand some tasks. After the tasks were revised, three expert researchers in the field of mathematics education evaluated the tasks, raising several concerns about test content. One thing the experts noted was that even though some tasks were designed to assess additive thinking, they were actually assessing multiplicative thinking. Through these multiple iterations of evaluations and revisions, I felt that I had enough evidence to believe that the final iteration of tasks were valid for assessing students' quantifications of angularity. However, because the final iteration of tasks were not given to a group of students, only face validity was provided.

After the final iteration of tasks was given to the sample of students, 22 students participated in a clinical interview. Throughout these interviews, students did not appear to have any misunderstandings of the tasks; they fully understood what was being asked. Therefore, this QAI is related to the construct it was designed to measure, students' quantifications of angularity.

Response Process

Response process was evaluated through the clinical interviews conducted in the Pilot Study and during Phase Two. When evaluating response process, the researcher looks for "evidence that connects how test takers may respond to a test item and how they actually respond to the item" (Krupa et al., 2019, p. 6). In the development of the tasks, I used prior research that examined students' actions and responses for solving tasks involving angles. For example, Hardison (2018) documented several motions that students use when comparing angles. I then used these descriptions to create a guide for assessing students' quantifications (see Table 3.11). From this, I then created my tasks to assess those particular actions.

During the first round of interviews in the Pilot Study, students did not respond to the tasks as I had hoped they would. Some students said that some tasks were confusing, or they did not understand the question. During the second round of interviews, after addressing these issues, students responded in the intended manner. There were no major issues with response process, as students understood what the tasks were asking and responded in a manner consistent with my hypotheses. During Phase Two interviews, again, students responded in the intended manner. Students understood what the tasks were asking and had no issues with understanding notation or language within the tasks. These results together provide evidence that this component of validity has been met for the QAI.

Internal Structure

Internal structure was evaluated during the Pilot Study, Phase One, and Phase Two. Internal structure is “the degree to which the relationship among test items and test components conform to the construct on which the proposed test score interpretations are based” (AERA et al., 2014, p. 16). Therefore, the tasks on the QAI were evaluated for their consistency and relationship to the quantifications of angularity framework. During the Pilot Study, the final iteration of tasks involved four tasks per quantification. This was to ensure there were an equal number of tasks for each quantification to help provide a balanced internal structure.

Further exploring the internal structure of the QAI, it is beneficial to examine how the quantifications scores correlate with one another. Examining the quantifications of angularity correlations, presented in Table 4.10, the largest correlations occur closest to the main diagonal. Moving further away from the main diagonal, the correlations become smaller creating a simplex correlation structure. This structure provides evidence that the measures of different stages of quantification form a hierarchical structure as theorized. In other words, these correlations show that measures that are closer together in the hierarchy tend to correlate more highly with one another, and measures that are further apart in the hierarchy tend to not be as correlated. This provides more evidence to support the internal structure validity component.

Results of interrater reliability also provide some evidence for internal structure. The Kappa scores for the two coders scoring of the QAI during Phase One (see Table 4.2) indicate almost perfect agreement between coders for all quantifications of angularity classifications (Landis & Koch, 1977). When looking at the Kappa scores for the Phase Two clinical interview coding, κ_ω scores were .809, again indicating almost perfect agreement (Landis & Koch, 1977). This provides some evidence for the reliability of these scores, which also provides some evidence for the validity of these classifications (Kane, 2013).

Relations to Other Variables

This component of validity was addressed by using the results of the 22 clinical interviews conducted in Phase Two. The goal is to "provide evidence about the degree to which these relationships are consistent with the construct underlying the proposed test score interpretations (AERA et al., 2014, p. 16). Therefore, I sought to compare students' "true" quantification of angularity to their QAI classification.

Students' classifications from the interviews, provided a criterion measure of students' quantifications by which to compare students' quantifications as determined by the QAI. These interviews provided a better representation of students' quantifications of angularity, as they enabled me to better understand students' thinking. As students worked through the tasks, I was able to ask follow up questions to truly understand their processes, providing a clearer evaluation of their quantifications of angularity.

When comparing students' quantifications based on the interviews to students' quantifications based on the QAI, results indicated a statistically significant positive relationship between students' interview quantification and their QAI quantification (G=.623, p<.05). A Spearman Rank Correlation was also calculated, indicating a positive correlation between students' interview quantification and their QAI quantification, which was statistically significant (r_s=.546, p<.01). These validity coefficients represent a strong association between students' classifications from the QAI and the interview (Cohen, 1992). This indicates that the classifications based on the QAI are similar to those inferred from the interviews.

Consequences of Testing

Consequences of testing were assessed by examining the results of the clinical interviews from the Pilot Study and Phase Two, and by observing students taking the QAI. This component assesses the "degree to which anticipated consequences from administering a test...align with an intended purpose of the test" (Krupa et al., 2019, p. 8). This study involved minimal risks, and per IRB guidelines, possible consequences of testing were limited in the design of this study. Students' participation was voluntary, it had no effect on their grades in school, and all their information would be blinded to maintain anonymity. As students were interviewed, there were no apparent negative consequences of testing. Students responded to the tasks in a manner similar to the hypothesized responses, with no unexpected behaviors. However, it is often the case that actual consequences of testing may not be known until after time has elapsed since the administration of the instrument (AERA et al., 2014). For now, results indicate there were no negative consequences of testing.

Insights, Contributions, and Implications for the Field

Several insights, contributions, and implications are provided as a result of this work. First, the major contribution this study offers is the quantifications of angularity framework and its connection to the measurement schemes for fractions framework. Secondly, this study offers a new instrument that can be used to assess students' quantifications of angularity. Finally, there are several curriculum and teaching implications that follow as a result of this study.

A New Conceptual Framework

The quantifications of angularity framework was developed from a synthesis of related literature. Hardison (2018) developed an initial framework that only included gross, extensive, ratio, and rate quantifications. After reading Piaget's (1965/1952) and Steffe's (1991) distinction between gross and intensive quantities, it seemed theoretically important to include an intensive quantification. I thought that moving from a gross to extensive was large developmental jump for understanding students' thinking. It can be difficult for students to move from using visual perceptions to compare angles to using units, especially when curriculum standards focus on the static image of angles as the intersection of two rays (VDOE, 2016b, 2016c). Therefore, I included the intensive quantification to help describe this developmental progression for how students quantify angularity. Instead of jumping from using visual comparisons to units, the intensive quantification describes a transitional stage where students have abstracted perceptual relationships and no longer rely on figurative material (Piaget, 1965/1952; Steffe, 1991).

In addition to creating new stages in the quantifications of angularity progression, this study also allowed for the exploration to distinguish the different stages of the quantifications of angularity. For example, in examining this developmental progression from extensive to ratio, it appeared that moving from additive units to multiplicative units in the context of circles was a major leap for students to make. It seemed as if there should be a stage of transition where students began using multiplicative units. With an extensive quantification, students use the partitioning and iterating operation (Hardison, 2018; Piaget et al., 1981/1960). This allows them to work with additive units. By developing the splitting operation, students can simultaneously partition and iterate, enabling them to reverse their thinking and treat those two operations as one (Hackenberg, 2010; Hackenberg et al., 2016; Norton, 2008; Steffe, 2002; Wilkins & Norton, 2011). With splitting, students are able to work with multiplicative units. Therefore, similar to the transition from an MSUF to MSPF, it seemed that the construction of splitting in the context of angles would be an important transitional stage between extensive and ratio. Results of this study show that splitting in the context of angles is necessary for the construction of a ratio and rate quantification. As a result, in this new framework, an angular splitting operation should be included and emphasized because it is an important developmental stage for helping describe students' quantifications of angularity (see Figure 5.1).

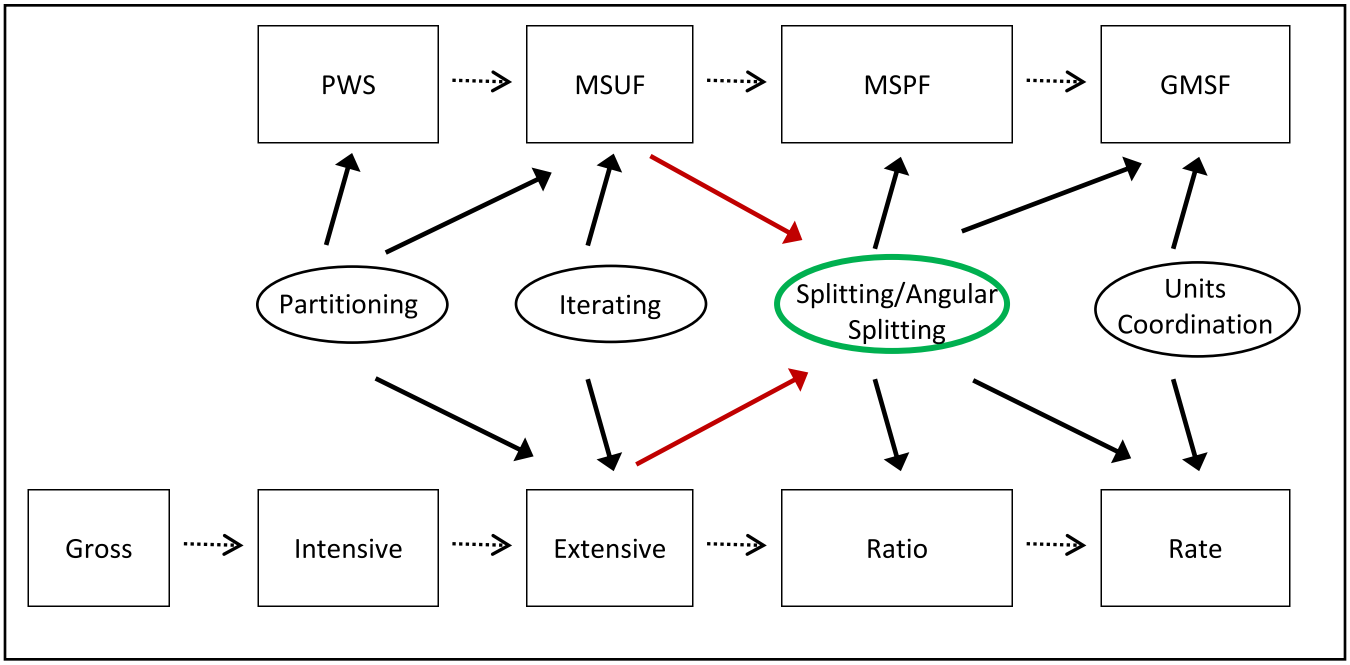
A Revised Reorganization Hypothesis

The purpose of this study was to investigate the relationship between students' measurement schemes for fractions and their quantifications of angularity. Throughout the literature, there seemed to be a connection between measurement, fractions, and students' notions of angle measurement. In the most conceptual sense, measurement involves the coordination of partitioning (a quantity into a unit) and iterating (that unit to recreate the whole) (Kieren, 1980; Piaget et al., 1981/1960). These are the same operations used for understanding and working with fractions (Hackenberg, 2007; Hackenberg et al., 2016; Lamon, 1996, 2012; Olive, 1999; Steffe, 2002, 2003; Steffe & Olive, 2010; von Glasersfeld, 1981; Wilkins & Norton, 2011, 2018). Moving into the context of angles, students also use the same operations to help conceptualize angle measurement as a fractional amount of a circle (Hardison, 2018; Moore, 2013; Thompson, 2008). Therefore, in my new reorganization hypothesis, I presented a model to help connect the measurement schemes for fractions framework with the quantifications of angularity framework (see Figure 2.25).

Results of this study indicated that students' measurement schemes for fractions are indeed related to their quantifications of angularity. As students construct a more sophisticated measurement scheme for fractions, they tend to construct more sophisticated quantifications of angularity. It was also determined that students need at least a MSUF, indicating a measurement concept, before they are able to construct an extensive quantification of angle. They also appear to construct an extensive quantification before they construct angular splitting (noted by the red arrows in Figure 5.1). Finally, they need angular splitting before they construct a ratio or rate quantification. All of these results provide support for the reorganization hypothesis and help to confirm the model. However, based on the results of this study, I revised that model to include angular splitting. In Figure 5.1, I have presented a preliminary revised model.

Figure 5.1

A Revised Reorganization Hypothesis



Results of this study indicated that angular splitting was related to fractional splitting. Based on the results it appears that splitting may co-develop across contexts, instead of transferring from one to the other. However, in order to develop angular splitting, students must know what it means to split the openness of an angle. From examining student work (see Figure 2.22 and 2.24), some students have constructed a splitting operation but are unable to then transfer this operation to the context of angles. I believe that students must first know what it means to split before they can split in the context of angles. Therefore, in the new model, the green circle surrounding the splitting and angular splitting operation represents this transfer between contexts. This also does not indicate that splitting and angular splitting are two separate operations, but it is used to help understand the role that angular splitting plays in this developmental progression of quantifications of angularity, to offer a better and more accurate classification of students' quantifications.

A New Instrument

In addition to a new conceptual framework to help describe students' quantifications of angularity, this study has allowed for the development of an instrument, the QAI, that offers valid measures of students' quantifications of angularity. To date, there are no quantitative assessments that specifically measure students' quantifications of angularity. In his work, Hardison (2018, 2019) has only focused on the qualitative nature of how students quantify angularity. In addition, most of the previous work examining students' concepts of angle and angle measurement have been qualitative (e.g., Clements & Burns, 2000; Hardison, 2018, 2019; Hardison & Lee, 2019; Kontorovich & Zazkis, 2016; Moore, 2013). Although qualitative work is necessary to understand how students quantify angularity, this quantitative survey offers an instrument that can be given in a single class period instead of individually interviewing each student.

