

GEOMETRICALLY-LINEAR AND NONLINEAR ANALYSIS OF
LINEAR VISCOELASTIC COMPOSITES USING
THE FINITE ELEMENT METHOD

By

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(ABSTRACT)

Over the past several decades, the use of composite materials has grown considerably. Typically, fiber-reinforced polymer-matrix composites are modeled as being linear elastic. However, it is well-known that polymers are viscoelastic in nature. Furthermore, the analysis of complex structures requires a numerical approach such as the finite element method. In the present work, a triangular flat shell element for linear elastic composites is extended to model linear viscoelastic composites. Although polymers are usually modeled as being incompressible, here they are modeled as compressible. Furthermore, the macroscopic constitutive properties for fiber-reinforced composites are assumed to be known and are not determined using the matrix and fiber properties along with the fiber volume fraction. Hygrothermo-rheologically simple materials are considered for which a change in the hygrothermal environment results in a horizontal shifting of the relaxation moduli curves on a log time scale, in addition to the usual hygrothermal loads. Both the temperature and moisture are taken to be prescribed. Hence, the heat energy generated by the viscoelastic deformations is not considered.

When the deformations and rotations are small under an applied load history, the usual engineering stress and strain measures can be used and the time history of a viscoelastic deformation process is determined using the original geometry of the structure. If, however, sufficiently large loads are applied, the deflections and rotations will be large leading to changes in the structural stiffness characteristics and possibly the internal loads carried throughout the structure. Hence, in such a case, nonlinear effects must be taken into account and the ap-

appropriate stress and strain measures must be used. Although a geometrically-nonlinear finite element code could always be used to compute geometrically-linear deformation processes, it is inefficient to use such a code for small deformations, due to the continual generation of the assembled internal load vector, tangent stiffness matrix, and deformation-dependent external load vectors. Rather, for small deformations, the appropriate deformation-independent stiffness matrices and load vectors to be used for all times can be determined once at the start of the analysis. Of course, the time-dependent viscoelastic effects need to be correctly taken into account in both types of analyses. The present work details both geometrically-linear and nonlinear triangular flat shell formulations for linear viscoelastic composites. The accuracy and capability of the formulations are shown through a range of numerical examples involving beams, rings, plates, and shells.

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Chapter 1

Introduction

1.1 Viscoelastic Materials

Viscoelastic materials experience both viscous and elastic phenomena as the name viscoelastic implies. Viscoelastic material response depends on both the current stress and the stress history up to the current time. Hence, viscoelastic analysis is also called hereditary elasticity. Some common materials which exhibit viscoelastic behavior are metals at high temperatures, plastics, and concrete.

One of the most distinguishing features of viscoelastic materials is their response to a constant stress, during which the strain does not assume a constant value as it would if the material were elastic. Rather, the strain continues to increase under such a load and ultimately approaches an asymptote. That is, the material creeps during a so-called creep test. Another distinguishing characteristic of viscoelastic materials is relaxation, in which the stress required to maintain a constant strain decreases as time evolves.

Appendix A provides a brief overview of some linear viscoelasticity basics. The constitutive law is written in several forms in the time domain and the representation of linear viscoelastic materials using spring and dashpot mechanical analogs is discussed.

1.2 Polymer-Matrix Composites

In recent years, the use of composite materials has grown in popularity, because their response characteristics can be tailored to meet specific design requirements, while allowing structural components to remain lightweight. Usually, composite structures are analyzed using a linear elastic material law. However, a linear elastic analysis may give inaccurate results for fiber-reinforced polymer-matrix composites, since it is well-known that polymers are viscoelastic in nature.¹ Furthermore, the viscoelastic response of polymers is affected by environmental conditions such as temperature and moisture.^{2,3}

For fiber-reinforced polymer-matrix composites, the polymer matrix is typically viscoelastic while the fibers are elastic. For polymers, the elastic part of the material response results from the polymer chains being stretched by the applied stress. The viscous response of polymers results from several relaxation mechanisms, such as motion of side chain groups, reorientation of chain segments relative to each other, and the translation of entire molecules past one another in the case of linear amorphous polymers in the rubbery-flow region.¹

Increasing the temperature and moisture results in the viscous response being accelerated, as has been experimentally observed.² For most amorphous polymers, experiments indicate that changing the temperature and moisture results in a simple horizontal shifting of the relaxation modulus on a log time scale. Such materials are referred to as being hygrothermo-rheologically simple. For these materials, the relaxation modulus (viscoelastic stress corresponding to a unit strain applied suddenly at $t = 0$ and then held constant) at the real temperature T , moisture H , and time t can be related to the relaxation modulus at the reference temperature T_{ref} , reference moisture H_{ref} , and reduced time ζ by

$$E(T, H, t) = E(T_{ref}, H_{ref}, \zeta) \quad (1.1)$$

where the reduced time and real time are related by the horizontal shift factor A_{TH} as follows:

$$\zeta = \int_0^t \frac{dt'}{A_{TH}[T(t'), H(t')]} \quad (1.2)$$

Although often found to be accurate, a simple horizontal shifting is not valid for multi-phase or semicrystalline polymers in general. For two-phase polymers, horizontal shifting

alone will still be accurate if there is one dominant phase. In addition to using horizontal shifting, vertical shifting can be included in accounting for the effects of physical aging or crystallization. Besides hastening its viscous response, increasing a polymer's temperature leads to an increase in the elastic stiffness of a rubbery network and a decrease in the density ρ that will decrease the relaxation modulus, as the relaxation modulus obviously depends upon the amount of matter per unit cross-sectional area.¹ This results in an additional vertical shifting for the rubbery-plateau region, with the vertical shift factor $\rho T / \rho_{ref} T_{ref}$ multiplying the right side of Eq. (1.1) in that region of response.¹

1.3 Time Integration Schemes

Linear viscoelastic stress analysis can be performed in the time, complex frequency, or Laplace domains. For general load histories (including hygrothermal loads) and viscoelastic material properties, direct time integration schemes appear to be the most robust. Furthermore, the incorporation of inertia terms is straightforward. When direct integration schemes were first formulated, they required relatively large amounts of storage in order to retain all the previous solutions needed to evaluate the current values of the viscoelastic memory loads.⁴ This deficiency in the method was remedied by the development of recursion relationships for these loads.⁵⁻⁷ One remaining disadvantage of numerically integrating viscoelastic equations directly in the time domain is that a large number of time steps is needed to generate long-term solutions. The time step size for accurate calculations may need to be relatively small due to possible error propagation. Given the wide availability of computational resources at the present time, these time step size constraints do not appear to be overly restrictive.

In most direct integration schemes, the hereditary integral form of the constitutive law is used to allow easy incorporation of the effects of temperature and moisture upon the material response rate. For hygrothermo-rheologically simple materials, the creep or relaxation kernel in the hereditary integral will be specified in terms of a reduced time scale. Usually, the time

domain is discretized and a numerical approximation is used for the hereditary integral terms corresponding to each time step. Although many techniques have been developed for this, typically one of the two terms of the product comprising the integrand is assumed to be constant over each time increment. The remaining integration over each time step can be evaluated exactly in some cases, while in other cases it must be computed numerically. Three such methodologies employed by Zak,⁵ White,⁶ and Taylor *et al.*⁷ will now be discussed in the context of a one-dimensional hygrothermo-rheologically simple linear viscoelastic constitutive equation.

The constitutive law corresponding to the work of Zak⁵ is written as follows:

$$\epsilon(t) = \int_0^t D(\zeta - \zeta') \frac{d\sigma(\tau)}{d\tau} d\tau \quad (1.3)$$

where ϵ is the strain, σ is the stress, $D(t)$ is the creep compliance, the reduced time ζ is related to the physical time t by Eq. (1.2), and the reduced time ζ' is similarly related to time τ . Following the method presented by Zak,⁵ the stress is considered to be constant over each time step, with its value over a particular time step equal to that corresponding to the final time of the increment. That is, the stress for $0 < t < t^p$ is written as

$$\sigma(t) = \sum_{i=1}^{p-1} \Delta\sigma^{i+1} u(t - t^i) \quad (1.4)$$

where

$$\Delta\sigma^{i+1} = \sigma(t^{i+1}) - \sigma(t^i) \quad (1.5)$$

and $u(t)$ is the unit step function. Here, the initial time is $t^1 = 0^-$, and $\sigma(t^1) = 0$. Using Eq. (1.4), the time derivative of the stress for $0 < \tau < t^p$ is

$$\frac{d\sigma(\tau)}{d\tau} = \sum_{i=1}^{p-1} \Delta\sigma^{i+1} \delta(\tau - t^i) \quad (1.6)$$

where $\delta(t)$ is the Dirac delta generalized function.

The strain at time t^p is thus approximated as follows:

$$\epsilon(t^p) \approx \int_0^{t^p} D(\zeta^p - \zeta') \left(\sum_{i=1}^{p-1} \Delta\sigma^{i+1} \delta(\tau - t^i) \right) d\tau \quad (1.7)$$

$$= \sum_{i=1}^{p-1} D(\zeta^p - \zeta^i) \Delta\sigma^{i+1} \quad (1.8)$$

Of course, Eq. (1.8) could have been written directly, once the stress history was assumed to be given by Eq. (1.4).

An alternative way to derive Eq. (1.8) is to approximate $D(\zeta^p - \zeta')$ to be constant over each time step, with its value for a particular time step equal to that at the beginning of the increment. That is,

$$\epsilon(t^p) = \sum_{i=1}^{p-1} \int_{t^i}^{t^{i+1}} D(\zeta^p - \zeta') \frac{d\sigma(\tau)}{d\tau} d\tau \quad (1.9)$$

$$\approx \sum_{i=1}^{p-1} D(\zeta^p - \zeta^i) \int_{t^i}^{t^{i+1}} \frac{d\sigma(\tau)}{d\tau} d\tau \quad (1.10)$$

Performing each integration appearing in Eq. (1.10) exactly gives Eq. (1.8).

For White's work,⁶ the relevant constitutive law is as follows:

$$\sigma(t) = E(0)\epsilon(t) - \int_0^t \frac{dE(\zeta - \zeta')}{d\tau} \epsilon(\tau) d\tau \quad (1.11)$$

where $E(t)$ is the relaxation modulus, as previously defined. The stress at time t^p is evaluated using the appropriate mean-value theorem for the integration over each time step, with the average value of the strain for each time step computed using a trapezoidal approximation as follows:⁴

$$\sigma(t^p) = E(0)\epsilon(t^p) - \sum_{i=1}^{p-1} \int_{t^i}^{t^{i+1}} \frac{dE(\zeta^p - \zeta')}{d\tau} \epsilon(\tau) d\tau \quad (1.12)$$

$$\approx E(0)\epsilon(t^p) - \sum_{i=1}^{p-1} \left(\frac{\epsilon(t^{i+1}) + \epsilon(t^i)}{2} \right) \int_{t^i}^{t^{i+1}} \frac{dE(\zeta^p - \zeta')}{d\tau} d\tau \quad (1.13)$$

$$= E(0)\epsilon(t^p) - \sum_{i=1}^{p-1} \left(\frac{\epsilon(t^{i+1}) + \epsilon(t^i)}{2} \right) (E(\zeta^p - \zeta^{i+1}) - E(\zeta^p - \zeta^i)) \quad (1.14)$$

where $t^1 = 0^+$.

To illustrate the method of Taylor *et al.*,⁷ still another form of the constitutive equation is used. The relevant material law is as follows:

$$\sigma(t) = E(\zeta)\epsilon(0) + \int_0^t E(\zeta - \zeta') \frac{d\epsilon(\tau)}{d\tau} d\tau \quad (1.15)$$

The time derivative of the strain is approximated as being constant over each time step, corresponding to a linear variation in the strain over each increment. The stress at time t^p

is then

$$\sigma(t^p) = E(\zeta^p)\epsilon(0) + \sum_{i=1}^{p-1} \int_{t^i}^{t^{i+1}} E(\zeta^p - \zeta') \frac{d\epsilon(\tau)}{d\tau} d\tau \quad (1.16)$$

$$\approx E(\zeta^p)\epsilon(0) + \sum_{i=1}^{p-1} \frac{\Delta\epsilon^{i+1}}{\Delta t^{i+1}} \int_{t^i}^{t^{i+1}} E(\zeta^p - \zeta') d\tau \quad (1.17)$$

where $t^1 = 0^+$ and $\Delta\epsilon^{i+1}$ and Δt^{i+1} are defined in a manner similar to that given in Eq. (1.5) for $\Delta\sigma^{i+1}$. In the approach given by Taylor *et al.*,⁷ the relaxation modulus is required to be expressed as a summation of decaying exponential terms and a constant. That is, the relaxation modulus must be given as a Prony series. Then, each integral appearing in Eq. (1.17) is evaluated exactly once the reduced time is approximated to vary linearly over each time step. For a hygrothermo-rheologically simple, linear viscoelastic material, this approximation corresponds to a constant horizontal shift factor over each time increment. Taylor *et al.*⁷ also studied the case where each integral in Eq. (1.17) is computed using a simple trapezoidal rule. For such a method, the current stress is given by

$$\sigma(t^p) = E(\zeta^p)\epsilon(0) + \sum_{i=1}^{p-1} \left(\frac{E(\zeta^p - \zeta^{i+1}) + E(\zeta^p - \zeta^i)}{2} \right) \Delta\epsilon^{i+1} \quad (1.18)$$

They found that for the case of a constant horizontal shift factor, the numerical integration technique corresponding to Eq. (1.18) required much smaller time increments for a given level of accuracy than did the technique which uses an exact integration of the integrals appearing in Eq. (1.17).

In the methods detailed by Zak,⁵ White,⁶ and Taylor *et al.*,⁷ the creep compliance or relaxation modulus is written as a Prony series. This allows the development of recursion relations for the hereditary integral terms corresponding to the previous time steps so that the current strain or stress can be computed using only quantities occurring at t^p , t^{p-1} , and t^{p-2} . The computational effort and storage requirements for calculating the current solution are related to the number of Prony series terms used, but are independent of the number of previous time steps.

If a Prony series for the creep compliance or relaxation modulus is not known, it is still possible to numerically integrate the viscoelastic hereditary integral terms using either

Eq. (1.8), (1.14), or (1.18). Obviously, the creep compliance or relaxation modulus must be known at the discrete times used to define the time steps employed in the numerical integration. These values could come from some other mathematical representation for the creep compliance or relaxation modulus, or from a table of experimentally measured values. Of course, the drawback of directly using Eq. (1.8), (1.14), or (1.18) without further modification is that the computational effort and memory storage required to evaluate the current value of the stress or strain depend upon the number of previous time steps. For the long-term response of linear viscoelastic problems involving more than a one-dimensional constitutive equation, the required computational effort and memory storage for such schemes may become excessive.

1.4 Geometrically-Linear Finite Element Analysis

White⁶ assumed a homogeneous, isotropic, thermo-rheologically simple material with a bulk modulus constant in time. In performing the stress analysis of solid propellant grain under transient thermal loads, a linear temperature variation was assumed over each element, with the reduced time for an element determined using its average nodal temperature.⁶

Taylor *et al.*⁷ developed finite element equations from the application of a variational principle for isotropic thermo-rheologically simple linear viscoelastic structures with time-dependent temperature fields. The temperature field and, hence, the thermal load vector were assumed to be independent of the displacement field.

Using the methodology developed by White,⁶ Srinatha and Lewis⁸ generated finite element codes for the axisymmetric thermo-viscoelastic analysis of homogeneous, isotropic, linear viscoelastic structures. Recall that the case of plane strain is included as a special case of axisymmetric analysis by considering the axis of symmetry to be located a relatively large distance away. Unlike White's work,⁶ the bulk modulus is not restricted to be constant in time (i.e., the volumetric behavior is allowed to be viscoelastic in nature). A one-way coupling is assumed between the thermal and structural responses, with the thermal field

determined independently and its effect included in the structural response problem in two ways. First, the thermal strain terms needed for the determination of the mechanical strains from the total strains are included. Secondly, the thermal field is assumed to affect the viscoelastic response rate in a thermo-rheologically simple manner.

Wang and Tsai⁹ performed isothermal quasi-static and dynamic finite element analyses of homogeneous, isotropic, linear viscoelastic Mindlin plates. The hereditary integrals appearing in the generalized stress-strain relations were evaluated using a composite trapezoidal rule. Recursion relations were derived for the computation of the viscoelastic memory loads. The Newmark method was utilized in incorporating the inertia term.

The integration method proposed by Taylor *et al.*⁷ for isotropic thermo-rheologically simple linear viscoelastic materials was extended to a general three-dimensional finite element model by Ben-Zvi¹⁰ under the assumption of a constant bulk modulus. The linear viscoelastic finite element equations for the increment in displacements presented by Ben-Zvi¹⁰ only require the user to supply a constitutive routine to a finite element code already incorporating an incremental displacement approach. Thus, the full capabilities of such a host code are maintained. Simplifications to two dimensions and extensions to include material and/or geometric nonlinearities can be made.

Krishna *et al.*¹¹ studied the quasi-static response of electronic packaging structures which have polymer films bonded to elastic substrates. The polymer film was assumed to be linearly viscoelastic and hygrothermo-rheologically simple. Thermal cycling and moisture diffusion were applied to packaging structures, with Fick's law used to determine the moisture distribution independent of the structural response problem. The formulation, including hygrothermal loads, was developed using the approximation scheme proposed by Taylor *et al.*⁷ to evaluate the hereditary integral in the constitutive law of the viscoelastic film.

Assuming the material for each layer to be thermo-rheologically simple, Lin and Hwang^{12,13} used the integration method of Taylor *et al.*⁷ to evaluate the thermo-viscoelastic response of laminated composites. A thermal load vector assuming the laminate temperature to be uniform at any instant of time was included. The contribution of the thermal load vector to

the memory load was evaluated in a similar manner to that used for the stiffness terms. The laminates were restricted to have symmetric stacking sequences and to be under a state of plane stress, with the mechanical loads restricted to be in-plane. Graphite-epoxy laminates subjected to creep, relaxation, and temperature load histories were studied.

Hilton and Yi^{14,15} analyzed the dynamic response of hygrothermo-rheologically simple linear viscoelastic composite beams and plates. The constant average acceleration method of the Newmark family was used to handle both the inertia term and the hereditary integral terms appearing in the assembled finite element equations. The finite difference expression for the current nodal accelerations was used directly in approximating the inertia term. The constant average acceleration method gives a linear variation over each time step for the nodal velocities appearing in the hereditary integral terms. The recursion relations needed to compute the viscoelastic memory terms in an efficient manner were also derived. The formulation for composite beams¹⁴ included both mechanical and hygrothermal loads, while the composite plate formulation¹⁵ only accounted for in-plane and transverse mechanical loads. Hygrothermal loads were added for the quasi-static analysis of hygrothermo-rheologically simple linear viscoelastic composite plates and shells by Yi *et al.*¹⁶ However, in Ref. 16, the required hereditary integral terms were evaluated using the technique developed by Taylor *et al.*,⁷ which assumes constant nodal velocities over each time step.

