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Relations between the notions of fundamental and minimal lengths, and duality, in a system with minimal length uncertainty relations are examined. Self-adjoint versions of operators relevant to the problem, and their spectra, are analyzed in detail. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2423220]

INTRODUCTION

While the ideas of *fundamental length* and *discretization* of space (*minimal length*) are vaguely related, they are logically and technically quite different. One of the aims of this paper is to discuss the interplay of these two ideas in a particular system, the other being preparation for investigation in Ref. 1 of the spectrum of hydrogen atom in this system. On the way we discuss carefully questions of domain and self-adjoint extension of various operators.

Skipping prehistory, during the last 50 years, the idea of an existence of fundamental length appeared repeatedly in analyses of gedanken experiments involving gravity and in string theory[3,4] (Ref. 2 has a review of this subject up to 1995). The analysis results often in a position uncertainty relations of the form

$$\Delta x \geq \frac{\hbar}{2}\left(\frac{1}{\Delta p} + \beta \Delta p\right), \tag{1}$$

which yields

$$\Delta x_{\min} = \hbar \sqrt{\beta}, \tag{2}$$

for minimal value of Δx, a quantity identified with fundamental length.

On the other hand, starting with Snyder,[5] discreteness of space, or space-time, brought in mainly with the hope of curing UV divergencies, was introduced through modification of kinematics of quantum mechanics, i.e., by modification of canonical commutation relations (CCR), and while the uncertainty relation (1) is "dynamical in its origin", in the work of Kempf, Mangano, and Mann[6] (KMM) it has also been implemented by a modification of CCR (cf. with an earlier work by Mead[7]).

Namely, Kempf *et al.*[6] introduced a two-parameter family,

$$[X_i, P_j] = i\hbar(\delta_{ij} + \beta P^2 \delta_{ij} + \beta' P_i P_j), \quad [P_i, P_j] = 0, \tag{3}$$

of deformations of CCR and of the Schrödinger (momentum) representation: the operators are acting in a Hilbert space of functions of $\mathbf{p} \in \mathbb{R}^D$, P_j is the (momentum) operator of multiplication by p_j, and the (noncommuting) position operators are

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$$X_j = i\hbar\left((1+\beta p^2)\partial_j + \beta' P_j\left(\sum_k P_k \partial_k\right)\right), \quad \partial_j = \frac{\partial}{\partial p_j}, \quad j = 1, \ldots, D. \tag{4}$$

A system defined by Eqs. (3) and (4) will be called a *KMM system* in the following.

A differential expression, such as Eq. (4), defines an operator only after its domain is specified. Different domains define different operators, all of which will be called, somewhat vaguely but conveniently, *versions* of the (ur-) operator X_j. Similarly, we will talk about versions of the R^2 operator Eq. (5).

As has been noted and extensively discussed in Ref. 6 (see also Ref. 8 and 9), the uncertainty relation [Eq. (1)] implies that the position operators, defined on the domain on which Eq. (1) holds, fail to be essentially self-adjoint in a very strong way (X_j's are examples of simple symmetric operators in the sense of Sec. 103 of Ref. 10; a closely related notion, that of "indeterminate operators," appears in a similar context in Ref. 7), and different self-adjoint versions of these operators may, and do here, have different spectra. Since spectra of self-adjoint operators and their functional calculus are essential in many problems of quantum mechanics—in the present context, spectra have been used in comparison of KMM system with standard quantum mechanics[1,6,11–14]—one has to face here the problem of the choice of the self-adjoint version.[6]

Whereas the introduction of fundamental length often eliminates ultraviolet (UV) divergencies, in many field theory models it leads also to a modification of infrared (IR) behavior ("UV/IR mixing"), and to duality—a strong form of UV/IR mixing.[15] Inequality (1), being invariant under the substitution $\Delta p \mapsto (\beta \Delta p)^{-1}$, suggests such a duality. We show that (a slight generalization of) KMM system has indeed a duality property: there exists a transformation relating the original model to another one, of the same form, but with high and low momenta interchanged: in Sec. II D we establish inversion invariance of the model and then trace the duality back to this invariance.

It is the problems of relation between fundamental and minimal lengths, domain problem, discreteness of spectrum, and then duality in a system defined by Eqs. (3) and (4) that are considered in the present paper.

For one degree of freedom, one has the position operator $X = i\hbar(1+\alpha p^2)d/dp$, $\alpha = \beta + \beta'$, which is initially defined on functions with suitable decay at infinity. This operator is analyzed in Refs. 6 and 8. Change of the variables $p \mapsto q = \arctan(\alpha p)$, $-\pi/2 < q < \pi/2$, maps X onto the operator $i\hbar\sqrt{\alpha}\partial_q$, self-adjoint versions of which are well known (see Sec. III A): they are naturally labeled by points $e^{i\theta}$ of the unit circle, with eigenfunctions satisfying twisted-periodic boundary conditions and $\{\hbar\sqrt{\alpha}(\theta/\pi+2n): n \in \mathbb{Z}\}$ as (the purely discrete and nondegenerate) spectrum, so that by picking up a self-adjoint version of the position operator one breaks translation invariance of the theory. This is akin to breaking gauge invariance by gauge fixing, invariance that is restored if one considers all the gauges. Moreover, the spacing of the spectrum, $2\hbar\sqrt{\alpha}$, is a gauge-independent quantity.

This attractive picture does not hold, however, in higher dimensions. For as shown in Sec. III in the simplest case of $\beta' = 0$, no natural extension of (X_j) has pure point spectrum.

This should be contrasted with properties of the *length operator R*—the square root of

$$R^2 = \sum_{j=1}^{D} X_j^2 = -\hbar^2 \sum_{j=1}^{D} \left((1+\beta p^2)\partial_j + \beta' P_j\left(\sum_k P_k \partial_k\right)\right)^2 \tag{5}$$

$(=: R^2_{(\beta,\beta')})$, the spectrum of any version of which is pure point. (The situation is reminiscent of that in loop quantum gravity,[16] where the spectra of area and volume operators are pure point, with no, however, topological interpretation of the spectrum in the present context, at least at this point). Moreover, since R^2 is a positive operator, it has a distinguished self-adjoint extension—Friedrichs' extension of its minimal version, which in the present context has many appealing properties. At the end of Sec. II C, it is argued that Friedrichs' extension is the "right one," both for R^2 and for some other operators, and notion of "physical states" introduced in Ref. 6 is discussed.

We note that for the KMM systems there seem to be no relation between the choice of self-adjoint extensions of the position operators and the "right" self-adjoint extension of R^2 (or of

harmonic oscillator): as can be easily seen in the case of one dimension for no self-adjoint extension of X is its square equal to Friedriechs' extension of R^2. The same holds in higher dimensions. Thus the problem of choosing the self-adjoint version of position operators, i.e., of the generating algebra of the system, is irrelevant to questions about its dynamics.

We also note that for the KMM systems there seem to be no relation between the choice of self-adjoint extensions of the position operators and the right self-adjoint version of R^2, and then of the Hamiltonian. For example, in the case of one dimension, it can easily be seen that there is no self-adjoint extension of X such that its square is equal to Friedriechs' extension of R^2. The situation is similar in higher dimensions. Thus the problem of choosing the self-adjoint version of position operators, i.e., of the generating algebra of the system, is irrelevant to questions about the dynamics of the system.

Our discussion of duality (Sec. II D) requires a slight generalization of Eqs. (3) and (4) in which $p^2\beta'$ is a function of p of the form $\beta_3+\beta_4 p^2$. It is convenient, however, to start with a still more general system, in which

$$[X_j,P_k]=i(f(P)\delta_{jk}+g(P)P_jP_k), \quad [P_i,P_j]=0, \tag{6}$$

where f and g are "arbitrary" functions of p, in representation

$$X_j=i\left(f(P)\frac{\partial}{\partial p_j}+g(P)P_j\sum_k P_k\frac{\partial}{\partial p_k}\right), \quad j=1,\ldots,D, \tag{7}$$

i.e., with the most general rotation invariant vector-valued first order differential operator $\mathbf{X}=(X_j)_{j=1,\ldots,D}$ (see the end of Sec. II D).

In general, the dual system is not isomorphic to the original one. Still, there is a two-parameter family of self-dual models—one parameter family, if the operators are brought to a dimensionless form. As one would expect, these are not small deformations of the operators of ordinary Quantum Mechanics.

We note that our results regarding position and length operators are in agreement with gedanken experiments involving gravity, which lead to fundamental length, not to minimal spacing for position operator. The later is usually obtained from the former by performing a "projection on coordinate axes," a questionable notion in a theory in which the position operators do not commute.

The paper is organized as follows. The stage is set in Sec. I, where we also show that the seemingly more general system introduced in Ref. 8 can be reduced to KMM system through a gauge transformation. The operator R^2—spectra of its self-adjoint versions, Friedrichs' extension, and duality—is analyzed in detail in Sec. II: after performing standard reduction with respect to rotations for each value of angular momentum one obtains a second order (real) ordinary differential operator, with deficiency indices 0, 1, or 2, depending on the values of the parameters. For each (β,β'), the reduced operators are essentially self-adjoint for large enough angular momenta. However, the number of angular momenta for which the reduced operator is not essentially self-adjoint tends to infinity as the parameter $\eta=\beta/\alpha$ tends to zero, even when both β and β' are small, indicating that $R^2_{(\beta,\beta')}$ is a rather singular deformation of $R^2_{(0,0)}$.

To compare R^2 with the Laplace operator $(=R^2_{(0,0)})$, we note that they have essentially the same (small p) behavior, as one would expect on inspection of Eq. (4). However, their UV behavior is quite different, leading to limit-circle behavior of R^2 at infinity. This again should be expected on the basis of Eq. (4): quasiclassically, considering iX_j to be generators of motion in $\mathbb{R}^D$, the polynomial character of the coefficients of ∂_j leads to motion which reaches infinity in finite time, and therefore one needs additional information to specify the operators. Thus we see again that it is the requirement that the deformation produces UV cutoff that leads to nonuniquecess of self-adjoint versions. Also, in contrast to the case of Laplace operator, some of the versions of R^2 are not rotation or conjugation invariant and in contrast to the case of harmonic oscillator, the deformation changes the energy spectrum in an essential way.

Section III contains a pretty detailed discussion of a natural family of self-adjoint versions of the position operators. While in dimension larger than 1, none of these versions has a pure point spectrum, the point spectrum of the one-dimensional case gets deformed into a kind of a band structure for higher dimensions.

There are three appendices: Appendix C, which has been already mentioned, Appendix A about reduction to the hypergeometric equation, and Appendix B in which we apply formulas of Appendix A to recalculate the spectrum of harmonic osciallator, computed earlier in Ref. 11, with added discussions of self-adjoint versions.

The mathematics of the present paper is kept (exaggeratedly) simple, a combination of explicit computations exploiting the hypergeometric equation and of a few well known theorems. That so much can be computed here is due to the fact that the coefficients of first order differential operators X_j are even quadratic functions of $\mathbf{p}$: this yields for the radial operators second order ordinary differential operators in the (complex) p^2 plane with three regular singular points (zero, infinity, and $p^2 = -1/\alpha$), and eigenvalue equation which is therefore reducible to the hypergeometric equation. The explicit formulas are used in our paper Ref. 1 on bound states of hydrogenlike atoms (see also Ref. 16). However, many of the results are valid for models that are not explicitly soluble.

I. PRELIMINARIES

We start with operators,[9]

$$X_j = X_j^{(\gamma)} = i\hbar\left((1 + \beta\mathbf{P}^2)\partial_j + \beta' P_j\left(\sum_k P_k\partial_k\right) + \gamma P_j\right), \quad j = 1, \ldots, D, \tag{8}$$

acting on functions of $\mathbf{p} \in \mathbb{R}^D$; here β and β' are non-negative constants, γ is an arbitrary real number, $\partial_k = \partial/\partial p_k$, and $P_j f(\mathbf{p}) = p_j f(\mathbf{p})$. The operators [Eq. (8)] are symmetric when acting in the Hilbert space $L^2(\mathbb{R}^D, w_\gamma(p)\mathrm{d}^D\mathbf{p})$ defined by the scalar product[11] (χ is denoted by κ in Ref. 11),

$$(f, g) \mapsto \int \mathrm{d}^D\mathbf{p}\, w_\gamma(p) f^*(\mathbf{p}) g(\mathbf{p}),$$

where

$$w_\gamma(p) = \frac{1}{(1 + \alpha\mathbf{p}^2)^{1+\chi(\gamma)}},$$

$$\alpha = \beta + \beta', \quad \chi(\gamma) = \frac{\beta'}{2\alpha}(D - 1) - \frac{\gamma}{\alpha},$$

Ref. 11. Since, as is shown below, the case of nonzero γ is related to that of $\gamma = 0$ by a gauge transformation, from now on we set γ equal to zero and consider the operators acting in the Hilbert space $L^2(\mathbb{R}^D, w_0(p)\mathrm{d}^D\mathbf{p})$ of measurable functions of $\mathbf{p}$ with scalar product,

$$(f|g) = \int \mathrm{d}^D\mathbf{p}\, w_0(p) f^*(\mathbf{p}) g(\mathbf{p}), \quad w_0(p) = \frac{1}{(1 + \alpha\mathbf{p}^2)^{1+\chi}}, \quad \chi = \frac{1}{2}(1 - \eta)\bar{D}. \tag{9}$$

Here and below

$$\alpha = \beta + \beta', \quad \eta = \frac{\beta}{\alpha}, \quad \text{and} \quad \bar{D} = D - 1.$$

In most of the present paper we assume that $D \geq 2$.

Passing to the dimensionless variable $\bar{\mathbf{p}} = \sqrt{\alpha}\mathbf{p}$, one obtains

$$X_j = i\sqrt{\alpha}\hbar\left[(1 + \eta\mathbf{p}^2)\partial_{\bar{p}_j} + (1 - \eta)\bar{p}_j\left(\sum_k \bar{p}_k\partial_{\bar{p}_k}\right)\right] = \sqrt{\alpha}\hbar\bar{X}_j, \tag{10}$$

and $R = \sqrt{\alpha}\hbar\sqrt{\Sigma_j\bar{X}_j^2} = \sqrt{\alpha}\hbar\bar{R}$, which defines the dimensionless operators $\bar{X}_j$ and $\bar{R}$; $\sqrt{\alpha}\hbar$ is closely related to the fundamental length of the theory (see Sec. II B). Furthermore, introducing (α, η) instead of (β, β') of Eq. (4) as the parameters, one notes that the $\bar{\mathbf{X}}$ and $\bar{R}$ depend on η but not on α, and that dropping from the scalar product [Eq. (9)], an overall factor of $\alpha^{-D/2}$, one ends up with Hilbert space with scalar product $(f, g) \mapsto \int d^D\bar{\mathbf{p}}(1+\bar{\mathbf{p}}^2)^{-1-\chi}f^*(\bar{\mathbf{p}})g(\bar{\mathbf{p}})$, which again depends on η but not on α.

These formulas explain the form of the α dependence of the eigenvalues [Eq. (46)] of R, of the change of variables [Eq. (17)] below, and of the fact that the Hilbert space depends on the parameters only through η. However, we work with the variable $\mathbf{p}$ in most of the paper.