The QAI has been evaluated and verified through the different phases of this study. The five components of validity have been addressed in the development of this instrument (see Validation section above for more details). Multiple iterations of task development was conducted to offer preliminary validity. Results from the main study offer more evidence for its validity and reliability. Overall, this instrument provides valid measures of students' quantifications of angularity, including the new angular splitting stage, and opens the doors to a more efficient method for understanding and classifying students' concepts of angle and angle measurement.

Curriculum and Teaching Implications

There a many different conceptions of fractions that are emphasized throughout research and curriculum standards, for example part-whole, quotient, measure, ratio, and operator

(Kieren, 1980). The most common concept emphasized in curriculum standards and textbooks in the US is part-whole (Watanabe, 2007). Although this concept of fractions is important, there are limitations to this understanding (Hackenberg et al., 2016; Thompson & Saldanha, 2003; Wilkins & Norton, 2018). When students do not have a measurement concept, they are limited in their construction of more sophisticated fraction schemes (Norton & Wilkins, 2010), as well as limited in transferring this concept to other contexts, such as angles.

Furthermore, although curriculum standards do focus on angle measurement as a fractional amount of a circle (CCSSI, 2010; SCDOE, 2018; TEA, 2012; VDOE, 2016a), research indicates that students most often are limited to a static conception of angles and as a result are limited in their understandings of angle measurement (Bütüner & Filiz, 2016; Kontorovich & Zazkis, 2016; Smith et al., 2014). For example, many textbooks and curriculum standards begin by defining an angle as the intersection of two rays or lines (CCSSI, 2010; Charles, 2005; Greenberg, 2008; Henderson & Taimina, 2005; Long, 2009; VDOE, 2016b, 2016c). Due to this reliance on an angle as a figure or visual object, students may think that angles have different measures because one appears larger than the other (Thompson et al., 2014). Students may also think that angle measurement can be determined by measuring the side lengths of the angles, is related to the size of the arc drawn to represent the angle (not the subtended arc), or is related to the linear distance between the sides of the angle (Barabash, 2017; Bütüner & Filiz, 2016; Clements & Battista, 1989, 1990; Clements, 2003; Piaget et al., 1981/1960). It is only later, for example in fifth grade in Virginia, that students learn about angle measurement as a fractional amount of a circle (VDOE, 2016a). By this time, it may be too late to help students rethink their understandings.

In mapping the CCSSI (2010) and VDOE standards (2016a, 2016b, 2016c), students begin with basic concepts of fractions and angles and then build to the more conceptual understandings. The majority of students in this study had not constructed a measurement meaning of fractions and also did not understand angle measurement as a fractional amount of a circle. It seems that without this basic understanding they are unable to move to the more conceptual understandings. Therefore, it seems that when teaching students about fractions and angle measurement it would better to start with a more conceptual approach. For example, research shows that part-whole understanding is important for fractions, but understanding fractions as measures better prepares students to solve problems involving fractions and also develop a deeper understanding of fractions and rational numbers (Lamon, 2007; Norton & Wilkins, 2009, 2012; Wilkins & Norton, 2018). Concerning angles, in their study, Clements and Burns (2000) found that when fourth grade students use dynamic actions for angles, they develop schemes that can then be utilized in later situations, allowing students to gain a deeper understanding of angle measurement and better connections between concepts. Other studies have also documented the importance of emphasizing angles as turns, instead of encouraging students to develop a notion of angles as a static figure (e.g., Browning et al., 2008; Clements & Battista, 1989, 1990; Mitchelmore & White, 2000; Yigit, 2014). Therefore, it is important to realize that even though the curriculum standards focus on beginning with basic understandings of fractions and angles, it is also important to emphasize the more conceptual understandings early on. This will prevent students from developing limited concepts of fractions, angles, and angle measurement.

Furthermore, research shows that students often view geometry as a disconnected subject (Melos & Martins, 2015). However, I have highlighted an important connection between measurement, fractions, and angles. This connection should be made explicitly for students, so they are better prepared to transfer their fraction operations into the context of angles. By understanding how measurement is related to fractions which is related to angle measurement, students may find it easier to solve tasks involving angles. They may also have a better understanding of the concepts, since they are all connected and not separate pieces of knowledge.

Future Work

Results of this study show that students' measurement schemes for fractions are related to their quantifications of angularity. Results also indicate that there is a developmental progression for the construction of fraction schemes and quantifications of angularity. As students progress in grade level, their fraction schemes and quantifications advance developmentally. As students move from sixth the seventh to eighth grade, the percentage of students who had constructed a more sophisticated fraction scheme or quantification increases, and the percentage of students who had constructed a less sophisticated fraction scheme or quantification decreases.

However, one limitation of this study is that it was correlational and cross-sectional. Therefore, we cannot draw causal inferences. While I was able to show a relationship between students' fraction schemes and quantifications of angularity, which suggests the developmental progression, this was only correlational. To further strengthen this understanding, a longitudinal study would be necessary to see how this progression develops over time. This would allow researchers to track students over time and to assess the development of their fraction schemes and quantifications of angularity. In addition, a constructivist teaching experiment would be beneficial to further understand how students quantify angularity, the developmental progression involved, and the role that splitting plays. This would help researchers better understand the proposed reorganization hypothesis of how students relate their concepts of measurement to the context of angles. A teaching experiment would also enable researchers to investigate the influence of the necessary underlying mental actions and operations. This future work should also examine the effects of curriculum standards, investigating how students interpret and understand angle measurement as a fractional amount of a circle in different curricular settings.

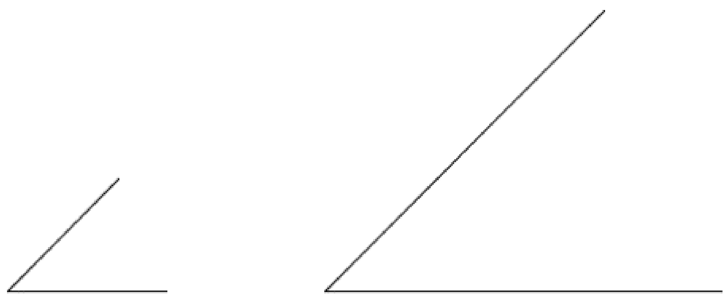
Finally, although this study demonstrated that the QAI provides a valid measure for assessing students' quantifications of angularity, the validity of the instrument needs to be further investigated. It would strengthen the validity of the instrument if each group of tasks were correlated to assess internal validity. Future research should also involve implementing the QAI with more students to again provide more variability and more data points to help assess validity. Further validation of the QAI could eventually result in it being used in PK-12 schools to help teachers assess students' quantifications of angularity and to provide insight into students' strengths and weaknesses concerning angles.

Appendix A

Quantifications of Angularity Instrument- Original Set of Tasks

1. Circle one below:

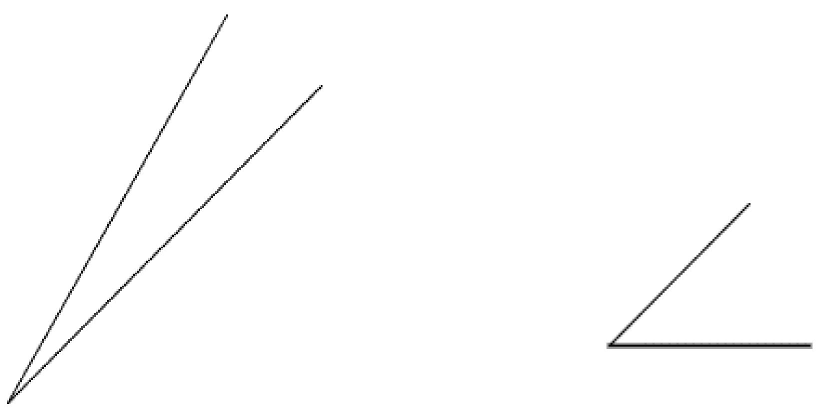
Angle 1 is | **a. larger than**
b. smaller than
c. equal to | **Angle 2. How do you know?**



Angle 1 Angle 2

2. Circle one below:

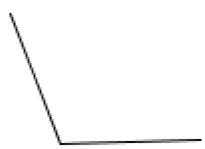
Angle 1 is | **a. larger than**
b. smaller than
c. equal to | **Angle 2. How do you know?**



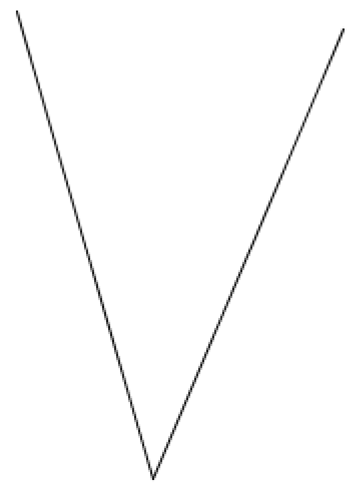
Angle 1 Angle 2

3. Circle one below:

Angle 1 is | **a. larger than** / **b. smaller than** / **c. equal to** | **Angle 2. How do you know?**



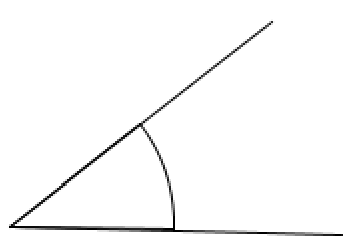
Angle 1



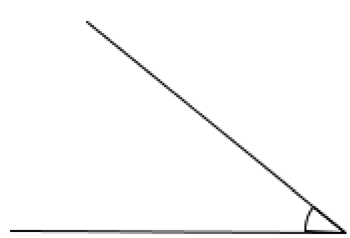
Angle 2

4. Circle one below:

Angle 1 is | **a. larger than** / **b. smaller than** / **c. equal to** | **Angle 2. How do you know?**

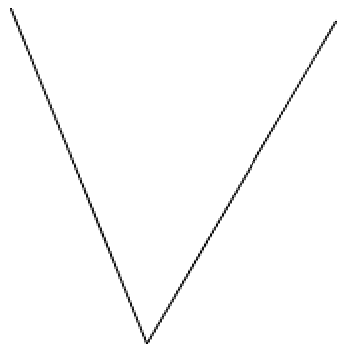


Angle 1



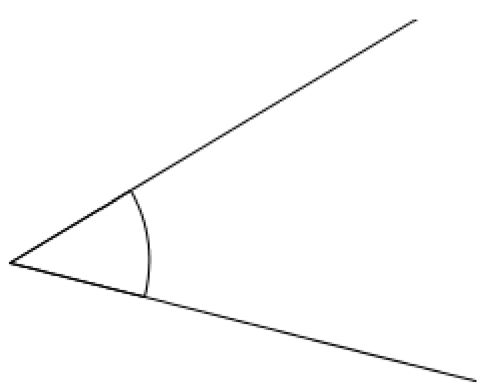
Angle 2

5. **Draw an angle that is the same size as the angle below. How do you know your angle is the same size?**

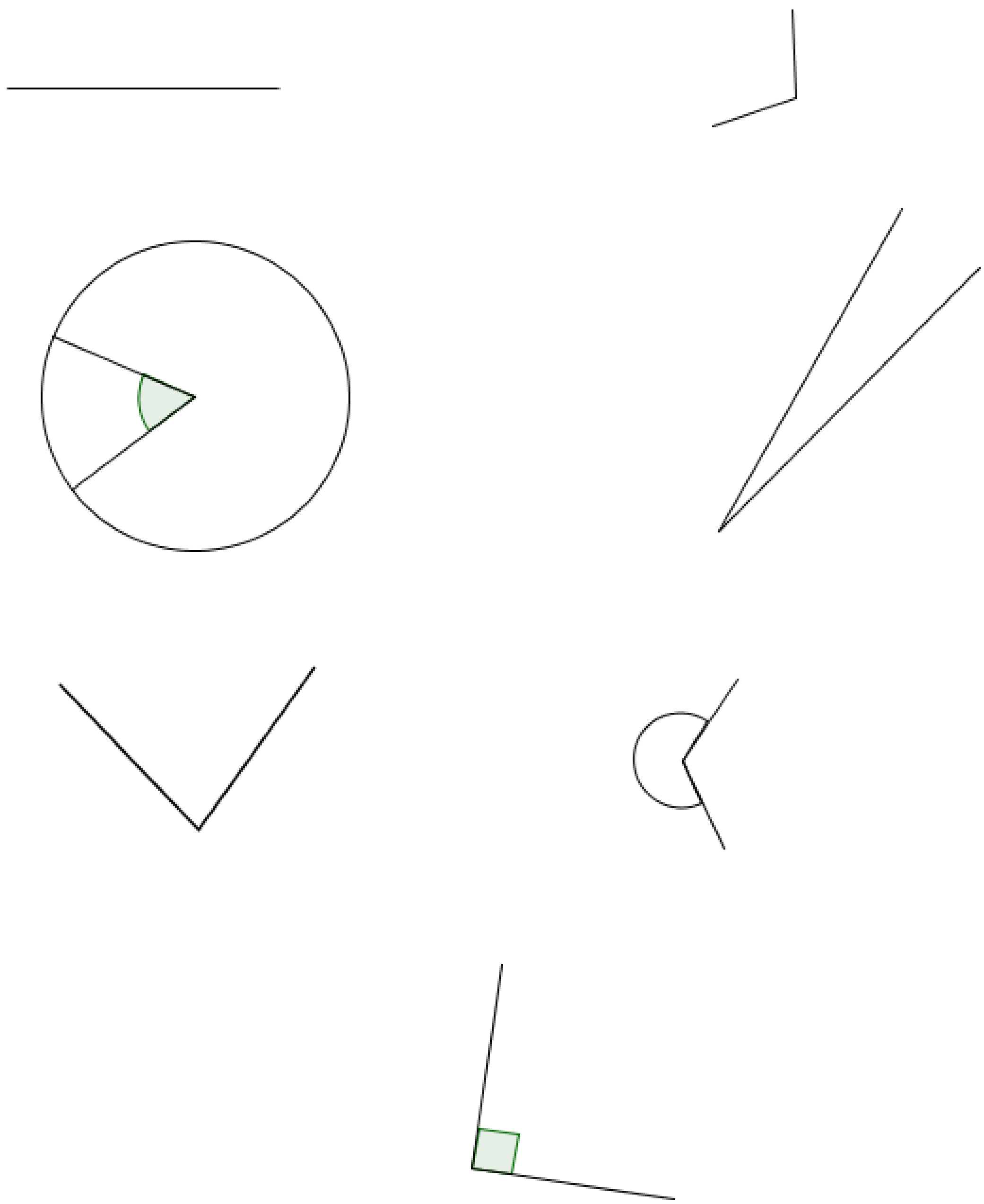


6. **Draw an angle that is larger than the angle below. How do you know your angle is larger?**

7. **Draw an angle that is smaller than the angle below. How do you know your angle is smaller?**

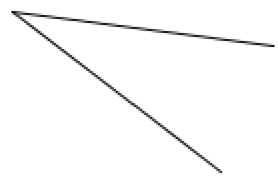


8. Order the angles below from smallest to largest (1-7). Label the SMALLEST ANGLE with a 1 and label the LARGEST with a 7. Explain your process.