Extending the formulation of Lin and Hwang,^{12,13} Lin and Yi¹⁷ evaluated the interlaminar stresses in linear viscoelastic composite laminates in a state of plane strain under mechanical and hygrothermal loads. The laminates were assumed to be hygrothermo-rheologically simple. In a later study, Yi and Hilton¹⁸ added Fick's law for diffusion to determine the in-plane and interlaminar stresses of hygrothermo-rheologically simple linear viscoelastic laminates subjected to moisture absorption and desorption.

1.5 Geometrically-Nonlinear Finite Element Analysis

For the regime of geometrically-nonlinear response, most finite element research has been performed for the isotropic viscoelastic case,^{19–24} while little work has been reported for the case of viscoelastic laminated composites.²⁵ However, additional research has been conducted on the large-deformation and stability analysis of viscoelastic composites using more traditional approaches.^{26–33}

In order to determine the large deformations of structures using the finite element method, an incremental approach typically is used. Usually, the second Piola-Kirchhoff (PK2) stress tensor is employed in conjunction with the nonlinear Green–Saint-Venant strain tensor in the description of the material behavior. Any known configuration can be used as the reference configuration for these tensors, which leads to two alternative methods. In the total Lagrangian method, the original configuration is chosen as the reference state, while in the updated Lagrangian method, the reference state is updated throughout the deformation process. As noted by Bathe,³⁴ the two approaches give identical results provided that the constitutive laws used in each are equivalent.

Yang and Lianis¹⁹ applied an incremental midpoint-tangent approach to study the quasi-static large deflection behavior of isotropic linear viscoelastic beams and frames. In their finite element approach, linear strain-displacement relations were used in each increment, with the nodal coordinates updated at the end of each increment to account for the effects of geometric nonlinearity. The stiffness matrix including the initial stress matrix was evaluated using estimates for the geometry and inplane forces at the midpoint of the time increment. No iterations were performed in evaluating the incremental displacements. The formulation is thus restricted to the case of small incremental strains and displacements. The linear viscoelastic material law was represented in a hereditary integral form, which was evaluated using a composite numerical integration scheme similar to that employed by White.⁶ However, recursion relations were not developed for the viscoelastic memory terms.

Key²⁰ developed a finite element method for the large-deflection dynamic analysis of axisymmetric solids. Several constitutive theories were employed, including elastic, elasto-

plastic, crushable foam, and linear viscoelastic. A linear relation was used between the second Piola-Kirchhoff (PK2) stresses and the nonlinear Green–Saint-Venant strains, thereby restricting the accuracy of the formulation in most cases to large deflections/rotations but small strains. The equations of motion were written in terms of the current configuration using Cauchy stresses. The explicit, conditionally-stable, central-difference time integration scheme was used in marching the equations of motion. For the viscoelastic analysis, the bulk modulus was treated as constant in time, while the shear modulus was represented as a three-parameter solid. Recursion relations were developed for the evaluation of the hereditary integrals used in representing the deviatoric PK2 stresses. Key²⁰ analyzed the wave propagation through a viscoelastic half-space which was subjected to a step change in pressure on its surface. The wave velocity was found to be well represented.

Shen *et al.*²¹ used a total Lagrangian approach in formulating a finite element code for the geometrically-nonlinear dynamic response of axisymmetric and three-dimensional isotropic linear viscoelastic solids. The constant average acceleration method of the Newmark family was used in representing both the acceleration appearing in the inertia term and the time derivative of the strain occurring in the hereditary integral of the constitutive law. The dynamic responses of a generic thick viscoelastic cylinder with a steel casing and a rocket motor with propellant, both under internal pressure, were studied.

Shen *et al.*²² presented both a total Lagrangian and updated Lagrangian approach for the three-dimensional finite element analysis of isotropic viscoelastic solids. The nonlinearity in the material law results from multiplying the hereditary integral that normally appears in the linear viscoelastic material law by a strain softening function. The material law was written for a total Lagrangian description. The resulting recursion relations derived for updating the stresses could be used directly in the total Lagrangian approach presented. However, to use these recursion relations in their updated Lagrangian approach, the recursion relations had to be transformed using the deformation gradient tensor from the original reference state to the current reference state. As will be shown in the present formulation, this additional complexity in the updated Lagrangian formulation can be avoided by judicious choice of

the reference states used in developing recursion relations from the material law. Shen *et al.*²² confirmed that the total Lagrangian and updated Lagrangian approaches gave identical results for a cantilever under a tip load. The effect of strain softening was to increase the deflections of a cantilever under various loads, as expected. The quasi-static response of a thick cylinder enclosed in a steel casing under a uniform internal pressure also was studied.

Roy and Reddy²³ analyzed the geometrically-nonlinear deformations of adhesive joints using an updated Lagrangian finite element formulation. The adhesive was modeled as nonlinear viscoelastic using a constitutive law proposed by Schapery.³⁵ The analysis of thermorheologically simple linear viscoelastic adhesives could be performed by setting the nonlinear parameters in Schapery's material law to unity.

Jenkins and Leonard²⁴ extended the method of White⁶ to the case of geometrically-nonlinear deformations. Using Prony series to represent the linear viscoelastic material behavior, the hereditary integral terms relating the Green–Saint-Venant strains and the PK2 stresses were evaluated using a composite numerical integration technique. Recursion relations were then employed for the evaluation of the viscoelastic memory terms. The numerical approximation of the viscoelastic material law was incorporated directly into the incremental finite element equations, which were cast in a total Lagrangian form. The modified Newton-Raphson method was used within each time increment to find the converged nonlinear solution. The Newmark method was utilized to handle the inertia term which was included for dynamic analysis. The dynamic in-plane and transverse deformations of viscoelastic membranes subjected to various loads were analyzed.

Marques and Creus²⁵ extended the linear elastic, total Lagrangian finite element formulation of Bathe³⁴ to include the effects of anisotropic hygrothermo-rheologically simple linear viscoelasticity. In the total Lagrangian formulation of Bathe,³⁴ the nonlinear terms in the principle of virtual work resulting from the product of the incremental PK2 stresses and the virtual nonlinear Green–Saint-Venant strains is linearized as follows. The virtual work of the incremental PK2 stresses is taken as the product of the incremental PK2 stresses and the virtual small-strain tensor. Also, the constitutive equation for the incremental PK2 stresses

is linearized such that the incremental PK2 stress tensor depends only upon the terms in the incremental strain tensor which are linear in the incremental displacements. Marques and Creus²⁵ decomposed the strains into instantaneous, deferred, thermal, and hygroscopic components. The deferred viscoelastic strains are given by hereditary integrals which are each evaluated as an incremental term plus a recursive term. The incremental viscoelastic strains for the current time step are evaluated using the stresses constant at their respective values from the end of the previous time step. The small and large deflection behavior of laminated graphite-epoxy plates and shells was studied. For a viscoelastic shell cap under a central point force which was applied as a creep load (i.e., the load is applied suddenly and then held constant), the critical elapsed time to snap-through was found to increase as the load magnitude decreased.

Using a quasi-elastic approach, Wilson and Vinson studied the linear viscoelastic buckling of laminated plates²⁶ and columns.²⁷ Their small deformation analyses included the effects of both transverse shear deformation (TSD) and transverse normal deformation (TND). In the quasi-elastic method,³⁶ the current viscoelastic response is determined by simply using the viscoelastic properties corresponding to an elapsed time equal to the current time in an elastic analysis. Using the quasi-elastic method allowed the determination of the time variation of the viscoelastic buckling curve for creep loading by the solution of a series of elastic eigenvalue analyses. This viscoelastic buckling curve gives the magnitude of the creep loading applied at $t = 0$ necessary to cause buckling at a given time t . Hence, using the quasi-elastic method simplified the viscoelastic buckling problem considerably for situations where a similar elastic structure would undergo bifurcation buckling with no prebuckling deformation, because viscoelastic stability was determined using eigenvalue analysis of geometrically-perfect elastic structures. For the analysis of such problems using a full viscoelastic approach, it would be necessary to examine if an initial deflection (caused by an imperfection in the structure or its loading) grows large under a given loading.

Using the theorem of minimum total potential energy, Wilson and Vinson^{26,27} formulated equilibrium equations. The resulting Euler-Lagrange equations were used to specify the

through-the-thickness variation of the translational deflections in terms of the midplane translational deflections. The elastic eigenvalue problems were then formulated using the Rayleigh-Ritz method. For both plates and columns, the viscoelastic stability curve was observed to decrease significantly (15% or more) as time evolved. The more prominent the matrix's role in the structural response, the more pronounced was the decay in the viscoelastic buckling load, since the fibers were taken to be elastic, while the matrix was treated as viscoelastic. The importance of the matrix response depended upon the stacking sequence and the importance of transverse shear and transverse normal deformations. It was found that neglecting TSD and TND for some viscoelastic buckling problems could lead to a significant overestimation of the magnitude of the viscoelastic stability curve.

Using the quasi-elastic method, Vinogradov²⁸ studied the viscoelastic buckling of asymmetric laminated beam-columns composed of two layers of isotropic linear viscoelastic materials bonded together. The axial loads were applied as creep loads at $t = 0$. A safe load limit which depended only upon long-term creep compliance values and geometric parameters was determined. For axial loads below this limit load, the transverse deflections initially introduced by a small distributed transverse load remained small for all times. It appeared that the buckling characteristics could be enhanced by tailoring the laminate composition.

Kim and Hong²⁹ studied the viscoelastic buckling of sandwich plates with cross-ply faces using the quasi-elastic approach. The core was assumed to be elastic, while both the adhesive and the surface plates were taken to be linear viscoelastic. The inplane viscoelastic buckling load varied with the face stacking sequence, adhesive strength, plate aspect ratio, and biaxial load ratio. The viscoelastic buckling mode was found to be time-dependent for relatively soft viscoelastic adhesives.

Huang studied the linear viscoelastic response of imperfect composite plates³⁰ and cylindrical panels³¹ under in-plane compressive loads. The edges were simply-supported, and symmetric cross-ply and antisymmetric angle-ply stacking sequences were considered. Geometric nonlinearity was included by using von Karman type strain-displacement relations. The Laplace transform of the compatibility equation was used to determine the viscoelastic

stress function. The transverse deflection was found by applying Galerkin's method to the moment equation. The resulting hereditary integral terms were evaluated using a composite numerical integration rule. The inplane loads were applied as creep loads at $t = 0$ in each case considered. Using a similar approach, quasi-elastic estimates for the viscoelastic deformation were also determined. The time history of viscoelastic deflection for both plates and cylindrical panels was shown to be sensitive to the size of the initial imperfection.

For composite plates, Huang³⁰ arrived at the following results. For loads large enough to cause instantaneous buckling, the viscoelastic deflection rate after buckling was relatively small. Thus, the internal moments resulting from the compressive edge loads and the transverse deflections varied slowly after load application. Hence, the quasi-elastic approach, which ignores load-history effects, gave very accurate results, because the internal loading approached a creep-loading situation for which the quasi-elastic method would give the exact viscoelastic result. For loads below the instantaneous buckling load, the accuracy of the quasi-elastic method depended upon the size of the initial imperfection. If the initial imperfection was relatively large, the viscoelastic deflection and, hence, the internal moments caused by the compressive edge loads increased rapidly after load application and then tapered off. Thus, the quasi-elastic method gave comparable results to those produced using the full viscoelastic analysis. If the initial imperfections were relatively small, a sharp viscoelastic buckling phenomenon was not indicated using the full viscoelastic analysis. Rather, the viscoelastic deflection grew relatively slowly after load application and then faster later on. However, for small initial imperfections, the quasi-elastic results still indicated a sharp buckling phenomenon for which a critical time could be defined. In general for the plate buckling problems studied, the quasi-elastic method overestimated the growth of the viscoelastic deflections, because it finds the current viscoelastic deflection by considering the internal load history to be the current internal loads applied as creep loads at $t = 0$. The amount of overestimation, of course, depended upon the applied external loading and the initial imperfection size.

Huang's study of cylindrical panels³¹ produced the following results. For creep loads

smaller than those necessary to cause snapping of an elastic panel with the initial viscoelastic properties, a delayed snapping phenomenon still may occur. The critical time to snap was found to be very sensitive to the load magnitude and initial imperfection size. For relatively small creep loads applied to panels with relatively large imperfections, snapping did not occur as the deflection progressed from the small to large deformation range. The quasi-elastic method was found to give good estimates for the viscoelastic deflection of panels with relatively large imperfections. For cylindrical panels with small imperfections, the quasi-elastic method greatly underestimated the critical time to snapping.

Touati and Cederbaum³² used Schapery's nonlinear viscoelastic material law³⁵ in studying the postbuckling response of laminated plates with initial imperfections. Geometric nonlinearity was included through the use of von Karman type strain-displacement relations. The nonlinear equilibrium equations were solved by first transforming them into a system of first-order nonlinear differential equations, which were then solved using Galerkin's method in conjunction with a higher-order Runge-Kutta method. The response of linear viscoelastic composite plates was determined by setting the nonlinear parameters in the material law to unity. The response of antisymmetric and symmetric cross-ply laminates subjected to uni-axial compression creep loads was studied. Using a nonlinear viscoelastic material law always led to larger deflections than using the corresponding linear viscoelastic constitutive equation. The magnitude of the difference between the deflections depended upon the number of layers, the ratio of the applied load to that necessary to cause instantaneous buckling, and the ratio of the plate thickness to the plate length.

Shalev and Aboudi³³ used higher-order shear deformation theory to study the postbuckling response of symmetric cross-ply laminated plates. The in-plane translations were taken to vary cubically through the thickness. Geometric nonlinearity was taken into account using von Karman type strain-displacement relations. Only creep loads large enough to cause instantaneous buckling of the viscoelastic plates were applied. The linear viscoelastic solution was developed using the correspondence principle³⁷ to find the solution in the Laplace domain. Numerical inversion was used to determine the time domain solution. As

expected, using higher-order shear deformation theory gave softer results for the viscoelastic deformations than did the classical lamination theory, which assumes no transverse shear deformation.

1.6 Scope of the Present Work

Although the main focus of the present work is the analysis of linear viscoelastic composite plates and shells, some additional work for isotropic beams and plane frames was performed in order to gain a preliminary understanding of numerical analysis of viscoelastic structures. In Chapter 2, several forms for the governing differential equation of a linear viscoelastic beam are developed in the time domain. Galerkin's method is then applied in the spatial domain and the resulting equations for the shape function amplitudes are presented. The numerical time integration scheme given by White⁶ is implemented with the appropriate hereditary integral form of the governing equations for the shape function amplitudes. For dynamic analysis, the inertia term is included and evaluated using the Newmark method. The numerical time integration scheme is then employed with a plane frame element.

In Chapter 3, the geometrically-linear, linear viscoelastic formulation of a triangular flat shell element for composites is presented. A direct integration scheme similar to that of Taylor *et al.*⁷ is employed to evaluate the hereditary integrals describing the linear viscoelastic behavior of hygrothermo-rheologically simple laminated plates and shells. Once again, the Newmark method is used for the inertia term when dynamic analysis is to be performed. Both deformation-independent mechanical loads and hygrothermal loads can be applied simultaneously.

The geometrically-nonlinear formulation of the triangular flat shell element for linear viscoelastic composites is developed in Chapter 4. Once again, a time integration scheme similar to that employed by Taylor *et al.*⁷ is implemented and the derivations of the internal load vector and tangent stiffness matrix are given. Some of the details of the iterative technique are then discussed. General load histories of point forces, deformation-dependent

pressure loads, and hygrothermal loads can be applied simultaneously.

Finally, a brief summary of the dissertation and some conclusions are presented in Chapter 5.

Chapter 2

Geometrically-Linear Analysis of Isotropic Linear Viscoelastic Beams and Plane Frames

In this chapter, the analysis of isotropic beams and plane frames is undertaken. The governing equation including inertia for a uniform isotropic viscoelastic beam is derived in several forms. The response in the spatial domain is first discretized using Galerkin's method. A numerical integration scheme for one form of the resulting time domain equations is presented. This time integration procedure is then applied to the decoupled version of the global finite element equations corresponding to a plane frame element. Numerical results for both quasi-static and dynamic analyses are presented along with exact solutions.

2.1 Derivation of Isotropic Beam Governing Equation

The governing equation for an isotropic viscoelastic beam, including inertia effects, will be derived using elementary Euler-Bernoulli beam theory. The positive senses for the coordinate axes, bending moment M , shear force V , distributed load p , and transverse deflection w are as shown in Fig. 2.1.

The derivation of the governing differential equation proceeds in a manner similar to that for the elastic case. Consider an elemental length of a beam which undergoes small deformations as it is subjected to pure bending as shown in Fig. 2.2. The beam is assumed to have a symmetric cross section with respect to the plane of bending. Because the strain on the top of the deformed element is compressive (negative), the strain on the bottom of the deformed element is tensile (positive), and the strain is continuous throughout the cross section, an axis will exist throughout the element where the strain is zero. This “neutral” axis is denoted as $\overline{AB}$ in the undeformed state and as $\overline{A'B'}$ in the deformed configuration. Consider also line $\overline{CD}$ which is at a distance “ z ” from the neutral axis. The length of the neutral axis in both the original and deformed states is given by $\rho\theta$ where ρ is the radius of curvature and θ is the angle subtended by the deformed element. The length of $\overline{C'D'}$ is simply $L_{C'D'} = (\rho - z)\theta$. The small strain corresponding to $\overline{CD}$ is then computed as

$$\epsilon_{CD} = \frac{L_{C'D'} - L_{CD}}{L_{CD}} = \frac{(\rho - z)\theta - \rho\theta}{\rho\theta} \quad (2.1)$$

Upon simplifying and recalling that the curvature $\kappa = 1/\rho$, ϵ_{CD} is given as

$$\epsilon_{CD} = -z\kappa \quad (2.2)$$

The curvature corresponding to the transverse deflection w at a given time can be found in elementary calculus books and is here given as

$$\kappa = \frac{\frac{\partial^2 w}{\partial x^2}}{\left[1 + \left(\frac{\partial w}{\partial x}\right)^2\right]^{3/2}} \quad (2.3)$$

For small $\partial w/\partial x$, the curvature is approximated as

$$\kappa \approx \frac{\partial^2 w}{\partial x^2} \quad (2.4)$$

Obviously, Eq. (2.2) holds for line $\overline{CD}$ at any distance “ z ” from the neutral axis. Hence, the subscript on ϵ_{CD} will be dropped and the bending strain is written as

$$\epsilon(x, t) = -z \frac{\partial^2 w}{\partial x^2}(x, t) \quad (2.5)$$

It should be noted that the usual assumption has been made that plane sections originally normal to the neutral axis remain planar and normal to the neutral axis throughout the deformation process.

