

Vision Based Guidance and Flight Control in Problems of Aerial Tracking

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(Abstract)

The use of visual sensors in providing the necessary information for the autonomous guidance and navigation of the unmanned-air vehicles (UAV) or micro-air vehicles (MAV) applications is inspired by biological systems and is motivated first of all by the reduction of the navigational sensor cost. Also, visual sensors can be more advantageous in military operations since they are difficult to detect. However, the design of a reliable guidance, navigation and control system for aerial vehicles based only on visual information has many unsolved problems, ranging from hardware/software development to pure control-theoretical issues, which are even more complicated when applied to the tracking of maneuvering unknown targets.

This dissertation describes guidance law design and implementation algorithms for autonomous tracking of a flying target, when the information about the target's current position is obtained via a monocular camera mounted on the tracking UAV (follower). The visual information is related to the target's relative position in the follower's body frame via the target's apparent size, which is assumed to be constant, but otherwise unknown to the follower. The formulation of the relative dynamics in the inertial frame requires the knowledge of the follower's orientation angles, which are assumed to be known. No information is assumed to be available about the target's dynamics. The follower's objective is to maintain a desired relative position irrespective of the target's motion.

Two types of guidance laws are designed and implemented in the dissertation. The first one is a smooth guidance law that guarantees asymptotic tracking of a target, the velocity of which

is viewed as a time-varying disturbance, the change in magnitude of which has a bounded integral. The second one is a smooth approximation of a discontinuous guidance law that guarantees bounded tracking with adjustable bounds when the target's acceleration is viewed as a bounded but otherwise unknown time-varying disturbance. In both cases, in order to meet the objective, an intelligent excitation signal is added to the reference commands.

These guidance laws are modified to accommodate measurement noise, which is inherently available when using visual sensors and image processing algorithms associated with them. They are implemented on a full scale non-linear aircraft model using conventional block backstepping technique augmented with a neural network for approximation of modeling uncertainties and atmospheric turbulence resulting from the closed-coupled flight of two aerial vehicles.

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*To my wife Loreta,
my daughter Lilit and my son Vahagn
for their love and support.*

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*A single lifetime, even though entirely devoted to the sky,
would not be enough for the study of so vast a subject.
A time will come when our descendants will be amazed that
we did not know things that are so plain to them.*

Seneca, Book 7, first century AD

Chapter 1

Introduction

The unmanned-air vehicles (UAV) or micro-air vehicles (MAV) are increasingly becoming an integral part of both military and commercial operations. A crucial aspect in the development of UAV or MAV is the reduction of the navigational sensor cost. These onboard sensors provide the necessary information for the autonomous guidance and navigation of the unmanned vehicle. Emerging visual sensor solutions show a lot of promise in replacing or augmenting the traditional inertial measurement units (IMU) or global positioning systems (GPS) in many mission scenarios. However, the design of a reliable guidance, navigation and control system for the aerial vehicles based only on visual information has many unsolved problems, ranging from hardware/software development to pure control-theoretical issues.

One of the most challenging of these problems is the design of a vision-based guidance system that is capable of tracking a maneuvering target using only visual information about the target. The main difference between the visual tracking problem and the standard tracking problems is the way the feedback signal is measured. Visual tracking is done via imaging sensors, which involve a projection of a 3-D object onto a 2-D plane, consequently

rendering the relative range between the two flying objects unobservable. Moreover, the visual measurements are outputs of some image processing algorithm that has its own time for convergence. This consequently leads to time-delay and noise in the feedback loop. In the case of a completely unknown target, its velocity and acceleration present time-varying disturbances in the relative dynamics. Thus, the guidance law for the follower vehicle in aerial tracking using only visual measurement needs to reject time-varying disturbances for an unobservable system in the presence of state constraints (limited field of view) with a noise-corrupted measurement subject to time-delays.

The target tracking problem (without the visual sensors) is a relatively old one, and there is an extensive literature in this field, addressing the topic from various perspectives. The fundamental issue in this problem is the lack of complete information about the target's dynamics. The mathematical model used to describe the scenario affects the control design choice. For example, in Cartesian coordinates the dynamical equations are linear, but the observation model is nonlinear [23, 77], while in the spherical coordinate system the observation model is linear at the expense of a highly nonlinear dynamical model [1, 3, 64]. In most cases, Extended Kalman filter (EKF) or modified EKF has been the main tool for extracting necessary information about target dynamics. Modifications of EKF include adaptive EKFs [25], multiple model adaptive estimators that identify the target maneuver based on some pre-stored models [11, 44], iterated extended Kalman filter (IEKF) that has an improved performance and a better accuracy [4], or two-step optimal estimators that divide the estimation into a “linear first step” using the standard Kalman filter and a “nonlinear second step” that treats states estimated via Kalman filter as measurements for the nonlinear least-square fit [23]. Convergence properties for EKF have been proven in [36] for linear deterministic models with unknown coefficients in a discrete-time setting. It is important to note that with the use of visual sensors the state space representation for the system is nonlinear either in the states (spherical coordinate system [1, 3, 64]) or in the measurement

equation (Cartesian coordinate system [23, 77]); hence, the results from [36] cannot be used without additional asymptotic analysis.

It has been noted that the relative range is unobservable from the bearing-only measurement (see for example [1]), and a special type of own-ship maneuver is required to overcome this issue. Another way is to look for additional information that can be extracted from the image processing algorithm. In [7], it has been suggested to use the maximum angle subtended by the target in the image plane, which is equivalent to the maximal image length. This angular measurement relates the unobservable range to the unknown target's geometric size, rendering the relative range observable if the target's geometry is known. Unfortunately, this is generally not the case, and special type of maneuvers are required from the follower to overcome the unobservability even if the target maintains constant velocity. In [74], an attempt has been made to find an optimal lateral motion (ownship maneuver) for the follower that can render the relative range observable. These maneuvers are known as excitation of the reference signal in the adaptive control framework. In [10], the notion of intelligent excitation has been introduced to replace the requirement for the persistent excitation, which is the common tool in adaptive control for achieving parameter convergence. The amplitude of this excitation depends on the tracking error, thus guaranteeing simultaneous parameter convergence (adaptive learning of the size of the target) and desired reference command tracking.

We note that visual tracking is not a new problem either. During the past two decades, visual sensors have been intensively used in ground robotics applications. In [21], a problem of an autonomous path following for a ground robot is considered. The path is assumed to be described by some unknown planar curve that the robot must follow. A camera fixed on the robot provides measurements of the lateral distance of the curve from the vehicle's longitudinal axis. The contour is approximated online using EKF based on some a priori

model of the curve. In [12], an adaptive kinematic controller is designed that asymptotically regulates a robot end-effector to a desired position and orientation using visual information from a camera fixed on the end-effector. It is assumed that the depth is not measurable and the camera calibration matrix is unknown. The controller is derived using the Lyapunov redesign technique. In [37], the problem of visual navigation of a mobile robot following the ground curve is formulated as controlling the shape of the curve in the image plane of the camera. It is shown that the system characterizing the image curve is finite dimensional and controllable for linear curvature curves only, regardless of the kinematics of the mobile robot base. In [72], the problem of the distributed leader-follower formation control for non-alcoholic mobile robots equipped with central-panoramic cameras is considered by specifying the desired formation in the image plane and translating the control problem into a separate visual servoing task for each follower. The approach uses motion segmentation techniques for estimating the position and velocities of each leader. The control strategies guarantee asymptotic tracking for the linear approximation, but suffer from degenerate configurations. For the nonlinear system these configurations can be avoided, but only input-to-state stability is guaranteed. In [18], a methodology for cooperative control of a group of nonholonomic robots is derived. An omnidirectional camera is used to provide necessary information to estimate the state of the formation at different levels, which enables centralized and decentralized cooperative control. In [72], the formation control problem is considered for a group of nonholonomic mobile robots with a paracatadioptric camera mounted on each of them. The proposed method is based on the translation of the formation control problem from the configuration space into the visual image plane for each of them. The position of each leader on the image plane of the follower is estimated from the optical flow across multiple frames. In [13], a Lyapunov-based adaptive control strategy is used for the robot end effector to track a desired path for both camera-in-hand and fixed camera configurations. The estimate of the constant unknown depth from the fixed reference point is used to derive a hybrid error

system that involves both pixel and Euclidean variables. The homography matrix obtained from the error system is decomposed afterwards to determine the rotational and translational motion components. In [75], a control strategy is presented for a ground vehicle to navigate through the unknown environment using the estimated map as a constraint for the locally optimal path planning.

In the meantime, relatively fewer results are reported on the use of visual sensors for aerial vehicle applications. The main challenge for extending the methods of visual tracking from ground robots to aerial vehicles is in the convergence speed of the image processing algorithms. Aerial vehicles have to respect the minimum stall speed requirement for sustained flight, which implies that the measurements for feedback need to be computed sufficiently fast. Most of the existing image processing algorithms work fine in an off-line setting but are poorly suited for real-time applications. Thus, for a sustained flight, the visual measurement is obtained with significant time-delay and noise. Another challenge in the extension of the methods for visual tracking is that for aerial vehicle applications with monocular cameras the range between two vehicles is not available as a measurement from the projection in the image plane. On the other hand, for ground robots, which move on the ground, the range information can be recovered by simple trigonometric considerations using the geometric parameters of the robot if the environment is known, as in the case of the planar motion. Hence, the aerial tracking problem lacks observability. One can avoid this problem by attaching two cameras to the wingtips of the aircraft, but for micro-unmanned vehicles this is neither efficient nor desirable.

In UAV applications, vision based algorithms have been used for path planning and vehicle's state estimation in partially known or unknown environments. In [62], a probabilistic approach is used for the online path planning of a UAV in a partially known 3D environment using visual information, the coordinates of the UAV obtained from GPS/INS and a

probabilistic model of the environment. For the unknown environment, Refs. [27, 75, 76] have used a combination of methods from the statistical learning theory [8], and Structure-From-Motion (SFM) algorithms to do map/image reconstruction and estimation of the vehicle's states in the absence of any tracking tasks. In [76] an implicit extended Kalman filter and epipolar constraint are used for the state estimation of the UAV. A brief overview of feature point tracking and SFM algorithms, which can be applied to the problem of aerial vehicles' state estimation one can find in [27]. In particular, it has been shown that the convergence of EKF can be improved when the states are propagated through the aircraft's dynamic model.

Vision-based algorithms have also been used in tracking known targets. In [58], a visual sensor is used to control the landing of a UAV. A geometric method is proposed to estimate the camera's angular and linear velocities relative to fixed feature points on the landing pad. In [60], design and implementation of a real-time computer vision algorithm is presented for a rotorcraft UAV to land on a known landing target that is designed to simplify the corner detection and correspondence matching in image processing. The algorithm includes some linear and nonlinear optimization schemes for model-based camera pose estimation. In [59], a multiple view motion estimation algorithm is presented for autonomous landing of a UAV that extends the previous results by incorporating all algebraic independent constraints among multiple views. In [73], a control algorithm is presented for a small UAV to track a static ground target using a rotating camera, when circling above the target.

The visual tracking problem is even more complicated in the case of a maneuvering unknown target, that is, a target that has nonzero acceleration. Existing results in this area mainly consider target motions with constant velocity [1, 7], or model the target's acceleration as a zero mean Gaussian process [74]. Adaptive output feedback control framework has been explored in [56, 57] under a certain set of assumptions about the target dynamics, when the

relative range between the aerial vehicles is measured. The derived controller guarantees tracking with ultimately bounded error, but this bound cannot be quantified.

In this dissertation we consider tracking of a maneuvering target, when the only information about the target is available through the visual sensor, that is the relative range between the aerial vehicles is not measurable. Thus, two challenges are addressed simultaneously: to estimate the relative position of the target with respect to the tracking UAV, and to reject the effects of the target's maneuvers, viewed as a time varying disturbance. The developed algorithms are based on the robust adaptive estimation and control theory in both state and output feedback settings for the multi-input multi-output systems that involve unknown parameters and time varying disturbances. In addition, the reference commands to be tracked depend on the unknown parameters. Therefore, along with the tracking error convergence analysis, the parameter convergence analysis is required to meet the control objectives.

The rest of the dissertation is organized as follows. Chapter 2 consists of basic definitions and results from the stability theory, adaptive output feedback control theory, approximation theory and differential equations theory with discontinuous right hand sides. The purpose is to produce a self-contained work by providing sufficient coverage of the material that is important from the point of view of a systematic and clear presentation of main ideas of this dissertation.

Chapter 3 presents the vision-based target tracking problem formulation using only visual information that consists of the coordinates in pixels of the image centroid on the image plane and the image maximal size in pixels. The latter measurement can be related to the relative range between two aerial vehicles by means of the target's size which is assumed to be unknown. Thus, the visual measurements are related to the relative position information up to an unknown scalar factor. Scaling the relative dynamics by this factor renders the

scaled relative position available for feedback. However, the challenge associated with the unobservable relative range translates into one associated with information deficit in formulating reference commands: they depend upon the unknown scaling parameter. Viewing the target's velocity or acceleration as a time-varying disturbance, the target tracking problem is formulated as an adaptive disturbance rejection control problem for a multi-input multi-output linear system with positive but unknown high frequency gain in each control channel. First, the target's velocity is assumed to be decomposed into a sum of a constant term and a time-varying term that has bounded integral of its 2-norm over time. Although somewhat restricting, any obstacle or collision avoidance maneuver satisfies this assumption, provided that after the maneuver the target velocity returns to some constant value in finite time, or asymptotically in infinite time, fast enough for the integral of the magnitude in velocity change to be convergent. Second, the target's acceleration is assumed to be bounded, but otherwise an unknown function of time, which is the most general case of motion for a mechanical system. In both cases, the reference commands depend on the unknown parameter.

Chapter 4 presents a solution to the first problem. First, the disturbance rejection guidance law is derived for known reference commands by employing adaptive estimation of the target's unknown size and nominal velocity. The derived smooth guidance law guarantees asymptotic tracking of the known reference commands irrespective of the learning process. However, in case of visual measurements, reference commands are unknown and parameter convergence is required for these estimates to be useful in the guidance law [65]. It is shown, that parameters converge to some constant values, but not necessarily to the true values. Intelligent excitation is used to ensure simultaneous parameter convergence and output tracking.

Chapter 5 presents a solution to the second problem, the problem of tracking a target with bounded, but otherwise unknown acceleration. In this case, the relative velocity between two aerial vehicles is estimated using the robust adaptive observer technique from [66], which is

simplified for the linear system of special structure describing the relative motion of the target with respect to the follower. The disturbance rejection guidance law is derived using adaptive bounding similar to [53] and input filtered transformations from [39]. For the transformed system, the virtual control, which guarantees asymptotic tracking of the estimated reference commands in the presence of intelligent excitation, is discontinuous. The guidance law that represents the inertial acceleration command for the follower aerial vehicle is designed using backstepping for the filter dynamics. This step requires the discontinuous virtual command to be replaced by a smooth approximation. The algorithm guarantees bounded tracking of given reference commands with adjustable bounds [67].

Chapter 6 presents the robustness analysis of derived in the previous two chapters guidance laws with respect to the measurement noise. The measurement noise is translated into an additive noise for the scaled relative position vector, which results in control problems with the noise in the regulated output. For both problems, state estimators are designed, the outputs of which are fed to guidance law and the adaptive estimation scheme, to accommodate the measurement noise. Through Lyapunov analysis, it is shown that the error signals remain uniformly ultimately bounded. It is shown that the estimator gains can be computed according to conventional Kalman filter scheme (see for example [9]).

Chapter 7 presents the flight control design to implement the two guidance laws derived in the previous chapters for a full scale nonlinear aircraft model, subject to the unmodeled dynamics and the atmospheric turbulence caused by the closed-coupled target tracking. The guidance laws are translated into proper reference commands using the model aircraft's dynamics. The actual control surface deflection commands are designed via block backstepping technique [33] using neural network (NN) approximation of the system's uncertainties [17].

Chapter 8 concludes the dissertation with a brief summary and directions for future work.

Two appendices are included in the dissertation. Appendix A presents the definition and

properties of the projection operator, which is used in definition of the adaptive laws.

Appendix B considers the problem of designing an adaptive observer for a class of multi-output non-autonomous nonlinear systems, in which i) unknown nonlinearities depend on system's state and input, ii) there are unknown multiplicative time-varying bounded parameters that have absolutely integrable derivative, and iii) there are unknown time-varying bounded additive disturbances. The adaptive observer utilizes the NN approximation properties of continuous functions and the adaptive bounding technique for rejecting disturbances similar to [53]. The linear part of the system is assumed to satisfy a slightly milder condition as in [14] than SPR. The resulting state observation error is shown to converge to zero asymptotically, while the parameter estimation error remains bounded. The adaptive output feedback scheme in Chapter 5 uses a particular case of this observer for the state estimation.

Throughout the manuscript, bold symbols are introduced for vectors, capital letters for matrices, small letters for scalars, $\|\cdot\|$ is introduced for 2-norm and $\|\cdot\|_F$ is introduced for Frobenius norm, that is, $\|A\|_F = \sqrt{\text{tr} A^\top A}$.

Chapter 2

Preliminaries

In this chapter we introduce all the necessary tools and results that are needed for the development of the target tracking guidance algorithms and their implementations on a full scale nonlinear model of aircraft's dynamics.

2.1 Lyapunov Stability and Ultimate Boundedness of Nonautonomous Systems

For trajectory tracking problems, the error dynamics are non-autonomous, even if the nominal system is autonomous to begin with. Here, we present the basic definitions and theorems for stability analysis of such systems. Towards this end, we consider the following general non-autonomous system dynamics:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{x} \in \mathbb{R}^n, \quad (2.1)$$

where $\mathbf{f} : [t_0, \infty) \times \mathcal{D} \rightarrow \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in $\mathbf{x}$ on $[t_0, \infty) \times \mathcal{D}$, $\mathcal{D}$ is an open set containing the origin, which is the equilibrium point of the system at t_0 .

To simplify the presentation of basic results, we introduce comparison functions.

Definition 1 (*[30], p.135*) A continuous function $\alpha : [0, a) \rightarrow \mathbb{R}^+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It belongs to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \rightarrow \infty$ as $r \rightarrow \infty$.

Definition 2 (*[30], p.135*) A continuous function $\beta : [0, a) \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ belongs to class $\mathcal{KL}$ if $\beta(r, s)$ is class $\mathcal{K}$ with respect to r for each fixed s , and $\beta(r, s)$ is decreasing in s for each fixed r and $\beta(r, s) \rightarrow 0$ as $s \rightarrow \infty$.

The next lemma demonstrates the connection between positive definite functions and comparison functions.

Lemma 1 Let $V(\mathbf{x})$ be continuous, positive definite and decrescent in a ball $\mathcal{B}_r \subset \mathbb{R}^n$. Then, there exist locally Lipschitz, class $\mathcal{K}$ functions α_1 and α_2 on $[0, r)$ such that

$$\alpha_1(\|\mathbf{x}\|) \leq V(\mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|)$$

for all $\mathbf{x} \in \mathcal{B}_r$. Moreover, if $V(\mathbf{x})$ is defined on all of $\mathbb{R}^n$ and is radially unbounded, then there exist class $\mathcal{K}_\infty$ functions α_1 and α_2 such that the above inequality holds for all $\mathbf{x} \in \mathbb{R}^n$.

The next lemma establishes necessary and sufficient conditions for stability of equilibria of nonautonomous systems in terms of class $\mathcal{K}$ and $\mathcal{KL}$ functions.

Lemma 2 (*[30], p.136*) The origin of the system (2.1) is

1) uniformly stable if and only if there exists a class $\mathcal{K}$ function $\alpha(\cdot)$ and a constant $c > 0$, independent of t_0 , such that

$$\|\mathbf{x}(t)\| \leq \alpha(\|\mathbf{x}(t_0)\|), \quad \forall t \geq t_0 \geq 0 \text{ and } \|\mathbf{x}(t_0)\| < c. \quad (2.2)$$

2) uniformly asymptotically stable if and only if there exist a class $\mathcal{KL}$ function $\beta(\cdot, \cdot)$ and a constant $c > 0$, independent of t_0 , such that

$$\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}(t_0)\|, t - t_0), \quad \forall t \geq t_0 \geq 0 \text{ and } \|\mathbf{x}(t_0)\| < c. \quad (2.3)$$

3) globally uniformly asymptotically stable if it is uniformly asymptotically stable and (2.3) holds for any $\mathbf{x}(t_0)$.

Remark 1 If in the definition of uniform asymptotic stability $\beta(\cdot, \cdot)$ takes the form $\beta(r, s) = kre^{-\lambda s}$, then one recovers the definition of exponential stability.

Theorem 1 [Lyapunov's Stability Theorem for Nonautonomous Systems.] ([30], p.138) Consider the system (2.1) with an equilibrium at the origin.

1) Let $V : [0, \infty) \times \mathcal{D} \rightarrow \mathbb{R}$ be a continuously differentiable function such that

$$W_1(\mathbf{x}) \leq V(t, \mathbf{x}) \leq W_2(\mathbf{x}) \quad (2.4)$$

$$\frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial \mathbf{x}} \right)^\top \mathbf{f}(\mathbf{x}, t) \leq 0 \quad (2.5)$$

for all $t \geq 0$ and $\mathbf{x} \in \mathcal{D}$, where W_1 and W_2 are continuous and positive definite. Then, $\mathbf{x} = 0$ is uniformly stable.

2) If the assumption in (2.5) can be verified as

$$\frac{\partial V}{\partial t} + \left(\frac{\partial V}{\partial \mathbf{x}} \right)^\top \mathbf{f}(\mathbf{x}, t) \leq -W_3(\mathbf{x}), \quad (2.6)$$

where W_3 is a continuous and positive definite function in $\mathcal{D}$, then $\mathbf{x} = 0$ is uniformly asymptotically stable.

3) Moreover, letting $\mathcal{B}_r = \{\mathbf{x} \mid \|\mathbf{x}\| \leq r\} \subset \mathcal{D}$ and $c < \min_{\|\mathbf{x}\|=r} W_1(\mathbf{x})$, every trajectory starting in $\{\mathbf{x} \in \mathcal{B}_r \mid W_2(\mathbf{x}) \leq c\}$ satisfies

$$\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}(t_0)\|, t - t_0), \quad \forall t \geq t_0 \geq 0$$

for some class $\mathcal{KL}$ function β .

4) If $\mathcal{D} = \mathbb{R}^n$, and $W_1(\mathbf{x})$ is radially unbounded, then $\mathbf{x} = 0$ is globally uniformly asymptotically stable.

5) Finally, if $V : [0, \infty) \times \mathcal{D} \rightarrow \mathbb{R}$ can be selected to verify

$$k_1 \|\mathbf{x}\|^p \leq V(t, \mathbf{x}) \leq k_2 \|\mathbf{x}\|^p, \quad t \in [0, \infty), \quad \mathbf{x} \in \mathcal{D} \quad (2.7)$$

$$\dot{V}(t, \mathbf{x}) \leq -k_3 \|\mathbf{x}\|^p, \quad t \in [0, \infty), \quad \mathbf{x} \in \mathcal{D} \quad (2.8)$$

for some positive constants k_1, k_2, k_3, p , where the norm and the power are the same in (2.7), (2.8), then the origin is locally (uniformly) exponentially stable. If V is continuously differentiable for all $[0, \infty) \times \mathbb{R}$, and (2.7), (2.8) hold for all $[0, \infty) \times \mathbb{R}$, then the origin is globally (uniformly) exponentially stable.

Next, we introduce Barbalat's lemma that helps to establish asymptotic stability of nonautonomous systems in cases when the derivative of the Lyapunov function is only negative semi-definite.

Lemma 3 [Barbalat's Lemma.] ([30], p.192). *If the differentiable function $f(t)$ converges to a finite limit as $t \rightarrow \infty$ and if $\dot{f}(t)$ is uniformly continuous, then $\dot{f}(t) \rightarrow 0$ as $t \rightarrow \infty$.*

In some cases, it is easier to use the following corollary of Barbalat's lemma:

Corollary 1 ([54], p.19) *If $f(t), \dot{f}(t) \in \mathcal{L}_\infty$ and $f(t) \in \mathcal{L}_p$ for some $p \in [1, \infty)$, then $f(t) \rightarrow 0$ as $t \rightarrow \infty$.*

Next, we introduce notions of boundedness and ultimate boundedness of the system (2.1).

Definition 3 ([30], p.211) *The solutions of the nonlinear system (2.1) are*

- *uniformly bounded if there exists a positive constant γ , independent of t_0 , such that for every $\delta \in (0, \gamma)$, there is $\beta = \beta(\delta) > 0$, independent of t_0 , such that $\|x_0\| \leq \delta$ implies $\|x(t)\| \leq \beta$, $t \geq t_0$.*
- *globally uniformly bounded if for every $\delta \in (0, \infty)$, there is $\beta = \beta(\delta) > 0$, independent of t_0 , such that $\|x_0\| \leq \delta$ implies $\|x(t)\| \leq \beta$, $t \geq t_0$.*
- *uniformly ultimately bounded with ultimate bound $b > 0$ if there exists $\gamma > 0$ such that, for every $\delta \in (0, \gamma)$, there exists $T = T(\delta, b) > 0$ such that $\|x_0\| \leq \delta$ implies $\|x(t)\| \leq b$, $t \geq T$.*
- *globally uniformly ultimately bounded, if for every $\delta \in (0, \infty)$, there exists $T = T(\delta, b) > 0$ such that $\|x_0\| < \delta$ implies $\|x(t)\| < b$, $t \geq T$.*

We notice that an ultimately bounded system is not necessarily Lyapunov stable. Also, in the case of autonomous systems, the word “uniformly” can be dropped, since the solution depends only upon $t - t_0$. Here, we present a statement of sufficient conditions for uniform ultimate boundedness and ultimate boundedness.

Theorem 2 ([30], p.211) *Consider the nonlinear system (2.1). Let $\mathcal{D}$ be a domain that contains the origin and $V : [0, \infty) \times \mathcal{D} \rightarrow \mathbb{R}$ be a continuously differentiable function, $\alpha_1(\cdot)$*

and $\alpha_2(\cdot)$ be class $\mathcal{K}$ functions, and $W : \mathcal{D} \rightarrow \mathbb{R}$ be a positive-definite function such that:

$$\alpha_1(\|\mathbf{x}\|) \leq V(t, \mathbf{x}) \leq \alpha_2(\|\mathbf{x}\|), \quad \mathbf{x} \in \mathcal{D} \subset \mathbb{R}^n, \quad (2.9)$$

$$\dot{V}(t, \mathbf{x}) \leq -W(\mathbf{x}), \quad \mathbf{x} \in \mathcal{D} \subset \mathbb{R}^n, \quad \|\mathbf{x}\| > \mu, \quad (2.10)$$

where

$$\mu < \alpha_2^{-1}(\alpha_1(r)), \quad (2.11)$$

and r is the radius of the ball $\mathcal{B}_r = \{\mathbf{x} : \|\mathbf{x}\| \leq r\} \subset \mathcal{D}$. Then, there exists a class $\mathcal{KL}$ function β such that for every initial state $\mathbf{x}(t_0)$, satisfying $\|\mathbf{x}(t_0)\| \leq \alpha_2^{-1}(\alpha_1(r))$ there is $T \geq 0$ (dependent on $\mathbf{x}(t_0)$ and μ) such that the solution of (2.1) satisfies

$$\|\mathbf{x}(t)\| \leq \beta(\|\mathbf{x}(t_0)\|, t - t_0), \quad \forall t_0 \leq t \leq t_0 + T \quad (2.12)$$

$$\|\mathbf{x}(t)\| \leq \alpha_1^{-1}(\alpha_2(\mu)), \quad \forall t \geq t_0 + T \quad (2.13)$$

Moreover, if $\mathcal{D} = \mathbb{R}^n$ and α_1 belongs to class K_∞ , then (2.12) and (2.13) hold for any initial state $\mathbf{x}(t_0)$ no matter how large μ is, i.e. the results are global.

2.2 Adaptive Output Feedback Control

In this section, we present basic definitions and results from the output feedback adaptive control framework.

Definition 4 ([63], p.127) A transfer function $H(s)$ is called positive real, if $\operatorname{Re}[H(s)] \geq 0$ for all $\operatorname{Re}[s] \geq 0$. It is strictly positive real (SPR), if $H(s - \epsilon)$ is positive real for some $\epsilon > 0$.

Theorem 3 ([63], p.128) A transfer function $H(s)$ is SPR, if and only if

- $H(s)$ is strictly stable transfer function;
- The real part of $H(s)$ is strictly positive along $j\omega$ axis, i.e.

$$\forall \omega \geq 0 \quad \operatorname{Re}[H(j\omega)] > 0.$$

From the theorem above, it follows that the system can not be SPR if it has a relative degree higher than one. For the SPR system, the following lemma is crucial in output feedback adaptive control design.

Lemma 4 [*Kalman-Yakubovich lemma.*] ([63], p.130) Consider a controllable linear time-invariant system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t).\end{aligned}$$

The transfer function $H(s) = C[s\mathbb{I} - A]^{-1}B$ is SPR if and only if there exist positive definite matrices P and Q such that

$$A^\top P + PA = -Q, \quad PB = C^\top.$$

Adaptive output feedback control with a global asymptotic stability proof can be done only if the transfer function of the error dynamics is SPR. The key result, that enables us to write adaptive laws for the systems with error dynamics being SPR and uncertainties being dependent only upon measured signals, is given by the following lemma.

Lemma 5 Consider the following system

$$\begin{aligned}\dot{e}(t) &= Ae(t) + b\lambda\tilde{\theta}^\top(t)v(t) \\ \tilde{y}(t) &= c^\top e(t),\end{aligned}\tag{2.14}$$

where $\tilde{y} \in \mathbb{R}$ is the only scalar measurable output signal ($e \in \mathbb{R}^n$ is not fully measurable), $H(s) = c^\top (s\mathbb{I} - A)^{-1}b$ is a strictly positive real transfer function, λ is an unknown constant with known sign, $\tilde{\theta}(t)$ is a $m \times 1$ vector function of time (usually modeling the parametric errors), and $v(t)$ is measurable $m \times 1$ vector. If the vector $\tilde{\theta}(t)$ varies according to

$$\dot{\tilde{\theta}}(t) = -\text{sgn}(\lambda)\gamma\tilde{y}(t)v(t) \quad (2.15)$$

with $\gamma > 0$ being a positive constant for adaptation rate, then $\tilde{y}(t)$ and $\tilde{\theta}(t)$ are globally bounded. Furthermore, if $v(t)$ is bounded, then

$$\tilde{y}(t) \rightarrow 0 \quad \text{as} \quad t \rightarrow \infty. \quad (2.16)$$

For the systems with a relative degree higher than one (is not SPR), an input filtered state transformation can be used. This transformation was introduced for the linear systems by Kreisselmeier [32] and for nonlinear systems by Marino and Tomei [39]. Here, we present a version of it given in [33] (p.348). Consider a single-input single output system

$$\begin{aligned} \dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + \phi(y) + \Phi(y)\mathbf{a} + \begin{bmatrix} \mathbf{0} \\ \mathbf{b} \end{bmatrix} \sigma(y)u \\ y(t) &= C\mathbf{x}(t), \end{aligned} \quad (2.17)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state of the system, $\phi(y) \in \mathbb{R}^n$, $\Phi(y) \in \mathbb{R}^{n \times q}$ and $\sigma(y) \in \mathbb{R}$ are known smooth functions of output y , $\sigma(y) \neq 0$, $\forall y \in \mathbb{R}$, $\mathbf{a} \in \mathbb{R}^q$ and $\mathbf{b} \in \mathbb{R}^{m+1}$ are unknown constant parameters with $B(s) = b_ms^m + \dots + b_1s + b_0$ being a Hurwitz polynomial, the sign of b_m is known and

$$A = \begin{bmatrix} 0 & I_{n-1} \\ 0 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 & 0_{1 \times n-1} \end{bmatrix}.$$

The following transformation (input filtered)

$$\mathbf{z} = \mathbf{x} - \begin{bmatrix} \mathbf{0} \\ \boldsymbol{\xi} + \Omega\boldsymbol{\theta} \end{bmatrix}, \quad \boldsymbol{\theta} = \begin{bmatrix} \mathbf{b} \\ \mathbf{a} \end{bmatrix} \in \mathbb{R}^p, \quad (2.18)$$

where $p = q + m + 1$, the vector $\boldsymbol{\xi} \in \mathbb{R}^{n-1}$ is given by the filter equation

$$\dot{\boldsymbol{\xi}} = A_l \boldsymbol{\xi} + B_l \boldsymbol{\phi}(y), \quad (2.19)$$

the matrix $\Omega \in \mathbb{R}^{(n-1) \times p}$ is given by the equation

$$\Omega = \begin{bmatrix} \mathbf{v}_m & \dots & \mathbf{v}_1 & \mathbf{v}_0 & \Xi \end{bmatrix}, \quad (2.20)$$

where $\mathbf{v}_j = A_l^j \boldsymbol{\eta}$, $j = 0, \dots, m$ and the vectors $\boldsymbol{\eta} \in \mathbb{R}^{n-1}$ and the matrix $\Xi \in \mathbb{R}^{(n-1) \times q}$ are defined by the filter equations

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= A_l \boldsymbol{\eta} + \mathbf{e}_{n-1} \sigma(y) u \\ \dot{\Xi} &= A_l \Xi + B_l \Phi(y), \end{aligned} \quad (2.21)$$

translates the system in (2.17) to the form

$$\begin{aligned} \dot{\mathbf{z}}(t) &= A \mathbf{z}(t) + l(\omega_0 + \boldsymbol{\omega}^\top \boldsymbol{\theta}) \\ y(t) &= C \mathbf{z}(t), \end{aligned} \quad (2.22)$$

where

$$\begin{aligned} \omega_0 &= \phi_1 + \xi_1 \\ \boldsymbol{\omega}_1 &= \begin{bmatrix} 0_{1 \times (m+1)} & \Phi_1 \end{bmatrix} + \Omega_1, \end{aligned} \quad (2.23)$$

ϕ_1 and ξ_1 are the first elements of the vectors $\boldsymbol{\phi}(y)$ and $\boldsymbol{\xi}$ respectively, Φ_1 and Ω_1 are the first rows of matrices $\Phi(y)$ and Ω respectively. In the above equations $\mathbf{e}_{n-1}$ is the $(n-1)$ -st coordinate vector, and matrices A_l and B_l are defined as

$$A_l = \begin{bmatrix} -\bar{l} & I_{n-2} \\ 0_{1 \times n-2} \end{bmatrix}, \quad B_l = \begin{bmatrix} -\bar{l} & I_{n-1} \end{bmatrix}, \quad \bar{l} = \begin{bmatrix} l_1 \\ \dots \\ l_{n-1} \end{bmatrix}, \quad l = \begin{bmatrix} 1 \\ \bar{l} \end{bmatrix},$$

where $\bar{l}$ is chosen such that the polynomial $L(s) = s^{n-1} + l_1 s^{n-2} + \dots + l_{n-1}$ is Hurwitz. The system in (2.22) is minimum phase, and its relative degree from the input $\omega_0 + \omega^\top \theta$ to the output y is one. It can be made SPR by the output injection or output feedback when designing an observer or a controller respectively.

2.3 Intelligent Excitation

The control problems considered in this dissertation involve unknown parameters in the reference commands of interest to tracked. Therefore, the tracking objective cannot be achieved without parameter convergence. In [10] Intelligent Excitation technique is introduced that guarantees parameter convergence, simultaneously keeping the excited reference commands in a small neighborhood of true ones, as opposed to the commonly used persistent excitation technique (see for example [42, 54, 63] for details). Here, we recall the definition and properties of this technique.

Consider the following single-input single-output system dynamics:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + b\lambda u(t) \\ y(t) &= c^\top x(t), \end{aligned} \tag{2.24}$$

where $x \in \mathbb{R}^n$ is the system state vector (measurable), $u \in \mathbb{R}$ is the control signal, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$ are known constant vectors, A is an unknown $n \times n$ matrix, $\lambda \in \mathbb{R}$ is an unknown constant with known sign, $y \in \mathbb{R}$ is the regulated output. The control objective is to regulate the output y to track $r(A, \lambda)$, where r is a known map $r : \mathbb{R}^{n \times n} \times \mathbb{R} \rightarrow \mathbb{R}$, dependent upon the unknown parameters, A and λ , of the system. When the classical model matching assumption is satisfied, that is there exist a Hurwitz matrix $A_m \in \mathbb{R}^{n \times n}$, a column vector $b_m \in \mathbb{R}^n$, ideal parameters $\theta_x^* \in \mathbb{R}^n$, $\theta_r^* \in \mathbb{R}$ such that (A_m, b_m) is controllable, $A_m - A = \lambda b \theta_x^{*\top}$, $\lambda b \theta_r^* = b_m$,

then the conventional adaptive model reference feedback/feedforward control signal

$$u(t) = \theta_x^\top(t)x(t) + \theta_r(t)k_g\hat{r}(t), \quad (2.25)$$

where

$$k_g = \lim_{s \rightarrow 0} \frac{1}{c^\top (sI - A_m)^{-1} b_m},$$

$\theta_x(t), \theta_r(t)$ are the estimates of the ideal parameters θ_x^*, θ_r^* , updated online according to adaptive laws

$$\begin{aligned} \dot{\theta}_x(t) &= \Gamma_x \text{Proj}(\theta_x(t), -x(t)e^\top(t)Pb \text{sgn}(\lambda)) \\ \dot{\theta}_r(t) &= \Gamma_r \text{Proj}(\theta_r(t), -k_g\hat{r}(t)e^\top(t)Pb \text{sgn}(\lambda)), \end{aligned} \quad (2.26)$$

where Γ_x and Γ_r are positive design gains and $\text{Proj}(\cdot, \cdot)$ is the projection operator (Appendix A), guarantees asymptotic tracking of the estimated reference command $\hat{r}(t) = r(\hat{A}(t), \hat{\lambda}(t))$. The corresponding Lyapunov function is defined as

$$V(e(t), \tilde{\theta}_x(t), \tilde{\theta}_r(t)) = e^\top(t)Pe(t) + |\lambda|\tilde{\theta}_x^\top(t)\Gamma_x^{-1}\tilde{\theta}_x(t) + \Gamma_r^{-1}|\lambda|\tilde{\theta}_r^2(t). \quad (2.27)$$

and has a derivative $\dot{V}(t) \leq -e^\top(t)Qe(t) \leq 0, \quad t \geq 0$.

The Intelligent excitation technique redefines the reference command to be tracked as follows:

$$\begin{aligned} \hat{r}(t) &= r(\hat{A}(\theta(t)), \hat{\lambda}(\theta(t))) + E_x(t) \\ E_x(t) &= k(t)e_x(t - jT), \quad \text{if } t \in [jT, (j+1)T), \quad j = 0, 1, 2, \dots \\ k(t) &= \begin{cases} k_0, & t \in [0, T) \\ \min \left\{ \Gamma_1 \int_{(j-1)T}^{jT} e^\top(\tau)Qe(\tau)d\tau, \Gamma_2 - \Gamma_3 \right\} + \Gamma_3, & t \in [jT, (j+1)T), \quad j \geq 1 \end{cases} \\ e_x(t) &= \sum_{i=1}^m \sin(\omega_i t), \quad t \in [0, T], \end{aligned} \quad (2.28)$$

where $\Gamma_1, \Gamma_2, \Gamma_3$ are positive design gains, k_0 is arbitrary value from $[\Gamma_3, \Gamma_2]$, T is the first

time instant for which $e_x(T) = 0$ and $\omega_1, \dots, \omega_m$ ensure that the matrix

$$\Omega_{pq} = \begin{cases} \operatorname{Re}(H_p(j\omega_{\lceil \frac{q}{2} \rceil})) & \text{if } q \text{ is odd} \\ \operatorname{Im}(H_p(j\omega_{\lceil \frac{q}{2} \rceil})) & \text{if } q \text{ is even} \end{cases}, \quad (2.29)$$

has full row rank, where $\lceil \frac{q}{2} \rceil$ denotes the smallest integer that is greater than $\frac{q}{2}$ and $H_p(s) = [1 \ ((sI - A_m)^{-1}b_mk_g)^\top]^\top$.

The main property of this technique is given by the following theorem.

Theorem 4 [10] *For the system in (2.24) and the adaptive controller with intelligent excitation in (2.25), (2.26), (2.28), there exists a finite $T_s > 0$ such that*

$$|y(t) - r(A, \lambda)| \leq \gamma(\Gamma_1, \Gamma_3, \bar{\epsilon}), \quad t \geq T_s. \quad (2.30)$$

where $\epsilon > 0$ is an arbitrary constant and

$$\lim_{\Gamma_1 \rightarrow \infty, \Gamma_3 \rightarrow 0, \bar{\epsilon} \rightarrow 0} \gamma(\Gamma_1, \Gamma_3, \bar{\epsilon}) = 0. \quad (2.31)$$

For practical implementation, due to the presence of noise and transient errors, it can be chosen $\Gamma_3 = 0$. The constant gain Γ_1 is inverse proportional to the bound of the parameter tracking error, so setting it large will increase the accuracy of parameter estimates. The gain Γ_2 is the amplitude of the excitation signal, which controls the rate of convergence.

2.4 Neural Network Approximation

In this section, we present some basic definitions and results from approximation theory that are used for approximation of uncertain nonlinear functions in flight dynamics.

Definition 5 *The family $\mathcal{F}$ of functions $f : \mathcal{U} \rightarrow \mathbb{R}$, where $\mathcal{U}$ is a metric space*

- separates points of $\mathcal{U}$ if for any pair $u_1 \neq u_2$ of $\mathcal{U}$ there exists a function $f \in \mathcal{F}$ such that $f(u_1) \neq f(u_2)$.
- vanishes at no point of $\mathcal{U}$ if for any point $u \in \mathcal{U}$ there exists an f such that $f(u) \neq 0$.

Definition 6 The set $\bar{\mathcal{A}}$ is the closure of $\mathcal{A}$, if it contains all the limit points of every convergent sequence composed of the points of $\mathcal{A}$.

Definition 7 The set $\mathcal{A}$ is dense in the set $\mathcal{G}$, if $\mathcal{G} \subseteq \bar{\mathcal{A}}$.

Let $\mathcal{U}$ be a metric space, and $\mathcal{C}[\mathcal{U}, \mathbb{R}]$ denote the set of continuous functions, mapping the metric space $\mathcal{U}$ to the real line $\mathbb{R}$.

Theorem 5 [51, p. 212] (**Stone-Weierstrass**) Let $\mathcal{U}$ be a compact metric space and $\mathcal{F}$ be a subalgebra of $\mathcal{C}[\mathcal{U}, \mathbb{R}]$. If $\mathcal{F}$ separates points of $\mathcal{U}$ and does not vanish at any point of $\mathcal{U}$, then $\mathcal{F}$ is dense in $\mathcal{C}[\mathcal{U}, \mathbb{R}]$.

Define S_K to be the collection of radial basis functions $q : \mathbb{R} \rightarrow \mathbb{R}^n$ represented by [45]

$$q(\mathbf{x}) = \sum_{i=1}^N w_i K\left(\frac{\mathbf{x} - \mathbf{z}_i}{\sigma}\right), \quad (2.32)$$

where N is a positive integer, $\sigma > 0$, $w_i \in \mathbb{R}$, and $\mathbf{z}_i \in \mathbb{R}^n$. The following theorem is taken from [45].

Theorem 6 Let $K : \mathbb{R}^n \rightarrow \mathbb{R}$ be an integrable bounded function such that K is continuous almost everywhere and $\int_{\mathbb{R}^n} K(\mathbf{x}) d\mathbf{x} \neq 0$. Then the family S_K is dense in $\mathcal{L}^p(\mathbb{R}^n)$ for every $p \in [1, \infty)$.

Since the family of continuous functions with compact support on $\mathbb{R}^n$ is dense in $\mathcal{L}^p(\mathbb{R}^n)$ [52] (p.69), the following theorem can be proven (see [45] and [46]):

Theorem 7 *Let $\mathcal{U} \in \mathbb{R}^n$ be a compact set. The family K_S of functions corresponding to single layer radial basis function networks (RBF) is dense in $\mathcal{C}[\mathcal{U}, \mathbb{R}]$.*

The above theorems make a basis for the universal approximation theorem, which for the case of linear parametrization is stated as follows:

Theorem 8 [22] *Given arbitrary $\varepsilon^* > 0$ and arbitrary compact set $\mathbf{x} \in \Omega \subset \mathbb{R}^n$ there exists a number m such that for arbitrary continuous function $\mathbf{f} : \Omega \rightarrow \mathbb{R}^k$ the following representation is true:*

$$\mathbf{f}(\mathbf{x}) = W^\top \Phi(\mathbf{x}) + \boldsymbol{\varepsilon}(\mathbf{x}), \quad \|\boldsymbol{\varepsilon}(\mathbf{x})\| < \varepsilon^*, \quad (2.33)$$

where W is a $(m \times k)$ -dimensional matrix of unknown constants, $\Phi(\mathbf{x})$ is a $(m \times 1)$ -dimensional vector of radial basis functions (RBF), $\boldsymbol{\varepsilon}(\mathbf{x})$ is uniformly bounded approximation error.

One choice for such functions is the Gaussian:

$$\Phi_i(x) = e^{-\|\mathbf{x} - \mathbf{x}_c\|^2 / 2\sigma^2}, \quad (2.34)$$

where $\mathbf{x}_c$ denotes the vector of centers, while σ denotes the width. It has been proven in [5] that $\varepsilon^* \simeq O\left(\frac{1}{\sqrt{m}}\right)$.

2.5 Differential Equations with Discontinuous Right Hand Sides

The robust adaptive algorithm developed in the sequel to reject time varying bounded disturbances uses an adaptive bounding technique similar to [53] that results in a discontinuous control signal. Therefore, the associated dynamic equations have discontinuous right hand sides, the solutions to which must be understood in Filippov's sense. Here, we introduce basic definitions and theorems for the analysis of such systems.

We recall Filippov solutions for differential equations with discontinuous right-hand sides. Consider the system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{x}(t)), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \quad (2.35)$$

where $\mathbf{f} : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is measurable and essentially locally bounded. Filippov solution is defined as follows.

Definition 8 [61] *A vector function $\mathbf{x}(\cdot)$ is called a solution of (2.35) on $[t_0, t_1]$, if $\mathbf{x}(\cdot)$ is absolutely continuous on $[t_0, t_1]$, and for almost all $t \in [t_0, t_1]$*

$$\dot{\mathbf{x}}(t) \in K[\mathbf{f}](t, \mathbf{x}), \quad (2.36)$$

where

$$K[\mathbf{f}](t, \mathbf{x}) = \bigcap_{\delta > 0} \bigcap_{\mu N = 0} \bar{co} \mathbf{f}(t, B(\mathbf{x}, \delta) - N), \quad (2.37)$$

and $B(\mathbf{x}, \delta)$ denotes a ball of radius δ centered at $\mathbf{x}$, $\bar{co}$ is the convex hull, while $\bigcap_{\mu N = 0}$ denotes the intersection over all sets N of Lebesgue measure zero. $\square$

It has been proven in [48] that the class of adaptive systems that involve switching functions of state, like sgn function, in the right-hand side, satisfy the conditions of existence

and uniqueness of solutions in Filippov sense. Moreover, away from the switching surface, the existence of a unique solution is proven in regular sense. Existence and uniqueness of solutions in Filippov sense is proven also in [53] for the first order adaptive systems defined by dynamics involving sgn functions. Admittedly, the structural form considered in [53] is quite specific. It remains to be proven that the governing equations discussed in this dissertation satisfy the structural assumptions explicitly defined in [48]. Given the success of the analysis in studies such as [53], we leave this proof for future work. That is, throughout this dissertation, it will be assumed that all feedback forms generate Filippov solutions. We will focus our attention on control synthesis and stability of the resultant systems.

Lyapunov stability theory and the related theorems have been extended to systems with Filippov solutions in [61] using the concept of Clarke's generalized gradient.