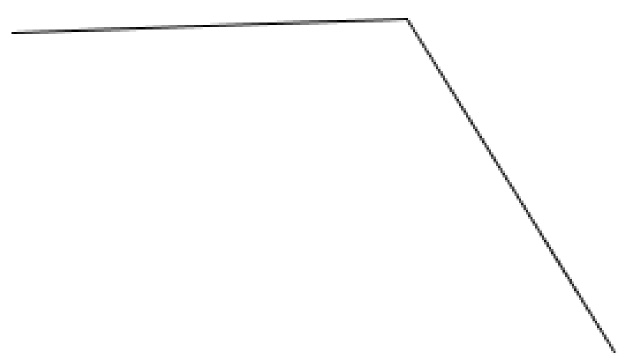
9. **Draw an angle that is three times larger than this angle:**

Explain your process:



10. **Draw an angle that is six times smaller than this angle:**

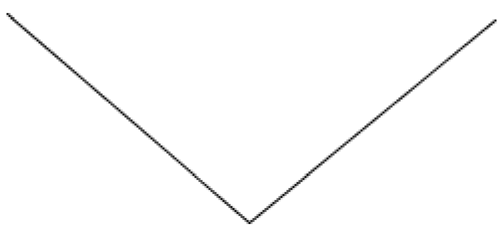
Explain your process:



11. Draw an angle that is five times larger than this angle:
Explain your process:

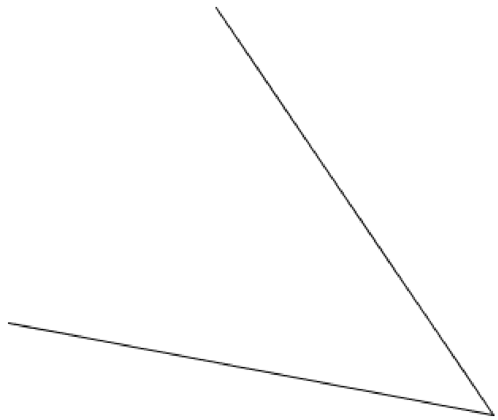
12. Draw an angle that is 1/7 as large as this angle:

Explain your process:



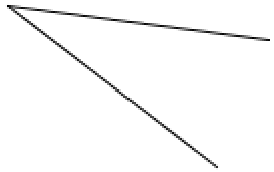
13. The angle below is five times as large as your angle. Draw your angle:

Explain your process:

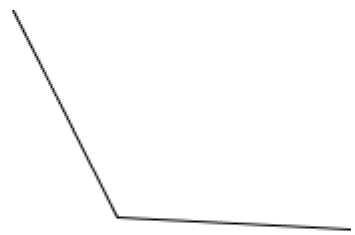


14. The angle below is three times as large as your angle. Draw your angle:

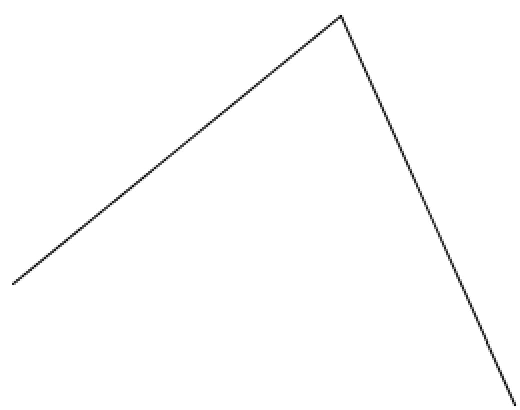
Explain your process:



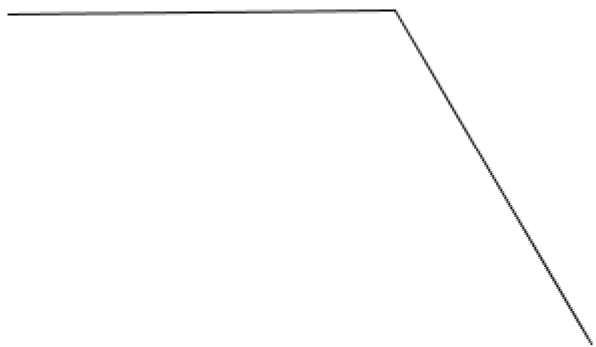
15. The angle below is six times as large as your angle. Draw your angle:
Explain your process:

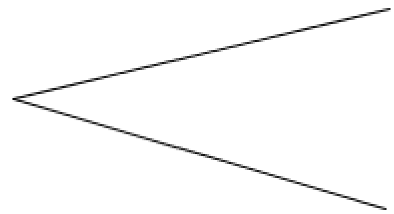


16. The angle below is three times as large as your angle. Draw your angle:
Explain your process:



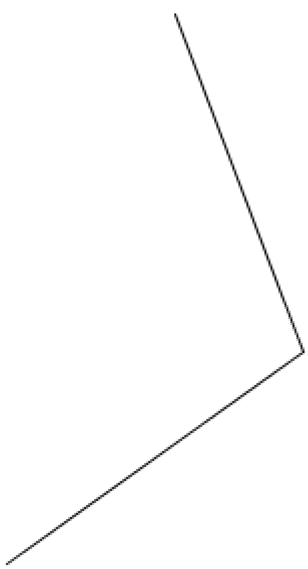
17. Measure Angle 1 in terms of Angle 2. Write your measurement of Angle 1 below. Measure Angle 2 in terms of Angle 1. Write your measurement below of angle 2 below.

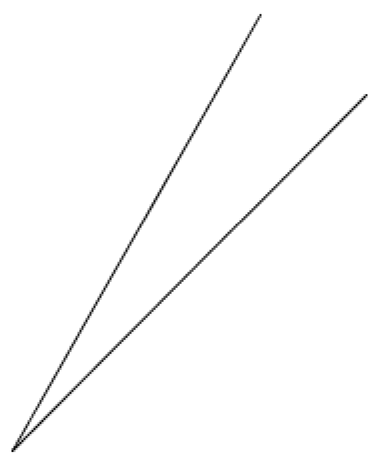




Angle 1 Angle 2

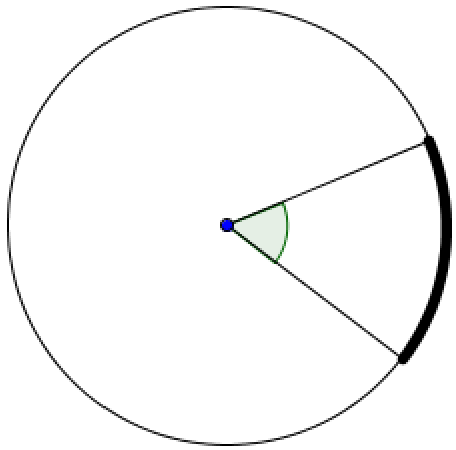
18. Measure Angle 1 in terms of Angle 2. Write your measurement of Angle 1 below. Measure Angle 2 in terms of Angle 1. Write your measurement below of angle 2 below.



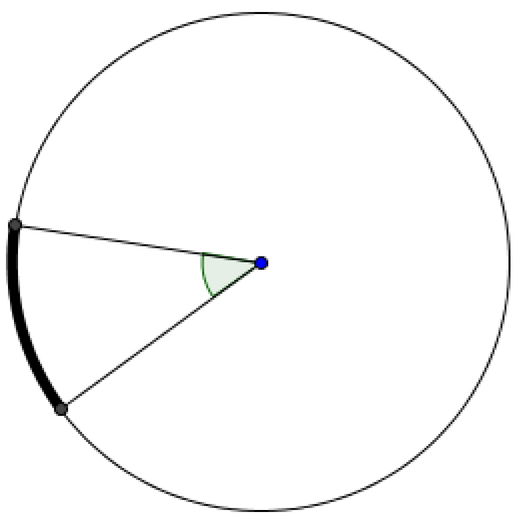


Angle 1 Angle 2

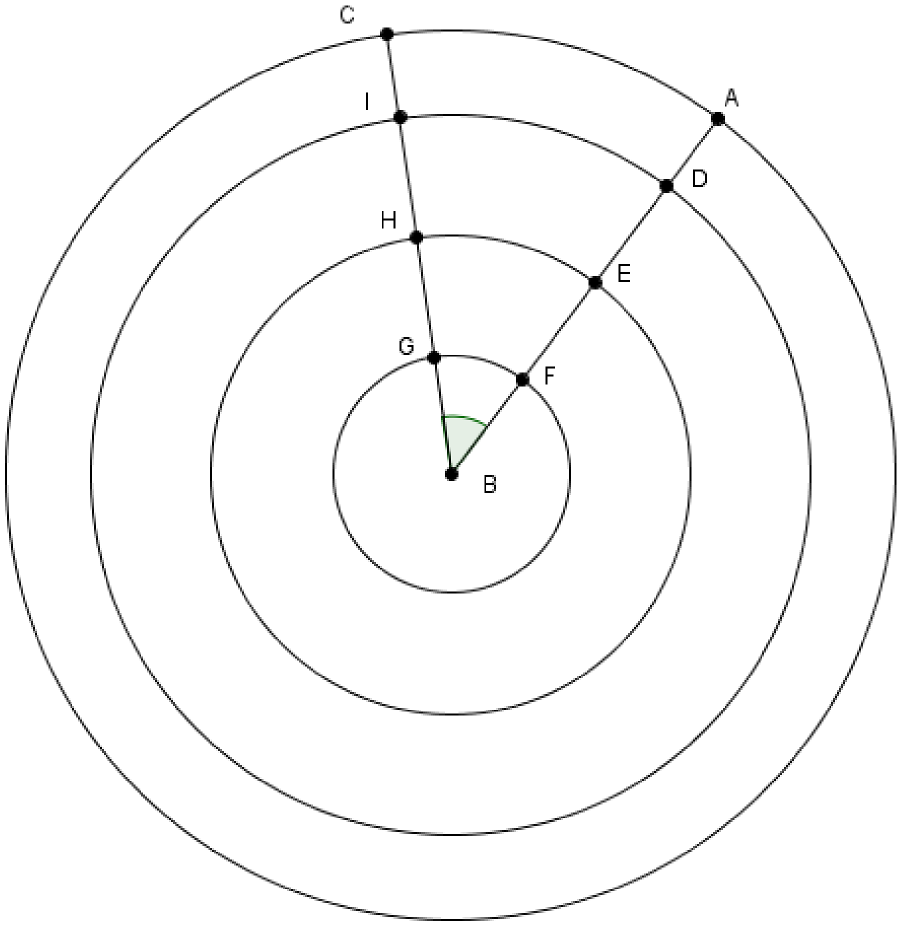
19. The circumference or total distance around the circle is 48cm. The thick arc is 8cm long. Determine the measure of the angle in relation to the circle.



20. The thick arc is 9cm long. The circumference or total distance around the circle is 72cm. Determine the measure of the angle in relation to the circle?



21. Compare the angles in the following diagram. Circle one answer.



∠GBF is

a. larger than
b. smaller than
c. equal to

∠IBD. How do you know?

∠ IBD is

a. larger than
b. smaller than
c. equal to

∠CBA. How do you know?