Several forms for the governing equation will be derived based upon different, but equivalent, forms for the linear viscoelastic constitutive law written in the time domain.

2.1.1 Differential Form

The material law is written at a distance “ z ” from the neutral axis as

$$\underline{P}(\sigma) = \underline{Q}(\epsilon) = -\underline{Q}\left(z \frac{\partial^2 w}{\partial x^2}\right) \quad (2.6)$$

where

$$\underline{P}(\cdot) = \sum_{k=0}^m p_k \frac{\partial^k}{\partial t^k}(\cdot) \quad \text{and} \quad \underline{Q}(\cdot) = \sum_{k=0}^n q_k \frac{\partial^k}{\partial t^k}(\cdot) \quad (2.7)$$

and p_k and q_k are material constants. Multiplying both sides of Eq. (2.6) by z and integrating over the cross section gives

$$\int_A \underline{P}(\sigma) z dA = - \int_A z \underline{Q}\left(z \frac{\partial^2 w}{\partial x^2}\right) dA \quad (2.8)$$

where A denotes the area of the cross section. Because the $\underline{P}$ and $\underline{Q}$ operators do not involve spatial derivatives and $\partial^2 w / \partial x^2$ is constant at a given x -location, Eq. (2.8) can be rewritten as

$$\underline{P}(M) = I \underline{Q}\left(\frac{\partial^2 w}{\partial x^2}\right) \quad (2.9)$$

where M is the bending moment given by $\int_A (-\sigma z) dA$ and I is the area moment of inertia given by $\int_A z^2 dA$.

Writing the inertia force using D’Alembert’s principle, the following equations are derived from force and moment balance on an elemental length of the beam:

$$\frac{\partial V}{\partial x} = p - \bar{\rho} A \frac{\partial^2 w}{\partial t^2} \quad (2.10)$$

$$\frac{\partial M}{\partial x} = V \quad (2.11)$$

where $\bar{\rho}$ is the material density. Using Eqs. (2.10) and (2.11), it is easily shown that

$$\frac{\partial^2 M}{\partial x^2} = p - \bar{\rho} A \frac{\partial^2 w}{\partial t^2} \quad (2.12)$$

Using Eqs. (2.9), (2.11), and (2.12) and assuming the beam to have a constant cross section and uniform material properties, the following two equations are derived:

$$\underline{P}(V) = \underline{I}\underline{Q}\left(\frac{\partial^3 w}{\partial x^3}\right) \quad (2.13)$$

$$\underline{P}\left(p - \bar{\rho} A \frac{\partial^2 w}{\partial t^2}\right) = \underline{I}\underline{Q}\left(\frac{\partial^4 w}{\partial x^4}\right) \quad (2.14)$$

Hence, Eq. (2.14) is the governing differential equation for the beam given in terms of w , the distributed load p , the beam's material properties, and some geometric parameters of the beam. In order to accurately model a real viscoelastic material, the generalized Kelvin chain or the generalized Maxwell ladder must be used. Thus, the highest-order time derivative appearing in Eq. (2.14) will most likely be greater than two. This would lead to specification of initial conditions which include more than the usual initial conditions of position and velocity. Flügge³⁸ describes a process by which the proper initial conditions on the higher-order derivatives can be determined.

2.1.2 Hereditary Integral Forms

In place of Eq. (2.6), one of the following hereditary integral forms for the constitutive law can be used:

$$\sigma(x, t) = E(t)\epsilon(x, 0) + \int_0^t E(t - \tau) \frac{\partial \epsilon(x, \tau)}{\partial \tau} d\tau \quad (2.15)$$

$$\sigma(x, t) = E(0)\epsilon(x, t) - \int_0^t \frac{dE(t - \tau)}{d\tau} \epsilon(x, \tau) d\tau \quad (2.16)$$

where E is the relaxation modulus for the beam material.

Performing the same steps that were used to derive Eq. (2.9), the following equation for M is derived using Eq. (2.15):

$$M(x, t) = IE(t) \frac{\partial^2 w}{\partial x^2}(x, 0) + I \int_0^t E(t - \tau) \frac{\partial^3 w(x, \tau)}{\partial x^2 \partial \tau} d\tau \quad (2.17)$$

Using Eqs. (2.11) and (2.12) and, once again, considering a beam of constant cross section and uniform material properties, the following equations result:

$$V(x, t) = IE(t) \frac{\partial^3 w}{\partial x^3}(x, 0) + I \int_0^t E(t - \tau) \frac{\partial^4 w(x, \tau)}{\partial x^3 \partial \tau} d\tau \quad (2.18)$$

$$\bar{\rho}A \frac{\partial^2 w}{\partial t^2}(x, t) + IE(t) \frac{\partial^4 w}{\partial x^4}(x, 0) + I \int_0^t E(t - \tau) \frac{\partial^5 w(x, \tau)}{\partial x^4 \partial \tau} d\tau = p(x, t) \quad (2.19)$$

If the constitutive law is written in the form of Eq. (2.16) instead of Eq. (2.15), the following equations result:

$$M(x, t) = IE(0) \frac{\partial^2 w}{\partial x^2}(x, t) - I \int_0^t \frac{dE(t - \tau)}{d\tau} \frac{\partial^2 w}{\partial x^2}(x, \tau) d\tau \quad (2.20)$$

$$V(x, t) = IE(0) \frac{\partial^3 w}{\partial x^3}(x, t) - I \int_0^t \frac{dE(t - \tau)}{d\tau} \frac{\partial^3 w}{\partial x^3}(x, \tau) d\tau \quad (2.21)$$

$$\bar{\rho}A \frac{\partial^2 w}{\partial t^2}(x, t) + IE(0) \frac{\partial^4 w}{\partial x^4}(x, t) - I \int_0^t \frac{dE(t - \tau)}{d\tau} \frac{\partial^4 w}{\partial x^4}(x, \tau) d\tau = p(x, t) \quad (2.22)$$

Using Eq. (2.15) or (2.16), the governing equation in terms of the transverse deflection, the applied distributed load, material properties, and geometric parameters for a uniform, isotropic beam is given by Eq. (2.19) or Eq. (2.22), respectively. In order to solve Eq. (2.19) or Eq. (2.22) exactly, only the beam's initial position and velocity must be specified.

2.2 Galerkin Analysis

2.2.1 Derivation of Time-Domain Equations

A method for solving Eq. (2.14), (2.19), or (2.22) along with the associated boundary and initial conditions is sought. One approach is to express the transverse deflection w by an expansion of the form

$$w(x, t) = \sum_{n=1}^{N_n} a_n(t) \phi_n(x) \quad (2.23)$$

where $\phi_n(x)$ is the n^{th} shape function and $a_n(t)$ is the corresponding time-dependent amplitude. Then using Eq. (2.23), Galerkin's method is applied for the spatial domain. The spatially-dependent terms will be integrated-out and the resulting equations will involve only

the a_n 's. If the ϕ_n 's are orthogonal to one another, the set of equations will be uncoupled. The uncoupled equations in the time domain can then be solved either exactly or numerically.

In the present work, governing equations for the a_n 's will be derived using Eqs. (2.14), (2.19), and (2.22). For convergence using Galerkin's method, the $\phi_n(x)$ must satisfy all boundary conditions, be linearly independent, and form a complete set.³⁹ A complete set of ϕ_n 's is one in which none of the admissible lower-order terms have been excluded in the solution process. For example, if $\phi_n(x) = x^n$, all n which lead to satisfaction of all boundary conditions up to the highest n chosen must be included in developing the approximate solution.

Here, the shape functions are chosen to be orthonormal. That is,

$$\int_0^L \phi_n(x) \phi_m(x) dx = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases} \quad (2.24)$$

The eigenvalues associated with the spatial eigenfunctions are defined by

$$\phi_n''''(x) = \lambda_n \phi_n(x) \quad (2.25)$$

where $(\cdot)'$ is used to represent $d(\cdot)/dx$.

Galerkin's method will now be applied to Eq. (2.22). The expansion for $w(x, t)$ given by Eq. (2.23) (with the index n changed to m) is substituted into Eq. (2.22). However, the governing equation will not be satisfied on a pointwise basis. Rather, substituting Eq. (2.23) into Eq. (2.22) will result in a residual R given by

$$\begin{aligned} R = & \bar{\rho}A \left(\sum_{m=1}^{N_n} \ddot{a}_m(t) \phi_m(x) \right) - I \int_0^t \frac{dE(t-\tau)}{d\tau} \left(\sum_{m=1}^{N_n} a_m(\tau) \phi_m''''(x) \right) d\tau \\ & + IE(0) \left(\sum_{m=1}^{N_n} a_m(t) \phi_m''''(x) \right) - p(x, t) \end{aligned} \quad (2.26)$$

where $(\cdot)$ denotes differentiation with respect to time. In Galerkin's method, the residual R is made orthogonal to each of the chosen shape functions. That is, the N_n equations needed to solve for the a_n 's are determined by multiplying R by $\phi_n(x)$, integrating over the domain, and setting the result to zero, where n takes the integer values from 1 to N_n successively.

Using Eqs. (2.24) and (2.25), the governing equation for $a_n(t)$ is found to be

$$\bar{\rho}A\ddot{a}_n(t) - \lambda_n I \int_0^t \frac{dE(t-\tau)}{d\tau} a_n(\tau) d\tau + \lambda_n I E(0) a_n(t) = \int_0^L p(x, t) \phi_n(x) dx \quad (2.27)$$

The initial conditions on $a_n(t)$ can easily be shown to be

$$\left. \frac{d^k a_n}{dt^k} \right|_{t=0} = \int_0^L \frac{\partial^k w}{\partial t^k}(x, 0) \phi_n(x) dx \quad \text{for } k = 0, 1, 2, \dots \quad (2.28)$$

Performing the same steps on Eqs. (2.14) and (2.19), alternate governing equations for $a_n(t)$ are derived to be

$$\bar{\rho}A \underline{P}(\ddot{a}_n(t)) + \lambda_n I \underline{Q}(a_n(t)) = \underline{P} \left(\int_0^L p(x, t) \phi_n(x) dx \right) \quad (2.29)$$

$$\bar{\rho}A\ddot{a}_n(t) + \lambda_n I \int_0^t E(t-\tau) \frac{da_n(\tau)}{d\tau} d\tau + \lambda_n I E(t) a_n(0) = \int_0^L p(x, t) \phi_n(x) dx \quad (2.30)$$

Hence, because orthogonal shapes are used, the $a_n(t)$ are determined independent of one another using either Eq. (2.27), (2.29), or (2.30) with the initial conditions given by Eq. (2.28).

2.2.2 Numerical Solution of One Form of the Time-Domain Equations

A numerical solution will be developed for Eq. (2.27) using the Newmark method for the inertia term and a composite numerical integration formula for the hereditary integral. Although Eq. (2.27) can be solved exactly using Laplace transforms, the solution technique to be employed here can be implemented with the finite element method in a straightforward manner.

The solution for a_n is assumed to have been determined up to time t^{p-1} . The solution one time step later (at $t = t^p$) is to be found. Without loss of generality, a single shape will be used from this point forward and the subscript n will be dropped from $a_n(t)$, $\phi_n(x)$, and λ_n .

First, Eq. (2.27) is written for $t = t^p$ as

$$\bar{\rho}A\ddot{a}(t^p) - \lambda I \int_0^{t^p} \frac{dE(t^p - \tau)}{d\tau} a(\tau) d\tau + \lambda I E(0) a(t^p) = \int_0^L p(x, t^p) \phi(x) dx \quad (2.31)$$

The hereditary integral will be approximated using the method employed by White.⁶ A composite numerical integration technique is used with the hereditary integral rewritten as

$$\int_0^{t^p} \frac{dE(t^p - \tau)}{d\tau} a(\tau) d\tau = \sum_{i=1}^{p-1} \int_{t^i}^{t^{i+1}} \frac{dE(t^p - \tau)}{d\tau} a(\tau) d\tau \quad (2.32)$$

where $t^1 = 0$. For the evaluation of the hereditary integral over the time step from t^i to t^{i+1} , the appropriate mean-value theorem is used, with the average value of $a(t)$ taken as that corresponding to a linear variation in $a(t)$ over the time interval.⁴ This gives

$$\begin{aligned} \int_{t^i}^{t^{i+1}} \frac{dE(t^p - \tau)}{d\tau} a(\tau) d\tau &\approx \frac{1}{2} (a(t^i) + a(t^{i+1})) \int_{t^i}^{t^{i+1}} \frac{dE(t^p - \tau)}{d\tau} d\tau \\ &= \frac{1}{2} (a(t^i) + a(t^{i+1})) (E(t^p - t^{i+1}) - E(t^p - t^i)) \end{aligned} \quad (2.33)$$

Hence, the governing equation for $a(t)$ is now

$$\begin{aligned} \bar{\rho} A \ddot{a}(t^p) - \lambda I \left[\sum_{i=1}^{p-1} \left(\frac{a(t^i) + a(t^{i+1})}{2} \right) (E(t^p - t^{i+1}) - E(t^p - t^i)) \right] \\ + \lambda I E(0) a(t^p) = \int_0^L p(x, t^p) \phi(x) dx \end{aligned} \quad (2.34)$$

The Newmark method will be used to replace the current value of $\ddot{a}$ by a finite difference expression. Uniform time steps will be used in marching the viscoelastic solution in time. The current acceleration is determined by rearranging the following equation:

$$a(t^p) = a(t^{p-1}) + \Delta t \dot{a}(t^{p-1}) + \frac{(\Delta t)^2}{2} [(1 - \gamma) \ddot{a}(t^{p-1}) + \gamma \ddot{a}(t^p)] \quad (2.35)$$

The current value of $\dot{a}$ is determined as follows:

$$\dot{a}(t^p) = \dot{a}(t^{p-1}) + \Delta t [(1 - \psi) \ddot{a}(t^{p-1}) + \psi \ddot{a}(t^p)] \quad (2.36)$$

It should be apparent that Eqs. (2.35) and (2.36) can be derived by using truncated Taylor series for $a(t^p)$ and $\dot{a}(t^p)$, respectively, with $\ddot{a}(t^{p-1})$ replaced by weighted averages of $\ddot{a}(t^{p-1})$ and $\ddot{a}(t^p)$. Reference 39 gives the values of γ and ψ corresponding to different methods. For instance, $\gamma = \psi = 1/2$ gives the constant average acceleration method, which is unconditionally stable for the elastic case.

Using the Newmark method, Eq. (2.34) can be rewritten as

$$Ka(t^p) = J^p \quad (2.37)$$

The constant K is determined as

$$K = \frac{2\bar{\rho}A}{\gamma(\Delta t)^2} + \frac{\lambda I}{2} (E(0) + E(\Delta t)) \quad (2.38)$$

On the other hand, J^p can be shown to be

$$\begin{aligned} J^p = & \int_0^L p(x, t^p) \phi(x) dx + \bar{\rho}A \left[\frac{2}{\gamma(\Delta t)^2} a(t^{p-1}) + \frac{2}{\gamma\Delta t} \dot{a}(t^{p-1}) + \left(\frac{1}{\gamma} - 1\right) \ddot{a}(t^{p-1}) \right] \\ & + \frac{\lambda I}{2} \left[\sum_{i=1}^{p-2} (a(t^i) + a(t^{i+1})) (E(t^p - t^{i+1}) - E(t^p - t^i)) + a(t^{p-1})(E(0) - E(\Delta t)) \right] \end{aligned} \quad (2.39)$$

The relaxation modulus is expressed as a Prony series as follows:

$$E(t) = E^\infty + \sum_{\rho=1}^N E_\rho e^{-\frac{t}{\lambda_\rho}} \quad (2.40)$$

Using this series expansion, the summation term in J^p can be simplified and ultimately written as a recurrence relation. That is,

$$\begin{aligned} & \sum_{i=1}^{p-2} \left[\frac{a(t^i) + a(t^{i+1})}{2} \right] [E(t^p - t^{i+1}) - E(t^p - t^i)] \\ &= \sum_{i=1}^{p-2} \sum_{\rho=1}^N E_\rho \left[\frac{a(t^i) + a(t^{i+1})}{2} \right] \left[\exp\left(\frac{-t^p + t^{i+1}}{\lambda_\rho}\right) - \exp\left(\frac{-t^p + t^i}{\lambda_\rho}\right) \right] \\ &= \sum_{\rho=1}^N \sum_{i=1}^{p-2} E_\rho \exp\left(\frac{-t^p}{\lambda_\rho}\right) \left[\frac{a(t^i) + a(t^{i+1})}{2} \right] \left[\exp\left(\frac{t^{i+1}}{\lambda_\rho}\right) - \exp\left(\frac{t^i}{\lambda_\rho}\right) \right] \\ &= \sum_{\rho=1}^N E_\rho \alpha_\rho^p \end{aligned} \quad (2.41)$$

where obviously

$$\alpha_\rho^p = \sum_{i=1}^{p-2} \exp\left(\frac{-t^p}{\lambda_\rho}\right) \left[\frac{a(t^i) + a(t^{i+1})}{2} \right] \left[\exp\left(\frac{t^{i+1}}{\lambda_\rho}\right) - \exp\left(\frac{t^i}{\lambda_\rho}\right) \right] \quad (2.42)$$

A recurrence relation for α_ρ^p is developed as follows:

$$\begin{aligned} \alpha_\rho^p = & \exp\left(\frac{-t^p}{\lambda_\rho}\right) \left[\exp\left(\frac{t^{p-1}}{\lambda_\rho}\right) - \exp\left(\frac{t^{p-2}}{\lambda_\rho}\right) \right] \left[\frac{a(t^{p-2}) + a(t^{p-1})}{2} \right] \\ & + \sum_{i=1}^{p-3} \exp\left(\frac{-t^p}{\lambda_\rho}\right) \left[\exp\left(\frac{t^{i+1}}{\lambda_\rho}\right) - \exp\left(\frac{t^i}{\lambda_\rho}\right) \right] \left[\frac{a(t^i) + a(t^{i+1})}{2} \right] \end{aligned} \quad (2.43)$$

Furthermore,

$$\begin{aligned} \alpha_\rho^p = & \exp\left(\frac{-t^p + t^{p-1}}{\lambda_\rho}\right) \left[1 - \exp\left(\frac{-t^{p-1} + t^{p-2}}{\lambda_\rho}\right) \right] \left[\frac{a(t^{p-2}) + a(t^{p-1})}{2} \right] \\ & + \exp\left(\frac{-t^p + t^{p-1}}{\lambda_\rho}\right) \sum_{i=1}^{p-3} \exp\left(\frac{-t^{p-1}}{\lambda_\rho}\right) \left[\exp\left(\frac{t^{i+1}}{\lambda_\rho}\right) - \exp\left(\frac{t^i}{\lambda_\rho}\right) \right] \left[\frac{a(t^i) + a(t^{i+1})}{2} \right] \end{aligned} \quad (2.44)$$

Finally, the recurrence relation is

$$\alpha_\rho^p = \exp\left(-\frac{\Delta t}{\lambda_\rho}\right) \left[\left(1 - \exp\left(-\frac{\Delta t}{\lambda_\rho}\right)\right) \left(\frac{a(t^{p-2}) + a(t^{p-1})}{2}\right) + \alpha_\rho^{p-1} \right] \quad (2.45)$$

Thus, the formula for J^p is now written as

$$\begin{aligned} J^p = & \int_0^L p(x, t^p) \phi(x) dx + \bar{\rho} A \left[\frac{2}{\gamma(\Delta t)^2} a(t^{p-1}) + \frac{2}{\gamma \Delta t} \dot{a}(t^{p-1}) + \left(\frac{1}{\gamma} - 1\right) \ddot{a}(t^{p-1}) \right] \\ & + \lambda I \left[\sum_{\rho=1}^N E_\rho \alpha_\rho^p + \frac{1}{2} a(t^{p-1}) (E(0) - E(\Delta t)) \right] \end{aligned} \quad (2.46)$$

Examining the governing equation for shape amplitude a given in Eq. (2.31), the physical problem requires initial conditions on a and $\dot{a}$. However, because the Newmark method is being employed, starting values of a , $\dot{a}$, and $\ddot{a}$ are needed. If $\partial w^2 / \partial t^2$ is known at the initial time, the starting value of $\ddot{a}$ can be determined using Eq. (2.28) with $k = 2$. However, if $\partial w^2 / \partial t^2$ is unknown at $t = 0$, then $\ddot{a}(0)$ can be determined by applying Eq. (2.31) at $t = 0$ with the result that

$$\ddot{a}(0) = \frac{1}{\bar{\rho} A} \left[\int_0^L p(x, 0) \phi(x) dx - \lambda I E(0) a(0) \right] \quad (2.47)$$

The solution procedure to obtain $a(t)$ involves four steps. First, the time step size and the Newmark parameters γ and ψ are chosen and K is calculated from Eq. (2.38). Next, $a(0)$

and $\dot{a}(0)$ and possibly $\ddot{a}(0)$ are initialized using Eq. (2.28). If needed, the condition on $\ddot{a}(0)$ is determined using Eq. (2.47). In the third step, α_ρ^1 and α_ρ^2 are set to zero. Furthermore, α_ρ^3 can be computed from Eq. (2.45) with $p = 3$. Finally, at each time step, the following quantities are computed in order: α_ρ^p from Eq. (2.45); J^p from Eq. (2.46); $a(t^p)$ from Eq. (2.37); $\ddot{a}(t^p)$ from Eq. (2.35); and $\dot{a}(t^p)$ from Eq. (2.36).