A. Gauge transformation

A gauge transformation simplifies somewhat calculations by removing the γ term from Eq. (8). Let u be a function of one variable, let G_u be the gauge transformation $\varphi \mapsto G_u\varphi$,

$$(G_u\varphi)(\mathbf{p}) = e^{u(p^2)}\varphi(\mathbf{p}),$$

and let $L^2(\mathbb{R}^D, w_0(p)|e^{-u(p^2)}|^2 d^D\mathbf{p})$ be the Hilbert space of functions ψ for which

$$\int d^D\mathbf{p}\, w_0(p)|e^{-u(p^2)}\psi(\mathbf{p})|^2 < \infty, \tag{11}$$

in which Eq. (11) is the square of the norm; G_u defines a unitary operator from $L^2(\mathbb{R}^D, w_0(p)|e^{-u(p^2)}|^2 d^D\mathbf{p})$ to $L^2(\mathbb{R}^D, w_0(p)d^D\mathbf{p})$. Then

$$G_u\mathbf{X}^{(0)}G_u^{-1}\psi(\mathbf{p}) = e^{u(p^2)}\mathbf{X}^{(0)}(e^{-u(p^2)}\psi(\mathbf{p})) = \mathbf{X}^{(0)}\psi(\mathbf{p}) - 2i\hbar(1 + \alpha p^2)\mathbf{p}u'(p^2)\psi,$$

which is equal to $\mathbf{X}^{(\gamma)}\psi$ if

$$-2i\hbar(1 + \alpha p^2)\mathbf{p}u'(p^2) = -i\hbar\gamma\mathbf{p}, \quad \text{i.e.,} \quad e^{-u(p^2)} = (1 + \alpha p^2)^{\gamma/2\alpha}. \tag{12}$$

This shows that for u of Eq. (12), G_u is a unitary map of $L^2(\mathbb{R}^D, w_\gamma(p)d^D\mathbf{p})$ onto $L^2(\mathbb{R}^D, w_0(p)d^D\mathbf{p})$, and that $G^{-1}X^{(0)}G = X^{(\gamma)}$. It is clear that one can replace here $\gamma\mathbf{p}$ with "any" function of the form $\gamma(p)\mathbf{p}$.

This gauge invariance explains why the spectrum of harmonic oscillator of Ref. 11 does not depend on γ and shows also that the eigenfunctions of the corresponding operators are related by G.

We note that if the number γ is purely imaginary, these calculations and formulas show that $\exp(-u(p^2))$ is of modulus one and that therefore Eq. (11) is the same as Eq. (9) with G_u being a unitary map of $L^2(\mathbb{R}^D, w_0(p)d^D\mathbf{p})$ into itself.

II. ANALYSIS OF R^2

We consider here in detail the second order differential operator [Eq. (5)], its self-adjoint versions, and their spectra. Since it commutes with rotations, in a standard way, it is tentatively reduced in Sec. II A to (second order) ordinary differential operators R_L^2 in the radial variable p. We then change the variable to obtain second order differential equation with three regular singular points. In Sec. II B we rewrite this equation as the hypergeometric equation, analyze it in terms of Jacobi polynomials, and then discuss self-adjoint versions of the reduced radial operator. Finally, in Sec. II C, we use all this to analyze self-adjoint versions of R^2, of which there are in general many, and argue that the version given by Friedrich's extension is the "right one".

A. Separation and change of variables

We start with the standard reduction process with respect to the group of rotations (p. 160 of Ref. 17, and Ref. 11, and references therein). First, the Hilbert space $L^2(\mathbb{R}^D, w_0(p)\mathrm{d}^D\mathbf{p})$ is factorized into radial and angular (spherical) parts

$$L^2(\mathbb{R}^D, w_0(p)\mathrm{d}^D\mathbf{p}) = \mathcal{R} \otimes \mathcal{S}, \tag{13}$$

where

$$\mathcal{R} = L^2(\mathbb{R}_+, w_0(p)p^{D-1}\mathrm{d}p), \quad \mathcal{S} = L^2(S^{D-1}),$$

and S^{D-1} is the unit sphere in $\mathbb{R}^D$, then $\mathcal{S}$ is decomposed into subspaces on which the action of rotations is irreducible, $\mathcal{S} = \oplus_L \mathcal{S}_L$, and finally $L^2(\mathbb{R}^D, w_0(p)\mathrm{d}^D\mathbf{p})$ and R^2 are decomposed,

$$L^2(\mathbb{R}^D, w_0(p)\mathrm{d}^D\mathbf{p}) = \oplus_L (\mathcal{R} \otimes \mathcal{S}_L), \quad R^2|_{\mathcal{R} \otimes \mathcal{S}_L} = R_L^2 \otimes \mathbf{1}_L, \quad R^2 = \oplus_L (R_L^2 \otimes \mathbf{1}_L), \tag{14}$$

where $L = \sqrt{\ell(\ell+D-2)}$, $\ell = 0, 1, \ldots, R_L^2$ is R^2 reduced at angular momentum L, and $\mathbf{1}_L$ is the identity operator in $\mathcal{S}_L$. At the end of this section we discuss ambiguities of this formulation in case the operators involved are not essentially self-adjoint.

In our case,

$$R_L^2 = -\hbar^2 \left[\left((1+\alpha p^2)\frac{\mathrm{d}}{\mathrm{d}p} \right)^2 + \bar{D}\left(\frac{1}{p} + \beta p \right)\left((1+\alpha p^2)\frac{\mathrm{d}}{\mathrm{d}p} \right) - \left(\frac{1}{p} + \beta p \right)^2 L^2 \right], \tag{15}$$

as can be checked by direct computation (see Sec. II D for a more general formula). Equation (15) agrees with the expression of Ref. 11 for reduced Hamiltonian of harmonic oscillator, and in fact can be easily derived from it.

Considered as an equation in complex domain, the eigenvalue equation corresponding to Eq. (15),

$$-\hbar^2 \left[\left((1+\alpha p^2)\frac{\mathrm{d}}{\mathrm{d}p} \right)^2 + \bar{D}\left(\frac{1}{p} + \beta p \right)\left((1+\alpha p^2)\frac{\mathrm{d}}{\mathrm{d}p} \right) - \left(\frac{1}{p} + \beta p \right)^2 L^2 \right]\Phi(p) = r^2\Phi(p), \tag{16}$$

where r^2 is an eigenvalue of R^2, has four (regular) singular points: $p = 0, \pm i/\sqrt{\alpha}, \infty$. A change of the variable to p^2 reduces the number of singular points to three. After that, a standard process, which is essentially unique, transforms Eq. (16) into the hypergeometric equation (HE) the relevant solutions of which are expressible in terms of Jacobi polynomials.[18]

We perform the reduction to HE by first bringing the three singularities to the standard position $(-1, 1, \infty)$ through the transformation to the variable z,

$$p = \frac{1}{\sqrt{\alpha}}\sqrt{\frac{1+z}{1-z}}, \quad -1 < z < 1, \quad z = \frac{\alpha p^2 - 1}{\alpha p^2 + 1}, \tag{17}$$

and then using the (standard) formulas of Appendix A reduce it the resulting equation to the hypergometric one. Erlier, Ref. 11 arrived at the transformation $p \mapsto z$ through a series of changes of variable. While some of these are not applicable in the present context, the transformation itself is very useful and, in a sense, canonical: it is determined by the singular points of Eq. (52) in the complex p^2 plane.

Namely, setting

$$\rho = \frac{r^2}{4\alpha\hbar^2} \quad (r = 2\hbar\sqrt{\alpha}\sqrt{\rho}), \quad D_1 = \frac{1}{2}\bar{D} - \frac{1}{2}, \quad D_2 = \frac{1}{2}\eta\bar{D} + \frac{1}{2}, \tag{18}$$

in the z variable, Eq. (15) is

$$R_L^2 = -4\alpha\hbar^2\left[(1-z^2)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + (D_1+D_2-(D_1-D_2+2)z)\frac{\mathrm{d}}{\mathrm{d}z} - \frac{L^2}{4}\frac{((1+\eta)-(1-\eta)z)^2}{1-z^2}\right],$$

(19)

while Eq. (16) can be written as

$$-(1-z^2)\Phi'' - (D_1+D_2-(D_1-D_2+2)z)\Phi' + \frac{L^2}{4}\frac{((1+\eta)-(1-\eta)z)^2}{1-z^2}\Phi = \rho\Phi.$$

(20)

Transformation to the z variable appeared earlier in [Eq. 59 of Ref 11] on harmonic oscillator, where it is obtained through a series of substitutions. While one of the intermediate substitutions does not make sense in the present context, the transformation $p \mapsto z$ does.

The differential operator of Eq. (19) is diagonalized in the Hilbert space obtained from $L^2(\mathbb{R}_+, p^{D-1}w_0(p)\mathrm{d}p)$ by transformation (17). Now,

$$\frac{p^{D-1}}{(1+\alpha p^2)^{1+\chi}}\mathrm{d}p = \frac{1}{2^{1+\chi}\alpha^{D/2}}(1+z)^{D_1}(1-z)^{-D_2}\mathrm{d}z.$$

Therefore for any real μ and ν we set

$$w^{(\mu,\nu)}(z) = (1-z)^\mu(1+z)^\nu, \quad \mathcal{H}^{(\mu,\nu)} = L^2([-1,1], w^{(\mu,\nu)}(z)\mathrm{d}z),$$

(21)

i.e., the scalar product in $\mathcal{H}^{(\mu,\nu)}$ is

$$(f|g)^{(\mu,\nu)} = \int \mathrm{d}z\, w^{(\mu,\nu)}(z) f^*(z) g(z),$$

(22)

and we let operator (20) act in $\mathcal{H} = \mathcal{H}(-D_2, D_1)$. We note that the Hilbert space $\mathcal{H}$ and the operator of Eq. (20) depend on η, but not on α.

Equation (20) can be put in a form which displays it as an eigenvalue equation for manifestly symmetric operator in $\mathcal{H}$,

$$-\frac{1}{w(z)}\frac{\mathrm{d}}{\mathrm{d}z}\left[p(z)\frac{\mathrm{d}\Phi(z)}{\mathrm{d}z}\right] + \frac{L^2}{4}\frac{((1+\eta)-(1-\eta)z)^2}{1-z^2}\Phi = \rho\Phi,$$

(23)

where $w = w^{(-D_2,D_1)}$ and $p(z) = (1+z)^{D_1+1}(1-z)^{-D_2+1}$.

Until this point our discussion has a tentative character since the operator R^2 decomposed in Eq. (14) has not yet been defined. Moreover, it appears that not all self-adjoint versions of R^2 are rotation invariant. Still all these versions can be described in terms of the reduced operators R_L^2, which is done in Sec. II C. We will now make more precise the setup for our analysis of R_L^2.

To analyze R_L^2 in the Hilbert space setup, one considers self-adjoint operators generated by the differential expressions (5) and (15), or, alternatively, (23). In a conservative approach to the later problem, one starts with a differential operator which has a rather small domain, so that the operator is transparently symmetric, and the differentiation has the usual elementary sense. This initial domain can be chosen in many ways, natural from one or another point of view, resulting in different *initial versions* of R^2. In case of Eq. (15) [Eq. (19)], one usually takes for this initial domain the family $C_0^\infty(]0,+\infty[)$ of all infinitely differentiable functions of p, each function vanishing in a neighborhood of zero and infinity $[C_0^\infty(]-1,+1[)]$, since the differential expressions are singular here at the boundary. This is the domain of a *fundamental* version $R_{L,\min}^2$ of R_L^2 (p. 248 of Ref. 19). Then one can consider the problem of finding self-adjoint extensions of the fundamental operator—*self-adjoint versions* of R_L^2 and their spectra discuss uniqueness of such an extension, and in case of nonuniqueness, as in the case at hand, to choose one extension or to discuss life with nonuniquely defined extension.

An alternative, but almost equivalent way, which is in a sense dual to the one described, is to start with the *maximal (differential) operator* (p. 248 of Ref. 19), corresponding to Eq. (5) or (23),

which is defined on the maximal natural domain on which these differential expressions make sense, yielding elements of the Hilbert space. The maximal version of the operator is equal to the adjoint of its fundamental version, it is almost unique, and is, in general, not symmetric. Self-adjoint versions of the operators are then obtained by restricting these maximal versions to suitable domains.

B. The reduced operators

Self-adjoint operators generated by holomorphic differential equations of form (23) are considered in Chap. 8 of Ref. 20 and other places; in fact Ref. 20 analyzes a system very close to ours, though not in the whole range of parameters needed here. Our discussion has the (dis)advantage of being very concrete, given a few basic facts about such systems, and in relating the problem to well known properties of Jacobi polynomials.

We start with a discussion of the simplest three-dimensional case,

$$D = 3, \quad L = 0, \quad \text{and } \beta' = 0(\eta = 1), \tag{24}$$

which, however, displays characteristic features of the problem. Equation (20) is now

$$-(1 - z^2)\Phi'' - (2 - z)\Phi' = \rho\Phi. \tag{25}$$

Equation (25) is a particular case of equation

$$(1 - z^2)u'' + [(b - a) - (c + 1)z]u' + \lambda(\lambda + c)u = 0, \quad c = a + b + 1, \tag{26}$$

or, equivalently, of the equation

$$-(1 - z^2)u''(z) + [(a - b) + (c + 1)z]u'(z) + (c^2/4)u(z) = \mu u(z), \quad \mu = (\lambda + c/2)^2. \tag{27}$$

For $\lambda = n$, $n = 0, 1, \ldots$, these equations are satisfied by Jacobi polynomials $P_n^{(a,b)}$ (Formula 8.962.1 of Ref. 21). The characteristic exponents of Eq. (27) are $(0, a)$ at $z = +1$ and $(0, b)$ at $z = -1$. Jacobi polynomials are square integrable with respect to the weight $w^{(a,b)}$ if and only if $a, b > -1$ and, in the later case, they form a complete orthogonal family in $\mathcal{H}^{(a,b)}$. Since $a + b = -1$ in case of Eq. (25), both a and b cannot be larger than -1 (in fact, here $a = -3/2$ and $b = 1/2$), and therefore Jacobi polynomials are not square integrable.

Square-integrable solutions of Eq. (25), which are expressible in terms of Jacobi polynomials are found in two steps: in step 1, substitution transforming it into an equation with non-negative a and b is performed and in step 2, square integrable solutions of the transformed equation [Eq. (29)] are found. The general case (arbitrary L, η) is analyzed in the same two steps.

Step 1. The substitution (gauge transformation) is here

$$\Phi(z) = (1 - z)^{3/2}\Psi(z). \tag{28}$$

It transforms Eq. (25) into

$$-(1 - z^2)\Psi''(z) + (1 + 4z)\Psi'(z) + (9/4)\Psi(z) = \rho\Psi(z), \tag{29}$$

which is Eq. (27) with

$$a = 3/2, \quad b = 1/2, \quad c = 3, \quad \text{and } \mu = \rho. \tag{30}$$

Step 2. Since a and b of Eq. (29) are > -1, Eq. (29) has Jacobi polynomials $P_n^{(3/2,1/2)}$ as a complete orthogonal family (in $\mathcal{H}^{(3/2,1/2)}$) of solutions, with eigenvalues $\rho_n = (n + 3/2)^2$, $n \geq 0$.