∠HBE is

a. larger than
b. smaller than
c. equal to

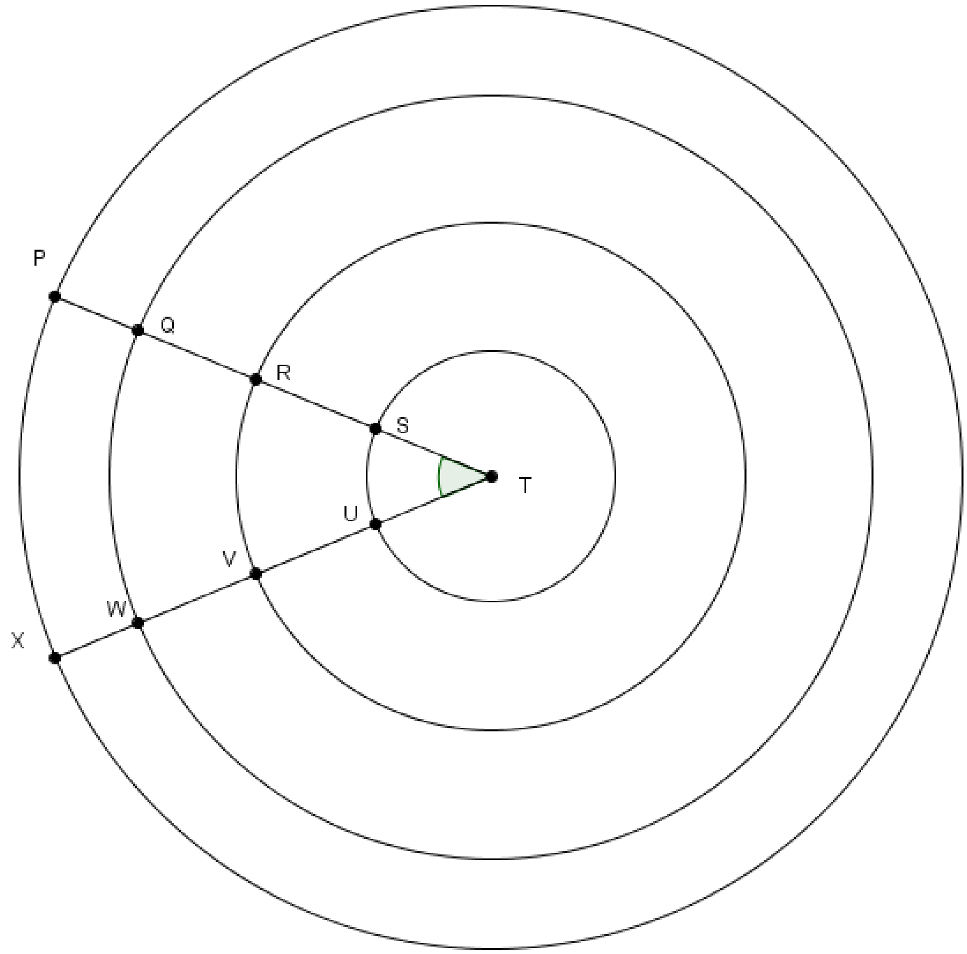
∠GBF. How do you know?

∠CBA is

a. larger than
b. smaller than
c. equal to

∠HBE. How do you know?

22. Compare the angles in the following diagram. Circle one answer.



∠PTV is

a. larger than
b. smaller than
c. equal to

∠RTX. How do you know?

∠ WTS is

a. larger than
b. smaller than
c. equal to

∠UTQ. How do you know?

∠XTQ is

a. larger than
b. smaller than
c. equal to

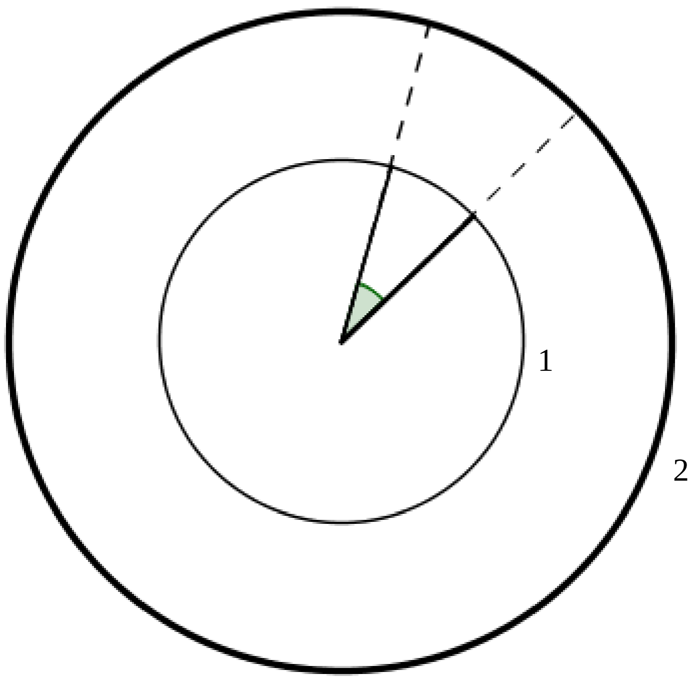
∠RTW. How do you know?

∠UTP is

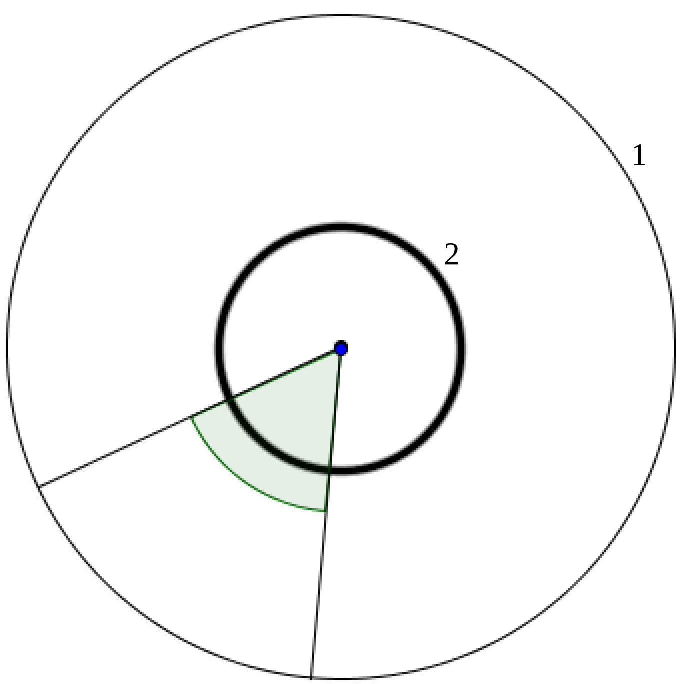
a. larger than
b. smaller than
c. equal to

∠STV. How do you know?

23. Suppose you are given the angle below. If the smaller circle 1 was enlarged to the dark circle 2, what would happen to the size of the angle?



24. Suppose you are given the angle below. If the larger circle 1 was shrunk to the dark circle 2, what would happen to the size of the angle?

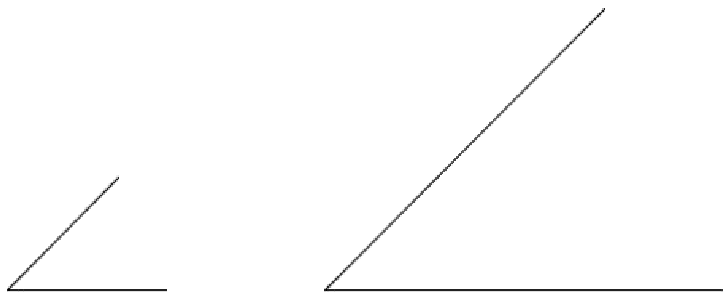


Appendix B

Quantifications of Angularity Instrument- Revised Set of Tasks

1. Circle one below:

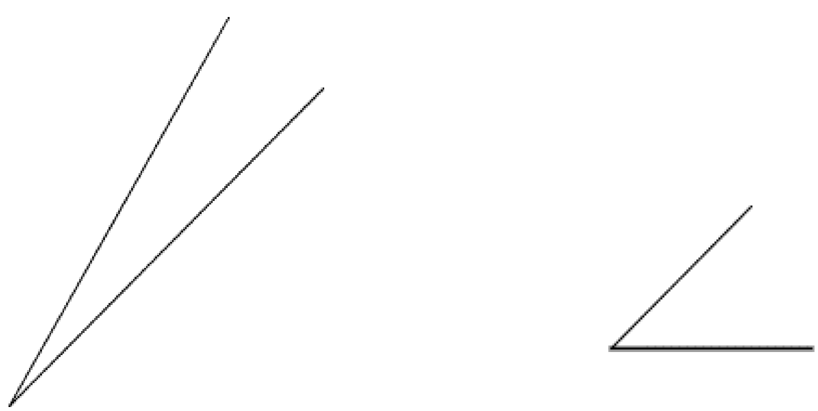
a. larger than
Angle 1 is b. smaller than Angle 2. How do you know?
c. equal to



Angle 1 Angle 2

2. Circle one below:

a. larger than
Angle 1 is b. smaller than Angle 2. How do you know?
c. equal to



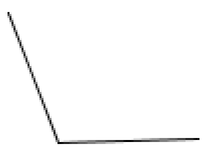
Angle 1 Angle 2

3. Circle one below:

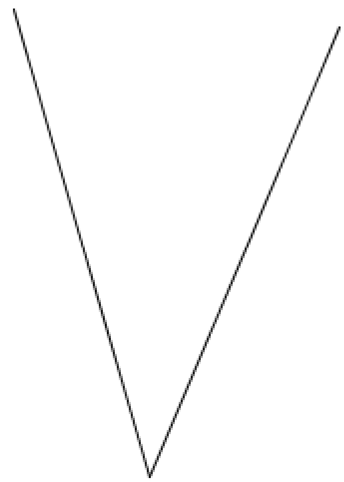
a. larger than

Angle 1 is b. smaller than Angle 2. How do you know?

c. equal to



Angle 1



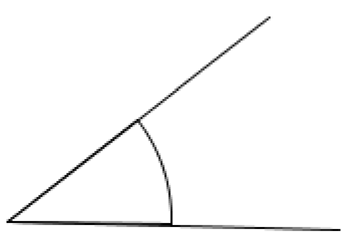
Angle 2

4. Circle one below:

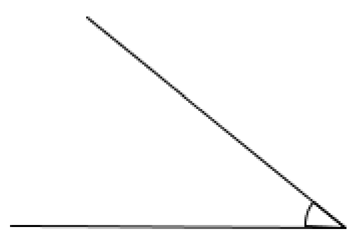
a. larger than

Angle 1 is b. smaller than Angle 2. How do you know?

c. equal to

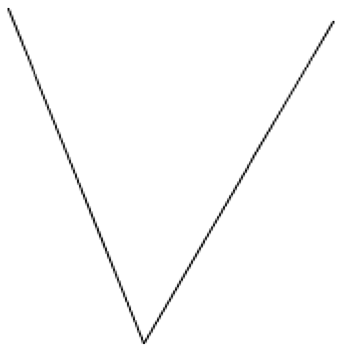


Angle 1



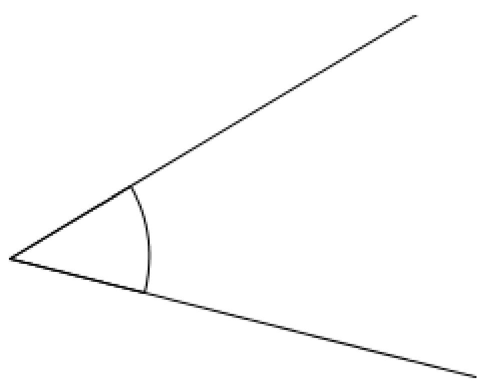
Angle 2

5. Draw an angle that is the same size as the angle below. How do you know your angle is the same size?

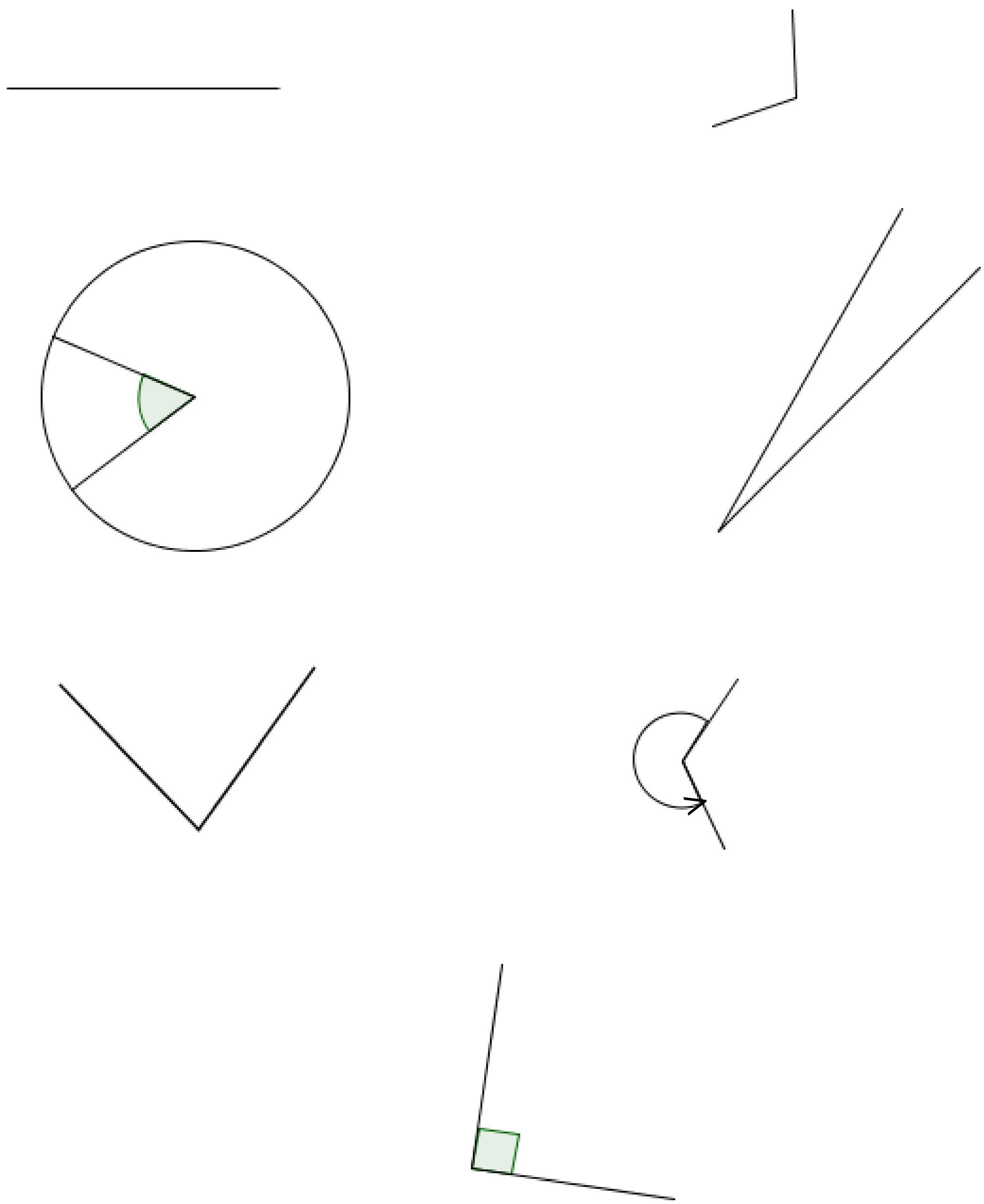


6. Draw an angle that is larger than the angle below. How do you know your angle is larger?

7. Draw an angle that is smaller than the angle below. How do you know your angle is smaller?

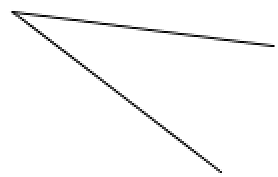


8. Order the angles below from smallest to largest (1-7). Label the SMALLEST ANGLE with a 1 and label the LARGEST with a 7. Explain your process.