Although not given in detail here, $\alpha_\rho^1 = 0$ and $\alpha_\rho^2 = 0$ can be shown to be correct by using them in the procedure detailed above and comparing the results to those computed using Eq. (2.34) with the current acceleration and velocity respectively determined using Eqs. (2.35) and (2.36).

The quasi-static solution (i.e., the solution ignoring inertia effects) is developed in the same manner as described above, by setting the density to zero in Eqs. (2.38) and (2.46). Furthermore, it is unnecessary to keep track of the velocity and acceleration using Eqs. (2.35) and (2.36).

The steps given above for dynamic and quasi-static Galerkin analyses have been coded using FORTRAN 77.

2.3 Plane Frame Finite Element Analysis

A viscoelastic plane frame element will be developed. The straight, two-node element is obtained by combining a linear bar element and a cubic beam element. The bar and beam element equations are derived in a local coordinate system. In the local coordinate system, the bar has two degrees of freedom (DOF) corresponding to the axial displacements at the two end nodes, whereas the cubic beam element has four DOF corresponding to the transverse deflection and rotation at each of the end nodes. Because Euler-Bernoulli beam theory is used, the rotations are expressed in terms of spatial derivatives of the transverse deflection of the beam. The combined element is then made valid for structural members at an arbitrary angle to the horizontal by transforming the element equations to the global coordinate system. The frame element code is restricted to the case where all elements

lie in a single plane. All together, the frame element has six DOF, corresponding to two translations and one rotation at each node.

Henceforth, the viscoelastic frame element code will be termed VFRAME. The VFRAME code can be used to model inextensible frame structures by prescribing a large axial rigidity. Also, the VFRAME code can be used to model truss structures by setting the area moment of inertia to zero to give the correct element stiffness matrices and then prescribing the rotational degrees of freedom to be zero. In the present formulation, only structures which have material homogeneity and homogeneous boundary conditions will be considered.

For an elastic structure, the one-dimensional constitutive law is written as

$$\sigma(x, t) = E\epsilon(x, t) \quad (2.48)$$

Using this material law, an elastic frame element can be derived. The derivation is straightforward and is not presented here. The interested reader is directed to Ref. 40 for the formulation details. After assemblage and imposition of boundary conditions, the resulting finite element equations are of the following form:

$$[M]\{\ddot{U}(t)\} + E[K]\{U(t)\} = \{Q(t)\} \quad (2.49)$$

where $[M]$ and $[K]$ are the reduced global mass and stiffness matrices, $\{U\}$ is the reduced vector of global displacements, and $\{Q\}$ is the reduced global force vector for mechanical loads. Here, a consistent mass matrix is used as opposed to a lumped mass matrix. Young's modulus E has been factored-out from the stiffness matrix in order to allow for easy derivation of the corresponding linear viscoelastic frame element equations.

If instead of Eq. (2.48), Eq. (2.16) is used, the resulting global finite element equations are for the linear viscoelastic case and are given by

$$[M]\{\ddot{U}(t)\} + E(0)[K]\{U(t)\} - \int_0^t \frac{dE(t-\tau)}{d\tau} [K]\{U(\tau)\} d\tau = \{Q(t)\} \quad (2.50)$$

where $[K]$ and $[M]$ are the same as those in Eq. (2.49). The goal is to decouple the system of equations so that the code developed for the time integration of the Galerkin amplitudes can be employed in marching the finite element equations in time.

Modal decomposition will first be derived for the elastic case. The decomposition procedure can then be used to decouple the viscoelastic system of equations, since $[M]$ and $[K]$ are the same in Eqs. (2.49) and (2.50).

For the elastic case, the modal frequencies and the corresponding mode shapes are derived as follows. In order to easily decouple both the elastic and viscoelastic systems, the modal frequencies and mode shapes are computed for an elastic structure with $E = 1$. First, the following substitution is made in the homogeneous form of Eq. (2.49) with $E = 1$:

$$\{U\} = \{\phi_r\} e^{i\omega_r t} \quad (2.51)$$

where $\{\phi_r\}$ is the r^{th} modal shape vector, ω_r is the r^{th} modal frequency, and i represents $\sqrt{-1}$. After dividing by $\exp(i\omega_r t)$, the resulting generalized eigenvalue problem is

$$[K]\{\phi_r\} = \omega_r^2 [M]\{\phi_r\} \quad (2.52)$$

Hence, ω_r is determined from

$$|[K] - \omega_r^2 [M]| = 0 \quad (2.53)$$

while $\{\phi_r\}$ is determined from

$$([K] - \omega_r^2 [M]) \{\phi_r\} = \{0\} \quad (2.54)$$

The eigenvalues and eigenvectors are arranged into ascending order (i.e., $\omega_1^2 \leq \omega_2^2 \leq \dots \leq \omega_{N_{eq}-1}^2 \leq \omega_{N_{eq}}^2$, where N_{eq} denotes the size of the system of equations).

It can be shown fairly easily for the case where no repeated eigenvalues occur that the modes are orthogonal with respect to the symmetric stiffness and mass matrices.⁴¹ Even if repeated eigenvalues occur, orthogonal modes can still be chosen.⁴¹ Orthogonality of the modes with respect to the stiffness and mass matrices is expressed mathematically as

$$\{\phi_r\}^T [K] \{\phi_s\} = \begin{cases} k_r & s = r \\ 0 & s \neq r \end{cases} \quad (2.55a)$$

$$\{\phi_r\}^T [M] \{\phi_s\} = \begin{cases} m_r & s = r \\ 0 & s \neq r \end{cases} \quad (2.55b)$$

where k_r and m_r are the r^{th} modal stiffness and mass, respectively, for the elastic case with $E = 1$. It can also be shown that⁴¹

$$k_r = \omega_r^2 m_r \quad (2.56)$$

Both the elastic and viscoelastic systems can be decoupled based upon the following equations:

$$[\Phi]^T [K] [\Phi] = [\bar{K}] \quad (2.57a)$$

$$[\Phi]^T [M] [\Phi] = [\bar{M}] \quad (2.57b)$$

where $[\Phi]$ is the modal matrix composed of the modal shape vectors arranged as columns and $[\bar{K}]$ and $[\bar{M}]$ are diagonal matrices with the i^{th} diagonal entry equal to k_i and m_i , respectively. To decouple the system of equations, the vector of nodal displacements $\{U(t)\}$ is rewritten in terms of principal coordinates $\{\eta(t)\}$ as follows:

$$\{U(t)\} = [\Phi] \{\eta(t)\} \quad (2.58)$$

Substituting Eq. (2.58) into Eq. (2.49) and pre-multiplying that result by $[\Phi]^T$ produces the following decoupled linear elastic system of equations:

$$[\bar{M}] \{\ddot{\eta}(t)\} + E[\bar{K}] \{\eta(t)\} = \{\bar{Q}\} \quad (2.59)$$

where the modal load vector $\{\bar{Q}\}$ is given by

$$\{\bar{Q}\} = [\Phi]^T \{Q\} \quad (2.60)$$

Likewise, substituting Eq. (2.58) into Eq. (2.50) and pre-multiplying by $[\Phi]^T$ produces the decoupled set of linear viscoelastic finite element equations which are as follows:

$$[\bar{M}] \{\ddot{\eta}(t)\} + E(0)[\bar{K}] \{\eta(t)\} - \int_0^t \frac{dE(t-\tau)}{d\tau} [\bar{K}] \{\eta(\tau)\} d\tau = \{\bar{Q}\} \quad (2.61)$$

The correct initial conditions on $\{\eta\}$ and $\{\dot{\eta}\}$ for the elastic or viscoelastic case are determined using

$$\{\eta(0)\} = [\Phi]^{-1} \{U(0)\} \quad (2.62a)$$

$$\{\dot{\eta}(0)\} = [\Phi]^{-1} \{\dot{U}(0)\} \quad (2.62b)$$

From Eq. (2.61), the governing equation for the j^{th} principal coordinate for the linear viscoelastic structure at time t^p is

$$m_j \ddot{\eta}_j(t^p) + E(0)k_j \eta_j(t^p) - k_j \int_0^{t^p} \frac{dE(t^p - \tau)}{d\tau} \eta_j(\tau) d\tau = q_j(t^p) \quad (2.63)$$

where q_j is the j^{th} component of $\{\bar{Q}\}$. Hence, the numerical solution technique developed for Eq. (2.31) can be used with the following substitutions made:

$$\begin{aligned} a(t^p) &\rightarrow \eta_j(t^p) & \int_0^L p(x, t^p) \phi(x) dx &\rightarrow q_j(t^p) \\ \bar{\rho}A &\rightarrow m_j & \lambda I &\rightarrow k_j \end{aligned} \quad (2.64)$$

If $\{\ddot{U}(0)\}$ is known, the starting value of $\ddot{\eta}_j$ to be used is determined as follows:

$$\{\ddot{\eta}(0)\} = [\Phi]^{-1} \{\ddot{U}(0)\} \quad (2.65)$$

If, however, $\{\ddot{U}(0)\}$ is unknown, the starting value of $\ddot{\eta}_j$ is determined by applying Eq. (2.63) at $t = 0$. Once $\{\eta(t)\}$ is known for the time range of interest, the vector of nodal displacements is recovered using Eq. (2.58).

The structural response is a superposition of the modal responses. Recall that k_r is related to m_r and ω_r as given in Eq. (2.56). It should be noted that ω_r has little physical meaning for the linear viscoelastic case, other than it is the r^{th} modal frequency for a similar elastic structure with $E = 1$. For a given external loading, the various modes may have differing degrees of importance and it may be possible to produce an accurate solution without considering all the modes. For the present code, all of the modes with modal frequencies up to a given level N^* will be retained in Eq. (2.58), where $N^* \leq N_{eq}$. The required number of modes needed for accurate calculations is, of course, problem dependent.

Although the formulation given above was implemented for the case of a plane frame element, it should be apparent that the formulation could be applied to any isotropic element that will be used to model a homogeneous linear viscoelastic structure undergoing small deformations.

2.4 Numerical Examples

Numerical and exact results are presented for two examples. In the first example, both the quasi-static and dynamic responses of a simply-supported beam subjected to a distributed creep load are determined. The second example is an arch under a vertical creep loading applied over its top. The effect of varying the length over which the load is applied is examined. The numerical results are shown to be very accurate.

2.4.1 Beam Under Uniform Distributed Load

Consider a beam under a uniform distributed load. The beam has a length L of 4.0 m, a width b of 0.12 m, and a height h of 0.2 m. The isotropic beam has a relaxation modulus which is represented using a three-parameter solid model as follows:

$$E(t) = 1.96 \times 10^7 + 7.84 \times 10^7 e^{-t/2.24} \text{ N/m}^2 \quad (2.66)$$

where t is in seconds. The material is taken to have a density of 1200 kg/m³.

All initial conditions for the beam are zero. That is, the transverse deflection $w(x, t)$ and all its derivatives with respect to time are zero at $t = 0$. The uniform distributed load has a magnitude $\bar{p} = 3 \text{ N/m}$ and is applied as a creep load at $t = 0$ (i.e., the load is applied suddenly at $t = 0$ and then held constant). The pinned boundary conditions on $w(x, t)$ and the internal bending moment $M(x, t)$ are

$$w(0, t) = w(L, t) = 0 \quad \text{and} \quad M(0, t) = M(L, t) = 0 \quad (2.67)$$

Both quasi-static and dynamic results will be computed using the Galerkin analysis and VFRAME codes. The exact solution for the quasi-static case under creep loading applied at $t = 0$ can be computed using the correspondence principle,³⁸ with the following result:

$$w(x, t) = m(x) D(t) \quad (2.68)$$

where $m(x)$ is the solution for a similar elastic structure with $E = 1$ and $D(t)$ is the creep compliance corresponding to $E(t)$ given in Eq. (2.66). For this problem, $m(x)$ is easily

determined and the transverse deflection is given as

$$w(x, t) = \frac{\bar{P}}{24I}(x^4 - 2Lx^3 + L^3x)D(t) \quad (2.69)$$

where I is the area moment of inertia ($bh^3/12$). Evaluating Eq. (2.69) at $x = L/2$ gives the exact midpoint transverse deflection for quasi-static analysis as

$$w_{mid}(t) = 6.3775 \times 10^{-3} - 5.1020 \times 10^{-3} e^{-t/11.2} \text{ m} \quad (2.70)$$

where t is once again in seconds.

For Galerkin analysis, $w(x, t)$ is expressed in the form given in Eq. (2.23). The shape functions are chosen as

$$\phi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L} \quad (2.71)$$

It should be clear that the ϕ_n 's result in a $w(x, t)$ which satisfies all of the boundary conditions listed in Eq. (2.67). Also, the ϕ_n 's are linearly independent and form a complete set, as required for convergence. Furthermore, the ϕ_n 's are orthonormal. Here, the eigenvalue associated with $\phi_n(x)$ defined by Eq. (2.25) is simply

$$\lambda_n = \left(\frac{n\pi}{L}\right)^4 \quad (2.72)$$

It should be clear that since all of the initial conditions on w and its derivatives are zero, all of the initial conditions on a_n and its derivatives are also zero.

The governing equation for the n^{th} shape amplitude $a_n(t)$ is given by Eq. (2.27), which for this example reduces to

$$28.8 \ddot{a}_n(t) - 1065.4 n^4 \int_0^t e^{-(t-\tau)/2.24} a_n(\tau) d\tau + 2983.12 n^4 a_n(t) = \frac{2.70095}{n} [(-1)^{n+1} + 1] u(t) \quad (2.73)$$

where $u(t)$ is the unit step function. Examining Eq. (2.73), it should be obvious that $a_n(t)$ is zero for even values of n , because not only are the initial conditions zero, but the forcing function is also zero in those cases. Hence, only odd values of n need to be considered in determining the Galerkin analysis solution.

An exact solution can be developed for $a_n(t)$ by taking the Laplace transform of Eq. (2.73), solving for $\hat{a}_n(s)$ (the Laplace transform of $a_n(t)$ with s as the Laplace variable), determining the partial fraction expansion for $\hat{a}_n(s)$, and then taking the inverse Laplace transform. Here, these calculations were performed using the commercial software package Mathematica.⁴² An approximate solution for $a_n(t)$ with and without the inclusion of inertia effects is computed using the methodology described in Subsection 2.2.2.

It should be clear from Eq. (2.73) that as n increases, the importance of the n^{th} shape will decrease. For the case where the inertia term is included and Eq. (2.73) is solved using Laplace transforms, the solution produced using a total of five terms (a_1 , a_3 , a_5 , a_7 , and a_9) matched the results produced using a total of 20 terms. Hence, a total of five terms will be used for Galerkin analysis and the dynamic solution for w_{mid} produced using Laplace transforms with Galerkin analysis will be considered to be exact.

The quasi-static and dynamic Galerkin analysis results for the midpoint transverse deflection w_{mid} are shown in Fig. 2.3. The numerical solutions are computed with a time step Δt of 0.01 seconds. Also shown in Fig. 2.3 are the exact quasi-static results produced using the correspondence principle and the exact dynamic results produced using Laplace transforms with Galerkin analysis. The differences between the numerical results and the exact results shown in Fig. 2.3 cannot be distinguished.

The dynamic results oscillate about the quasi-static results with an amplitude of vibration that decreases as time evolves, because of the viscoelastic damping. In general, the amplitude of the oscillations depends upon the material density, the rate at which the loading is applied, and the amount of viscoelastic damping.

This example is also analyzed using ten VFRAME elements. For a straight beam undergoing small deformations, the equations governing the axial and transverse motion are uncoupled. After the boundary conditions are imposed, the size of the viscoelastic finite element equations governing the transverse motion is 20x20. Hence, up to 20 modes can be used in determining the finite element transverse response. However, it is found that a total of 5 modes is sufficient to produce accurate results for w_{mid} in both the quasi-static and

dynamic cases. The VFRAME results produced using 5 modes and a time step of 0.01 sec are shown in Fig. 2.4, along with the exact quasi-static and dynamic solutions. Once again, the differences between the exact and numerical results cannot be distinguished.