Moreover, since in our case $b < 1$, there is another family of solutions of Eq. (25) expressible in terms of Jacobi polynomials: the gauge transformation $\Psi \mapsto \tilde{\Psi}$,

$$\Psi(z) = (1 + z)^{-1/2}\tilde{\Psi}(z), \tag{31}$$

transforms Eq. (29) into

$$- (1 - z^2)\tilde{\Psi}''(z) + (2 + 3z)\tilde{\Psi}'(z) + \Theta(z) = \mu\tilde{\Psi}(z), \quad \mu = \rho, \tag{32}$$

($a = 3/2$, $b = -1/2$, and $c = 2$), which again has Jacobi polynomials, $P_n^{(3/2, -1/2)}$ this time, as a complete orthogonal family of solutions (in $\mathcal{H}^{(3/2, -1/2)}$). Performing now transformation inverse to Eq. (31), one obtains two complete orthogonal (in $\mathcal{H}^{(3/2, 1/2)}$) families,

$$\Psi_n^{(1)}(z) = P_n^{(3/2, 1/2)}(z) \quad \text{eigenvalues } (n + 3/2)^2, \tag{33}$$

$$\Psi_n^{(2)}(z) = (1 + z)^{-1/2} P_n^{(3/2, -1/2)}(z) \quad \text{eigenvalues } (n + 1)^2, \tag{34}$$

and, finally, transformation (28) yields two complete orthogonal (in $\mathcal{H} = \mathcal{H}^{(-3/2, 1/2)}$) families of eigenfunctions of R^2,

$$\Phi_n^{(1)}(z) = (1 - z)^{3/2} P_n^{(3/2, 1/2)}(z) \quad \text{eigenvalues } (n + 3/2)^2,$$

$$\Phi_n^{(2)}(z) = (1 - z)^{3/2}(1 + z)^{-1/2} P_n^{(3/2, -1/2)}(z) \quad \text{eigenvalues } (n + 1)^2, \quad n = 0, 1, 2, \dots .$$

These two families of eigenfunctions diagonalize two different self-adjoint extensions of the fundamental operator $R_{0,\min}^2$—extensions, which have different spectra. They are two members of a one-parameter family of all self-adjoint extensions of $R_{0,\min}^2$ for $\eta = 1$, extensions which can be characterized by their (different) asymptotics at $z = -1$—the point at which the nonzero exponent b is smaller than 1 (see below).

We will pass now to determination of spectra of R_L^2 at all angular momenta L and for all values of η.

Step 1. Let

$$a(\eta, L) = \sqrt{D_2^2 + \eta^2 L^2}, \quad b(\eta, L) = \sqrt{D_1^2 + L^2}, \quad c = a + b + 1, \tag{35}$$

$$\sigma_1 = -\frac{1}{2}D_1 + \frac{1}{2}b(\eta, L), \quad \sigma_2 = \frac{1}{2}D_2 + \frac{1}{2}a(\eta, L) \tag{36}$$

[$D_i = D_i(\eta)$, $i = 1, 2$, are as in Eq. (18)]; the notation does not make explicit the dependence of a, b, c, and σ on D. Let

$$\mu = \frac{1}{4}(\chi^2 + (1 - \eta)^2 L^2) + \rho, \quad \text{i.e., } \rho = \mu - \frac{1}{4}(\chi^2 + (1 - \eta)^2 L^2) \tag{37}$$

[η as in Eq. (9)]; it can be checked by direct calculation that the substitution

$$\Phi(z) = (1 + z)^{\sigma_1}(1 - z)^{\sigma_2}\Psi(z) \tag{38}$$

transforms Eq. (20) into Eq. (27) with $a = a(\eta, L)$, $b = b(\eta, L)$, and μ as in Eqs. (35) and (37) (see Appendix A).

In terms of Hilbert spaces and operators, the substitution (38) defines an isomorphism $U: \Psi \mapsto \Phi$ of $\mathcal{H}^{(a,b)}$ onto $\mathcal{H}$, which sends the minimal version $J_{\min}$ of the operator defined by left hand side of Eq. (27) to $R_{L,\min}^2$,

$$U^{-1}J_{\min}U = R_{L,\min}^2, \quad J_{\min} = J_{\min}^{(a,b)}, \quad \mathcal{D}(J_{\min}) = C_0^\infty(]-1, 1[) \subset \mathcal{H}^{(a,b)}. \tag{39}$$

Thus, the problems of finding self-adjoint extensions of $R_{L,\min}^2$ and of $J_{\min}$ are equivalent. We will therefore discuss the problem for $J_{\min}$; the corresponding properties of $R_{L,\min}^2$ are summarized in the theorem of the end of this section.

Step 2. We start by considering four special families $\{\mathcal{F}^\varkappa\}_{\varkappa=(\varkappa_+, \varkappa_-)}$, $\varkappa_+ = \pm$, $\varkappa_- = \pm$, of solutions of Eq. (27) with $a, b \geq 0$, families that can be expressed in terms of Jacobi polynomials. The case of

$a > 0$ and $b = 0$, which appears only in two dimensions, stands apart and is considered separately below. Substitution (38) yields then solutions of Eq. (20).

$\mathcal{F}^{(+,+)}$ is just the family $\{P_n^{(a,b)}\}_{n=0,1,\ldots} = \{P_n^{(a,b)}\}_{n \geq 0}$ of Jacobi polynomials. More generally, for any $\varkappa$, let

$$\mathcal{F}^{\varkappa} = \{\Psi_n^{\varkappa}\}_{n \geq 0}, \quad \text{where } \Psi_n^{\varkappa}(z) = (1-z)^{(a^{\varkappa}-a)/2}(1+z)^{(b^{\varkappa}-b)/2}P_n^{(a^{\varkappa},b^{\varkappa})}(z), \tag{40}$$

and $a^{\varkappa} = \varkappa_+ a$ and $b^{\varkappa} = \varkappa_- b$. We will say that *the family $\mathcal{F}^{\varkappa}$ is square integrable* if its elements are in $\mathcal{H}^{(a,b)}$. The number $\mathfrak{n}(a,b)$ of square-integrable families is equal to deficiency indices of $J_{\min}$. While we do not write down solutions of the equations $(J_{\min})^*\Phi = \pm i\Phi$, i.e., the elements of deficiency subspaces $\mathcal{K}_{\pm}$—these can be expressed in terms of the hypergeometric function, we note that the solutions of the indicial equations [Eqs. (A13) and (A14)] do not depend on the eigenvalue ρ of Eq. (20), and therefore boundary behavior of these equations, which determines their sqare integrability, is the same as of the elements of the corresponding families $\mathcal{F}^{\varkappa}$.

Since the substitution $\Psi \mapsto \Psi^{\varkappa}$, $\Psi(z) = (1-z)^{(a^{\varkappa}-a)/2}(1+z)^{(b^{\varkappa}-b)/2}\Psi^{\varkappa}(z)$ transforms Eq. (27) into

$$-(1-z^2)(\Psi^{\varkappa})''(z) + [(a^{\varkappa} - b^{\varkappa}) + (c^{\varkappa}+1)z](\Psi^{\varkappa})'(z) + (c^{\varkappa}/2)^2 \Psi^{\varkappa}(z) = \mu \Psi^{\varkappa}(z), \tag{41}$$

where $c^{\varkappa} = a^{\varkappa} + b^{\varkappa} + 1$, of which $P_n^{(a^{\varkappa},b^{\varkappa})}$ are solutions, $\Psi_n^{\varkappa}$ is a solution of Eq. (27) with

$$\mu = \mu_n^{\varkappa} = (n + c^{\varkappa}/2)^2. \tag{42}$$

In case of $b = 0$ (which appears for $D = 2$ only), i.e., in case of equal characteristic exponents at $z = -1$, in addition to $\mathcal{F}^{(+,+)}$, also the family $\mathcal{F}^{(+,0)} := \{\Psi_n^{(+,0)} : n \geq 0\}$ is square integrable; here $\Psi_n^{(+,0)}(z)$ is the $u_2((1+z)/2)$ of Sections 9.153-3 of Ref. 20, with $\alpha = -n$, $\beta = n + c$, and $m = 0$. The values of μ's and ρ's are here the same as for the $\mathcal{F}^{(+,+)}$. For $0 < a < 1$, the same holds for $\mathcal{F}^{(-,0)}$.

Since for real a and b, $\{P_n^{(a^{\varkappa},b^{\varkappa})} : n \geq 0\}$ is a complete orthogonal family in $\mathcal{H}^{(a^{\varkappa},b^{\varkappa})}$ if and only if $a^{\varkappa}, b^{\varkappa} > -1$, one obtains that $\{\Psi_n^{\varkappa}\}_{n \geq 0}$ is a complete orthogonal family in $\mathcal{H}^{(a,b)}$ under the same conditions on a, b, and that it then defines a self-adjoint extension $J^{\varkappa}$ of $J_{\min}$: since the functions $\Psi_n^{\varkappa}$ are smooth on $]-1,1[$, they are in domain of the adjoint operator $J_{\min}^*$ and then the closure of the restriction of $J_{\min}^*$ to linear span of $\{\Psi_n^{\varkappa}\}_{n \geq 0}$ is an obviously self-adjoint operator, which is an extension of $J_{\min}$. We call these extensions of $J_{\min}$, and then of $R_{L,\min}^2$, *pure extensions*: their behavior at the boundary points of the interval $]-1,+1[$ is given by the exponents $(\sigma_1^{\varkappa}, \sigma_2^{\varkappa})$, whereas other extensions are described by more complicated ("mixed") boundary conditions (see below).

It follows from Chap. 8 of Ref. 19, and Ref. 17, that for any (a,b) deficiency indices of $J_{\min}$ are equal and that they can assume the values 0, 1, or 2; we will say that $J_{\min}$ (and $R_{L,\min}^2$) is of *type I, II, or III* if deficiency indices of $J_{\min}$ (and $R_{L,\min}^2$) are $(0,0)$, $(1,1)$, or $(2,2)$, respectively. By general theory (Chap. 8 of Ref. 19, and Ref. 17, if $J_{\min}$ (and $R_{L,\min}^2$) is of type I then it is essentially self-adjoint and out of the four families $\mathcal{F}^{\varkappa}$, only the family $\mathcal{F}^{(+,+)}$ is square integrable; if $J_{\min}$ (and $R_{L,\min}^2$) is of type II then two of the four families $\mathcal{F}^{\varkappa}$ are square integrable and one has a one-parameter family of self-adjoint extensions, parametrized naturally by $U(1)$ (a circle) and if $J_{\min}$ (and $R_{L,\min}^2$) is of type III, all four families $\mathcal{F}^{\varkappa}$ are square integrable and one has a four-parameter family of self-adjoint extensions, parametrized naturally by $U(2)$—the family of 2×2 unitary matrices.

We will discuss now how the type of $J_{\min}$ (and of $R_{L,\min}^2$) depends on the parameters a and b. Type II is split into II$_+$, II$_-$, and II$_-^0$ subtypes according to at which end of the interval $[-1,+1]$ one has to impose boundary conditions, i.e., at which end of the interval one has limit-circle (lc) case: at $+1$ for type II$_+$ and at -1 for type II$_-$. Subtypes II$_-^0$ and III^0 corresponding to $b = 0$, a case that occurs only in two dimensions, have eigenvectors which are not expressible in terms of Jacobi polynomials, and have other special properties. Subtypes that do not appear in analysis of $R_{L,\min}^2$ are omitted. (In fact, only types I and II$_+$ appear in the decomposition of $R_{\min}^2$.)

I: $a, b \geq 1$; $\mathcal{F}^{(+,+)}$ is the only square-integrable family. II$_+$: $1 > a > 0$, $b \geq 1$; $\mathcal{F}^{(+,+)}$ and $\mathcal{F}^{(-,+)}$ are square integrable (lc at $+1$). II$_-$: $a \geq 1$, $1 > b > 0$; $\mathcal{F}^{(+,+)}$ and $\mathcal{F}^{(+,-)}$ are square integrable (lc at -1).

$\mathrm{II}^0_-: a \geq 1$, $b=0$; $\mathcal{F}^{(+,+)}$ and $\mathcal{F}^{(+,0)}$ are square integrable (lc at -1). III: $1 > a, b > 0$; $\mathcal{F}^{(+,+)}$, $\mathcal{F}^{(+,-)}$, $\mathcal{F}^{(-,+)}$, and $\mathcal{F}^{(-,-)}$ are square integrable (lc at ± 1). $\mathrm{III}^0: 1 > a > 0$, $b=0$; $\mathcal{F}^{(+,+)}$ and $\mathcal{F}^{(+,0)}$ are square integrable (lc at ± 1), deficiency indices $(1, 1)$, one has one-parameter family of self-adjoint extensions of $J_{\min}$. The case $a=0$ and $b=0$, which can be analyzed in a similar way, does not appear in the analysis of $R^2_{L,\min}$, and is therefore omitted.

To get a feeling for self-adjoint extensions of $J_{\min}$ different from the pure ones, we describe now, somewhat loosely, description of these extensions for Eq. (29) in terms of boundary conditions; we follow here closely Chap. 8 of Ref. 19.

Since in this case, $a=3/2$ and $b=1/2$, we are dealing with type II_-. The families $\mathcal{F}^{(+,+)}$ and $\mathcal{F}^{(+,-)}$ are displayed in Eqs. (33) and (34), respectively. Let $J=(J_{\min})^*$—the maximal operator defined by the differential equation (Chap. 8 of Ref. 18); let $p(z)=(1-z)^{a+1}(1+z)^{b+1}$, and for any two $u, v \in \mathcal{D}(J)$ let

$$B_u(v) = \lim_{z \searrow -1} p(z)(u(z)^* v'(z) - u'(z)^* v(z))$$

(the limit exists and is denoted by $[u,v]_{-1}$ on p. 249 of Ref. 19). Then for any real solution u of Eq. (29) the self-adjoint extension J_u of $J_{\min}$ is the restriction of J to $\mathcal{D}_u := \{v \in \mathcal{D}(J) : B_u(v) = 0\}$—the set of functions satisfying B_u-boundary conditions. As is clear from the definition, the form B_u depends only on asymptotics of u at $z=-1$. Here one has two "pure asymptotics:" one defined by any function of $\mathcal{F}^{(+,+)}$ and the other by functions of $\mathcal{F}^{(+,-)}$.

Thus, let u_1 (u_2) be any element of $\mathcal{F}^{(+,+)}$ ($\mathcal{F}^{(+,-)}$), $u_1(z) = 2^{-a-1} P_0^{(3/2,1/2)}(z) = 2^{-a-1}$ and $u_2(z) = 2^{-a-1}(1+z)^{-1/2} P_0^{(3/2,-1/2)}(z) = 2^{-a-1}(1+z)^{-1/2}$, for example. Setting $u^{(\theta)} = (\cos 2\pi\theta) u_1 + (\sin 2\pi\theta) u_2$ for any $\theta \in \mathbb{R}/\mathbb{Z}$, one obtains the one-parameter family of $B_u(\theta)$-boundary conditions and the corresponding one-parameter family of the self-adjoint extensions $J_u(\theta)$ of $J_{\min}$. The equation $B_u(v) = 0$ gives then that the eigenvalues λ satisfy

$$\frac{(3/2 + \lambda)\sin \pi\lambda}{(2 + \lambda)(1 + \lambda)} \cos 2\pi\theta - 2 \cos \pi\lambda \sin 2\pi\theta = 0. \tag{43}$$

For $\theta = 0$ (u_1-boundary condition) this yields again the spectrum (33) whereas for $\theta = 1/4$ (u_2-boundary condition) one obtains the spectrum (34). For other values of θ, the spectrum is given by Eq. (43) and is not computable in closed form. However, we note certain stability property of the u_1-boundary condition: for any $\theta \neq 1/4$ the values of $\lambda_n(\theta)$ approach fast $\lambda_n(0)$ as $n \to \infty$. The situation here is typical for the case of $(1,1)$ deficiency indices. In the case of $(2, 2)$ deficiency indices, the description of spectrum is more complicated since one has to deal with mixed boundary conditions.

Clearly, Jacobi polynomials satisfy the $B^{(1)}$ boundary condition and do not satisfy $B^{(2)}$ boundary condition. Therefore the extension $J^{(+,+)}$ of $J_{\min}$ is identical with J_{u_1}, in agreement with our discussion of the spectrum.