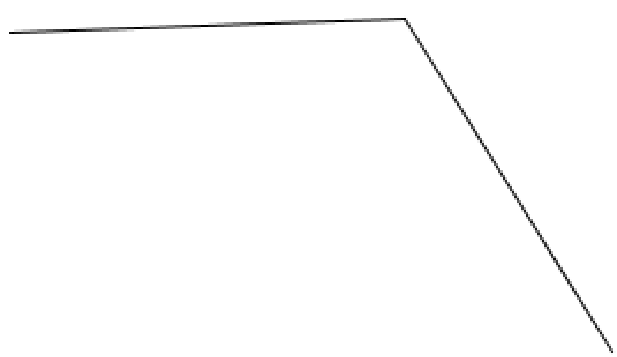
9. Draw an angle that is three times larger than this angle:

Explain your process:



10. Draw an angle that is six times smaller than this angle:

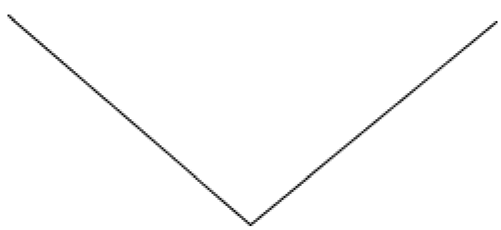
Explain your process:



11. Draw an angle that is five times larger than this angle:
Explain your process:

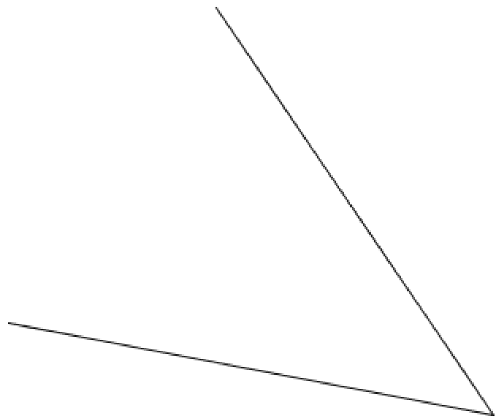
12. Draw an angle that is 1/7 as large as this angle:

Explain your process:



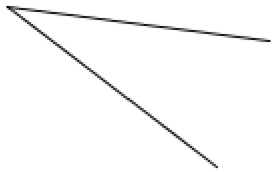
13. The angle below is five times as large as your angle. Draw your angle:

Explain your process:

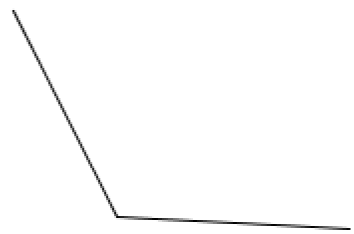


14. The angle below is three times as large as your angle. Draw your angle:

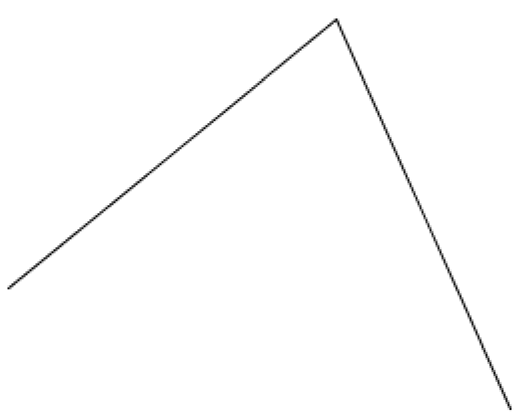
Explain your process:



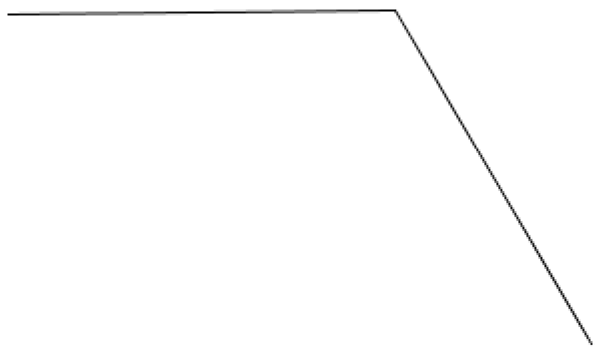
15. The angle below is six times as large as your angle. Draw your angle:
Explain your process:



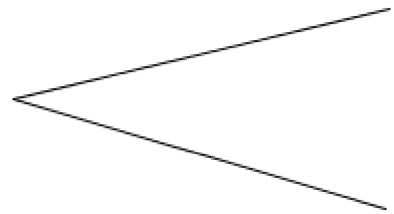
16. The angle below is three times as large as your angle. Draw your angle:
Explain your process:



17. Use **Angle 2 to measure Angle 1. Write your measurement of Angle 1 below.**
Use Angle 1 to measure Angle 2. Write your measurement of Angle 2 below.

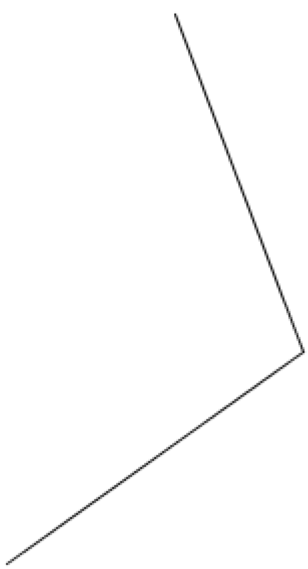


Angle 1

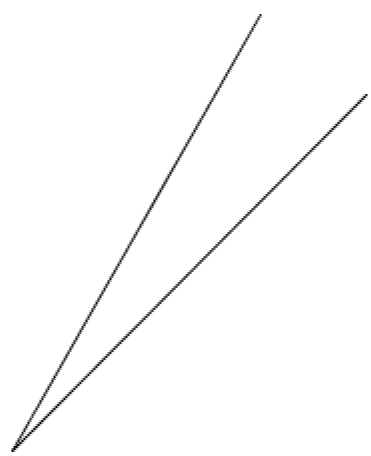


Angle 2

18. Use **Angle 2 to measure Angle 1. Write your measurement of Angle 1 below.**
Use Angle 1 to measure Angle 2. Write your measurement of Angle 2 below.

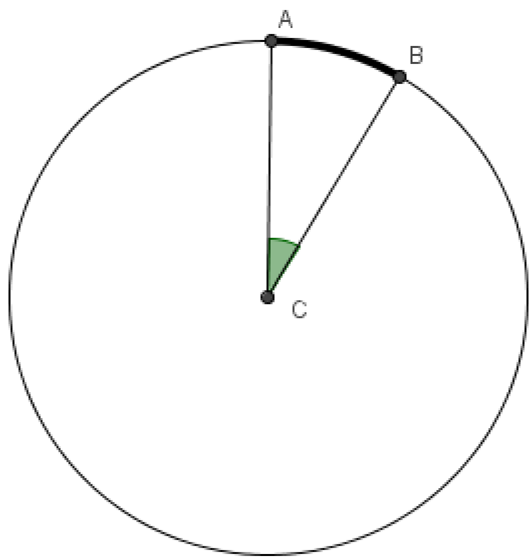


Angle 1

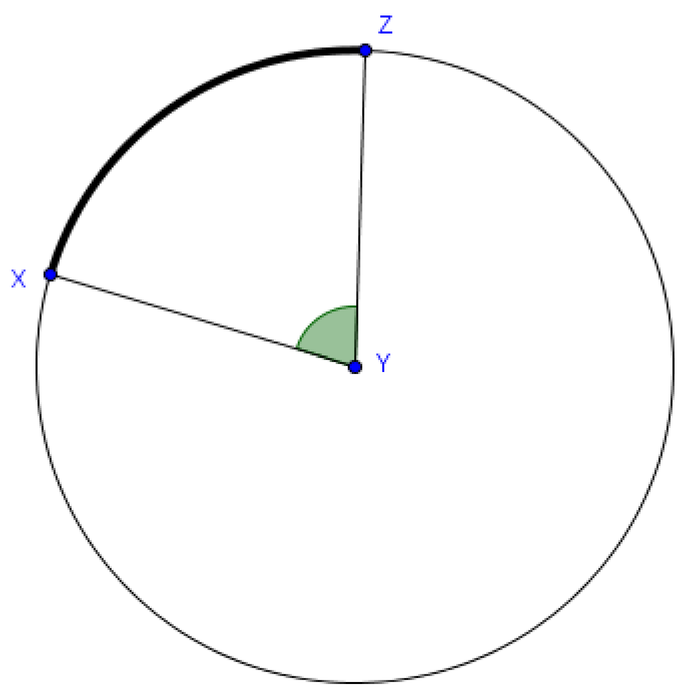


Angle 2

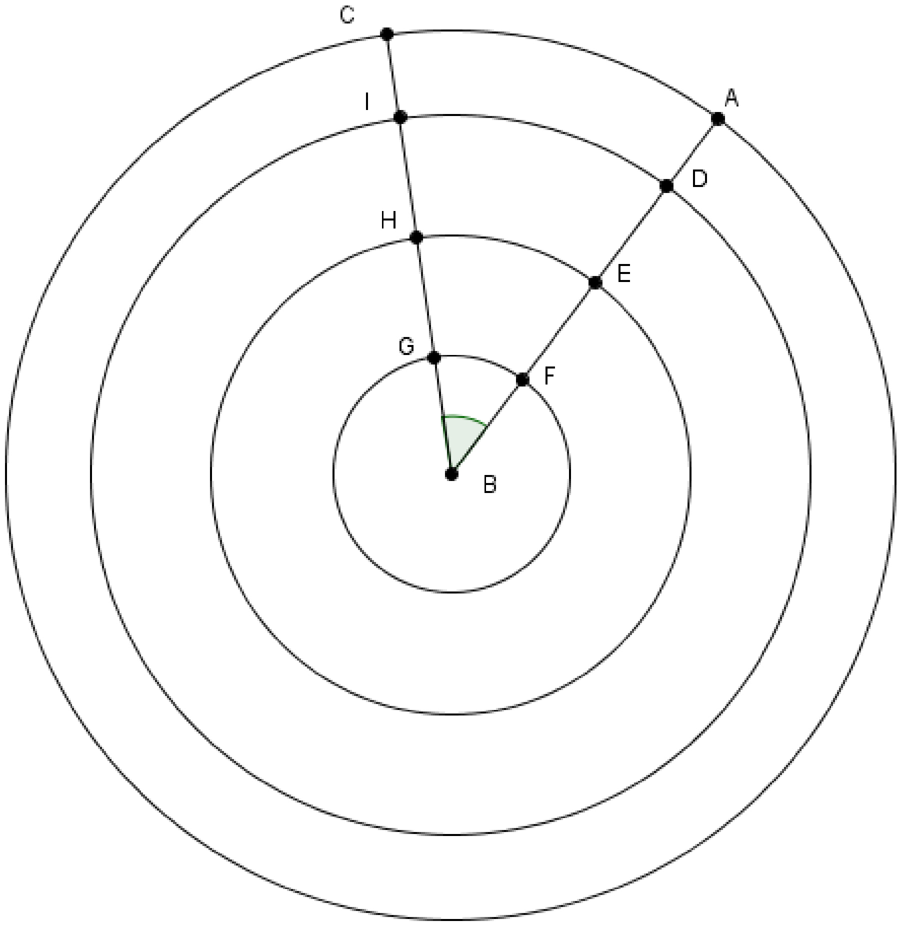
19. The circumference or total distance around the circle is 48cm. Arc AB is 8cm long. Determine the measure of ∠ACB in relation to the circle.



20. Arc XZ is 9cm long. The circumference or total distance around the circle is 72cm. Determine the size of ∠XYZ in relation to the circle.