2.4.2 Arch Under Symmetric Loading

The arch has a circular cross-section with an inner radius of 0.28 in. and a thickness of 0.02 in. The arch centerline is semi-circular with a radius R of 60.0 in. The ends of the arch are fully clamped. The arch is taken to be composed of a material with the following relaxation modulus:

$$E(t) = \left(26.9920 + 1.8201 e^{-t} + 1.1612 e^{-\frac{t}{10}} + 1.7135 e^{-\frac{t}{100}} + 1.6264 e^{-\frac{t}{1000}} \right) \times 10^7 \text{ psi} \quad (2.74)$$

where t is in hours.

The arch is subjected to a total vertical load of $P_0 = 20.0$ lb distributed over its top from $\theta = -\theta_t$ to $\theta = \theta_t$ where θ is as shown in Fig. 2.5. The following values of θ_t are considered: 0° , 5° , 10° , and 15° . Note that $\theta_t = 0^\circ$ corresponds to the case of a point load applied at the arch apex. In the other cases, the force per unit of horizontal projected length is uniform and has a magnitude p which is determined from the total vertical load as follows:

$$p = \frac{P_0}{2R \sin \theta_t} \quad (2.75)$$

The corresponding force per unit arc length is simply $p \cos \theta$. In each case, the load is applied suddenly at $t = 0$ and then held constant. Only the quasi-static responses will be determined.

The arch deflections in the x - and y -directions are denoted by u and v , respectively, while the rotation of the arch centerline is denoted by θ_z . The vertical deflection of the arch apex, v_{mid} , can be determined exactly using Castigliano's theorem in conjunction with the correspondence principle. Because of the symmetry in the geometry and loading of the arch, it is only necessary to model one-half of the arch as shown in Fig. 2.6. For the exact solution, it is necessary to consider the reaction force S_0 and moment M_0 corresponding to

the symmetry boundary conditions of

$$u|_{\theta=0} = 0 \quad (2.76)$$

$$\theta_z|_{\theta=0} = 0 \quad (2.77)$$

Values for S_0 and M_0 will be determined as part of the solution process. Furthermore, in order to determine v_{mid} , a vertical force must be applied at the arch apex. For $\theta_t = 0^\circ$, the vertical force F shown in Fig. 2.6 is simply P_0 , while for the other values of θ_t , F is a fictitious force.

The total strain energy for the corresponding elastic structure is

$$U = \int_0^{\pi/2} \frac{M^2}{2EI} R d\theta + \int_0^{\pi/2} \frac{S^2}{2EA} R d\theta \quad (2.78)$$

where E is the Young's modulus, I is the area moment of inertia, A is the cross-sectional area, and M and S respectively are the internal bending moment and axial force distributions in the arch. The positive sign conventions for M and S are as shown in Fig. 2.7. Using simple equilibrium considerations, the bending moment distribution is

$$M = \begin{cases} M_0 + S_0 R(1 - \cos \theta) - F R \sin \theta - \frac{1}{2} p R^2 \sin^2 \theta & 0 \leq \theta \leq \theta_t \\ M_0 + S_0 R(1 - \cos \theta) - F R \sin \theta - p R^2 (\sin \theta_t \sin \theta - \frac{1}{2} \sin^2 \theta_t) & \theta_t \leq \theta \leq \pi/2 \end{cases} \quad (2.79)$$

while the axial force distribution is

$$S = \begin{cases} -S_0 \cos \theta - F \sin \theta - p R \sin^2 \theta & 0 \leq \theta \leq \theta_t \\ -S_0 \cos \theta - F \sin \theta - p R \sin \theta_t \sin \theta & \theta_t \leq \theta \leq \pi/2 \end{cases} \quad (2.80)$$

Equations (2.79) and (2.80) are substituted into Eq. (2.78) and the equations necessary to solve for S_0 and M_0 are determined by enforcing Eqs. (2.76) and (2.77) as follows:

$$u|_{\theta=0} = \frac{\partial U}{\partial S_0} = 0 \quad (2.81)$$

$$\theta_z|_{\theta=0} = -\frac{\partial U}{\partial M_0} = 0 \quad (2.82)$$

After M_0 and S_0 have been found, the elastic solution for v_{mid} is calculated as follows:

$$v_{mid} = \begin{cases} -\frac{\partial U}{\partial F} \Big|_{F=P_0, p=0} & \theta_t = 0^\circ \\ -\frac{\partial U}{\partial F} \Big|_{F=0, p=P_0/(2R \sin \theta_t)} & \theta_t = 5^\circ, 10^\circ, 15^\circ \end{cases} \quad (2.83)$$

The elastic solutions for the various loading cases are given in Table 2.1. The results given in Table 2.1 indicate that the magnitude of the center deflection decreases as the length over which the total load is distributed increases.

The time history of v_{mid} for each viscoelastic case is computed using the correspondence principle,³⁷ which for these creep loading situations gives the viscoelastic solution from the corresponding elastic solution by making the following substitution:

$$\frac{P_0}{E} \rightarrow P_0 D(t) \quad (2.84)$$

where $D(t)$ is the creep compliance.

The symmetry in the geometry and loading is also utilized in computing the results using the VFRAME code. Thirty-six elements, 40 modes, and a time step of 1.0 hr are used to compute the responses for one-half of the arch, with the results for the $\theta_t = 15^\circ$ case remaining essentially unchanged when 72 elements, 80 modes, and a time step of 0.1 hr are used. Shown in Fig. 2.8 are the exact and finite element results for the various load cases. The accuracy of the VFRAME results is excellent.

Table 2.1: Elastic solutions for v_{mid} for an arch under various load distributions.

θ_t	$\frac{v_{mid}E}{P_0}$	$v_{mid}/(v_{mid})_{\theta_t=15^\circ}$
0°	-1.64378×10^6	1.13083
5°	-1.61868×10^6	1.11357
10°	-1.55134×10^6	1.06724
15°	-1.45360×10^6	1.0

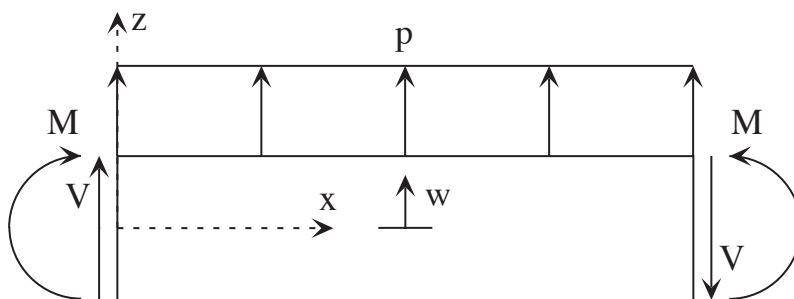


Figure 2.1: Sign conventions for isotropic viscoelastic beam.

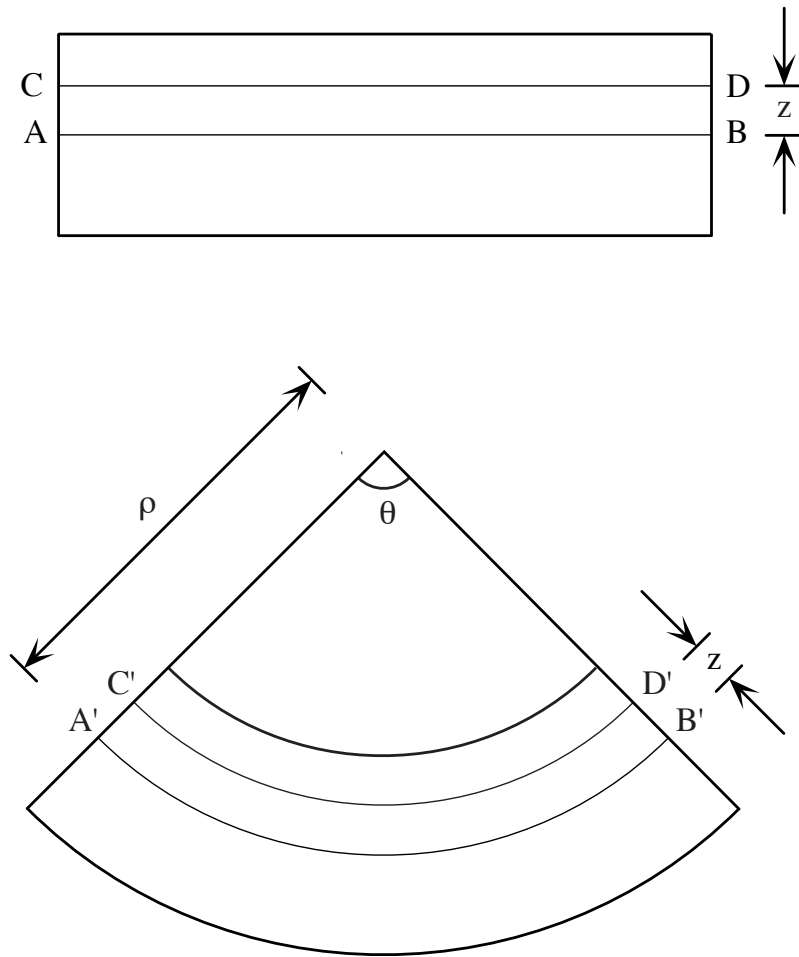


Figure 2.2: Elemental length of a beam under pure bending.

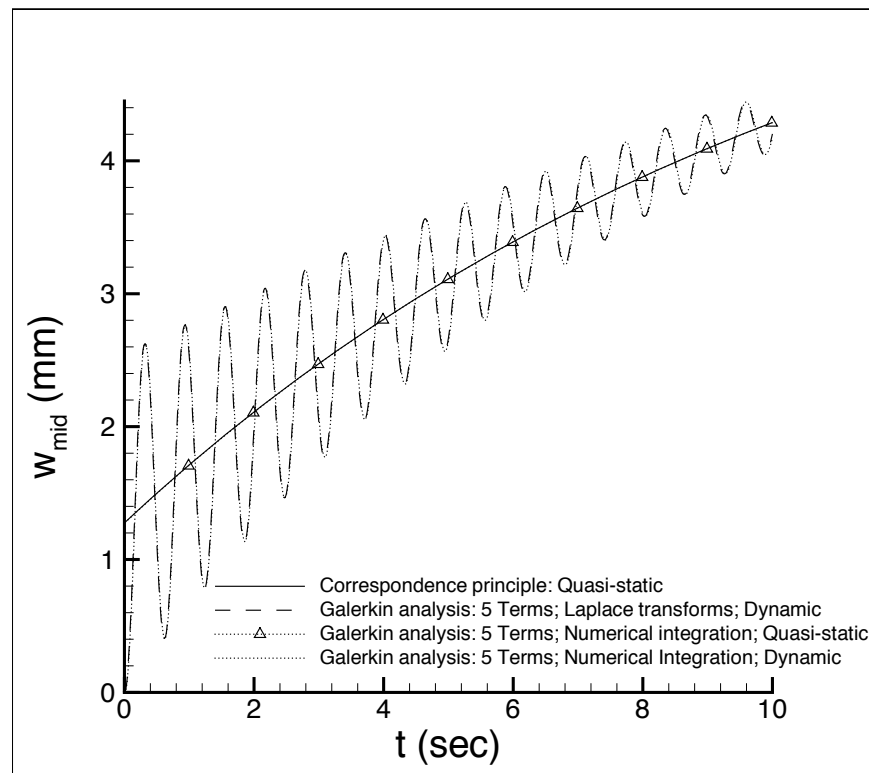


Figure 2.3: Galerkin analysis results for pinned-pinned beam under uniform distributed load.

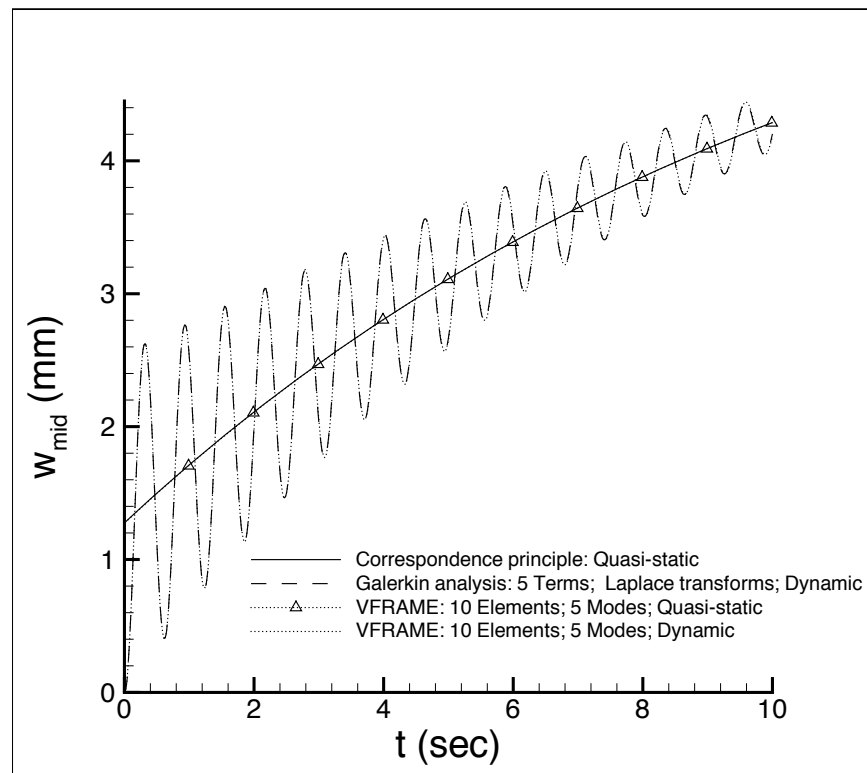


Figure 2.4: VFRAME results for pinned-pinned beam under uniform distributed load.

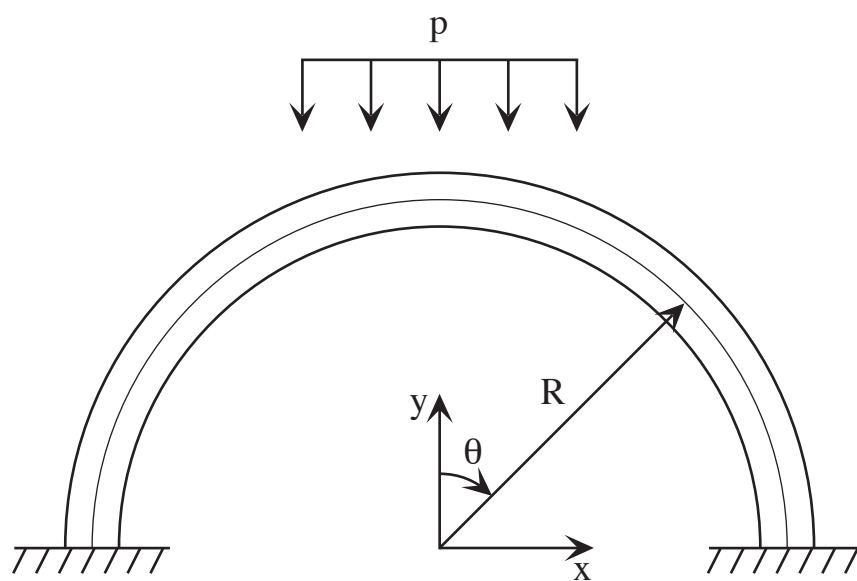


Figure 2.5: Geometry and loading of a viscoelastic arch.

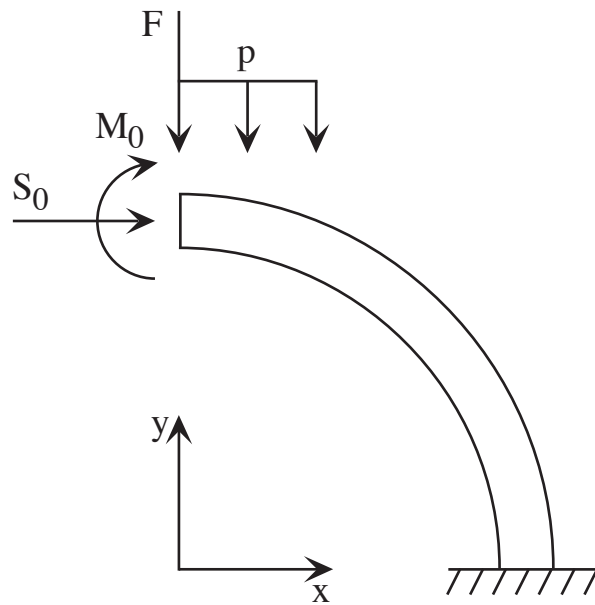


Figure 2.6: Loading and portion of the arch to be considered in determining v_{mid} exactly.

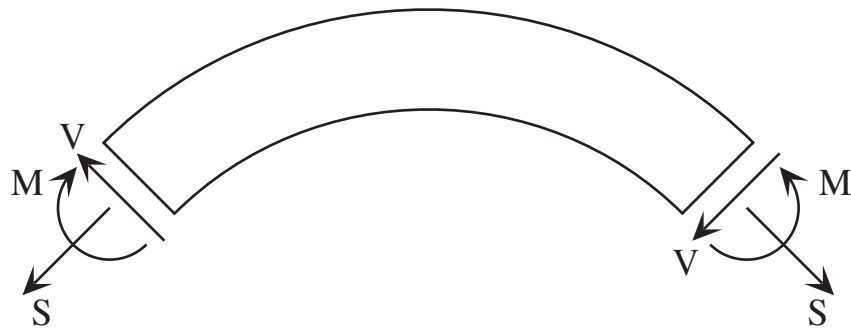


Figure 2.7: Positive sign conventions for the bending moment M , axial force S , and shear force V .

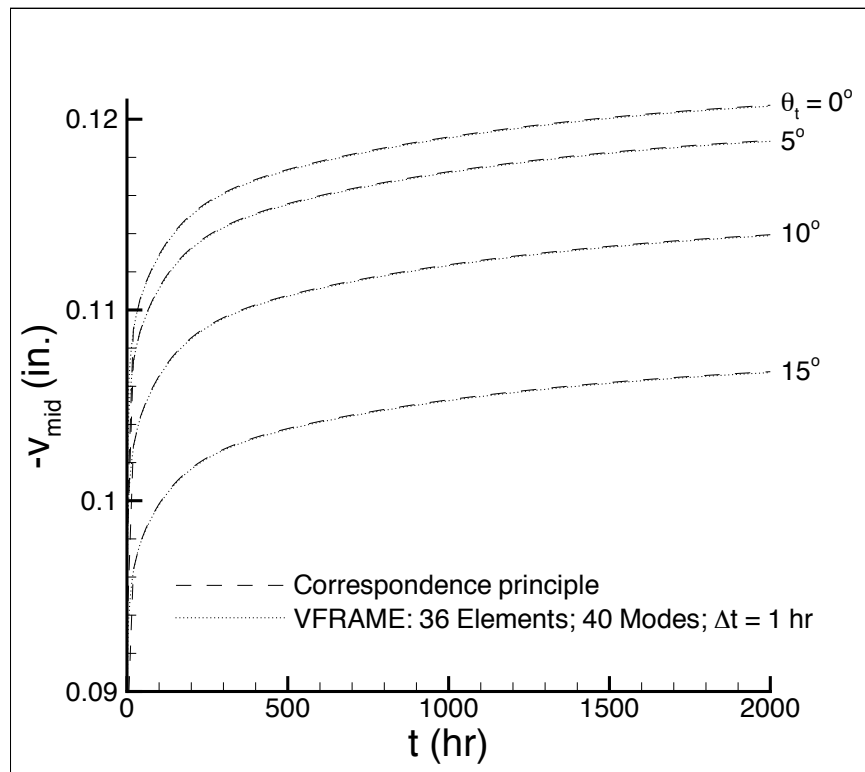


Figure 2.8: Time histories of v_{mid} for an arch under various creep loadings applied at $t = 0$. The total vertical load in each case is the same.