The extension $J^{(+,+)}$ of $J_{\min}$ is characterized by the fact that it has in its domain a function equal to 1 in a neighborhood of -1 and 0 in a neighborhood of 1. It is also clear that our second family of eigenfunctions diagonalizes the B_2-extension.

We pass now to a description of self-adjoint extensions of $R^2_{L,\min}$ at different angular momenta L and for different values of the parameter η. It follows from Eq. (35) that the type of $R^2_{L,\min}$ depends on (η, L) but not on α, and that for η fixed the values of both a and b form a discrete set.

As is clear from Eq. (35), $a(\eta, L)^2$ is a quadratic form in η, L (and also in D), which is a strictly increasing function of each of these (non-negative) parameters. Let $\eta_{\mathrm{cr}}(L)$ be the ($\bar{D}$-dependent) positive solution of the equation $a(\eta, L) = 1$,

$$\eta_{\mathrm{cr}}(L) := \frac{3}{\bar{D} + 2\sqrt{\bar{D}^2 + 3L^2}}. \tag{44}$$

Proposition: *If $L = 0$ and $\eta < \eta_{\mathrm{cr}}(0)$ $(= 1/\bar{D}, \beta' > (\bar{D}-1)\beta)$ then $R^2_{L,\min}$ is of* type III *for $D = 3$ and*

of type III^0 *for D=2; if, on the other hand, L=0 and* $\eta \geqslant \eta_{cr}(0)$ *then* $R^2_{L,\min}$ *is of* type II$_-$ *for D* =3 *and of* type II$^0_-$ *for D=2.*

If L>0, for any dimension D, $R^2_{L,\min}$ *is* type II$_+$ *for* $\eta < \eta_{cr}(L)$ *and of* type I *otherwise. If D* $\geqslant 4$ *then the statement is true also for L=0.*

The proof of Proposition, which we skip, consists of calculation of ranges of a and b of Eq. (35) as functions of (η, L) and application of the above classification into types.

Proposition yields self-adjoint extensions of $R^2_{L,\min}$ at each angular momentum L for different values of η. To pass from it to a description of self-adjoint extensions of $R^2_{\min}$ itself, we first fix η and discuss $R^2_{L,\min}$ for different values of L. One then obtains the following.

Assume that $\eta \leqslant 1/\bar{D}$ and let $L_{cr}=L_{cr}(\eta)$ be the ($\bar{D}$-dependent) positive solution of the equation $a(\eta, L)=1$,

$$L_{cr}(\eta) = \frac{(1-\bar{D}\eta)(3+\bar{D}\eta)}{4\eta^2}. \tag{45}$$

Note that $L_{cr}(\eta)$ increases from 0 to $+\infty$ as η decreases from $1/\bar{D}$ to 0, and that the conditions $a<1$, $\eta < \eta_{cr}(L)$, and $L < L_{cr}(\eta)$ are equivalent.

Theorem: *For D=3, if* $\eta < 1/2$ ($=\bar{D}^{-1}$), *i.e.,* $\beta < \beta'$, *then* $R^2_{L,\min}$ *is of* type III *for L=0, of* type II$_+$ *for* $0 < L < L_{cr}(\eta)$, *and of* type I *otherwise; if* $\eta \geqslant 1/2$ *then* $R^2_{L,\min}$ *is of* type II$_-$ *for L=0 and of* type I *otherwise.*

For D=2, if $\eta < 1$ ($=\bar{D}^{-1}$) *then* $R^2_{L,\min}$ *is of* type III^0 *for L=0, of* type II$_+$ *for* $0 < L < L_{cr}(\eta)$, *and of* type II$_-$ *otherwise; if* $\eta \geqslant 1$ *then* $R^2_{L,\min}$ *is of* type II$_-$ *for L=0 and of* type I *otherwise.*

For D $\geqslant 4$, *if* $\eta < 1/\bar{D}$ *then* $R^2_{L,\min}$ *is of* type II$_+$ *for* $0 < L < L_{cr}(\eta)$ *and of* type I *otherwise; if* $\eta \geqslant 1/\bar{D}$ *then* $R^2_{L,\min}$ *is of* type I *for any L.*

C. Self-adjoint versions of R^2

We will now use results of the preceding section about the reduced operators to describe self-adjoint versions of R^2.

Let $R^2_{\min}$ be the minimal version of R^2, with domain $C^\infty_0(\mathbb{R}^D)$, and let $R^2_{\min,0}$ be its restriction to $C^\infty_0(\mathbb{R}^D\backslash\{0\})$ (Schwartz versions of these domains can also be used, with the same final results). $R^2_{\min,0}$ is directly related to the reduced operators since its restriction to $\mathcal{R} \otimes \mathcal{S}_L$ of Eq. (14) is equal to $R^2_{L,\min} \otimes 1_L$.

We note first that, since $R^2_{\min,0}$ is an extension of the algebraic sum $\Sigma_L(R^2_{L,\min} \otimes 1_L)$, $(R^2_{\min,0})^*$ is a restriction of $(\Sigma_L(R^2_{L,\min} \otimes 1_L))^*$ ($=\oplus_L(R^2_{L,\min})^* \otimes 1_L$). It follows that, for any choice of a self-adjoint extensions R^2_L (all L), the closure of $\Sigma_L R^2_L \otimes 1_L$ defines a self-adjoint extension of $R^2_{\min,0}$. All rotation invariant self-adjoint extensions are of this form. Among these extensions are the pure ones: writing $\mathcal{F}^\varkappa(\eta, L)$ for $\mathcal{F}^\varkappa$ when $a=a(\eta,L)$ and $b=b(\eta,L)$, fixing η and then choosing for each L a square-integrable family $\mathcal{F}^{\varkappa(L)}(\eta,L)$, one obtains a unique self-adjoint extension of $R^2_{\min,0}$ domain of which contains elements functions of the form $\Psi \otimes \Theta$, $\Psi \in \mathcal{F}^{\varkappa(L)}(\eta,L)$ and $\Theta \in \mathcal{S}_L$.

Similarly, for deficiency subspaces $\mathcal{K}_\pm$ (Sec. X.1 of Ref. 17, and Ref. 19), one obtains

$$\mathcal{K}_\pm(R^2_{\min,0}) = \oplus_L \mathcal{K}_\pm(R^2_{L,\min}) \otimes \mathcal{S}_L, \quad \mathcal{K}_\pm(R^2_{\min}) = \oplus_L \mathcal{K}_\pm\left(R^2_{\min}\big\lfloor_{\mathcal{R} \otimes \mathcal{S}_L}\right).$$

This, in conjunction with the Theorem, shows that deficiency indices $n(R^2_{\min,0})$ of $R^2_{\min,0}$ are equal to

$$n(R_{\min,0}^2) = \begin{cases} 1 + \displaystyle\sum_{0 \leqslant L < L_{\mathrm{cr}}(\eta)} \dim(\mathcal{S}_L) & \text{for } D = 2,3 \\[2ex] \displaystyle\sum_{0 \leqslant L < L_{\mathrm{cr}}(\eta)} \dim(\mathcal{S}_L) & \text{for } D \geqslant 4. \end{cases}$$

The term $\Sigma_{0 \leqslant L < L_{\mathrm{cr}}(\eta)} \dim(\mathcal{S}_L)$ comes from the fact that one has lc behavior of $R_{L,\min}^2$ at $p = \infty$ ($z = 1$), while the 1 in $1 + \Sigma_{0 \leqslant L < L_{\mathrm{cr}}(\eta)} \dim(\mathcal{S}_L)$ is the result of lc behavior of $R_{0,\min}^2$ at $p = 0$ ($z = -1$).

The later disappears when one passes to $R_{\min}^2$ and one obtains that

$$n(R_{\min}^2) = \sum_{0 \leqslant L < L_{\mathrm{cr}}(\eta)} \dim(\mathcal{S}_L)$$

for any η. For when $\eta = 1$ ($\beta' = 0$) (more generally, for $\eta \geqslant 1/\bar{D}$), the deficiency indices of $R_{0,\min}^2$ are equal to 1 for $D = 2,3$ and to 0 for $D \geqslant 4$. For $D = 2,3$ the nonuniqueness of self-adjoint extension of $R_{0,\min}^2$ is due to the fact that at zero angular momentum, when the term $(1 + \beta p^2) L^2 / p^2$ is absent from Eq. (16), operator (19) is regular at 0. As in the case of Laplace operator, to which R^2 is reduced for $\beta = 0$, this nonuniqueness disappears when one passes to $R_{\min}^2$. This can be seen from the fact that, as integration by part shows, $\Phi_{0,n}(z)$ with $\chi_- = -$ is not in the domain of $(R_{\min}^2)^*$. (In fact, enlarging the initial domain of $R_{\min,0}^2$ by just one $C_0^\infty(\mathbb{R}^D)$-function, which is nonzero at 0, one obtains an essentially self-adjoint operator.)

According to von Neumann's theory of self-adjoint extensions,[16] this yields a large family of self-adjoint extensions of $R_{\min}^2$, a family that is naturally parametrized by $U(n(R_{\min}^2))$. However, many of these extensions lack one or more of natural properties one would want R^2 to have. Here are the following two such properties, rotation (i) and conjugation (ii) invariances

(i) Let A be a symmetric operator which commutes with a group $\mathcal{G}$ of unitary operators. Then, as is easy to see, its adjoint also commutes with operators of $\mathcal{G}$, and therefore each of the deficiency subspaces $\mathcal{K}_\pm$ of A is invariant under $\mathcal{G}$ (in notation and terminology, we are following here, Sec. X.1 of Ref. [17]). Let K be a unitary operator from $\mathcal{K}_+$ to $\mathcal{K}_-$ and let A_K be corresponding self-adjoint extension of A (Theorem X.2 of Ref. [17]) Then, as is easy to see, $U A_K U^{-1} = A_{U K U^{-1}}$ for any $U \in \mathcal{G}$. Thus, A_K commutes with $\mathcal{G}$ if and only if K does. The situation where the adjoint A^* of A is $\mathcal{G}$-invariant but some of self-adjoint extensions of A are not, may seem at first paradoxical. The point is that whereas A^* is $\mathcal{G}$-invariant, the self-adjoint extensions of A are obtained by restriction of A^* to suitable domains, and it is the domains that are in general not $\mathcal{G}$-invariant. (All this is obvious and hardly new, but we have no reference for a discussion of covariance properties of self-adjoint extensions.)

It follows that for rotation-invariant extension R_K^2 of $R_{\min}^2$, K maps $\mathcal{K}_+(R_{\min}^2) \frown (\mathcal{R} \otimes \mathcal{S}_L)$ onto $\mathcal{K}_-(R_{\min}^2) \frown (\mathcal{R} \otimes \mathcal{S}_L)$ for each L. Thus the rotation invariant versions of R^2 are naturally parametrized by $\Pi_{0 \leqslant L < L_{\mathrm{cr}}(\eta)} U(1)$, an $\#\{L : 0 \leqslant L < L_{\mathrm{cr}}(\eta)\}$-parameter family.

(ii) The self-adjoint version of R^2 should be invariant under suitable conjugation on the Hilbert space, pointwise complex conjugation of functions of **p**, for example. All rotation-invariant extensions of $R_{\min}^2$, but not of $R_{\min,0}^2$, fullfill this condition.

One self-adjoint extension, Friedrichs' extension, Fr (R^2) (Ref. [22], Theorem X.23 of Ref. [17], and Ref. [19]), natural in the present context, has both of these invariance properties. And while this is not the only self-adjoint version of R^2 which is both rotation and conjugation invariant, there are many of these, it has a number of other good properties:

(f1) It is canonically defined for any symmetric operator which is semibounded, for the non-negative $R_{\min}^2$ in particular, and, as is not hard to see, for any semibounded symmetric operator invariant under a group of unitary transformations its Friedrichs' extension is also invariant under the group. The same holds for conjugation-invariance. Friedrichs' version of R^2 is equal to $\Sigma_i X_i^* X_i$, where X_i are the closed versions of the position operators (p. 180 of Ref. [17])—a natural definition of R^2. (Note, however, that already in one dimension, $X^* X$ is not equal to the square of a self-adjoint extension of coordinate operator!)

(f2) In general, i.e., for any η, Friedrichs' extension of $J_{\min}$, and then of $R_{\min}^2$, is the extension defined by the family $\mathcal{F}^{(+,+)}$; this is proved in Appendix C. Friedrichs' extension shares with the other pure extensions the property of nondegeneracy of spectrum.

Eigenfunctions and eigenvalues of the Friedrichs' version of R_L^2 of (19) are

$$\Phi_{L,n}(z) = (1 + z)^{\sigma_1}(1 - z)^{\sigma_2}P_n^{(a,b)}(z), \quad r_{L,n}^2 = 4\alpha\hbar^2\rho_{L,n}, \tag{46}$$

where

$$\rho_{L,n} = \left(n + \frac{c}{2}\right)^2 - \frac{1}{4}(\chi^2 + (1 - \eta)^2L^2) = \left(n + \frac{c}{2}\right)^2 - \frac{1}{16}\left(\frac{\beta'}{\beta + \beta'}\right)^2((D - 1)^2 + 4L^2), \tag{47}$$

$n = 0, 1, \ldots$, z is as in Eq. (17), a, b, and c as in Eq. (35), $\sigma_{1,2}$ as in Eq. (36). (To obtain the eigenvalues and eigenfunctions for extensions defined by the families $\mathcal{F}^\varkappa$ one has to replace here everywhere, including $\sigma_\pm$, a by $a^\varkappa$, b by $b_\varkappa$, and c by $c^\varkappa$.)

For fixed n, $\rho_{L,n}$ is an increasing function of L, as can be seen both from the explicit formula [Eq. (46)] and directly from Eq. (16). Hence, $R_{\min}^2$ and its Friedrichs' extension are bounded below by the lowest eigenvalue $r_{0,0}^2 = 4\alpha\hbar^2\rho_{0,0} = \hbar^2\alpha(1 + \bar{D})(1 + \eta\bar{D}) > 0$. On the other hand, other pure extensions may have zero as an eigenvalue or may have negative spectrum. For example, $\rho_{0,0}^{(-,+)} = 0$—so for $(R^2)^{(-,+)}$ there is no minimal length, while $\rho_{0,0}^{(-,-)} = \frac{1}{4}(\bar{D} - 1)(\bar{D}\eta - 1)$, which is negative if $\eta < 1/\bar{D}$.

(f3) Friedrichs' extension has a number of stability properties, not shared by other extensions. For example, any other extension not defined by pure boundary conditions yields spectrum with the same asymptotic behavior for large n as that of Fr (R^2).

(f4) Eigenfunctions of Fr (R^2) have the weakest singularities at the boundary points. For example, $p\Phi_n^{(+,+)}$, any n, is square integrable while $p\Phi_n^{(-,+)}$ is not. The case of $p^2\Phi$ is more complicated: if $a \neq 1$ then $p^2\Phi_n^{(+,+)}$ are square integrable if and only if one has limit-point case at infinity (and $p^{-2}\Phi_n^{(+,+)}$ are square integrable if and only if one has limit-point case at zero). This is true for both R^2 and harmonic oscillator (see Appendix B). This indicates, in particular, that to analyze the Hamiltonian in the whole range of the parameter η, one has to use quadratic form technique both in case of hydrogenlike atom and harmonic oscillator. We will return to this point in Ref. 1.