Definition 9 [16] *For a locally Lipschitz function $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ define the generalized gradient of V at $(t, \mathbf{x})$ by*

$$\partial V(t, \mathbf{x}) = \bar{co} \left\{ \lim_{(t_i, \mathbf{x}_i) \rightarrow (t, \mathbf{x})} \nabla V(t_i, \mathbf{x}_i) \mid (t_i, \mathbf{x}_i) \notin \Omega_V \right\} \quad (2.38)$$

where Ω_V is the set of measure zero where the gradient of V is not defined. $\square$

Next, we recall two more definitions on the generalized directional derivative and the regular function.

Definition 10 [16] *The generalized directional derivative at the point $\mathbf{x}$ in the direction $\mathbf{v}$ is defined as*

$$f^0(\mathbf{x}; \mathbf{v}) = \lim_{\mathbf{y} \rightarrow \mathbf{x}} \sup_{h \downarrow 0} \frac{f(\mathbf{y} + h\mathbf{v}) - f(\mathbf{y})}{h}. \quad (2.39)$$

$\square$

Note that the definition of the Clarke's generalized directional derivative differs only subtly from the usual Gateaux or Frechet derivative. The difference is that the Clarke's derivative is defined as the "limit to the base point", $\mathbf{y} \rightarrow \mathbf{x}$ is taken. In the conventional Gateaux derivative the base point $\mathbf{x}$ is held fixed in the limiting process. In other words,

$$f'(\mathbf{x}; \mathbf{v}) = \lim_{h \downarrow 0} \frac{f(\mathbf{x} + h\mathbf{v}) - f(\mathbf{x})}{h}. \quad (2.40)$$

Definition 11 [16] *$f(t, \mathbf{x}) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is called regular if*

- 1) *for all $\mathbf{v}$, the usual one-sided directional derivative $f'(\mathbf{x}; \mathbf{v})$ exists;*
- 2) *for all $\mathbf{v}$, $f'(\mathbf{x}; \mathbf{v}) = f^0(\mathbf{x}; \mathbf{v})$.* □

Smooth functions are regular functions according to this definition.

Theorem 9 [16] *(Chain Rule) Let $\mathbf{x}(\cdot)$ be a Filippov solution to $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$ on an interval containing t^* and $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ be Lipschitz continuous and, in addition, a regular function. Then, $V(t, \mathbf{x})$ is absolutely continuous, $(d/dt)V(t^*, \mathbf{x})$ exists almost everywhere and*

$$\frac{d}{dt}V(t^*, \mathbf{x}) \in^{a.e.} \dot{V}(t^*, \mathbf{x}), \quad (2.41)$$

where

$$\dot{V}(t^*, \mathbf{x}) = \bigcap_{\xi \in \partial V(t^*, \mathbf{x})} \xi^\top \begin{bmatrix} K[\mathbf{f}](t^*, \mathbf{x}) \\ 1 \end{bmatrix}. \quad (2.42)$$

□

Now, we recall the Lyapunov stability theorem due to Shevitz and Paden for systems with discontinuous right-hand sides.

Theorem 10 [61] Let $\mathbf{f}(t, \mathbf{x})$ be essentially locally bounded and $0 \in K[\mathbf{f}](t, \mathbf{0})$ in a domain $\Omega \supset \{t \mid t_0 \leq t < \infty\} \times \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| < r\}$. Also, let $V : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a regular function satisfying $V(t, \mathbf{0}) = \mathbf{0}$ and $0 < V_1(\|\mathbf{x}\|) \leq V(t, \mathbf{x}) \leq V_2(\|\mathbf{x}\|)$ for $\mathbf{x} \neq \mathbf{0}$ in Ω for some class $\mathcal{K}$ functions V_1, V_2 .

- 1) Then $\dot{V}(t, \mathbf{x}) \leq 0$ in Ω implies that $\mathbf{x}(t) \equiv \mathbf{0}$ is a uniformly stable solution of (2.35).
- 2) If in addition, there exists a class $\mathcal{K}$ function $\omega(\cdot)$ in Ω with the property $\dot{V}(t, \mathbf{x}) \leq -\omega(\mathbf{x}) < 0$, then the solution $\mathbf{x}(t) \equiv \mathbf{0}$ is uniformly asymptotically stable. $\square$

Chapter 3

Problem Formulation

In this chapter we introduce the basic assumptions for the vision based target tracking problem, give the equations of relative motion of two aerial vehicles, relate the relative position vector to the visual measurements and formulate the objectives.

3.1 Relative Dynamics

Equations of the motion of a flying target can be given by:

$$\begin{aligned}\dot{\mathbf{R}}_{\text{T}}(t) &= \mathbf{V}_{\text{T}}(t), & \mathbf{R}_{\text{T}}(0) &= \mathbf{R}_{\text{T}_0} \\ \dot{\mathbf{V}}_{\text{T}}(t) &= \mathbf{a}_{\text{T}}(t), & \mathbf{V}_{\text{T}}(0) &= \mathbf{V}_{\text{T}_0},\end{aligned}\tag{3.1}$$

where $\mathbf{R}_{\text{T}}(t) = [x_{\text{T}}(t) \ y_{\text{T}}(t) \ z_{\text{T}}(t)]^{\top}$, $\mathbf{V}_{\text{T}}(t) = [V_{\text{T}_x}(t) \ V_{\text{T}_y}(t) \ V_{\text{T}_z}(t)]^{\top}$ and $\mathbf{a}_{\text{T}}(t) = [a_{\text{T}_x}(t) \ a_{\text{T}_y}(t) \ a_{\text{T}_z}(t)]^{\top}$ are respectively the position, velocity and acceleration vectors of the target's center of gravity in the inertial frame $F_E = (x_E, y_E, z_E)$ chosen according to the flat and non-rotating Earth convention adopted in the flight dynamics, that is the origin of F_E

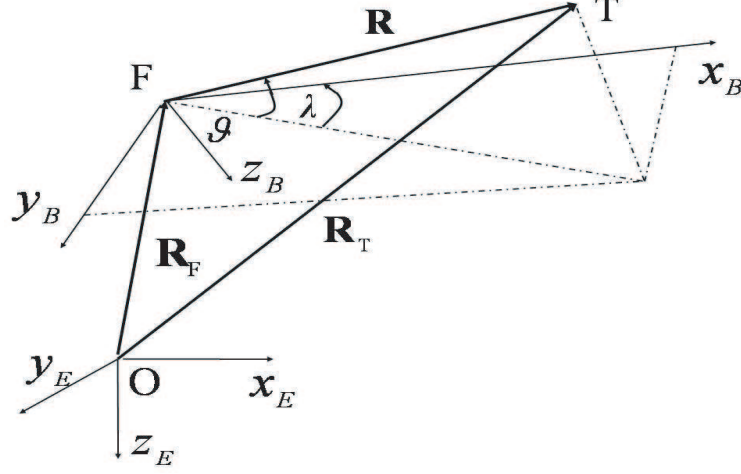


Figure 3.1: Coordinate frames and angles definition

is fixed to an arbitrary point on the surface of the Earth, x_E axis points due North, y_E axis points due East and z_E axis points towards the center of the Earth. The follower's dynamics are given by

$$\begin{aligned}\dot{\mathbf{R}}_F(t) &= \mathbf{V}_F(t), & \mathbf{R}_F(0) &= \mathbf{R}_{F_0} \\ \dot{\mathbf{V}}_F(t) &= \mathbf{a}_F(t), & \mathbf{V}_F(0) &= \mathbf{V}_{F_0},\end{aligned}\tag{3.2}$$

where $\mathbf{R}_F(t) = [x_F(t) \ y_F(t) \ z_F(t)]^\top$, $\mathbf{V}_F(t) = [V_{F_x}(t) \ V_{F_y}(t) \ V_{F_z}(t)]^\top$ and $\mathbf{a}_F(t) = [a_{F_x}(t) \ a_{F_y}(t) \ a_{F_z}(t)]^\top$ are respectively the follower's position, velocity and acceleration vectors in the same inertial frame.

We assume that the follower can measure its own states and can get visual information about the target via a camera that is fixed on the follower with its optical axis parallel to the follower's longitudinal x_B -axis of the body frame $F_B = (x_B, y_B, z_B)$, and its optical center fixed at the follower's center of gravity. Then the body frame can be chosen to be aligned with the camera frame (Fig. 3.1). This assumption is for the simplification of notations and is not restrictive, since otherwise a fixed coordinate transformation from the camera frame

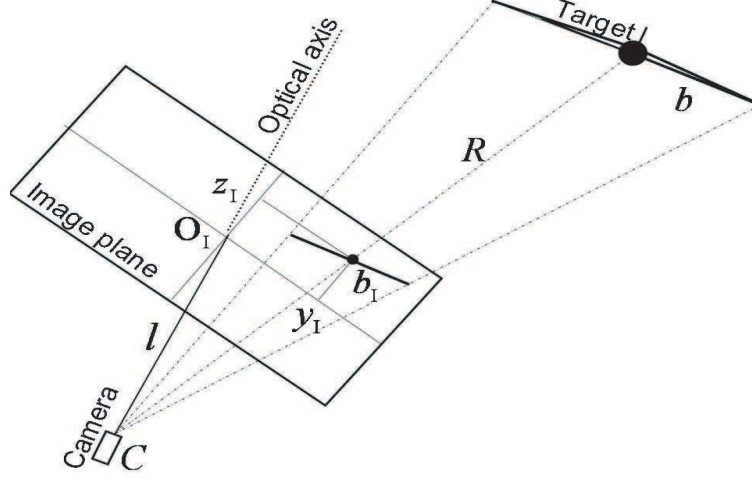


Figure 3.2: Camera frame and measurements

to the body frame can be incorporated. It is assumed that the image processing algorithm associated with the visual sensor provides three measurements in real time. These are the pixel coordinates of the image centroid (y_I, z_I) in the image plane I and the image maximum length b_I in pixels (Fig. 3.2). Assuming that the camera focal length l is known, the bearing angle λ and the elevation angle ϑ (see Fig. 3.1) can be expressed via the measurements through the geometric relationships

$$\begin{aligned}\tan \lambda &= \frac{y_I}{l}, \\ \tan \vartheta &= \frac{z_I}{\sqrt{l^2 + y_I^2}}.\end{aligned}\tag{3.3}$$

The target's relative position with respect to the follower is given by the inertial vector (see Fig. 3.1)

$$\mathbf{R} = \mathbf{R}_T - \mathbf{R}_F.\tag{3.4}$$

Hence, the relative dynamics in the inertial frame is given by the equation

$$\begin{aligned}\dot{\mathbf{R}}(t) &= \mathbf{V}(t) \triangleq \mathbf{V}_T(t) - \mathbf{V}_F(t), \quad \mathbf{R}(0) = \mathbf{R}_0 \\ \dot{\mathbf{V}}(t) &= \mathbf{a}(t) \triangleq \mathbf{a}_T(t) - \mathbf{a}_F(t), \quad \mathbf{V}(0) = \mathbf{V}_0,\end{aligned}\tag{3.5}$$

where the initial conditions are given by $\mathbf{R}_0 = \mathbf{R}_{T_0} - \mathbf{R}_{F_0}$ and $\mathbf{V}_0 = \mathbf{V}_{T_0} - \mathbf{V}_{F_0}$.

3.2 Measurement Transformation

We notice that the relative position $\mathbf{R}(t)$, the relative velocity $\mathbf{V}(t)$ and the relative acceleration $\mathbf{a}(t)$ are not measurable, since no information about the target's motion is assumed except for the visual measurements y_I , z_I , b_I . However, the relative position $\mathbf{R}$ can be related to the measurements y_I , z_I , b_I as follows. From the geometric considerations, we observe that the relative range $R(t) = \|\mathbf{R}(t)\|$ can be expressed as

$$R = \frac{b}{b_I} \sqrt{l^2 + y_I^2 + z_I^2}, \quad (3.6)$$

where $b > 0$ is the maximum size of the target that is assumed to be a constant. Introducing a notation

$$a_I = \frac{1}{b_I} \sqrt{l^2 + y_I^2 + z_I^2}, \quad (3.7)$$

the relative position can be expressed in the follower's body frame as follows

$$\mathbf{R}^B = R \mathbf{T}_{IB} = b a_I \mathbf{T}_{IB}, \quad (3.8)$$

where the vector $\mathbf{T}_{IB}$ is defined as

$$\mathbf{T}_{IB} = \begin{bmatrix} \cos \vartheta \cos \lambda \\ \cos \vartheta \sin \lambda \\ -\sin \vartheta \end{bmatrix}. \quad (3.9)$$

Using the coordinate transformation matrix $L_{B/E}$ from the inertial frame F_E to the follower's body frame F_B is given by:

$$L_{B/E} = \begin{bmatrix} \cos \theta \cos \psi & \cos \theta \sin \psi & -\sin \theta \\ \sin \phi \sin \theta \cos \psi - \cos \phi \sin \psi & \sin \phi \sin \theta \sin \psi + \cos \phi \cos \psi & \sin \phi \cos \theta \\ \cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi & \cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi & \cos \phi \cos \theta \end{bmatrix},$$

where ϕ , η , ψ are the Euler angles associated with the frame F_B [20] (p. 313), the inertial relative position vector $\mathbf{R}$ can be written as

$$\mathbf{R} = L_{B/E}^\top \mathbf{R}^B = ba_I L_{B/E}^\top \mathbf{T}_{IB}. \quad (3.10)$$

In this form, the relative position is computable up to the unknown scaling factor b . Therefore, we introduce the scaled relative position vector

$$\mathbf{r}(t) = \frac{\mathbf{R}(t)}{b},$$

the dynamics of which are written as

$$\begin{aligned} \dot{\mathbf{r}}(t) &= \mathbf{v}(t) \triangleq \frac{\mathbf{V}(t)}{b}, \quad \mathbf{r}(0) = \mathbf{r}_0, \\ \dot{\mathbf{v}}(t) &= \frac{\mathbf{a}_T(t) - \mathbf{a}_F(t)}{b}, \quad \mathbf{v}(0) = \mathbf{v}_0, \end{aligned} \quad (3.11)$$

where $\mathbf{r}_0 = \frac{\mathbf{R}_0}{b}$ and $\mathbf{v}_0 = \frac{\mathbf{V}_0}{b}$. The vector $\mathbf{r}$ is related to the visual measurements via the following algebraic equation

$$\mathbf{r} = a_I L_{B/E}^\top \mathbf{T}_{IB}, \quad (3.12)$$

and hence, is available for feedback. However, the system in (3.11) still involves the unmeasurable signal $\mathbf{v}(t)$ and the unknown acceleration $\mathbf{a}_T(t)$. In addition, the unknown constant parameter b appears in it.

3.3 Objectives

The follower's objective is to maintain a desired relative position with respect to the target, which can be given either in the inertial frame F_E in the form of the vector command $\mathbf{R}_c(t)$ or in the follower's body frame in the form of relative range command $R_c(t)$ and the bearing and

elevation commands $\lambda_c(t)$ and $\vartheta_c(t)$ respectively. In the latter case the reference commands need to be translated to the inertial frame as follows

$$\mathbf{R}_c(t) = R_c(t) L_{B/E}^\top \mathbf{T}_{IBc}, \quad (3.13)$$

where

$$\mathbf{T}_{IBc} = \begin{bmatrix} \cos \vartheta_c \cos \lambda_c \\ \cos \vartheta_c \sin \lambda_c \\ -\sin \vartheta_c \end{bmatrix}. \quad (3.14)$$

In general, the reference commands are functions of time that are assumed to be bounded and smooth. In particular, for the formation flight we set $\mathbf{R}_c = \text{constant}$. For the target interception problem the relative range command is set to zero: $R_c = \|\mathbf{R}_c\| = 0$. In any case these commands must be scaled by the unknown parameter b to match the scaled relative dynamics in (3.11).

The follower's objective is split in two steps: first, to design a guidance law in the form of inertial velocity command or inertial acceleration command, and second, to design the actual control surface deflections $\mathbf{u} = [\delta_T \ \delta_e \ \delta_a \ \delta_r]^\top$ for the follower in order to implement the guidance law from the first step. Here, as usual, δ_T , δ_e , δ_a and δ_r are respectively the throttle, elevator, aileron and rudder deflections of the follower. To this end we formulate two guidance problems. The first one is formulated for the system

$$\dot{\mathbf{r}}(t) = \frac{\mathbf{V}_T(t) - \mathbf{V}_F(t)}{b}, \quad \mathbf{r}(0) = \mathbf{r}_0, \quad (3.15)$$

viewing the followers velocity $\mathbf{V}_F(t)$ as a control input and the target's velocity $\mathbf{V}_T(t)$ as an unknown time varying disturbance.

Problem 1. Design a control input $\mathbf{V}_F(t)$ such that the scaled relative position vector $\mathbf{r}(t)$ tracks the reference command $\mathbf{r}_c(t)$ regardless of the realization of the target's velocity

$\mathbf{V}_T(t)$, which is assumed to be bounded with the bounded derivative, and the change in magnitude of which has a bounded integral.

From the control theoretical point of view, this problem can be considered in the state feedback framework for a multi-input multi-output linear system with positive but unknown high frequency gain in each control channel and unknown disturbances. In addition, the reference command $\mathbf{r}_c(t)$ depends upon unknown parameter b . The solution to this problem is the inertial velocity command for the follower that guarantees an asymptotic tracking. The advantage of this approach is that the system is of lower order and the control algorithm presented below generates smooth velocity command. The disadvantage is that it is applicable to the restricted class of target's motions (see Assumption 1 in Section 4.1). Also, implementation of the velocity commands for the full scale non-linear dynamic model involves the commands' derivatives that are not available and have to be computed numerically.

The second problem is formulated for the scaled relative dynamics in (3.11), viewing the followers acceleration $\mathbf{a}_F(t)$ as a control input and the target's acceleration $\mathbf{a}_T(t)$ as an unknown time varying disturbance.

Problem 2. Design a control input $\mathbf{a}_F(t)$ such that the scaled relative position vector $\mathbf{r}(t)$ tracks the reference command $\mathbf{r}_c(t)$ regardless of the realization of target acceleration $\mathbf{a}_T(t)$, assumed to be bounded. $\square$

This problem has the following specifications. The scaled relative position $\mathbf{r}(t)$ is available for feedback; the scaled relative velocity $\mathbf{v}(t)$ is not available; the high frequency gain is positive, but otherwise unknown parameter; the dynamics is subject to bounded but otherwise unknown time-varying disturbance $\mathbf{d}(t) = \frac{1}{b}\mathbf{a}_T(t)$; the reference command $\mathbf{r}_c(t)$ depends upon unknown parameter. The problem is thus cast into an output feedback framework. The advantage of this approach is that it can be used in any tracking scenario, and it produces an inertial acceleration command for the follower, the implementation of which does

not require command differentiation. The disadvantage is that the control algorithm involves additional structures in the form of filters and an observer, and therefore has more complex architecture. In addition, the obtained acceleration command involves approximations of discontinuities signals by smooth functions, resulting in bounded tracking with regulated bounds.

Remark 2 *We notice that with two cameras the target's relative position vector $\mathbf{R}^B$ can be calculated from both camera measurements, thus removing the unknown parametrization and reducing the problem to a simple disturbance rejection for a known system. In this case, the relative range is proportional to the separation of two cameras, while in the case of one camera, it is proportional to the target's maximum size, as it can be seen from the relationship in (3.7). Thus, if the follower has two cameras, the relative range can be estimated with sufficient accuracy if the follower has relatively large wing span, whereas in the case of a MAV or a missile, the second camera may not add any useful information.* $\square$

In this dissertation only a monocular camera is assumed.

Chapter 4

Velocity Command

In this chapter, we consider Problem 1 for the system in (3.15). We first present an adaptive disturbance rejection guidance law for tracking known given reference commands following the framework in [40]. Then, we demonstrate that dependent upon parameter convergence, the proposed guidance law solves the problem of visual tracking. The parameter convergence can be achieved in the presence of some type of excitation in the reference signal.

4.1 Disturbance Rejection Guidance Law for Known Reference Commands

Let the dynamics of the system is given by the equation in (3.15), where $\mathbf{V}_F(t)$ is viewed as a control input and $\mathbf{V}_T(t)$ is viewed as a time varying disturbance, subject to the following assumption.

Assumption 1 . Assume that the target's inertial velocity $\mathbf{V}_T(t)$ is a bounded function of

time and has bounded time derivative, i.e. $\mathbf{V}_T(t), \dot{\mathbf{V}}_T(t) \in \mathcal{L}_\infty$. Furthermore, assume that any maneuver made by the target is such that the velocity returns to some constant value in finite time or asymptotically in infinite time with a rate sufficient for the integral of the magnitude of velocity change to be finite. $\square$

Since b is just a constant, for the convenience in the stability proof, this assumption is formulated for the function $\frac{1}{b}\mathbf{V}_T(t)$ as follows:

$$\frac{1}{b}\mathbf{V}_T(t) = \mathbf{d} + \boldsymbol{\delta}(t), \quad (4.1)$$

where $\mathbf{d} \in \mathbb{R}^3$ is an unknown but otherwise constant vector and $\boldsymbol{\delta}(t) \in \mathcal{L}_2$ is an unknown time-varying term. For example, any obstacle or collision avoidance can be viewed as such a maneuver, provided that after the maneuver the target velocity returns to a constant in finite time or asymptotically in infinite time subject to $\boldsymbol{\delta}(t) \in \mathcal{L}_2(\mathbb{R}^3)$. Since $\mathbf{V}_T(t), \dot{\mathbf{V}}_T(t) \in \mathcal{L}_\infty$, it follows that $\boldsymbol{\delta}(t), \dot{\boldsymbol{\delta}}(t) \in \mathcal{L}_\infty$. Therefore, applying Corollary 1 from Section 2.1, we conclude that

$$\boldsymbol{\delta}(t) \rightarrow 0, \quad t \rightarrow \infty. \quad (4.2)$$

Let $\mathbf{r}_c(t)$ be a continuously differentiable bounded known reference signal of interest to track. This corresponds to the case when the target's size is known. The controller developed in [40] rejects time-varying disturbances $\boldsymbol{\delta}(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. Here, using the Projection-based classical adaptive laws (see Appendix A), we estimate the constant disturbance $\mathbf{d}$ and reject $\boldsymbol{\delta}(t)$ simultaneously. The tracking guidance law is defined by

$$\begin{aligned} \mathbf{V}_F(t) &= \hat{b}(t)\mathbf{g}(t) \\ \mathbf{g}(t) &= k\mathbf{e}(t) + \hat{\mathbf{d}}(t) - \dot{\mathbf{r}}_c(t), \end{aligned} \quad (4.3)$$

where $\mathbf{e}(t) = \mathbf{r}(t) - \mathbf{r}_c(t) \in \mathbb{R}^3$ is the tracking error, $\hat{b}(t) \in \mathbb{R}$ and $\hat{\mathbf{d}}(t) \in \mathbb{R}^3$ are the estimates of the unknown parameter b and the nominal disturbance $\mathbf{d}$ respectively, $k > 0$ is the control

gain. Substituting the guidance law from (4.3) into relative dynamics in (3.15), the error dynamics can be written as

$$\dot{\mathbf{e}}(t) = \mathbf{d} + \boldsymbol{\delta}(t) - k\mathbf{e}(t) - \hat{\mathbf{d}}(t) - \frac{\tilde{b}(t)}{b}\mathbf{g}(t). \quad (4.4)$$

The adaptive laws for the estimates $\hat{b}(t)$ and $\hat{\mathbf{d}}(t)$ are given as follows:

$$\begin{aligned} \dot{\hat{b}}(t) &= \sigma \text{Proj}(\dot{\hat{b}}(t), \mathbf{e}^\top(t)\mathbf{g}(t)), \quad \hat{b}(0) = \hat{b}_0 > 0 \\ \dot{\hat{\mathbf{d}}}(t) &= G \text{Proj}(\dot{\hat{\mathbf{d}}}(t), \mathbf{e}(t)), \quad \hat{\mathbf{d}}(0) = \hat{\mathbf{d}}_0, \end{aligned} \quad (4.5)$$

where $\sigma > 0$ is a constant (adaptation gain), G is a positive definite matrix (adaptation gain), and $\text{Proj}(\cdot, \cdot)$ denotes the Projection operator [49] (see Appendix A).

Now we are ready to state the main theorem.

Theorem 11 . *The adaptive guidance law given by (4.3) and (4.5) guarantees global uniform ultimate boundedness of all error signals in the system (4.4), (4.5) and asymptotic tracking of the reference trajectory $\mathbf{r}_c(t)$.* $\square$

Proof. Letting $\tilde{b}(t) = \hat{b}(t) - b$ and $\tilde{\mathbf{d}}(t) = \hat{\mathbf{d}}(t) - \mathbf{d}$, the error dynamics take the form

$$\dot{\mathbf{e}}(t) = -k\mathbf{e}(t) - \tilde{\mathbf{d}}(t) + \boldsymbol{\delta}(t) - \frac{\tilde{b}(t)}{b}\mathbf{g}(t). \quad (4.6)$$

Consider the following Lyapunov function candidate

$$V(\mathbf{e}, \tilde{b}, \tilde{\mathbf{d}}) = \frac{1}{2}\mathbf{e}^\top(t)\mathbf{e}(t) + \frac{1}{2\sigma b}\tilde{b}^2(t) + \frac{1}{2}\tilde{\mathbf{d}}^\top(t)G^{-1}\tilde{\mathbf{d}}(t). \quad (4.7)$$

Its derivative along the trajectories of the system (4.3)-(4.4) has the form

$$\begin{aligned} \dot{V}(t) &= -k\mathbf{e}^\top(t)\mathbf{e}(t) - \mathbf{e}^\top(t)\tilde{\mathbf{d}}(t) + \frac{\tilde{b}(t)}{\sigma b}\dot{\tilde{b}}(t) - \frac{\tilde{b}(t)}{b}\mathbf{e}^\top(t)\mathbf{g}(t) \\ &+ \tilde{\mathbf{d}}^\top(t)G^{-1}\dot{\tilde{\mathbf{d}}}(t) + \mathbf{e}^\top(t)\boldsymbol{\delta}(t) = -k\mathbf{e}^\top(t)\mathbf{e}(t) + \mathbf{e}^\top(t)\boldsymbol{\delta}(t) \\ &+ \frac{\tilde{b}(t)}{b}[-\mathbf{e}^\top(t)\mathbf{g}(t) + \text{Proj}(\dot{\hat{b}}(t), \mathbf{e}^\top(t)\mathbf{g}(t))] \\ &+ \tilde{\mathbf{d}}^\top(t)[- \mathbf{e}(t) + \text{Proj}(\dot{\hat{\mathbf{d}}}(t), \mathbf{e}(t))]. \end{aligned} \quad (4.8)$$

From the properties of the projection operator (see Appendix A) the following inequalities can be written:

$$\begin{aligned}
b_{\min} &\leq \hat{b}(t) \leq b_{\max} \\
\tilde{b}(t) \left[-\mathbf{g}^\top(t) \mathbf{e}(t) + \text{Proj} \left(\hat{b}(t), \mathbf{g}^\top(t) \mathbf{e}(t) \right) \right] &\leq 0 \\
\|\hat{\mathbf{d}}(t)\| &\leq d_{\max} \\
\tilde{\mathbf{d}}^\top(t) \left[-\mathbf{e}(t) + \text{Proj} \left(\hat{\mathbf{d}}(t), \mathbf{e}(t) \right) \right] &\leq 0,
\end{aligned} \tag{4.9}$$

Using the inequalities in (4.9), $\dot{V}(t)$ can be upper bounded as follows:

$$\dot{V}_1(t) \leq -k \mathbf{e}^\top(t) \mathbf{e}(t) + \mathbf{e}^\top(t) \boldsymbol{\delta}(t), \tag{4.10}$$

which upon completing the squares yields

$$\dot{V}_1(t) \leq -k_1 \|\mathbf{e}(t)\|^2 + c_1^2 \|\boldsymbol{\delta}(t)\|^2, \tag{4.11}$$

where c_1 is a positive constant such that $k_1 = k - \frac{1}{4c_1^2} > 0$. Since $\boldsymbol{\delta}(t) \in \mathcal{L}_\infty$, there exists a positive constant ρ such that $c_1 \|\boldsymbol{\delta}(t)\| \leq \rho$. Also, from the inequalities in (4.9) it follows that $\|\tilde{b}(t)\| \leq \tilde{b}_*$ and $\|\tilde{\mathbf{d}}(t)\| \leq \tilde{d}_*$, where $\tilde{b}_*$ and $\tilde{d}_*$ are some positive constants. From the relationship in (4.11) it follows that $\dot{V}(t) \leq 0$ outside the compact set

$$\Omega = \{(\mathbf{e}, \tilde{b}, \tilde{\mathbf{d}}) : \|\mathbf{e}\| \leq \frac{\rho}{\sqrt{k_1}}, \|\tilde{b}\| \leq \tilde{b}_*, \|\tilde{\mathbf{d}}\| \leq \tilde{d}_*\}$$

Theorem 2 from Section 2.1 ensures that all the error signals $\mathbf{e}(t)$, $\tilde{b}(t)$, $\tilde{\mathbf{d}}(t)$, $\mathbf{g}(t)$ in the system (4.3), (4.4) are uniformly ultimately bounded. From (4.6) it follows that $\dot{\mathbf{e}}(t)$ is bounded. Integrating (4.11), we get

$$\begin{aligned}
\int_0^t k_1 \|\mathbf{e}(\tau)\|^2 d\tau &\leq V_1(0) - V_1(t) + \int_0^t c_1^2 \|\boldsymbol{\delta}(\tau)\|^2 d\tau \\
&\leq V_1(0) + \int_0^t c_1^2 \|\boldsymbol{\delta}(\tau)\|^2 d\tau.
\end{aligned} \tag{4.12}$$

Since $\delta(t) \in \mathcal{L}_2$, from (4.12) we have

$$\lim_{t \rightarrow \infty} \int_0^t k_1 \|\mathbf{e}(\tau)\|^2 d\tau < \infty. \quad (4.13)$$

Thus, $\mathbf{e}(t) \in \mathcal{L}_2(\mathbb{R}^3) \cap \mathcal{L}_\infty(\mathbb{R}^3)$. Also, the error dynamics in (4.6) imply that $\dot{\mathbf{e}}(t) \in \mathcal{L}_\infty(\mathbb{R}^3)$. Application of Corollary 1 from Section 2.1 ensures that $\mathbf{e}(t) \rightarrow 0$ as $t \rightarrow \infty$, and therefore, $\mathbf{r}(t) \rightarrow \mathbf{r}_c(t)$ as $t \rightarrow \infty$. The proof is complete. $\square$

4.2 Visual Guidance Law

In the case of visual measurement, as discussed above, $\mathbf{r}_c(t)$ depends upon unknown parameters, and hence, is not available for the guidance law in (4.3). For the given reference commands $\mathbf{R}_c(t)$, one can consider estimated reference command $\hat{\mathbf{r}}_c(t)$ using $\hat{b}(t)$ defined as follows:

$$\hat{\mathbf{r}}_c(t) = \frac{1}{\hat{b}(t)} \mathbf{R}_c(t). \quad (4.14)$$

From the inequality in (4.9) it follows that $\hat{b}(t)$ is bounded away from zero, hence the estimated reference command $\hat{\mathbf{r}}_c(t)$ is well defined. Moreover, it is continuously differentiable, with its derivative calculated as follows:

$$\dot{\hat{\mathbf{r}}}_c(t) = \frac{1}{\hat{b}(t)} \dot{\mathbf{R}}_c(t) - \frac{1}{\hat{b}^2(t)} \dot{\hat{b}}(t) \mathbf{R}_c(t). \quad (4.15)$$

According to Theorem 11, the guidance law in (4.3) with $\mathbf{r}_c(t)$ replaced by $\hat{\mathbf{r}}_c(t)$ and $\dot{\mathbf{r}}_c(t)$ replaced by $\dot{\hat{\mathbf{r}}}_c(t)$ guarantees the convergence of $\mathbf{r}(t)$ to $\hat{\mathbf{r}}_c(t)$. However, the convergence of $\mathbf{r}(t)$ to $\mathbf{r}_c(t)$ is not guaranteed, unless $\hat{\mathbf{r}}_c(t)$ converges to $\mathbf{r}_c(t)$, which can take place in the presence of parameter convergence, i.e. when the parameter estimate $\hat{b}(t)$ converges to the true value b . Thus, the visual tracking problem can be solved if one ensures that $\hat{b}(t) \rightarrow b$

as $t \rightarrow \infty$. A discussion on this is provided in the next section. First, we need the following auxiliary result. We need to prove that as $t \rightarrow \infty$ the Lyapunov function $V(t)$ has a finite limit.

Lemma 6 . *There exists a constant $\bar{V} \geq 0$ such that*

$$\lim_{t \rightarrow \infty} V(\mathbf{e}(t), \tilde{\mathbf{b}}(t), \tilde{\mathbf{d}}(t)) = \bar{V} .$$

Proof. Introduce a function $s(t)$ as follows:

$$s(t) = -\dot{V}_1(t) - k_1 \|\mathbf{e}(t)\|^2 + c_1^2 \|\boldsymbol{\delta}(t)\|^2 . \quad (4.16)$$

From the upper bound in (4.11), it follows that $s(t) \geq 0$. This implies that $\int_0^t s(\tau) d\tau$ is a monotonous (non-decreasing) function of t . Integration of the equation in (4.16) results in

$$\int_0^t s(\tau) d\tau = -V_1(t) + V_1(0) + \int_0^t [-k_1 \|\mathbf{e}(\tau)\|^2 + c_1^2 \|\boldsymbol{\delta}(\tau)\|^2] d\tau . \quad (4.17)$$

It follows from Theorem 11 that $V_1(t)$ is bounded, while $\mathbf{e}(t), \boldsymbol{\delta}(t) \in \mathcal{L}_2$. Therefore, $\int_0^t s(\tau) d\tau$ is bounded. Hence, $\lim_{t \rightarrow \infty} \int_0^t s(\tau) d\tau$ exists. Therefore,

$$\begin{aligned} \bar{V}_1 &= \lim_{t \rightarrow \infty} V_1(t) \\ &= V_1(0) - \lim_{t \rightarrow \infty} \int_0^t s(\tau) d\tau + \lim_{t \rightarrow \infty} \int_0^t [-k_1 \|\mathbf{e}(\tau)\|^2 + c_1^2 \|\boldsymbol{\delta}(\tau)\|^2] d\tau \end{aligned} \quad (4.18)$$

also exists. The proof is complete. $\square$

4.3 Parameter Convergence

Theorem 11 guarantees convergence of the tracking error $\mathbf{e}(t)$ to zero but not of the estimation errors $\tilde{\mathbf{b}}(t)$ or $\tilde{\mathbf{d}}(t)$. Instead, it guarantees convergence of $\dot{\tilde{\mathbf{b}}}(t)$ and $\dot{\tilde{\mathbf{d}}}(t)$ to zero. In this

section, we prove that if $\dot{\hat{\mathbf{r}}}_c(t) \rightarrow 0$ as $t \rightarrow \infty$, then the parameter estimates $\hat{b}(t)$ and $\hat{\mathbf{d}}(t)$ in the system (4.3)-(4.4) converge to some constant values $\bar{b}$ and $\bar{\mathbf{d}} = [\bar{d}_x \ \bar{d}_y \ \bar{d}_z]^\top$ respectively, but not necessarily to the true values b and $\mathbf{d}$. To prove this result, we need the following Extended Barbalat's lemma from [38] :

Lemma 7 (*Extended Barbalat's Lemma*). *Let $\varphi(t)$ be a real piecewise continuous function of real variable t and defined for $t \geq t_0$, $t_0 \in \mathbb{R}$. Assume that $\varphi(t)$ can be decomposed as $\varphi(t) = \varphi_1(t) + \varphi_2(t)$, where $\varphi_1(t)$ is uniformly continuous for $t \geq t_0$ and $\varphi_2(t)$ satisfies $\lim_{t \rightarrow \infty} \varphi_2(t) = 0$. If $\lim_{t \rightarrow \infty} \int_{t_0}^t \varphi(\tau) d\tau$ exists and is finite, then $\lim_{t \rightarrow \infty} \varphi(t) = 0$.* $\square$

A proof of this lemma is given in [38].

Lemma 8 *The adaptive controller in (4.3) and (4.5) guarantees convergence of parameter estimates $\hat{b}(t)$ and $\hat{\mathbf{d}}(t)$ to constant values, provided that $\dot{\hat{\mathbf{r}}}_c(t) \rightarrow 0$ as $t \rightarrow \infty$.* $\square$

Proof. From Theorem 11 it follows that all the signals in the system are bounded; therefore, $\dot{\hat{b}}(t)$ and $\dot{\hat{\mathbf{d}}}(t)$ are bounded, implying uniform continuity of the signals $\hat{b}(t)$, $\tilde{b}(t)$, $\hat{\mathbf{d}}(t)$ and $\tilde{\mathbf{d}}(t)$. The error dynamics in (4.4) can be written in the form

$$\dot{\mathbf{e}}(t) = \boldsymbol{\varphi}_1(t) + \boldsymbol{\varphi}_2(t), \quad (4.19)$$

where

$$\begin{aligned} \boldsymbol{\varphi}_1(t) &= -\tilde{\mathbf{d}}(t) - \frac{\tilde{b}(t)}{b} \hat{\mathbf{d}}(t), \\ \boldsymbol{\varphi}_2(t) &= \left(-1 - \frac{\tilde{b}(t)}{b} \right) k \mathbf{e}(t) + \boldsymbol{\delta}(t) + \frac{\tilde{b}(t)}{b} \dot{\hat{\mathbf{r}}}_c(t). \end{aligned}$$

The function $\boldsymbol{\varphi}_1(t)$ is uniformly continuous, since it is a sum of uniformly continuous functions. Also, since $\mathbf{e}(t) \rightarrow 0$, $\boldsymbol{\delta}(t) \rightarrow 0$, and $\dot{\hat{\mathbf{r}}}_c(t) \rightarrow 0$, then $\boldsymbol{\varphi}_2(t) \rightarrow \mathbf{0}$ as $t \rightarrow \infty$. Since the

error $\mathbf{e}(t)$ converges to zero as $t \rightarrow \infty$, the application of the Extended Barbalat's lemma implies that $\dot{\mathbf{e}}(t) \rightarrow 0$ as $t \rightarrow \infty$. Hence, as $t \rightarrow \infty$, the error dynamics in (4.19) reduce to

$$\tilde{\mathbf{d}}(t) + \frac{\tilde{b}(t)}{b} \hat{\mathbf{d}}(t) \rightarrow \mathbf{0}. \quad (4.20)$$

With $\hat{\mathbf{d}}(t) = \mathbf{d} + \tilde{\mathbf{d}}(t)$, the relationship in (4.20) can be transformed into

$$\hat{b}(t)\tilde{\mathbf{d}}(t) + \tilde{b}(t)\mathbf{d} \rightarrow \mathbf{0}, \quad t \rightarrow \infty. \quad (4.21)$$

Fix arbitrary $\varepsilon > 0$. The limit in (4.21) implies that there exists a time instance $T_1(\varepsilon)$ such that for all $t > T_1(\varepsilon)$

$$\|\hat{b}(t)\tilde{\mathbf{d}}(t) + \tilde{b}(t)\mathbf{d}\| < \varepsilon. \quad (4.22)$$

We recall that according to the inequality in (4.9) $\hat{b}(t) \geq b_{\min} > 0$ for every $t > 0$. Let $t > T_1(\varepsilon)$. The relationship in (4.22) can be written component-wise, e.g. for x , as follows:

$$|\hat{b}(t)\tilde{d}_x(t) + \tilde{b}(t)d_x| < \varepsilon. \quad (4.23)$$

Upon some algebra in (4.23), the following inequality can be derived:

$$\frac{\tilde{b}^2(t)}{\hat{b}^2(t)} d_x^2 - \zeta_x(\varepsilon) < \tilde{d}_x^2(t) < \frac{\tilde{b}^2(t)}{\hat{b}^2(t)} d_x^2 + \zeta_x(\varepsilon), \quad (4.24)$$

where

$$\zeta_x(\varepsilon) = \frac{|\varepsilon^2 + 2\varepsilon d_x|}{b_{\min}^2}$$

and obviously $\zeta_x(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. The same type of inequalities can be written for the other two components of $\tilde{\mathbf{d}}(t)$. From Lemma 6 it follows that for the same ε chosen above, there exists a time instance $T_2(\varepsilon)$ such that for $t > T_2(\varepsilon)$

$$\left| \frac{1}{2\sigma b} \tilde{b}^2(t) + \frac{1}{2} \tilde{\mathbf{d}}^\top(t) G^{-1} \tilde{\mathbf{d}}(t) - \bar{V} \right| < \varepsilon. \quad (4.25)$$

For the given ε , the time instances $T_1(\varepsilon)$ and $T_2(\varepsilon)$ may be different. Choosing the largest one and denoting

$$T(\varepsilon) = \max[T_1(\varepsilon), T_2(\varepsilon)],$$

all of the above and following conclusions remain valid for all $t > T(\varepsilon)$.

Let G be a diagonal matrix with positive elements g_1, g_2, g_3 on the diagonal. This is usually the case for the adaptive gains in the update laws. Then, taking into account the inequality in (4.24), it follows from the inequality in (4.25) that for all $t > T(\varepsilon)$

$$\left| \frac{1}{2\sigma b} \tilde{b}^2(t) + \frac{1}{2} \mathbf{d}^\top G^{-1} \mathbf{d} \frac{\tilde{b}^2(t)}{\tilde{b}^2(t)} - \bar{V} \right| < \varepsilon + \zeta(\varepsilon), \quad (4.26)$$

where $\zeta(\varepsilon) = \frac{1}{2}[g_1\zeta_x(\varepsilon) + g_2\zeta_y(\varepsilon) + g_3\zeta_z(\varepsilon)]$, which implies that

$$(b + \tilde{b}(t))^2 \tilde{b}^2(t) + c_1 \tilde{b}^2(t) - c_2 (b + \tilde{b}(t))^2 \rightarrow 0, \quad (4.27)$$

where $c_1 = b\sigma \mathbf{d}^\top G^{-1} \mathbf{d} > 0$, $c_2 = 2\sigma b \bar{V} > 0$. Consider the function

$$p(\xi) = (b + \xi)^2 \xi^2 + c_1 \xi^2 - c_2 (b + \xi)^2.$$

It is a monic polynomial of fourth order in ξ , and takes a negative value at $\xi = 0$. Therefore, it has at least two real roots. In general, it can be represented in the case of four real roots as

$$p(\xi) = (\xi - b_1)(\xi - b_2)(\xi - b_3)(\xi - b_4)$$

or in the case of two real roots as

$$p(\xi) = (\xi - b_1)(\xi - b_2)q(\xi),$$

where $q(\xi)$ is a monic quadratic with no real roots. Thus, there exists some positive constant ν , such that $q(\xi) \geq \nu$ for all values of ξ . First, consider the case of four real roots. The

relationship in (4.27) implies that for an arbitrary positive number ε , there exists a time instance $T_3(\varepsilon)$, such that for all $t > T_3(\varepsilon)$

$$|(\tilde{b}(t) - b_1)(\tilde{b}(t) - b_2)(\tilde{b}(t) - b_3)(\tilde{b}(t) - b_4)| < \varepsilon \quad (4.28)$$

or equivalently

$$|\tilde{b}(t) - b_1||\tilde{b}(t) - b_2||\tilde{b}(t) - b_3||\tilde{b}(t) - b_4| < \varepsilon. \quad (4.29)$$

This implies that

$$|\tilde{b}(t) - b_k| = \min_i [|\tilde{b}(t) - b_i|, \quad i = 1, \dots, 4] < \varepsilon^{1/4}. \quad (4.30)$$

Since ε is an arbitrary small number, the inequality in (4.30) implies that $\tilde{b}(t) \rightarrow b_k$ as $t \rightarrow \infty$. Notice, that since $\tilde{b}(t)$ is a continuous function, it cannot simultaneously converge to two distinct constants. That is, only one of the factors in (4.29) can be arbitrary small. The case of two real roots can be handled similarly. This implies that the parameter error $\tilde{b}(t)$ converges to some constant value, and so does the parameter estimate $\hat{b}(t)$, that is, $\hat{b}(t) \rightarrow \bar{b}$ as $t \rightarrow \infty$, where $\bar{b}$ is some constant. From (4.21) it follows that $\hat{\mathbf{d}}(t)$ converges to some constant vector $\bar{\mathbf{d}}$ as well. $\square$

Remark 3 *In the special case of formation flight the reference command $\mathbf{R}_c$ is constant. Also, from Theorem 11 it follows that $\dot{\tilde{b}}(t) \rightarrow 0$ as $t \rightarrow \infty$, implying that $\dot{\hat{b}}(t) \rightarrow 0$. Then, the relationships in (4.15) imply that $\dot{\hat{\mathbf{r}}}_c(t) \rightarrow 0$ as $t \rightarrow \infty$.* $\square$

Lemma 8 states that the parameter estimates converge to a point within the set described by the equation $\bar{b}\bar{\mathbf{d}} = b\mathbf{d}$, if $\dot{\hat{\mathbf{r}}}_c(t) \rightarrow 0$ as $t \rightarrow \infty$. However, $\bar{b} = b$ and $\bar{\mathbf{d}} = \mathbf{d}$ are not guaranteed. Thus, for the convergence of parameter estimates to true values, we need to provide some excitation in the form of sinusoidal components in the reference signal to prevent $\dot{\hat{\mathbf{r}}}_c(t) \rightarrow 0$ [63]. Next, lemma shows that in the special case of formation flight,

exciting only the reference command $\mathbf{R}_c$ in one direction is sufficient for the convergence of all parameters. Thus, for the given constant reference command $\mathbf{R}_c$ the, "excited" reference command is defined as follows:

$$\mathbf{R}_c^*(t) = \mathbf{R}_c + a \sin(\omega t) \mathbf{E}, \quad (4.31)$$

where a is the excitation amplitude, ω is the excitation frequency and the vector $\mathbf{E}$ specifies the direction of the excitation. For instance, we can excite only the reference command in the x direction by setting $\mathbf{E} = [1 \ 0 \ 0]^\top$.

Lemma 9 *If $\mathbf{R}_c$ is constant, then for the reference command $\mathbf{R}_c^*(t)$, the adaptive controller in (4.3) guarantees the convergence of parameter errors to zero.* $\square$

Proof. The adaptive controller in (4.3) guarantees the convergence of the tracking error to zero as $t \rightarrow \infty$ (Theorem 11), which consequently implies that $\dot{\hat{b}}(t) \rightarrow 0$. According to the equations in (4.15), the derivative of the estimated reference command $\dot{\hat{\mathbf{r}}}_c(t)$ is represented as a sum of two terms:

$$\dot{\hat{\mathbf{r}}}_c(t) = \dot{\hat{\mathbf{r}}}_{c1}(t) + \dot{\hat{\mathbf{r}}}_{c2}(t),$$

where

$$\begin{aligned} \dot{\hat{\mathbf{r}}}_{c1}(t) &= \frac{1}{\hat{b}(t)} a \omega \cos(\omega t) \mathbf{E} \\ \dot{\hat{\mathbf{r}}}_{c2}(t) &= -\frac{\dot{\hat{b}}(t)}{\hat{b}^2(t)} \mathbf{R}_c^*(t). \end{aligned}$$

It is easy to see that $\dot{\hat{\mathbf{r}}}_{c1}(t)$ is uniformly continuous, since $\frac{1}{\hat{b}(t)}$ and $\cos(\omega t)$ are uniformly continuous, and $\dot{\hat{\mathbf{r}}}_{c2}(t) \rightarrow 0$ as $t \rightarrow \infty$, since $\hat{b}(t)$ is bounded away from zero and $\dot{\hat{b}}(t) \rightarrow 0$. Thus, the error dynamics in (4.4) can be represented as

$$\dot{\mathbf{e}}(t) = \boldsymbol{\varphi}_1(t) + \boldsymbol{\varphi}_2(t), \quad (4.32)$$

where

$$\boldsymbol{\varphi}_1(t) = -\tilde{\mathbf{d}}(t) - \frac{\tilde{b}(t)}{b}[\hat{\mathbf{d}}(t) - \dot{\hat{\mathbf{r}}}_{c1}(t)]$$

is uniformly continuous, while

$$\boldsymbol{\varphi}_2(t) = (-1 - \frac{\tilde{b}(t)}{b})k\mathbf{e}(t) + \boldsymbol{\delta}(t) + \frac{\tilde{b}(t)}{b}\dot{\hat{\mathbf{r}}}_{c2}(t) \rightarrow \mathbf{0}$$

as $t \rightarrow \infty$. Application of Extended Barbalat's lemma implies that $\dot{\mathbf{e}}(t) \rightarrow \mathbf{0}$ as $t \rightarrow \infty$.