Chapter 3

Geometrically-Linear Analysis of Linear Viscoelastic Composites

In this chapter, the geometrically-linear finite element formulation for the analysis of linear viscoelastic composite plates and shells is presented. The triangular flat shell element that is employed has been used for the small-deformation analysis of linear elastic composites by Kapania and Mohan.⁴³ First, a brief overview of the element is presented. Then, the equations necessary to determine the deflections, stresses, and strains are developed. Various numerical examples are presented next to validate the formulation and to show the types of results that can be computed. A large portion of this work is presented in Ref. 44.

3.1 Triangular Flat Shell Element Overview

Flat shell elements are obtained as the superposition of a membrane element and a bending element. The shell behavior (geometric coupling of membrane and bending behavior between elements) results from transforming the element stiffness matrices and load vectors to a single global coordinate system. The present flat shell element combines the Discrete-Kirchhoff Theory (DKT) plate bending element⁴⁵ with a membrane element having the same nodal degrees of freedom (DOF) as the Allman Triangle (AT).⁴⁶

The DKT plate bending element begins with shape functions for the rotations β_x and β_y of the normal to the undeformed mid-surface. Note that β_x is the rotation of the normal about the positive y -axis, while β_y is the rotation of the normal about the negative x -axis. The bending displacement field is then expressed as

$$u_b = z \beta_x(x, y), \quad v_b = z \beta_y(x, y), \quad \text{and} \quad w_b = w(x, y) \quad (3.1)$$

where z is the through-the-thickness coordinate. The assumption of thin structures is incorporated in two ways. The transverse shear energy is neglected so that the strain energy is composed only of the strain energy due to bending. This bending energy involves only the bending curvatures, which are expressed in terms of the spatial derivatives of β_x and β_y . In order to determine the finite element equations in terms of the chosen nodal DOF of the element, which are w , $\theta_x = w_{,y}$, and $\theta_y = -w_{,x}$, the rotations β_x and β_y of the normal at the nodes need to be related to w , θ_x , and θ_y at the nodes. This is accomplished by applying the Kirchhoff hypothesis that the transverse shear is zero ($\beta_x = -w_{,x}$ and $\beta_y = -w_{,y}$) along the edges of the element. Note that here $(\cdot)_{,x}$ and $(\cdot)_{,y}$ have been used to represent $\partial(\cdot)/\partial x$ and $\partial(\cdot)/\partial y$, respectively.

Following Ertas *et al.*,⁴⁷ the transformation suggested by Cook⁴⁸ is used to transform the well-known Linear Strain Triangle (LST) element into a membrane element having two inplane translational DOF and one drilling DOF at each node. The inplane strain-displacement relations are derived as follows. The LST shape functions are used to derive the strain-displacement relations in terms of the DOF of the LST element. The resulting relations are then expressed in terms of the DOF of the Allman triangle by mapping the two inplane translational DOF at the mid-side nodes of the LST element into inplane translational and drilling DOF at the corner nodes.^{43,49} The DOF associated with the transformed LST element and DKT plate bending element are indicated in Fig. 3.1.

Combining the bending and membrane elements gives the elemental translational displacements $\bar{U}$, $\bar{V}$, and $\bar{W}$ along the x -, y -, and z -axes represented in the following form:

$$\bar{U}(x, y, z) = u(x, y) + z\beta_x(x, y) \quad (3.2)$$

$$\bar{V}(x, y, z) = v(x, y) + z\beta_y(x, y) \quad (3.3)$$

$$\bar{W}(x, y, z) = w(x, y) \quad (3.4)$$

where u, v , and w are midplane translations. Altogether, the triangular flat shell element has three translations and three rotations at each corner node for a total of 18 degrees of freedom.

Henceforth, the geometrically-linear, linear viscoelastic element to be developed will be referred to as TVATDKT-L (Thermo-Viscoelastic, Allman Triangle, Discrete Kirchhoff Theory triangle, geometrically-Linear).

3.2 Formulation

For an elastic composite under plane stress including the effects of thermal strains, $\{\epsilon^T\}$, and hygroscopic strains, $\{\epsilon^H\}$, the stress and strain in a layer relative to a local x-y coordinate system are related through the transformed reduced stiffness matrix, $[\bar{Q}]$, as follows:

$$\{\sigma\} = [\bar{Q}]\{\epsilon - \epsilon^T - \epsilon^H\} \quad (3.5)$$

For a hygrothermo-rheologically simple linear viscoelastic composite, the corresponding relationship is

$$\{\sigma(t)\} = \int_0^t [\bar{Q}(\zeta(t) - \zeta'(\tau))] \left\{ \frac{\partial \epsilon(\tau)}{\partial \tau} - \frac{\partial \epsilon^T(\tau)}{\partial \tau} - \frac{\partial \epsilon^H(\tau)}{\partial \tau} \right\} d\tau \quad (3.6)$$

where $[\bar{Q}(t)]$ is a matrix of transformed reduced relaxation moduli and reduced time scales are used in the stiffness operator to account for the temperature and moisture-level dependence of the material response rate. The equations defining the reduced time scales will be given later in the formulation.

The element strains, $\{\epsilon(t)\}$, are expressed as

$$\begin{aligned} \{\epsilon(t)\} &= \{e(t)\} + z\{\kappa(t)\} \\ &= [B_{at}]\{u(t)\} + z[B_{dkt}]\{u(t)\} \end{aligned} \quad (3.7)$$

where $[B_{at}]$ and $[B_{dkt}]$ are the strain-displacement matrices for the membrane and bending elements, respectively, and $\{u\}$ is the nodal displacement vector for the complete shell element. The thermal and hygroscopic strains for a layer are written, respectively, as

$$\{\epsilon^T(t)\} = \theta_T(t)\{\alpha\} \quad \text{and} \quad \{\epsilon^H(t)\} = \theta_H(t)\{\bar{\beta}\} \quad (3.8)$$

where $\{\alpha\}$ and $\{\bar{\beta}\}$ are the transformed coefficients of thermal and hygroscopic expansion, respectively, and θ_T and θ_H are the deviation of the temperature and moisture from thermal-strain free and hygroscopic-strain free states, respectively. The present formulation is restricted to the case where the structure's temperature and moisture content are independent of the structural response, uniform at any instant of time, and known *a priori*. Equations (3.7) and (3.8) hold for both the elastic and viscoelastic cases.

For the elastic case including inertia, the finite element equations before time discretization are written as

$$[M]\{\ddot{U}\} + [K]\{U\} = \{F\} + \{F^T\} + \{F^H\} \quad (3.9)$$

where $[M]$, $[K]$, $\{F\}$, $\{F^T\}$, $\{F^H\}$, and $\{U\}$ are the assembled global mass matrix, stiffness matrix, mechanical load vector, thermal load vector, hygroscopic load vector, and nodal displacement vector, respectively. The corresponding quantities on the element level are denoted with lower-case letters.

For the hygrothermo-rheologically simple linear viscoelastic case, the assembled global finite element equations before time discretization become

$$\begin{aligned} [M]\{\ddot{U}\} + \int_0^t [K(\zeta - \zeta')] \left\{ \frac{\partial U}{\partial \tau} \right\} d\tau &= \{F\} + \int_0^t \{F^T(\zeta - \zeta')\} \frac{\partial \theta_T}{\partial \tau} d\tau \\ &+ \int_0^t \{F^H(\zeta - \zeta')\} \frac{\partial \theta_H}{\partial \tau} d\tau \end{aligned} \quad (3.10)$$

where the element stiffness kernel and hygrothermal load kernels are

$$\begin{aligned} [k(\zeta - \zeta')] &= \int_{\tilde{A}} \left([B_{at}]^T [A(\zeta - \zeta')] [B_{at}] + [B_{at}]^T [B(\zeta - \zeta')] [B_{dkt}] \right. \\ &\quad \left. + [B_{dkt}]^T [B(\zeta - \zeta')] [B_{at}] + [B_{dkt}]^T [D(\zeta - \zeta')] [B_{dkt}] \right) d\tilde{A} \\ \{f^T(\zeta - \zeta')\} &= \int_{\tilde{A}} [B_{at}]^T \left(\int_{z_{min}}^{z_{max}} [\bar{Q}(\zeta - \zeta')]\{\alpha\} dz \right) d\tilde{A} + \end{aligned} \quad (3.11)$$

$$\int_{\tilde{A}} [B_{dkt}]^T \left(\int_{z_{min}}^{z_{max}} z [\bar{Q}(\zeta - \zeta')] \{\alpha\} dz \right) d\tilde{A} \quad (3.12)$$

$$\begin{aligned} \{f^H(\zeta - \zeta')\} &= \int_{\tilde{A}} [B_{at}]^T \left(\int_{z_{min}}^{z_{max}} [\bar{Q}(\zeta - \zeta')] \{\bar{\beta}\} dz \right) d\tilde{A} + \\ &\int_{\tilde{A}} [B_{dkt}]^T \left(\int_{z_{min}}^{z_{max}} z [\bar{Q}(\zeta - \zeta')] \{\bar{\beta}\} dz \right) d\tilde{A} \end{aligned} \quad (3.13)$$

where $\tilde{A}$ is the area of the element and

$$[A(\zeta - \zeta'); B(\zeta - \zeta'); D(\zeta - \zeta')] = \int_{z_{min}}^{z_{max}} (1; z; z^2) [\bar{Q}(\zeta - \zeta')] dz \quad (3.14)$$

The numerical technique for the solution of these equations is similar to the approaches used by Taylor *et al.*,⁷ Lin and Hwang,^{12,13} Hilton and Yi,^{14,15} and Yi *et al.*¹⁶ The elastic transformed reduced stiffness matrix is written as

$$[\bar{Q}] = [T]^{-1} \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} ([T]^{-1})^T \quad (3.15)$$

$$= Q_1 [D_1] + Q_2 [D_2] + Q_3 [D_3] + Q_4 [D_4] \quad (3.16)$$

where the following contracted notation for the reduced stiffnesses is used:

$$Q_1 = Q_{11}, \quad Q_2 = Q_{12}, \quad Q_3 = Q_{22}, \quad Q_4 = Q_{66} \quad (3.17)$$

and $[D_1]$ - $[D_4]$ are determined using

$$[T] = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \quad (3.18)$$

where $c = \cos \theta$, $s = \sin \theta$, and θ is the angle between the fiber direction and the local x-axis. Note that $[D_i]$ is the transformed reduced stiffness matrix for an elastic laminate with $Q_i = 1$ and the other Q_r 's equal to zero.

For linear viscoelasticity, each of the reduced stiffnesses (Q_1, Q_2, Q_3 , and Q_4) is expressed in terms of Prony series as follows:

$$Q_r(t) = Q_r^\infty + \sum_{\rho=1}^{N_r} Q_{r\rho} e^{-\lambda_{r\rho} t} \quad \text{for } r = 1, 2, 3, 4 \quad (3.19)$$

where the $\lambda_{r\rho}$'s denote relaxation times governing the material response characteristics.

Direct substitution of Eq. (3.19) into Eq. (3.15) would lead to $(4 + \sum_{r=1}^4 N_r)$ transformed reduced stiffness matrices needed for the description of a lamina's behavior. This ultimately would result in $(4 + \sum_{r=1}^4 N_r)$ global stiffness matrices and thermal and hygroscopic load vectors that would need to be stored.¹²⁻¹⁶ However, substitution of Eq. (3.19) into Eq. (3.16) results in the kernel $[\bar{Q}(\zeta - \zeta')]$ of Eq. (3.6) being expressed in terms of only four matrices $([D_1]-[D_4])$ as follows:

$$[\bar{Q}(\zeta - \zeta')] = \sum_{r=1}^4 Q_r(\zeta_r - \zeta'_r) [D_r] = \sum_{r=1}^4 \left(Q_r^\infty + \sum_{\rho=1}^{N_r} Q_{r\rho} e^{-\frac{\zeta_r - \zeta'_r}{\lambda_{r\rho}}} \right) [D_r] \quad (3.20)$$

where the effect of changing the temperature T and moisture-level H from the chosen reference values T_{ref} and H_{ref} upon the material response rate is according to the hygrothermo-rheologically simple postulate as follows:

$$Q_r(T, H, t) = Q_r(T_{ref}, H_{ref}, \zeta_r) \quad (3.21)$$

and the reduced times ζ_r and ζ'_r are defined by

$$\zeta_r = \int_0^t \frac{dt'}{A_r[T(t'), H(t')]} \quad \text{and} \quad \zeta'_r = \int_0^\tau \frac{dt'}{A_r[T(t'), H(t')]} \quad (3.22)$$

Note that each reduced stiffness has been allowed to have its own reduced time scale denoted by ζ_r with associated horizontal shift factor A_r , in order to model the possibility that each may be affected differently by the hygrothermal environment.

Similar to the kernel $[\bar{Q}(\zeta - \zeta')]$ in the material law, the global stiffness matrix kernel can be written as

$$[K(\zeta - \zeta')] = \sum_{r=1}^4 \left(Q_r^\infty + \sum_{\rho=1}^{N_r} Q_{r\rho} e^{-\frac{\zeta_r - \zeta'_r}{\lambda_{r\rho}}} \right) [K_r] \quad (3.23)$$

where $[K_i]$ is the elastic global stiffness matrix for the case in which $Q_i = 1$ and the other three Q_r 's are zero. Likewise, the kernels for the global thermal and hygroscopic loads are written as

$$\{F^T(\zeta - \zeta')\} = \sum_{r=1}^4 \left(Q_r^\infty + \sum_{\rho=1}^{N_r} Q_{r\rho} e^{-\frac{\zeta_r - \zeta'_r}{\lambda_{r\rho}}} \right) \{F_r^T\} \quad (3.24)$$

$$\{F^H(\zeta - \zeta')\} = \sum_{r=1}^4 \left(Q_r^\infty + \sum_{\rho=1}^{N_r} Q_{r\rho} e^{-\frac{\zeta_r - \zeta'_r}{\lambda_{r\rho}}} \right) \{F_r^H\} \quad (3.25)$$

where $\{F_i^T\}$ and $\{F_i^H\}$ are the elastic global thermal and hygroscopic load vectors, respectively, for the case where $Q_i = 1$ and the other three Q_r 's are zero.

The integration of a sample hereditary integral over a time step is carried out as follows. Approximating $\{\partial U/\partial \tau\}$ to be constant over a time step and equal to $\{\Delta U^p\}/\Delta t^p$ where $\Delta(\cdot) = (\cdot)^p - (\cdot)^{p-1}$, the following approximation is made:

$$\int_{t^{p-1}}^{t^p} e^{-\frac{\zeta_r^p - \zeta_r'}{\lambda_{r\rho}}} Q_{r\rho}[K_r] \left\{ \frac{\partial U}{\partial \tau} \right\} d\tau \approx \bar{S}_{r\rho}^p Q_{r\rho}[K_r] \{\Delta U^p\} \quad (3.26)$$

where

$$\bar{S}_{r\rho}^p = \frac{1}{\Delta t^p} \int_{t^{p-1}}^{t^p} e^{-\frac{\zeta_r^p - \zeta_r'}{\lambda_{r\rho}}} d\tau \quad (3.27)$$

Taylor *et al.*⁷ demonstrated the importance of accurately evaluating $\bar{S}_{r\rho}^p$ in controlling numerical error. For the case where A_r is constant over a time step, $\bar{S}_{r\rho}^p$ can be evaluated exactly as⁷

$$\bar{S}_{r\rho}^p = \frac{1}{\Delta \zeta_r^p} \lambda_{r\rho} \left[1 - e^{-\frac{\Delta \zeta_r^p}{\lambda_{r\rho}}} \right] \quad (3.28)$$

Presumably, a sufficiently small time step size will be used so that Eq. (3.28) represents an acceptable approximation to Eq. (3.27) for a given variation in the hygrothermal environment.

The Newmark method is used to represent the current acceleration in terms of the current displacement and the previous displacements, velocities, and accelerations. The current acceleration, $\{\ddot{U}^p\}$, is calculated by rearranging the following:

$$\{U^p\} = \{U^{p-1}\} + \Delta t^p \{\dot{U}^{p-1}\} + \frac{1}{2}(\Delta t^p)^2 \left[(1 - \gamma)\{\ddot{U}^{p-1}\} + \gamma\{\ddot{U}^p\} \right] \quad (3.29)$$

and the current velocity is given by

$$\{\dot{U}^p\} = \{\dot{U}^{p-1}\} + (1 - \psi)\Delta t^p \{\ddot{U}^{p-1}\} + \psi\Delta t^p \{\ddot{U}^p\} \quad (3.30)$$

where γ and ψ are scalars corresponding to specific schemes.³⁹

Altogether, the equation to be solved for the current displacement vector, $\{U^p\}$, is

$$\left[\frac{2}{\gamma(\Delta t^p)^2} [M] + \sum_{r=1}^4 \left(Q_r^\infty + \sum_{\rho=1}^{N_r} \bar{S}_{r\rho}^p Q_{r\rho} \right) [K_r] \right] \{U^p\} =$$

$$\begin{aligned}
& \{F^p\} + [M] \left\{ \frac{2}{\gamma(\Delta t^p)^2} \{U^{p-1}\} + \frac{2}{\gamma\Delta t^p} \{\dot{U}^{p-1}\} + \left(\frac{1}{\gamma} - 1\right) \{\ddot{U}^{p-1}\} \right\} \\
& - \sum_{r=1}^4 \sum_{\rho=1}^{N_r} \{R_{r\rho}^p\} + \sum_{r=1}^4 \left(\sum_{\rho=1}^{N_r} \bar{S}_{r\rho}^p Q_{r\rho} \right) [K_r] \{U^{p-1}\} \\
& + \sum_{r=1}^4 Q_r^\infty \left\{ \theta_T^p \{F_r^T\} + \theta_H^p \{F_r^H\} \right\} \\
& + \sum_{r=1}^4 \left(\sum_{\rho=1}^{N_r} \bar{S}_{r\rho}^p Q_{r\rho} \right) \left\{ \Delta\theta_T^p \{F_r^T\} + \Delta\theta_H^p \{F_r^H\} \right\}
\end{aligned} \tag{3.31}$$

where

$$\begin{aligned}
\{R_{r\rho}^p\} = & \int_0^{t^{p-1}} Q_{r\rho} e^{-\frac{\zeta_r^p - \zeta_r'}{\lambda_{r\rho}}} [K_r] \left\{ \frac{\partial U}{\partial \tau} \right\} d\tau - \int_0^{t^{p-1}} Q_{r\rho} e^{-\frac{\zeta_r^p - \zeta_r'}{\lambda_{r\rho}}} \frac{\partial \theta_T}{\partial \tau} \{F_r^T\} d\tau \\
& - \int_0^{t^{p-1}} Q_{r\rho} e^{-\frac{\zeta_r^p - \zeta_r'}{\lambda_{r\rho}}} \frac{\partial \theta_H}{\partial \tau} \{F_r^H\} d\tau
\end{aligned} \tag{3.32}$$

Writing the expression for $\{R_{r\rho}^{p-1}\}$ and approximating the hereditary integrals from t^{p-2} to t^{p-1} in a manner similar to that shown in Eq. (3.26), the following recurrence relation is obtained for $\{R_{r\rho}^p\}$:

$$\{R_{r\rho}^p\} = e^{-\frac{\Delta\zeta_r^p}{\lambda_{r\rho}}} \left[\{R_{r\rho}^{p-1}\} + \bar{S}_{r\rho}^{p-1} Q_{r\rho} \left([K_r] \{\Delta U^{p-1}\} - \Delta\theta_T^{p-1} \{F_r^T\} - \Delta\theta_H^{p-1} \{F_r^H\} \right) \right] \tag{3.33}$$

Hence, the displacement solution for the current time step is completely determined in terms of quantities from only the previous two time steps.