We note that Sec. III of Ref. 6 proposed to solve this problem of domain of operators by restricting it to physical states. At the beginning of Sec. III of Ref. 6 one reads that "… physical states always lie in the common domain $D_{\mathbf{x},\mathbf{x}^2,\mathbf{p},\mathbf{p}^2}$ of the symmetric operators $\mathbf{x}, \mathbf{x}^2, \mathbf{p}, \mathbf{p}^2$." According-ing to this definition, for suitable values of the parameters, the eigenfunctions of the harmonic oscillator Hamiltonian do not define physical states, which, presumably, runs against the author's intention.

This suggests to adopt the Friedrichs' version as the right one. It is Friedrichs' version of R^2 that was used in Refs. 1 and 6, and it is Friedrichs' version of harmonic oscillator Hamiltonian that was analyzed in Refs. 6 and 11 (see Appendix B here).

D. A generalization and duality

In vector form, the position of the system defined by Eqs. (6) and (7) are

$$\mathbf{X} = i(f\partial_{\mathbf{p}} + g\mathbf{p}(\mathbf{p} \cdot \partial_{\mathbf{p}})), \tag{48}$$

where the rotation invariant functions f and g satisfy the condition

$$h(p) := f(p) + p^2g(p) > 0, \tag{49}$$

a condition that can be somewhat relaxed, allowing h to be negative or to have zeros. A term $\gamma(p)\mathbf{P}$ can be added to Eq. (7) and then removed from it by a gauge transformation, as in Sec. I A. For completeness, we write down the commutators $[X_j, X_k]$,

$$[X_j, X_k] = (h(p)f'(p) - pf(p)g(p))(\hat{p}_k\partial_j - \hat{p}_j\partial_k) + h(p)p^2g'(p)(\hat{p}_k - \hat{p}_j)\hat{p}_j\hat{p}_k(\hat{\mathbf{p}}\cdot\partial), \qquad (50)$$

since they differ in form from those of KMM system, and at the end of the section we sketch a proof of the fact that $f\partial_{\mathbf{p}} + g\mathbf{p}(\mathbf{p}\cdot\partial_{\mathbf{p}})$ is indeed the most general rotation invariant vector field.

Direct computation shows that the operators [Eq. (48)] defined on the common domain $C_0^\infty(\mathbb{R}^D)$ are symmetric in $L^2(\mathbb{R}^D, w(p)\mathrm{d}^D\mathbf{p})$, if and only if the weight function w satisfies the differential equation

$$h(p)w'(p) + (\bar{D}pg(p) + h'(p))w(p) = 0 \quad (\bar{D} = D-1),$$

which, assuming that h has no zeros, has

$$w(p) = \frac{C}{h(p)}\exp\left(-\bar{D}\int \mathrm{d}p\,\frac{pg(p)}{h(p)}\right), \quad C \in \mathbb{R}, \qquad (51)$$

as its general solution. For $C = \mathrm{sign}(h)$, one obtains the unique, up to a multiplicative positive factor, weight function, making X_j into symmetric operators. Specializing to $f(p) = \hbar(1 + \beta p^2)$, $g(p) = \hbar\beta'$ $(h(p) = \hbar(1 + \alpha p^2))$—the case of KMM system, one obtains the weight function of Eq. (9).

Furthermore, for $\mathbf{X}$ of Eq. (48), again direct computation shows that $\mathbf{X}^2$ reduced at angular momentum L is equal to

$$R_L^2 = -\left[\left(h\frac{\mathrm{d}}{\mathrm{d}p}\right)^2 + \frac{f\bar{D}}{p}\left(h\frac{\mathrm{d}}{\mathrm{d}p}\right) - \frac{f^2L^2}{p^2}\right]. \qquad (52)$$

We consider now the transformation, $\mathbf{p} \mapsto \mathbf{q} = -(1/p^2)\mathbf{p}$. We call it duality transformations since it relates IR and UV regions. Since the transformation commutes with rotations, and Eq. (48) is the most general rotation invariant vector field, the transform of $\mathbf{X}$ must be again of form (48). Explicit computation of the transform is given below. We note that more general transfromation of the form $\mathbf{q} = u(p)\mathbf{p}$ could also be used.

Direct computation shows that Eqs. (6) and (7) imply that

$$[X_j, Q_k] = i(\widetilde{f}(q)\delta_{jk} + q^{-2}\widetilde{g}(q)Q_jQ_k), \quad \mathbf{X} = i\widetilde{f}(q)\partial_{\mathbf{q}} + i\widetilde{g}(q)\mathbf{q}(\mathbf{q}\cdot\partial_{\mathbf{q}}), \qquad (53)$$

where $\mathbf{Q}$ is the operator of multiplication by $\mathbf{q}$, $\partial_{\mathbf{q}} = (\partial\mathbf{p}/\partial\mathbf{q})\partial_{\mathbf{p}} = 2\mathbf{p}\,(\mathbf{p}\cdot\partial_{\mathbf{p}}) - p^2\partial_{\mathbf{p}}$ (and therefore $\mathbf{q}\cdot\partial_{\mathbf{q}} = -\mathbf{p}\cdot\partial_{\mathbf{p}}$),

$$\widetilde{f}(q) = -q^2f(1/q), \quad q^2\widetilde{g}(q) = 2q^2h(1/q) - g(1/q),$$

so that $\widetilde{h}(q) := \widetilde{f}(q) + q^2\widetilde{g}(q) = q^2h(1/q)$.

Also, it can be checked by direct computation that

$$\frac{1}{h(p)}\exp\left(-\bar{D}\int \mathrm{d}p\,\frac{pg(p)}{h(p)}\right)\mathrm{d}^D\mathbf{p} = \frac{1}{\widetilde{h}(q)}\exp\left(-\bar{D}\int \mathrm{d}q\,\frac{q\widetilde{g}(q)}{\widetilde{h}(q)}\right)\mathrm{d}^D\mathbf{q} \qquad (54)$$

(up to a multiplicative constant), in agreement with the fact that the weight is determined by the requirement that the operators X_j are symmetric. In other words, with a suitable choice of integration constants of Eq. (54), the change of the variable map $\varphi \mapsto \mathfrak{D}\varphi$, $\mathfrak{D}\varphi(\mathbf{q}) = \varphi(\mathbf{p})$, $\mathbf{p} = -(1/q^2)\mathbf{q}$ defines an isomorphism of the Hilbert spaces $L^2(\mathbb{R}^D, w(p)\mathrm{d}^D\mathbf{p})$ and $L^2(\mathbb{R}^D, \widetilde{w}(q)\mathrm{d}^D\mathbf{q})$ $(\widetilde{w}(q) = \widetilde{h}(q)^{-1}\exp(-\bar{D}\int\mathrm{d}q\,q\widetilde{g}(q)/\widetilde{h}(q)))$, under which $\mathbf{X}$ is mapped onto $\widetilde{\mathbf{X}} = \mathfrak{D}X\mathfrak{D}^{-1}$, where

$$\widetilde{\mathbf{X}} = i\widetilde{f}(q)\partial_{\mathbf{q}} + i\widetilde{g}(q)\mathbf{q}(\mathbf{q}\cdot\partial_{\mathbf{q}}), \quad \partial_{\mathbf{q}} = \partial/\partial\mathbf{q}.$$

The map $\mathfrak{D}$ is involutive, in a suitable sense and, obviously, commutes with rotations, so that it defines an isomorphism of $L^2(\mathbb{R}_+, w_0(p)p^{D-1}\mathrm{d}p)$ and $L^2(\mathbb{R}_+, \widetilde{w}_0(q)q^{D-1}\mathrm{d}q)$, which is denoted again

by $\mathfrak{D}$; defining $\widetilde{R}_L^2$ as $\mathfrak{D}R_L^2\mathfrak{D}^{-1}$, one has that $\widetilde{R}_L^2$ is equal to is $\widetilde{\mathbf{X}}^2$ reduced at angular momentum L, and that

$$\widetilde{R}_L^2 = \left(\widetilde{h}\frac{\mathrm{d}}{\mathrm{d}q}\right)^2 + \frac{\widetilde{f}\overline{D}}{q}\left(\widetilde{h}\frac{\mathrm{d}}{\mathrm{d}q}\right) - \frac{\widetilde{f}^2 L^2}{q^2}.$$

Thus R_L^2 and $\widetilde{R}_L^2$ have the same spectra and their corresponding eigenfunctions are related $\mathfrak{D}$.

Our main interest is in the *general quadratic case* of

$$f(p) = \beta_1 + \beta_2 p^2, \quad p^2 g(p) = \beta_3 + \beta_4 p^2, \tag{55}$$

so that $h(p) = \alpha_1 + \alpha_2 p^2$, where $\alpha_1 = \beta_1 + \beta_3$ and $\alpha_2 = \beta_2 + \beta_4$; we set

$$\alpha = \frac{\alpha_2}{\alpha_1}, \quad \beta = \frac{\beta_2}{\beta_1}, \quad \eta = \frac{\eta_2}{\eta_1} = \frac{\beta}{\alpha} = \frac{\beta_2 \alpha_1}{\beta_1 \alpha_2}, \quad \eta_1 = \frac{\beta_1}{\alpha_1}, \quad \eta_2 = \frac{\beta_2}{\alpha_2}. \tag{56}$$

Equation (51) yields now

$$w(p) = (1 + \alpha p^2)^{\overline{D}(\eta_2 - \eta_1)/2 - 1} p^{(\eta_1 - 1)\overline{D}}, \tag{57}$$

which for Kempf system ($\eta_1 = 1$, $\eta_2 = \eta$) reduces to Eq. (9). (The inversion map $\mathbf{p} \mapsto (1/p^2)\mathbf{p}$ would lead from positive to negative α_1, α_2.)

In the quadratic case of Eq. (55), transformation (17) to the variable z yields

$$R_L^2 = -4\alpha_1 \alpha_2 \left[(1 - z^2)\frac{\mathrm{d}^2}{\mathrm{d}z^2} + (D_+ + (D_- - 2)z)\frac{\mathrm{d}}{\mathrm{d}z} - \frac{(\eta_+ + \eta_- z)^2}{4(1 - z^2)}L^2 \right],$$

where

$$D_\pm = D_2 \pm D_1, \quad D_1 = \frac{1}{2}\eta_1\overline{D} - \frac{1}{2}, \quad D_2 = \frac{1}{2}\eta_2\overline{D} + \frac{1}{2}, \quad \eta_\pm = \eta_2 \pm \eta_1. \tag{58}$$

Hence the eigenvalue problem $R_L^2 \Phi = r^2 \Phi$ takes again the form of Eq. (A12) of Appendix A, with L replaced by $\eta_1 L$, $\overline{D}$ replaced by $\eta_1 \overline{D}$, and

$$\rho = \frac{r^2}{4\alpha_1 \alpha_2}, \quad a = \sqrt{D_2^2 + \eta_2^2 L^2}, \quad b = \sqrt{D_1^2 + \eta_1^2 L^2},$$

Eq. (36) yields again an expression of $\sigma_{1,2}$ in terms of $a, b, D_{1,2}$, while Eq. (A20) is now

$$\rho = \mu - \frac{1}{4}L^2(\eta_1 - \eta_2)^2 - \frac{1}{4}\chi^2.$$

Eigenfunctions and eigenvalues of Friedrichs' version of R_L^2 are see [see Eq. (46)]:

$$\Phi_{L,n}^{(a,b)}(z) = (1 + z)^{\sigma_1}(1 - z)^{\sigma_2} P_n^{(a,b)}(z), \quad r_{L,n}^2 = 4\alpha_1 \alpha_2 \rho_{L,n}, \tag{59}$$

where

$$\rho_{L,n} = \left(n + \frac{c}{2}\right)^2 - \frac{1}{4}(\chi^2 + \eta_-^2 L^2) = \left(n + \frac{c}{2}\right)^2 - \frac{1}{16}(\overline{D}^2 + 4L^2)\eta_-^2, \tag{60}$$

$n = 0, 1, \ldots$, z is as in Eq. (17), a, b, c as in Eq. (35), and $\sigma_{1,2}$ as in Eq. (36).

In the general quadratic case,

$$\widetilde{f}(q) = -\beta_2 - \beta_1 q^2, \quad q^2 \widetilde{g}(q) = (2\alpha_2 - \beta_4) + (2\alpha_1 - \beta_3)q^2, \quad \widetilde{h}(q) = \alpha_2 + \alpha_1 q^2.$$

Writing $\widetilde{f}(q) = \widetilde{\beta}_1 + \widetilde{\beta}_2 q^2$, $q^2 \widetilde{g}(p) = \widetilde{\beta}_3 + \widetilde{\beta}_4 q^2$, $\widetilde{h}(q) = \widetilde{\alpha}_1 + \widetilde{\alpha}_2 q^2$, one has

$$\widetilde{\beta}_1 = -\beta_2, \quad \widetilde{\beta}_2 = -\beta_1, \quad \widetilde{\beta} = \beta^{-1}, \quad \widetilde{\alpha}_1 = \alpha_2, \quad \widetilde{\alpha}_2 = \alpha_1, \quad \widetilde{\alpha} = \alpha^{-1},$$

$$\widetilde{\eta}_1 = -\eta_2, \quad \widetilde{\eta}_2 = -\eta_1, \quad \widetilde{\eta} = \eta^{-1}. \tag{61}$$

Hence in the general quadratic case the dual is again quadratic, with both $\widetilde{f}$ and $\widetilde{h}$ obtained by permutation $1 \leftrightarrow 2$, and by a change of sign, in case of f.

The duality transformation is particularly simple in terms of the z variable: performing the substitution $q = \widetilde{\alpha}^{-1/2}((1+z)/(1-z))^{1/2}$ one obtains

$$-4\widetilde{\alpha}_1\widetilde{\alpha}_2\left[(1-z^2)\frac{d^2}{dz^2} + (\widetilde{D}_+ + (\widetilde{D}_- - 2)z)\frac{d}{dz} - \frac{(\widetilde{\eta}_+ + \widetilde{\eta}_- z)^2}{4(1-z^2)}L^2\right],$$

which in view of Eqs. (61) and (58) is

$$\widetilde{R}_L^2 = -4\alpha_1\alpha_2\left[(1-z^2)\frac{d^2}{dz^2} + (-D_+ + (D_- - 2)z)\frac{d}{dz} - \frac{(\eta_+ + \eta_- z)^2}{4(1-z^2)}L^2\right].$$

This shows that in terms of the z variable the duality transformation is $z \mapsto -z$. (Since the spectrum of R_L^2 is nondegenerate, this implies that $\Phi_{L,n}^{(\widetilde{a},\widetilde{b})}(z)$ is proportional to $\Phi L, n(a,b)(-z)$. Indeed, since $\widetilde{D}_1 = -D_2$, $\widetilde{D}_2 = -D_1$, $\widetilde{\sigma}_{1,2} = \sigma_{2,1}$, $\widetilde{a} = b$, $\widetilde{b} = a$, and $\widetilde{c} = c$, the identity $P_n^{(a,b)}(-x) = (-1)^n P_n^{(b,a)}(x)$ shows that $\Phi_{L,n}^{(\widetilde{a},\widetilde{b})}(-z) = (-1)^n \Phi_{L,n}^{(a,b)}(z)$.)