Therefore, the error dynamics reduce to

$$-\tilde{\mathbf{d}}(t) - \frac{\tilde{b}(t)}{b}[\hat{\mathbf{d}}(t) - \dot{\hat{\mathbf{r}}}_{c1}(t)] \rightarrow \mathbf{0}, \quad t \rightarrow \infty. \quad (4.33)$$

Without loss of generality, we let $\mathbf{E} = [1 \ 0 \ 0]^\top$. Consider the x component of the vector relationship in (4.33):

$$\tilde{d}_x(t) + \frac{\tilde{b}(t)}{b}[\hat{d}_x(t) - \frac{a\omega}{\hat{b}(t)}\cos(\omega t)] \rightarrow 0, \quad (4.34)$$

With the help of $\hat{\mathbf{d}}(t) = \mathbf{d} + \tilde{\mathbf{d}}(t)$ and $\hat{b}(t) = b + \tilde{b}(t)$, it can be transformed into the form

$$\hat{b}(t)\tilde{d}_x(t) + \tilde{b}(t)[d_x - \frac{a\omega}{\hat{b}(t)}\cos(\omega t)] \rightarrow 0, \quad (4.35)$$

which is equivalently written in inner product form as follows

$$\begin{bmatrix} \hat{b}(t) \\ d_x - \frac{a\omega}{\hat{b}(t)}\cos(\omega t) \end{bmatrix}^\top \begin{bmatrix} \tilde{d}_x(t) \\ \tilde{b}(t) \end{bmatrix} \rightarrow 0, \quad (4.36)$$

where the vector

$$\mathbf{h}(t) = \begin{bmatrix} \hat{b}(t) \\ d_x - \frac{a\omega}{\hat{b}(t)}\cos(\omega t) \end{bmatrix}$$

is persistently exciting, i.e. there exist positive constants α_1 , α_2 , T such that the inequalities

$$\alpha_1 \geq \int_t^{t+T} (\mathbf{h}^\top(\tau)\boldsymbol{\xi})^2 d\tau \geq \alpha_2 \quad (4.37)$$

hold for all $\boldsymbol{\xi} = [\xi_1 \ \xi_2]^\top$ with $\|\boldsymbol{\xi}\| = 1$ and $t \geq t_0$. Indeed, the left side of the inequality in (4.37) follows from the fact that all the signals in the system are bounded; hence, there exists a positive constant a_1 such that $\|\mathbf{h}(t)\| \leq a_1$. Therefore,

$$\int_t^{t+T} (\mathbf{h}^\top(\tau)\boldsymbol{\xi})^2 d\tau \leq \int_t^{t+T} \|\mathbf{h}^\top(\tau)\|^2 \|\boldsymbol{\xi}\|^2 d\tau \leq a_1^2 T.$$

So α_1 can be chosen equal to $a_1^2 T$. To prove the right hand side of the inequality in (4.37), we write

$$\begin{aligned} I(t, T) &= \int_t^{t+T} (\mathbf{h}^\top(\tau)\boldsymbol{\xi})^2 d\tau \\ &= \int_t^{t+T} (\xi_1 \hat{b}(\tau) + \xi_2 d_x)^2 d\tau \\ &+ a^2 \omega^2 \xi_2^2 \int_t^{t+T} \frac{1}{\hat{b}^2(\tau)} \cos^2(\omega\tau) d\tau \\ &- 2a\omega \xi_1 \xi_2 \int_t^{t+T} \cos(\omega\tau) d\tau \\ &- 2a\omega \xi_2^2 d_x \int_t^{t+T} \frac{1}{\hat{b}(\tau)} \cos(\omega\tau) d\tau. \end{aligned} \quad (4.38)$$

From the inequality in (4.9) we deduce the following lower bound for the integral $I(t, T)$:

$$\begin{aligned} I(t, T) &\geq \int_t^{t+T} (\xi_1 \hat{b}(\tau) + \xi_2 d_{0x})^2 d\tau \\ &+ a^2 \omega^2 \xi_2^2 \frac{1}{b_{\max}^2} \int_t^{t+T} \cos^2(\omega\tau) d\tau \\ &- 2a\omega \xi_1 \xi_2 \int_t^{t+T} \cos(\omega\tau) d\tau \\ &- 2a\omega \xi_2^2 d_x \frac{1}{b_{\min}} \int_t^{t+T} \cos(\omega\tau) d\tau. \end{aligned} \quad (4.39)$$

Choosing $T = \frac{2\pi}{\omega} m$, where m is any positive integer, renders

$$\int_t^{t+T} \cos(\omega\tau) d\tau = 0, \quad \int_t^{t+T} \cos^2(\omega\tau) d\tau = \frac{T}{2}.$$

Therefore,

$$I(t, T) \geq \int_t^{t+T} (\xi_1 \hat{b}(\tau) + \xi_2 d_{0x})^2 d\tau + \frac{a^2 \omega^2 \xi_2^2}{2b_{\max}^2} T.$$

It is easy to see that if $\xi_2 = 0$, then $\xi_1 = 1$

$$I(t, T) \geq \int_t^{t+T} \hat{b}^2(\tau) d\tau \geq T b_{\min}^2,$$

otherwise

$$I(t, T) \geq \frac{a^2 \omega^2 \xi_2^2}{2b_{\max}^2} T.$$

Hence, for every ξ there exists a positive constant $\alpha_2 > 0$ such that $I(t, T) \geq \alpha_2$. Thus, (4.37) holds, and therefore, it follows from (4.36) that $\tilde{d}_x(t) \rightarrow 0$, $\tilde{b}(t) \rightarrow 0$ as $t \rightarrow \infty$. Then, from the remaining two components of the vector relationship in (4.20), it follows that $\tilde{d}_y(t) \rightarrow 0$, $\tilde{d}_z(t) \rightarrow 0$. Hence, $\hat{b}(t) \rightarrow b$ and $\hat{\mathbf{r}}_c(t) \rightarrow \frac{1}{b} \mathbf{R}_c^*(t)$. The proof is complete. $\square$

Remark 4 *In applications, a persistently exciting signal is usually undesirable because it deteriorates the tracking of the constant reference command. Moreover, in visual control, the moving target can be easily lost from the field of view. A recently developed technique of intelligent excitation allows the controlling of the amplitude of the excitation signal, dependent upon the convergence of the output tracking and parameter errors [10]. It also ensures simultaneous convergence of the parameter and output tracking error to zero. Below in simulations, we incorporate this technique to ensure parameter convergence.* $\square$

Remark 5 *For the target interception problem, the reference command for the relative range is zero. Therefore $\mathbf{r}_c = 0$ and, hence $\dot{\mathbf{r}}_c = 0$. Then, the guidance law*

$$\begin{aligned} \mathbf{V}_F(t) &= \hat{b}(t) \mathbf{g}(t) \\ \mathbf{g}(t) &= k \mathbf{r}(t) + \hat{\mathbf{d}}(t) \\ \dot{\hat{b}}(t) &= \sigma \text{Proj}(\hat{b}(t), \mathbf{r}^\top(t) \mathbf{g}(t)) \\ \dot{\hat{\mathbf{d}}}(t) &= G \text{Proj}(\hat{\mathbf{d}}(t), \mathbf{r}(t)) \end{aligned} \tag{4.40}$$

guarantees boundedness of all the signals and convergence of the relative states to zero: $\mathbf{r} \rightarrow 0$ as $t \rightarrow \infty$. There is no need to require parameter convergence, and hence, the need for introducing an excitation signal disappears. $\square$

4.4 Simulations

To demonstrate the performance of the proposed algorithm, two simulation examples are considered in this section: a formation flight scenario and a target interception scenario. In both simulations no camera model was used. Instead, to demonstrate the performance of the derived guidance law, it is assumed that the scaled relative position is available for feedback. The signal $\hat{b}(t)$ is generated according to the adaptive law for the estimate $\hat{b}(t)$ in (4.3) with the initial estimation of $\hat{b}(0) = 4ft$. Excitation signal is introduced following Ref. [10] with the following amplitude:

$$a = \begin{cases} k_1, & t \in [0, T) \\ \min\{k_2 \int_{t-T}^t \mathbf{e}^\top(\tau) \mathbf{e}(\tau) d\tau, k_1 - k_3\} + k_3, & t \geq T, \end{cases}$$

where $T = \frac{2\pi}{\omega}$ is the period of the excitation signal $a \sin(\omega t)$, $k_i > 0$, $i = 1, 2, 3$ are design constants that are set to $T = 3sec$, $k_1 = 0.5$, $k_2 = 10$, $k_3 = 0.0002$. In the simulation scenario the target of the length $b = 8ft$ starts the motion with a velocity of $V_{Tx} = 50ft/sec$,

$V_{Ty} = V_{Tz} = 0$ and follows the following velocity profile:

$$V_{Tx}(t) = \begin{cases} 50, & t \in [0, 3] \\ 25(1 + \cos(\frac{\pi}{30}(t - 3))), & t \in (3, 33] \\ 0, & t \in (33, 43] \\ 30(1 - \cos(\frac{\pi}{30}(t - 43))), & t \in (43, 73] \\ 60 - (t - 73), & t \in (73, 83] \\ 25(1 + \cos(\frac{\pi}{30}(t - 83))), & t \in (83, 113] \\ 0, & t > 113 \end{cases}$$

$$V_{Ty}(t) = \begin{cases} 0, & t \in [0, 3] \\ 25(1 - \cos(\frac{\pi}{30}(t - 3))), & t \in (3, 33] \\ 50 + (t - 33), & t \in (33, 43] \\ 30(1 + \cos(\frac{\pi}{30}(t - 43))), & t \in (43, 73] \\ 0, & t \in (73, 83] \\ -25(1 - \cos(\frac{\pi}{30}(t - 83))), & t \in (83, 113] \\ -50, & t > 113 \end{cases}$$

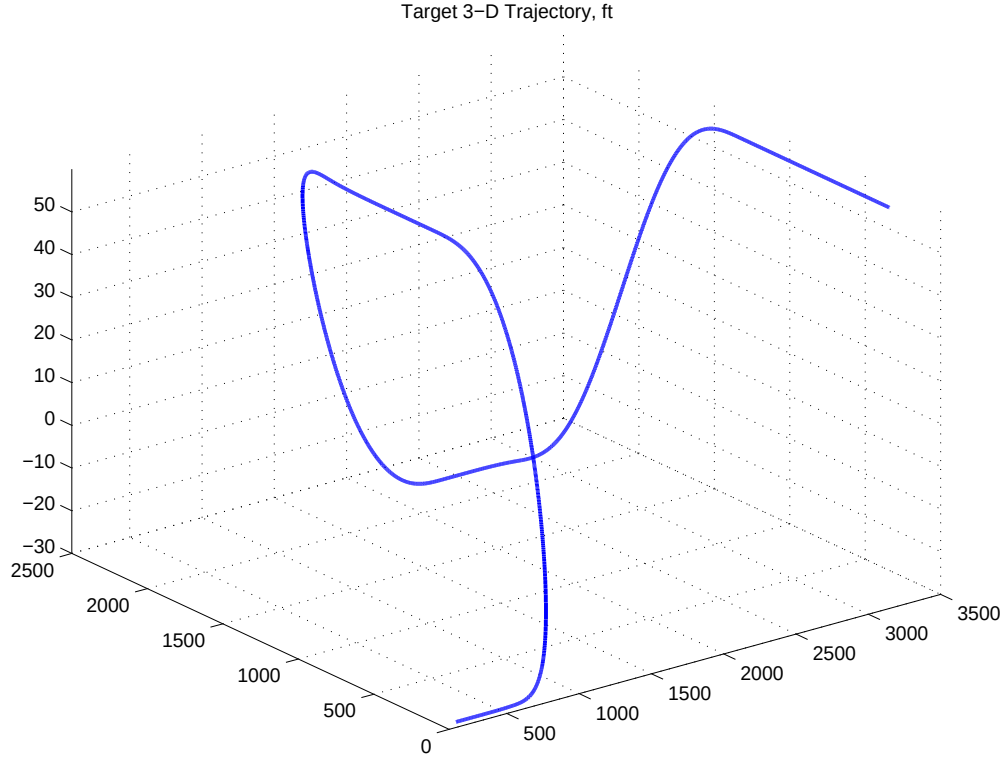


Figure 4.1: Target's 3D trajectory

$$V_{Tz}(t) = \begin{cases} 0, & t \in [0, 8] \\ -3(1 - \cos(\frac{\pi}{15}(t - 8))), & t \in (8, 38] \\ 0, & t \in (36, 50] \\ 3(1 - \cos(\frac{\pi}{15}(t - 50))), & t \in (50, 80] \\ 0, & t \in (80, 90] \\ -3(1 - \cos(\frac{\pi}{15}(t - 90))), & t \in (90, 120] \\ 0, & t > 120 \end{cases}$$

These maneuvers can be related to obstacle avoidance or other objectives that the target might be pursuing. The 3-D trajectory of the target is presented in Figure 4.2 and the

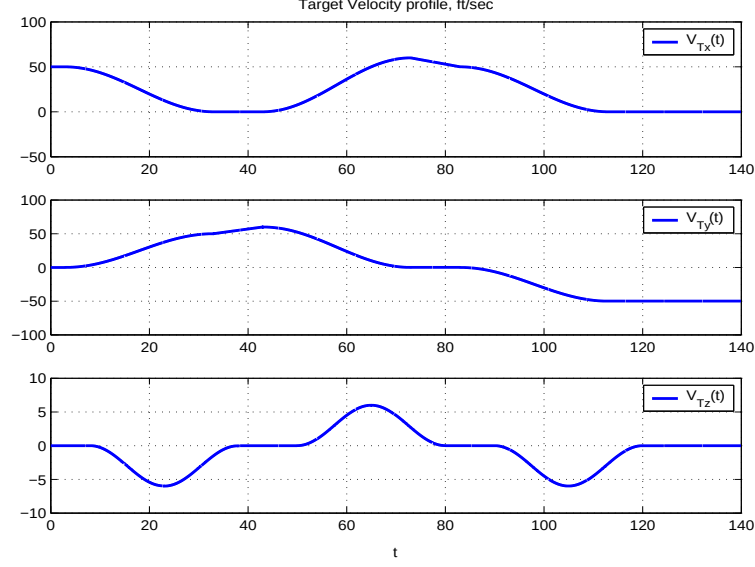


Figure 4.2: Target's velocity profile

velocity profile is presented in in Figure 4.5. Thus, the target's velocity can be represented as $\mathbf{V}_T(t) = b(\mathbf{d} + \boldsymbol{\delta}(t))$, where $\mathbf{d} = [0 \ -6.25 \ 0]^\top ft/sec$ is a constant term and $\boldsymbol{\delta}(t)$ is a time-varying term that captures all the maneuvers. It is easy to see that $\|\boldsymbol{\delta}\|_\infty$ is finite. We recall that $\dot{\boldsymbol{\delta}}(t)$ represents the target's acceleration, and, hence, during the maneuvers it is bounded. Also, $\boldsymbol{\delta}(t) \in \mathcal{L}_2$, since all the maneuvers are made on the finite interval of time. Thus, the target's velocity satisfies the assumptions of Theorem 11. In the formation flight scenario, the follower is commanded to maintain a relative position $\mathbf{R}_c = [32 \ 8 \ 0]^\top ft$. The initial position of the target is chosen to be $\mathbf{R}_{T_0} = [200 \ 50 \ 30]^\top ft$, and the follower initially is at the origin of the inertial frame. In order to reduce the transient, a command shaping filter is used with the poles of -0.1 and with the initial condition at $\mathbf{r}(0) = \frac{1}{b}\mathbf{R}_{T_0}$, which is available through the visual measurements. The guidance law is implemented according to (4.3) with $k = 3.5$ and $\sigma = 39$, $G = 40$. Simulation results are shown in Figures 4.3, 4.4, 4.5. Figure 4.5 shows that the guidance law is able to capture the target's velocity profile after a

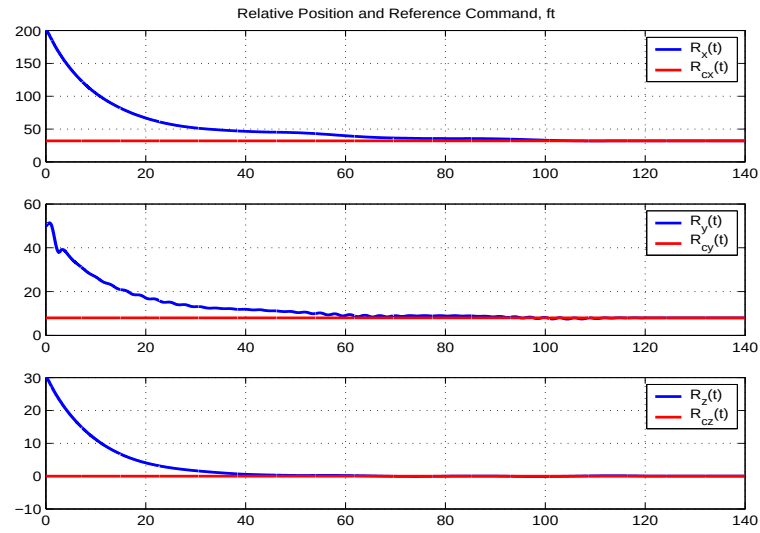


Figure 4.3: Relative position versus reference command, velocity guidance

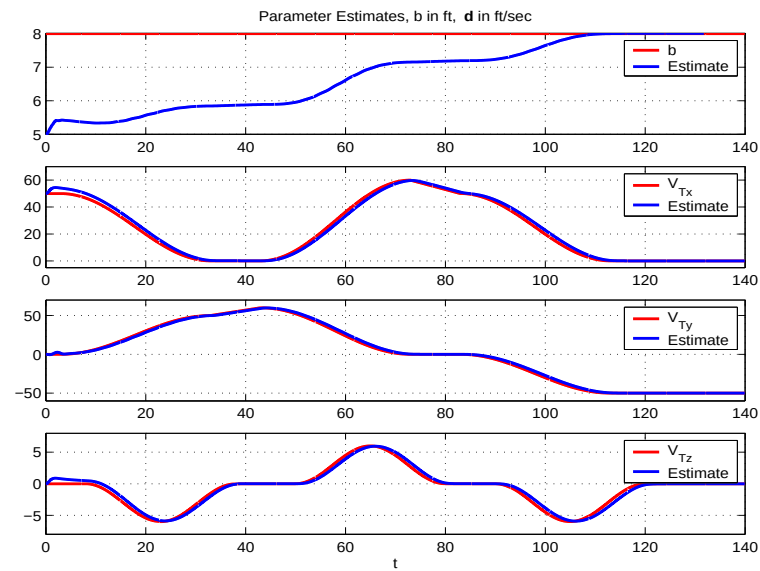


Figure 4.4: Parameter convergence, velocity guidance

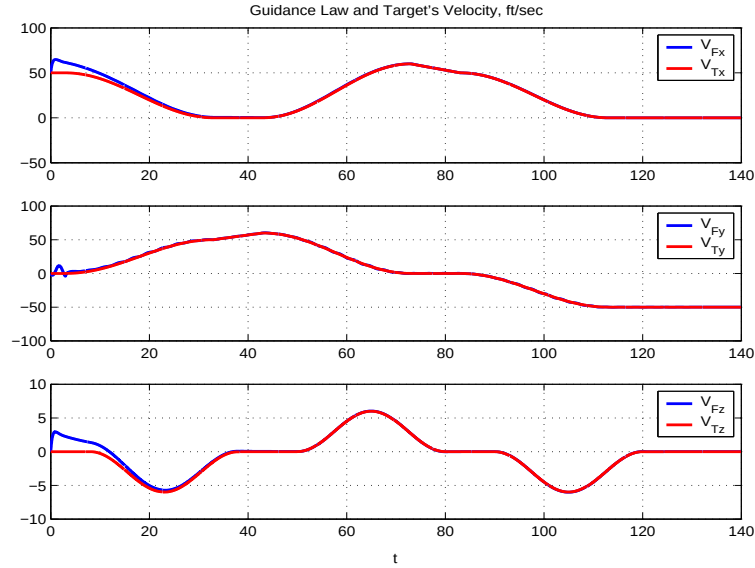


Figure 4.5: Guidance law versus target's velocity, velocity guidance

short transient. The parameter convergence is shown in Figure 4.4. The large fluctuations in the estimation of $\mathbf{d}$ are due to the presence of target acceleration during the maneuvers. The target's size estimation gradually converges to the true value. The system output tracking is demonstrated in Figure 4.3.

For the target interception simulation the settings are used except for the reference command, which is equal to zero. The guidance law is generated according to Remark 5. The interception is assumed to take place if $\|\mathbf{R}(t)\| < 2.4 \text{ ft}$, which is reasonable for 8 ft wingspan target. After the criterion is met at $t = 44.8 \text{ sec}$, the simulation is stopped. The output tracking is displayed in Figure 4.6. Figure 4.7 displays the guidance law versus the target's velocity.

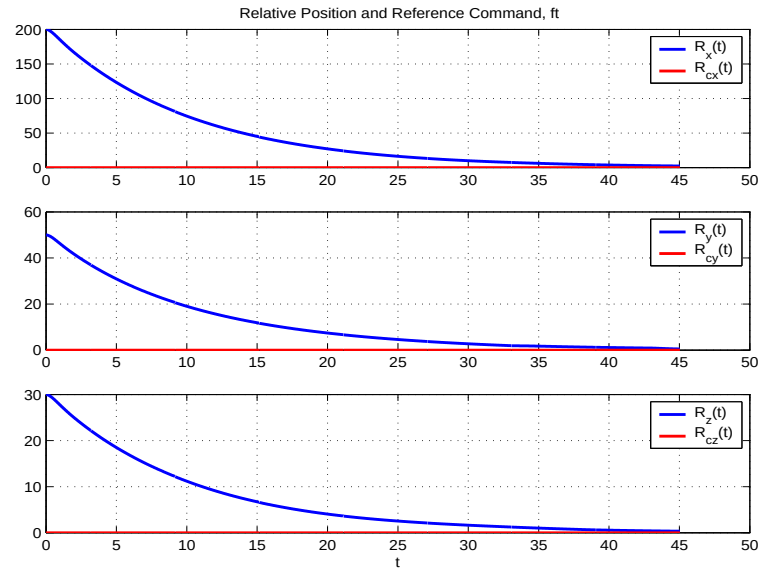


Figure 4.6: Relative position versus reference command in interception, velocity guidance

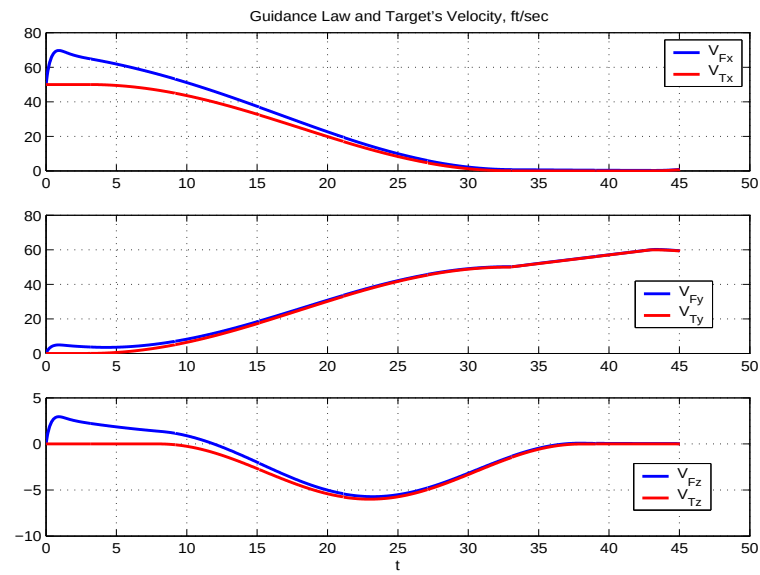


Figure 4.7: Followers velocity versus target's velocity in interception, velocity guidance

Chapter 5

Acceleration Command

In this chapter, we consider **Problem 2** for the system in (3.11), which can be written in the matrix form as follows:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + \frac{1}{b}B[\mathbf{a}_T(t) - \mathbf{a}_F(t)], \quad \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{y}(t) &= C\mathbf{x}(t),\end{aligned}\tag{5.1}$$

where $\mathbf{x}(t) = [r_x(t) \ \dot{r}_x(t) \ r_y(t) \ \dot{r}_y(t) \ r_z(t) \ \dot{r}_z(t)]^\top$ is the system state, b is the unknown parameter, $\mathbf{a}_F(t) = [a_{Fx}(t) \ a_{Fy}(t) \ a_{Fz}(t)]^\top$ is the follower's control input, $\mathbf{a}_T(t) = [a_{Tx}(t) \ a_{Ty}(t) \ a_{Tz}(t)]^\top$ is the bounded disturbance, $\mathbf{y}(t) = [r_x(t) \ r_y(t) \ r_z(t)]^\top$ is the regulated output, which is available for feedback,

$$A = \begin{bmatrix} \mathcal{A} & 0 & 0 \\ 0 & \mathcal{A} & 0 \\ 0 & 0 & \mathcal{A} \end{bmatrix}, \quad B = \begin{bmatrix} \mathcal{B} & 0 & 0 \\ 0 & \mathcal{B} & 0 \\ 0 & 0 & \mathcal{B} \end{bmatrix}, \quad C = \begin{bmatrix} \mathcal{C} & 0 & 0 \\ 0 & \mathcal{C} & 0 \\ 0 & 0 & \mathcal{C} \end{bmatrix},$$

where the matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ are in controllable-observable-canonical forms:

$$\mathcal{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The reference command for the system in (5.1) is given by

$$\mathbf{y}_c(t) = \frac{1}{b} \mathbf{R}_c(t).$$

The control design problem formulated above implies defining an adaptive output feedback disturbance rejection controller for the linear system in (5.1) that has vector relative degree $[2 \ 2 \ 2]$. As stated in Section 2.2, the adaptive output feedback control problem can be solved only for systems that have strictly positive real (SPR) transfer functions, which is not the case if the relative degree is higher than one in each control channel (for an overview on this fundamental limitation the reader can refer to [78]). To proceed with the control problem formulated above, we apply input filtered state transformation introduced in Section 2.2 for the single-input single-output that keeps the output unchanged while transforming the system into one with relative degree 1 in each control channel. The difference is that we need to also filter the disturbance along with the input to achieve the desired property. Since the disturbance is unknown, this step does not seem to be implementable, but we later show that this step is needed only for analysis purposes.

5.1 Input Filtered Transformation

Consider the following transformation of coordinates for the system in (5.1):

$$\mathbf{z}(t) = \mathbf{x}(t) - \frac{1}{b} B[\mathbf{w}(t) + \mathbf{d}(t)], \quad (5.2)$$

in which $\mathbf{w}(t), \mathbf{d}(t)$ are generated via stable filters

$$\begin{aligned} \dot{\mathbf{w}}(t) &= \Lambda \mathbf{w}(t) - \mathbf{a}_F(t), & \mathbf{w}(0) &= \mathbf{0} \\ \dot{\mathbf{d}}(t) &= \Lambda \mathbf{d}(t) + \mathbf{a}_T(t), & \mathbf{d}(0) &= \mathbf{0}, \end{aligned} \quad (5.3)$$

where

$$\Lambda = \begin{bmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_3 \end{bmatrix},$$

and $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$ are design constants. Here we notice that the bounded disturbance $\mathbf{a}_T(t)$ in the second filter in (5.3) will generate a bounded output $\mathbf{d}(t)$, since the matrix Λ is Hurwitz. Thus, there exist unknown constants $d_{Ti} > 0$, $i = 1, 2, 3$ such that $|d_i(t)| \leq d_{Ti}$ for all $t \geq 0$. In the synthesis approach below, we will use the adaptive bounding technique from [47] to adapt to these unknown constants d_{Ti} .

Differentiating both sides of (5.2) and taking into account (5.3) and (5.1), we obtain the following equation

$$\dot{\mathbf{z}}(t) = A\mathbf{z}(t) + \frac{1}{b}[AB - B\Lambda][\mathbf{w}(t) + \mathbf{d}(t)]. \quad (5.4)$$

It is straightforward to verify that the pair $(A, \bar{B})$ is controllable, where $\bar{B} = AB - B\Lambda = \text{diag}(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$, and $\mathcal{B}_i = [1 \ \lambda_i]^\top$, $i = 1, 2, 3$. Since $CB = 0$, the output in (5.1) can be equivalently written as $\mathbf{y}(t) = C\mathbf{z}(t)$. Thus, the system in (5.1) is transformed into one with the transmission zeros in the left half plane and with the well defined vector relative degree of $[1 \ 1 \ 1]$ from the input $\mathbf{w}(t) + \mathbf{d}(t)$ to the output $\mathbf{y}(t)$:

$$\begin{aligned} \dot{\mathbf{z}}(t) &= A\mathbf{z}(t) + \frac{1}{b}\bar{B}[\mathbf{w}(t) + \mathbf{d}(t)] \\ \mathbf{y}(t) &= C\mathbf{z}(t). \end{aligned} \quad (5.5)$$

Here, we notice that by means of the transformations in (5.2), (5.3) the control objective can be realized in two steps. First, we need to design a control input $\mathbf{w}_c(t)$ for the system in (5.5) in the presence of the unknown parameter $\frac{1}{b}$ and bounded disturbance $\mathbf{d}(t)$ to adaptively track the reference command $\mathbf{y}_c(t)$. Next, we design a control input $\mathbf{a}_F(t)$ for the system in (5.3) to track the reference command $\mathbf{w}_c(t)$ using conventional block backstepping [33].

5.2 Reference Model

The controller $\mathbf{w}(t)$ for the system in (5.5) is designed by augmenting the conventional model reference adaptive control scheme with a robustifying term to compensate for the unknown disturbance $\mathbf{d}(t)$. Towards this end, first we design a reference model

$$\begin{aligned}\dot{\mathbf{z}}_m(t) &= \mathbf{A}_m \mathbf{z}(t) + \mathbf{B}_m \mathbf{y}_c(t) \\ \mathbf{y}_m(t) &= \mathbf{C} \mathbf{z}_m(t).\end{aligned}\tag{5.6}$$

Since the matrices $\mathbf{A}, \bar{\mathbf{B}}, \mathbf{C}$ are block diagonal, it is natural to select a block diagonal matrix

$$\mathbf{K} = \begin{bmatrix} \mathcal{K}_1 & 0 & 0 \\ 0 & \mathcal{K}_2 & 0 \\ 0 & 0 & \mathcal{K}_3 \end{bmatrix}$$

such that

$$\mathbf{A}_m = \mathbf{A} - \bar{\mathbf{B}}\mathbf{K} = \begin{bmatrix} \mathcal{A}_{m1} & 0 & 0 \\ 0 & \mathcal{A}_{m2} & 0 \\ 0 & 0 & \mathcal{A}_{m3} \end{bmatrix},$$

where $\mathcal{A}_{mi} = \mathcal{A} - \mathcal{B}_i \mathcal{K}_i$, ($i = 1, 2, 3$), and

$$\mathbf{B}_m = \bar{\mathbf{B}}\mathbf{K}\mathbf{C}^\top = \begin{bmatrix} \mathcal{B}_{m1} & 0 & 0 \\ 0 & \mathcal{B}_{m2} & 0 \\ 0 & 0 & \mathcal{B}_{m3} \end{bmatrix},$$

where $\mathcal{B}_{mi} = \mathcal{B}_i \mathcal{K}_i \mathcal{C}^\top$, ($i = 1, 2, 3$). Here $\mathcal{K}_i = [k_{i1} \ k_{i2}]$, ($i = 1, 2, 3$). It is straightforward to verify that if $k_{i1} > 0, k_{i1} + \lambda_i k_{i2} > 0$, ($i = 1, 2, 3$), then $\mathbf{A}_m$ is Hurwitz and the transfer function from the input $y_{ci}(t)$ to the output $y_{mi}(t)$ is

$$G_i(s) = k_{i1} \frac{s + \lambda_i}{s^2 + (k_{i1} + \lambda_i k_{i2})s + \lambda_i k_{i1}}.\tag{5.7}$$

It follows that $G_i(s) \rightarrow 1$ as $s \rightarrow 0$, that is $\mathbf{y}_m(t) \rightarrow \mathbf{y}_c$ as $t \rightarrow \infty$, provided that $\mathbf{y}_c$ is a constant. It can be shown by straightforward computation that $G_i(s)$ is SPR if

$$k_{i1} + \lambda_i k_{i2} \geq \lambda_i.$$

Then, according to Kalman-Yakubovich-Popov lemma, there exist $P_1 = P_1^\top > 0$ and $Q_1 > 0$ such that

$$A_m^\top P_1 + P_1 A_m = -Q_1, \quad P_1 \bar{B} = C^\top. \quad (5.8)$$

5.3 Stabilizing Virtual Controller Design

Consider a virtual controller

$$\begin{aligned} \mathbf{w}_c(t) &= \hat{b}(t)\mathbf{h}(t) - S(\cdot)\hat{\mathbf{d}}(t) \\ \mathbf{h}(t) &= -K\hat{\mathbf{z}}(t) + KC^\top \mathbf{y}_c(t), \end{aligned} \quad (5.9)$$

where $\hat{\mathbf{z}}(t)$ is the estimate of the state $\mathbf{z}(t)$, governed by the following dynamics

$$\begin{aligned} \dot{\hat{\mathbf{z}}}(t) &= A_m \hat{\mathbf{z}}(t) + B_m \mathbf{y}_c(t) + L[\mathbf{y}(t) - \hat{\mathbf{y}}(t)] \\ \hat{\mathbf{y}}(t) &= C\hat{\mathbf{z}}(t), \end{aligned} \quad (5.10)$$

in which $\hat{b}(t)$ is the estimate of the unknown parameter b , $\hat{\mathbf{d}}(t)$ is the estimate of the unknown bound $\mathbf{d}_T$ and $S(\cdot)$ is a robustifying matrix-function to be defined shortly. At first we need to show that $\mathbf{w}_c(t)$ in (5.9) stabilizes the $\mathbf{z}$ -dynamics in (5.5) in the absence of filters from (5.3). Towards that end, we directly substitute it into (5.5):

$$\begin{aligned} \dot{\mathbf{z}}(t) &= A\mathbf{z}(t) + \frac{1}{b}\bar{B} \left\{ (b + \tilde{b}(t)) [-K\hat{\mathbf{z}}(t) + KC^\top \mathbf{y}_c(t)] + \mathbf{d}(t) - S(\cdot)\hat{\mathbf{d}}(t) \right\} \\ &= A_m \mathbf{z}(t) + B_m \mathbf{y}_c(t) + \bar{B}K\tilde{\mathbf{z}}(t) + \frac{1}{b}\bar{B} \left[\tilde{b}(t)\mathbf{h}(t) + \mathbf{d}(t) - S(\cdot)\hat{\mathbf{d}}(t) \right], \end{aligned} \quad (5.11)$$

where $\tilde{b}(t) = \hat{b}(t) - b$ is the parameter estimation error and $\tilde{\mathbf{z}}(t) = \mathbf{z}(t) - \hat{\mathbf{z}}(t)$ is the state estimation error, the dynamics of which can be derived straightforwardly:

$$\begin{aligned}\dot{\tilde{\mathbf{z}}}(t) &= (A - LC)\tilde{\mathbf{z}}(t) + \frac{1}{b}\bar{B} \left[\tilde{b}(t)\mathbf{h}(t) + \mathbf{d}(t) - S(\cdot)\hat{\mathbf{d}}(t) \right] \\ \tilde{\mathbf{y}}(t) &= C\tilde{\mathbf{z}}(t).\end{aligned}\tag{5.12}$$

The gain matrix L is chosen to render $A_o = A - LC$ Hurwitz and to ensure that the transfer matrix $G_o(s) = C(sI - A_o)^{-1}\bar{B}$ is SPR. Naturally, it can be chosen in block diagonal form:

$$L = \begin{bmatrix} L_1 & 0 & 0 \\ 0 & L_2 & 0 \\ 0 & 0 & L_3 \end{bmatrix}, \quad L_i = \begin{bmatrix} l_{i1} \\ l_{i2} \end{bmatrix}, \quad i = 1, 2, 3.$$

Then

$$A_o = \begin{bmatrix} \mathcal{A}_{o1} & 0 & 0 \\ 0 & \mathcal{A}_{o2} & 0 \\ 0 & 0 & \mathcal{A}_{o3} \end{bmatrix}, \quad \mathcal{A}_{oi} = \mathcal{A}_i - L_i C, \quad i = 1, 2, 3$$

and

$$G_o(s) = \begin{bmatrix} G_{o1}(s) & 0 & 0 \\ 0 & G_{o2}(s) & 0 \\ 0 & 0 & G_{o3}(s) \end{bmatrix}, \quad G_{oi}(s) = \frac{s + \lambda_i}{s^2 + l_{i1}s + l_{i2}}, \quad i = 1, 2, 3.$$

Hence, A_o is Hurwitz and $G_o(s)$ is SPR, if A_{o1} , A_{o2} , A_{o3} are Hurwitz and $G_{o1}(s)$, $G_{o2}(s)$, $G_{o3}(s)$ are SPR, which can be ensured by choosing $l_{i1} \geq \lambda_i$ and $l_{i2} > 0$. Then, following Kalman-Yakubovich-Popov lemma, there exist $P_2 = P_2^\top > 0$ and $Q_2 = Q_2^\top > 0$ such that

$$A_o^\top P_2 + P_2 A_o = -Q_2, \quad P_2 \bar{B} = C^\top.\tag{5.13}$$

Denoting the tracking error by $\mathbf{e}(t) = \mathbf{z}(t) - \mathbf{z}_m(t)$, the error dynamics are written as

$$\begin{aligned}\dot{\mathbf{e}}(t) &= A_m \mathbf{e}(t) + \bar{B} K \tilde{\mathbf{z}}(t) + \frac{1}{b}\bar{B} \left[\tilde{b}(t)\mathbf{h}(t) + \mathbf{d}(t) - S(\cdot)\hat{\mathbf{d}}(t) \right] \\ \mathbf{e}_c(t) &= C\mathbf{e}(t)\end{aligned}\tag{5.14}$$

Introducing the parameter estimation $\tilde{\mathbf{d}}(t) = \hat{\mathbf{d}}(t) - \mathbf{d}_T$, we combine both the tracking and the state estimation errors dynamics in (5.14) and (5.12) in one equation

$$\begin{aligned}\dot{\boldsymbol{\xi}}_a(t) &= A_a \boldsymbol{\xi}_a(t) + \frac{1}{b} B_a \left[\tilde{b}(t) \mathbf{h}(t) + \mathbf{d}(t) - S(\cdot) \mathbf{d}_T - S(\cdot) \tilde{\mathbf{d}}(t) \right] \\ \mathbf{y}_a(t) &= C_a \boldsymbol{\xi}_a(t),\end{aligned}\tag{5.15}$$

where

$$\boldsymbol{\xi}_a = \begin{bmatrix} \mathbf{e} \\ \tilde{\mathbf{z}} \end{bmatrix}, \quad \mathbf{y}_a = \begin{bmatrix} \mathbf{e}_c \\ \tilde{\mathbf{y}} \end{bmatrix}, \quad A_a = \begin{bmatrix} A_m & \bar{B}K \\ 0_{6 \times 6} & A_o \end{bmatrix}, \quad B_a = \begin{bmatrix} \bar{B} \\ \bar{B} \end{bmatrix}, \quad C_a = \begin{bmatrix} C & C \end{bmatrix}.$$

Now we can define the matrix-function $S(\cdot)$ as follows:

$$S(\mathbf{y}_a) = \begin{bmatrix} \text{sgn}(y_{a1}) & 0 & 0 \\ 0 & \text{sgn}(y_{a2}) & 0 \\ 0 & 0 & \text{sgn}(y_{a3}) \end{bmatrix},\tag{5.16}$$

where the $\text{sgn}(\cdot)$ is defined as

$$\text{sgn}(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}.\tag{5.17}$$

Notice that the function $S(\mathbf{y}_a)$ is discontinuous, and consequently the combined error dynamics have discontinuous right hand sides. Thus, the solutions are treated in the Filippov's sense and the extension of Lyapunov theory presented in Section 2.5 is applied.

From the relationships in (5.8) and (5.13) it follows that

$$A_a^\top P_a + P_a A_a = -Q_a, \quad P_a B_a = C_a^\top,\tag{5.18}$$

where

$$P_a = \begin{bmatrix} P_1 & 0_{6 \times 6} \\ 0_{6 \times 6} & P_2 \end{bmatrix}, \quad Q_a = \begin{bmatrix} Q_1 & 0_{6 \times 6} \\ 0_{6 \times 6} & Q_2 \end{bmatrix}.$$

To complete the controller design, we need the adaptive laws for the online update of parameter estimates $\hat{b}(t)$ and $\hat{\mathbf{d}}(t)$. These are given as follows:

$$\dot{\hat{\mathbf{d}}}(t) = GS(\mathbf{y}_a(t))\mathbf{y}_a(t), \quad \hat{\mathbf{d}}(0) = \mathbf{0} \quad (5.19)$$

$$\dot{\hat{b}}(t) = \rho \text{Proj} \left(\hat{b}(t), -\mathbf{y}_a^\top(t)\mathbf{h}(t) \right), \quad \hat{b}(0) = b_0 > 0, \quad (5.20)$$

where $\rho > 0$ and $G > 0$ are constant adaptation gains, $\text{Proj}(\cdot, \cdot)$ is the projection operator [49] (see Appendix A).

5.4 Stability Analysis

Theorem 12 *The robust adaptive virtual control $\mathbf{w}_c(t)$ in (5.9), along with the adaptive laws in (5.19) and (5.20), guarantees uniform ultimate boundedness of all error signals and asymptotic convergence of $\boldsymbol{\xi}_a(t)$ to zero.* $\square$

Proof. We choose candidate Lyapunov function dependent on all the states of (5.15), (5.19) and (5.20):

$$V_1(\boldsymbol{\xi}_a(t), \tilde{b}(t), \tilde{\mathbf{d}}(t)) = \boldsymbol{\xi}_a^\top(t) P_a \boldsymbol{\xi}_a(t) + \frac{1}{b} \left(\frac{\tilde{b}^2(t)}{\rho} + \tilde{\mathbf{d}}^\top(t) G^{-1} \tilde{\mathbf{d}}(t) \right). \quad (5.21)$$

The derivative of $V_1(t)$ along the trajectories of the system (5.15), (5.19) and (5.20) has the form

$$\begin{aligned} \dot{V}_1(t) &= \boldsymbol{\xi}_a^\top(t) [A_a^\top P_a + P_a A_a] \boldsymbol{\xi}_a(t) + \frac{2}{b} \left(\frac{\tilde{b}(t)}{\rho} \dot{\tilde{b}}(t) + \tilde{\mathbf{d}}^\top(t) G^{-1} \dot{\tilde{\mathbf{d}}}(t) \right) \\ &+ 2\boldsymbol{\xi}_a^\top(t) P_a \frac{1}{b} B_a \left[\tilde{b}(t)\mathbf{h}(t) + \mathbf{d}(t) - S(\mathbf{y}_a)\mathbf{d}_T - S(\mathbf{y}_a)\tilde{\mathbf{d}}(t) \right] \\ &\leq -\boldsymbol{\xi}_a^\top(t) Q_a \boldsymbol{\xi}_a(t) + \frac{2\tilde{b}(t)}{b} \left[\mathbf{y}_a^\top(t)\mathbf{h}(t) + \frac{1}{\rho} \dot{\tilde{b}}(t) \right] \\ &+ \frac{2}{b} \tilde{\mathbf{d}}^\top(t) \left[-S(\mathbf{y}_a)\mathbf{y}_a(t) + G^{-1} \dot{\tilde{\mathbf{d}}}(t) \right] + \frac{2}{b} \boldsymbol{\xi}_a^\top(t) P_a B_a [\mathbf{d}(t) - S(\mathbf{y}_a)\mathbf{d}_T]. \end{aligned} \quad (5.22)$$

We notice that $\dot{V}_1(t)$ is discontinuous due to the discontinuity of $S(\mathbf{y}_a(t))$ on the switching manifold $\Omega_S = \{\boldsymbol{\xi}_a : C_a \boldsymbol{\xi}_a = \mathbf{y}_a = 0\}$. Following [61], we need to calculate $\dot{V}_1(t)$ on the switching manifold Ω_S and away from it. However, in both cases we obtain the same upper bound for $\dot{V}_1(t)$ since

$$\boldsymbol{\xi}_a^\top(t) P_a B_a = \boldsymbol{\xi}_a^\top(t) C_a^\top = \mathbf{y}_a^\top(t),$$

and the last term in (5.22) can be upper bounded as follows

$$\mathbf{y}_a^\top [\mathbf{d}(t) - S(\mathbf{y}_a(t)) \mathbf{d}_T] = \sum_{i=1}^3 [y_{ai}(t) d_i(t) - |y_{ai}(t)| d_{Ti}] \leq 0. \quad (5.23)$$

Upon substitution of the adaptive laws from (5.19) and (5.20) and using the properties of the projection operator from (5.24) (see Appendix A):

$$\begin{aligned} b_{\min} \leq \hat{b}(t) \leq b_{\max} \\ \tilde{b}(t) \left[\mathbf{y}_a^\top(t) \mathbf{h}(t) + \text{Proj} \left(\hat{b}(t), -\mathbf{y}_a^\top(t) \mathbf{h}(t) \right) \right] \leq 0. \end{aligned} \quad (5.24)$$

we conclude that

$$\dot{V}_1(t) \leq -\boldsymbol{\xi}_a^\top(t) Q_a \boldsymbol{\xi}_a(t) \leq -\lambda_{\min}(Q_a) \|\boldsymbol{\xi}_a(t)\|^2, \quad (5.25)$$

where $\lambda_{\min}(Q_a)$ denotes the minimum eigenvalue of the matrix Q_a . Since the inequalities in (5.23) and (5.24) hold for any value of $\mathbf{y}_a(t)$, the inequality in (5.25) holds globally, implying that the signals $\boldsymbol{\xi}_a(t)$, $\tilde{b}(t)$ and $\tilde{\mathbf{d}}(t)$ are globally bounded. It follows that $V_1(t)$ is also bounded. Integration of the inequality in (5.25) results in

$$\lim_{t \rightarrow \infty} \int_0^t \lambda_{\min}(Q_a) \|\boldsymbol{\xi}_a(\tau)\|^2 d\tau \leq \lim_{t \rightarrow \infty} (V_1(0) - V_1(t)) < \infty. \quad (5.26)$$

Since $\mathbf{y}_c(t)$ is bounded, the reference state $\mathbf{z}_m(t)$ is bounded, which implies boundedness of the state $\mathbf{z}(t)$ and, consequently, the estimate $\hat{\mathbf{z}}(t)$. Therefore, $\mathbf{h}(t)$ is bounded. Then, the combined error dynamics in (5.15) imply that $\dot{\boldsymbol{\xi}}_a(t)$ is bounded. Thus, $\boldsymbol{\xi}_a(t) \in \mathcal{L}_\infty \cap \mathcal{L}_2$ and $\dot{\boldsymbol{\xi}}_a(t) \in \mathcal{L}_\infty$. Application of Corollary 1 from Section 2.1 ensures that $\boldsymbol{\xi}_a(t) \rightarrow 0$ as $t \rightarrow \infty$ [54]. The proof is complete. $\square$

5.5 Parameter Convergence

In the case of visual measurement, as discussed above, $\mathbf{y}_c(t)$ depends upon the unknown parameters, and hence is not available for the virtual control design in (5.9). For the given bounded reference commands $\mathbf{R}_c(t)$ one can use the estimated reference input $\hat{\mathbf{y}}_c(t)$ in (5.9) as follows: $\hat{\mathbf{y}}_c(t) = \frac{\mathbf{R}_c(t)}{\hat{b}(t)}$, where $\hat{b}(t)$ is obtained from (5.20). Notice that Theorem 12 is true for any bounded $\mathbf{y}_c(t)$, and hence it holds if one replaces $\mathbf{y}_c(t)$ by $\hat{\mathbf{y}}_c(t)$. Since the objective is the tracking of $\mathbf{y}_c(t)$, one needs to ensure that $\hat{\mathbf{y}}_c(t)$ converges to $\mathbf{y}_c(t)$, which will hold if $\hat{b}(t)$ converges to the true value of b .