21. Compare the angles in the following diagram. Circle one answer.



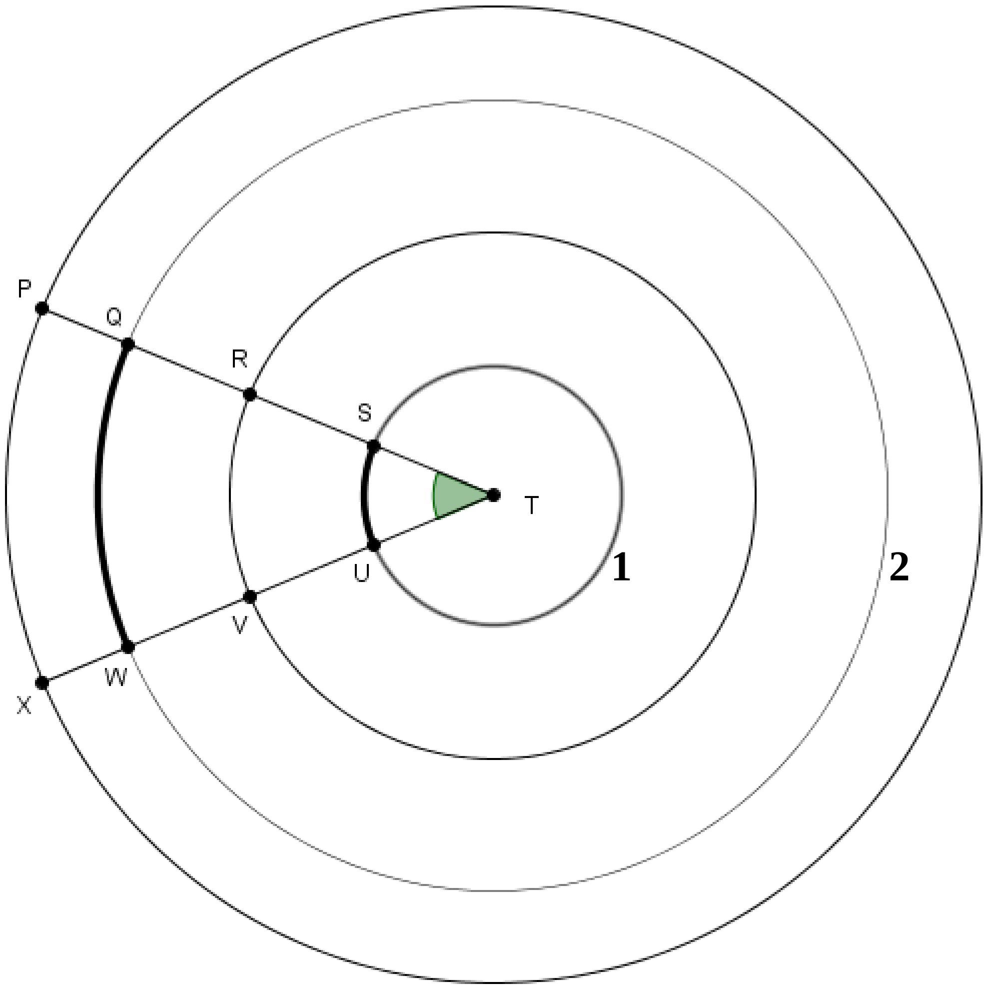
	a. larger than	
∠GBF is	**b. smaller than**	**∠IBD. How do you know?**
	c. equal to	

	a. larger than	
∠IBF is	**b. smaller than**	**∠ABH. How do you know?**
	c. equal to	

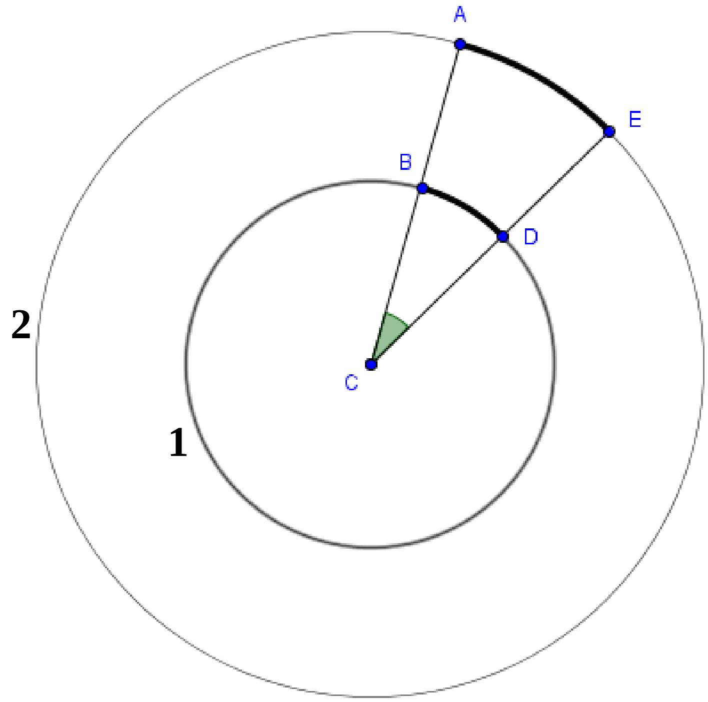
	a. larger than	
∠HBE is	**b. smaller than**	**∠CBA. How do you know?**
	c. equal to	

	a. larger than	
∠GBD is	**b. smaller than**	**∠EBC. How do you know?**
	c. equal to	

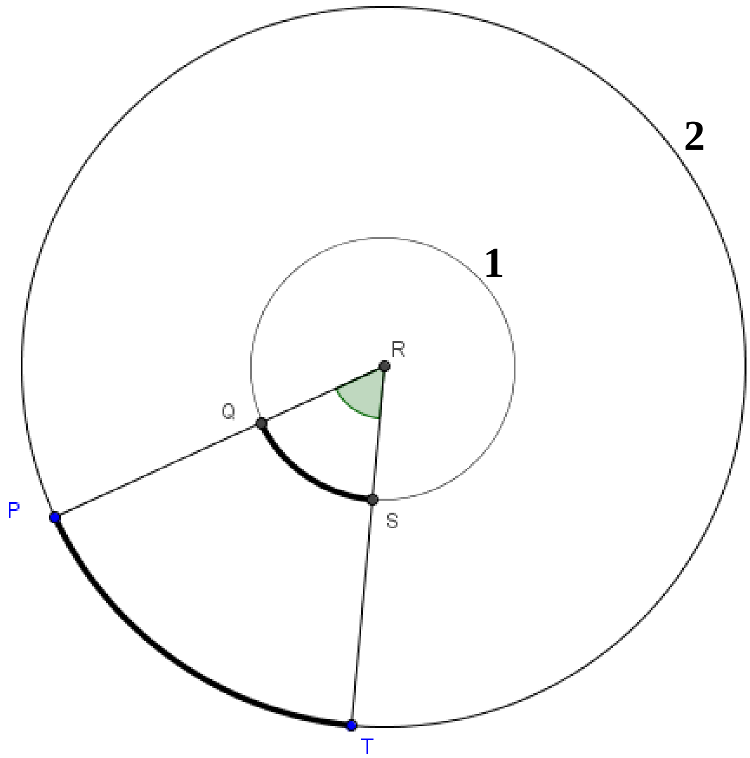
22. Circle 2 has a circumference of 36 cm and Arc QW is 9cm long. If Arc SU is 3cm long, what is the circumference of Circle 1?



23. Circle 2 has a circumference of 60cm and Arc AE is 10cm long. What is the measure of ∠BCD in relation to Circle 1?



24. Circle 1 has a circumference of 56cm and Arc QS is 7cm long. If the circumference of Circle 2 is 91cm, how long is Arc PT?



Appendix C

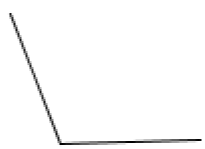
Quantifications of Angularity Instrument- Final Version

1. Circle one option below:

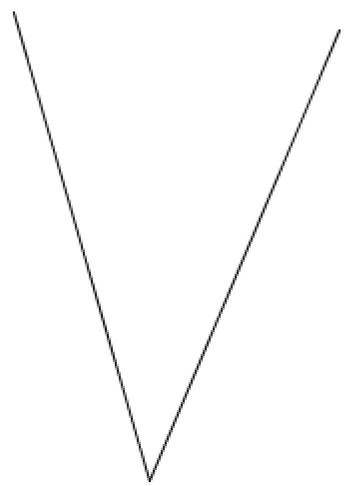
a. larger than

Angle 1 is b. smaller than Angle 2. How do you know?

c. equal to



Angle 1



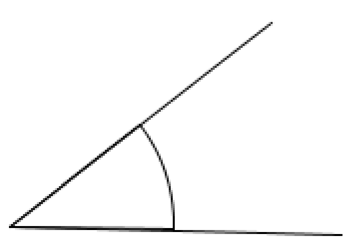
Angle 2

2. Circle one option below:

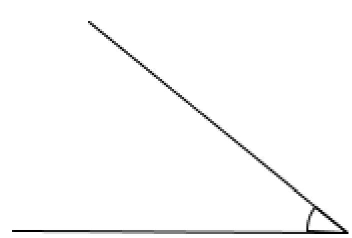
a. larger than

Angle 1 is b. smaller than Angle 2. How do you know?

c. equal to



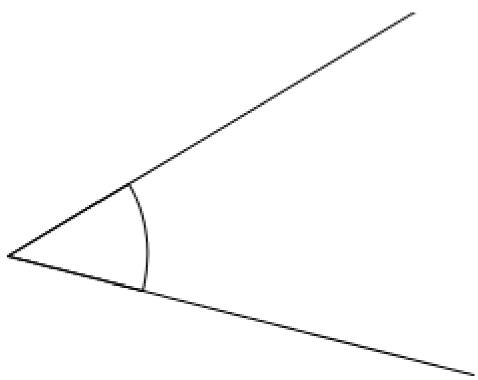
Angle 1



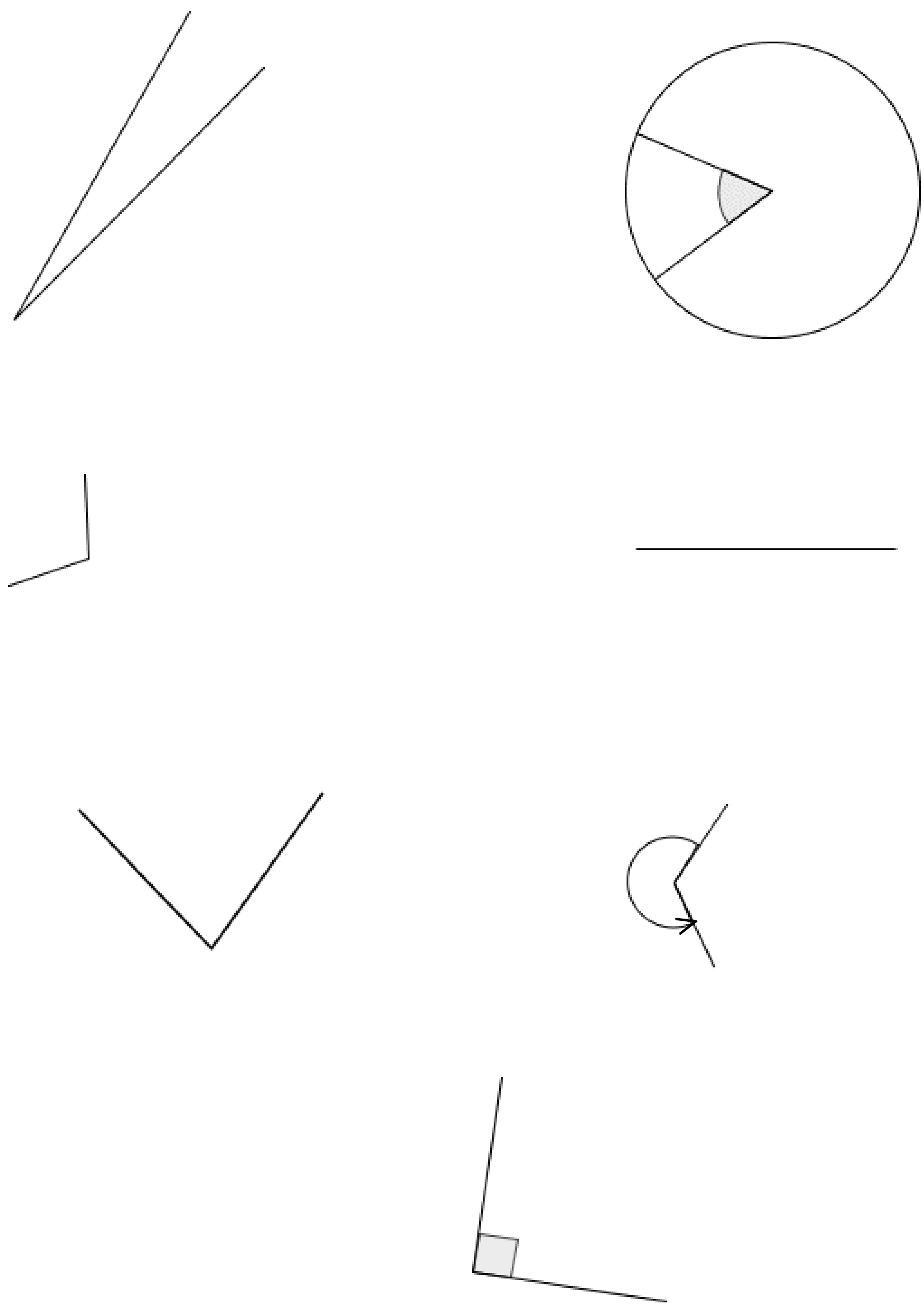
Angle 2

3. Draw an angle that is larger than the angle below. How do you know your angle is larger?

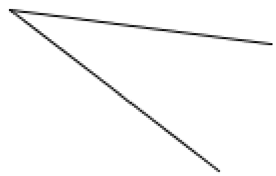
4. Draw an angle that is smaller than the angle below. How do you know your angle is smaller?