One can reduce the number of parameters and give the duality transformation dimensionally "right form" by passing to dimensionless variables and operators [cf. Eq. (10)]: let $\overline{\mathbf{p}} = \sqrt{\alpha}\mathbf{p}$, let $\overline{\mathbf{P}}$ be the operator of multiplication by $\overline{p}$ and $\overline{\mathbf{X}} = (\alpha_1\alpha_2)^{-1/2}\mathbf{X}$. Then

$$\overline{\mathbf{X}} = i((\eta_1 + \eta_2\overline{p}^2)\partial_{\overline{\mathbf{p}}} + ((1-\eta_1) + (1-\eta_2)\overline{p}^2)\hat{\mathbf{p}}(\hat{\mathbf{p}} \cdot \partial_{\mathbf{p}})).$$

$$[\overline{X}_j, \overline{P}_k] = i(\overline{f}\delta_{jk} + \overline{g}\overline{P}_j\overline{P}_k), \quad \overline{f}(\overline{p}) = (\eta_1 + \eta_2\overline{p}^2), \quad \overline{p}^2\overline{g}(\overline{p}) = (1-\eta_1) + (1-\eta_2)\overline{p}^2. \tag{62}$$

This reduces the four-parameter system [Eq. (55)] to a two-parameter one.

The duality transformation yields now

$$\widetilde{f}(\widetilde{p}) = -\eta_2 - \eta_1\widetilde{p}^2, \quad \widetilde{p}^2\widetilde{g}(\widetilde{p}) = (1+\eta_2) + (1+\eta_1)\widetilde{p}^2,$$

i.e., writing $\widetilde{f}(\widetilde{p}) = \widetilde{\eta}_1 + \widetilde{\eta}_2 ep^2$ and $\widetilde{p}^2\widetilde{g}(\widetilde{p}) = (1-\widetilde{\eta}_1) + (1-\widetilde{\eta}_2)\widetilde{p}^2$, one has

$$\widetilde{\eta}_1 = -\eta_2, \quad \widetilde{\eta}_2 = -\eta_1. \tag{63}$$

Thus both $\widetilde{f}$ and $\widetilde{g}$ are obtained by permutation $1 \leftrightarrow 2$ followed by a change of sign.

We note that for general values of parameters, the system $(\overline{\mathbf{X}}, \overline{\mathbf{P}})$ is not equivalent to its dual $(\widetilde{\mathbf{X}}, \widetilde{\mathbf{P}})$, since the commutation relations of these two systems are different, i.e., the system $(\overline{\mathbf{X}}, \overline{\mathbf{P}})$ is not self-dual, in general. Taking Eq. (61) into account, one obtains that $(\overline{\mathbf{X}}, \overline{\mathbf{P}})$ [and $(\mathbf{X}, \mathbf{P})$] is self-dual if and only if $\widetilde{\eta}_1 = \eta_1 = -\eta_2, \widetilde{\eta}_2 = \eta_2 = -\eta_1$, i.e., when $\overline{\mathbf{X}}$ is of the form $\overline{\mathbf{X}} = i(\xi(1-\overline{p}^2)\partial_{\mathbf{p}} + ((1+p^2) - \xi(1-\overline{p}^2))\hat{\mathbf{p}}(\hat{\mathbf{p}} \cdot \partial_{\overline{\mathbf{p}}}))$, where $\xi = \eta_1$. Thus one obtains an (essentially) one-parameter family of quadratic self-dual models.

We now sketch a proof of the fact that Eq. (48) is the most general first order vector valued rotation invariant differential operator. Let $\mathbf{X} = A(\mathbf{p})\partial_{\mathbf{p}}$, i.e., $X_i = \Sigma_j A_{ij}(\mathbf{p})\partial_j$ be such an operator . Then the problem is reduced to description of matrix-valued functions $\mathbf{p} \mapsto A(\mathbf{p})$, such that, for any orthogonal matrx R,

$$A(R\mathbf{p}) = R^{-1}A(\mathbf{p})R \quad \text{for all } R \in G = O(n). \tag{64}$$

Since R is orthogonal, Eq. (64) implies that also ${}^t A(R\mathbf{p}) = R^{-1}({}^t A(\mathbf{p}))R$ for all $R \in G$.

Proposition: Equation (64) implies that

$$A(\mathbf{p}) = f(p)\mathbf{1} + g(p)\mathbf{p} \otimes \mathbf{p} = f(p)\mathbf{1} + p^2 g(p)\hat{\mathbf{p}} \otimes \hat{\mathbf{p}}.$$

▷ Fix $\mathbf{w}$, one can, and we will, take it normalized and apply this identity to R in the isotropy subgroup $G_{\mathbf{w}}$ of $\mathbf{w}$, which is isomorphic to $O(n-1)$. It follows that A commutes with $G_{\mathbf{w}}$, which implies that A is a linear combination of identity operator and of the operator of projection on $\mathbf{w}$, which can be shown as follows.

Let $c = (\mathbf{w}|A(\mathbf{w})\mathbf{w})$, $B = A(\mathbf{w}) - c\mathbf{w} \otimes \mathbf{w}$. Then B and ${}^t B$ commute with $G_{\mathbf{w}}$. Since $(\mathbf{w}|B\mathbf{w}) = c - c = 0$ and $(\mathbf{w}|{}^t B\mathbf{w}) = (B\mathbf{w}|\mathbf{w}) = 0$, both $B\mathbf{w}$ and ${}^t B\mathbf{w}$ are perpendicular to $\mathbf{w}$. Since both $B\mathbf{w}$ and ${}^t B\mathbf{w}$ are $G_{\mathbf{w}}$-invariant, both of these vectors are zero and it follows that the orthogonal complement $[\mathbf{w}]^{\perp}$ of $[\mathbf{w}]$ is B invariant and the action of $G_{\mathbf{w}}$ in $[\mathbf{w}]^{\perp}$ is irreducible, the restriction of B to $[\mathbf{w}]^{\perp}$ is a multiple of identity. ◁

III. POSITION OPERATORS

For simplicity, only the case of $\beta' = 0$ ($\eta = 1$) is being considered here. In this case Eq. (3) is

$$[X_i, P_j] = i\hbar(1 + \beta P^2)\delta_{ij}, \quad \mathbf{X} = i\hbar(1 + \beta \mathbf{P}^2)\partial_{\mathbf{p}}, \quad i,j = 1, \dots, D. \tag{65}$$

Change of variables $\mathbf{p} \mapsto (q, \mathbf{q})$—note that q, $\mathbf{q}$ are different here from those of Sect. II B:

$$q(\mathbf{p}) = \frac{1}{\sqrt{1 + \beta\sum_{i=2}^{D} p_i^2}} \arctan \frac{\sqrt{\beta}p_1}{\sqrt{1 + \beta\sum_{i=2}^{D} p_i^2}},$$

$$\mathbf{q} = (q_1, \dots, q_{D-1}) = (\sqrt{\beta}p_2, \dots, \sqrt{\beta}p_D), \tag{66}$$

maps $\mathbb{R}^D$ onto

$$\Omega = \{(q, \mathbf{q}) \in \mathbb{R}^D : |q| < Q(\mathbf{q})\}, \quad \text{where } Q(\mathbf{q}) = \frac{\pi}{2\sqrt{1 + \mathbf{q}^2}} \tag{67}$$

and transforms X_1 into the operator $\hbar\sqrt{\beta}A$, where

$$A = i\partial, \quad \partial = \frac{\partial}{\partial q}, \tag{68}$$

acting in Hilbert space $\mathfrak{H}$ with the scalar product

$$(\Phi, \Psi) \mapsto \int_{\Omega} \mathrm{d}q\mathrm{d}^{D-1}\mathbf{q}\Phi^*(q, \mathbf{q})\Psi(q, \mathbf{q}). \tag{69}$$

More precisely, A stands for the minimal version of the differential operator [Eq. (68)], with $C_0^{\infty}(\Omega)$ as its domain. Thus A is the familiar differentiation operator, with the caveat that the range of the variable q is $\mathbf{q}$-dependent, yielding a "kinematic coupling" of various degrees of freedom.

To clarify the effect of this kinematic coupling on the spectrum of A, let us perform direct integral decomposition of $\mathfrak{H}$ with respect to $\mathbf{q}$,

$$\int_{\mathbb{R}^{D-1}}^{\oplus} \mathfrak{H}(\mathbf{q})\mathrm{d}^{D-1}\mathbf{q}, \quad \text{where } \mathfrak{H}(\mathbf{q}) = L^2(-Q(\mathbf{q}), Q(\mathbf{q})). \tag{70}$$

Then A acts in each $\mathfrak{H}(\mathbf{q})$, i.e., A is (in a loose sense) a direct integral of $A(\mathbf{q})$, $\mathbf{q} \in \mathbb{R}^{D-1}$. Each $A(\mathbf{q})$ is still of form (68) but the spectrum of (self-adjoint extension of) $A(\mathbf{q})$ is now, in general, different for different values of $Q(\mathbf{q})$, so that eigenvectors of (self-adjoint extension of) $A(\mathbf{q})$ can

be combined into an eigenvector of (self-adjoint extension of) A only under special circumstances.

While this description is basically right, the situation is a little bit more complicated since A is not essentially self-adjoint.

To proceed with our analysis we start with the familiar case of "one degree of freedom."

Let a be a positive number, let ∂^a be the minimal version of the operator of derivative defined in $L^2(-a,a)$, and let $i\partial^{a,\xi}$ be the self-adjoint extension of $i\partial^a$ defined by the boundary conditions $\varphi(a)=\xi\varphi(-a)$, where ξ is a complex number of modulus (B 141 of Ref. 16). The spectrum of $i\partial^{a,\xi}$ is determined by the equation $e^{-2i\lambda a}=\xi$, i.e., it is pure point, consisting of eigenvalues

$$\lambda_n^{a,\xi} = \lambda_0^{a,\xi} + (\pi/a)n, \quad \lambda_0^{a,\xi} = -(\arg \xi)/(2a), \quad 0 \leqslant \arg \xi < 2\pi, \quad n \in \mathbb{Z}, \tag{71}$$

with (non-normalized) eigenfunctions $\Phi_n^{a,\xi}(q)=\exp[-i\lambda_n^{a,\xi}q]$. The eigenfunction expansion yields direct sum decomposition of $\mathfrak{H}=L^2(-a,a)$

$$\mathfrak{H} = \underset{n\in\mathbb{Z}}{\oplus} \mathfrak{H}_n, \quad \mathfrak{H}_n = \{\Phi_n^{a,\xi}\psi : \psi \in \mathbb{C}\},$$

with $\mathcal{D}(i\partial^{a,\xi})=\{\Sigma_n\Phi_n^{a,\xi}\psi_n \in \mathfrak{H} : \Sigma_n|\lambda_n^{a,\xi}\Phi_n^{a,\xi}\psi_n|^2<\infty\}$ and

$$i\partial^{a,\xi}\left(\sum_n \Phi_n^{a,\xi}\psi_n\right) = \lambda_n^{a,\xi}\Phi_n^{a,\xi}\psi_n.$$

Setting $a=\pi/2$, one recovers here a result of Ref. 6.

An alternative (von Neumann's) approach to the problem of obtaining self-adjoint extensions of a symmetric operator A (p. 141 of Ref. 21), is in considering first the deficiency subspaces $\mathcal{K}_\pm$ of A, which consist of solutions of the equations $A^*\Phi=\pm i\Phi$, then labeling self-adjoint extensions of A by unitary operators from $\mathcal{K}_+$ to $\mathcal{K}_-$. Let K be such a unitary operator and let A_K be the corresponding self-adjoint extension of A. Then $\mathcal{D}(A_K)=\{\Phi+\Phi^++K\Phi^+ : \Phi \in \mathcal{D}(\bar{A}), \Phi^+ \in \mathcal{K}_+\}$ and A_K is the restriction of A^* to $\mathcal{D}(A_K)$,

$$A_K(\Phi + \Phi^+ + K\Phi^+) = \bar{A}\Phi + i(\Phi^+ - K\Phi^+), \tag{72}$$

where $\bar{A}$ is the closure of A.

Since for A of Eq. (68), $\mathcal{D}(A^*)$ is the family of $\Phi \in \mathfrak{H}$ which are absolutely continuous and of square integrable derivative, with $A^*\Phi=i\partial\Phi(q)$, one has here

$$\mathcal{K}_\pm = \Phi_\psi^\pm \in \mathfrak{H} : \Phi_\psi^\pm(q) = e^{\pm q}\psi, \quad \psi \in \mathbb{C}, \tag{73}$$

K is given by a complex number of modulus one, say κ,

$$K\Phi_\psi^+ = \kappa\Phi_\psi^-,$$

and therefore Eq. (72) becomes

$$A_K(\Phi + e^q\psi + \kappa e^{-q}\psi) = i\partial\Phi + i(e^q\psi - \kappa e^{-q}\psi),$$

where Φ is an absolutely continuous function vanishing at $\pm a$, with square integrable derivative.

Setting

$$\Psi = \Phi + e^q\psi + \kappa e^{-q}\psi, \tag{74}$$

one obtains that $\Psi(a)=(e^a+\kappa e^{-a})\psi$, $\Psi(-a)=(e^{-a}+\kappa e^a)\psi$, so that

$$\Psi(a) = \xi\Psi(-a), \quad \text{where } \xi = \frac{e^a + \kappa e^{-a}}{e^{-a} + \kappa e^a}, \quad \left(\text{i.e., } \kappa = -\frac{\xi e^{-a} - e^a}{\xi e^a - e^{-a}}\right). \tag{75}$$

And conversely, if $\Psi \in \mathcal{D}(A^*)$ and $\Psi(a)=\xi\Psi(-a)$ then Φ defined by

$$\Psi(q) = \Phi(q) + e^q\psi + \kappa e^{-q}\psi, \quad \text{where } \psi = \frac{\xi e^a - e^{-a}}{e^{2a} - e^{-2a}}\Psi(-a), \tag{76}$$

satisfies $\Phi(a) = 0 = \Phi(-a)$ and therefore belongs to $\mathcal{D}(A_K)$ (p. 141 of Ref. 21) This shows that $A_K = i\partial^{a,\xi}$ if ξ and κ are related by Eq. (75).

With little change, formulas (73) and (74) define the general self-adjoint extension of A for any number D of degrees of freedom. Namely, the deficiency subspaces $\mathcal{K}_{\pm}$ of A consist of solutions of the equations

$$i\partial_q\Phi(q,\mathbf{q}) = \pm i\Phi(q,\mathbf{q}),$$

i.e.,

$$\mathcal{K}_{\pm} = \{\Phi^{\pm}_{\psi} \in \mathfrak{H} : \Phi^{\pm}_{\psi}(q,\mathbf{q}) = e^{\pm q}\psi(\mathbf{q}) \text{ for some measurable } \psi\}. \tag{77}$$

Since

$$\int_{\Omega} |e^{\pm q}\psi(\mathbf{q})|^2 dq d^{D-1}\mathbf{q} = \left(\int_{\mathbb{R}^{D-1}} |\psi(\mathbf{q})|^2 \sinh 2Q(\mathbf{q}) d^{D-1}\mathbf{q}\right), \tag{78}$$

denoting by $\mathcal{K}$ the Hilbert space of functions of $\mathbf{q} \in \mathbb{R}^{D-1}$ with square of the norm given by Eq. (78), one obtains that $\psi \mapsto \Phi^+_{\psi}$ ($\psi \mapsto \Phi^-_{\psi}$) is an isometry of $\mathcal{K}$ onto $\mathcal{K}_+$ ($\mathcal{K}_-$). Thus the deficiency indices of A are infinite and self-adjoint extensions can be parametrized by unitary operators from $\mathfrak{K}_+$ to $\mathfrak{K}_-$ or, equivalently, by unitary operators on $\mathfrak{K}$: if K is a unitary operator from $\mathcal{K}_+$ to $\mathcal{K}_-$, then for the corresponding unitary operator on $\mathcal{K}$, which will be again denoted by K, we have

$$K\Phi^+_{\psi} = \Phi^-_{K\psi}.$$

The self-adjoint extension A_K of A corresponding to K (Theorem X.2 of Ref. 14) is defined by

$$A_K(\Phi + \Phi^+_{\psi} + \Phi^-_{K\psi}) = \bar{A}\Phi + i(\Phi^+_{\psi} - \Phi^-_{K\psi}),$$

where Φ is in the domain of the closure $\bar{A}$ of A, i.e.,

$$\mathcal{D}(A_K) = \{\Psi \in \mathcal{D}(A^*) : \Psi(q,\mathbf{q}) = \Phi(q,\mathbf{q}) + e^q\psi(\mathbf{q}) + e^{-q}(K\psi)(\mathbf{q}), \Phi \in \mathcal{D}(\bar{A})\}.$$

We return now to the "general decomposable case," i.e., to the case of

$$(K\psi)(\mathbf{q}) = \kappa(\mathbf{q})\psi(\mathbf{q}), \quad |\kappa(\mathbf{q})| = 1, \tag{79}$$

where κ is a measurable function. Writing $A_{(\kappa)}$ for A_K, one has

$$\mathcal{D}(A_{(\kappa)}) = \{\Psi \in \mathcal{D}(A^*) : \Psi(q,\mathbf{q}) = \Phi(q,\mathbf{q}) + e^q\psi(\mathbf{q}) + \kappa(\mathbf{q})e^{-q}\psi(\mathbf{q}), \Phi \in \mathcal{D}(\bar{A})\}.$$

$A_{(\kappa)}$ will now be identified with self-adjoint extension of A defined by suitable boundary conditions.