From the adaptive law in (5.19) it follows that $\hat{\mathbf{d}}(t)$ is a nondecreasing function, and since it is initialized at 0, then following Theorem 12 it is bounded above, therefore it converges to some finite limit $\bar{\mathbf{d}} > 0$ as $t \rightarrow \infty$, which need not be the true value of $\mathbf{d}_T$. Also, we note that the Lyapunov function $V_1(t)$ is nonincreasing and bounded below, so it also has a limit as $t \rightarrow \infty$. Since all the variables except for $\tilde{b}(t)$ on the right hand side of (5.21) are convergent, $\tilde{b}(t)$ also converges. Therefore, $\exists \bar{b}$, such that $\hat{b}(t) \rightarrow \bar{b}$ as $t \rightarrow \infty$. To ensure that $\bar{b} = b$, a common approach is to introduce an excitation signal in the reference input. In realistic applications, however, a persistently exciting signal is not desirable. Therefore we use the intelligent excitation technique from [10]. This technique modifies the reference input by adding a sinusoidal signal $k(t) \sin(\omega t)$ to the reference input $R_c(t)$, the amplitude $k(t)$ of which depends on the tracking error as follows:

$$k(t) = \begin{cases} k_0, & t \in [0, T) \\ \min \left[k_1 \int_{(j-1)T}^{jT} \|\boldsymbol{\xi}(\tau)\|^2 d\tau, k_2 - k_3 \right] + k_3, & t \in [jT, (j+1)T) \end{cases} \quad (5.27)$$

where $k_i > 0$, $i = 0, 1, 2, 3$ are design constants, k_3 being a sufficiently small number¹, $j = 1, 2, \dots$ and $T = \frac{2\pi}{\omega}$. Since the Lyapunov proofs are true for any continuous and

¹Commonly referred to as regularization term to ensure boundedness away from zero

bounded reference input, they are valid also for the modified reference input. The new reference commands are obtained according to

$$\hat{\mathbf{y}}_c(t) = \frac{1}{\hat{b}(t)} [\mathbf{R}_c(t) + k(t) \sin(\omega t) \mathbf{E}],$$

where the vector $\mathbf{E}$ specifies the direction of the excitation. For instance, we can excite only the reference command in x direction by setting $\mathbf{E} = [1 \ 0 \ 0]^\top$. As it is proved in [10], this will guarantee convergence of the estimate $\hat{b}(t)$ to the true value b . Therefore, the estimated reference input $\hat{\mathbf{y}}_c(t)$ will converge to the reference input $\mathbf{y}_c(t)$, in which $\mathbf{R}_c(t)$ is replaced by $\mathbf{R}_c(t) + k(t) \sin(\omega t)$. It is easy to see that $k(t) \rightarrow k_3$ as $\boldsymbol{\xi}_a(t) \rightarrow 0$. Thus, as the tracking error goes to zero, the amplitude of the excitation is proportional to k_3 . Hence, one can make the *excited* command to be as close to $\mathbf{R}_c(t)$ as needed by choosing a sufficiently small k_3 .

Remark 6 *For the target interception problem, the reference command is $R_c = 0$. Therefore, $\mathbf{y}_c = 0$. Then, the virtual control*

$$\mathbf{w}_c(t) = -\hat{b}(t)K\hat{\mathbf{z}}(t) - S(\mathbf{y}_a)\hat{\mathbf{d}}(t), \quad (5.28)$$

along with the adaptive laws

$$\begin{aligned} \dot{\hat{b}}(t) &= \rho \text{Proj} \left(\hat{b}(t), -\mathbf{y}_a^\top(t)K\hat{\mathbf{z}}(t) \right), \quad \hat{b}(0) = b_0 > 0 \\ \dot{\hat{\mathbf{d}}}(t) &= GS(\mathbf{y}_a)\mathbf{y}_a(t), \quad \hat{\mathbf{d}}(0) = \mathbf{0}, \end{aligned} \quad (5.29)$$

guarantees the boundedness of all the signals and the convergence of the output to zero: $\mathbf{y}(t) \rightarrow 0$ as $t \rightarrow \infty$. There is no need to require parameter convergence, and hence, the need for introducing an excitation signal vanishes. $\square$

Remark 7 *An important aspect of visual control is to ensure that the target always stays in the follower's field of view, i.e. the designed controller has to guarantee the inequality*

$|\lambda(t)| \leq \lambda_m$ for all $t \geq 0$, where λ_m is the maximal bearing angle of the camera. This in turn implies that the following inequalities need to hold for all $t \geq 0$: $\left| \frac{r_y^B(t)}{r_x^B(t)} \right| \leq \tan(\lambda_m)$ and $r_x^B(t) \geq 0$, or in terms of z coordinates $|z_3(t)| \leq z_1(t) \tan(\lambda_m)$, $z_1(t) \geq 0$. With a proper choice of the reference model, the field of view requirement can be translated into a condition on the initial value of the associated Lyapunov function, which can be guaranteed via the choice of appropriate adaptive gains using the conservative bounds on the unknown parameters. $\square$

5.6 Backstepping

We still need to determine the acceleration command $\mathbf{a}_F(t)$ to ensure that $\mathbf{w}(t)$ tracks the virtual control $\mathbf{w}_c(t)$. Recall that $\mathbf{w}(t)$ is generated via the filter equation:

$$\dot{\mathbf{w}}(t) = \Lambda \mathbf{w}(t) - \mathbf{a}_F(t). \quad (5.30)$$

We notice that the function $S(\mathbf{y}_a)$, and hence, the virtual control $\mathbf{w}_c(t)$ is not differentiable. Therefore, it can not be used in the backstepping procedure. One way to circumvent the situation is to replace the discontinuous function $S(\mathbf{y}_a)$ with a smooth approximation $S_\chi(\mathbf{y}_a)$ with the entries defined as follows:

$$s_{\chi ii}(y_{ai}) = \begin{cases} s_{ii}(y_{ai}), & \mathbf{y}_a \notin \Omega_\chi \\ \frac{3}{2\chi} y_{ai} - \frac{1}{2\chi^3} y_{ai}^3, & \mathbf{y}_a \in \Omega_\chi \end{cases}, \quad (5.31)$$

where

$$\Omega_\chi = \{\mathbf{y}_a : |y_{ai}| \leq \chi, i = 1, 2, 3\},$$

and $\chi > 0$ is a design parameter. When $|y_{ai}| = \chi$, the following relationships hold

$$s_{\chi ii}(y_{ai}) = s_{ii}(y_{ai}), \quad \frac{\partial s_{\chi ii}(y_{ai})}{\partial y_{ai}} = \frac{\partial s_{ii}(y_{ai})}{\partial y_{ai}} = 0. \quad (5.32)$$

Replacing $S(\mathbf{y}_a)$ in (5.9) with $S_\chi(\mathbf{y}_a)$ results in a smooth virtual control

$$\mathbf{w}_c(t) = \hat{b}(t)\mathbf{h}(t) - S_\chi(\mathbf{y}_a)\hat{\mathbf{d}}(t). \quad (5.33)$$

However, this approximation will complicate the backstepping procedure, since its derivative will involve $\dot{\mathbf{y}}_a(t)$, and hence, $\dot{\boldsymbol{\xi}}_a(t)$, resulting in modifications of the adaptive laws and more complex control architecture. Therefore, we replace $S_\chi(\mathbf{y}_a)$ with its filtered version $S_f(t)$, where

$$\dot{S}_f(t) = A_f S_f(t) - S_\chi(\mathbf{y}_a(t)), \quad S_f(0) = 0, \quad (5.34)$$

where A_f is a diagonal matrix with the entries $a_{fi} < 0$, $i = 1, 2, 3$. Since $|s_{\chi i}(y_{ai}(t))| \leq 1$ for any $\mathbf{y}_a(t)$, the following bound can be written immediately:

$$|s_{fi}(t)| = \left| \int_0^t \exp[a_{fi}(t - \tau)] s_{\chi i}(y_{ai}(\tau)) d\tau \right| \leq -a_{fi}^{-1} (1 - \exp[a_{fi}t]) \leq -a_{fi}^{-1}, \quad (5.35)$$

that is $S_f(t)$ is bounded for any $\mathbf{y}_a(t)$. Introducing the error $\mathbf{e}_w(t) = \mathbf{w}(t) - \mathbf{w}_f(t)$, where

$$\mathbf{w}_f(t) = \hat{b}(t)\mathbf{h}(t) - S_f(t)\hat{\mathbf{d}}(t), \quad (5.36)$$

and differentiating, we can write the filter dynamics in (5.3) as follows:

$$\dot{\mathbf{e}}_w(t) = \Lambda \mathbf{w}(t) - \mathbf{a}_F(t) - \dot{\mathbf{w}}_f(t), \quad (5.37)$$

where the signal $\dot{\mathbf{w}}_f(t)$ is computed as

$$\begin{aligned} \dot{\mathbf{w}}_f(t) &= -\dot{\hat{b}}(t) [-K\hat{\mathbf{z}}(t) + KC^\top \mathbf{y}_c(t)] - \hat{b}(t) [-K\dot{\hat{\mathbf{z}}}(t) + KC^\top \dot{\mathbf{y}}_c(t)] \\ &\quad - S_f(t)\dot{\hat{\mathbf{d}}}(t) + [A_f(S_f(t) - S(\mathbf{y}_a))]\hat{\mathbf{d}}(t), \end{aligned}$$

and is available for feedback. Therefore, the guidance law $\mathbf{a}_F(t)$ can be designed as

$$\mathbf{a}_F(t) = \Lambda \mathbf{w}_f(t) - \dot{\mathbf{w}}_f(t) + \mathbf{y}_a(t), \quad (5.38)$$

so that the filter dynamics reduce to

$$\dot{\mathbf{e}}_w(t) = \Lambda \mathbf{e}_w(t) - \mathbf{y}_a(t). \quad (5.39)$$

Since

$$\mathbf{w}(t) - \mathbf{w}_c(t) = \mathbf{e}_w(t) - S_f(t)\hat{\mathbf{d}}(t) + S(\mathbf{y}_a(t))\hat{\mathbf{d}}(t),$$

the error dynamics in (5.15) can be written in the following form

$$\begin{aligned} \dot{\boldsymbol{\xi}}_a(t) &= A_a \boldsymbol{\xi}_a(t) + \frac{1}{b} B_a \left[\tilde{b}(t) \mathbf{h}(t) + \mathbf{d}(t) - S(\mathbf{y}_a(t))\hat{\mathbf{d}}(t) + \mathbf{w}(t) - \mathbf{w}_c(t) \right] \\ &= A_a \boldsymbol{\xi}_a(t) + \frac{1}{b} B_a \left[\tilde{b}(t) \mathbf{h}(t) + \mathbf{d}(t) - S_f(t)\hat{\mathbf{d}}(t) + \mathbf{e}_w(t) \right]. \end{aligned} \quad (5.40)$$

Here, we can prove only ultimate boundedness of the state estimation and tracking errors, provided that the parameter errors remain bounded. The projection operator in (5.20) guarantees boundedness of $\hat{b}(t)$, and therefore $\tilde{b}(t)$ is bounded. To provide the boundedness for $\hat{\mathbf{d}}(t)$, hence for the error $\tilde{\mathbf{d}}(t)$, we have to modify the adaptive law in (5.19). Here, we again choose a projection based adaptive law

$$\dot{\hat{\mathbf{d}}}(t) = G \text{Proj} \left(\hat{\mathbf{d}}(t), S(\mathbf{y}_a(t)) \mathbf{y}_a(t) \right), \quad \hat{\mathbf{d}}(0) = \mathbf{0}, \quad (5.41)$$

that guarantees the inequalities [49]

$$\begin{aligned} \|\hat{\mathbf{d}}(t)\| &\leq d_{\max} \\ \tilde{\mathbf{d}}^\top(t) \left[-S(\mathbf{y}_a(t)) \mathbf{y}_a(t) + \text{Proj} \left(\hat{\mathbf{d}}(t), S(\mathbf{y}_a(t)) \mathbf{y}_a(t) \right) \right] &\leq 0. \end{aligned} \quad (5.42)$$

Since the boundedness of $S_f(t)$ has been established for any signal $\mathbf{y}_a(t)$, its dynamics are not included in the formal proof of boundedness of error signal $\boldsymbol{\xi}_a(t)$. This can be justified by the fact that the error dynamics in (5.40) can be viewed as a non-autonomous system, where $S_f(t)$ is a bounded continuous time-varying signal. We have the following theorem.

Theorem 13 *The guidance law in (5.38), along with the adaptive laws in (5.20) and (5.41), guarantees uniform ultimate boundedness of all error signals of the systems in (5.39) and (5.40). $\square$*

Proof. Consider the following Lyapunov function candidate

$$V_2(\boldsymbol{\xi}_a(t), \tilde{b}(t), \tilde{\mathbf{d}}(t)) = V_1(\boldsymbol{\xi}_a(t), \tilde{b}(t), \tilde{\mathbf{d}}(t), \mathbf{e}_w(t)) + \frac{1}{b} \mathbf{e}_w^\top(t) \mathbf{e}_w(t). \quad (5.43)$$

Its derivative along the solutions of the systems in (5.20), (5.39), (5.40) and (5.41) can be computed following the same steps as above:

$$\begin{aligned} \dot{V}_2(t) &= \boldsymbol{\xi}_a^\top(t) [A_a^\top P_a + P_a A_a] \boldsymbol{\xi}_a(t) + \frac{2}{b} \left(\frac{\tilde{b}(t)}{\rho} \dot{\tilde{b}}(t) + \tilde{\mathbf{d}}^\top(t) G^{-1} \dot{\tilde{\mathbf{d}}}(t) \right) \\ &+ 2\boldsymbol{\xi}_a^\top(t) P_a \frac{1}{b} B_a \left[\tilde{b}(t) \mathbf{h}(t) + \mathbf{d}(t) - S_f(t) \hat{\mathbf{d}}(t) + \mathbf{e}_w(t) \right] - \frac{2}{b} \mathbf{e}_w^\top(t) \Lambda \mathbf{e}_w(t) \\ &= -\boldsymbol{\xi}_a^\top(t) Q_a \boldsymbol{\xi}_a(t) + \frac{2\tilde{b}(t)}{b} \left[\mathbf{y}_a^\top(t) \mathbf{h}(t) + \frac{1}{\rho} \dot{\tilde{b}}(t) \right] \\ &+ \frac{2}{b} \tilde{\mathbf{d}}^\top(t) \left[-S(\mathbf{y}_a(t)) \mathbf{y}_a(t) + G^{-1} \dot{\tilde{\mathbf{d}}}(t) \right] + \frac{2}{b} \boldsymbol{\xi}_a^\top(t) P_a B_a [\mathbf{d}(t) - S(\mathbf{y}_a(t)) \mathbf{d}_T] \\ &+ \frac{2}{b} \mathbf{y}_a^\top(t) [S(\mathbf{y}_a(t)) - S_\chi(\mathbf{y}_a(t)) + S_\chi(\mathbf{y}_a(t)) - S_f(t)] \hat{\mathbf{d}}(t) - \frac{2}{b} \mathbf{e}_w^\top(t) \Lambda \mathbf{e}_w(t) \\ &\leq -\boldsymbol{\xi}_a^\top(t) Q_a \boldsymbol{\xi}_a(t) - \frac{2}{b} \mathbf{e}_w^\top(t) \Lambda \mathbf{e}_w(t) \\ &+ \frac{2}{b} \mathbf{y}_a^\top(t) [S(\mathbf{y}_a(t)) - S_\chi(\mathbf{y}_a(t)) + S_\chi(\mathbf{y}_a(t)) - S_f(t)] \hat{\mathbf{d}}(t). \end{aligned} \quad (5.44)$$

Outside the compact set Ω_χ , we have $S(\mathbf{y}_a) = S_\chi(\mathbf{y}_a)$ and $\dot{S}_\chi(\mathbf{y}_a(t)) = 0$. Therefore, with the notation $E_S(t) = S_f(t) - S_\chi(\mathbf{y}_a(t))$, (5.34) reduces to

$$\dot{E}_S(t) = A_f E_S(t), \quad (5.45)$$

the solution of which can be written as $E_S(t) = \exp(A_f t) E_S(0)$. The following bound can be immediately derived:

$$\|E_S(t)\|_F = \exp(a_f t) \|E_S(0)\|_F, \quad (5.46)$$

where $a_f = \max\{a_{fi}, i = 1, 2, 3\}$. Therefore, the derivative of the Lyapunov function candidate in (5.43) can be upper bounded as follows

$$\dot{V}_2(t) \leq -\lambda_{\min}(Q_a)\|\boldsymbol{\xi}_a(t)\|^2 - \frac{\bar{\lambda}}{b}\|\mathbf{e}_w(t)\|^2 + \frac{2d_{\max}}{b}\exp(a_ft)\|\boldsymbol{\xi}_a(t)\|\|E_S(0)\|_F, \quad (5.47)$$

where $\bar{\lambda} = \min\{\lambda_1, \lambda_2, \lambda_3\}$ and $d_{\max}$ is the bound for the estimate $\hat{\mathbf{d}}(t)$, guaranteed by the projection operator. Completing the squares in (5.47) results in

$$\dot{V}_2(t) \leq -(\lambda_{\min}(Q_a) - c_1^2)\|\boldsymbol{\xi}_a(t)\|^2 - \frac{\bar{\lambda}}{b}\|\mathbf{e}_w(t)\|^2 + c_2 \exp(2a_ft), \quad (5.48)$$

where c_1 is chosen such that $\lambda_{\min}(Q_a) - c_1^2 > 0$ and $c_2 = \frac{d_{\max}^2\|E_S(0)\|_F^2}{b^2c_1^2}$. From the relationship in (5.48), it follows that $\dot{V}_2(t) \leq 0$ outside the compact set

$$\Omega_1 = \left\{ \|\boldsymbol{\xi}_a\| \leq \sqrt{\frac{c_2}{\lambda_{\min}(Q_a) - c_1^2}}, \mathbf{e}_w = \mathbf{0}, |\tilde{b}| \leq \tilde{b}^*, \|\tilde{\mathbf{d}}\| \leq \tilde{d}^* \right\}, \quad (5.49)$$

where the bounds $\tilde{b}^*$ and $\tilde{d}^*$ are guaranteed by the projection operator. Thus, the error signals $\boldsymbol{\xi}_a(t)$, $\tilde{b}(t)$, $\tilde{\mathbf{d}}(t)$, $\mathbf{e}_w(t)$ are uniformly ultimately bounded. Rearranging the equation in (5.48) and integrating we obtain

$$\begin{aligned} & \int_0^t \left[(\lambda_{\min}(Q_a) - c_1^2)\|\boldsymbol{\xi}_a(\tau)\|^2 + \frac{\bar{\lambda}}{b}\|\mathbf{e}_w(\tau)\|^2 \right] d\tau \\ & \leq V_2(0) - V_2(t) + \int_0^t c_2 \exp[-2\lambda_{\min}(A_f)\tau] d\tau < \infty, \end{aligned} \quad (5.50)$$

Thus, $\boldsymbol{\xi}_a(t) \in \mathcal{L}_2(\mathbb{R}^6) \cap \mathcal{L}_\infty(\mathbb{R}^6)$ and $\mathbf{e}_w(t) \in \mathcal{L}_2(\mathbb{R}^3) \cap \mathcal{L}_\infty(\mathbb{R}^3)$. Also, from the boundedness of error signals and reference command it follows that $\mathbf{h}(t)$ is bounded. Since the matrix $S_f(t)$ is bounded, it follows that $\mathbf{w}_f(t)$ and $\dot{\mathbf{w}}_f(t)$ are bounded and therefore, $\mathbf{a}_F(t)$ is bounded. Then, the error dynamics in (5.39) and (5.40) imply that $\dot{\boldsymbol{\xi}}_a(t) \in \mathcal{L}_\infty(\mathbb{R}^6)$ and $\dot{\mathbf{e}}_w(t) \in \mathcal{L}_\infty(\mathbb{R}^3)$. Application of Barbalat's lemma ensures that $\boldsymbol{\xi}_a(t) \rightarrow 0$ and $\mathbf{e}_w(t) \rightarrow 0$ as $t \rightarrow \infty$ [54]. Therefore, there exists time instant $t_\chi > 0$ such that $\mathbf{y}_a(t_\chi)$ enters the set Ω_χ and remains inside thereafter for $t \geq t_\chi$ for arbitrary χ selected in (5.31).

Inside the compact set Ω_χ , we have

$$2\mathbf{y}_a^\top(t) [S(\mathbf{y}_a) - S_\chi(\mathbf{y}_a)] \hat{\mathbf{d}}(t) = \chi \sum_{i=1}^3 \left[2 \frac{|y_{ai}|}{\chi} - 3 \frac{|y_{ai}|^2}{\chi^2} + \frac{|y_{ai}|^4}{\chi^4} \right] \hat{d}_i(t). \quad (5.51)$$

The term in the square bracket in (5.51) is positive for $|y_{ai}(t)| \leq \chi$ and has a maximum value of $c_3 = \frac{1}{4}(6\sqrt{3} - 9)$. Therefore,

$$2\mathbf{y}_a^\top(t) [S(\mathbf{y}_a) - S_\chi(\mathbf{y}_a)] \hat{\mathbf{d}}(t) \leq c_4 \chi, \quad (5.52)$$

where $c_4 = 3c_3 d_{\max}$. Since $\|S_\chi(\mathbf{y}_a(t))\|_F \leq \sqrt{3}$ by definition, from the equation in (5.34) it follows that

$$\|E_S(t)\|_F = \left\| \int_0^t e^{-A_f(t-\tau)} S_\chi(\mathbf{y}_a(\tau)) d\tau - S(\mathbf{y}_a(t)) \right\|_F \leq \sqrt{3} (1 - a_f^{-1}). \quad (5.53)$$

Therefore, inside the set Ω_χ , the derivative of $V_2(t)$ can be further upper bounded as follows:

$$\begin{aligned} \dot{V}_2(t) &\leq -\lambda_{\min}(Q_a) \|\boldsymbol{\xi}_a(t)\|^2 + c_4 \chi + 2c_5 \|\boldsymbol{\xi}_a(t)\| - \frac{\lambda}{b} \|\mathbf{e}_w(t)\|^2 \\ &\leq -(\lambda_{\min}(Q_a) - c_6^2) \|\boldsymbol{\xi}_a(t)\|^2 + \frac{c_5^2}{c_6^2} + c_4 \chi - \frac{\lambda}{b} \|\mathbf{e}_w(t)\|^2, \end{aligned} \quad (5.54)$$

where $c_5 = (\|A_f^{-1}\|_F + 1) d_{\max}$ and c_6 is chosen such that $c_7 = \lambda_{\min}(Q_a) - c_6^2 > 0$. The inequality in (5.54) implies that $\dot{V}_2(t) \leq 0$ outside the compact set

$$\Omega_2 = \left\{ \|\boldsymbol{\xi}_a\| \leq \sqrt{\frac{c_8}{c_7}}, \mathbf{e}_w = \mathbf{0}, |\tilde{b}| \leq \tilde{b}^*, |\tilde{\mathbf{d}}| \leq \tilde{d}^* \right\}, \quad (5.55)$$

where $c_8 = \frac{c_5^2}{c_6^2} + c_4 \chi$. Theorem 2 from Section 2.1 ensures that the trajectories of the closed loop system in (5.20), (5.39), (5.40) and (5.41) are uniformly ultimately bounded. Decreasing the parameter χ will decrease the size of the set Ω_χ , and hence the bounds on the components of the output tracking error $\mathbf{e}_c(t)$. However, the bounds on the remaining components of $\boldsymbol{\xi}_a(t)$ that are in the null space of C_a are not affected directly, and are defined by the ultimate bound of the closed loop system, which can be determined following the steps in [66] (see Appendix B. The proof is complete. $\square$)

5.7 Simulations

For the simulation we use the same scenario as in Chapter 4. The target's acceleration profile is displayed in Figure 5.1. The matrix Λ for the filter equation in (5.3) is chosen as $\Lambda = \text{diag}(2, 2, 2)$. The reference model is chosen with the matrices $A_m = A - \bar{B}K$, where $K = \text{diag}(\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3)$ with $\mathcal{K}_i = [1.75 \ 0.5]$, ($i = 1, 2, 3$), and $B_m = \bar{B}KC^\top = \text{diag}(\mathcal{B}_{m1}, \mathcal{B}_{m2}, \mathcal{B}_{m3})$, where $\mathcal{B}_{mi} = [1.75 \ 3.45]^\top$, ($i = 1, 2, 3$). So, the reference model has the poles at $-1.3625 \pm 1.2624j$, and its transfer matrix

$$G(s) = 1.75 \frac{s + 2}{s^2 + 2.75s + 3.5} \mathbb{I}_{3 \times 3}$$

is SPR. The observer is designed with the gain matrix

$$L = \begin{bmatrix} 2 & 1.6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1.6 \end{bmatrix}^\top,$$

which makes $A - LC$ Hurwitz and satisfies the SPR condition. The excitation signal is introduced with same parameters as in Chapter 4. The approximation of the signum function is done according to relationships in (5.31) with $\chi = 1.5$. The guidance law, defined in (5.38) along with (5.20), (5.31), (5.33), (5.34) and (5.41), is able to capture the profile of the target's acceleration with some transient, which is displayed in Figure 5.2. The reference command tracking performance can be seen in Figure 5.3, and parameter convergence is given in Figure 5.4.

It can be seen from the simulations that the performance of the velocity guidance law presented in Chapter 4 is better than the performance of the acceleration guidance law, presented in this chapter. This can be explained by the smoothness of the former guidance law, and by the approximation of the signum function in the virtual control in (5.33).

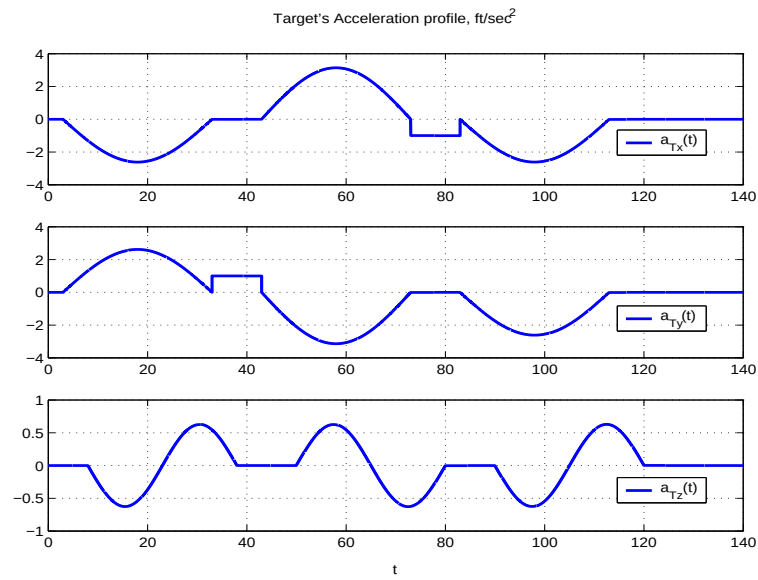


Figure 5.1: Target's acceleration profile

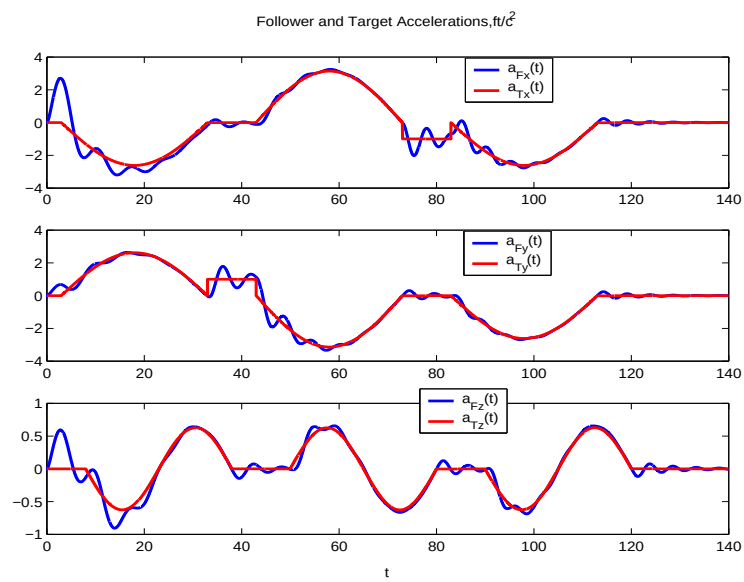


Figure 5.2: Guidance law vs target's acceleration, acceleration guidance

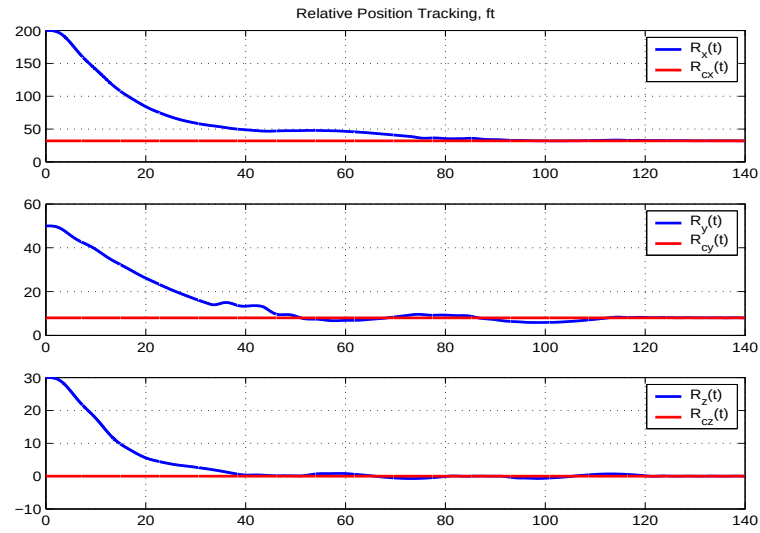


Figure 5.3: Relative position versus reference command, acceleration guidance

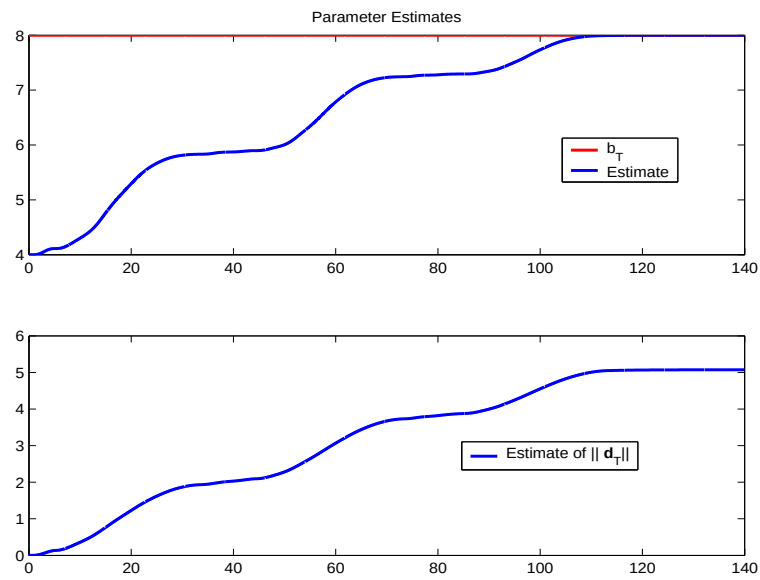


Figure 5.4: Parameter convergence, acceleration guidance

Chapter 6

Target Tracking in the Presence of Measurement Noise

In this chapter, we develop state estimation algorithms, which are used in the feedback loops of the previously developed guidance laws to compensate for the effects of the measurement noise, assumed to be bounded. It is shown that the guidance laws guarantee the ultimate boundedness of the corresponding closed-loop systems. The estimation gain matrices are designed using conventional Kalman scheme.

6.1 Measurement Noise Definition

The guidance law derived in the previous chapters assumes availability of perfect measurements from the visual sensor. However, in realistic applications these measurements are corrupted by noise. Therefore, instead of ideal measurements $y_I(t)$, $z_I(t)$, $b_I(t)$ one should

consider

$$\begin{aligned} y_I^*(t) &= y_I(t) + n_y(t) \\ z_I^*(t) &= z_I(t) + n_z(t) \\ b_I^*(t) &= b_I(t) + n_b(t), \end{aligned} \quad (6.1)$$

where $\mathbf{n}_I(t) \triangleq [n_y(t) \ n_z(t) \ n_b(t)]^\top$ represents the vector of the noise components in the corresponding measurements. In this section, we analyze the performance of the above derived guidance law in the presence of noisy measurements $y_I^*(t)$, $z_I^*(t)$, $b_I^*(t)$. Then the expressions in (3.8) and (3.7) in this case take the form:

$$\begin{aligned} \tan \lambda^*(t) &= \frac{y_I(t) + n_y(t)}{l} \\ \tan \vartheta^*(t) &= \frac{z_I(t) + n_z(t)}{\sqrt{l^2 + (y_I(t) + n_y(t))^2}} \\ a_I^*(t) &= \frac{\sqrt{l^2 + (y_I(t) + n_y(t))^2 + (z_I(t) + n_z(t))^2}}{b_I(t) + n_b(t)}. \end{aligned} \quad (6.2)$$

The measurements of the scaled state vector $\mathbf{r}(t)$ are expressed in the following form

$$\begin{aligned} \mathbf{r}^*(t) &= L_{E/B} a_I^*(t) \mathbf{T}_{BI}^*(\lambda^*(t), \vartheta^*(t)) \\ &\triangleq L_{E/B} \mathbf{f}(y_I^*(t), z_I^*(t), b_I^*(t)), \end{aligned} \quad (6.3)$$

where the vector $\mathbf{T}_{BI}^*(\lambda^*(t), \vartheta^*(t))$ is defined similar to $\mathbf{T}_{BI}$ in (3.9) with λ , ϑ replaced by λ^* , ϑ^* respectively. When the noise level is small, one can expand the expressions in (6.3) into Taylor series around the ideal measurements $y_I(t)$, $z_I(t)$, $b_I(t)$:

$$\mathbf{r}^*(t) = \mathbf{r}(t) + L_{E/B} \nabla \mathbf{f}(y_I(t), z_I(t), b_I(t)) \mathbf{n}_I(t), \quad (6.4)$$

where $\nabla \mathbf{f}(y_I(t), z_I(t), b_I(t))$ is the Jacobian of the vector function $\mathbf{f}(y_I(t), z_I(t), b_I(t))$ with respect to arguments $y_I(t)$, $z_I(t)$, $b_I(t)$. For the uncorrelated, zero-mean, Gaussian, white noise $\mathbf{n}_I(t)$ with the correlation matrix

$$E\{\mathbf{n}_I(t) \mathbf{n}_I^\top(\tau)\} = K_{\mathbf{n}_I} \delta(t - \tau),$$

where $E\{\cdot\}$ denotes the mathematical expectation,

$$K_{\mathbf{n}_I} = \begin{bmatrix} \bar{n}_y & 0 & 0 \\ 0 & \bar{n}_z & 0 \\ 0 & 0 & \bar{n}_b \end{bmatrix}$$

with $\bar{n}_y$, $\bar{n}_z$, $\bar{n}_b$ being positive constants, and $\delta(\cdot)$ is the Dirac delta function, the vector

$$\mathbf{n}_r(t) = L_{E/B} \nabla \mathbf{f}(y_I(t), z_I(t), b_I(t)) \mathbf{n}_I(t) \quad (6.5)$$

also represents a zero-mean Gaussian, white noise process with the correlation matrix

$$E\{\mathbf{n}_r(t) \mathbf{n}_r^\top(\tau)\} = S_n(t) K_{\mathbf{n}_I} S_n^\top(\tau) \delta(t - \tau), \quad (6.6)$$

where

$$S_n(t) = L_{E/B}(t) \nabla \mathbf{f}(y_I(t), z_I(t), b_I(t)). \quad (6.7)$$

Therefore, in the sequel we will assume that the scaled state vector $\mathbf{r}(t)$ is available with the additive, zero-mean, Gaussian, white noise:

$$\mathbf{r}^*(t) = \mathbf{r}(t) + \mathbf{n}_r(t) \quad (6.8)$$

with the correlation matrix

$$E\{\mathbf{n}_r(t) \mathbf{n}_r^\top(\tau)\} = K_{\mathbf{n}_r}(t, \tau) \delta(t - \tau), \quad (6.9)$$

where $K_{\mathbf{n}_r}(t, \tau) > 0$ for all $t, \tau > 0$.

6.2 Velocity Control with Noisy Measurements

In this section, we consider **Problem 1** from the Chapter 3, when the visual measurements are corrupted by a noise leading to the output measurements in (6.8) and (6.9).

6.2.1 Error dynamics modification

To compensate for the measurement noise we introduce a state estimator for the scaled relative position vector $\mathbf{r}(t)$ as follows:

$$\dot{\hat{\mathbf{r}}}(t) = \hat{\mathbf{d}}(t) - \mathbf{g}(t) + L_v[\mathbf{r}^*(t) - \hat{\mathbf{r}}(t)], \quad (6.10)$$

where the function $\mathbf{g}(t)$ and hence the guidance law are modified to be

$$\begin{aligned} \mathbf{V}_F(t) &= \hat{b}(t)\mathbf{g}(t) \\ \mathbf{g}(t) &= k[\hat{\mathbf{r}}(t) - \mathbf{r}_c(t)] + \hat{\mathbf{d}}(t) - \dot{\hat{\mathbf{r}}}_c(t). \end{aligned} \quad (6.11)$$

and the gain matrix L_v is chosen to minimize the mean square estimation error

$$J_n = E\{\tilde{\mathbf{r}}^\top(t)\tilde{\mathbf{r}}(t)\}, \quad (6.12)$$

where $\tilde{\mathbf{r}}(t) = \mathbf{r}(t) - \hat{\mathbf{r}}(t)$ is the estimation error. The estimation error dynamics can be written as follows

$$\dot{\tilde{\mathbf{r}}}(t) = -L_v\tilde{\mathbf{r}}(t) - \tilde{\mathbf{d}}(t) - \frac{\tilde{b}(t)}{b}\mathbf{g}(t) + \boldsymbol{\delta}(t) - L_v\mathbf{n}_r(t). \quad (6.13)$$

As in the case of the ideal measurements, the tracking error is defined as $\mathbf{e}(t) = \mathbf{r}(t) - \mathbf{r}_c(t)$. With the guidance law presented in (6.11), the tracking error dynamics are modified as follows:

$$\dot{\mathbf{e}}(t) = -k\mathbf{e}(t) + k\tilde{\mathbf{r}}(t) - \tilde{\mathbf{d}}(t) + \boldsymbol{\delta}(t) - \frac{\tilde{b}(t)}{b}\mathbf{g}(t). \quad (6.14)$$

We combine both error dynamics into one equation

$$\dot{\boldsymbol{\xi}}_v(t) = A_v\boldsymbol{\xi}_v(t) + B_v \left[\boldsymbol{\delta}(t) - \tilde{\mathbf{d}}(t) - \frac{\tilde{b}(t)}{b}\mathbf{g}(t) \right] - D_v\mathbf{n}_r(t), \quad (6.15)$$

where

$$\boldsymbol{\xi}_v = \begin{bmatrix} \mathbf{e} \\ \tilde{\mathbf{r}} \end{bmatrix}, \quad A_v = \begin{bmatrix} -kI_{3 \times 3} & kI_{3 \times 3} \\ 0_{3 \times 3} & -L_v \end{bmatrix}, \quad B_v = \begin{bmatrix} I_{3 \times 3} \\ I_{3 \times 3} \end{bmatrix}, \quad D_v = \begin{bmatrix} 0_{3 \times 3} \\ L_v \end{bmatrix}.$$

The adaptive laws are modified as follows:

$$\begin{aligned} \dot{\hat{\mathbf{b}}}(t) &= \sigma \text{Proj} \left(\hat{\mathbf{b}}(t), \mathbf{g}(t)^\top B_v^\top P_v \boldsymbol{\xi}_v^*(t) \right) \\ \dot{\hat{\mathbf{d}}}(t) &= G \text{Proj} \left(\hat{\mathbf{d}}(t), B_v^\top P_v \boldsymbol{\xi}_v^*(t) \right), \end{aligned} \quad (6.16)$$

where

$$\boldsymbol{\xi}_v^* = \begin{bmatrix} \mathbf{r}^* - \mathbf{r}_c \\ \mathbf{r}^* - \hat{\mathbf{r}} \end{bmatrix} = \begin{bmatrix} \mathbf{e} + \mathbf{n}_r \\ \tilde{\mathbf{r}} + \mathbf{n}_r \end{bmatrix} = \boldsymbol{\xi}_v + B_v \mathbf{n}_r,$$

and P_v is a symmetric positive definite matrix that solves the Lyapunov equation

$$A_v^\top P_v + P_v A_v = -Q_v, \quad (6.17)$$

for the Hurwitz matrix A_v and any symmetric positive definite matrix Q_v . Next we prove that for any bounded $\mathbf{n}_r(t)$, all the signals in (6.15) and (6.16) are uniformly ultimately bounded.

6.2.2 Stability analysis

Now we show that the error system in (6.15) and (6.16) is ultimately bounded for any bounded $\mathbf{n}_r(t)$, for which the solution to the augmented error equation in (6.15) exists.

Theorem 14 . *If $\mathbf{n}_r(t)$ is bounded, the systems in (6.15) and (6.16) are uniformly ultimately bounded. Moreover, if $\mathbf{n}_r(t) = 0$, the combined error $\boldsymbol{\xi}_v(t)$ asymptotically converges to zero. In addition, if the estimated reference command $\hat{\mathbf{r}}_c(t)$ is intelligently exciting, then the parameter errors $\tilde{\mathbf{b}}(t)$ and $\tilde{\mathbf{d}}(t)$ converge to zero as well.* $\square$

Proof. Consider the following Lyapunov function candidate

$$V_1(\boldsymbol{\xi}_v, \tilde{b}, \tilde{\mathbf{d}}) = \boldsymbol{\xi}_v^\top(t) P_v \boldsymbol{\xi}_v(t) + \frac{1}{b\sigma} \tilde{b}^2(t) + \tilde{\mathbf{d}}^\top(t) G^{-1} \tilde{\mathbf{d}}(t).$$

Its derivative along the trajectories of the system (6.15)-(6.16) has the form

$$\begin{aligned} \dot{V}_1(t) &= \boldsymbol{\xi}_v^\top(t) (A_v^\top P_v + P_v A_v) \boldsymbol{\xi}_v(t) \\ &+ 2\boldsymbol{\xi}_v^\top(t) P_v B_v \left[\boldsymbol{\delta}(t) - \tilde{\mathbf{d}}(t) - \frac{\tilde{b}(t)}{b} \mathbf{g}(t) \right] \\ &+ 2\boldsymbol{\xi}_v^\top(t) P_v D_v \mathbf{n}_r(t) + \frac{2}{\sigma b} \tilde{b}(t) \dot{\tilde{b}}(t) + 2\tilde{\mathbf{d}}^\top(t) G^{-1} \dot{\tilde{\mathbf{d}}}(t). \end{aligned} \quad (6.18)$$

Substituting the adaptive laws from (6.16) and taking into account the definition of $\boldsymbol{\xi}_v^*$ we obtain

$$\begin{aligned} \dot{V}_1(t) &= -\boldsymbol{\xi}_v^\top(t) Q_v \boldsymbol{\xi}_v(t) + 2\boldsymbol{\xi}_v^\top(t) P_v B_v \boldsymbol{\delta}(t) + 2\boldsymbol{\xi}_v^\top(t) P_v D_v \mathbf{n}_r(t) \\ &+ 2\frac{\tilde{b}(t)}{b} \left[-\mathbf{g}^\top(t) B_v^\top P_v \boldsymbol{\xi}_v^*(t) + \text{Proj} \left(\hat{b}(t), \mathbf{g}^\top(t) B_v^\top P_v \boldsymbol{\xi}_v^*(t) \right) \right] \\ &+ \tilde{\mathbf{d}}^\top(t) \left[-B_v^\top P_v \boldsymbol{\xi}_v^*(t) + \text{Proj} \left(\hat{\mathbf{d}}(t), B_v^\top P_v \boldsymbol{\xi}_v^*(t) \right) \right] \\ &+ 2 \left(\frac{\tilde{b}(t)}{b} \mathbf{g} + \tilde{\mathbf{d}}(t) \right)^\top B_v^\top P_v B_v \mathbf{n}_r(t). \end{aligned} \quad (6.19)$$

From the properties of the projection operator (see Appendix A), the following inequalities can be written:

$$\begin{aligned} b_{\min} &\leq \hat{b}(t) \leq b_{\max} \\ \tilde{b}(t) \left[-\mathbf{g}^\top(t) B_v^\top P_v \boldsymbol{\xi}_v^*(t) + \text{Proj} \left(\hat{b}(t), \mathbf{g}^\top(t) B_v^\top P_v \boldsymbol{\xi}_v^*(t) \right) \right] &\leq 0 \\ \|\hat{\mathbf{d}}(t)\| &\leq d_{\max} \\ \tilde{\mathbf{d}}^\top(t) \left[-B_v^\top P_v \boldsymbol{\xi}_v^*(t) + \text{Proj} \left(\hat{\mathbf{d}}(t), B_v^\top P_v \boldsymbol{\xi}_v^*(t) \right) \right] &\leq 0, \end{aligned} \quad (6.20)$$

which also imply that $\|\tilde{b}(t)\| \leq \tilde{b}_v^*$ and $\|\tilde{\mathbf{d}}(t)\| \leq \tilde{d}_v^*$, where $\tilde{b}_v^*$ and $\tilde{d}_v^*$ are some positive constants. Therefore, the derivative of V_1 can be upper bounded as follows:

$$\begin{aligned} \dot{V}_1(t) &\leq -\lambda_{\min}(Q_v)\|\boldsymbol{\xi}_v(t)\|^2 + 2\left(\frac{\tilde{b}_v^*}{b}\|\mathbf{g}\| + \tilde{d}_v^*\right)\|B_v^\top P_v B_v\|_F\|\mathbf{n}_r(t)\| \\ &\quad + 2\|\boldsymbol{\xi}_v(t)\|(\|P_v D_v\|_F\|\mathbf{n}_r(t)\| + \|P_v B_v\|_F\|\boldsymbol{\delta}(t)\|), \end{aligned} \quad (6.21)$$

where $\lambda_{\min}(Q_v)$ denotes the minimum eigenvalue of the matrix Q_v . Since $\dot{\mathbf{R}}_c(t)$ is assumed to be bounded, the function $\mathbf{g}(t)$ can be upper bounded as follows:

$$\begin{aligned} \|\mathbf{g}(t)\| &\leq \|k[\hat{\mathbf{r}}(t) - \mathbf{r}_c(t)] + \hat{\mathbf{d}}(t) - \dot{\mathbf{r}}_c(t)\| \\ &\leq \|k[\mathbf{e}(t) + \tilde{\mathbf{r}}(t)] - \dot{\mathbf{r}}_c(t)\| + d_{\max}. \end{aligned} \quad (6.22)$$

Since $\mathbf{r}_c(t)$ is bounded, the boundedness of $\hat{b}(t)$ and $\hat{b}(t)$ imply that $\dot{\mathbf{r}}_c(t)$ is bounded. Thus, there exist positive constants c_1, c_2 such that $\|\mathbf{g}\| \leq c_1\|\boldsymbol{\xi}_v(t)\| + c_2$. Therefore, the inequality in (6.21) can be written as

$$\begin{aligned} \dot{V}_1(t) &\leq -\lambda_{\min}(Q_v)\|\boldsymbol{\xi}_v(t)\|^2 + 2c_3\|\mathbf{n}_r(t)\| \\ &\quad + 2c_4\|\boldsymbol{\xi}_v(t)\|\|\mathbf{n}_r(t)\| + 2c_5\|\boldsymbol{\xi}_v(t)\|\|\boldsymbol{\delta}(t)\|, \end{aligned} \quad (6.23)$$

where

$$\begin{aligned} c_3 &= 2\left(\frac{\tilde{b}_v^*}{b}c_2 + \tilde{d}_v^*\right)\|B_v^\top P_v B_v\|_F, \\ c_4 &= \|P_v B_v\|_F, \\ c_5 &= \frac{\tilde{b}_v^*}{b}c_1\|B_v^\top P_v B_v\|_F + \|P_v D_v\|_F. \end{aligned}$$

Completing the squares in (6.23) yields

$$\dot{V}_1(t) \leq -k_v\|\boldsymbol{\xi}_v(t)\|^2 + c_4^2 c_6^2\|\mathbf{n}_r(t)\|^2 + c_5^2 c_7^2\|\boldsymbol{\delta}(t)\|^2 + c_3\|\mathbf{n}_r(t)\|, \quad (6.24)$$

where c_6, c_7 are positive constants such that

$$k_v = \lambda_{\min}(Q_v) - \frac{1}{c_6^2} - \frac{1}{c_7^2} > 0.$$

Since $\boldsymbol{\delta}(t) \in \mathcal{L}_\infty$, for any bounded $\mathbf{n}_r(t)$ there exists some $\rho_v > 0$ such that

$$c_4^2 c_6^2 \|\mathbf{n}_r(t)\|^2 + c_5^2 c_7^2 \|\boldsymbol{\delta}(t)\|^2 + c_3 \|\mathbf{n}_r(t)\| \leq \rho_v.$$

From the relationship in (6.24) it follows that $\dot{V}_1(t) \leq 0$ outside the compact set

$$\Omega_1 = \{(\boldsymbol{\xi}_v, \tilde{b}, \tilde{\mathbf{d}}) : \|\boldsymbol{\xi}_v\| \leq \frac{\rho_v}{\sqrt{k_v}}, \|\tilde{b}\| \leq \tilde{b}_v^*, \|\tilde{\mathbf{d}}\| \leq \tilde{d}_v^*\}$$

Theorem 2 from Section 2.1 ensures that all the signals $\boldsymbol{\xi}_v(t)$, $\tilde{b}(t)$, $\tilde{\mathbf{d}}(t)$, $\mathbf{g}(t)$ in the system (6.15)-(6.16) are uniformly ultimately bounded. From (6.15) it follows that $\dot{\boldsymbol{\xi}}_v(t)$ is bounded as well.