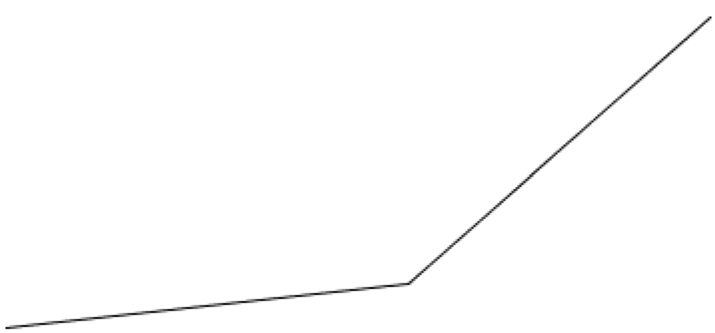
5. **Order the angles below from smallest to largest (1-7). Label the SMALLEST ANGLE with a 1 and label the LARGEST with a 7. Explain your process.**



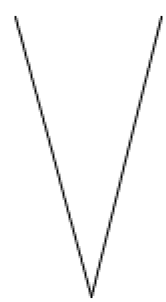
6. **Draw an angle that is four times larger than this angle.**
Explain your process.



7. **How many times smaller is Angle 2 than Angle 1?**



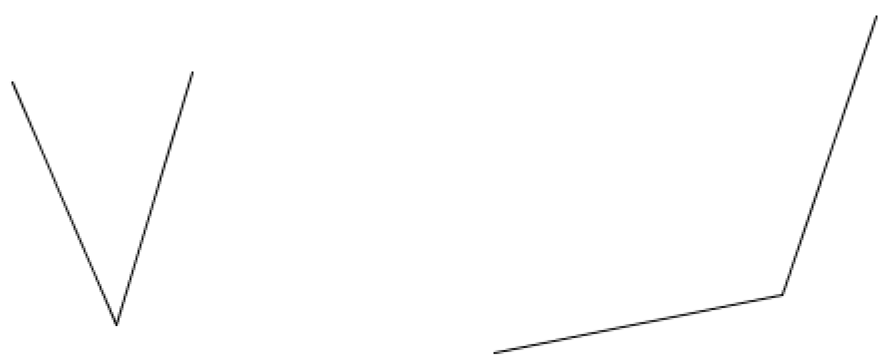
Angle 1



Angle 2

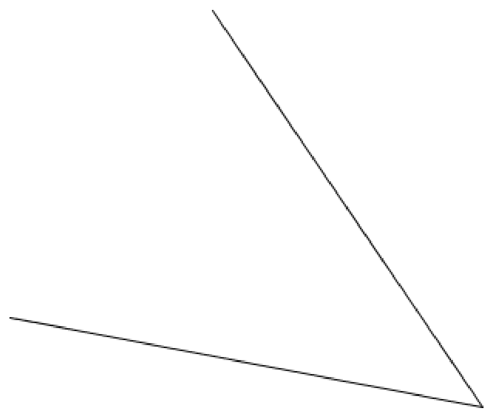
8. Draw an angle that is six times larger than this angle.
Explain your process.

9. How many times will Angle 1 fit into Angle 2?
Explain your process.

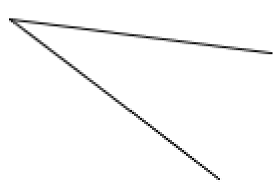


Angle 1 Angle 2

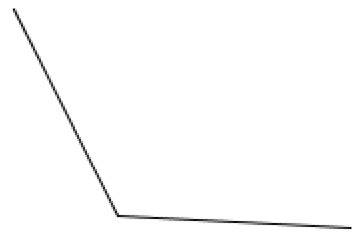
10. The angle below is five times as large as your angle. Draw your angle. Explain your process.



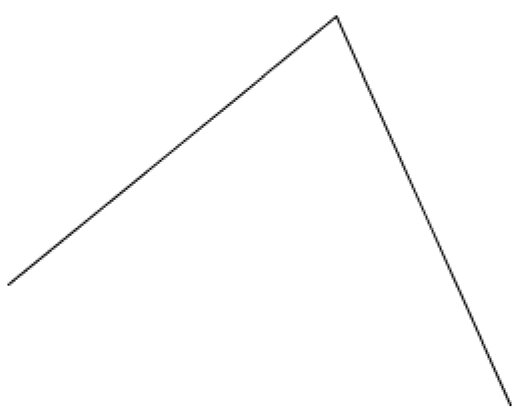
11. The angle below is three times as large as your angle. Draw your angle. Explain your process.



12. The angle below is four times as large as your angle. Draw your angle. Explain your process.

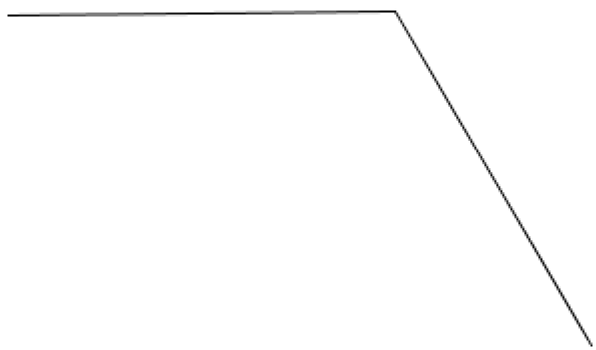


13. The angle below is six times as large as your angle. Draw your angle. Explain your process.

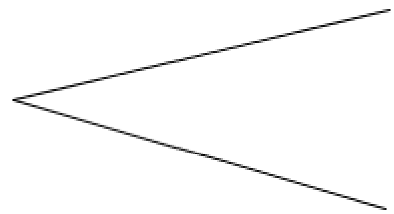


14. **What fraction is Angle 2 of Angle 1?**

What fraction is Angle 1 of Angle 2?



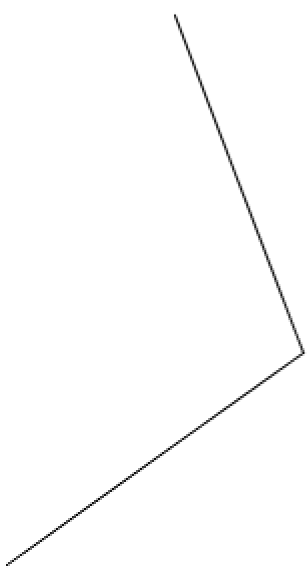
Angle 1



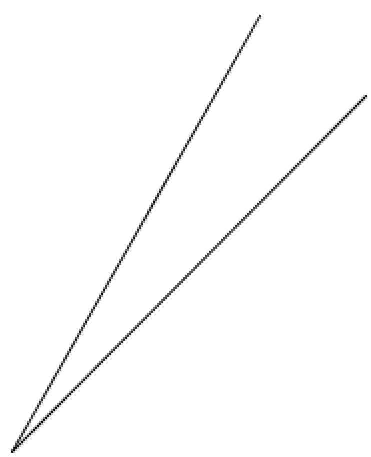
Angle 2

15. **What fraction is Angle 2 of Angle 1?**

What fraction is Angle 1 of Angle 2?



Angle 1

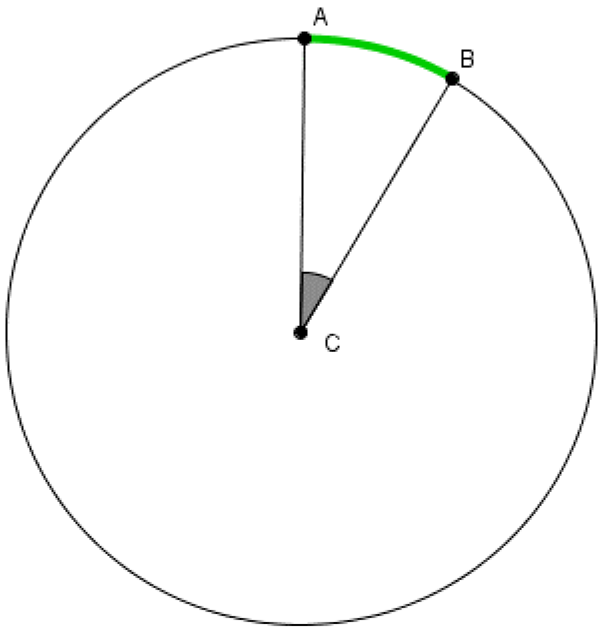


Angle 2

16. The circumference (total distance around the circle) is 49 cm. (Not drawn to scale)

The GREEN part of the circle is 7 cm long.

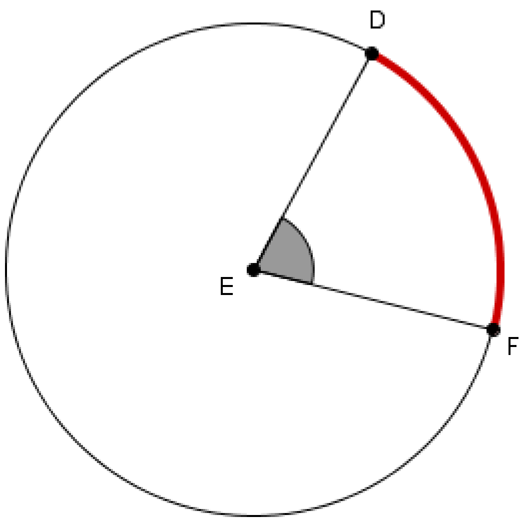
What is the measure of ∠ACB in relation to the circle?



17. The circumference (total distance around the circle) is 96 cm. (Not drawn to scale)

∠DEF is 1/8 of the circle.

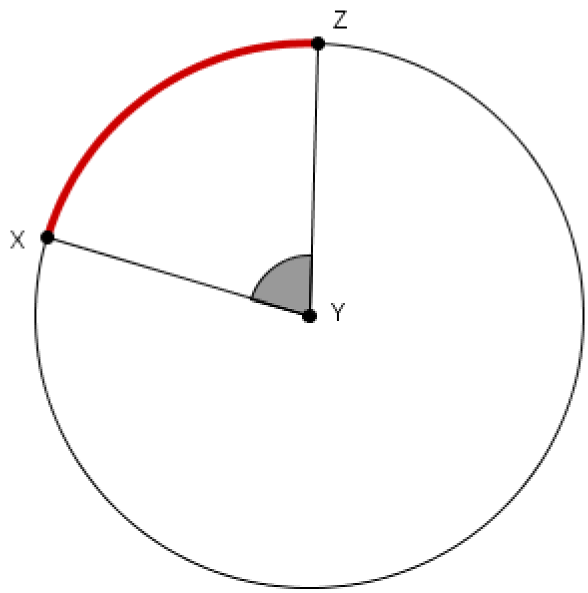
How long is the RED part of the circle?



18. The RED part of the circle is 8 cm long. (Not drawn to scale)

The circumference (total distance around the circle) is 72 cm.

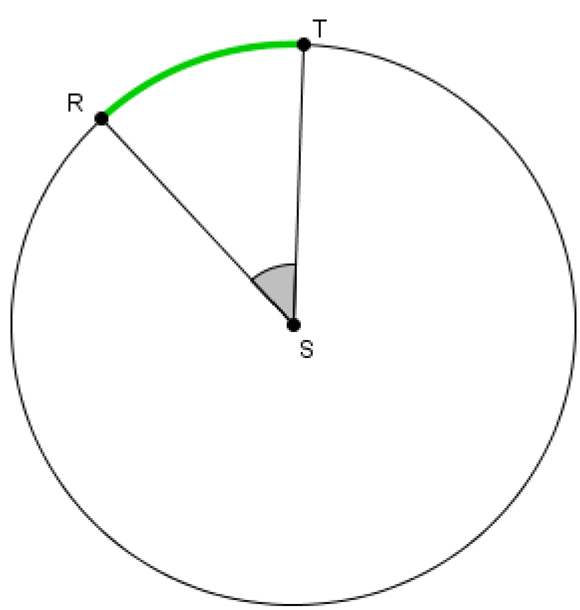
What is the size of ∠XYZ in relation to the circle?



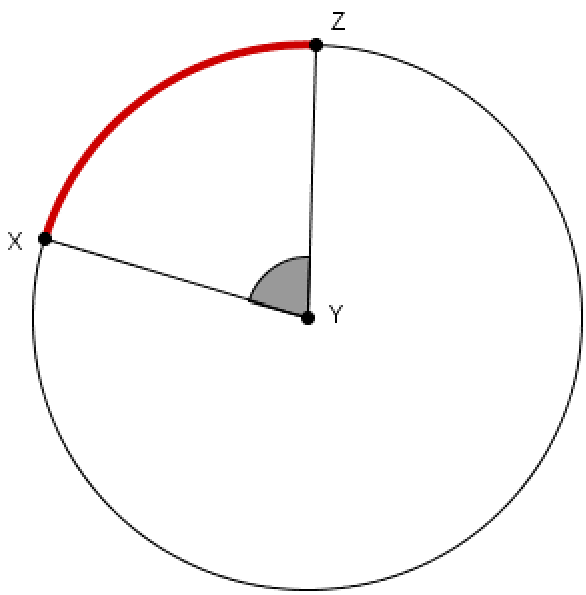
19. The GREEN part of the circle is 7 cm long. (Not drawn to scale)

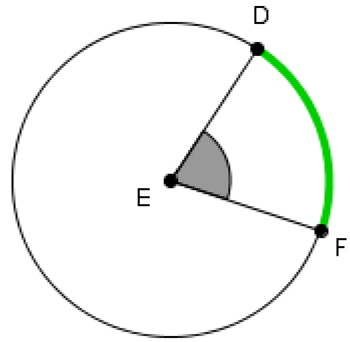
∠RST is 1/6 of the circle.