Let the (measurable) function ξ be defined by

$$\xi(\mathbf{q}) = \frac{e^{Q(\mathbf{q})} + \kappa(\mathbf{q})e^{-Q(\mathbf{q})}}{e^{-Q(\mathbf{q})} + \kappa(\mathbf{q})e^{Q(\mathbf{q})}}, \tag{80}$$

so that $\kappa(\mathbf{q}) = -(\xi(\mathbf{q})e^{-Q(\mathbf{q})} - e^{Q(\mathbf{q})})/(\xi(\mathbf{q})e^{Q(\mathbf{q})} - e^{-Q(\mathbf{q})})$ and $|\xi(\mathbf{q})| = 1$; the Hilbert space $\mathfrak{H}$ of Eq. (69) admits direct sum decomposition,

$$\mathfrak{H} = \underset{n \in \mathbb{Z}}{\oplus} \mathfrak{H}_n, \quad \mathfrak{H}_n = \{\Phi \in \mathfrak{H} : \Phi(q, \mathbf{q}) = e^{-iq\lambda_n^{Q(\mathbf{q}), \xi(\mathbf{q})}} \psi(\mathbf{q}), \quad \psi \in L^2(\mathbb{R}^{D-1})\}. \tag{81}$$

Furthermore, let A_ξ be the self-adjoint that acts on $\mathfrak{H}_n$ as an operator of multiplication by $\lambda_n^{Q(\mathbf{q}), \xi(\mathbf{q})}$, i.e.,

$$D(A_\xi) = \left\{ \sum_{n \in \mathbb{Z}} \Phi_n : \Phi_n \in \mathfrak{H}_n, \sum_{n \in \mathbb{Z}} \int |\lambda_n^{Q(\mathbf{q}), \xi(\mathbf{q})} \Phi_n(q, \mathbf{q})|^2 dq d^{D-1} \mathbf{q} < \infty \right\} \tag{82}$$

and

$$A_\xi \left(\sum_{n \in \mathbb{Z}} \Phi_n \right) = \sum_{n \in \mathbb{Z}} \lambda_n^{Q, \xi} \Phi_n. \tag{83}$$

Formulas (81)–(83) organize the spectrum of A_ξ into *series* and *bands*: when $\mathbf{q}$ is fixed, the (generalized) eigenvalues $(\lambda_n^{Q(\mathbf{q}), \xi(\mathbf{q})})_{n \in \mathbb{Z}}$ form a ($\mathbf{q}$) series, while for fixed n the (generalized, in general) eigenvalues $(\lambda_n^{Q(\mathbf{q}), \xi(\mathbf{q})})_{\mathbf{q} \in \mathbb{R}^{D-1}}$ form an n band. Different bands may, and often will, overlap; spacing of eigenvalues within a series is equal $\pi/Q(\mathbf{q})$, and the corresponding spacing of eigenvalues of X_1 is $\hbar \pi \sqrt{\beta}/Q(\mathbf{q})$—a reminder of the minimal length of the case of one degree of freedom, where the spectrum is discrete with spacing $\hbar \sqrt{\beta}$.

Repeating the argument that led above to identification of A_K with $i\partial^{q, \xi}$, one obtains that $A_{(\kappa)} = A_\xi$: for, for $\Psi \in \mathcal{D}(A_{(\kappa)})$ one can see that $\Psi(Q(\mathbf{q}), \mathbf{q}) = \xi(\mathbf{q})\Psi(-Q(\mathbf{q}), \mathbf{q})$, that for Ψ satisfying the later condition, one has

$$\Psi(q, \mathbf{q}) = \Phi(q, \mathbf{q}) + e^q \psi(\mathbf{q}) + e^{-q} \kappa(\mathbf{q}) \psi(\mathbf{q}),$$

$$\text{where } \psi(\mathbf{q}) = \{[\xi(\mathbf{q})e^{Q(\mathbf{q})} - e^{-Q(\mathbf{q})}]/[e^{2Q(\mathbf{q})} - e^{-2Q(\mathbf{q})}]\}\Psi(-Q(\mathbf{q}), \mathbf{q}),$$

and $\Phi \in \mathcal{D}(A^*)$. Hence it follows that $\mathfrak{H}_n \subset \mathcal{D}(A_{(\kappa)})$, and then that $\mathcal{D}(A_\xi) \subset \mathcal{D}(A_{(\kappa)})$, and that therefore $A_{(\kappa)} = A_\xi$ since both $A_{(\kappa)}$ and A_ξ are self-adjoint operators.

The spectral representation [Eq. (83)] yields the following:

Proposition: *In case of continuous ξ (or κ), the spectrum of A_ξ consists of the range of $\lambda_\cdot^{Q(\cdot), \xi(\cdot)}$,*

$$\sigma(A_\xi) = \{\lambda_n^{Q(\mathbf{q}), \xi(\mathbf{q})} : \mathbf{q} \in \mathbb{R}^{D-1}, n \in \mathbb{Z}\},$$

and for general measurable ξ of the essential range of $\lambda_\cdot^{Q(\cdot), \xi(\cdot)}$ (P. 229 of Ref. 23). Furthermore, A_ξ has no singular spectrum and λ is an eigenvalue of A_ξ if and only if there exists $n \in \mathbb{Z}$ and a non-negligible subset of $\mathbb{R}^{D-1}$, such that, $\lambda_n^{Q(\mathbf{q}), \xi(\mathbf{q})} = \lambda$ for $\mathbf{q}$ in the subset or, equivalently, the set of $\mathbf{q} \in \mathbb{R}^{D-1}$, for which

$$e^{-2i\lambda Q(\mathbf{q})} = \xi(\mathbf{q})$$

has nonzero (Lebesgue) measure.

As is not hard to see, that the above implies that for each continuous ξ, the continuous spectrum of A_ξ contains intervals extending to $+\infty$ and $-\infty$—in fact, this is true for any measurable ξ, and that therefore, unlike in the one-dimensional case, the spectrum of A_ξ is never pure point.

We will now discuss the spectrum of A_ξ in a number of special cases.

A. Globally periodic and antiperiodic boundary conditions

These are the only cases in which ξ and κ are the same, namely, equal to ± 1, respectively, for all $\mathbf{q}$. However, the spectra in these two cases are different, as will be shown now.

In the case of $\kappa = \mathbf{1}$, i.e., $K = \text{id}$, $\xi = \mathbf{1}$, arg $\xi = \mathbf{0}$,

$$\sigma(A_K) = -\infty, -2 \cup \{0\} \cup [2, +\infty[, \tag{84}$$

with $\lambda = 0$ being the only eigenvalue, albeit of infinite multiplicity.

For, Eq. (71) yields

$$\lambda_n^{Q(\mathbf{q}),\mathbf{1}} = \pi n/Q(\mathbf{q}),$$

which is a continuous function of $\mathbf{q}$. Hence, the 0 band consists of one point, 0, while the n band is $[2n, +\infty[$ for $n > 0$ and $]-\infty, -2n]$ for $n < 0$. This implies that $\mathfrak{H}_0$ is the eigenspace corresponding to eigenvalue $\lambda = 0$ [see Eqs. (81)–(83)]. On the other hand, for $\lambda \neq 0$ and n fixed, solutions $\mathbf{q}$ of the equation $\pi n/Q(\mathbf{q}) = \lambda$, $\lambda \geq 2n$, form a sphere of radius $\sqrt{\lambda^2/(2n)^2 - 1}$ [see Eq. (66)]—a set of measure zero. Putting this together, one obtains Eq. (84), the fact that zero is an eigenvalue of finite multiplicity for $D = 2$ and of infinite multiplicity for $D \geq 3$, since for $2n \leq \lambda$ any measure supported by the sphere of radius $\sqrt{\lambda^2/(2n)^2 - 1}$ yields a generalized eigenvector of A_K with generalized eigenvalue λ.

For antiperiodic boundary conditions ($\xi = -\mathbf{1}$, arg $\xi = \pi$, $\kappa = -1$, $K = -$id) the situation is different. Now

$$\lambda_n^{Q(\mathbf{q}),\xi} = (2n - 1)\pi/(2Q(\mathbf{q})), \quad n \in \mathbb{Z},$$

n-band is

$$[2n - 1, +\infty[\text{ for } n > 0 \quad \text{and} \quad]-\infty, 2n - 1[\text{ for } n \leq 0,$$

and the spectrum,

$$\sigma(A_{-\mathrm{id}}) = -\infty, -1 \cup [1, +\infty[,$$

is purely continuous.

Similar analysis works in the generic scalar case, when K is the operator of multiplication by a number κ of modulus one, different from ± 1.

B. General decomposable case

By Proposition, a number μ is an eigenvalue of A_ξ if and only if there exists an integer, say m, and a non-negligible $M \subset \mathbb{R}^{D-1}$, such that

$$\mu = \lambda_m^{Q(\mathbf{q}),\xi(\mathbf{q})}, \quad \xi(\mathbf{q}) = \mathrm{e}^{-2i\mu Q(\mathbf{q})} \quad \text{for all } \mathbf{q} \in M. \tag{85}$$

Moreover, multiplicity of the eigenvalue μ is always infinite since there is one-to-one correspondence between eigensubspace of A_ξ corresponding to eigenvalue μ and the set of elements of $\mathcal{K}$ which are zero on the complement of M.

Furthermore, as is easy to see, any eigenvector of A_ξ with eigenvalue different from μ are a.e. zero on M.

It follows that for general ξ, A_ξ has the following structure: There is a pairwise disjoint family $(\Lambda_\iota)_{\iota \in I}$ of non-negligible subsets of $\mathbb{R}^{D-1}$, such that, for each $\iota \in I$ there is a unique integer n_ι and real λ_ι with the following properties.

- $\{\lambda_\iota : \iota \in I\}$ is the set of all the eigenvalues of A_ξ; each of the eigenvalues is of infinite multiplicity.
- $\xi(\mathbf{q}) = \exp(-2i\lambda_\iota Q(\mathbf{q}))$ for $\mathbf{q} \in \Lambda_\iota$ and the function $\mathbf{q} \mapsto 2Q(\mathbf{q})\lambda_\iota + \arg \xi(\mathbf{q})$ is integer valued and constant on each Λ_ι. Let n_ι be its value on Λ_ι.
- The number $(n - \arg \xi(\mathbf{q}))/(2Q(\mathbf{q}))$ is in continuous spectrum for any $n \neq n_\iota$ for a.a. $\mathbf{q} \in \Lambda_\iota$ and for any integer n for a.a. $\mathbf{q} \in \mathbb{R}^{D-1} \backslash (\cup \Lambda_\iota)$.

The simplest case of this construction is obtained as follows: pick a real number μ, define $\xi(\mathbf{q})$ for any $\mathbf{q} \in \mathbb{R}^{D-1}$ by the second identity of Eq. (85), and then $m \in \mathbb{Z}$ by the first one. The spectrum of A_ξ consists of the series

$$\lambda_n = \mu + n\pi/Q(\mathbf{q}), \quad n \in \mathbb{Z},$$

with the m-band reduced to the eigenvalue μ, the rest of the spectrum being continuous.

A slight modification of this construction yields an infinite number of eigenvalues: choose an infinite family of real numbers, say, $(\lambda_j)_{j=1}^{\infty}$, a nonzero subset of $\mathbb{R}^{D-1}$, say, Λ, together with its decomposition into nonzero subsets, $\Lambda = \bigcup_{j=1}^{\infty}\Lambda_j$, then define a ξ, such that, $\xi(\mathbf{q}) = e^{-2i\lambda_j Q(\mathbf{q})}$ on Λ_j. Note that for so defined ξ, the eigenspace corresponding to λ_j is a subspace of $\mathfrak{H}_{n_j}$ of infinite dimension, where $n_j = (2\lambda Q(\mathbf{q}) + \arg\xi(\mathbf{q}))/(2\pi)$, and that one can obtain in this way many self-adjoint extensions of A with an infinite number of eigenvalues and, in addition, continuous spectrum. In fact, it is not hard to see that continuous spectrum is present in each of the decomposable extensions of A.

ACKNOWLEDGMENTS

APPENDIX A: REDUCTION OF THE HYPERGEOMETRIC EQUATION

For convenience of the reader, and the author, we collect here some formulas, mostly from, Ref. 18 (p. 28–30)] on gauge transformations of second order differential equations, used in Sec. II. Since it is simpler and more enlightening to consider general Sturm-Liouville operators of Fuchsian type, with three regular singular point, it is this framework that is adopted in most of this appendix.

Equation (20) is Fuchsian with regular singular points at -1, $+1$, and ∞. Such an equation can be written as

$$\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + q_1(z)\frac{\mathrm{d}u}{\mathrm{d}z} + q_2(z)u(z) = 0, \tag{A1}$$

with

$$q_1(z) = \frac{A_1}{z+1} + \frac{A_2}{z-1}, \quad q_2(z) = \frac{B_1}{(z+1)^2} + \frac{B_2}{(z-1)^2} + \frac{B_\infty}{(z+1)(z-1)}.$$

The (*indicial*) equations for the *(characteristic) exponents* (σ_i, τ_i) of Eq. (A1) are

$$r^2 + (A_1 - 1)r + B_1 = 0 \quad \text{for } (\sigma_1, \tau_1) \text{ at } -1, \tag{A2}$$

$$r^2 + (A_2 - 1)r + B_2 = 0 \quad \text{for } (\sigma_2, \tau_2) \text{ at } +1, \tag{A3}$$

$$r^2 + (-A_1 - A_2 + 1)r + B_1 + B_2 + B_\infty = 0 \quad \text{for } (\sigma_\infty, \tau_\infty) \text{ at } \infty. \tag{A4}$$

The exponents (σ_i, τ_i) will be indexed so that $\sigma_i \geqslant \tau_i$, $i = -, +, \infty$.