The following initial conditions are used:

$$\begin{aligned}
\{U^0\} &= \{0\}, \quad \{\dot{U}^0\} = \{0\}, \quad \{\ddot{U}^0\} = \{0\}, \quad \bar{S}_{r\rho}^0 = 1, \\
\theta_T^0 &= 0, \quad \theta_H^0 = 0, \quad \{R_{r\rho}^0\} = \{0\}
\end{aligned} \tag{3.34}$$

For the case where the temperature and moisture are constant over a time step, the reduced time increment, $\Delta\zeta_r^p$, is simply

$$\Delta\zeta_r^p = \frac{1}{A_r(T^p, H^p)} \Delta t^p \tag{3.35}$$

If the hygrothermal environment varies over a time step, $\Delta\zeta_r^p$ can be evaluated by numerically integrating $1/A_r$ over the time step as indicated by Eq. (3.22).

Note that only four stiffness matrices and thermal and hygroscopic load vectors (i.e., $[K_r]$, $\{F_r^T\}$, and $\{F_r^H\}$ for $r = 1-4$) are needed in marching the solution forward in time, regardless

of the number of exponential terms used to represent the four independent relaxation moduli Q_1 - Q_4 . However, the number of viscoelastic memory terms is exactly equal to the number of exponential terms ($\sum_{r=1}^4 N_r$) used in representing the relaxation moduli.

In general, the number of Prony series terms needed to represent the material response depends upon the time period that will be considered in the structural analysis. Furthermore, for a chosen time range, a number of various Prony series with slightly different coefficients and time constants can be determined. The particular choice that is made for each Prony series is not extremely critical provided that together the Prony series represent the material behavior to a sufficient level of accuracy over the time period of interest.

The quasi-static equation for the current displacement is similar to Eq. (3.31), but with all of the terms multiplied by the mass matrix deleted. Either a lumped or what is termed a “consistent” mass matrix may be used in the calculations when inertia is to be included. If a lumped mass matrix is chosen, the mass associated with each translational DOF is simply one-third of the elemental mass. In that case, the rotary inertia associated with each node is neglected. If, however, a “consistent” mass matrix is chosen, the mass matrix including all terms corresponding to the rotary inertias is formulated as follows. The development of a consistent mass matrix for the transformed LST element presents no difficulties. However, since explicit shape functions for the transverse displacement w are not used in the derivation of the DKT element, a further approximation in the derivation of a “consistent” bending mass matrix must be made. As described by Kapania and Mohan,⁴³ in addition to using the shape functions for θ_x and θ_y , a nine-term polynomial in area coordinates is introduced to approximate the transverse displacement w over an element, allowing a “consistent” bending mass matrix to be found.

Once the displacements for the time period of interest have been determined, the strains at a particular point in an element can be computed directly from Eq. (3.7). Using the constitutive law given by Eq. (3.6), the stress at a given point at a given time t^p is then evaluated as

$$\{\sigma(t^p)\} = \sum_{r=1}^4 Q_r^\infty [D_r] \{ \{e^p\} + z\{\kappa^p\} - \theta_T^p \{\alpha\} - \theta_H^p \{\bar{\beta}\} \}$$

$$\begin{aligned}
& + \sum_{r=1}^4 \sum_{\rho=1}^{N_r} \bar{S}_{r\rho}^p Q_{r\rho}[D_r] \{ \{ \Delta e^p \} + z \{ \Delta \kappa^p \} - \Delta \theta_T^p \{ \alpha \} - \Delta \theta_H^p \{ \bar{\beta} \} \} \\
& + \sum_{r=1}^4 \sum_{\rho=1}^{N_r} \{ W_{r\rho}^p \}
\end{aligned} \tag{3.36}$$

where

$$\{ W_{r\rho}^p \} = \int_0^{t^{p-1}} e^{-\frac{\zeta_r^p - \zeta_r'}{\lambda_{r\rho}}} Q_{r\rho}[D_r] \left\{ \left\{ \frac{\partial e}{\partial \tau} \right\} + z \left\{ \frac{\partial \kappa}{\partial \tau} \right\} - \frac{\partial \theta_T}{\partial \tau} \{ \alpha \} - \frac{\partial \theta_H}{\partial \tau} \{ \bar{\beta} \} \right\} d\tau \tag{3.37}$$

$$\begin{aligned}
& = e^{-\frac{\Delta \zeta_r^p}{\lambda_{r\rho}}} \left[\{ W_{r\rho}^{p-1} \} + \bar{S}_{r\rho}^{p-1} Q_{r\rho}[D_r] \left(\{ \Delta e^{p-1} \} + z \{ \Delta \kappa^{p-1} \} \right. \right. \\
& \quad \left. \left. - \Delta \theta_T^{p-1} \{ \alpha \} - \Delta \theta_H^{p-1} \{ \bar{\beta} \} \right) \right]
\end{aligned} \tag{3.38}$$

with the initial condition

$$\{ W_{r\rho}^0 \} = \{ 0 \} \tag{3.39}$$

Shown in Fig. 3.2 is a flowchart for the procedure to calculate the nodal displacements, velocities, and accelerations using the TVATDKT-L code. The procedure to calculate the elemental stresses and strains at a given point are shown in Fig. 3.3. Note that the global nodal displacements, $\{U\}$, for the total time considered are found first, with the elemental stresses and strains found afterwards, if desired. The relevant equation numbers have been indicated.

3.3 Numerical Examples

Various example problems are presented to demonstrate the capability and verify the accuracy of the TVATDKT-L element. First, the quasi-static and dynamic responses of an isotropic cantilever subjected to a tip load are determined. Numerical results for the quasi-static deflection and stress response of a simply-supported viscoelastic $[0/90]_s$ composite plate under a uniform pressure load are then presented. The strain response of a free viscoelastic composite plate subjected to a uniform edge load applied instantaneously at $t = 0$

and removed instantaneously at $t = 24$ hr follows. Several stacking sequences are considered. Next, the thermal response of one of these plates is studied. The results for two values of uniform temperature change are calculated. Finally, numerical results are presented for two composite viscoelastic cylindrical panels which are both subjected to a uniform pressure load.

3.3.1 Cantilever Under Tip Load

A tip load $P_0 = 0.1$ N is suddenly applied at $t = 0$ and held constant thereafter. The beam has a length of 1.0 m, a width of 0.3 m, and a thickness of 0.0254 m. The material is represented as a 3-parameter solid with the relaxation modulus given by

$$E(t) = 1.96 \times 10^7 + 7.84 \times 10^7 e^{-t/2.24} \text{ N/m}^2 \quad (3.40)$$

where t is in seconds. The material density is 2200 kg/m^3 .

Shown in Fig. 3.4 are the quasi-static and dynamic results for the cantilever tip deflection. The exact quasi-static solution for the tip deflection is computed using the correspondence principle³⁸ as

$$w_{tip} = \frac{P_0 L^3}{3I} D(t) \quad (3.41)$$

where L and I are the beam's length and cross-sectional moment of inertia, respectively, and $D(t)$ is the creep compliance. Setting $\nu = 0$ and computing the $Q_r(t)$'s using the expression given by Eq. (3.40) for $E(t)$ allows the flat shell element code to be used to model the cantilever beam. Quasi-static and dynamic results for the tip deflection are computed using 30 TVATDKT-L elements. For the dynamic case, a lumped mass matrix is employed in the TVATDKT-L code. The dynamic results are also computed using the VFRAME code developed in Chapter 2. A total of 10 VFRAME elements is used. For the dynamic case, the constant average acceleration method ($\gamma = \psi = 1/2$) with $\Delta t = 0.01$ sec is used with the VFRAME and TVATDKT-L formulations. Implementing the constant average acceleration method with $\Delta t = 0.001$ sec did not appreciably change either the VFRAME or TVATDKT-L results. The quasi-static TVATDKT-L results are also computed using $\Delta t = 0.01$ sec. In

Fig. 3.4, the differences between the TVATDKT-L and exact results for the quasi-static case and between the TVATDKT-L and VFRAME dynamic results using the constant average acceleration method cannot be distinguished. The maximum acceleration corresponding to w_{tip} computed for the dynamic case using the TVATDKT-L formulation and the constant average acceleration method is 101.25 mm/sec².

A poor choice of γ and ψ for the Newmark method can lead to undesired numerical damping for a given time step size, as exhibited by the dynamic viscoelastic results shown in Fig. 3.5a produced using the TVATDKT-L formulation with the backward difference method ($\gamma = 2$, $\psi = 3/2$) and $\Delta t = 0.01$ sec. Also shown in Fig. 3.5a for comparison are the dynamic results computed using the constant average acceleration method with the same time step size.

In order to examine the Newmark method in the absence of viscous damping, dynamic results are also computed for the elastic case with Young's modulus taken to be equal to the initial value of the relaxation modulus. The exact dynamic response in such an elastic case is a vibration about the static equilibrium point with an amplitude of vibration that remains constant. Here, the transverse deflection at the tip has a static equilibrium value of 0.8303 mm, which can be computed using Eq. (3.41) with $D(0) = 1/E(0)$. Examining Fig. 3.5b, it appears that results produced using the constant average acceleration and backward difference methods both exhibit numerical phenomena. The constant average acceleration results exhibit a slight "beat" phenomenon in which the amplitude of the vibrations periodically increases and decreases. Although the results are not shown, using a smaller time step of 0.001 sec with the constant average acceleration method did not diminish this beat phenomenon. Rather, the beat frequency increased slightly, while the amount by which the amplitude varies over time remained essentially equal to that which occurs when the larger time step size is used. Once again, the backward difference method results exhibit significant decay in the vibration amplitude due to algorithmic damping.

3.3.2 Square Composite Plate Under Pressure Load

The length of each side is 10 m, while the plate thickness is 0.1 m. The cross-ply stacking sequence is $[0/90]_s$. The boundary conditions are taken as simply supported. A uniform pressure load of $q_0 = 1 \text{ N/m}^2$ is suddenly applied at $t = 0$ and held constant thereafter. The material relaxation moduli are assumed as follows:

$$E_1 = 9.8 \times 10^7 \text{ N/m}^2 \quad (3.42)$$

$$E_2(t) = (1.96 + 7.84 e^{-t/2.24} + e^{-t/10}) \times 10^5 \text{ N/m}^2 \quad (3.43)$$

$$\nu_{12} = 0.25 \quad (3.44)$$

$$G_{12}(t) = (0.98 + 3.92 e^{-t/2.24} + 0.5 e^{-t/10}) \times 10^5 \text{ N/m}^2 \quad (3.45)$$

where t is in seconds. This will give the fiber-dominated relaxation moduli Q_1 as elastic (constant in time) and the other three relaxation moduli, Q_2 - Q_4 , which are matrix-dominated, as viscoelastic. The relaxation moduli are approximated as follows:

$$Q_1 = \frac{E_1}{1 - \nu_{21}\nu_{12}} \quad (3.46a)$$

$$Q_2(t) = \frac{\nu_{12}E_2(t)}{1 - \nu_{21}\nu_{12}} \quad (3.46b)$$

$$Q_3(t) = \frac{E_2(t)}{1 - \nu_{21}\nu_{12}} \quad (3.46c)$$

$$Q_4(t) = G_{12}(t) \quad (3.46d)$$

$$\nu_{21} = \frac{E_2(0)}{E_1} \nu_{12} \quad (3.46e)$$

Although the major and minor Poisson's ratios are treated as constant in this and the other composite examples, it should be noted that the TVATDKT-L formulation is not limited to such an approximation. Rather, the TVATDKT-L formulation only requires that Prony series be given for Q_1 - Q_4 .

The exact quasi-static response for the transverse deflection can be determined once again

using the correspondence principle³⁷ and is given by

$$w(x, y, t) = \mathcal{L}^{-1} \left\{ \frac{16q_0}{\pi^6 s^2} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\frac{1}{mn} \sin \frac{m\pi x}{L} \sin \frac{n\pi y}{L}}{\hat{D}_{11}(\frac{m}{L})^4 + 2(\hat{D}_{12} + 2\hat{D}_{66})(\frac{m}{L})^2(\frac{n}{L})^2 + \hat{D}_{22}(\frac{n}{L})^4} \right\} \quad (3.47)$$

where $\mathcal{L}$ is the Laplace operator, $(\cdot)$ denotes the Laplace transform of $(\cdot)$, and s is the Laplace variable. A total of 36 terms in the Fourier series was sufficient to give accurate results for the initial and final ($t = \infty$) viscoelastic midpoint deflection, as using 100 terms only changed both these results by slightly more than 0.0012%. Hence, using a total of 36 terms to compute the “exact” solution for all times considered is acceptable and is performed. In this example and all other examples where needed, partial fraction expansions in the Laplace domain and inverse Laplace transforms are computed exactly utilizing the commercial software package Mathematica.⁴²

Using the resulting expression for $w(x, y, t)$, the bending curvature vector $\{\kappa\}$ is found and the stresses are computed as follows:

$$\begin{aligned} \{\sigma(t)\} &= \int_0^t z[\bar{Q}(t - \tau)] \left\{ \frac{\partial \kappa}{\partial \tau} \right\} d\tau \\ &= \mathcal{L}^{-1} \left\{ sz[\hat{\bar{Q}}(s)] \{\hat{\kappa}(s)\} \right\} \end{aligned} \quad (3.48)$$

The TVATDKT-L results for the midpoint transverse deflection, w_{mid} , and the midpoint stress σ_x at $z = z_{max}$ using 72, 200, and 512 elements with $\Delta t = 0.1$ sec are shown in Figs. 3.6 and 3.7, respectively, along with exact values computed by applying the correspondence principle.³⁷ For the 200-element mesh, using $\Delta t = 0.01$ sec did not significantly change the results. The TVATDKT-L results appear to converge to the exact solution, with convergence from below for w_{mid} and convergence from above for the midpoint σ_x at $z = z_{max}$.

3.3.3 Laminated Plate Under Edge Load

The plate is free on all edges. The plate dimensions are 1.0×2.0 in. Each layer has a thickness of 0.02 in. with the following material properties (matching a graphite-epoxy) at the instant

the plate is loaded:

$$\begin{aligned} E_1 &= 18 \text{ Msi}, & G_{12} &= 0.9 \text{ Msi}, & \alpha_1 &= 0.2 \times 10^{-6} / ^\circ\text{F}, \\ E_2 &= 1.4 \text{ Msi}, & \nu_{12} &= 0.34, & \alpha_2 &= 16.0 \times 10^{-6} / ^\circ\text{F} \end{aligned} \quad (3.49)$$

Once again, E_1 is taken as fiber-dominated and constant in time. Hence, Q_1 will also be constant in time. The reduced relaxation moduli Q_2 - Q_4 are taken to have the same normalized time variation as follows:

$$Q_r(t) = Q_r(0) f(t) \quad \text{for } r = 2, 3, 4 \quad (3.50)$$

where

$$f(t) = f_0 + \sum_{i=1}^{10} f_i e^{-\frac{t}{\lambda_i}} \quad (3.51)$$

The values given in Table 3.1 for f_i and λ_i are those listed in Lin and Hwang¹² which are a data fit to relaxation data presented in Crossman *et al.*² The constant value of Q_1 and the initial values of Q_2 - Q_4 are computed using $t = 0$ in Eqs. (3.46a)-(3.46e).

A uniform edge load of $N_x (= \int_{z_{min}}^{z_{max}} \sigma_x dz)$ equal to 276.48 lb/in. is applied at $t = 0$ and held constant until $t = 24$ hr, at which point it is suddenly removed.

Due to symmetry, only one-fourth of the plate is analyzed using 2 elements. The quasi-static strain results for several stacking sequences ($[\pm 45/0/90]_s$, $[\pm 45_2]_s$, and $[(0/90)_2]_s$) are shown in Fig. 3.8a for $\Delta t = 100$ sec. As expected for this loading, the $[\pm 45_2]_s$ plate shows the most time dependence in its response due to the fact that the matrix for this stacking sequence carries a larger percentage of the load than those for the other stacking sequences.

The exact quasi-static strain solution for the $[\pm 45_2]_s$ laminate is computed as follows. For this laminate, the only non-zero stress in each layer is simply

$$\sigma_x(t) = 1728[u(t) - u(t - 24 \text{ hr})] \text{ psi} \quad (3.52)$$

where $u(t)$ is the unit step function. The Laplace transform of the constitutive law is then used to calculate $\hat{\epsilon}_x(s)$. This result can be simplified to give

$$\epsilon_x(t) = \mathcal{L}^{-1} \left\{ \frac{\hat{\sigma}_x}{s} \left[\frac{1}{4\hat{Q}_4} + \frac{1}{\hat{Q}_1 + 2\hat{Q}_2 + \hat{Q}_3} \right] \right\} \quad (3.53)$$

As shown in Fig. 3.8b, the TVATDKT-L results computed using $\Delta t = 10, 100$, and 1000 sec are all in excellent agreement with the exact solution.