If $\mathbf{n}_r(t) = 0$, the inequality in (6.24) reduces to

$$\dot{V}_1(t) \leq -k_v \|\boldsymbol{\xi}_v(t)\|^2 + c_5^2 c_7^2 \|\boldsymbol{\delta}(t)\|^2, \quad (6.25)$$

which can be rearranged and integrated to yield

$$k_v \int_0^t \|\boldsymbol{\xi}_v(\tau)\|^2 d\tau \leq V_1(0) + c_5^2 c_7^2 \int_0^t \|\boldsymbol{\delta}(\tau)\|^2 d\tau. \quad (6.26)$$

Since $\boldsymbol{\delta}(t) \in \mathcal{L}_2$, from (6.26) we have

$$\lim_{t \rightarrow \infty} \int_0^t \|\boldsymbol{\xi}_v(\tau)\|^2 d\tau < \infty. \quad (6.27)$$

Thus, $\boldsymbol{\xi}_v(t) \in \mathcal{L}_2(\mathbb{R}^6) \cap \mathcal{L}_\infty(\mathbb{R}^6)$. Also, the error dynamics in (6.15) imply that $\dot{\boldsymbol{\xi}}_v(t) \in \mathcal{L}_\infty(\mathbb{R}^6)$. Application of Corollary 1 from Section 2.1 ensures that $\boldsymbol{\xi}_v(t) \rightarrow 0$ as $t \rightarrow \infty$.

If the estimated reference command $\hat{\mathbf{r}}_c(t)$ is exciting in at least one component, then the application of Lemma 8 will guarantee asymptotic convergence of the parameter errors to zero. The proof is complete. $\square$

6.2.3 State estimator gain design

To complete the guidance design we show that the estimator gain matrix L_v can be chosen according to Kalman's scheme (see for example [9]). To this end we write the overall closed-

loop error dynamics for the state vector

$$\boldsymbol{\xi}_l = [\mathbf{e}^\top \quad \tilde{\mathbf{r}}^\top \quad \tilde{\mathbf{d}}^\top \quad \tilde{\mathbf{b}}^\top]^\top \in \mathbb{R}^{10}$$

in the time-varying linear system form:

$$\begin{aligned} \dot{\boldsymbol{\xi}}_l(t) &= A_l(t)\boldsymbol{\xi}_l(t) - D_l\mathbf{n}_r(t) + B_\delta\boldsymbol{\delta}(t) \\ \tilde{\mathbf{r}}(t) &= C_l\boldsymbol{\xi}_l(t), \end{aligned} \tag{6.28}$$

where

$$A_l(t) = \begin{bmatrix} A_v & -\frac{1}{b}B_v & -\frac{1}{b}B_v\mathbf{g}(t) \\ G_1(t) & 0_{3 \times 3} & 0 \\ G_2(t) & 0_{1 \times 3} & 0 \end{bmatrix}, \quad D_l = \begin{bmatrix} D_v \\ 0_{3 \times 3} \\ 0_{1 \times 3} \end{bmatrix}, \quad C_l = \begin{bmatrix} 0_{3 \times 3} \\ I_{3 \times 3} \\ 0_{3 \times 3} \\ 0_{1 \times 3} \end{bmatrix}^\top, \quad B_\delta = \begin{bmatrix} B_v \\ 0_{3 \times 3} \\ 0_{1 \times 3} \end{bmatrix},$$

and the 3×6 matrix the $G_1(t)$ and 1×6 matrix $G_2(t)$ are determined through the definition of the projection operator in the adaptive laws in (6.16). Their explicit expressions are not presented here. Although the closed-loop error dynamics are non-linear, the linear form in (6.28) can be justified by the fact that all the signals in the systems in (6.15)-(6.16) are bounded for all $t \geq 0$ as stated by Theorem 14 [29] (p. 626).

The necessary condition for the estimate $\hat{\mathbf{r}}(t)$ to be optimal in the sense of minimizing the performance index J_n in (6.12) is that the estimation error must be orthogonal to the measurement data (see for example [9], p.233), that is

$$E\{\tilde{\mathbf{r}}(t)\mathbf{r}^{*\top}(\tau)\} = 0 \tag{6.29}$$

for all $\tau \leq t$. Then the estimation error correlation matrix $K_{\tilde{\mathbf{r}}}(t, \tau)$ can be computed as follows. Taking into account the orthogonality of the estimator's state for all $\tau \leq t$ and the

estimation error at time t , and the necessary condition in (6.29) we write

$$\begin{aligned}
K_{\tilde{\mathbf{r}}}(t, \tau) &= E\{\tilde{\mathbf{r}}(t)\tilde{\mathbf{r}}^\top(\tau)\} = E\{\tilde{\mathbf{r}}(t)[\mathbf{r}(\tau) - \hat{\mathbf{r}}(\tau)]^\top\} \\
&= E\{\tilde{\mathbf{r}}(t)[\mathbf{r}^*(\tau) - \mathbf{n}_r(\tau) - \hat{\mathbf{r}}(\tau)]^\top\} \\
&= -E\{\tilde{\mathbf{r}}(t)\mathbf{n}_r^\top(\tau)\}.
\end{aligned} \tag{6.30}$$

The output $\tilde{\mathbf{r}}(t)$ of the linear system in (6.28) can be written as

$$\tilde{\mathbf{r}}(t) = C_l \Phi_l(t, 0) \boldsymbol{\xi}_l(0) + C_l \int_0^t \Phi_l(t, s) (-D_l \mathbf{n}_r(s) + B_\delta \boldsymbol{\delta}(s)) ds, \tag{6.31}$$

where $\Phi_l(t, \tau)$ is the state transition matrix of the system in (6.28). Substituting $\tilde{\mathbf{r}}(t)$ from (6.31) into (6.30) we get

$$\begin{aligned}
K_{\tilde{\mathbf{r}}}(t, \tau) &= -E\{C_l \Phi_l(t, 0) \boldsymbol{\xi}_l(0) \mathbf{n}_r^\top(\tau)\} \\
&+ E\left\{C_l \int_0^t \Phi_l(t, s) D_l \mathbf{n}_r(s) ds \mathbf{n}_r^\top(\tau)\right\} \\
&- E\left\{C_l \int_0^t \Phi_l(t, s) B_\delta \boldsymbol{\delta}(s) ds \mathbf{n}_r^\top(\tau)\right\}.
\end{aligned} \tag{6.32}$$

Taking into account the zero-mean property of the noise, the equation in (6.32) reduces to

$$\begin{aligned}
K_{\tilde{\mathbf{r}}}(t, \tau) &= C_l \int_0^t \Phi_l(t, s) D_l E\{\mathbf{n}_r(s) \mathbf{n}_r^\top(\tau)\} ds \\
&= C_l \int_0^t \Phi_l(t, s) D_l K_{\mathbf{n}_r}(s, \tau) \delta(s - \tau) ds.
\end{aligned}$$

Since $\Phi_l(t, t) = \mathbb{I}$ and

$$\int_0^t f(s) \delta(s - t) ds = f(t)$$

for any integrable function $f(t)$, we conclude that

$$K_{\tilde{\mathbf{r}}}(t, t) = C_l D_l K_{\mathbf{n}_r}(t, t) = L_v(t) K_{\mathbf{n}_r}(t, t), \tag{6.33}$$

and therefore the estimator gain matrix can be computed as

$$L_v(t) = K_{\tilde{\mathbf{r}}}(t, t) K_{\mathbf{n}_r}^{-1}(t, t). \tag{6.34}$$

It can be verified that the evolution of the estimation error correlation matrix is described by the differential equation (see for example [9], p. 71)

$$\begin{aligned}\dot{K}_{\boldsymbol{\xi}}(t) &= A_l(t)K_{\boldsymbol{\xi}}(t) + K_{\boldsymbol{\xi}}(t)A_l^\top(t) + D_l K_{\mathbf{n}_r}(t)D_l^\top \\ K_{\hat{\mathbf{r}}}(t) &= C_l K_{\boldsymbol{\xi}}(t)C_l^\top\end{aligned}\tag{6.35}$$

Remark 8 *If we assume a constant correlation matrix $K_{\mathbf{n}_r}$ for the measurement noise $\mathbf{n}_r(t)$, there exists a steady state estimator gain in two cases: in the case of target interception that is solved without an excitation signal and in the case of formation flight with the constant reference command $\mathbf{R}_c$ when the intelligent excitation is used. In these two cases it can be shown that $A_l(t) \rightarrow \bar{A}_l$ as $t \rightarrow \infty$, where $\bar{A}_l$ is a constant matrix. Therefore, the steady state solution can be found according to equations*

$$\begin{aligned}0 &= \bar{A}_l K_{\boldsymbol{\xi}} + K_{\boldsymbol{\xi}} \bar{A}_l^\top + D_l K_{\mathbf{n}_r} D_l^\top \\ L_v &= K_{\hat{\mathbf{r}}} K_{\mathbf{n}_r}^{-1} = C_l K_{\boldsymbol{\xi}} C_l^\top K_{\mathbf{n}_r}^{-1}.\end{aligned}\tag{6.36}$$

□

6.2.4 Simulations

Here we use the same simulation scenario as in Chapter 4 to show the performance of the algorithm with the state estimation presented in this section. The estimator gain matrix is chosen suboptimal according to equation in (6.36) and is equal to $L = 1.7\mathbb{I}_{3 \times 3}$. First we run the simulation with the same adaptive gains as in Chapter 4 and without any noise, to show the performance of the guidance law with the estimator, which is displayed in Figure 6.1. Figure 6.2 displays the reference command tracking performance and the parameter estimate convergence is displayed in Figure 6.3. In all three figures, a slight deterioration can be observed compared to the performance in ideal case.

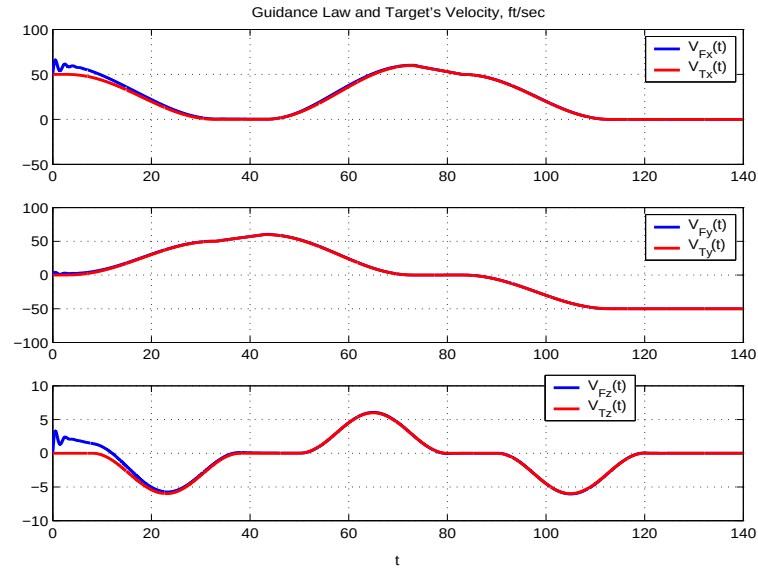


Figure 6.1: Guidance law with the estimator versus target's velocity without noise, velocity guidance

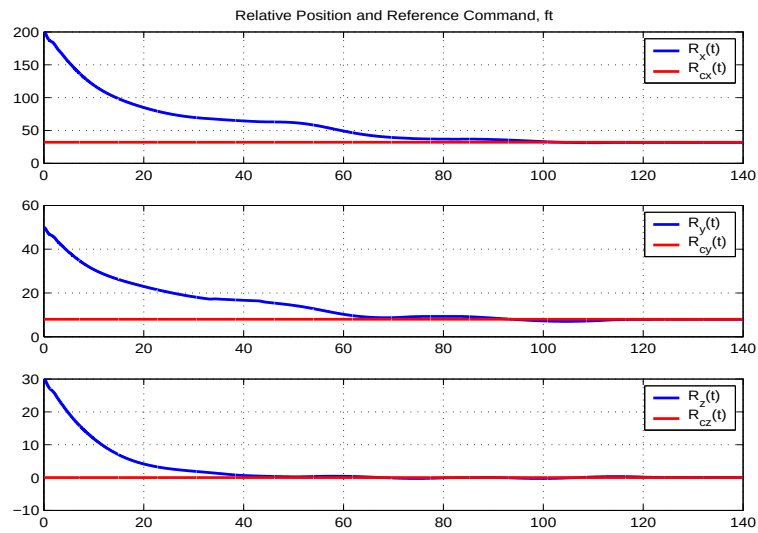


Figure 6.2: Relative position versus reference command with the estimator and without noise, velocity guidance

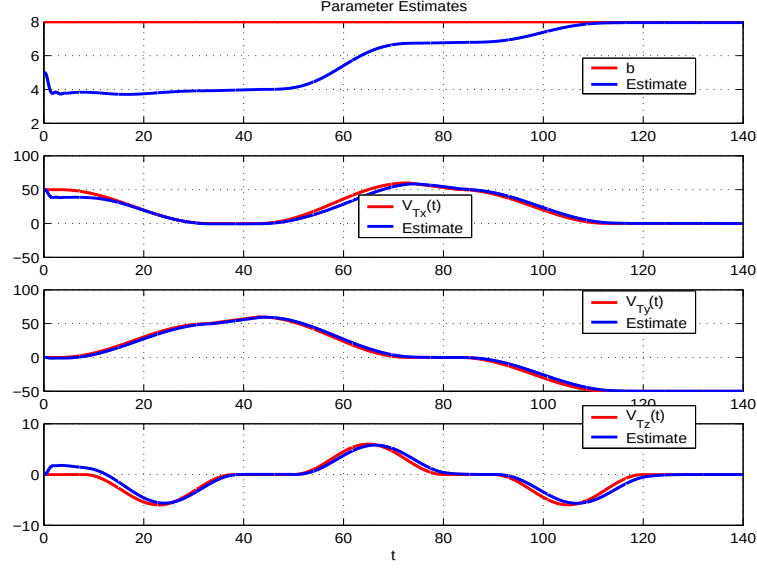


Figure 6.3: Parameter estimates with the estimator and without noise, velocity guidance

Next a white noise signal is added to the scaled relative position vector $\mathbf{r}(t)$ according to the equation in (6.3), with signal to noise ratio $SNR = 25$. The performance of the modified guidance law given by the equations in (6.11) and (6.16) is presented in Figures 6.4, 6.5, 6.6.

6.3 Acceleration Control with Noisy Measurements

In this section, we consider Problem 2 from Chapter 3, when the visual measurements are corrupted by noise that gives rise to the output equation given by (6.8) and (6.9). Recall that Problem 2 is considered for the system

$$\begin{aligned}
 \dot{\mathbf{z}}(t) &= \mathbf{A}\mathbf{z}(t) + \frac{1}{b}\bar{\mathbf{B}}[\mathbf{w}(t) + \mathbf{d}(t)] \\
 \dot{\mathbf{w}}(t) &= -\Lambda\mathbf{w}(t) - \mathbf{a}_F(t) \\
 \mathbf{y}(t) &= \mathbf{C}\mathbf{z}(t),
 \end{aligned} \tag{6.37}$$

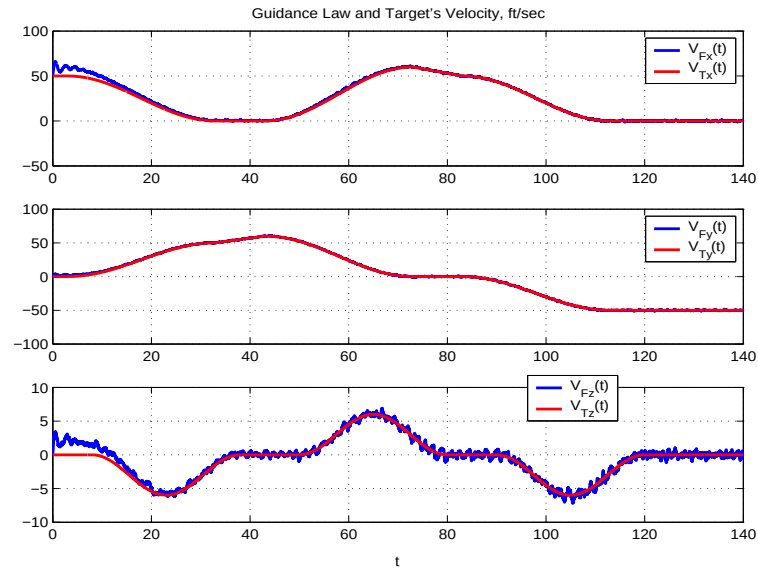


Figure 6.4: Guidance law with noisy measurements versus target's velocity, velocity guidance, SNR=25

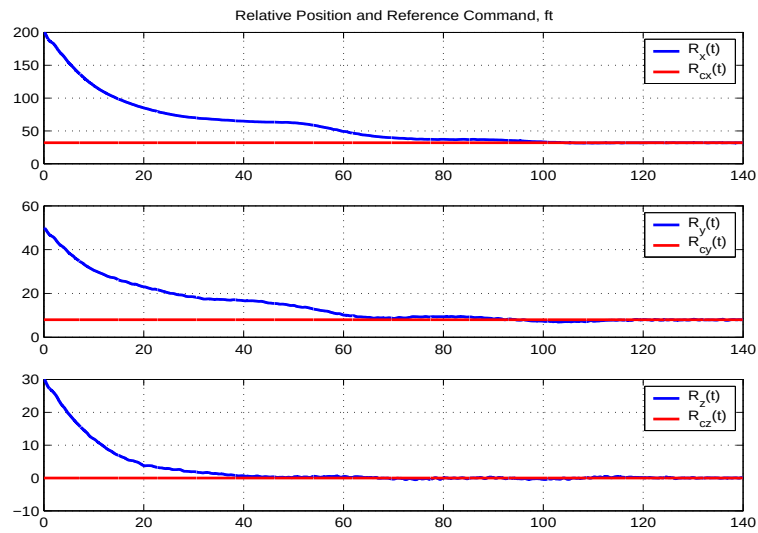


Figure 6.5: Relative position versus reference command with noisy measurements, velocity guidance, SNR=25

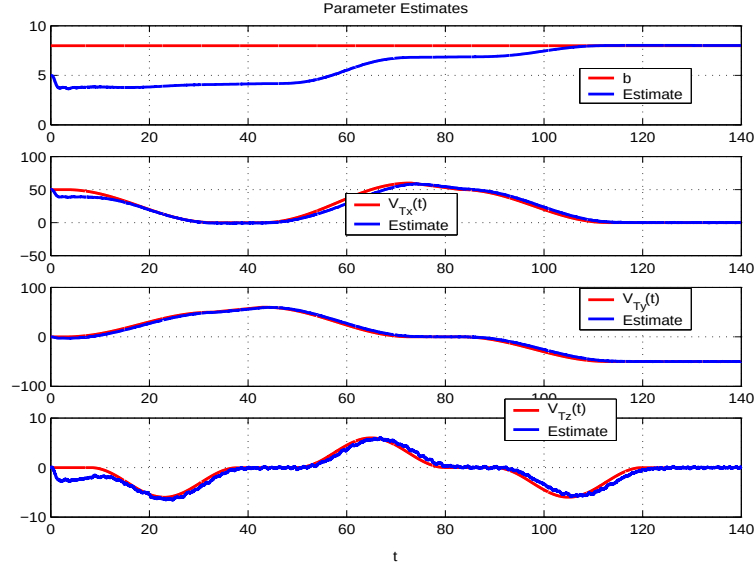


Figure 6.6: Parameter estimates with noisy measurements, velocity guidance, SNR=25

where the matrices A , $\bar{B}$, C are defined in Chapter 5, and the reference model to be tracked is given by the equations

$$\begin{aligned}\dot{\mathbf{z}}_m(t) &= A_m \mathbf{z}(t) + B_m \mathbf{y}_c(t) \\ \mathbf{y}_m(t) &= C \mathbf{z}_m(t),\end{aligned}\tag{6.38}$$

where

$$A_m = A - \bar{B}K, \quad B_m = \bar{B}KC^\top,$$

K being the gain matrix satisfying the the SPR and Hurwitz requirements (see Chapter 5 for the details).

6.3.1 Error dynamics modification

First we notice that the observer dynamics in (5.10) will be driven by the noise measurement

$$\mathbf{y}^*(t) = \mathbf{r}^*(t) = \mathbf{y}(t) + \mathbf{n}_r(t)$$

instead of $\mathbf{y}(t)$. So the observer dynamics are written as follows:

$$\begin{aligned}\dot{\hat{\mathbf{z}}}^*(t) &= A_m \hat{\mathbf{z}}^*(t) + B_m \hat{\mathbf{y}}_c(t) + L[\mathbf{y}^*(t) - \hat{\mathbf{y}}^*(t)] \\ \hat{\mathbf{y}}^*(t) &= C \hat{\mathbf{z}}^*(t),\end{aligned}\tag{6.39}$$

where L is the observer gain matrix that satisfies the SPR and Hurwitz requirements. In the guidance law, given by the equation (5.38) along with (5.31), (5.34), (5.36) and (5.38), the ideal measurement $\mathbf{y}(t)$ will be replaced by $\mathbf{y}^*(t)$. For clarity, here we present the complete guidance law:

$$\begin{aligned}\mathbf{a}_F(t) &= \Lambda \mathbf{w}_f^*(t) - \dot{\mathbf{w}}_f^*(t) + \mathbf{y}_a^*(t) \\ \mathbf{w}_f^*(t) &= \hat{b}(t) \mathbf{h}^*(t) - S_f^*(t) \hat{\mathbf{d}}(t) \\ \mathbf{h}^*(t) &= -K \hat{\mathbf{z}}^*(t) + K C^\top \hat{\mathbf{y}}_c(t) \\ \dot{S}_f^*(t) &= A_f S_f^*(t) - S_\chi(\mathbf{y}_a^*(t)),\end{aligned}\tag{6.40}$$

where $S_\chi(\mathbf{y}_a^*)$ is defined according to (5.16) and (5.31), with $\mathbf{y}_a(t)$ being replaced with $\mathbf{y}_a^*(t)$, the latter being defined as follows:

$$\begin{aligned}\mathbf{y}_a^* &= C_a \boldsymbol{\xi}_a^* = C \mathbf{e}^* + C \tilde{\mathbf{z}}^* \\ &= \mathbf{y}^* - \mathbf{y}_m + \mathbf{y}^* - \hat{\mathbf{y}}^* \\ &= \mathbf{e}_c + \mathbf{n}_r + \tilde{\mathbf{y}} + \mathbf{n}_r \\ &= C_a \boldsymbol{\xi}_a + 2\mathbf{n}_r = \mathbf{y}_a + 2\mathbf{n}_r.\end{aligned}\tag{6.41}$$

The adaptive laws are modified by replacing $\mathbf{y}_a(t)$ with $\mathbf{y}_a^*(t)$ and $\mathbf{h}(t)$ with $\mathbf{h}^*(t)$:

$$\begin{aligned}\dot{\hat{b}}(t) &= \rho \text{Proj} \left(\hat{b}(t), -\mathbf{y}_a^{*\top}(t) \mathbf{h}^*(t) \right), \quad \hat{b}(0) = b_0 > 0 \\ \dot{\hat{\mathbf{d}}}(t) &= G \text{Proj} \left(\hat{\mathbf{d}}(t), S(\mathbf{y}_a^*) \mathbf{y}_a^*(t) \right), \quad \hat{\mathbf{d}}(0) = \mathbf{0}.\end{aligned}\tag{6.42}$$

The error dynamics, given by the equations (5.15) and (5.39), now take the form

$$\begin{aligned}\dot{\boldsymbol{\xi}}_a(t) &= A_a \boldsymbol{\xi}_a(t) + \frac{1}{b} B_a \left[\tilde{b}(t) \mathbf{h}^*(t) + \mathbf{d}(t) - S_f^*(t) \hat{\mathbf{d}}(t) + \mathbf{e}_w(t) \right] - D_a \mathbf{n}_r(t) \\ \dot{\mathbf{e}}_w(t) &= -\Lambda \mathbf{e}_w(t) - \mathbf{y}_a^*(t) \\ \mathbf{y}_a^*(t) &= C_a \boldsymbol{\xi}_a(t) + 2\mathbf{n}_r(t),\end{aligned}\tag{6.43}$$

where $D_a = \begin{bmatrix} 0_{3 \times 6} & L^\top \end{bmatrix}^\top$.

6.3.2 Stability analysis

Now we show that the error system in (6.43) and (6.42) is ultimately bounded for any bounded $\mathbf{n}_r(t)$, for which the solution exists.

Theorem 15 *For any bounded $\mathbf{n}_r(t)$, the guidance law in (6.40), along with the adaptive laws in (6.42), guarantees the uniform ultimate boundedness of the error signals $\boldsymbol{\xi}_a(t)$, $\tilde{b}(t)$ and $\tilde{\mathbf{d}}(t)$.* $\square$

Proof. Consider the Lyapunov function candidate

$$V_2(\boldsymbol{\xi}_a(t), \tilde{b}(t), \tilde{\mathbf{d}}(t), \mathbf{e}_w(t)) = \boldsymbol{\xi}_a^\top(t) P_a \boldsymbol{\xi}_a(t) + \frac{1}{b} \left(\frac{\tilde{b}^2(t)}{\rho} + \tilde{\mathbf{d}}^\top(t) G^{-1} \tilde{\mathbf{d}}(t) + \mathbf{e}_w^\top(t) \mathbf{e}_w(t) \right) \tag{6.44}$$

The derivative of $V_2(t)$ along the trajectories of the system (6.43) and (6.42) can be computed as follows:

$$\begin{aligned}
\dot{V}_2(t) &= \boldsymbol{\xi}_a^\top(t)(A_a^\top P_a + P_a A_a)\boldsymbol{\xi}_a(t) - 2\boldsymbol{\xi}_a^\top(t)P_a D_a \mathbf{n}_r(t) \\
&+ \frac{2}{b}\boldsymbol{\xi}_a^\top(t)P_a B_a \left[\tilde{b}(t)\mathbf{h}^*(t) + \mathbf{d}(t) - S_f^*(t)\hat{\mathbf{d}}(t) + \mathbf{e}_w(t) \right] \\
&+ \frac{2}{b} \left(\frac{\tilde{b}(t)}{\rho} \dot{\tilde{b}}(t) + \tilde{\mathbf{d}}^\top(t)G^{-1}\dot{\tilde{\mathbf{d}}}(t) + \mathbf{e}_w^\top(t)\dot{\mathbf{e}}_w(t) \right) \\
&= -\boldsymbol{\xi}_a^\top(t)Q_a\boldsymbol{\xi}_a(t) + \frac{2\tilde{b}(t)}{b} \left[\mathbf{y}_a^{*\top}(t)\mathbf{h}^*(t) + \frac{1}{\rho}\dot{\tilde{b}}(t) \right] \\
&+ \frac{2\tilde{b}(t)}{b}\tilde{\mathbf{d}}^\top(t) \left[-S(\mathbf{y}_a^*)\mathbf{y}_a^*(t) + G^{-1}\dot{\tilde{\mathbf{d}}}(t) \right] \\
&- \frac{4}{b}\mathbf{n}_r^\top(t) \left[\tilde{b}(t)\mathbf{h}^*(t) + \mathbf{d}(t) - S(\mathbf{y}_a^*)\hat{\mathbf{d}}(t) + \mathbf{e}_w^*(t) \right] \\
&- 2\boldsymbol{\xi}_a^\top(t)P_a D_a \mathbf{n}_r(t) + \frac{2}{b}\mathbf{y}_a^{*\top}(t) [\mathbf{d}(t) - S(\mathbf{y}_a^*)\mathbf{d}_T] \\
&+ \frac{2}{b}\mathbf{y}_a^\top(t) [S(\mathbf{y}_a^*) - S_f^*(t)] \hat{\mathbf{d}}(t) - \frac{2}{b}\mathbf{e}_w^\top(t)\Lambda\mathbf{e}_w(t), \tag{6.45}
\end{aligned}$$

Upon substitution of the adaptive laws from (6.42) and using the properties of the projection operator from (5.24) and (5.42), we conclude that

$$\begin{aligned}
\dot{V}_2(t) &\leq -\boldsymbol{\xi}_a^{*\top}(t)Q_a\boldsymbol{\xi}_a^*(t) - \frac{2}{b}\mathbf{e}_w^\top(t)\Lambda\mathbf{e}_w(t) \\
&- \frac{4}{b}\mathbf{n}_r^\top(t) \left[\tilde{b}(t)\mathbf{h}^*(t) + \mathbf{d}(t) - S(\mathbf{y}^*)\hat{\mathbf{d}}(t) + \mathbf{e}_w(t) \right] \\
&- 2\boldsymbol{\xi}_a^\top(t)P_a D_a \mathbf{n}_r(t) + \frac{2}{b}\mathbf{y}_a^{*\top}(t) [\mathbf{d}(t) - S(\mathbf{y}_a^*)\mathbf{d}_T] \\
&+ \frac{2}{b}\mathbf{y}_a^\top(t) [S(\mathbf{y}_a^*) - S_f^*(t)] \hat{\mathbf{d}}(t). \tag{6.46}
\end{aligned}$$

Taking into account the relationship

$$\mathbf{y}_a^{*\top}(t)\mathbf{d}(t) - \mathbf{y}_a^{*\top}(t)S(\mathbf{y}_a^*)\mathbf{d}_T = \sum_{i=1}^3 [y_{ai}^*(t)d_i(t) - |y_{ai}^*(t)|d_{Ti}] \leq 0, \tag{6.47}$$

the inequality in (6.46) can be reduced to

$$\begin{aligned}
\dot{V}_2(t) &\leq -\boldsymbol{\xi}_a^\top(t)Q_a\boldsymbol{\xi}_a(t) - 2\boldsymbol{\xi}_a^\top(t)P_aD_a\mathbf{n}_r(t) \\
&\quad - \frac{4}{b}\mathbf{n}_r^\top(t)\left[\tilde{b}(t)\mathbf{h}^*(t) + \mathbf{d}(t) - S(\mathbf{y}^*)\hat{\mathbf{d}}(t) + \mathbf{e}_w(t)\right] \\
&\quad + \frac{2}{b}\mathbf{y}_a^\top(t)[S(\mathbf{y}_a^*) - S_f^*(t)]\hat{\mathbf{d}}(t) - \frac{2}{b}\mathbf{e}_w^\top(t)\Lambda\mathbf{e}_w(t). \tag{6.48}
\end{aligned}$$

The first square bracketed term in (6.48) can be upper bounded by the norm of the error signals $\boldsymbol{\xi}_a(t)$ and $\mathbf{e}_w(t)$ as follows

$$\|\tilde{b}(t)\mathbf{h}^*(t) + \mathbf{d}(t) - S(\mathbf{y}^*)\hat{\mathbf{d}}(t) + \mathbf{e}_w(t)\| \leq \|\mathbf{e}_w(t)\| + c_8\|\boldsymbol{\xi}_a(t)\| + c_9, \tag{6.49}$$

where c_8 and c_9 are positive constants. Indeed, the inequalities in (5.24) and (5.42) imply boundedness of the signals $\tilde{b}(t)$ and $\hat{\mathbf{d}}(t)$ by some constants $\tilde{b}^*$ and $\tilde{d}^*$ respectively, the signal $\mathbf{d}(t)$ is bounded by the assumption, and $S(\mathbf{y}^*)$ is bounded by definition in (5.16). Since

$$\hat{\mathbf{z}}^*(t) = \mathbf{z}(t) - \tilde{\mathbf{z}}(t) = \mathbf{z}_m(t) + \mathbf{e}(t) - \tilde{\mathbf{z}}(t),$$

it follows that

$$\|\mathbf{h}^*(t)\| \leq \|K\|_F(\|\boldsymbol{\xi}_a(t)\| + \|\mathbf{z}_m(t)\|) + \|KC^\top\hat{\mathbf{y}}_c(t)\|.$$

From the boundedness of $\hat{\mathbf{y}}_c(t)$ and $\mathbf{z}_m(t)$, it follows that there exist positive constants c_8 and c_9 such that (6.49) holds. Therefore, $\dot{V}_2(t)$ can be further upper bounded as follows:

$$\begin{aligned}
\dot{V}_2(t) &\leq -\lambda_{\min}(Q_a)\|\boldsymbol{\xi}_a(t)\|^2 - \frac{2}{b}\lambda_{\min}(\Lambda)\|\mathbf{e}_w(t)\|^2 \\
&\quad + \frac{4}{b}\|\mathbf{n}_r(t)\|(\|\mathbf{e}_w(t)\| + c_8\|\boldsymbol{\xi}_a(t)\| + c_9) \\
&\quad - 2\|P_aD_a\|_F\|\boldsymbol{\xi}_a(t)\|\|\mathbf{n}_r(t)\| + \frac{2}{b}\mathbf{y}_a^\top(t)[S(\mathbf{y}_a^*) - S_f^*(t)]\hat{\mathbf{d}}(t). \tag{6.50}
\end{aligned}$$

Recall that for $\mathbf{y}_a^* \notin \Omega_\chi$ we have $S(\mathbf{y}_a^*) = S_\chi(\mathbf{y}_a^*)$ and the inequality in (5.46) holds, while for $\mathbf{y}_a^* \in \Omega_\chi$ the inequality in (5.53) holds. Therefore, $\dot{V}_2(t)$ can be further upper bounded

as follows. For $\mathbf{y}_a^* \notin \Omega_\chi$

$$\begin{aligned} \dot{V}_2(t) &\leq -\lambda_{\min}(Q_a)\|\boldsymbol{\xi}_a(t)\|^2 - \frac{2}{b}\lambda_{\min}(\Lambda)\|\mathbf{e}_w(t)\|^2 \\ &\quad + \frac{4}{b}\|\mathbf{n}_r(t)\|(\|\mathbf{e}_w(t)\| + c_8\|\boldsymbol{\xi}_a(t)\| + c_9) \\ &\quad + 2\|P_a D_a\|_F\|\boldsymbol{\xi}_a(t)\|\|\mathbf{n}_r(t)\| + 2c_{10}\exp(a_f t)\|\boldsymbol{\xi}_a(t)\|, \end{aligned} \quad (6.51)$$

where $c_{10} = \frac{d_{\max}\|E_S(0)\|_F}{b}$. Completing the squares in (6.51) yields

$$\begin{aligned} \dot{V}_2(t) &\leq -(\lambda_{\min}(Q_a) - c_{11}^2 - c_{12}^2)\|\boldsymbol{\xi}_a(t)\|^2 \\ &\quad - \left(\frac{2}{b}\lambda_{\min}(\Lambda) - c_{13}^2\right)\|\mathbf{e}_w(t)\|^2 + \frac{4c_9}{b}\|\mathbf{n}_r(t)\| \\ &\quad + c_{14}\|\mathbf{n}_r(t)\|^2 + \frac{c_{10}^2}{c_{12}^2}\exp(2a_f t), \end{aligned} \quad (6.52)$$

where the constants c_{11} , c_{12} and c_{13} are chosen such that

$$c_{15} = \lambda_{\min}(Q_a) - c_{11}^2 - c_{12}^2 > 0,$$

$$\frac{2}{b}\lambda_{\min}(\Lambda) - c_{13}^2 > 0,$$

and the notation has been introduced

$$c_{14} = \frac{1}{b^2} \left[\frac{(2c_8 + b\|P_a D_a\|_F)^2}{c_{11}^2} + \frac{4}{c_{13}^2} \right].$$

The inequality in (6.52) implies that $\dot{V}_2(t) \leq 0$ outside the compact set

$$\Omega_2 = \left\{ \|\boldsymbol{\xi}_a\| \leq \sqrt{\frac{c_{16}}{c_{15}}}, \mathbf{e}_w = \mathbf{0}, |\tilde{b}| \leq \tilde{b}^*, \|\tilde{\mathbf{d}}\| \leq \tilde{d}^* \right\}, \quad (6.53)$$

where

$$c_{16} = c_{14}c_{17}^2 + \frac{4c_9}{b}c_{17} + \frac{c_{10}^2}{c_{12}^2}$$

with c_{17} being the norm bound of $\mathbf{n}_r(t)$. Thus, the signals $\boldsymbol{\xi}_a(t)$, $\tilde{b}(t)$, $\tilde{\mathbf{d}}(t)$ and $\mathbf{e}_w(t)$ are uniformly ultimately bounded. For $\mathbf{y}_a^* \in \Omega_\chi$, we have the inequality

$$\begin{aligned} \dot{V}_2(t) &\leq -\lambda_{\min}(Q_a)\|\boldsymbol{\xi}_a(t)\|^2 - \frac{2}{b}\lambda_{\min}(\Lambda)\|\mathbf{e}_w(t)\|^2 \\ &+ \frac{4}{b}\|\mathbf{n}_r(t)\|(\|\mathbf{e}_w(t)\| + c_8\|\boldsymbol{\xi}_a(t)\| + c_9) \\ &+ 2\|P_a D_a\|_F\|\boldsymbol{\xi}_a(t)\|\|\mathbf{n}_r(t)\| + \frac{2}{b}\sqrt{3}(1 - a_f^{-1})d_{\max}\|\boldsymbol{\xi}_a(t)\|, \end{aligned} \quad (6.54)$$

which upon completion of squares yields

$$\begin{aligned} \dot{V}_2(t) &\leq -(\lambda_{\min}(Q_a) - c_{11}^2 - c_{12}^2)\|\boldsymbol{\xi}_a(t)\|^2 \left(\frac{2}{b}\lambda_{\min}(\Lambda) - c_{13}^2 \right) \|\mathbf{e}_w(t)\|^2 \\ &+ \frac{4c_9}{b}\|\mathbf{n}_r(t)\| + c_{14}\|\mathbf{n}_r(t)\|^2 + \frac{3}{b^2c_{12}^2}(1 - a_f^{-1})^2. \end{aligned} \quad (6.55)$$

The inequality in (6.55) implies that $\dot{V}_2(t) \leq 0$ outside the compact set

$$\Omega_3 = \left\{ \|\boldsymbol{\xi}_a\| \leq \sqrt{\frac{c_{17}}{c_{15}}}, \mathbf{e}_w = \mathbf{0}, |\tilde{b}| \leq \tilde{b}^*, \|\tilde{\mathbf{d}}\| \leq \tilde{d}^* \right\}, \quad (6.56)$$

where

$$c_{16} = c_{14}c_{17}^2 + \frac{4c_9}{b}c_{17} + \frac{3}{b^2c_{12}^2}(1 - a_f^{-1})^2.$$

Theorem 2 from Section 2.1 ensures that all the signals $\boldsymbol{\xi}_a(t)$, $\tilde{b}(t)$, $\tilde{\mathbf{d}}(t)$ and $\mathbf{e}_w(t)$ are uniformly ultimately bounded. The proof is complete. $\square$

6.3.3 Gain matrix design

The observer gain matrix L_a can be chosen to minimize the mean square estimation error

$$J_n = E\{\tilde{\mathbf{z}}^\top(t)\tilde{\mathbf{z}}(t)\}. \quad (6.57)$$

For the linear estimate in (6.39) to be optimal the estimation error must be orthogonal (in the probabilistic sense) to the measurement data at all the time (see for example [9], p.233),

that is

$$E\{\tilde{\mathbf{z}}(t)\mathbf{y}^{*\top}(\tau)\} = 0 \quad (6.58)$$

for all $\tau \leq t$. Then, using the standard procedure (see for example [9], p.241), the estimation error correlation matrix $K_{\tilde{\mathbf{z}}}(t, \tau)$ can be computed as:

$$K_{\tilde{\mathbf{z}}}(t, t)C = L_a K_{\mathbf{n}_r}(t, t), \quad (6.59)$$

and therefore, the estimator gain matrix can be computed as

$$L_a = K_{\tilde{\mathbf{z}}}(t, t)C K_{\mathbf{n}_r}^{-1}(t, t). \quad (6.60)$$

It can be verified that the evolution of the estimation error correlation matrix is described by the differential equation (see for example [9], p. 71)

$$\dot{K}_{\tilde{\mathbf{z}}}(t) = A_m K_{\tilde{\mathbf{z}}}(t) + K_{\tilde{\mathbf{z}}}(t)A_m^\top - K_{\tilde{\mathbf{z}}}(t)C^\top K_{\mathbf{n}_r}(t)^{-1}C K_{\tilde{\mathbf{z}}}(t). \quad (6.61)$$

If we assume a constant correlation matrix $K_{\mathbf{n}_r}$ for the measurement noise $\mathbf{n}_r(t)$, the steady state estimator gain can be computed using the standard Ricatti algebraic equation:

$$\begin{aligned} 0 &= A_m K_{\tilde{\mathbf{z}}} + K_{\tilde{\mathbf{z}}}A_m^\top - K_{\tilde{\mathbf{z}}}C^\top K_{\mathbf{n}_r}^{-1}C K_{\tilde{\mathbf{z}}} \\ L_a &= K_{\tilde{\mathbf{z}}}C K_{\mathbf{n}_r}^{-1}. \end{aligned} \quad (6.62)$$

In either case, one needs to make sure that the conditions $l_{i1} \geq \lambda_i$ and $l_{i2} > 0$ are satisfied. Otherwise a suboptimal gain matrix can be chosen.

6.3.4 Simulations

Here we use the same simulation scenario as in Chapter 5 to show the performance of the algorithm when the measurements are corrupted by the noise. The design matrices for the

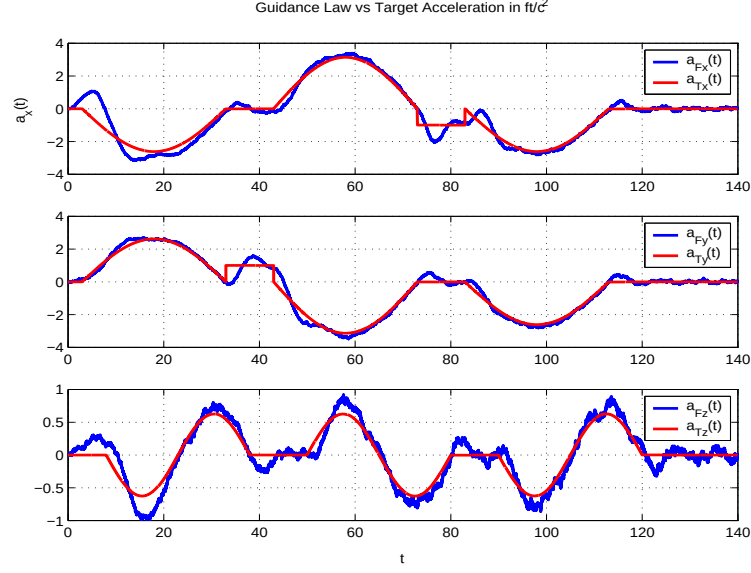


Figure 6.7: Guidance law with noisy measurements versus target's acceleration, acceleration guidance, SNR=25

controller and observer are the same as in Chapter 5. The only difference is the white noise with $SNR = 25$, added to the scaled relative position vector $\mathbf{r}(t)$ according to the equation in (6.3). The performance of the modified guidance law given by the equations in (6.40) is presented in Figures 6.7, 6.8 and 6.9. As can be seen from the Figures 6.8 and 6.9, the algorithm shows robust performance to the measurement noise, only the guidance law gets corrupted by the noise, but still is acceptable, as it is seen in Figure 6.7.

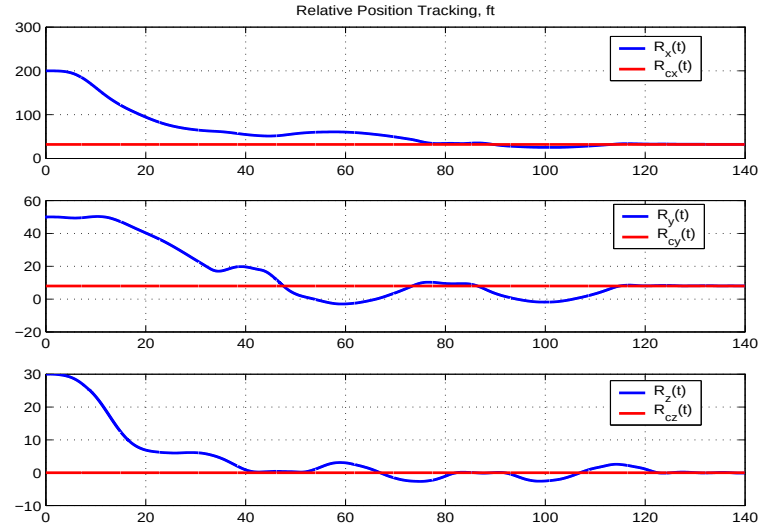


Figure 6.8: Relative position versus reference command with noisy measurements, acceleration guidance, SNR=25

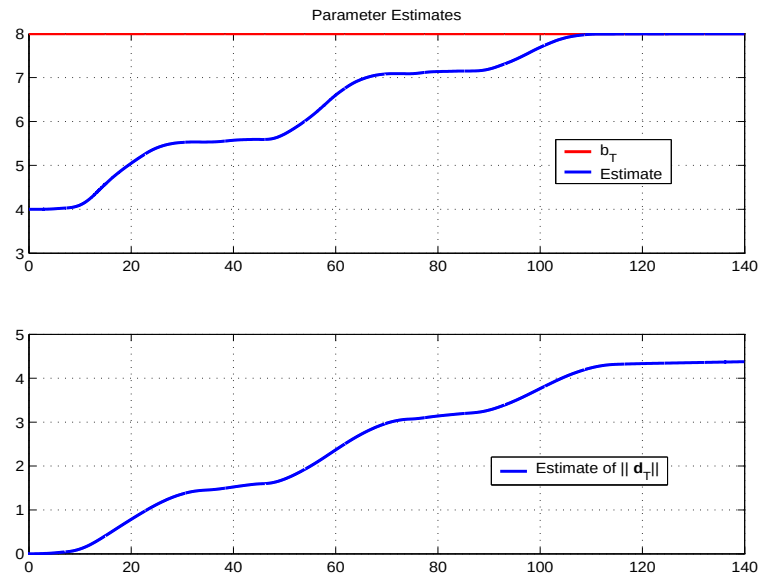


Figure 6.9: Parameter estimates with noisy measurements, acceleration guidance, SNR=25

Chapter 7

Flight Control Design

In this chapter we present the implementation of guidance laws derived in preceding three Chapters .