The (gauge) transformation

$$u(z) = (1 + z)^{\sigma_1}(1 - z)^{\sigma_2}v(z) \tag{A5}$$

changes the exponents to

$$(0, \tau_1 - \sigma_1), \quad (0, \tau_2 - \sigma_2), \quad \text{and} \quad (\sigma_\infty + \sigma_1 + \sigma_2, \tau_\infty + \sigma_1 + \sigma_2) \tag{A6}$$

at -1, $+1$, and ∞, respectively. The corresponding differential equation for v—the hypergeometric equation with singular points -1, $+1$ instead of the standard 0, 1 can be written as

$$(1 - z^2)v''(z) + [(b - a) - (c + 1)z]v'(z) + \lambda(\lambda + c)v(z) = 0, \quad c = a + b + 1, \tag{A7}$$

or

$$-(1 - z^2)v''(z) + [(a - b) + (c + 1)z]v'(z) + (c^2/4)v(z) = \mu v(z), \quad \mu = (\lambda + c/2)^2, \tag{A8}$$

where

$$b = \sigma_1 - \tau_1 \quad a = \sigma_2 - \tau_2, \tag{A9}$$

$$\mu - \frac{1}{4}c^2 + (\sigma_\infty + \sigma_1 + \sigma_2)(\tau_\infty + \sigma_1 + \sigma_2) = 0. \tag{A10}$$

Under our assumption of $\sigma_i \geq \tau_i$, one has $a, b \geq 0$—this inequality motivates the choice of gauge transformation (A5), and therefore Eq. (A7) has a family of solutions, Jacobi polynomials,

$$P_n^{(a,b)}(z), \quad \lambda_n = n, \quad n = 0, 1, \ldots, \tag{A11}$$

which form a complete orthogonal family in the Hilbert space $\mathcal{H}^{(a,b)}$ of Sec. II A (Formula 8.962.1 of Ref. 21).

In our case, Eq. (A1) is obtained from Eq. (20). Considering Eq. (20) for general real values of $D_\pm$, L, η, on has

$$q_1(z) = \frac{1}{1 - z^2}(D_1 + D_2 - (D_1 - D_2 + 2)z) = \frac{D_1 + 1}{1 + z} + \frac{D_2 - 1}{1 - z},$$

$$q_2(z) = \frac{1}{1 - z^2}\left(\rho - \frac{1}{4}\frac{(1 + \eta - (1 - \eta)z)^2}{1 - z^2}L^2\right) = \frac{1}{2}\frac{2\rho - L^2\eta}{(1 - z)(1 + z)} - \frac{1}{4}\frac{\eta^2 L^2}{(1 - z)^2} - \frac{1}{4}\frac{L^2}{(1 + z)^2}, \tag{A12}$$

and

$$A_1 = 1 + D_1, \quad A_2 = 1 - D_2, \quad B_1 = -\frac{1}{4}L^2, \quad B_2 = -\frac{1}{4}\eta^2 L^2, \quad B_\infty = \frac{1}{2}L^2\eta - \rho.$$

We proceed now with calculation of the product $(\sigma_\infty + \sigma_1 + \sigma_2)(\tau_\infty + \sigma_1 + \sigma_2)$, which is needed for Eq. (A10).

The indicial equations are

$$r^2 - D_1 r - \frac{1}{4}L^2 = 0 \quad \text{for } (\sigma_1, \tau_1) \text{ at } -1, \tag{A13}$$

$$r^2 + D_2 r - \frac{1}{4}\eta^2 L^2 = 0 \quad \text{for } (\sigma_2, \tau_2) \text{ at } +1, \tag{A14}$$

$$r^2 + (D_2 - D_1 - 1)r - \frac{1}{4}L^2(1 - \eta)^2 - \rho = 0 \quad \text{for } (\sigma_\infty, \tau_\infty) \text{ at } \infty. \tag{A15}$$

First two of these equations yield

$$(\sigma_1, \tau_1) = -\frac{1}{2}D_1 \pm \frac{1}{2}\sqrt{D_1^2 + L^2}, \quad (\sigma_2, \tau_2) = \frac{1}{2}D_2 \pm \frac{1}{2}\sqrt{D_2^2 + \eta^2 L^2},$$

and then—see (A9), a and b are

$$a = \sqrt{D_2^2 + \eta^2 L^2}, \quad b = \sqrt{D_1^2 + L^2}, \tag{A16}$$

so that

$$\sigma_1 = -\frac{1}{2}D_1 + \frac{1}{2}b, \quad \sigma_2 = \frac{1}{2}D_2 + \frac{1}{2}a. \tag{A17}$$

Setting

$$\chi = 1 - (D_2 - D_1), \tag{A18}$$

$\chi = (1 - \eta)\bar{D}/2$ in case of Eq. (4), from the third of the inidicial equations, [Eq. (A16)], one obtains

$$\sigma_\infty + \tau_\infty = \chi \text{ and } \sigma_\infty \tau_\infty = -\frac{1}{4}L^2(1-\eta)^2 - \rho,$$

$$\sigma_\infty + \tau_\infty = D_1 - D_2 + 1 = \chi - 2 \text{ and } \sigma_\infty \tau_\infty = -\frac{1}{4}L^2(1-\eta)^2 - \rho,$$

while

$$\sigma_1 + \sigma_2 = \frac{1}{2}(D_2 - D_1) + \frac{1}{2}\sqrt{D_1^2 + L^2} + \frac{1}{2}\sqrt{D_2^2 + \eta^2 L^2} = \frac{1}{2}(c - \chi). \tag{A19}$$

Hence

$$(\sigma_\infty + \sigma_1 + \sigma_2)(\tau_\infty + \sigma_1 + \sigma_2) = \sigma_\infty \tau_\infty + (\sigma_1 + \sigma_2)(\sigma_\infty + \tau_\infty) + (\sigma_1 + \sigma_2)^2$$

$$= -\frac{1}{4}L^2(1-\eta)^2 - \rho + \frac{1}{2}(c - \chi)\chi + \frac{1}{4}(c - \chi)^2$$

And, finally, from Eq. (A10),

$$\mu - \frac{1}{4}c^2 - \frac{1}{4}L^2(1-\eta)^2 - \rho + \frac{1}{2}(c - \chi)\chi + \frac{1}{4}(c - \chi)^2 = 0,$$

i.e.,

$$\rho = \mu - \frac{1}{4}L^2(1-\eta)^2 - \frac{1}{4}\chi^2. \tag{A20}$$

We now summarize results of calculations of this appendix. The gauge transformation (A5) of Eq. (A12), with $u = \Phi$ and $\sigma_\pm$ given by Eq. (A17), yields Eqs. (A7) and (A8), with the above a, b, and c. These equations have a family [Eq. (A7)] of solutions with $\rho = \rho_n$ given by Eq. (A20), where $\mu = \mu_n = (n + c/2)^2$.

APPENDIX B: HARMONIC OSCILLATOR

Performing reduction of the Hamiltonian,

$$H = \frac{1}{2M}\mathbf{P}^2 + \frac{1}{2}M\omega^2\mathbf{X}^2,$$

with respect to rotations and then substitution [Eq. (17)], one obtains the reduced Hamiltonian

$$H_L = \frac{1}{2M\alpha}\frac{1+z}{1-z} - 2\alpha\hbar^2 M\omega^2 \left[(1-z^2)\frac{d^2}{dz^2} + (D_2 + D_1 + (D_2 - D_1 - 2)z)\frac{d}{dz} \right.$$
$$\left. - \frac{((1+\eta)-(1-\eta)z)^2}{4(1-z^2)}L^2 \right] ,$$

so that the eigenvalue equation $H_L\Phi = E\Phi$—our Eq. (A1) can be written as

$$\Phi'' + \frac{1}{1-z^2}(D_2 + D_1 + (D_2 - D_1 - 2)z)\Phi' + \frac{1}{1-z^2}\left(\varepsilon - \frac{(1+\eta-(1-\eta)z)^2}{4(1-z^2)}L^2 - \nu\frac{1+z}{1-z} \right)\Phi = 0,$$

where

$$\varepsilon = \frac{E}{\delta\varpi^2}, \quad \nu = \frac{1}{\delta^2\varpi^2}, \quad \varpi = \hbar\omega, \quad \delta = 2\alpha M,$$

and $D_{\pm}$ are as in Eq. (18). Comparing this with Eq. (A12), one can see that of among the coefficients A_i, B_i, only B_2 is changed, to $-\eta^2 L^2/4 - \nu$, so that one has now

$$A_2 = 1 - D_2, \quad A_1 = D_1 + 1, \quad B_2 = -\frac{1}{4}\eta^2 L^2 - \nu, \quad B_1 = -\frac{1}{4}L^2, \quad B_\infty = \frac{1}{2}L^2\eta - \varepsilon,$$

The first two of the indicial equations [Eqs. (A2) and (A3)] yield now

$$(\sigma_2, \tau_2) = \frac{1}{2}D_2 \pm \frac{1}{2}\sqrt{D_2^2 + \eta^2 L^2 + 4\nu}, \quad (\sigma_1, \tau_1) = -\frac{1}{2}D_1 \pm \frac{1}{2}\sqrt{D_1^2 + L^2}, \tag{B1}$$

and then, by Eq. (A9),

$$a = a(\eta, \nu) = \sqrt{D_2^2 + \eta^2 L^2 + 4\nu}, \quad b = b(\eta, \nu) = \sqrt{D_1^2 + L^2};$$

and since relations (A17) and (A18) are unchanged, the third indicial equation [Eq. (A9)], can be written as

$$r^2 - \chi r - \frac{1}{4}L^2(1-\eta)^2 - \varepsilon - \nu = 0.$$

Continuing the calculations as in Appendix A, one obtains the following modification of (A20):

$$\mu - \varepsilon - \frac{1}{4}L^2(1-\eta)^2 - \nu - \frac{1}{4}\chi^2 = 0. \tag{B2}$$

Hence Friedrichs' extension has eigenvectors

$$\Phi_{L,n}(z) = (1+z)^{\sigma_1}(1-z)^{\sigma_2}P_n^{(a,b)}(z), \quad \ell, n = 0, 1, \dots ,$$

with eigenvalues

$$E_{L,n} = \delta\varpi^2\left[\left(n+\frac{c}{2}\right)^2 - \left(\frac{\beta'}{4\alpha}\right)^2(4L^2 + \bar{D}^2) \right] - \frac{1}{\delta}. \tag{B3}$$

In view of the discussion of Sec. II C, the curve $a(\eta, \nu) = 1$ in the (η, ν) space separates the values of parameters for which one has lc case at $p = \infty(z = 1)$ from those for which one case lp

case. Similary for the equation $b(\eta, \nu) = 1$ for lc and lp cases at $p = 0$ $(z = -1)$. One can also take over the classification of Sec. II C into types.

APPENDIX C: FRIEDRICHS' EXTENSION OF THE HYPERGEOMETRIC OPERATOR

It is proved here that, as claimed in Sec. II C, *for any* $a, b > 0$, $J^{(+,+)}$ *is the Friedrichs' extension of* J_0. Most likely, this can be deduced from the material of the exercises of Ref. 20 but since we do not have a reference and the proof that follows is simple enough, we include it here for completeness.

One needs to show that the domain of Friedrichs' extension of J_0 contains the family $\mathcal{F}^{(+,+)}$ defining $J^{(+,+)}$. Since elements of $\mathcal{F}^{(+,+)}$ are polynomial functions on $]-1, +1[$, it is enough to show that any bounded smooth function f is in the domain of Friedrichs' extension.

The minimal operator J_0 defined by left hand side of Eq. (27) on $C_0^\infty(]-1, 1[)$ can also be written as

$$J_0 u = -\frac{1}{w}(pu')' + \frac{c^2}{4}u, \quad u \in C_0^\infty(]-1, 1[),$$

where, see Eq. (23),

$$w(z) = (1-z)^a(1+z)^b, \quad p(z) = (1-z)^{a+1}(1+z)^{b+1}.$$

Let χ be C^∞-function on $\mathbb{R}$, which is equal zero in a neighborhood of 0, say, for $x < 1$, equal to 1 at infinity, say, for $x > 2$, and such that $0 \leq \chi(z) \leq 1$ for all $z \in \mathbb{R}$; such a function is constructed in a standard way and, obviously, $|\chi'(z)| \leq h$ for all z, for some positive h. Furthermore, for $\varepsilon > 0$, let

$$\chi_\varepsilon(z) = \chi((1+z)/\varepsilon)\chi((1-z)/\varepsilon);$$

χ_ε is zero in a neighborhood of -1 and $+1$ and $\chi_\varepsilon(z) = 1$ for $-1 + 2\varepsilon \leq z \leq 1 - 2\varepsilon$. Obviously,

$$|\chi_\varepsilon'(z)| \leq h/\varepsilon \quad \text{for all } z.$$

Since $\lim_{\varepsilon \to 0} \|\chi_\varepsilon f - f\|^{(a,b)}$, where $\|\cdot\|^{(a,b)}$ is the norm defined by the scalar product [Eq. (22)], it is enough to show that

$$\left(g_\varepsilon \middle| -\frac{1}{w}(pg_\varepsilon')'\right)^{(a,b)} \to 0 \text{ as } \varepsilon \to 0, \quad \text{where} \quad g_\varepsilon = \chi_\varepsilon f - f = (\chi_\varepsilon - 1)f.$$

It is sufficient to consider here real g_ε, which we do to simplify the notation. Then

$$\left(g_\varepsilon \middle| -\frac{1}{w}(pg_\varepsilon')'\right)^{(a,b)} = -\int_{-1}^{+1} ((p(z)g_\varepsilon'(z))')g_\varepsilon(z)dz = \int_{-1}^{+1} g_\varepsilon'(z)p(z)g_\varepsilon'(z)dz,$$

and since $g_\varepsilon'(z) = (\chi_\varepsilon)'f + (\chi_\varepsilon - 1)f'$, it follows that $g_\varepsilon'(z) = 0$ for $-1 + 2\varepsilon \leq z \leq 1 - 2\varepsilon$. Therefore

$$\int_{-1}^{+1} g_\varepsilon'(z)p(z)g_\varepsilon'(z)dz = \int_{-1}^{-1+2\varepsilon} g_\varepsilon'(z)p(z)g_\varepsilon'(z)dz + \int_{1-2\varepsilon}^{+1} g_\varepsilon'(z)p(z)g_\varepsilon'(z)dz.$$

Since $(\chi_\varepsilon - 1)f' \to 0$ in $L^2([-1, 1])$, it is enough to consider the integral of $(\chi_\varepsilon)'fp(\chi_\varepsilon)'f$. Now, for $b > 0$,

$$\left| \int_{-1}^{-1+2\varepsilon} (\chi_\varepsilon)' f(z) p(z) (\chi_\varepsilon)' f(z) \mathrm{d}z \right| \leq \frac{hM}{\varepsilon^2} \left| \int_{-1}^{-1+2\varepsilon} (1+z)^{a+1}(1-z)^{b+1} \mathrm{d}z \right|$$

$$\leq 2^{b+1} \frac{hM}{\varepsilon^2} \left| \int_{-1}^{-1+2\varepsilon} (1+z)^{a+1} \mathrm{d}z \right| = 2^{b+a+3} hM \frac{\varepsilon^a}{2+a},$$

:

$$\frac{1}{a+2} \lim_{z \to -1+} \left((1+z)^{a+2} + 4 \times 2^a \varepsilon^{a+2} \right) = \frac{2^{a+2}}{a+2} \varepsilon^{a+2},$$

where M is an upper bound of $|f(z)|$ for $|z| \leq 1$, which tends to zero with ε. The integral $\int_{1-2\varepsilon}^{+1}$ is dealt with in the same way.

Obviously, the argument works under much more general circumstances since it depends only on behavior of p at the end points of the interval $[-1,1]$, namely, on the fact that $\int_{-1}^{-1+\varepsilon}|g(z)|\mathrm{d}z$ and $\int_{1-\varepsilon}^{1}|g(z)|\mathrm{d}z$ are $o(\varepsilon^2)$ as $\varepsilon \searrow 0$.

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