What is the circumference (total distance around the circle)?



20.





The RED part of the circle is 5 cm long.

The circumference (total distance around the circle) is 45 cm.

The GREEN part of the circle is 4 cm long.

The circumference (total distance around the circle) is 24 cm.

Given the figures above, choose one option below (figures are not drawn to scale):

	a. larger than	
∠XYZ is	**b. smaller than**	**∠DEF. How do you know?**
	c. equal to	

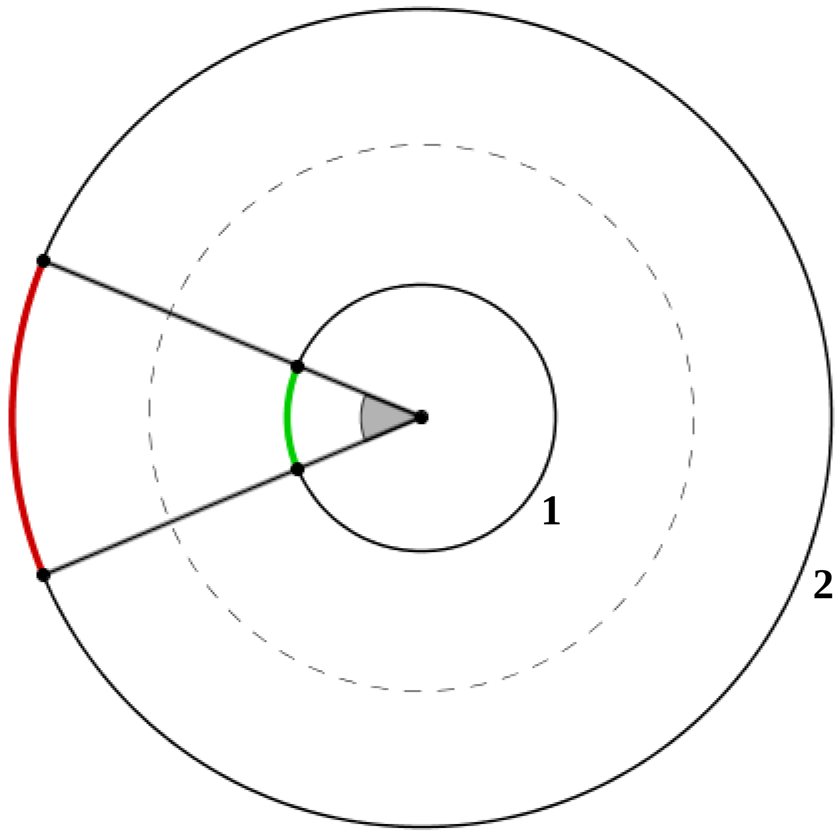
21. Circle 2 has a circumference (total distance around the circle) of 36 cm.

The RED part of Circle 2 is 9 cm long.

The GREEN part of Circle 1 is 3 cm long.

What is the circumference (total distance around the circle) of Circle 1?

(Figure is not drawn to scale)

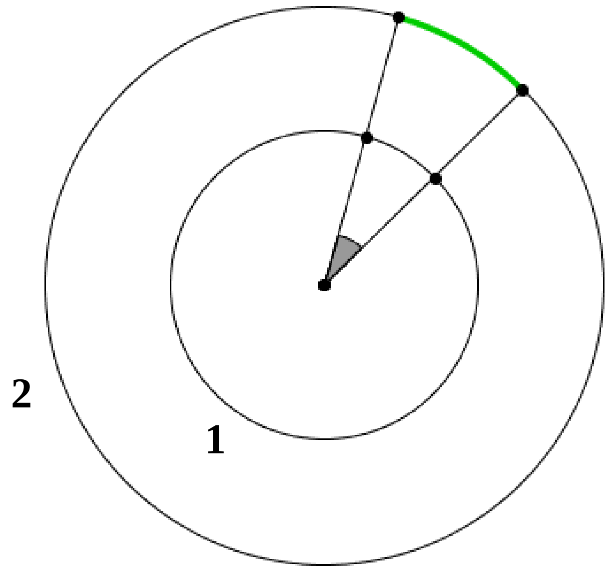


22. Circle 2 has a circumference (total distance around the circle) of 50 cm.

The GREEN part of Circle 2 is 10 cm long.

What is the measure of the angle in relation to Circle 1?

(Figure is not drawn to scale)



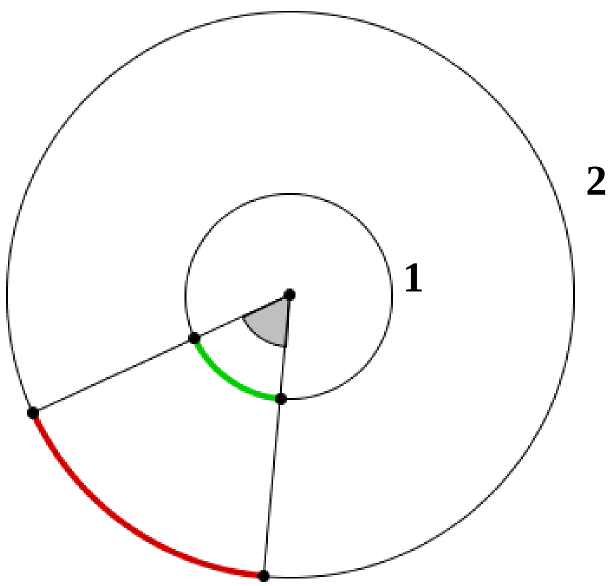
23. Circle 1 has a circumference (total distance around the circle) of 56 cm.

The GREEN part of Circle 1 is 7 cm long.

The circumference (total distance around the circle) of Circle 2 is 96 cm.

How long is the RED part of Circle 2?

(Figure is not drawn to scale)



Appendix D

Table of Revisions between First Iterations and Final Iteration

Old Tasks		New tasks		Revisions:
Numbers	*Classification/ Old Title*	*Numbers*	*Classification/ New Title*	
1-4	Gross	1-5	Gross/ Intuitive (Intensive)	We combined the old tasks 1-7 to create a new set of tasks designed to assess gross/confirm intensive, as well as to further confirm intensive. We felt that old 1-7 were repetitive and assessed similar concepts, so we kept old 3, 4, 6, & 7. We also thought that #8 assessed additional/special aspects of intensive, so we kept it. We rearranged the order of items and added some notation on new #5 so students would know the line was part of the ordering, and that one of the angles was an obtuse angle.
5-8	Intensive			
9-12	Extensive	6-9	Additive	We revised the old tasks to make partitioning and iterating clearer. For example, the new iterating tasks require students to iterate an even number of times (4 times and six times larger). The partitioning tasks require students to partition into an odd number (1/5 and 1/3 as large). We did not want students to have 1/6 or 1/4 because some students in the pilot said to split it in half, and then split each half in half to create 4.
13-16	Splitting	10-13	Splitting	We revised old #16 to make new #13 say four times a large, so we did not have three times as large in two tasks.
17-18	Ratio	14-15	Multiplicative	We revised old #17 & 18 because no student was able to understand the question. We rewrote it to explicitly ask students to find what fraction is Angle 1 of Angle 2.

Table of Revisions between First Iterations and Final Iteration Continued

Old Tasks		New tasks		Revisions:
Numbers	*Classification/ Old Title*	*Numbers*	*Classification/ New Title*	
19-20	Ratio	16-19	Ratio	We changed the formatting and wording of old #19 & 20 to make it more clearly written. We also created two new tasks to maintain consistency with the grouping of four tasks per quantification. We further changed the pictures so they were not drawn to scale so students could not simply iterate the angle to figure it out. We also included color to help students understand what parts of the circle we were referencing.
21-24	Rate	20-23	Rate	We combined old 21 & 22 into a single task. We included two comparisons where the side lengths were the same and two where the side lengths were not. We also revised the angles with different side lengths so that none of their arcs would line up. For example <EBI and <GBD would have the same arc at ID, causing some students to say they were equal. We later realized this task did not provide a valid assessment of rate and so it was eliminated. We also eliminated old 23 & 24 as they were too easy for students. We realized these tasks did not truly assess a rate quantification, so we created three new tasks. We finally created four new tasks to assess students' rate quantification by including tasks similar to the ratio ones.

Appendix E

Clinical Interview Questions

1. How would you define an angle?

2. Here is my angle [angle create with a pair of long sticks]. Can you make the same angle with your sticks? [student has short sticks]. How do you know?
   a. Can you make an angle that is larger than mine with your sticks? How do you know? [Task 3]
   b. Can you make an angle that is smaller than mine with your sticks? How do you know? [Task 4]

3. Given this task [Task 5], how would you solve it?

4. Given this task [Task 6], how would you solve it?

5. Given this task [Task 9], how would you solve it?

6. Given this angle [one green] and this angle [one orange piece], which angle is larger? By how much? (Use fraction circles for this task)
   a. Given this angle [one orange piece], how much larger is it than this angle [one light blue piece]? (Use fraction circles for this task)
   b. Given this angle [green piece] how much smaller is it than this angle [red circle]? (Use fraction circles for this task)

7. Given this task [Task 10], how would you solve it?

8. Given this task [Task 14], how would you solve it?

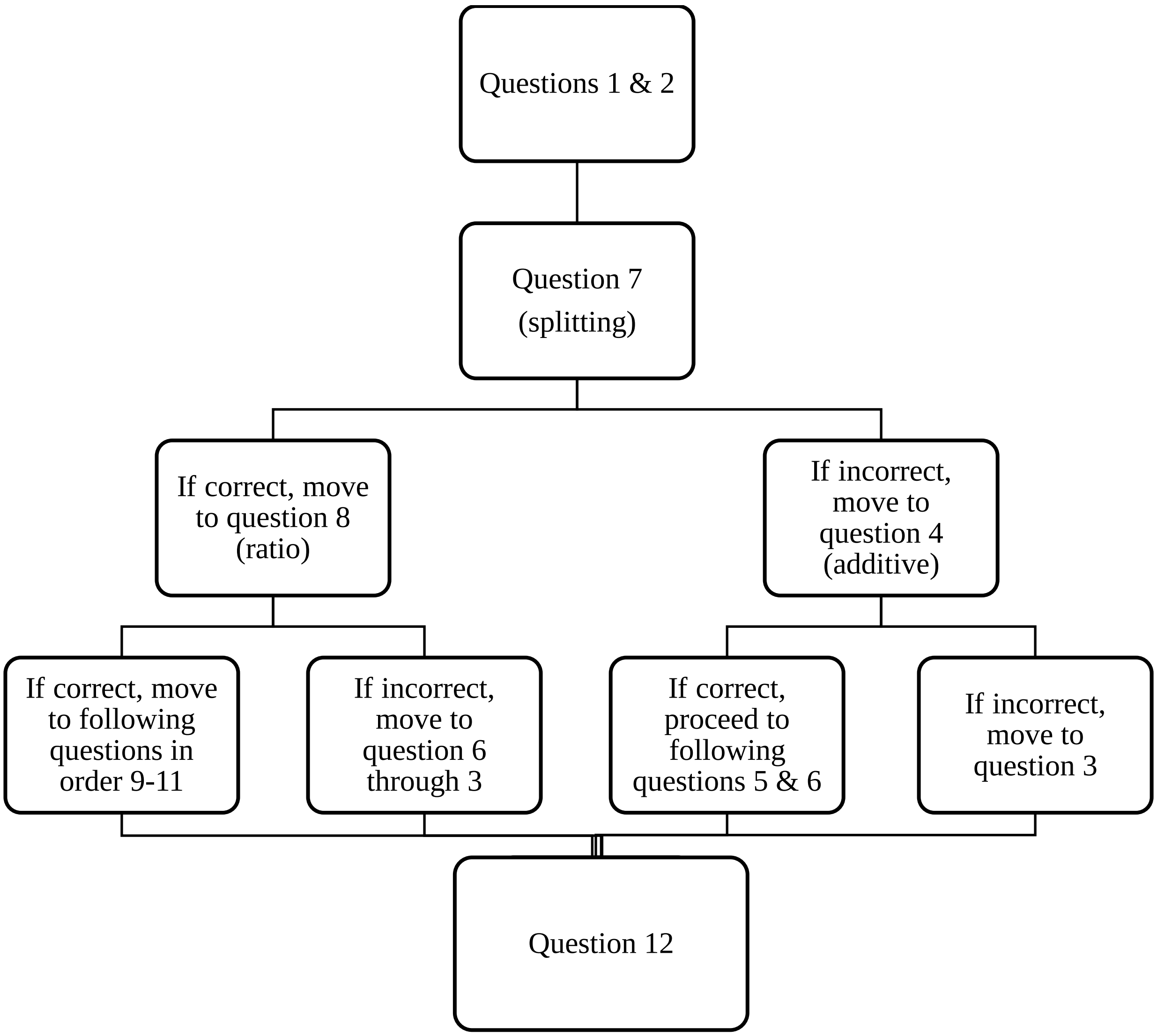
9. Given this task [Task 17] how would you solve it?

10. Given this task [Task 19] how would you solve it?

11. Given this task [Task 20] how would you solve it?

12. Suppose I handed you an angle and said that it had a measure of one degree. How would you check or prove it was in fact one degree?

Logical Order of Tasks:



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