7.1 Reference Command Formulation

The guidance law $\mathbf{V}_F(t)$ in (6.11) that guarantees tracking of the maneuvering target subject to Assumption 1, represents the follower's desired inertial velocity vector. The aircraft's actual control surface deflections must be defined to ensure that the aircraft center of gravity at every instance of time has that inertial velocity $\mathbf{V}_F(t)$. Since $\mathbf{V}_F(t)$ is a smooth function of t , we can differentiate it to obtain the inertial acceleration $\mathbf{a}_F(t)$ for the center of gravity moving with the velocity $\mathbf{V}_F(t)$. Since the analytic differentiation will involve unknown disturbances through the definitions in (6.14), it can be done numerically using for example technique from [41].

The guidance law $\mathbf{a}_F(t)$ in (6.40) represents the desired inertial acceleration vector of the

follower's center of gravity. The inertial velocity of the aircraft's center of gravity is readily obtained by integrating the acceleration:

$$\mathbf{V}_F(t) = \mathbf{V}_F(0) + \int_0^t \mathbf{a}_F(\tau) d\tau.$$

Thus for both guidance laws, we can make use of inertial velocity and acceleration vectors. Therefore, the airspeed

$$V_c(t) = \|\mathbf{V}_F(t)\|, \quad (7.1)$$

the flight path angle

$$\gamma(t) = -\sin^{-1} \left(\frac{V_{Fz}(t)}{V_c(t)} \right), \quad (7.2)$$

and the azimuth angle

$$\chi(t) = \tan^{-1} \left(\frac{V_{Fy}(t)}{V_{Fx}(t)} \right) \quad (7.3)$$

can be straightforwardly computed. Next, we compute the aircraft attitude angles. To this end, recall that the inertial forces acting on the aerial vehicle can be calculated from Newton's second law

$$\mathbf{F} = m\mathbf{a}_F, \quad (7.4)$$

where m is the aircraft's mass. The inertial force can be transformed into the wind axis force as follows

$$\mathbf{F}^W = L_{W/E} \mathbf{F}, \quad (7.5)$$

where $L_{W/E}$ is the coordinate transformation matrix from the inertial frame to the wind frame. It is defined similar to $L_{B/E}$ in (3.10) by replacing the body axis Euler angles ϕ, θ, ψ by the wind axis Euler angles μ, γ, χ . The angles γ, χ have been already computed, but the

wind axis roll angle μ still needs to be determined. Using the wind axis force components [20], we can write the following equation

$$\begin{bmatrix} T \cos \alpha \cos \beta - D - mg \sin \gamma \\ -T \cos \alpha \sin \beta - C + mg \sin \mu \cos \gamma \\ -T \sin \alpha - L + mg \cos \mu \cos \gamma \end{bmatrix} = \mathbf{F}^W, \quad (7.6)$$

where T is the thrust, D is the drag, C is the side force, L is the lift, α is the angle of attack, β is the sideslip angle and g is the gravity acceleration. In these three equations we have seven unknowns: T , D , C , L , α , β , μ . One more equation can be obtained from the condition that all the turns made by the aircraft need to be perfectly coordinated [68], that is, the side force is zero. In this case, we can assume that the sideslip angle can be neglected. Therefore, the second equation in (7.6) reduces to

$$mg \sin \mu \cos \gamma = F_y^W, \quad (7.7)$$

from which μ can be computed as follows

$$\tan \mu = \frac{a_{Fx} \sin \chi - a_{Fy} \cos \chi}{a_{Fx} \sin \gamma \cos \chi + a_{Fy} \sin \gamma \sin \chi + (a_{Fz} - g) \cos \gamma}$$

Neglecting β in the remaining two equations in (7.6) reduces them into

$$\begin{aligned} T \cos \alpha - D - mg \sin \gamma &= F_x^W \\ -T \sin \alpha - L + mg \cos \mu \cos \gamma &= F_z^W, \end{aligned} \quad (7.8)$$

where F_x^W and F_z^W are the wind axis force components that are now available. Here we recall that the control surface deflections are primarily moment generators for the conventional aircraft with no direct force control; therefore, the dependence of the lift and drag forces on these deflections can be neglected: $C_D = C_D(\alpha)$, $C_L = C_L(\alpha, \dot{\alpha}, q)$, where $C_D = \frac{D}{\bar{q}S}$ and $C_L = \frac{L}{\bar{q}S}$ are respectively the drag and lift coefficients, $\bar{q} = \frac{1}{2}\rho V_c^2$ is the dynamic pressure,

ρ is the air density and q is the pitch rate. The presence of q in the lift coefficient requires differentiation of wind axis Euler angles and can be done numerically as it is done in [41]. In that case solving the two equations in (7.8) will result in a differential equation for α . Here we make a simplifying assumption that the lift dependence on the angle of attack rate and pitch rate is negligible so that $C_L = C_L(\alpha) = C_{L0} + C_{L\alpha}\alpha$. The drag force can be expressed using the parabolic drag polar equation $C_D = C_{D0} + KC_L^2$. From the equations in (7.8) we can easily derive the following equation to be solved for α :

$$\begin{aligned} -D \sin \alpha - L \cos \alpha &= (F_x^W + mg \sin \gamma) \sin \alpha \\ &+ (F_z^W - mg \cos \mu \cos \gamma) \cos \alpha. \end{aligned} \quad (7.9)$$

With $\beta = 0$, the transformation matrix $L_{B/W}$ from the wind to body axis reduces to

$$L_{B/W} = \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix}. \quad (7.10)$$

Using the relationship $L_{B/E} = L_{B/W}L_{W/E}$, the body axis Euler angles can be computed as follows:

$$\begin{aligned} \phi_c &= \tan^{-1} \left(\frac{L_{B/E}(2,3)}{L_{B/E}(3,3)} \right) \\ \theta_c &= -\sin^{-1} (L_{B/E}(1,3)) \\ \psi_c &= \tan^{-1} \left(\frac{L_{B/E}(1,2)}{L_{B/E}(1,1)} \right), \end{aligned} \quad (7.11)$$

where $L_{B/E}(i,j)$ denotes the (i,j) -th entry of the matrix $L_{B/E}$.

Remark 9 *We can avoid the differentiation of the guidance command $\mathbf{V}_F(t)$ by assuming that it represents the inertial velocity of the model that moves with zero sideslip and zero angle of attack, that is the longitudinal axis of the body (stability) frame of the model is*

aligned with the inertial velocity vector $\mathbf{V}_F(t)$ all the time. In this case

$$\theta_c = \gamma_c = -\sin^{-1} \left(\frac{V_{Fx}}{V_c} \right) \quad (7.12)$$

and

$$\psi_c = \chi_c = \tan^{-1} \left(\frac{V_{Fy}}{V_{Fx}} \right), \quad (7.13)$$

and the aircraft is required to track the reference commands V_c, θ_c, ψ_c . Since there are four control inputs to be selected, we can also satisfy an additional performance specification. For example, we can require the aircraft turns to be perfectly coordinated. This requirement imposes a constraint on the roll angle [68] (p.190) that can be used to specify the roll reference command:

$$\phi_c = \tan^{-1} \left(\mathfrak{S} \frac{\dot{\psi} V \cos \beta}{g \cos \alpha} \right), \quad (7.14)$$

where

$$\mathfrak{S} = \frac{(a - b^2) + b \tan \alpha + \sqrt{c(1 - b^2 + \mathfrak{S}^2 \sin^2 \beta)}}{a^2 - b^2(1 + c \tan^2 \alpha)},$$

V is the airspeed,

$$a = 1 - \frac{\dot{\psi} V}{g} \tan \alpha \sin \beta,$$

$$b = \frac{\sin \gamma}{\cos \beta},$$

$$c = 1 + \left(\frac{\dot{\psi} V}{g} \right)^2 \cos^2 \beta.$$

This approach is incorporated in the simulation example. □

7.2 Aircraft Model

Consider the dynamic equations of the aircraft written in a combined wind and body axis [68]:

$$\begin{aligned}
 \dot{V}(t) &= \frac{1}{m}[-D - mg \sin \gamma(t) + T \cos \beta(t) \cos \alpha(t)] \\
 \dot{\alpha}(t) &= q(t) - p(t) \cos \alpha(t) \tan \beta(t) \\
 &\quad - r(t) \sin \alpha(t) \tan \beta(t) - q_w(t) \sec \beta(t) \\
 \dot{\beta}(t) &= r_w(t) + p(t) \sin \alpha(t) - r(t) \cos \alpha(t) \\
 \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} &= P(t) \begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix} \\
 J \begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \\ \dot{r}(t) \end{bmatrix} &= - \begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix} \times J \begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix} + \begin{bmatrix} \mathcal{L} \\ \mathcal{M} \\ \mathcal{N} \end{bmatrix}
 \end{aligned} \tag{7.15}$$

with

$$\begin{aligned}
 P(t) &= \begin{bmatrix} 1 & \sin \phi(t) \tan \theta(t) & \cos \phi(t) \tan \theta(t) \\ 0 & \cos \phi(t) & -\sin \phi(t) \\ 0 & \sin \phi(t) \sec \theta(t) & \cos \phi(t) \sec \theta(t) \end{bmatrix} \\
 r_w &= \frac{1}{mV}[-C + mg \sin \mu \cos \gamma - T \sin \beta \cos \alpha] \\
 q_w &= \frac{1}{mV}[L - mg \cos \mu \cos \gamma + T \sin \alpha] \\
 J &= \begin{bmatrix} J_{xx} & 0 & -J_{xz} \\ 0 & J_{yy} & 0 \\ -J_{xz} & 0 & J_{zz} \end{bmatrix},
 \end{aligned} \tag{7.16}$$

where p is the roll rate, r is the yaw rate, J is the follower's inertia matrix. The body-axis moments are expressed in the form

$$\begin{aligned}\mathcal{L} &= \mathcal{L}_n + \mathcal{L}_T + \Delta_l \\ \mathcal{M} &= \mathcal{M}_n + \mathcal{M}_T + \Delta_m \\ \mathcal{N} &= \mathcal{N}_n + \mathcal{N}_T + \Delta_n,\end{aligned}\tag{7.17}$$

where $\mathcal{L}_n, \mathcal{M}_n, \mathcal{N}_n$ are respective nominal aerodynamic moments that are known and linear in control surface deflections, $\mathcal{L}_T, \mathcal{M}_T, \mathcal{N}_T$ are body-axis moments due to engine thrust, and $\Delta_l, \Delta_m, \Delta_n$ represent uncertainties in the aerodynamic moments not accounted for in the nominal moments. They are associated with the modeling of the aircraft dynamics and the turbulent effect of the closed coupled formation flight due to the tracking of the target. In general, these are unknown functions of states $\boldsymbol{\xi} = [V \ \alpha \ \beta \ p \ q \ r]^\top$ of the system in (7.15) and control $\mathbf{u} = [\delta_T \ \delta_a \ \delta_e \ \delta_r]^\top$: $\Delta_l(\boldsymbol{\xi}, \mathbf{u}), \Delta_m(\boldsymbol{\xi}, \mathbf{u}), \Delta_n(\boldsymbol{\xi}, \mathbf{u})$. The aerodynamic forces D, C, L also are assumed to have nominal known parts D_n, C_n, L_n and unknown parts $\Delta_D(\boldsymbol{\xi}, \mathbf{u}), \Delta_C(\boldsymbol{\xi}, \mathbf{u}), \Delta_L(\boldsymbol{\xi}, \mathbf{u})$. The control algorithm presented bellow will compensate for the uncertainties in the aerodynamic drag force and aerodynamic moments that are assumed to be bounded and continuous functions of $\boldsymbol{\xi}, \mathbf{u} \in \mathcal{D}$, where $\mathcal{D}$ is a compact set of respective possible initial conditions. We use Theorem 8 to approximate these unknown functions by linear in parameters RBF neural networks in some compact domain of interest Ω_ξ :

$$\begin{aligned}\frac{1}{m}\Delta_D(\boldsymbol{\xi}, \mathbf{u}) &= \mathbf{W}_D^\top \boldsymbol{\Phi}(\boldsymbol{\xi}, \mathbf{u}) + \epsilon_D(\boldsymbol{\xi}, \mathbf{u}) \\ J^{-1} \begin{bmatrix} \Delta_l(\boldsymbol{\xi}, \mathbf{u}) \\ \Delta_m(\boldsymbol{\xi}, \mathbf{u}) \\ \Delta_n(\boldsymbol{\xi}, \mathbf{u}) \end{bmatrix} &= \begin{bmatrix} \mathbf{W}_l^\top \boldsymbol{\Phi}(\boldsymbol{\xi}, \mathbf{u}) + \epsilon_l(\boldsymbol{\xi}, \mathbf{u}) \\ \mathbf{W}_m^\top \boldsymbol{\Phi}(\boldsymbol{\xi}, \mathbf{u}) + \epsilon_m(\boldsymbol{\xi}, \mathbf{u}) \\ \mathbf{W}_n^\top \boldsymbol{\Phi}(\boldsymbol{\xi}, \mathbf{u}) + \epsilon_n(\boldsymbol{\xi}, \mathbf{u}) \end{bmatrix}\end{aligned}\tag{7.18}$$

$$\|\epsilon_D(\boldsymbol{\xi}, \mathbf{u})\| \leq \epsilon_D^*, \quad \|\epsilon_l(\boldsymbol{\xi}, \mathbf{u})\| \leq \epsilon_l^*,$$

$$\|\epsilon_m(\boldsymbol{\xi}, \mathbf{u})\| \leq \epsilon_m^*, \quad \|\epsilon_n(\boldsymbol{\xi}, \mathbf{u})\| \leq \epsilon_n^*,$$

where $\mathbf{W}_i \in \mathbb{R}^{N_i}$, $i = D, l, m, n$ are the vectors of unknown constant parameters and $\Phi(\boldsymbol{\xi}, \mathbf{u})$ is the vector of RBFs. Also, we assume that the thrust and nominal aerodynamic moments are linear in controls and can be represented as follows

$$\begin{aligned}
T &= \frac{1}{2} \rho V_F^2 S C_{T_{\delta_T}} \delta_T \\
\mathcal{L} &= \frac{1}{2} \rho V_F^2 S b [C_l(\beta, p, r) + C_{l_{\delta_a}} \delta_a + C_{l_{\delta_r}} \delta_r] \\
\mathcal{N} &= \frac{1}{2} \rho V_F^2 S b [C_n(\beta, p, r) + C_{n_{\delta_a}} \delta_a + C_{n_{\delta_r}} \delta_r] \\
\mathcal{M} &= \frac{1}{2} \rho V_F^2 S \bar{c} [C_m(M, \alpha, q) + C_{m_{\delta_e}} \delta_e].
\end{aligned} \tag{7.19}$$

Here ρ is the air density, S is the aircraft's reference area, b_F is the follower's wing span, $\bar{c}$ is the mean-aerodynamic chord, $C_{T_{\delta_T}}, C_{l_{\delta_a}}, C_{l_{\delta_r}}, C_{n_{\delta_a}}, C_{n_{\delta_r}}, C_{m_{\delta_e}}$ are the corresponding control effectiveness and C_l, C_n, C_m are the aerodynamic coefficients.

7.3 Control Design

In this section, we derive the control surface deflection commands $\mathbf{u} = [\delta_T \ \delta_e \ \delta_a \ \delta_r]^\top$ to track the desired motion. Since there are four control inputs, we chose V, ϕ, θ, ψ as the regulated outputs to track the desired velocity and attitude commands $V_c, \phi_c, \theta_c, \psi_c$ in (7.1), (7.3), (7.2), (7.8). For this purpose we first derive desired laws for $X_T, \mathcal{L}^n, \mathcal{M}^n, \mathcal{N}^n$ for the dynamics in (7.15) to track the reference commands $[V_c(t), \phi_c(t), \theta_c(t), \psi_c(t)]$. These can be inverted afterwards to obtain the actual throttle and control surface deflections using the relationships in (7.19).

7.3.1 Airspeed control

We start with the first equation in (7.15) to design the tracking control law for the velocity command $V_c(t)$ and set

$$\begin{aligned} T(t) &= \frac{1}{\cos(\alpha(t))\cos(\beta(t))} [mg \sin(\gamma(t)) \\ &+ D^n(\boldsymbol{\xi}(t), \mathbf{u}(t) + m(\hat{\mathbf{W}}_D^\top(t) \boldsymbol{\Phi}(\boldsymbol{\xi}(t), \mathbf{u}(t)) \\ &- mk_V(V(t) - V_c(t)) + m\dot{V}_c(t))] , \end{aligned}$$

where $k_V > 0$ is the desired time constant (design parameter), and $\hat{\mathbf{W}}_D(t)$ is the estimate of the unknown weight vector $\mathbf{W}_D$, which is updated online. Defining the tracking error as $e_V = V(t) - V_c(t)$, the error dynamics can be written as

$$\begin{aligned} \dot{e}_V(t) &= -k_V e_V(t) - \tilde{\mathbf{W}}_D^\top(t) \boldsymbol{\Phi}(\boldsymbol{\xi}(t), \mathbf{u}(t)) \\ &- \epsilon_D(\boldsymbol{\xi}(t), \mathbf{u}(t)) , \end{aligned} \quad (7.20)$$

where $\tilde{\mathbf{W}}_D(t) = \hat{\mathbf{W}}_D(t) - \mathbf{W}_D$ is the parameter error. The adaptive law for the estimate $\hat{\mathbf{W}}_D(t)$ is designed using the following Lyapunov function candidate

$$V_1(e_V(t), \tilde{\mathbf{W}}_D(t)) = \frac{1}{2} e_V^2(t) + \tilde{\mathbf{W}}_D^\top(t) G_D^{-1} \tilde{\mathbf{W}}_D(t) , \quad (7.21)$$

where $G_D > 0$ is the adaptive gain. It is straightforward to verify that the derivative of V_1 along the trajectory of the system in (7.20) satisfies the inequality

$$\dot{V}_1(e_V(t), \tilde{\mathbf{W}}_D(t)) \leq -k_V e_V^2(t) - e_V(t) \epsilon_D(\boldsymbol{\xi}(t), \mathbf{u}(t)) , \quad (7.22)$$

if we set

$$\dot{\hat{\mathbf{W}}}_D(t) = G_D \text{Proj} \left(\hat{\mathbf{W}}_D(t), e_V(t) \boldsymbol{\Phi}(\boldsymbol{\xi}(t), \mathbf{u}(t)) \right) , \quad (7.23)$$

where $\text{Proj}(\cdot, \cdot)$ is the projection operator (see Appendix A) and guarantees boundedness of the estimate $\hat{\mathbf{W}}_D(t)$ and, hence, the parameter error $\tilde{\mathbf{W}}_D(t)$. The inequality in (7.22) implies that the tracking error is uniformly ultimately bounded.

7.3.2 Orientation control

Next, we use the block backstepping technique from [33] to design the pseudo-controls $p_c(t)$, $q_c(t)$, $r_c(t)$ for the Euler angles dynamic equations to track the reference commands $\phi_c(t)$, $\theta_c(t)$, $\psi_c(t)$. To this end, we set

$$\begin{bmatrix} p_c(t) \\ q_c(t) \\ r_c(t) \end{bmatrix} = H(t) \begin{bmatrix} -k_\phi(\phi(t) - \phi_c(t)) + \dot{\phi}_c(t) \\ -k_\theta(\theta(t) - \theta_c(t)) + \dot{\theta}_c(t) \\ -k_\psi(\psi(t) - \psi_c(t)) + \dot{\psi}_c(t) \end{bmatrix},$$

where $k_\phi > 0$, $k_\theta > 0$, k_ψ are the desired time constants (design parameters), while

$$H(t) = \begin{bmatrix} 1 & 0 & -\sin \theta(t) \\ 0 & \cos \phi(t) & \sin \phi(t) \cos \theta(t) \\ 0 & -\sin \phi(t) & \cos \phi(t) \cos \theta(t) \end{bmatrix}.$$

Then, defining the tracking errors as

$$e_\phi(t) = \phi(t) - \phi_c(t), \quad e_\theta(t) = \theta(t) - \theta_c(t), \quad e_\psi(t) = \psi(t) - \psi_c(t),$$

the corresponding error dynamics result in the system

$$\begin{aligned} \dot{e}_\phi(t) &= -k_\phi e_\phi(t) \\ \dot{e}_\theta(t) &= -k_\theta e_\theta(t) \\ \dot{e}_\psi(t) &= -k_\psi e_\psi(t), \end{aligned} \tag{7.24}$$

which is exponentially stable. Therefore, denoting the corresponding errors by

$$e_p(t) = p(t) - p_c(t), \quad e_q(t) = q(t) - q_c(t), \quad e_r(t) = r(t) - r_c(t)$$

and designing the nominal aerodynamic moments as

$$\begin{aligned}
 \begin{bmatrix} \mathcal{L}_n(t) \\ \mathcal{M}_n(t) \\ \mathcal{N}_n(t) \end{bmatrix} &= \begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix} \times J \begin{bmatrix} p(t) \\ q(t) \\ r(t) \end{bmatrix} \\
 -JP(t) \begin{bmatrix} e_\phi(t) \\ e_\theta(t) \\ e_\psi(t) \end{bmatrix} &- J \begin{bmatrix} \hat{\mathbf{W}}_l^\top \Phi(\boldsymbol{\xi}, \mathbf{u}) \\ \hat{\mathbf{W}}_m^\top \Phi(\boldsymbol{\xi}, \mathbf{u}) \\ \hat{\mathbf{W}}_n^\top \Phi(\boldsymbol{\xi}, \mathbf{u}) \end{bmatrix} \\
 +J \begin{bmatrix} -k_p e_p(t) + \dot{p}_c(t) \\ -k_q e_q(t) + \dot{q}_c(t) \\ -k_r e_r(t) + \dot{r}_c(t) \end{bmatrix}, & \quad (7.25)
 \end{aligned}$$

where $k_p > 0, k_r > 0, k_q > 0$ are the desired time constants (design parameters) and $\hat{\mathbf{W}}_l(t)$, $\hat{\mathbf{W}}_m(t)$, $\hat{\mathbf{W}}_n(t)$ are respectively the estimates of the unknown weight vectors $\mathbf{W}_l$, $\mathbf{W}_m$, $\mathbf{W}_n$ that are updated online, the overall error dynamics can be written as follows

$$\begin{aligned}
 \begin{bmatrix} \dot{e}_\phi(t) \\ \dot{e}_\theta(t) \\ \dot{e}_\psi(t) \end{bmatrix} &= \begin{bmatrix} -k_\phi e_\phi(t) \\ -k_\theta e_\theta(t) \\ -k_\psi e_\psi(t) \end{bmatrix} + P(t) \begin{bmatrix} e_p(t) \\ e_q(t) \\ e_r(t) \end{bmatrix} \\
 \begin{bmatrix} \dot{e}_p(t) \\ \dot{e}_q(t) \\ \dot{e}_r(t) \end{bmatrix} &= \begin{bmatrix} -k_p e_p(t) \\ -k_q e_q(t) \\ -k_r e_r(t) \end{bmatrix} - P(t) \begin{bmatrix} e_\phi(t) \\ e_\theta(t) \\ e_\psi(t) \end{bmatrix} \\
 &- \begin{bmatrix} \tilde{\mathbf{W}}_l^\top(t) \Phi(\boldsymbol{\xi}(t), \mathbf{u}(t)) - \epsilon_l(\boldsymbol{\xi}(t), \mathbf{u}(t)) \\ \tilde{\mathbf{W}}_m^\top(t) \Phi(\boldsymbol{\xi}(t), \mathbf{u}(t)) - \epsilon_m(\boldsymbol{\xi}(t), \mathbf{u}(t)) \\ \tilde{\mathbf{W}}_n^\top(t) \Phi(\boldsymbol{\xi}(t), \mathbf{u}(t)) - \epsilon_n(\boldsymbol{\xi}(t), \mathbf{u}(t)) \end{bmatrix}, \quad (7.26)
 \end{aligned}$$

where

$$\tilde{\mathbf{W}}_l(t) = \hat{\mathbf{W}}_l(t) - \mathbf{W}_l, \quad \tilde{\mathbf{W}}_m(t) = \hat{\mathbf{W}}_m(t) - \mathbf{W}_m, \quad \tilde{\mathbf{W}}_n(t) = \hat{\mathbf{W}}_n(t) - \mathbf{W}_n$$

are the parameter estimation errors.

Consider the following Lyapunov function candidate for the error dynamics in (7.26):

$$\begin{aligned}
V_2(e_\phi(t), e_\theta(t), e_\psi(t), e_p(t), e_q(t), e_r(t), \tilde{\mathbf{W}}_l(t), \tilde{\mathbf{W}}_m(t), \tilde{\mathbf{W}}_n(t)) = \\
+ \frac{1}{2}e_\phi^2(t) + \frac{1}{2}e_\theta^2(t) + \frac{1}{2}e_\psi^2(t) + \frac{1}{2}e_p^2(t) + \frac{1}{2}e_q^2(t) \\
+ \frac{1}{2}e_r^2(t) + \tilde{\mathbf{W}}_l^\top(t)G_p^{-1}\tilde{\mathbf{W}}_l(t) + \tilde{\mathbf{W}}_m^\top(t)G_q^{-1}\tilde{\mathbf{W}}_m(t) + \tilde{\mathbf{W}}_n^\top(t)G_r^{-1}\tilde{\mathbf{W}}_n(t),
\end{aligned} \tag{7.27}$$

where $G_l > 0, G_m > 0, G_n > 0$ are the adaptive gains. If we define the adaptive laws for the estimates $\hat{\mathbf{W}}_l(t)$, $\hat{\mathbf{W}}_m(t)$ and $\hat{\mathbf{W}}_n(t)$ as

$$\begin{aligned}
\dot{\hat{\mathbf{W}}}_l(t) &= G_p \text{Proj} \left(\hat{\mathbf{W}}_l(t), e_p(t) \Phi(\boldsymbol{\xi}(t), \mathbf{u}(t)) \right) \\
\dot{\hat{\mathbf{W}}}_m(t) &= G_q \text{Proj} \left(\hat{\mathbf{W}}_m(t), e_q(t) \Phi(\boldsymbol{\xi}(t), \mathbf{u}(t)) \right) \\
\dot{\hat{\mathbf{W}}}_n(t) &= G_r \text{Proj} \left(\hat{\mathbf{W}}_n(t), e_r(t) \Phi(\boldsymbol{\xi}(t), \mathbf{u}(t)) \right),
\end{aligned} \tag{7.28}$$

then it can be verified that

$$\begin{aligned}
\dot{V}_2(t) &\leq -k_\phi e_\phi^2(t) - k_\theta e_\theta^2(t) - k_\psi e_\psi^2(t) \\
&\quad - k_p e_p^2(t) - e_p(t) \epsilon_l(\boldsymbol{\xi}(t), \mathbf{u}(t)) \\
&\quad - k_q e_q^2(t) - e_q(t) \epsilon_m(\boldsymbol{\xi}(t), \mathbf{u}(t)) \\
&\quad - k_r e_r^2(t) - e_r(t) \epsilon_n(\boldsymbol{\xi}(t), \mathbf{u}(t)).
\end{aligned} \tag{7.29}$$

Taking into account the uniform bounds on the function approximation errors in (7.18), the following upper bound can be written

$$\begin{aligned}
\dot{V}_2(t) &\leq -k_\phi e_\phi^2(t) - k_\theta e_\theta^2(t) - k_\psi e_\psi^2(t) \\
&\quad - |e_p(t)| (k_p |e_p(t)| - \epsilon_l^*) \\
&\quad - |e_q(t)| (k_q |e_q(t)| - \epsilon_m^*) \\
&\quad - |e_r(t)| (k_r |e_r(t)| - \epsilon_n^*).
\end{aligned} \tag{7.30}$$

It follows that $\dot{V}_2 \leq 0$ outside the compact region

$$\begin{aligned} \Omega = \{ & |e_p(t)| \leq k_p^{-1}\epsilon_l^*, \quad |e_q(t)| \leq k_q^{-1}\epsilon_m^*, \\ & |e_r(t)| \leq k_r^{-1}\epsilon_n^*, \quad \|\tilde{\mathbf{W}}_i(t)\| \leq W_i^* \quad i = l, m, n \}, \end{aligned} \quad (7.31)$$

where the bounds W_i^* are guaranteed by the projection operator used in the adaptive laws (7.28).

The throttle and control surface deflections are easily found by solving the equations in (7.20), (7.25) for $\delta_T, \delta_a, \delta_e, \delta_r$, since the nominal aerodynamic moments are assumed to have linear representations in stability derivatives. The derivation of the control laws is summarized in the following theorem.

Theorem 16 *The aircraft thrust and the aerodynamic moments defined in (7.20), (7.25) along with the adaptive laws in (7.23) and (7.28) guarantee tracking of the reference commands $V_c(t), \phi_c(t), \theta_c(t), \psi_c(t)$ with uniformly ultimately bounded errors.* $\square$

If the approximation errors are zero, i.e. $\epsilon_i(x) = 0, i = D, l, m, n$, then it follows from (7.22) and (7.30), that for all $\boldsymbol{\xi}$, $\dot{V}_2(t) \leq 0$ and $\dot{V}_1(t) \leq 0$. Using Barbalat's lemma, one can show asymptotic tracking of the acceleration command $\mathbf{a}_F$.

Remark 10 *The control laws in (7.20), (7.25) contain the derivatives of the reference commands $V_c(t), \phi_c(t), \theta_c(t), \psi_c(t)$ that involve unknown disturbances $\mathbf{d}(t)$ through the guidance law in (6.11) or in (6.40). One can pre-filter the reference commands $V_c(t), \phi_c(t), \theta_c(t), \psi_c(t)$ by a stable low-pass filter. For instance, for the $V(t)$ dynamics, one can set*

$$\dot{V}_m(t) = -\lambda_V(V_m(t) - V_c(t)), \quad (7.32)$$

where $\lambda_V > 0$ is a constant. In the control law in (7.20) one can use $V_m(t)$ instead of $V_c(t)$, and replace the derivative $\dot{V}_m(t)$ by the right hand side of equation in (7.32). To remove the

steady state error, one can define the integral error $V_I(t) = \int_{t_0}^t V_m(\tau) - V_c(\tau)$ and write the extended dynamics:

$$\begin{aligned}\dot{V}_I(t) &= V_m(t) - V_c(t) \\ \dot{V}_m(t) &= -2\zeta_u\omega_V(V_m(t) - V_c(t)) - \omega_V^2 V_I, \end{aligned} \quad (7.33)$$

where ζ_V , ω_V are the desired damping ratio and frequency (design parameters). Again the command $V_c(t)$ can be replaced by $V_m(t)$ in the control law in (7.20), its derivative being calculated according to the dynamics in (7.33). $\square$

Remark 11 If the UAV is equipped with a built in autopilot, which is capable of way points command tracking, the guidance laws in (6.11) and in (6.40) can be integrated respectively once and twice to obtain an inertial position commands that can be provided to the existing autopilot as input. $\square$

7.4 Simulations

The proposed algorithm is simulated for the small UAV model with the following parameters: $m = 0.6293 \text{ sl}$, $J = [0.123000; 00.17470; 000.2553] \text{ sl.ft}^2$, $S = 11.5 \text{ ft}^2$, $c = 1.2547 \text{ ft}$, $b_F = 9.1 \text{ ft}$. For target model, a UAV is chosen with a wing span (maximal size) of 8 ft . The target's is the same as in the previous chapters with the speed range from 35 ft/s to 60 ft/s and acceleration up to 5 ft/s^2 . In simulations, no camera model is used. Instead, the tracking error is formed according to the equation

$$\mathbf{e}(t) = \frac{1}{b} C(\mathbf{x}_T(t) - \mathbf{x}_F(t)) - \mathbf{z}_m(t) + \mathbf{n}(t). \quad (7.34)$$

where $\mathbf{n}(t)$ is the white gaussian measurement noise. The initial conditions for the reference model and observer are set identical and equal to

$$\mathbf{z}_m(0) = \hat{\mathbf{z}}_m(0) = \frac{1}{b}C(\mathbf{x}_T(0) - \mathbf{x}_F(0)).$$

The initialization is done with $\hat{b}_0 = 4 \text{ ft}$. The target starts its motion from the straight level flight with the airspeed of 50 ft/s from the position $\mathbf{x}_T(0) = [200 \ 50 \ 30]^\top$, where the origin of the inertial frame is placed in the follower's initial position. The follower is initially in straight level flight with the same speed and is commanded to maintain the relative coordinates $\mathbf{R}_c = [32 \ 8 \ 0]^\top \text{ ft}$. The excitation signal is added to the R_{cy} coordinate with the amplitude defined according to (2.28), where $T = \frac{2\pi}{\omega}$ is the period of the excitation signal $a \sin(\omega t)$, $k_i > 0$, $i = 1, 2, 3$ are design constants set to $T = 3 \text{ sec}$, $k_1 = 0.4$, $k_2 = 50$ and k_3 is set to zero.

Two simulations are run. First, we run a simulation with the velocity guidance law derived in Chapter 6 and given by the equations (6.10), (6.11) and (6.16), where the design parameters are the same as in Chapter 6. The measurement noise level is set to $SNR = 35$. To maintain realistic control bounds during the transient, the following saturations are imposed on the control surface deflections: elevator -30° , aileron -30° and rudder -20° . However, the bounds are never hit in simulation.

Figure 7.2 shows that the guidance law is able to capture the target's velocity profile after a short transient. The resulting deflection commands are displayed in Figure 7.5. The output tracking is displayed in Figure 7.1. The overall UAV responses to the commands are represented in Figures 7.3 and 7.4. The angles of attack and sideslip are in the acceptable range although are not controlled directly (Figure 7.6). The angular rates have a good response after a small initial transient.

Next, we run a simulation with the acceleration guidance law derived in Chapter 6 and given

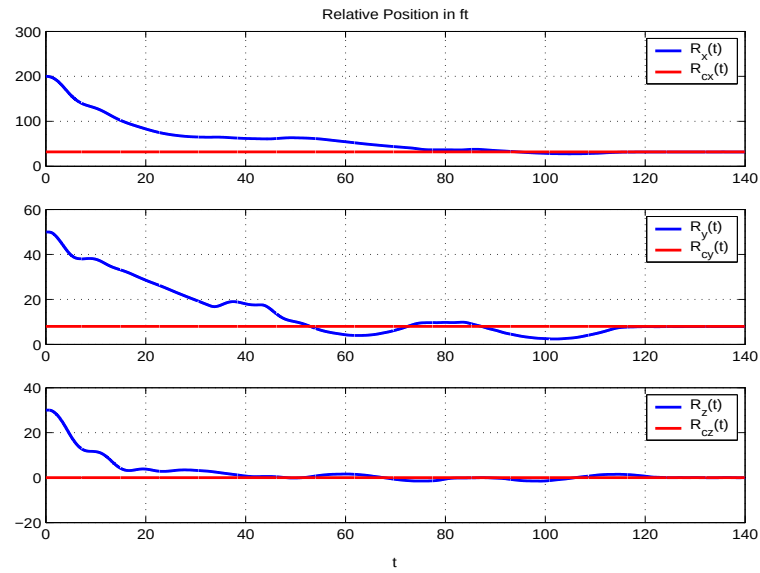


Figure 7.1: Target's and Follower's relative position, velocity guidance

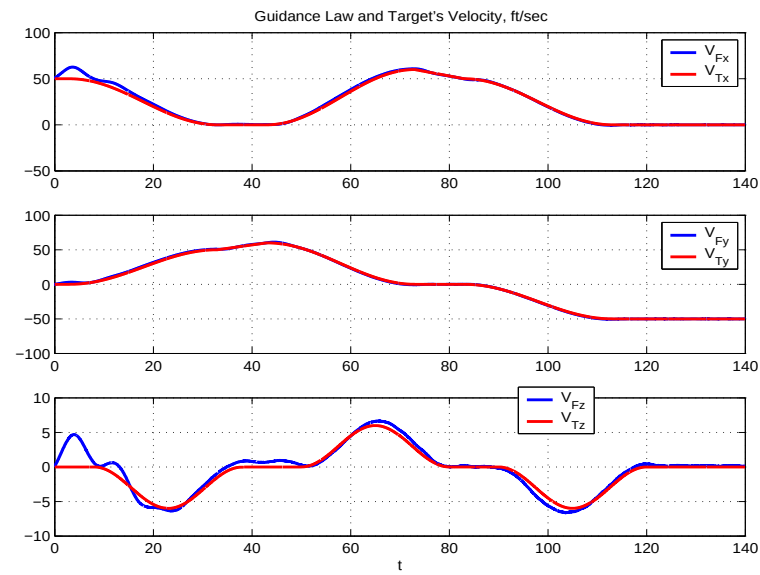


Figure 7.2: Guidance law versus target's inertial velocities, velocity guidance

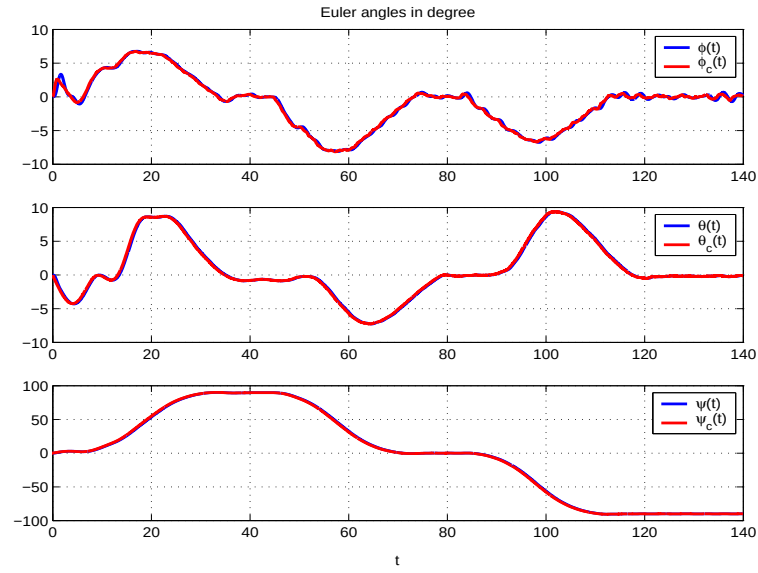


Figure 7.3: Euler angle responses, velocity guidance

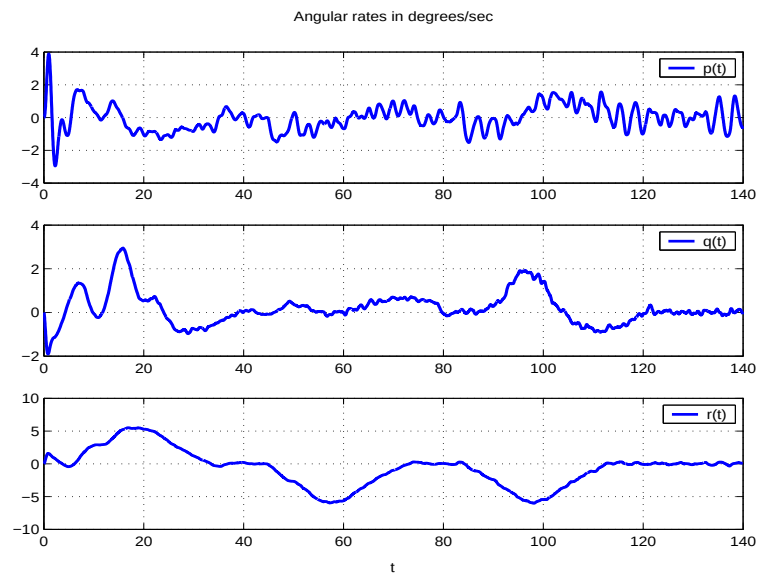


Figure 7.4: Angular rate responses, velocity guidance

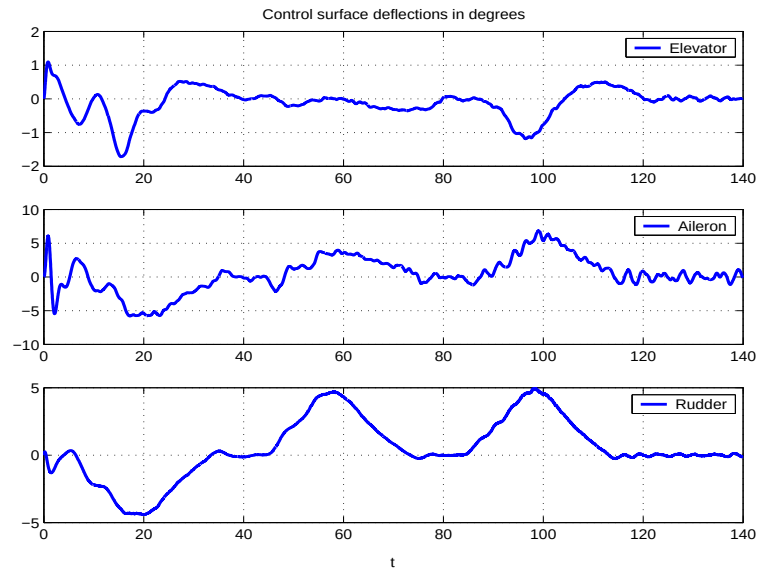


Figure 7.5: Control surface deflections, velocity guidance

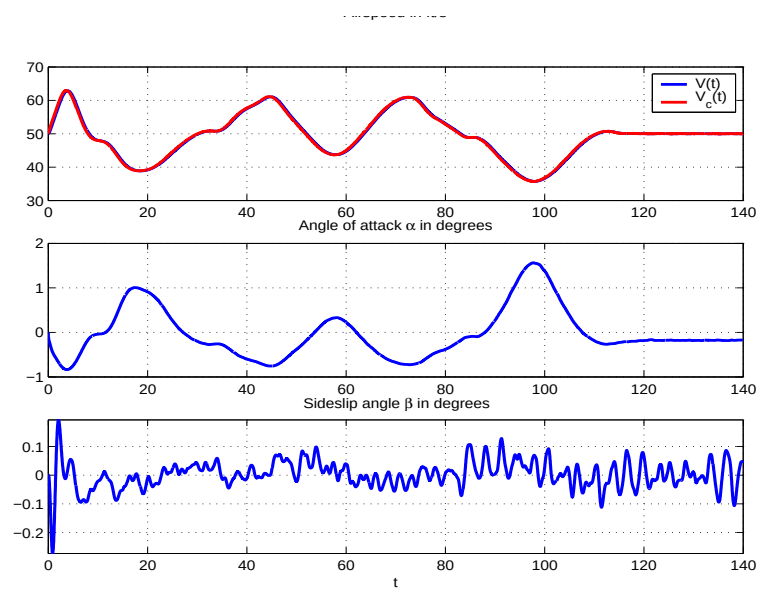


Figure 7.6: Airspeed, angle of attack and sideslip, velocity guidance

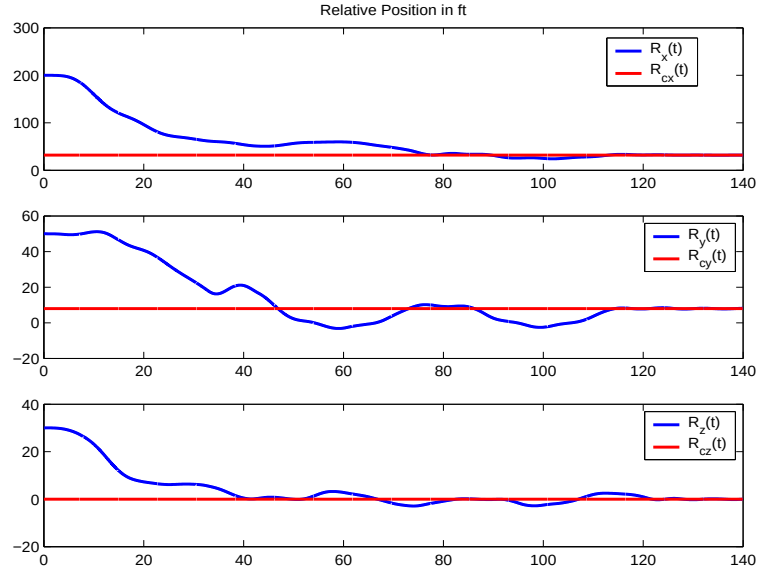


Figure 7.7: Target's and follower's relative position, acceleration guidance

by the equations in (6.40), where the design parameters are the same as in Chapter 6. The measurement noise level is set to $SNR = 25$. To maintain realistic control bounds during the transient, the following saturations are imposed on the control surface deflections: elevator- 30° , aileron- 30° and rudder 20° . However, this bounds where never hit in simulation.

Figure 7.9 shows that the guidance law closely follows the target's acceleration profile. The resulting deflection commands are displayed in Figure 7.12. The output tracking is displayed in Figure 7.7. The overall UAV responses to the commands are represented in Figures 7.10 and 7.11. The angles of attack and sideslip are in the acceptable range although are not controlled directly (Figure 7.13).

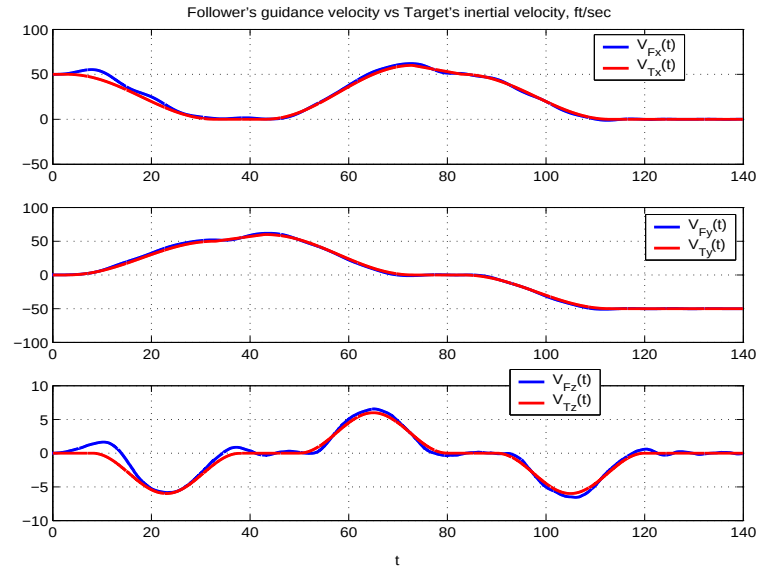


Figure 7.8: Target's and follower's inertial velocities, acceleration guidance

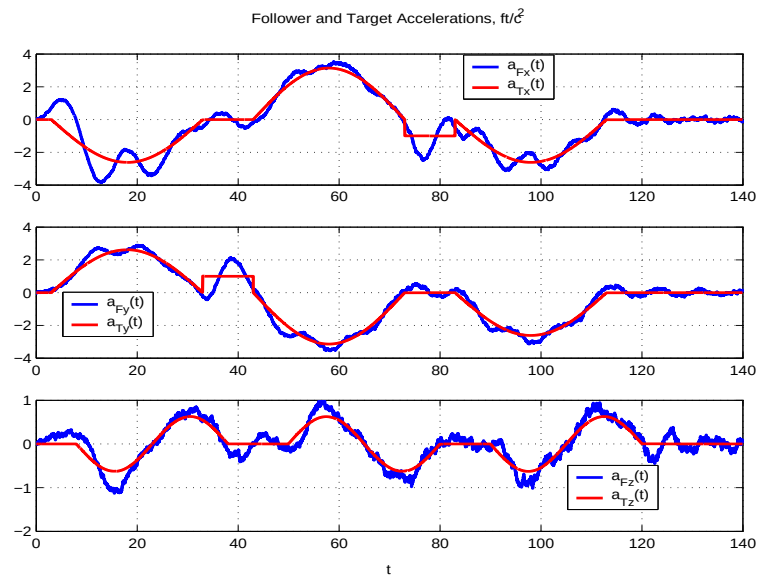


Figure 7.9: Guidance law versus target's inertial accelerations, acceleration guidance

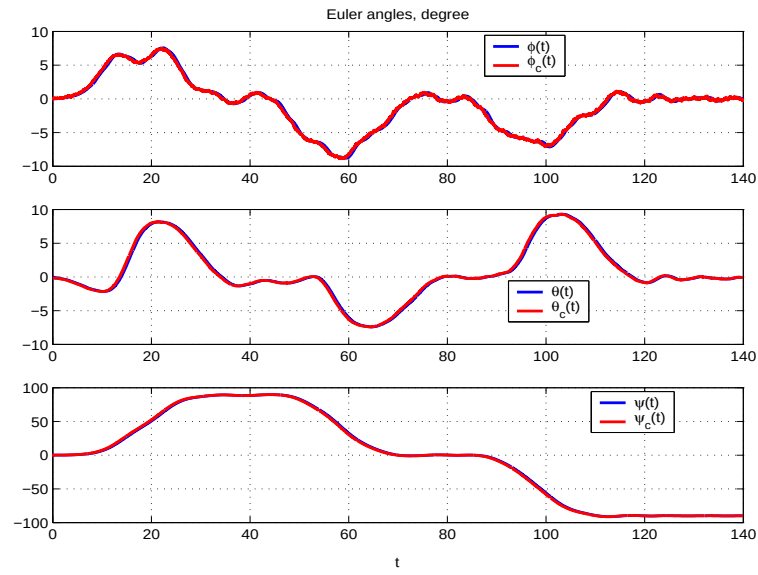


Figure 7.10: Euler angle responses, acceleration guidance

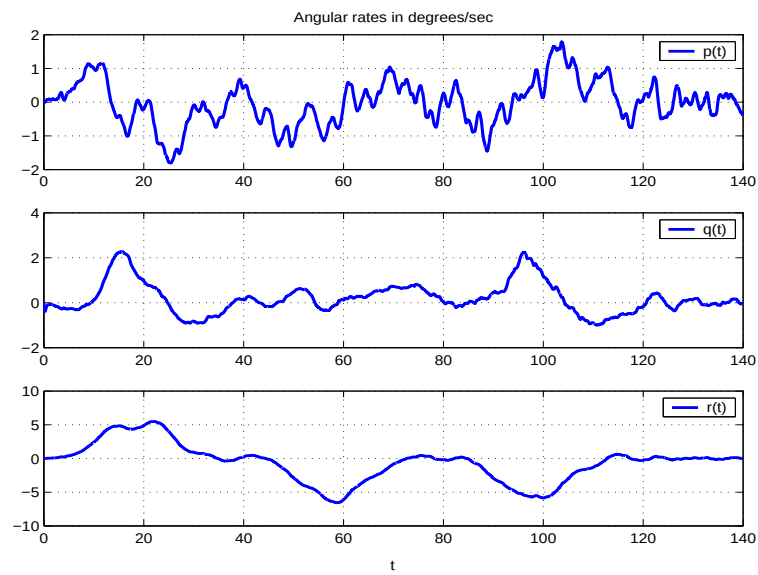


Figure 7.11: Angular rate responses, acceleration guidance

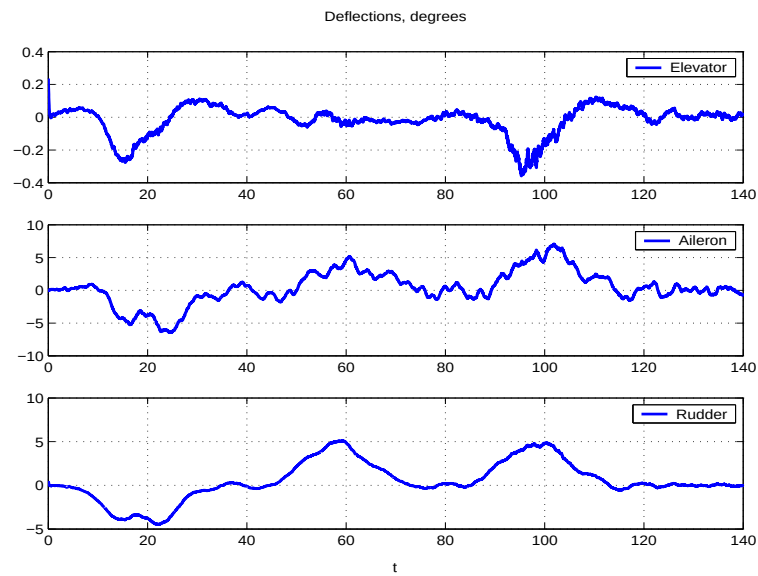


Figure 7.12: Control surface deflections, acceleration guidance

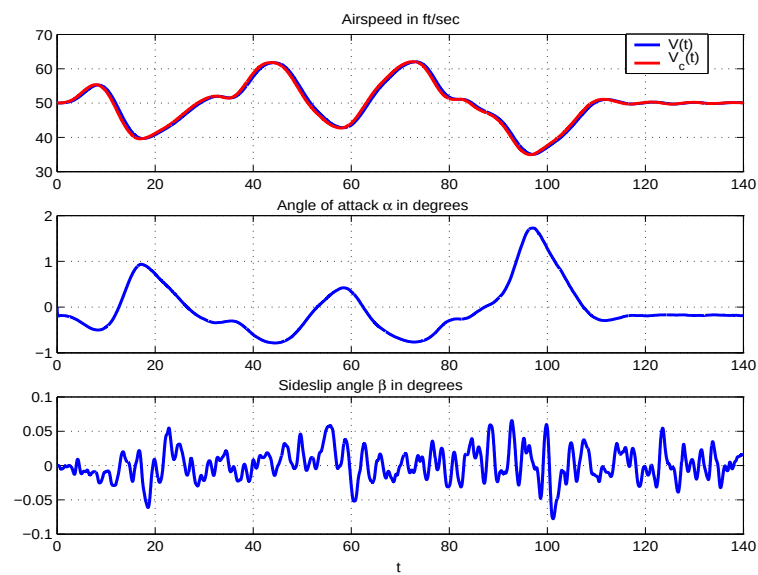


Figure 7.13: Airspeed, angle of attack and sideslip, acceleration guidance

Chapter 8

Concluding Remarks

In this dissertation, a maneuvering target tracking problem is considered using only visual information from the fixed monocular camera. Although the relative range between two flying aerial vehicles is unobservable from the visual measurements, which consist of the pixel coordinates of the image centroid and the image maximum size in pixels, the associated mathematical modeling enables us to cast the problem into adaptive control framework, with reference commands depending on the unknown parameters.

8.1 Summary

In Chapter 3, the relative dynamics of the flying target with respect to the follower aircraft is formulated, and geometric relationships that relate the visual measurements to the relative position vector are presented. Viewing the target's velocity and acceleration as time varying disturbances, two problems are formulated for the different classes of the target's motion. The first class assumes the target's maneuvers with a velocity that can be decomposed into

a constant plus an $\mathcal{L}_\infty \cap \mathcal{L}_2$ term. This problem can be cast into a state feedback control problem framework subject to unknown parameters and disturbances. The second class assumes only that the target has a bounded acceleration, but the resulting control problem has to be considered in the output feedback framework, again subject to unknown parameters and disturbances.

In chapter 4, a robust adaptive control framework is presented, which solves the first problem. Although no direct measurement of the relative range is available, it is shown that the measurements of the image length and image centroid coordinates, obtained through the visual sensor, enable the follower to maintain the desired relative position with respect to the target, provided that the reference command has an additive intelligent excitation signal. It is shown that the latter is not required for the target interception problem.

In chapter 5, a solution to the second problem is presented, and it requires input filtered state transformation to cope with the output feedback framework. An adaptive bounding technique is used for the transformed system to design a discontinuous guidance law that guarantees the asymptotic tracking of the reference commands in the presence of the additive intelligent excitation signal, which is not needed when intercepting a target. In order to backstep to the original system, this discontinuous signal is approximated by a smooth one, which guarantees bounded tracking with the adjustable bounds.

In chapter 6, both guidance laws are modified to accommodate for the measurement noise from the visual sensor. The modifications are based on the state estimators, which are shown to be stable, with gains that can be chosen suboptimal according to Kalman's scheme.

Finally, in chapter 7, an algorithm is presented for designing the aircraft's actual control surface deflection commands to implement the guidance laws presented in the previous chapters. This algorithm is based on the full scale aircraft non-linear model and uses the conventional adaptive block backstepping technique, taking into account modeling uncertainties and un-

known atmospheric turbulent effects.

8.2 Directions for Future Research

Target's Size. In the developments of the guidance laws, the target's apparent size b is assumed to be a constant. However, in a realistic scenario, b is not a constant and the dependence of b on the relative orientation has quite a complicated character as can be seen from the following considerations. To simplify the notations, we assume that the target's length and wing span are much greater than its thickness, that is, the target is assumed to be flat. Let the vectors $\mathbf{P}_l^T$, $\mathbf{P}_r^T$, $\mathbf{P}_n^T$ and $\mathbf{P}_t^T$ denote respectively the left wing tip, the right wing tip, the nose and the tail of the target given in its body frame $F_T = (x_T, y_T, z_T)$. Then, the target's length is $l_T = \|\mathbf{P}_n - \mathbf{P}_t\|$ and the wing span is $b_T = \|\mathbf{P}_r - \mathbf{P}_l\|$. These vectors are translated into the followers body frame by means of the transformation matrix $L_{F/T} = LF_{B/E}LT_{E/B}$, where $LF_{B/E}$ is the coordinate transformation matrix from the inertial frame to the follower's body frame, F_B and $LT_{E/B}$ is the coordinate transformation matrix from the target's body frame F_T to the inertial frame. In the followers body frame F_B (when its origin is coincident with the origin of F_T), we have

$$\mathbf{P}_i^F = LF_{B/E}LT_{E/B}\mathbf{P}_i^T, \quad i = l, r, n, t. \quad (8.1)$$

Then, the apparent maximum size of the target is given by

$$b = \max \left\{ \|\mathbf{P}_i^F - \mathbf{P}_j^F\|, \quad i, j = l, r, n, t, \quad i \neq j \right\}. \quad (8.2)$$

As it can be seen from equation (8.2), the target's apparent size depends not only on the target's motion, but also on the target's relative orientation with respect to the follower, which is not measurable, and more research is required to related the relative orientation to the available measurements. Although the resulting function $b(\cdot)$ is bounded, that is, there

exist $b_{min} > 0$ and $b_{max} > 0$ such that $b_{min} \leq b \leq b_{max}$, the rate of the change of b depends on the angular rates of both the target and the follower, and any assumption imposed on $\dot{b}$ will introduce circularity in control formulation. The problem modeling requires additional research in this direction as well.

Field of View. The guidance laws derived in this dissertation implicitly assume that the target is always in the camera's field of view, that is, the visual information is available at all times. However, in realistic scenarios, the target can be lost from the field of view because of several reasons: bad transient processes, when the target makes sharp turnings, when it is occluded by some parts of the aircraft, whitening effects, etc. In such cases, special care must be taken to keep the system functioning. One way to relax the situation is to use a gimbaled camera that will provide an additional control authority to keep the target in the center of the field of view, thus, increasing the system's robustness to the target's "disappearance". In any case, this problem requires more research.

Realistic Control Bounds. In some tracking scenarios, when the target makes aggressive maneuvers, the guidance command may require control signals that exceed the actual control surface deflections limits. Therefore, special care must be taken to avoid instability in such cases. The approach that can be used here to achieve bounded tracking in the presence of saturation has been developed in [35].

Group Tracking. An interesting extension to the presented work can be the autonomous tracking of multiple targets by a platoon of aerial vehicles, or even by a heterogenous team of autonomous vehicles, using only visual information about the targets' positions when only visual communication is allowed between the teammates. The resulting control problem can be cast into the hybrid systems framework involving high level decision making algorithms along with low level guidance design procedures.

Appendix A

Projection Operator

The projection operator introduced in Ref. [49] ensures boundedness of the parameter estimates by definition. We recall the main definition from Ref. [49].

Definition 12 [49] *Consider a convex compact set with a smooth boundary given by:*

$$\Omega_c \triangleq \{\boldsymbol{\theta} \in \mathbb{R}^n \mid f(\boldsymbol{\theta}) \leq c\}, \quad 0 \leq c \leq 1, \quad (\text{A.1})$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a smooth convex function. For any given $\mathbf{y} \in \mathbb{R}^n$ the Projection operator is defined as:

$$\text{Proj}(\boldsymbol{\theta}, \mathbf{y}) = \begin{cases} \mathbf{y}, & \text{if } f(\boldsymbol{\theta}) < 0 \\ \mathbf{y}, & \text{if } f(\boldsymbol{\theta}) \geq 0, \nabla f^\top \mathbf{y} \leq 0 \\ \mathbf{y} - \frac{\nabla f \nabla f^\top \mathbf{y}}{\|\nabla f\|^2} f(\boldsymbol{\theta}), & \text{if } f(\boldsymbol{\theta}) \geq 0, \nabla f^\top \mathbf{y} > 0 \end{cases}, \quad (\text{A.2})$$

□

The properties of the Projection operator are given by the following lemma:

Lemma 10 [49] *Let $\boldsymbol{\theta}^* \in \Omega_0 = \{\boldsymbol{\theta} \in \mathbb{R}^n \mid f(\boldsymbol{\theta}) \leq 0\}$, and let the parameter $\boldsymbol{\theta}(t)$ evolve according to the following dynamics*

$$\dot{\boldsymbol{\theta}}(t) = \text{Proj}(\boldsymbol{\theta}(t), \mathbf{y}), \quad \boldsymbol{\theta}(t_0) \in \Omega_c. \quad (\text{A.3})$$

Then

$$\boldsymbol{\theta}(t) \in \Omega_c \quad (\text{A.4})$$

for all $t \geq t_0$, and

$$h_{\boldsymbol{\theta}, \mathbf{y}} \triangleq (\boldsymbol{\theta}(t) - \boldsymbol{\theta}^*)^\top [\text{Proj}(\boldsymbol{\theta}(t), \mathbf{y}) - \mathbf{y}] \leq 0. \quad (\text{A.5})$$

From Definition 12 it follows that $\mathbf{y}$ is not altered if $\boldsymbol{\theta}$ belongs to the set Ω_0 . In the set $\{0 \leq f(\boldsymbol{\theta}) \leq 1\}$, $\text{Proj}(\boldsymbol{\theta}, \mathbf{y})$ subtracts a vector normal to the boundary of Ω_c so that we get a smooth transformation from the original vector field $\mathbf{y}$ to an inward or tangent vector field for $c = 1$. Therefore, on the boundary of Ω_c , $\dot{\boldsymbol{\theta}}(t)$ always points toward the inside of Ω_c and $\boldsymbol{\theta}(t)$ never leaves the set Ω_c , which is the first property.

From the convexity of function $f(\boldsymbol{\theta})$ it follows that for any $\boldsymbol{\theta}^* \in \Omega_0$ and $\boldsymbol{\theta} \in \Omega_c$, the inequality

$$l_{\boldsymbol{\theta}} \triangleq (\boldsymbol{\theta} - \boldsymbol{\theta}^*)^\top \nabla f(\boldsymbol{\theta}) \geq 0$$

holds. Then, from Definition 12 it follows that

$$h_{\boldsymbol{\theta}, \mathbf{y}} = \begin{cases} 0, & \text{if } f(\boldsymbol{\theta}) < 0 \\ 0, & \text{if } f \geq 0, \nabla f^\top \mathbf{y} \leq 0 \\ \underbrace{l_{\boldsymbol{\theta}}}_{\geq 0} \underbrace{\nabla f^\top \mathbf{y}}_{\geq 0} \underbrace{f(\boldsymbol{\theta})}_{\geq 0}, & \text{if } f \geq 0, \nabla f^\top \mathbf{y} > 0 \end{cases}$$

The second property follows.

The function $f_d(\hat{\mathbf{d}})$ in the adaptive law for the estimate $\hat{\mathbf{d}}(t)$ is defined as usual by

$$f_d(\hat{\mathbf{d}}) = \frac{\hat{\mathbf{d}}^\top \hat{\mathbf{d}} - d_{\max}^2}{\epsilon_d}, \quad (\text{A.6})$$

since the unknown parameter $\mathbf{d}$ can take both positive and negative values in the range $\|\mathbf{d}\| \leq d_{\max}$, where $d_{\max}$ is the available conservative norm bound. According to Lemma 10 it follows that the adaptive law in the second equation in (4.5) guarantees the following inequalities:

$$\begin{aligned} \|\hat{\mathbf{d}}(t)\| &\leq d_{\max} \\ \tilde{\mathbf{d}}^\top(t)[- \mathbf{e}(t) + \text{Proj}(\hat{\mathbf{d}}(t), \mathbf{e}(t))] &\leq 0, \end{aligned} \quad (\text{A.7})$$

provided that $\hat{\mathbf{d}}(0) \in \Omega_{\mathbf{d}}^0$.

We can impose some conservative lower and upper bounds on the parameter b in the following form: $0 < b_{\min} \leq b \leq b_{\max}$. Therefore, the function $f_b(\hat{b})$ in the adaptive law for the estimate $\hat{b}$ can be defined as

$$f_b(\hat{b}) = \frac{\left(b - \frac{b_{\max} + b_{\min}}{2}\right)^2 - \left(\frac{b_{\max} - b_{\min}}{2}\right)^2}{\epsilon_b}, \quad (\text{A.8})$$

where ϵ_b denotes the convergence tolerance of our choice. According to Lemma 10 it follows that the adaptive law in the first equation in (4.5) guarantees the following inequalities:

$$\begin{aligned} b_{\min} &\leq \hat{b}(t) \leq b_{\max} \\ \tilde{b}(t)[- \mathbf{e}^\top(t) \mathbf{g}(t) + \text{Proj}(\hat{b}(t), \mathbf{e}^\top(t) \mathbf{g}(t))] &\leq 0 \end{aligned} \quad (\text{A.9})$$

provided that $\hat{b}(0) \in \Omega_b^0$.

From the above definitions and property 1) it follows that ϵ_θ specifies the maximum tolerance that projection operator would allow $\boldsymbol{\theta}(t)$ to exceed as compared to $\theta_{\max}$. This can be easily seen for example in the case of the parameter estimate $\hat{\mathbf{d}}$ by solving $f_d(\hat{\mathbf{d}}) \leq 1$ for $\hat{\mathbf{d}}$, which results in $\hat{\mathbf{d}}^\top \hat{\mathbf{d}} \leq \epsilon_d + d_{\max}^2$.

Appendix B

Robust Adaptive Observer Design for Uncertain Systems with Bounded Disturbances

This Appendix presents a robust adaptive observer design methodology for a class of uncertain nonlinear systems in the presence of time-varying unknown parameters with absolutely integrable derivatives, and non-vanishing disturbances. Using the universal approximation property of radial basis function neural networks and the adaptive bounding technique, the developed observer achieves asymptotic convergence of state estimation error to zero, while ensuring boundedness of parameter errors. A comparative simulation study is presented by the end of this appendix.

Design of adaptive observers for nonlinear systems is one of the active areas of research, the significance of which cannot be underestimated in problems of system identification, failure detection or output feedback control. Extended Kalman filter (EKF) and its modifications

have been the main tools for a long time for addressing the state estimation problem in the presence of nonlinearities [1, 3, 4, 11, 23, 25, 44, 64, 77]. However, convergence proofs for EKF have been derived only for limited class of systems or under assumptions, like in [36] for linear deterministic models with unknown coefficients in discrete-time setting.

For uncertain nonlinear systems, adaptive observers, without involving EKFs, have been introduced in [6, 28, 34, 39, 70, 79, 80] to estimate the unknown parameters along with the state variables where no a priori information about the unknown parameters is available. While establishing global results, these approaches are applicable only to systems transformable to output feedback form, i.e. to a form in which the nonlinearities depend only on the measured outputs and inputs of the system.

In [14, 15, 31, 43, 69, 71, 81], neural network (NN) based identification and estimation schemes were proposed that relaxed the restrictive assumptions on the system dynamics, but obtained local results. In these estimation schemes, the main challenge was in defining a feedback signal, which could be employed in the adaptive laws for the NN weights. This problem was solved by imposing conditions on the system's linear part to behave like a strictly positive real (SPR) filter [2, 14], or by prefiltering the system's input to make the system SPR like [31, 81]. Both ways enabled writing NN adaptive laws in terms of the available measurement error signal. In [71], an approach is laid out for general nonlinear processes without relying on the SPR condition and ensuring ultimately bounded estimation errors.

The observer design problems are more challenging in the presence of unknown time-varying disturbances. In the presence of bounded non-differentiable disturbances, adaptive observers have been developed in [2, 31, 38] guaranteeing bounded state and parameter estimation errors. It has been shown that the state estimation error converges to zero only if the disturbances vanish. In [26], the disturbances are assumed to be generated by an exponentially stable system, such that observability of the combined system is not violated.

The proposed observer is a generalization of the existing ones in the sense of the class of systems under consideration and in the sense of the observer performance. It utilizes the NN approximation properties of continuous functions and the adaptive bounding technique for rejecting the disturbances similar to [53]. The linear part of the system is assumed to satisfy a slightly milder condition as in [14] than SPR. The resulting state observation error is shown to converge to zero asymptotically, while the parameter estimation error remains bounded. The results are most closely related to the results in [2,14,31,38] and extend those by proving asymptotic convergence of the state estimation error to zero. In case of constant unknown parameters and known nonlinearities, in the absence of disturbances, the results of [14] are recovered. Using adaptive bounding, the presented observer achieves asymptotic estimation for any bounded disturbance, whereas the one designed in [38] achieves the asymptotic estimation only for vanishing disturbances.

B.1 Problem Statement

Consider the system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B[\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) + \sum_{k=1}^p \mathbf{g}_k(\mathbf{x}(t), \mathbf{u}(t))\theta_k(t) + \mathbf{d}(t)] \\ \mathbf{y}(t) &= C\mathbf{x}(t),\end{aligned}\tag{B.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state of the system, $\mathbf{u} \in \mathbb{R}^p$ is the control input, $\mathbf{y} \in \mathbb{R}^r$, $q < n$ is the measured output, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $C \in \mathbb{R}^{q \times n}$ are known constant matrices, $\mathbf{f} : \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}^m$ and $\mathbf{g}_k : \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}^m$, $k = 1, \dots, p$ are vectors of unknown functions, $\theta_k : \mathbb{R} \rightarrow \mathbb{R}$, $k = 1, \dots, p$ are time-varying unknown parameters, and $\mathbf{d} : \mathbb{R} \rightarrow \mathbb{R}^m$ is an unknown disturbance.

Assumption 2 On any compact subset of $\mathbb{R}^n \times \mathbb{R}^r$, the functions $\mathbf{f}(\mathbf{x}, \mathbf{u})$, $\mathbf{g}_k(\mathbf{x}, \mathbf{u})$, $k = 1, \dots, p$, and their first partial derivatives with respect to $\mathbf{x}$, $\mathbf{u}$ are continuous in $\mathbf{x}$, $\mathbf{u}$. $\square$

Assumption 3 The process in (B.1) evolves on a compact set, i.e. $(\mathbf{x}(t), \mathbf{u}(t)) \in \Omega_{\mathbf{x}} \times \Omega_{\mathbf{u}}$ for all $t \geq t_0$, where $\Omega_{\mathbf{x}} \times \Omega_{\mathbf{u}} \subset \mathbb{R}^n \times \mathbb{R}^r$ is a compact set. $\square$

Assumption 4 For each $k = 1, \dots, p$, $\theta_k(t)$ is piecewise continuous and has known constant sign. Moreover, $\theta_k(t) \in \mathcal{L}_{\infty}$, while $\dot{\theta}_k(t) \in \mathcal{L}_1 \cap \mathcal{L}_{\infty}$; that is there exist positive constants β_k , δ_k , σ_k , $k = 1, \dots, p$ and $\gamma > 0$ such that

$$\delta_k \leq |\theta_k(t)| \leq \beta_k, \quad |\dot{\theta}_k(t)| \leq \sigma_k, \quad k = 1, \dots, p \quad (\text{B.2})$$

for all $t \geq t_0$. $\square$

Assumption 5 The disturbance is a piecewise continuous bounded function, $\mathbf{d}(t) \in \mathcal{L}_{\infty}$; that is there exists positive constant $\gamma > 0$ such that

$$\|\mathbf{d}(t)\| \leq \gamma \quad (\text{B.3})$$

for all $t \geq t_0$. $\square$

Assumption 6 The pair A , C is detectable, and in addition, there exists a symmetric positive definite matrix P , a positive constant $\rho > 0$ and a gain matrix L such that

$$(A - LC)^{\top} P + P(A - LC) + \rho PP + \rho I < 0$$

$$B^{\top} P = C^*, \quad (\text{B.4})$$

where C^* lies in the span of the rows of C . $\square$

Remark 12 *Assumptions 1 and 2 are well-known regularity assumptions. Assumption 3 implies that the parameters $\theta_k(t)$ converge to constant values. Assumption 4 is quite natural and is common in the robust control or observer design literature. Assumption 5 is the most restrictive one, that essentially forces the system be SPR like. In [50], necessary and sufficient conditions are given for matrix P to satisfy the matrix inequality (B.4). These conditions require existence of a gain matrix L such that $A - LC$ is Hurwitz and*

$$\min_{\omega \in \mathbb{R}^+} \sigma_{\min}(A - LC - i\omega I) > \rho \quad (\text{B.5})$$

where $\sigma_{\min}(\cdot)$ denotes the minimum singular value of the matrix, and $i = \sqrt{-1}$. $\square$

The objective is to design an observer for the system in (B.1) to ensure asymptotic estimation of the states of the system in the presence of the unknown nonlinearities, time-varying parameters and disturbances.

B.2 Neural Network Approximation

Following Theorem 8, consider the following approximations for the unknown vector functions $\mathbf{f}(\mathbf{x}, \mathbf{u})$ and $\mathbf{g}_k(\mathbf{x}, \mathbf{u})$, $k = 1, \dots, p$ on the compact set $\Omega_x \times \Omega_u$:

$$\begin{aligned} \mathbf{f}(\mathbf{x}, \mathbf{u}) &= W_0^\top \Phi_0(\mathbf{x}, \mathbf{u}) + \boldsymbol{\epsilon}_0(\mathbf{x}, \mathbf{u}), \quad \|\boldsymbol{\epsilon}_0(\mathbf{x}, \mathbf{u})\| \leq \varepsilon_0 \\ \mathbf{g}_k(\mathbf{x}, \mathbf{u}) &= W_k^\top \Phi_k(\mathbf{x}, \mathbf{u}) + \boldsymbol{\epsilon}_k(\mathbf{x}, \mathbf{u}), \quad \|\boldsymbol{\epsilon}_k(\mathbf{x}, \mathbf{u})\| \leq \varepsilon_k, \quad k = 1, \dots, p, \end{aligned} \quad (\text{B.6})$$

where $W_k^\top \in \mathbb{R}^{m \times r_k}$, $k = 0, \dots, p$, are unknown constant weight matrices with their Frobenius norms bounded $\|W_k\|_F \leq W_k^*$, W_k^* represent our conservative knowledge of the ideal parameters, $\Phi_k(\mathbf{x}, \mathbf{u}) \in \mathbb{R}^{r_k}$ are properly chosen vectors of Gaussian Radial Basis Functions (RBF), $\boldsymbol{\epsilon}_k(\mathbf{x}, \mathbf{u})$ are the vectors of approximation errors, ε_k are known conservative bounds

for these errors, and r_k is the number of RBFs in approximation of the corresponding function. Using relationships from (B.6), the system in (B.1) can be written as follows

$$\begin{aligned}\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B \left[W_0^\top \Phi_0(\mathbf{x}(t), \mathbf{u}(t)) + \sum_{k=1}^p W_k^\top \Phi_k(\mathbf{x}(t), \mathbf{u}(t)) \theta_k(t) + \mathbf{h}(t, \mathbf{x}(t), \mathbf{u}(t)) \right] \\ \mathbf{y}(t) &= C\mathbf{x}(t),\end{aligned}\tag{B.7}$$

where

$$\mathbf{h}(t, \mathbf{x}(t), \mathbf{u}(t)) = \boldsymbol{\epsilon}_0(\mathbf{x}(t), \mathbf{u}(t)) + \mathbf{d}(t) + \sum_{k=1}^p \boldsymbol{\epsilon}_k(\mathbf{x}(t), \mathbf{u}(t)) \theta_k(t). \tag{B.8}$$

Since all the terms on the right hand side in (B.8) are bounded, there exists a positive constant α such that

$$\|\mathbf{h}(t, \mathbf{x}(t), \mathbf{u}(t))\| \leq \alpha, \quad \forall t > 0, \tag{B.9}$$

From Assumptions 4 and 5 and the conservative bounds on NN reconstruction error it follows that $\alpha \leq \alpha^* = \varepsilon_0 + \sum_{k=1}^p \varepsilon_k \beta_k + \gamma$.

B.3 Observer Design

Consider the following state observer

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= A\hat{\mathbf{x}}(t) + B \left[\hat{W}_0^\top(t) \Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t)) \right. \\ &\quad \left. + \sum_{k=1}^p \hat{W}_k^\top(t) \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t)) \hat{\theta}_k(t) + \mathbf{s}(\tilde{\mathbf{x}}(t)) \hat{\alpha}(t) \right] + L\tilde{\mathbf{y}}(t) \\ \tilde{\mathbf{y}}(t) &= \mathbf{y}(t) - C\hat{\mathbf{x}}(t),\end{aligned}\tag{B.10}$$

where $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ is the state observation error, $\hat{W}_k(t)$, $k = 0, \dots, p$, $\hat{\theta}_k(t)$, $k = 1, \dots, p$ and $\hat{\alpha}(t)$ are the estimates of the unknown parameters W_k , $k = 0, \dots, p$, $\theta_k(t)$, $k = 1, \dots, p$

and α respectively, and

$$\mathbf{s}(\tilde{\mathbf{x}}(t)) = \begin{cases} \frac{C^* \tilde{\mathbf{x}}(t)}{\|C^* \tilde{\mathbf{x}}(t)\|}, & \text{if } C^* \tilde{\mathbf{x}}(t) \neq \mathbf{0} \\ \mathbf{0}, & \text{if } C^* \tilde{\mathbf{x}}(t) = \mathbf{0}. \end{cases} \quad (\text{B.11})$$

The observer gain L is chosen to ensure that the matrix $\bar{A} = A - LC$ is Hurwitz. It follows then from Assumption 6 that there exists a matrix $Q = Q^\top > 0$ satisfying the equation

$$\bar{A}^\top P + P\bar{A} + \rho PP + \rho I = -Q, \quad (\text{B.12})$$

and a matrix $\bar{Q} = \bar{Q}^\top > 0$ satisfying the Lyapunov equation

$$\bar{A}^\top P + P\bar{A} = -\bar{Q}. \quad (\text{B.13})$$

Consider the following adaptive laws:

$$\begin{aligned} \dot{\hat{W}}_0(t) &= G_0 \text{Proj} \left(\hat{W}_0(t), \Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t))(C^* \tilde{\mathbf{x}}(t))^\top \right) \\ \dot{\hat{W}}_k(t) &= G_k \text{Proj} \left(\hat{W}_k(t), \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t))(C^* \tilde{\mathbf{x}}(t))^\top \text{sgn}(\theta_k(t)) \right), \quad k = 1, \dots, p \\ \dot{\hat{\theta}}_k(t) &= \nu_k \text{Proj} \left(\hat{\theta}_k(t), (C^* \tilde{\mathbf{x}}(t))^\top \hat{W}_k^\top(t) \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t)) \right), \quad k = 1, \dots, p \\ \dot{\hat{\alpha}}(t) &= \sigma \text{Proj}(\hat{\alpha}(t), \|C^* \tilde{\mathbf{x}}(t)\|), \end{aligned} \quad (\text{B.14})$$

where the matrices $G_k > 0$, $k = 0, \dots, p$ and constants $\sigma > 0$, $\nu_k > 0$, $k = 1, \dots, p$ are the adaptation gains. In application of the Projection operator, we use the known conservative bounds $\|W\|_F \leq W_k^*$, $k = 0, \dots, p$, $\beta_k \leq \beta_k^*$, $k = 1, \dots, p$ and $\alpha \leq \alpha^*$. Therefore, the adaptive laws in (B.14) guarantee the following inequalities according to Lemma 10:

$$\begin{aligned} \|\hat{W}_k(t)\|_F &\leq W_k^*, \quad k = 0, \dots, p \\ |\hat{\theta}_k(t)| &\leq \beta_k, \quad k = 1, \dots, p \\ |\hat{\alpha}(t)| &\leq \alpha^*. \end{aligned} \quad (\text{B.15})$$

Remark 13 We note that though $\tilde{\mathbf{x}}(t)$ is not available, the feedback signal $C^*\tilde{\mathbf{x}}(t)$ in (B.11), (B.14) can be computed via $\tilde{\mathbf{y}}(t)$. Indeed, since matrix C^* lies in the span of the rows of C , there exists a matrix T such that $C^* = TC$. To determine T , consider the singular value decomposition of C :

$$C = U \begin{bmatrix} \Sigma & 0_{r_c \times (n-r_c)} \\ 0_{(m-r_c) \times r_c} & 0_{(m-r_c) \times (n-r_c)} \end{bmatrix} H^\top, \quad (\text{B.16})$$

where U is a $q \times q$ orthogonal matrix, $\Sigma = \text{diag}[\sigma_1(C), \dots, \sigma_{r_c}(C)]$ with $\sigma_k(C)$, $k = 1, \dots, r_c$ being the singular values of C , and H is a $n \times n$ orthogonal matrix, where $r_c \triangleq \text{rank}(C) \leq q$, [19]. Then the pseudo inverse $C^\dagger$ is given by

$$C^\dagger = H \begin{bmatrix} \Sigma^{-1} & 0_{r_c \times (m-r_c)} \\ 0_{(n-r_c) \times r_c} & 0_{(n-r_c) \times (m-r_c)} \end{bmatrix} U^\top. \quad (\text{B.17})$$

It is straightforward to verify that $T = C^*C^\dagger$ satisfies the equation $C^* = TC$. Therefore

$$C^*\tilde{\mathbf{x}}(t) = TC\tilde{\mathbf{x}}(t) = T\tilde{\mathbf{y}}(t), \quad (\text{B.18})$$

where $\tilde{\mathbf{y}}(t)$ is available as a measurement. □

The error dynamics are written as follows

$$\begin{aligned} \dot{\tilde{\mathbf{x}}}(t) &= (A - LC)\tilde{\mathbf{x}}(t) + B \left[W_0^\top \Phi_0(\mathbf{x}(t), \mathbf{u}(t)) + \sum_{k=1}^p W_k^\top \Phi_k(\mathbf{x}(t), \mathbf{u}(t))\theta_k(t) + \mathbf{h}(t, \mathbf{x}(t), \mathbf{u}(t)) \right] \\ &\quad - B \left[\hat{W}_0^\top(t) \Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t)) + \sum_{k=1}^p \hat{W}_k^\top(t) \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t))\hat{\theta}_k(t) + \mathbf{s}(\tilde{\mathbf{x}}(t))\hat{\alpha}(t) \right]. \end{aligned} \quad (\text{B.19})$$

We note that

$$\begin{aligned}
W_0^\top \Phi_0(\mathbf{x}(t), \mathbf{u}(t)) - \hat{W}_0^\top(t) \Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t)) &= \left[W_0 - \hat{W}_0(t) \right]^\top \Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t)) \\
&\quad + W_0^\top [\Phi_0(\mathbf{x}(t), \mathbf{u}(t)) - \Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t))], \\
W_k^\top \Phi_k(\mathbf{x}(t), \mathbf{u}(t)) \theta_k(t) - \hat{W}_k^\top(t) \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t)) \hat{\theta}_k(t) &= \left[W_k - \hat{W}_k(t) \right]^\top \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t)) \theta_k(t) \\
&\quad + W_k^\top [\Phi_k(\mathbf{x}(t), \mathbf{u}(t)) - \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t))] \theta_k(t) + \hat{W}_k^\top(t) \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t)) [\theta_k(t) - \hat{\theta}_k(t)], \\
k &= 1, \dots, p.
\end{aligned} \tag{B.20}$$

Denoting the parameter errors as $\tilde{W}_k(t) = \hat{W}_k(t) - W_k$, $k = 0, \dots, p$, $\tilde{\theta}_k(t) = \hat{\theta}_k(t) - \theta_k(t)$, $k = 1, \dots, p$, $\tilde{\alpha}(t) = \hat{\alpha}(t) - \alpha$, and taking into account the equalities in (B.20) the error dynamics take the form

$$\begin{aligned}
\dot{\tilde{\mathbf{x}}}(t) &= \bar{A} \tilde{\mathbf{x}}(t) + B \{ W_0^\top [\Phi_0(\mathbf{x}(t), \mathbf{u}(t)) - \Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t))] \\
&\quad - \tilde{W}_0^\top(t) \Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t)) + \sum_{k=1}^p W_k^\top [\Phi_k(\mathbf{x}(t), \mathbf{u}(t)) - \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t))] \theta_k(t) \\
&\quad - \sum_{k=1}^p \tilde{W}_k^\top(t) \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t)) \theta_k(t) - \sum_{k=1}^p \hat{W}_k^\top(t) \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t)) \tilde{\theta}_k(t) \\
&\quad + \mathbf{h}(t, \mathbf{x}(t), \mathbf{u}(t)) - \mathbf{s}(\tilde{\mathbf{x}}(t)) \alpha - \mathbf{s}(\tilde{\mathbf{x}}(t)) \tilde{\alpha}(t) \}.
\end{aligned} \tag{B.21}$$

From (B.15) and the definitions of parameter errors it follows, that there exist positive constants $\tilde{W}_0^*, \dots, \tilde{W}_p^*, \tilde{\beta}_1^*, \dots, \tilde{\beta}_p^*, \tilde{\alpha}^*$ such that the following inequalities hold

$$\begin{aligned}
\|\tilde{W}_k(t)\| &\leq \tilde{W}_k^*, \quad k = 0, \dots, p \\
|\tilde{\theta}_k(t)| &\leq \tilde{\beta}_k^*, \quad k = 1, \dots, p \\
|\tilde{\alpha}_k(t)| &\leq \tilde{\alpha}^*.
\end{aligned} \tag{B.22}$$

Remark 14 We notice that the observer dynamics in (B.10) and the error dynamics in (B.21) have discontinuous right-hand sides due to the presence of function $\mathbf{s}(\tilde{\mathbf{x}})$, defined in

(B.11). However, they satisfy the conditions of existence and uniqueness of the Filippov solutions as stated in Definition 8. Moreover, following [48], we notice that the set

$$S = \{\tilde{\mathbf{x}} \mid C^* \tilde{\mathbf{x}} = 0\} \quad (\text{B.23})$$

is the switching manifold, which is smooth. Away from this manifold both dynamics have smooth right-hand sides and the solutions exist in regular sense. $\square$

B.4 Stability Analysis

In this section, we show through Lyapunov's direct method that the parameter estimation errors $\tilde{W}_k(t)$, $k = 0, \dots, p$, $\tilde{\theta}_k(t)$, $k = 1, \dots, p$, $\tilde{\alpha}(t)$ are ultimately bounded, and the observation error $\tilde{\mathbf{x}}(t)$ converges to zero as $t \rightarrow \infty$. Consider the following composite error vector

$$\boldsymbol{\zeta} = \begin{bmatrix} \tilde{\mathbf{x}}^\top & \text{vec}(\tilde{W}_0)^\top & \dots & \text{vec}(\tilde{W}_p)^\top & \tilde{\theta}_1 \dots \tilde{\theta}_p & \tilde{\alpha} \end{bmatrix}^\top. \quad (\text{B.24})$$

In the extended error space consider the largest ball

$$B_R \triangleq \{\boldsymbol{\zeta} \mid \|\boldsymbol{\zeta}\| \leq R\}, \quad R > 0, \quad (\text{B.25})$$

such that for every $\boldsymbol{\zeta} \in B_R$ one has $(\mathbf{x}, \mathbf{u}) \in \Omega_x \times \Omega_u$. Introduce the following gain matrices:

$$\begin{aligned} T_1 &\triangleq \text{diag}\{P, F_0^{-1}, \delta_1 F_1^{-1}, \dots, \delta_p F_p^{-1}, \nu_1^{-1}, \dots, \nu_1^{-1}, \sigma_1^{-1}\}, \\ T_2 &\triangleq \text{diag}\{P, F_0^{-1}, \beta_1 F_1^{-1}, \dots, \beta_p F_p^{-1}, \nu_1^{-1}, \dots, \nu_1^{-1}, \sigma_1^{-1}\}, \end{aligned} \quad (\text{B.26})$$

where $F_k = \text{diag}\{G_k, \dots, G_k\} \in \mathbb{R}^{mr_k \times mr_k}$, $k = 0, \dots, p$, and the matrix $P = P^\top > 0$ satisfies the relationships in (B.4). For the stability analysis, we consider the following positive definite function as a Lyapunov function candidate:

$$V(t, \boldsymbol{\zeta}) = \tilde{\mathbf{x}}^\top P \tilde{\mathbf{x}} + \frac{1}{\sigma} \tilde{\alpha}^2 + \text{tr} \left(\tilde{W}_0^\top G_0^{-1} \tilde{W}_0 \right) + \sum_{k=1}^p \text{tr} \left(\tilde{W}_k^\top G_k^{-1} \tilde{W}_k \right) |\theta_k(t)| + \sum_{k=1}^p \frac{1}{\nu_k} \tilde{\theta}_k^2. \quad (\text{B.27})$$

Notice that the bounds in (B.2) immediately imply:

$$\zeta^\top T_1 \zeta \leq V(t, \zeta) \leq \zeta^\top T_2 \zeta. \quad (\text{B.28})$$

Assumption 7 *Let*

$$R > \gamma_0 \sqrt{\frac{\lambda_{\max}(T_2)}{\lambda_{\min}(T_1)}} \geq \gamma_0, \quad (\text{B.29})$$

where $\lambda_{\max}(T_2)$ is the maximum eigenvalue of T_2 and $\lambda_{\min}(T_1)$ is the minimum eigenvalue of T_1 defined in (B.26), while

$$\gamma_0 = \max \left\{ \sqrt{\frac{\gamma_1}{\lambda_{\min}}}, \tilde{W}_k^*, k = 0, \dots, p, \tilde{\theta}_k^*, k = 1, \dots, p, \tilde{\alpha}^* \right\} \quad (\text{B.30})$$

and

$$\gamma_1 = \sum_{k=1}^p \eta_k \sigma_k, \quad (\text{B.31})$$

$$\eta_k = \frac{(\tilde{W}_k^*)^2}{\lambda_{\min}(G_k)} + \frac{2\beta_k}{\nu_k}, \quad (\text{B.32})$$

$$\lambda_{\min} = \min(\lambda_{\min}(Q), \lambda_{\min}(\bar{Q})). \quad (\text{B.33})$$

□

Define a compact set

$$\Omega_{\gamma_2} = \{\zeta \in B_R \mid \zeta^\top T_1 \zeta \leq \gamma_2\}, \quad (\text{B.34})$$

where

$$\gamma_2 \triangleq \min_{\|\zeta\|=R} \zeta^\top T_1 \zeta = R^2 \lambda_{\min}(T_1). \quad (\text{B.35})$$

Theorem 17 *Let Assumptions 2-7 hold. Then, if the initial error $\zeta(t_0)$ belongs to the compact set Ω_{γ_2} , the observer in (B.10) along with the adaptive laws in (B.14), ensures that the parameter estimation errors are uniformly ultimately bounded and $\tilde{\mathbf{x}}(t) \rightarrow 0$ as $t \rightarrow \infty$. □*

Proof. Consider the Lyapunov function candidate $V(t, \zeta(t))$ defined in (B.27). It is absolutely continuous and is differentiable with respect to time everywhere, but not continuously. Its derivative has discontinuity on the switching manifold S . According to Definition 11 it is a regular function, and, hence, the chain rule from Theorem 9 can be applied. Therefore, the function $V(t, \zeta(t))$ can be used for the stability analysis of the error dynamics as stated in Theorem 10. The generalized gradient of $V(t, \zeta(t))$ contains only one element everywhere, which is shown in [61] to be the regular gradient. Thus, the derivative of $V(t, \zeta(t))$ can be computed in the regular sense everywhere, except for the points of the switching manifold. On the switching manifold the derivative has a jump and needs to be computed differently. Away from S , $\dot{V}(t, \zeta(t))$ is computed as follows:

$$\begin{aligned}
\dot{V}(t, \zeta(t)) &= \tilde{\mathbf{x}}^\top(t)(\bar{A}^\top P + P\bar{A})\tilde{\mathbf{x}}(t) + 2\tilde{\mathbf{x}}^\top(t)PB\{W_0^\top[\Phi_0(\mathbf{x}(t), \mathbf{u}(t)) - \Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t))] \\
&\quad - \tilde{W}_0^\top(t)\Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t)) + \sum_{k=1}^p W_k^\top[\Phi_k(\mathbf{x}(t), \mathbf{u}(t)) - \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t))]\theta_k(t) \\
&\quad - \sum_{k=1}^p \tilde{W}^\top(t)\Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t))\theta_k(t) - \sum_{k=1}^p \hat{W}^\top(t)\Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t))\tilde{\theta}_k(t) \\
&\quad + \mathbf{h}(t, \mathbf{x}(t), \mathbf{u}(t)) - \mathbf{s}(\tilde{\mathbf{x}}(t))\alpha - \mathbf{s}(\tilde{\mathbf{x}}(t))\tilde{\alpha}(t)\} + \frac{2}{\sigma}\tilde{\alpha}(t)\dot{\tilde{\alpha}}(t) + \sum_{k=1}^p \frac{2}{\nu_k}\tilde{\theta}_k(t)\dot{\tilde{\theta}}_k(t) \\
&\quad + 2\text{tr}\left(\tilde{W}_0^\top(t)G_0^{-1}\dot{\tilde{W}}_0(t)\right) + 2\sum_{k=1}^p \text{tr}\left(\tilde{W}_k^\top(t)G_k^{-1}\dot{\tilde{W}}_k(t)\right)|\theta_k(t)| \\
&\quad + \sum_{k=1}^p \text{tr}\left(\tilde{W}_k^\top(t)G_k^{-1}\tilde{W}_k(t)\right)\dot{\theta}_k(t)\text{sgn}(\theta_k(t)). \tag{B.36}
\end{aligned}$$

We recall that according to Assumption 6, $PB = (C^*)^\top$. Also, by the parameter error definitions $\dot{\tilde{\theta}}_k(t) = \dot{\hat{\theta}}_k(t) - \dot{\theta}_k(t)$. Taking into account the above equations and the trace property $\text{tr}(X^\top Y) = \text{tr}(XY^\top)$, where X and Y are matrices of compatible dimensions, and

substituting the adaptive laws from (B.14) we obtain

$$\begin{aligned}
\dot{V}(t, \zeta(t)) &= \tilde{\mathbf{x}}^\top(t) (\bar{A}^\top P + P \bar{A}) \tilde{\mathbf{x}}(t) + 2\tilde{\mathbf{x}}^\top(t) P B [W_0^\top (\Phi_0(\mathbf{x}(t), \mathbf{u}(t)) - \Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t)))] \\
&- 2\text{tr} \left\{ \tilde{W}_0^\top(t) \left[\Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t)) (C^* \tilde{\mathbf{x}}(t))^\top - \text{Proj} \left(\hat{W}_0(t), \Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t)) (C^* \tilde{\mathbf{x}}(t))^\top \right) \right] \right\} \\
&+ 2\tilde{\mathbf{x}}^\top(t) P B \sum_{k=1}^p W_k^\top (\Phi_k(\mathbf{x}(t), \mathbf{u}(t)) - \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t))) \theta_k(t) \\
&- 2 \sum_{k=1}^p \text{tr} \{ \tilde{W}_k^\top(t) [\Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t)) (C^* \tilde{\mathbf{x}}(t))^\top \theta_k(t) \\
&- \text{Proj} \left(\hat{W}_k(t), \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t)) (C^* \tilde{\mathbf{x}}(t))^\top \text{sgn}(\theta_k(t)) \right) |\theta_k(t)|] \} \\
&- \sum_{k=1}^p \frac{2}{\nu_k} \tilde{\theta}_k(t) \dot{\theta}_k(t) - 2 \sum_{k=1}^p \tilde{\theta}_k(t) [(C^* \tilde{\mathbf{x}}(t))^\top \hat{W}_k^\top(t) \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t)) \\
&- \text{Proj} \left(\hat{\theta}_k(t), (C^* \tilde{\mathbf{x}}(t))^\top \hat{W}_k^\top(t) \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t)) \right)] \\
&+ 2(C^* \tilde{\mathbf{x}}(t))^\top [\mathbf{h}(t, \mathbf{x}(t), \mathbf{u}(t)) - \mathbf{s}(\tilde{\mathbf{x}}(t))\alpha] \\
&- 2\tilde{\alpha}(t) [(C^* \tilde{\mathbf{x}}(t))^\top \mathbf{s}(\tilde{\mathbf{x}}(t)) - \text{Proj}(\hat{\alpha}(t), \|C^* \tilde{\mathbf{x}}(t)\|)] \\
&+ \sum_{k=1}^p \text{tr} \left(\tilde{W}_k^\top(t) G_k^{-1} \tilde{W}_k(t) \right) \dot{\theta}_k(t) \text{sgn}(\theta_k(t)). \tag{B.37}
\end{aligned}$$

Taking into account the second property of the projection operator from Lemma 10, the following inequalities can be written

$$\begin{aligned}
&\tilde{W}_0^\top(t) \left[\Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t)) (C^* \tilde{\mathbf{x}}(t))^\top - \text{Proj} \left(\hat{W}_0(t), \Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t)) (C^* \tilde{\mathbf{x}}(t))^\top \right) \right] \geq 0 \\
&\sum_{k=1}^p \tilde{W}_k^\top(t) [\Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t)) (C^* \tilde{\mathbf{x}}(t))^\top \theta_k(t) \\
&- \text{Proj} \left(\hat{W}_k(t), \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t)) (C^* \tilde{\mathbf{x}}(t))^\top \text{sgn}(\theta_k(t)) \right) |\theta_k(t)|] = \\
&\sum_{k=1}^p |\theta_k(t)| \tilde{W}_k^\top(t) [\Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t)) (C^* \tilde{\mathbf{x}}(t))^\top \text{sgn}(\theta_k(t)) \\
&- \text{Proj} \left(\hat{W}_k(t), \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t)) (C^* \tilde{\mathbf{x}}(t))^\top \text{sgn}(\theta_k(t)) \right)] \geq 0 \\
&\sum_{k=1}^p \tilde{\theta}_k(t) [(C^* \tilde{\mathbf{x}}(t))^\top \hat{W}_k^\top(t) \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t)) \\
&- \text{Proj} \left(\hat{\theta}_k(t), (C^* \tilde{\mathbf{x}}(t))^\top \hat{W}_k^\top(t) \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t)) \right)] \geq 0 \\
&\tilde{\alpha}(t) [(C^* \tilde{\mathbf{x}}(t))^\top \mathbf{s}(\tilde{\mathbf{x}}(t)) - \text{Proj}(\hat{\alpha}(t), \|C^* \tilde{\mathbf{x}}(t)\|)] \geq 0. \tag{B.38}
\end{aligned}$$

Therefore the following upper bound for the derivative of the Lyapunov function candidate can be derived:

$$\begin{aligned}
\dot{V}(t, \zeta(t)) &\leq \tilde{\mathbf{x}}^\top(t)(\bar{A}^\top P + P\bar{A})\tilde{\mathbf{x}}(t) - \sum_{k=1}^p \frac{2}{\nu_k} \tilde{\theta}_k(t) \dot{\theta}_k(t) \\
&+ 2\tilde{\mathbf{x}}^\top(t)PBW_0^\top [\Phi_0(\mathbf{x}(t), \mathbf{u}(t)) - \Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t))] \\
&+ 2\tilde{\mathbf{x}}^\top(t)PB \sum_{k=1}^p W_k^\top [\Phi_k(\mathbf{x}(t), \mathbf{u}(t)) - \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t))] \theta_k(t) \quad (\text{B.39}) \\
&+ \sum_{k=1}^p \text{tr} \left(\tilde{W}_k^\top(t) G_k^{-1} \tilde{W}_k(t) \right) \text{sgn}(\theta_k(t)) \dot{\theta}_k(t).
\end{aligned}$$

In the derivation of the above upper bound the following inequality is used

$$(C^* \tilde{\mathbf{x}}(t))^\top \mathbf{h}(t, \mathbf{x}(t), \mathbf{u}(t)) \leq \|C^* \tilde{\mathbf{x}}(t)\| \|\mathbf{h}(t, \mathbf{x}(t), \mathbf{u}(t))\| \leq \|C^* \tilde{\mathbf{x}}(t)\|_\alpha = (C^* \tilde{\mathbf{x}}(t))^\top \mathbf{s}(\tilde{\mathbf{x}}(t)) \quad (\text{B.40})$$

which follows from Schwartz inequality ([55], pp.144), inequality in (B.9) and definition (B.11). Since the Gaussian basis functions are Lipschitz in $\mathbf{x}$, the following bound can be derived:

$$\begin{aligned}
&2\tilde{\mathbf{x}}^\top(t)PBW_0^\top [\Phi_0(\mathbf{x}(t), \mathbf{u}(t)) - \Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t))] \\
&+ 2\tilde{\mathbf{x}}^\top(t)PB \sum_{k=1}^p W_k^\top [\Phi_k(\mathbf{x}(t), \mathbf{u}(t)) - \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t))] \theta_k(t) \leq \\
&2\|B\|_F \left(\lambda_0 \|W_0\|_F + \sum_{k=1}^p \lambda_k \beta_k \|W_k\|_F \right) \|P\tilde{\mathbf{x}}(t)\| \|\tilde{\mathbf{x}}(t)\| \leq \\
&\rho [\tilde{\mathbf{x}}^\top(t)PP\tilde{\mathbf{x}}(t) + \tilde{\mathbf{x}}^\top(t)\tilde{\mathbf{x}}(t)], \quad (\text{B.41})
\end{aligned}$$

where $\lambda_k, k = 0, \dots, p$ are the Lipschitz constants for the corresponding basis functions, and

$$\rho = \|B\|_F \left(\lambda_0 W_0^* + \sum_{k=1}^p \lambda_k \beta_k^* W_k^* \right). \quad (\text{B.42})$$

Therefore, $\dot{V}(t, \zeta(t))$ can be further upper bounded as follows

$$\begin{aligned}
\dot{V}(t, \zeta(t)) &\leq \tilde{\mathbf{x}}^\top(t) (\bar{A}^\top P + P\bar{A} + \rho PP + \rho I) \tilde{\mathbf{x}}(t) \\
&+ \sum_{k=1}^p \left[\text{tr} \left(\tilde{W}_k^\top(t) G_k^{-1} \tilde{W}_k(t) \right) \text{sgn}(\theta_k(t)) - \frac{2}{\nu_k} \tilde{\theta}_k(t) \right] \dot{\theta}_k(t). \quad (\text{B.43})
\end{aligned}$$

Taking into account inequalities in (B.22), the last two terms in the inequality (B.43) can be bounded as follows

$$\sum_{k=1}^p \left[\text{tr} \left(\tilde{W}_k^\top(t) G_k^{-1} \tilde{W}_k(t) \right) \text{sgn}(\theta_k(t)) - \frac{2}{\nu_k} \tilde{\theta}_k(t) \right] \dot{\theta}_k(t) \leq \sum_{k=1}^p \eta_k |\dot{\theta}_k(t)|, \quad (\text{B.44})$$

where positive constants η_k are defined in (B.32). Using the equation in (B.13), the following bound can be derived for the derivative of the Lyapunov function candidate, away from the switching manifold S :

$$\dot{V}(t, \zeta(t)) \leq -\tilde{\mathbf{x}}^\top(t) Q \tilde{\mathbf{x}}(t) + \sum_{k=1}^p \eta_k |\dot{\theta}_k(t)|. \quad (\text{B.45})$$

On the switching manifold S in (B.23), one has

$$\tilde{\mathbf{x}}^\top(t) P B = (C^* \tilde{\mathbf{x}}(t))^\top = 0, \quad (\text{B.46})$$

and the adaptive laws in (B.14) are reduced to

$$\dot{\hat{\alpha}}(t) = 0, \quad \dot{W}_k(t) = 0, \quad k = 0, \dots, p, \quad \dot{\hat{\theta}}_k(t) = 0, \quad k = 1, \dots, p. \quad (\text{B.47})$$

Therefore

$$\dot{\hat{\alpha}}(t) = 0, \quad \dot{W}_k(t) = 0, \quad k = 0, \dots, p, \quad \dot{\hat{\theta}}_k(t) = -\dot{\theta}_k(t) \quad k = 1, \dots, p \quad (\text{B.48})$$

and the derivative of the Lyapunov function candidate can be written in the form

$$\begin{aligned} \dot{V}(t, \zeta(t)) &= \tilde{\mathbf{x}}^\top(t) (\bar{A}^\top P + P \bar{A}) \tilde{\mathbf{x}}(t) - \sum_{k=1}^p \frac{2}{\nu_k} \tilde{\theta}_k(t) \dot{\theta}_k(t) \\ &\quad + \sum_{k=1}^p \text{tr} \left(\tilde{W}_k^\top(t) G_k^{-1} \tilde{W}_k(t) \right) \dot{\theta}_k(t) \text{sgn}(\theta_k(t)) \\ &\leq \tilde{\mathbf{x}}^\top(t) (\bar{A}^\top P + P \bar{A}) \tilde{\mathbf{x}}(t) + \sum_{k=1}^p \eta_k |\dot{\theta}_k(t)|. \end{aligned} \quad (\text{B.49})$$

Thus on the switching manifold S the derivative of the Lyapunov function candidate satisfies the inequality

$$\dot{V}(t, \zeta(t)) \leq -\tilde{\mathbf{x}}^\top(t) \bar{Q} \tilde{\mathbf{x}}(t) + \sum_{k=1}^p \eta_k |\dot{\theta}_k(t)|. \quad (\text{B.50})$$

Therefore, for the derivative of $V(t)$ we have the following bounds

$$\dot{V}(t, \zeta(t)) \leq \begin{cases} -\lambda_{\min}(Q) \|\tilde{\mathbf{x}}(t)\|^2 + \gamma_1, & \tilde{\mathbf{x}} \in S \\ -\lambda_{\min}(\bar{Q}) \|\tilde{\mathbf{x}}(t)\|^2 + \gamma_1, & \tilde{\mathbf{x}} \notin S \end{cases}, \quad (\text{B.51})$$

where γ_1 is defined in (B.31). It is easy to see that $\dot{V}(t, \zeta(t)) < 0$ outside of the compact set

$$\Omega = \{\|\tilde{\mathbf{x}}\| \leq \sqrt{\frac{\gamma_1}{\lambda_{\min}}}, \|\tilde{W}_k\|_F \leq \tilde{W}_k^*, k = 0, \dots, p, |\tilde{\theta}_k| \leq \tilde{\theta}_k^*, k = 1, \dots, p, |\tilde{\alpha}| \leq \tilde{\alpha}^*\}.$$

To complete the proof consider the ball

$$B_{\gamma_0} = \{\zeta \in B_R \mid \|\zeta\| < \gamma_0\} \quad (\text{B.52})$$

in the space of the error vector ζ outside of which $\dot{V}(t, \zeta(t)) < 0$. Notice from (B.29) that $B_{\gamma_0} \subset B_R$. Let $\Gamma \triangleq \max_{\|\zeta\|=\gamma_0} \zeta^\top T_2 \zeta = \gamma_0^2 \lambda_{\max}(T_2)$. Introduce the set:

$$\Omega_{\gamma_0} = \{\zeta \mid \zeta^\top T_2 \zeta \leq \Gamma\}. \quad (\text{B.53})$$

The condition in (B.29) ensures that $\Omega_{\gamma_0} \subset \Omega_{\gamma_2}$. Thus, if the initial error $\zeta_0 = \zeta(t_0)$ belongs to Ω_{γ_2} , then $\zeta(t) \in B_R$ for all $t \geq 0$. Hence all the signals in the system are ultimately bounded. To prove the convergence of the observation error to zero we integrate the inequality in (B.45), resulting in

$$\begin{aligned} \int_{t_0}^t \tilde{\mathbf{x}}^\top(\tau) Q \tilde{\mathbf{x}}(\tau) d\tau &\leq V(t_0) - V(t) + \int_{t_0}^t \sum_{k=1}^p \eta_k |\dot{\theta}_k(\tau)| d\tau \\ &\leq V(t_0) + \int_{t_0}^t \sum_{k=1}^p \eta_k |\dot{\theta}_k(\tau)| d\tau. \end{aligned} \quad (\text{B.54})$$

Since $\dot{\theta}_k(t) \in \mathcal{L}_1$, $k = 1, \dots, p$, the inequality in (B.54) implies that

$$\lim_{t \rightarrow \infty} \int_{t_0}^t \|\tilde{\mathbf{x}}(\tau)\|^2 d\tau < \infty. \quad (\text{B.55})$$

The same limiting relationship we get by integrating the inequality in (B.50). Thus $\tilde{\mathbf{x}}(t) \in \mathcal{L}_\infty \cap \mathcal{L}_2$ everywhere. Also, since all the terms on the right-hand side of equation (B.19) are bounded, it follows that $\dot{\tilde{\mathbf{x}}}(t) \in \mathcal{L}_\infty$. Application of Barbalat's lemma ensures that $\tilde{\mathbf{x}}(t) \rightarrow 0$ as $t \rightarrow \infty$ [54]. The proof is complete. $\square$

Remark 15 *Assumption 7 may be interpreted as implying both an upper and lower bound for the adaptation gains. Define $\bar{\gamma} \triangleq \max\{\lambda_{\max}(G_0), \beta_1 \lambda_{\max}(G_1) \dots, \beta_p \lambda_{\max}(G_p), \nu_1, \dots, \nu_p, \sigma\}$, $\underline{\gamma} \triangleq \min\{\lambda_{\min}(G_0), \delta_1 \lambda_{\min}(G_1) \dots, \delta_p \lambda_{\min}(G_p), \nu_1, \dots, \nu_p, \sigma\}$. Then an upper bound for the adaptation gains results when $\underline{\gamma} \lambda_{\max}(P) > 1$, for which the relation in (B.29) reduces to $\bar{\gamma} < R^2/(\gamma_0^2 \lambda_{\max}(P))$. A lower bound for the adaptation gains results when $\bar{\gamma} \lambda_{\min}(P) < 1$, for which the relation in (B.29) reduces to $\underline{\gamma} > \gamma_0^2/(R^2 \lambda_{\min}(P))$.* $\square$

Remark 16 *The proposed observer requires to find matrices L and P such that the relationships in (B.4) are satisfied, where the value of ρ is calculated according to equation in (B.42) with $\|W_k\|_F$ β_k replaced with their conservative bounds W_k^* and β_k^* respectively. We do not get into details of this, in general, nontrivial task and refer the interested reader to [50], where a detailed guideline can be found and the MATLAB code can be obtained from the author.*

Remark 17 *If the system can be made SPR by output injection, that is L can be chosen such that the triple $(A - LC, B, C)$ is SPR, then instead of relationships in (B.4) one can use Kalman-Yakubovich-Popov [63], which guarantees the existence of positive definite symmetric*

matrices P_0, Q_0 such that

$$\begin{aligned} (A - LC)^\top P_0 + P_0(A - LC) &= -Q_0 \\ B^\top P_0 &= C_0. \end{aligned} \quad (\text{B.56})$$

In this case, the expression $C^* \tilde{\mathbf{x}}(t)$ is replaced with $\tilde{\mathbf{y}}(t)$ in all equations. The stability proof is simplified as follows. The inequality in (B.41) can be written in the form

$$\begin{aligned} & 2\tilde{\mathbf{x}}^\top(t)PBW_0^\top [\Phi_0(\mathbf{x}(t), \mathbf{u}(t)) - \Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t))] \\ + & 2\tilde{\mathbf{x}}^\top(t)PB \sum_{k=1}^p W_k^\top [\Phi_k(\mathbf{x}(t), \mathbf{u}(t)) - \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t))] \theta_k(t) \leq \\ & 2\|\tilde{\mathbf{y}}(t)\| \left(\lambda_0 \|W_0\|_F + \sum_{k=1}^p \lambda_k \beta_k \|W_k\|_F \right) \|\tilde{\mathbf{x}}(t)\| \leq \rho \|\tilde{\mathbf{x}}(t)\|^2, \end{aligned} \quad (\text{B.57})$$

where

$$\rho = \lambda_0 W_0^* + \sum_{k=1}^p \lambda_k \beta_k W_k^*. \quad (\text{B.58})$$

Therefore, away from the switching surface S , instead of inequality in (B.45) we obtain

$$\dot{V}(t, \zeta(t)) \leq -(\lambda_{\min}(Q_0) - \rho) \|\tilde{\mathbf{x}}(t)\|^2 + \sum_{k=1}^p \eta_k |\dot{\theta}_k(t)|, \quad (\text{B.59})$$

and, on the surface S , the inequality in (B.50) holds with $\bar{Q}$ replaced with Q_0 . Thus, the conclusion of Theorem remains valid if Q_0 is chosen such that $\lambda_{\min}(Q_0) > \rho$. In this case, the design task is to choose L to satisfy the SPR condition with the required Q_0 , which is a simpler task than the previous one. This is incorporated in the simulation example.

Remark 18 In observer's practical implementations, a chattering can be generated when the continuous signals are replaced by sampled ones. The frequent sampling results in a better performance, but is time consuming. On the other hand, the larger sampling time results in

a cheaper computation, but may generate chattering because of the signum function in the observer dynamics. To avoid possible chattering, one can replace the discontinuous function $\mathbf{s}(\tilde{\mathbf{x}})$ by its continuous approximation $\mathbf{s}_\chi(\tilde{\mathbf{x}})$:

$$\mathbf{s}_\chi(\tilde{\mathbf{x}}(t)) = \begin{cases} \frac{C^* \tilde{\mathbf{x}}(t)}{\|C^* \tilde{\mathbf{x}}(t)\|}, & \text{if } \|C^* \tilde{\mathbf{x}}(t)\| > \chi \\ \frac{C^* \tilde{\mathbf{x}}(t)}{\chi}, & \text{if } \|C^* \tilde{\mathbf{x}}(t)\| \leq \chi, \end{cases} \quad (\text{B.60})$$

where $\chi > 0$ is a design parameter. Then the error dynamics in (B.21) are continuous and can be written as

$$\begin{aligned} \dot{\tilde{\mathbf{x}}}(t) = & (A - LC)\tilde{\mathbf{x}}(t) + B\{W_0^\top [\Phi_0(\mathbf{x}(t), \mathbf{u}(t)) - \Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t))] \\ & - \tilde{W}_0^\top(t)\Phi_0(\hat{\mathbf{x}}(t), \mathbf{u}(t)) + \sum_{k=1}^p W_k^\top [\Phi_k(\mathbf{x}(t), \mathbf{u}(t)) - \Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t))]\theta_k(t) \\ & - \sum_{k=1}^p \tilde{W}_k^\top(t)\Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t))\theta_k(t) - \sum_{k=1}^p \hat{W}_k^\top(t)\Phi_k(\hat{\mathbf{x}}(t), \mathbf{u}(t))\tilde{\theta}_k(t) \\ & + \mathbf{h}(t, \mathbf{x}(t), \mathbf{u}(t)) - \mathbf{s}(\tilde{\mathbf{x}}(t))\alpha - \mathbf{s}(\tilde{\mathbf{x}}(t))\tilde{\alpha}(t) + [\mathbf{s}(\tilde{\mathbf{x}}(t)) - \mathbf{s}_\chi(\tilde{\mathbf{x}}(t))]\hat{\alpha}(t)\}. \end{aligned} \quad (\text{B.61})$$

The Lyapunov function candidate $V(t, \zeta)$ in (B.27) is smooth and its derivative can be upper bounded using the same steps as above:

$$\dot{V}(t, \zeta(t)) \leq -\tilde{\mathbf{x}}^\top(t)Q\tilde{\mathbf{x}}(t) + \sum_{k=1}^p \eta_k |\dot{\theta}_k(t)| + (C^* \tilde{\mathbf{x}}(t))^\top [\mathbf{s}(\tilde{\mathbf{x}}(t)) - \mathbf{s}_\chi(\tilde{\mathbf{x}}(t))]\hat{\alpha}(t) \quad (\text{B.62})$$

Outside the boundary layer

$$S_\chi = \{\tilde{\mathbf{x}} \mid \|C^* \tilde{\mathbf{x}}\| \leq \chi\} \quad (\text{B.63})$$

the derivative of the Lyapunov function candidate $\dot{V}(t, \zeta(t))$ has the same bounds as in (B.50), and application of Theorem 17 implies that $\tilde{\mathbf{x}}(t)$ tends asymptotically towards zero, while all other error signals remain bounded. Therefore, there exists time instant $\bar{t} > 0$ such that $\tilde{\mathbf{x}}(\bar{t})$ enters the boundary layer S_χ and remains inside thereafter. Inside the boundary

layer S_χ we have

$$\|\mathbf{s}(\tilde{\mathbf{x}}(t)) - \mathbf{s}_\chi(\tilde{\mathbf{x}}(t))\| \leq 1. \quad (\text{B.64})$$

Since, according to (B.15), $|\hat{\alpha}(t)| \leq \hat{\alpha}^*$ for all $t > t_0$, the derivative of the Lyapunov function candidate can be upper bounded as follows:

$$\dot{V}(t, \boldsymbol{\zeta}(t)) \leq -\tilde{\mathbf{x}}^\top(t) Q \tilde{\mathbf{x}}(t) + \gamma_1 + \gamma_3, \quad (\text{B.65})$$

where $t \geq \bar{t}$, $\gamma_3 = \chi \hat{\alpha}^* > 0$, and γ_1 is the same as in (B.31). It is straightforward to verify that $\dot{V}(t, \boldsymbol{\zeta}(t)) < 0$ outside the compact set

$$\begin{aligned} \Omega_\chi = \{ \|\tilde{\mathbf{x}}\| \leq \sqrt{\frac{\gamma_1 + \gamma_2}{\lambda_{\min}(Q)}}, \|\tilde{W}_k\|_F \leq \tilde{W}_k^*, \quad k = 0, \dots, p, \\ |\tilde{\theta}_k| \leq \tilde{\theta}_k^*, \quad k = 1, \dots, p, \quad |\tilde{\alpha}| \leq \tilde{\alpha}^* \}. \end{aligned} \quad (\text{B.66})$$

Thus the trajectories of the closed loop system in (B.14) and (B.61) are bounded. Decreasing the parameter χ will decrease the size of the boundary layer S_χ , and hence reduce the bounds on the components of $\tilde{\mathbf{x}}(t)$ that are in the range of matrix C^* . However, the bounds on the components of $\tilde{\mathbf{x}}(t)$ that are in the null space of C^* are not affected directly, and are defined by the ultimate bound of the closed loop system, which can be determined following the methodology in ([30], p.567). Towards that end, notice that the Lyapunov function candidate $V(t, \boldsymbol{\zeta})$ in (B.27) satisfies the inequality

$$V_1(\tilde{\mathbf{x}}, \tilde{W}_0, \dots, \tilde{W}_p, \tilde{\theta}_1, \dots, \tilde{\theta}_p, \tilde{\alpha}) \leq V(t, \boldsymbol{\zeta}) \leq V_2(\tilde{\mathbf{x}}, \tilde{W}_0, \dots, \tilde{W}_p, \tilde{\theta}_1, \dots, \tilde{\theta}_p, \tilde{\alpha}), \quad (\text{B.67})$$

where

$$\begin{aligned} V_1(\tilde{\mathbf{x}}, \tilde{W}_0, \dots, \tilde{W}_p, \tilde{\theta}_1, \dots, \tilde{\theta}_p, \tilde{\alpha}) &= \lambda_{\min}(P) \|\tilde{\mathbf{x}}(t)\|^2 + \frac{1}{\lambda_{\max}(G_0)} \|\tilde{W}_0(t)\|^2 \\ &+ \sum_{k=1}^p \frac{\delta_k}{\lambda_{\max}(G_k)} \|\tilde{W}_k(t)\|^2 + \frac{1}{\sigma} \tilde{\alpha}^2(t) + \sum_{k=1}^p \frac{1}{\nu_k} \tilde{\theta}_k^2(t), \end{aligned} \quad (\text{B.68})$$

$$\begin{aligned}
V_2(\tilde{\mathbf{x}}, \tilde{W}_0, \dots, \tilde{W}_p, \tilde{\theta}_1, \dots, \tilde{\theta}_p, \tilde{\alpha}) &= \lambda_{\max}(P) \|\tilde{\mathbf{x}}(t)\|^2 + \frac{1}{\lambda_{\min}(G_0)} \|\tilde{W}_0(t)\|^2 \\
&+ \sum_{k=1}^p \frac{\beta_k}{\lambda_{\min}(G_k)} \|\tilde{W}_k(t)\|^2 + \frac{1}{\sigma} \tilde{\alpha}^2(t) + \sum_{k=1}^p \frac{1}{\nu_k} \tilde{\theta}_k^2(t)
\end{aligned}$$

are class $\mathcal{K}$ functions of their arguments. Define the level sets

$$\begin{aligned}
\Omega_V = \{(\tilde{\mathbf{x}}, \tilde{W}_0, \dots, \tilde{W}_p, \tilde{\theta}_1, \dots, \tilde{\theta}_p, \tilde{\alpha}) \in \mathbb{R}^n \times \mathbb{R}^{m \times r_0} \times \dots \times \mathbb{R}^{m \times r_p}, \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_p \times \mathbb{R} \mid \\
V(\tilde{\mathbf{x}}, \tilde{W}_0, \dots, \tilde{W}_p, \tilde{\theta}_1, \dots, \tilde{\theta}_p, \tilde{\alpha}) \leq \eta_{V_2}\},
\end{aligned}$$

$$\begin{aligned}
\Omega_{V_1} = \{(\tilde{\mathbf{x}}, \tilde{W}_0, \dots, \tilde{W}_p, \tilde{\theta}_1, \dots, \tilde{\theta}_p, \tilde{\alpha}) \in \mathbb{R}^n \times \mathbb{R}^{m \times r_0} \times \dots \times \mathbb{R}^{m \times r_p}, \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_p \times \mathbb{R} \mid \\
V_1(\tilde{\mathbf{x}}, \tilde{W}_0, \dots, \tilde{W}_p, \tilde{\theta}_1, \dots, \tilde{\theta}_p, \tilde{\alpha}) \leq \eta_{V_2}\},
\end{aligned}$$

$$\begin{aligned}
\Omega_{V_2} = \{(\tilde{\mathbf{x}}, \tilde{W}_0, \dots, \tilde{W}_p, \tilde{\theta}_1, \dots, \tilde{\theta}_p, \tilde{\alpha}) \in \mathbb{R}^n \times \mathbb{R}^{m \times r_0} \times \dots \times \mathbb{R}^{m \times r_p}, \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_p \times \mathbb{R} \mid \\
V_2(\tilde{\mathbf{x}}, \tilde{W}_0, \dots, \tilde{W}_p, \tilde{\theta}_1, \dots, \tilde{\theta}_p, \tilde{\alpha}) \leq \eta_{V_2}\},
\end{aligned}$$

where

$$\eta_{V_2} = \frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}(\gamma_1 + \gamma_3) + \frac{1}{\lambda_{\min}(G_0)}(\tilde{W}_0^*)^2 + \sum_{k=1}^p \frac{\beta_k}{\lambda_{\min}(G_k)}(\tilde{W}_k^*)^2 + \frac{1}{\sigma}(\tilde{\alpha}^*)^2 + \sum_{k=1}^p \frac{1}{\nu_k}(\tilde{\theta}_k^*)^2.$$

From the inequality in (B.67) and definition of η_{V_2} it follows that $\Omega_{\chi} \subset \Omega_{V_2} \subset \Omega_V \subset \Omega_{V_1}$.

Then the ultimate bound for the observation error can be found from the inequality

$$V_1(\tilde{\mathbf{x}}, \tilde{W}_0^*, \dots, \tilde{W}_p^*, \tilde{\theta}_1^*, \dots, \tilde{\theta}_p^*, \tilde{\alpha}^*) \leq \eta_{V_2} \quad (\text{B.70})$$

and is equal to

$$\eta_{\mathbf{x}} = \sqrt{\frac{1}{\lambda_{\min}(P)} \left[\frac{\lambda_{\max}(P)}{\lambda_{\min}(Q)}(\gamma_1 + \gamma_3) + \eta_W \right]}, \quad (\text{B.71})$$

where

$$\eta_W = \left[\frac{1}{\lambda_{\min}(G_0)} - \frac{1}{\lambda_{\max}(G_0)} \right] (\tilde{W}_0^*)^2 + \sum_{k=1}^p \left[\frac{\beta_k}{\lambda_{\min}(G_k)} - \frac{\delta_k}{\lambda_{\max}(G_k)} \right] (\tilde{W}_k^*)^2. \quad (\text{B.72})$$

Remark 19 *The approach can be used in the problem of vision based tracking of a maneuvering unknown target, the geometric parameters of which can be viewed as time-varying unknown parameters from the follower's perspective [7, 65]. These parameters are positive and bounded, and moreover, when the tracking is achieved, they are seen as constant quantities. So, they satisfy the assumptions imposed on $\theta_j(t)$. The target's acceleration can be viewed as a time-varying unknown bounded disturbance. The dynamics of the follower always contain state dependent unknown nonlinear terms due to approximations adopted for modeling. Also, the relative dynamics of the target-follower motion as a result of closed coupled motion are affected by the unknown aerodynamic forces and moments that depend on the states. The visual sensors can give measurements that contain information on the relative positions but not velocities. Thus, the dynamics of visual tracking can be described by equations similar to (B.1), and therefore, the system's state can be asymptotically reconstructed from the visual measurements, if the Assumption 6 can be verified. Verification of Assumption 6 depends upon the modeling perspective and the placement of the visual sensor. This is a challenging problem, and presents the next step towards solving the visual tracking problem of a maneuvering target.*

B.5 Simulations

Consider a bounded process that is described by the open-loop system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \quad x_1(0) = 0 \\ \dot{x}_2(t) &= f(x_1(t), x_2(t)) + g(x_1(t), x_2(t))\theta(t) + d(t), \quad x_2(0) = 0 \\ y(t) &= c_1x_1(t) + c_2x_2(t), \end{aligned} \tag{B.73}$$

where $f(x_1, x_2)$ and $g(x_1, x_2)$ are unknown functions, $\theta(t)$ is the unknown time-varying parameter and $d(t)$ is the unknown bounded time-varying disturbance given by

$$\begin{aligned} f(x_1, x_2) &= -5x_1^3 + 0.9x_2^2 \\ g(x_1, x_2) &= \sin(-0.4x_1 - 0.5x_2^3) \\ \theta(t) &= 1.5 - 0.5 \exp(-0.5t) \\ d(t) &= 0.9 \sin(t) \\ c_1 &= 0.4, \quad c_2 = 0.9. \end{aligned} \tag{B.74}$$

For the system in (B.73)

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.4 & 0.9 \end{bmatrix}.$$

It is straightforward to show that the triple $(A - LC, B, C)$ can be made SPR by the choice of the observer gain L . Therefore, we choose L according to Remark 17 to satisfy equations in (B.56) for a symmetric positive definite matrix Q_0 such that $\lambda_{\min}(Q_0) > \rho$, where ρ is calculated according to equation in (B.58). The process is bounded on the square region $\Omega_x = \{(x_1, x_2) : -1 \leq x_1 \leq 1, -1 \leq x_2 \leq 1\}$, over which nine normalized radial basis functions ([24], p.299) $\Phi_i(x_1, x_2) = \exp \left\{ -\frac{(x_1 - x_{1i})^2 + (x_2 - x_{2i})^2}{2\sigma^2} \right\}$ are defined, with the centers (x_{1i}, x_{2i}) at the grid points with a unit step and the width $\sigma = \frac{2}{3}$. The Lipschitz constant for the RBF is found to be $\lambda = \sigma^{-1} \exp(-0.5)$. The ideal NN weights $\mathbf{W}_0$ and $\mathbf{W}_1$ are determined by minimizing the squared approximation error over the set of training data. Therefore, they can be represented by the least-squared solution [24] (p.280):

$$\begin{aligned} \mathbf{W}_0 &= (\Xi^\top \Xi)^{-1} \Xi^\top \mathbf{F} \\ \mathbf{W}_1 &= (\Xi^\top \Xi)^{-1} \Xi^\top \mathbf{G} \end{aligned} \tag{B.75}$$

where Ξ is the matrix of the values of RBFs over the grid points where the training data are taken, $\mathbf{F}$ and $\mathbf{G}$ are the vector of training data of $f(x_1, x_2)$ and $g(x_1, x_2)$ respectively. Using

201 × 201 uniform grid over the square Ω_x , the equations in (B.75) and in (B.58) result in $\|\mathbf{W}_0\| = 1.0420$, $\|\mathbf{W}_0\| = 0.2938$, $\rho = 1.3490$. Assuming that a conservative bound $\rho = 1.4$ is available, the equations (B.56) are solved for $Q_0 = 1.5I_{2 \times 2}$ resulting in $L = [0.6468 \ 1.1323]^\top$, which places the closed loop poles at $[-0.6389 + 0.2115i, \ -0.6389 - 0.2115i]$.

For the system in (B.73) three types of observers are implemented from the same initial conditions $\hat{x}_1(0) = 0.5$, $\hat{x}_2(0) = -0.2$. First, a linear observer is designed using the gain matrix L found above:

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= A\hat{\mathbf{x}}(t) + L(y(t) - \hat{y}(t)) \\ \hat{y}(t) &= C\hat{\mathbf{x}}(t),\end{aligned}\tag{B.76}$$

The performance of the linear observer, when applied to the nonlinear system in (B.73), is displayed in Figure B.1.

Next, the robust adaptive observer from [38] is implemented. For this purpose we transform the system in (B.73) using the state transformation $\xi_1 = c_1x_1 + c_2x_2$, $\xi_2 = c_1x_2$. The transformed system is written as

$$\begin{aligned}\dot{\xi}_1(t) &= \xi_2(t) + c_2(\bar{f}(\xi_1, \xi_2) + \bar{g}(\xi_1, \xi_2)\theta(t) + d(t)) \\ \dot{\xi}_2(t) &= c_1(\bar{f}(\xi_1, \xi_2) + \bar{g}(\xi_1, \xi_2)\theta(t) + d(t)) \\ y(t) &= \xi_1(t),\end{aligned}\tag{B.77}$$

where the functions $\bar{f}(\xi_1, \xi_2)$ and $\bar{g}(\xi_1, \xi_2)$ are obtained respectively from $f(x_1, x_2)$ and $g(x_1, x_2)$ by the inverse transformation

$$x_1 = \frac{1}{c_1}\xi_1 - \frac{c_2}{c_1^2}\xi_2, \quad x_2 = \frac{1}{c_1}\xi_2$$

and have the form

$$\bar{f}(\xi_1, \xi_2) = 0.9\frac{1}{c_1^2}\xi_2^2 - 5\left(\frac{1}{c_1}\xi_1 - \frac{c_2}{c_1^2}\xi_2\right)^3,$$

$$\bar{g}(\xi_1, \xi_2) = \sin \left(-0.4 \left(\frac{1}{c_1} \xi_1 - \frac{c_2}{c_1^2} \xi_2 \right) - 0.5 \frac{1}{c_1^3} \xi_2^3 \right).$$

If one assumes that the signals $\psi_0(t) = \bar{f}(\xi_1(t), \xi_2(t))$ and $\psi(t) = \bar{g}(\xi_1(t), \xi_2(t))$ are known and can be used in the observer equation and in the adaptive law, the system (B.77) will satisfy the assumptions for adaptive observer design from [38]. The latter is implemented with the equivalent poles for the linear part as above, and with the adaptive gain $\nu = 3.9$ for the parameter update law. The observation error is bounded, but does not converge to zero due to the presence of *non-vanishing* disturbance, as it is shown in Figure B.2.

Finally, the observer in (B.10) is implemented with the same poles for the linear part as above, and with the adaptive part according to equations in (B.14) with the following parameters: $r_0 = r_1 = 9$, $G_0 = 1.9$; $G_1 = 1.6$, $\nu = 2.9$, $\sigma = 3.8$. We run three simulations. For the first run we choose the integration step equal to 0.0005. Figure B.3 displays the performance of the observer showing asymptotic observation. When we increase the step size to 0.02, the signum function generates a chattering, which can be seen in Figure B.4. For a better demonstration of the chattering a zoomed version is presented in Figure B.5. To avoid chattering, the sgn function in $s = \text{sgn}(0.4\tilde{x}_1(t) + 0.9\tilde{x}_2(t))$, is approximated as follows

$$s = \begin{cases} 1, & 0.4\tilde{x}_1(t) + 0.9\tilde{x}_2(t) > 0.05 \\ \frac{0.4\tilde{x}_1(t) + 0.9\tilde{x}_2(t)}{0.05}, & |0.4\tilde{x}_1(t) + 0.9\tilde{x}_2(t)| \leq 0.05 \\ -1, & 0.4\tilde{x}_1(t) + 0.9\tilde{x}_2(t) < -0.05 \end{cases}$$

With the same step size 0.02 the observer shows the performance that matches that of the exact observer with the much smaller step size of 0.0005, as can be seen in Figure B.6.

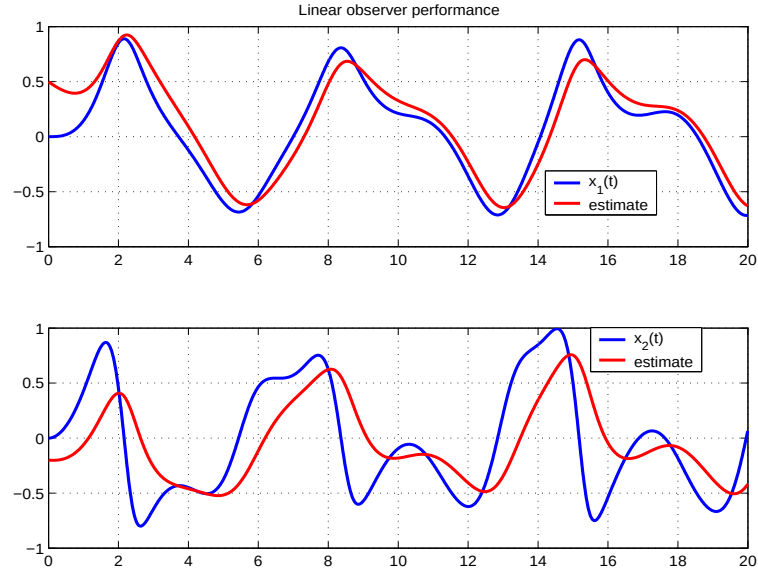


Figure B.1: Performance of the linear observer applied to the nonlinear system, no convergence is observed.

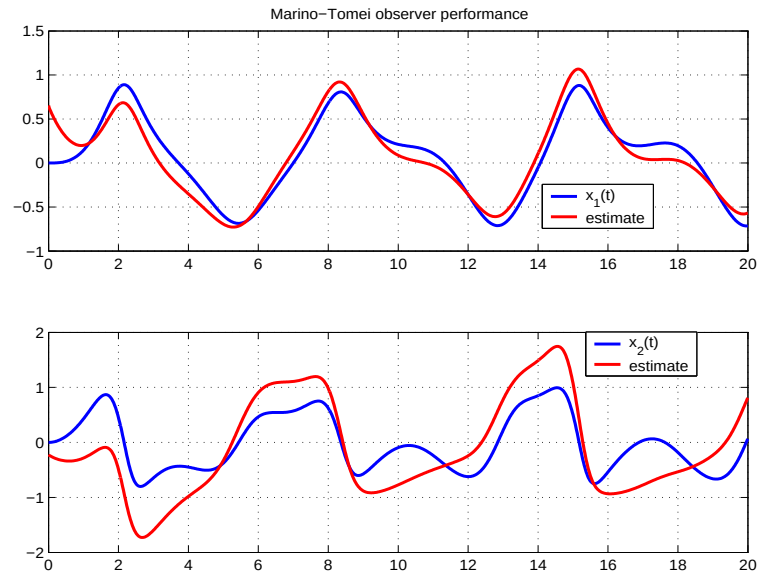


Figure B.2: Performance of the observer from [38], only bounded estimation is achieved.

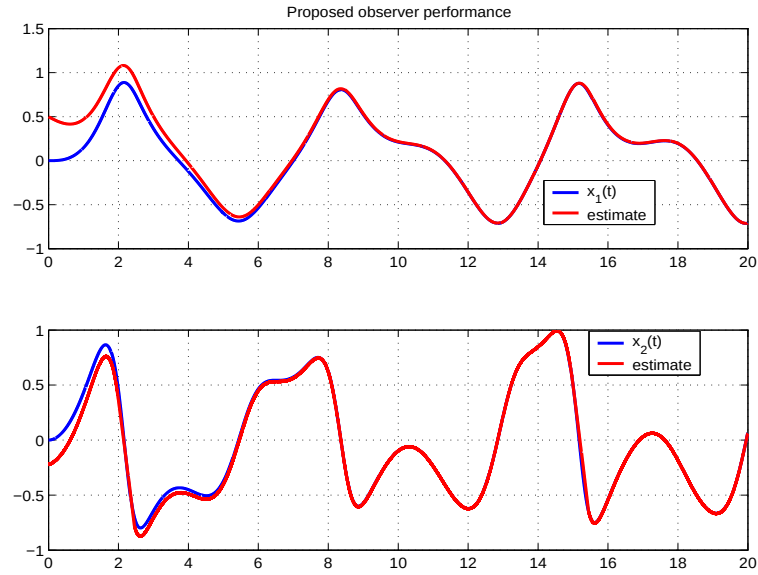


Figure B.3: Performance of the proposed observer with the integration step size of 0.0005, asymptotic convergence is achieved.

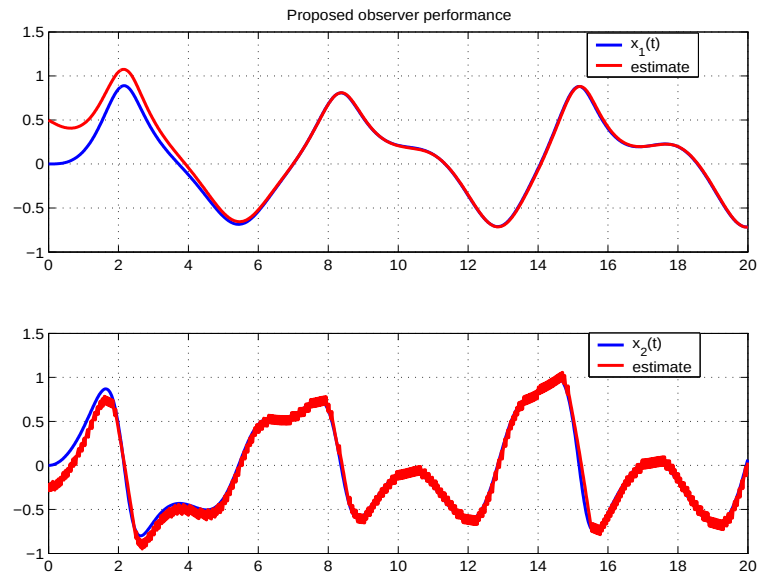


Figure B.4: Performance of the proposed observer with integration step size of 0.02, chattering is generated in $\hat{x}_2$, the dynamics of which involves a signum function.

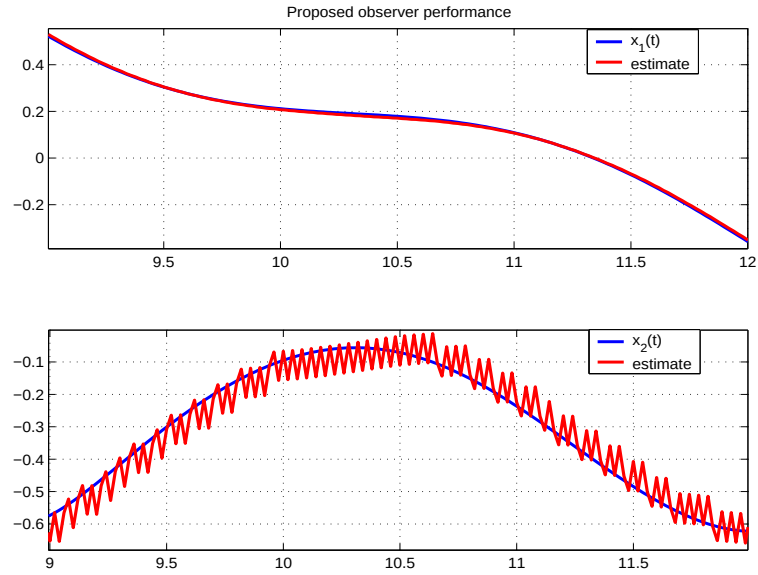


Figure B.5: Performance of the proposed observer with integration step size of 0.02, zoomed for the better presentation of the chattering in $\hat{x}_2$.

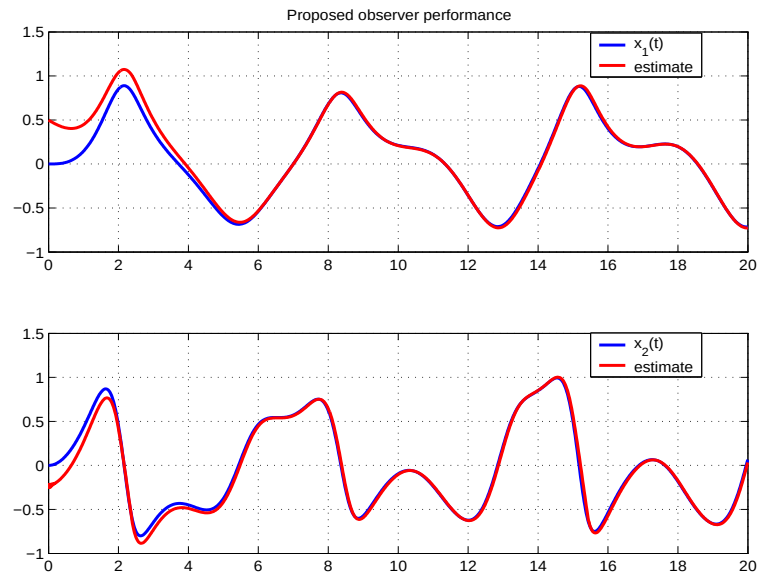


Figure B.6: Performance of the proposed observer with the approximation of signum function by a continuous one with $\chi = 0.05$.

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Vita

Vahram Stepanyan was born on July 7, 1958 in Armenia. In June of 1980, Vahram graduated from the Yerevan State University, Yerevan, Armenia with a Master of Science degree in Applied Mathematics and Mechanical Engineering. He spent the next three years doing postgraduate study in the department of Applied Mechanics at Moscow State University, Moscow, Russia, where he finished all the course work toward his PhD. From 1984 to 2000 Vahram worked in the department of Theoretical Mechanics at Yerevan State University first as a laboratory head, then as an assistant professor in 1991. In December, 1994 he successfully defended his dissertation on the topic "Stochastic differential games with many players" to receive a PhD in Applied Mathematics from the department of Mathematics at Yerevan State University. In June 2000 Vahram joined the department of Electrical and Computer Engineering at Brigham Young University, Provo, Utah as a visiting professor. In August 2001, he joined the department of Mathematics at Boise State University, Boise, Idaho as a visiting professor. In August 2002, he returned to the department of Electrical and Computer Engineering at Brigham Young University as a PhD student working on the information consensus in the cooperative control of a multi agent team. In August 2003, Vahram joined the department of Aerospace and Ocean Engineering at Virginia Polytechnic Institute and State University (Virginia Tech), Blacksburg, Virginia to continue his PhD studies in vision based target tracking problems. In December 2005, Vahram received MS

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