3.3.4 Laminated Plate Under Thermal Load

Now consider the $[\pm 45/0/90]_s$ plate of the previous example to be subjected to a uniform change in temperature applied suddenly at $t = 0$, with the temperature held constant thereafter. The temperature for zero thermal strain is taken as 350°F . Temperature changes of -275°F and -190°F are considered. The shift factors for Q_2 , Q_3 , and Q_4 are taken to be the same and are given as 1.0 and $1.51357\text{E-}5$ for $T = 75^\circ\text{F}$ and $T = 160^\circ\text{F}$, respectively, which are taken from Lin and Hwang.¹² Once again, two elements are used to model one-fourth of the plate, since using a higher number of elements does not alter the solution greatly. Shown in Figs. 3.9a and 3.9b are the results for the stress in the 0° layer and the strain normalized by their initial values, along with some of the results given by Lin and Hwang.¹³ In the present analysis, a time step size of $\Delta t = 400$ sec is used. From the TVATDKT-L calculations, the initial σ_x and ϵ_x values respectively are -5380.0 psi and -4.55512×10^{-4} for $\Delta T = -275^\circ\text{F}$, while for $\Delta T = -190^\circ\text{F}$, the initial stress and strain values are -3717.1 psi and -3.14717×10^{-4} , respectively. Lin and Hwang¹³ computed initial stress values of -5380 psi and -3717 psi for $\Delta T = -275^\circ\text{F}$ and $\Delta T = -190^\circ\text{F}$, respectively, with an initial strain value of -4.56×10^{-4} for $\Delta T = -275^\circ\text{F}$. Good agreement exists between the present and previously published results. It is observed that the thermo-rheologically simple linear viscoelastic response of the plate is a nonlinear function of the plate's temperature. Note that if the results were plotted versus the reduced time ζ instead of the physical time t , the normalized curves corresponding to $\Delta T = -275^\circ\text{F}$ and $\Delta T = -190^\circ\text{F}$ would coincide.

3.3.5 Composite Cylindrical Panels Under Pressure Load

Two cylindrical panels differing only in the panel half-angle are subjected to a uniform pressure load $q_0 = 0.3$ psi which is applied suddenly at $t = 0$ and held constant thereafter. The pressure is applied on the concave side of the panels pointing away from the center of curvature of the panel cross sections. The length of both panels is 80 in., while the arc length of the other side is either 41.89 in. or 62.83 in., corresponding respectively to half-angles of $\phi = 12^\circ$ and $\phi = 18^\circ$ and a radius of 100 in. On each edge, both cylindrical panels rest on

diaphragms which are rigid in their plane, but perfectly flexible otherwise. These diaphragms are perpendicular to the panel surface along its edges. The cross-ply stacking sequence is $[0/90]_s$ with the thickness of each layer taken to be 0.08 in., giving a total laminate thickness of 0.32 in. The properties at the initial time match those given in Eq. (3.49) for a graphite-epoxy. Once again, Q_1 is assumed to be fiber-dominated and therefore constant in time, as computed using Eqs. (3.46a) and (3.46e). However, the Q_2 - Q_4 relaxation functions are determined by using Eqs. (3.46b)-(3.46e) with²⁵

$$E_2(t) = (116.333 + 6.02203 e^{-t/287.154} + 10.8854 e^{-t/5512.77} + 6.75662 e^{-t/113,384}) \times 10^4 \text{ psi} \quad (3.54)$$

$$G_{12}(t) = (75.9301 + 3.86299 e^{-t/287.154} + 6.3555 e^{-t/5512.77} + 3.85226 e^{-t/113,384}) \times 10^4 \text{ psi} \quad (3.55)$$

where t is in minutes.

A closed-form solution for this problem using shallow shell theory is computed as follows. Consider the curvilinear shell coordinate system to be as shown in Fig. 3.10a. The first shell coordinate ξ_1 varies along the direction having zero curvature, while the second shell coordinate ξ_2 varies along the direction having constant non-zero curvature. The third shell coordinate ξ_3 (the shell normal) is determined from the right-hand rule and points away from the center of curvature of the ξ_1 -constant arcs. As chosen, the shell coordinates are principal coordinates.

Let v_1 , v_2 , and v_3 be the translational displacements in the directions of the three shell coordinates. Using the Love-Kirchhoff hypothesis and other approximations appropriate for thin elastic laminated shallow shells as described by Leissa and Qatu,⁵⁰ the governing equations in terms of v_1 , v_2 , and v_3 for a shallow elastic cylindrical panel with a symmetric cross-ply stacking sequence under uniform pressure are

$$A_{11} v_{1,11} + A_{12} \left(v_{2,12} + \frac{v_{3,1}}{R_2} \right) + A_{66} (v_{1,22} + v_{2,12}) = 0 \quad (3.56)$$

$$A_{12} v_{1,12} + A_{22} \left(v_{2,22} + \frac{v_{3,2}}{R_2} \right) + A_{66} (v_{1,12} + v_{2,11}) = 0 \quad (3.57)$$

$$\begin{aligned}
A_{12} \frac{v_{1,1}}{R_2} + A_{22} \left(\frac{v_{2,2}}{R_2} + \frac{v_3}{R_2^2} \right) + D_{11} v_{3,1111} \\
+ 2(D_{12} + 2D_{66}) v_{3,1122} + D_{22} v_{3,2222} = q_3
\end{aligned} \tag{3.58}$$

Here R_2 is the radius associated with the panel and $(\cdot)_{,i}$ denotes differentiation with respect to shell coordinate ξ_i .

The uniform pressure load q_3 is written as

$$q_3 = q_0 = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{16q_0}{\pi^2 mn} \sin \frac{m\pi\xi_1}{l_1} \sin \frac{n\pi\xi_2}{l_2} \tag{3.59}$$

The boundary conditions are

$$\xi_1 = 0, l_1 : \quad v_2 = v_3 = 0 \quad \text{and} \quad N_{11} = 0, M_{11} = 0 \tag{3.60}$$

$$\xi_2 = 0, l_2 : \quad v_1 = v_3 = 0 \quad \text{and} \quad N_{22} = 0, M_{22} = 0 \tag{3.61}$$

where N_{ij} and M_{ij} are the usual force and force-couple resultants in the directions of the shell coordinates. Using the constitutive law, the strain-displacement relations, and the conditions on v_1 - v_3 given above, the boundary conditions can be written strictly in terms of v_1 - v_3 as

$$\xi_1 = 0, l_1 : \quad v_2 = v_3 = 0 \quad \text{and} \quad v_{1,1} = 0, v_{3,11} = 0 \tag{3.62}$$

$$\xi_2 = 0, l_2 : \quad v_1 = v_3 = 0 \quad \text{and} \quad v_{2,2} = 0, v_{3,22} = 0 \tag{3.63}$$

The boundary conditions and governing equations are satisfied exactly by the following Fourier series expansions:

$$v_1 = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} I_{mn} \cos \frac{m\pi\xi_1}{l_1} \sin \frac{n\pi\xi_2}{l_2} \tag{3.64}$$

$$v_2 = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} J_{mn} \sin \frac{m\pi\xi_1}{l_1} \cos \frac{n\pi\xi_2}{l_2} \tag{3.65}$$

$$v_3 = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} K_{mn} \sin \frac{m\pi\xi_1}{l_1} \sin \frac{n\pi\xi_2}{l_2} \tag{3.66}$$

The constant coefficients I_{mn} , J_{mn} , and K_{mn} for the elastic case are determined by substituting Eqs. (3.64), (3.65), and (3.66) into Eqs. (3.56), (3.57), and (3.58) and solving. The

viscoelastic results are then computed using the correspondence principle³⁷ by replacing A_{ij} and D_{ij} by $s\hat{A}_{ij}(s)$ and $s\hat{D}_{ij}(s)$, respectively, and q_0 by q_0/s , and then taking the inverse Laplace transform. A total of 49 Fourier terms is used to compute the shallow shell theory results, with the results for $(v_3)_{mid}$ at $t = 0$ for the $\phi = 12^\circ$ and $\phi = 18^\circ$ panels respectively being slightly less than 0.007% and 0.12% different than the results computed using 100 Fourier terms.

For both panels, the quarter of the panel for which $0 \leq \xi_1 \leq l_1/2$ and $0 \leq \xi_2 \leq l_2/2$ is modeled with symmetry boundary conditions enforced on the two edges $\xi_1 = l_1/2$ and $\xi_2 = l_2/2$. Of course, results for the portions of the panels not modeled can be determined using the correct symmetry conditions with the available results. The global coordinate system for the finite element analyses is shown in Fig. 3.10b. Note that the origin of the global and shell coordinate systems are the same. Also, the straight panel edges lie in the x - y plane. Furthermore, at $\xi_2 = l_2/2$, the ξ_1 -, ξ_2 -, and ξ_3 -axes are parallel to the global x -, y -, and z -axes, respectively. The geometric or essential boundary conditions to be applied in the finite element analyses are then

$$\xi_1 = 0 : \quad v = w = 0 \quad (3.67)$$

$$\xi_1 = l_1/2 : \quad u = 0, \theta_y = \theta_z = 0 \quad (3.68)$$

$$\xi_2 = 0 : \quad u = 0, w \cos \phi - v \sin \phi = 0 \quad \text{or} \quad w = v \tan \phi \quad (3.69)$$

$$\xi_2 = l_2/2 : \quad v = 0, \theta_x = \theta_z = 0 \quad (3.70)$$

where u , v , and w respectively are the translations in the x -, y - and z -directions and θ_x , θ_y , and θ_z are the corresponding rotations.

A special technique must be employed to enforce the constraint $w = v \tan \phi$ along the edge $\xi_2 = 0$. Note that it is not possible to chose a different global coordinate system so that all of the geometric boundary conditions for the finite element analysis involve a single deflection. In order to minimize the required modifications for the TVATDKT-L code, a method which preserves the symmetry of the needed stiffness matrices is desired. Reddy³⁹ has described such a method. However, using the method presented by Reddy,³⁹ it would be

necessary to shift the rows and columns of the stiffness matrices before accounting for the desired constraints between some of the system degrees of freedom. Ultimately, this shifting allows for a straightforward reduction in the system size equal to the number of constraints enforced. Here, a method similar to that given by Reddy³⁹ will be used, but it will not be necessary to shift the rows and columns of the stiffness matrices.

The global degrees of freedom of the finite element system $\{U\}$ are rewritten in terms of $\{\bar{U}\}$ as follows:

$$\{U\} = [B]\{\bar{U}\} \quad (3.71)$$

Note that $\{U\}$ and $\{\bar{U}\}$ have the same dimensions. For the problem being studied, $[B]$ is equal to the appropriately-sized identity matrix with the rows corresponding to the w degrees of freedom along the edge $\xi_2 = 0$ modified to reflect the necessary constraints. Consider one of these w degrees of freedom to be U_M . The global degrees of freedom are numbered such that for the node which has $w = U_M$, $v = U_{M-1}$. Hence, the M^{th} row of $[B]$ is as follows:

$$\begin{aligned} B_{Mj} &= 0 \quad \text{for } j = 1, 2, \dots, M-3, M-2, M, M+1, \dots, N_{eq} \\ B_{M,M-1} &= \tan \phi \end{aligned} \quad (3.72)$$

where N_{eq} is the size of the assembled finite element equations.

For these panels, there are no hygroscopic loads and only quasi-static analysis is being considered so that Eq. (3.10) at $t = t^p$ is as follows:

$$\int_0^{t^p} [K(\zeta^p - \zeta')] \left\{ \frac{\partial U}{\partial \tau} \right\} d\tau = \{F^p\} \quad (3.73)$$

Using the expansion given in Eq. (3.23), the viscoelastic system of equations is

$$\int_0^{t^p} \sum_{r=1}^4 \left(Q_r^\infty + \sum_{\rho=1}^{N_r} Q_{r\rho} e^{-\frac{\zeta_r^p - \zeta_r'}{\lambda_{r\rho}}} \right) [K_r] \left\{ \frac{\partial U}{\partial \tau} \right\} d\tau = \{F^p\} \quad (3.74)$$

Now, the necessary constraints are enforced by replacing $\{U\}$ in terms of $\{\bar{U}\}$ in Eq. (3.74) and pre-multiplying that result by $[B]^T$ to give the following:

$$\int_0^{t^p} \sum_{r=1}^4 \left(Q_r^\infty + \sum_{\rho=1}^{N_r} Q_{r\rho} e^{-\frac{\zeta_r^p - \zeta_r'}{\lambda_{r\rho}}} \right) [\bar{K}_r] \left\{ \frac{\partial \bar{U}}{\partial \tau} \right\} d\tau = \{\bar{F}^p\} \quad (3.75)$$

where

$$[\bar{K}_r] = [B]^T [K_r] [B] \quad (3.76)$$

and

$$\{\bar{F}^p\} = [B]^T \{F^p\} \quad (3.77)$$

Using the previously developed solution technique will give the following system of equations to be solved for $\{\bar{U}^p\}$:

$$\left[\sum_{r=1}^4 \left(Q_r^\infty + \sum_{\rho=1}^{N_r} \bar{S}_{r\rho}^p Q_{r\rho} \right) [\bar{K}_r] \right] \{\bar{U}^p\} = \{\bar{F}^p\} - \sum_{r=1}^4 \sum_{\rho=1}^{N_r} \{\bar{R}_{r\rho}^p\} + \sum_{r=1}^4 \left(\sum_{\rho=1}^{N_r} \bar{S}_{r\rho}^p Q_{r\rho} \right) [\bar{K}_r] \{\bar{U}^{p-1}\} \quad (3.78)$$

where

$$\{\bar{R}_{r\rho}^p\} = \int_0^{t^{p-1}} Q_{r\rho} e^{-\frac{\zeta_r^p - \zeta_r'}{\lambda_{r\rho}}} [\bar{K}_r] \left\{ \frac{\partial \bar{U}}{\partial \tau} \right\} d\tau \quad (3.79)$$

$$= e^{-\frac{\Delta \zeta_r^p}{\lambda_{r\rho}}} \left[\{\bar{R}_{r\rho}^{p-1}\} + \bar{S}_{r\rho}^{p-1} Q_{r\rho} [\bar{K}_r] \{\Delta \bar{U}^{p-1}\} \right] \quad (3.80)$$

For each U_M which is replaced in terms of $\bar{U}_{M-1}$, the M^{th} column of $[B]$ is zeros and, hence, the M^{th} row and column of $[\bar{K}_r]$ will be zeros and the M^{th} component of $\{\bar{F}^p\}$ will also be zero. Hence, in order to prevent a singular matrix on the right side of Eq. (3.78), the diagonal entry on the M^{th} row of $[\bar{K}_r]$ is set to unity (i.e., $[\bar{K}_r]_{(M,M)} = 1$). Because $\{\bar{R}_{r\rho}^0\} = \{0\}$ and the M^{th} component of $\{\bar{F}^p\}$ is zero, the value of $\bar{U}_M$ will always be computed to be zero. Despite this, the effect of U_M still has been properly accounted for during the solution process. If desired, U_M can be simply computed from $\bar{U}_{M-1}$ after the solution procedure is complete. The necessary changes to the TVATDKT-L code for this example have been coded.

For the $\phi = 12^\circ$ panel, the TVATDKT-L results are computed using 450 elements for the quarter-panel and a time step of $\Delta t = 100$ min. The time histories of midpoint transverse deflection $(v_3)_{mid}$ for the 12° half-angle panel computed using shallow shell theory and the TVATDKT-L code are shown in Fig. 3.11. The mesh size of 450 elements and time step of 100 min used in producing the TVATDKT-L results are converged, as using 968 elements and $\Delta t = 25$ min did not significantly alter the TVATDKT-L results for $(v_3)_{mid}$. Shown

in Fig. 3.12 are plots of the closed-form and TVATDKT-L results for the v_3 deflection at the cross section with $\xi_1 = x = 0.5 l_1$ for $t = 100$ min and $t = 600$ hr. The “y” coordinate in Fig. 3.12 refers to a point’s location in the undeformed panel. The undeformed mesh and the deformed mesh at $t = 600$ hr are shown in Fig. 3.13. Note that the deformations are scaled by a factor of 200 in Fig. 3.13b in order to better ascertain the deformation pattern. The panel deforms such that a single extremum exists at the panel center for the time range considered here. However, it is apparent from Fig. 3.12 that the creep growth of the deflections is not spatially uniform, due to the applied boundary conditions and the anisotropic material properties. The applied pressure causes the two ξ_2 -constant edges to be pulled towards the panel center, while the two ξ_1 -constant edges are pushed away from the panel center, but to a lesser extent. Because of the applied boundary conditions, the four corners do not translate. The agreement between the closed-form and TVATDKT-L results for this panel is very good.

Shown in Fig. 3.14 are the results for $(v_3)_{mid}$ for the 18° half-angle panel computed using shallow shell theory and the TVATDKT-L code with 1800 elements and a time step of 100 min. This mesh size is based upon a mesh convergence study. Although a relatively fine mesh is used, the results produced using a coarser mesh of 968 elements and $\Delta t = 100$ min are fairly close to those computed using a 1800 elements and the same time step size. For instance, $(v_3)_{mid}$ at $t = 600$ hr differed by approximately 0.5% between these two sets of results. Furthermore, the time step of 100 min is sufficient for convergence, as using a time step of 25 min did not significantly alter the results produced using 1800 elements.

The undeformed mesh and the deformed mesh at $t = 600$ hr for the $\phi = 18^\circ$ panel are shown in Fig. 3.15. The deformations have been multiplied by a factor of 300 in producing the deformed mesh plot. Shown in Fig. 3.16 are graphs of the closed-form and TVATDKT-L values for the v_3 deflection for the cross section at $\xi_1 = x = 0.5 l_1$ at $t = 100$ min and $t = 600$ hr. Once again, “y” in these plots refers to the undeformed configuration. The deformed shape of the shell at any given time is similar to that shown in Fig. 3.15b. Similar to the 12° half-angle panel, the creep growth of the viscoelastic deflections is spatially

nonuniform, as indicated in Fig. 3.16. The differences between the shallow shell theory and finite element results for this panel are somewhat larger than those for the $\phi = 12^\circ$ panel. This is to be expected, as the accuracy of shallow shell theory decreases as the panel half-angle increases. However, the TVATDKT-L and shallow shell theory results for this panel are still comparable.

Table 3.1: Components of normalized function giving viscoelastic graphite-epoxy property time variation.

i	f_i	λ_i (sec)
0	0.06698253	
1	0.0729459	$8.174141919 \times 10^{15}$
2	0.0696426	$4.976486103 \times 10^{14}$
3	0.150514	$1.477467149 \times 10^{13}$
4	0.148508	$4.761315266 \times 10^{11}$
5	0.146757	$1.799163029 \times 10^{10}$
6	0.102892	5.253922053×10^8
7	0.114155	1.846670914×10^7
8	0.071036	5.288067476×10^5
9	0.0484272	1.494783951×10^4
10	0.00813977	5.516602214×10^2

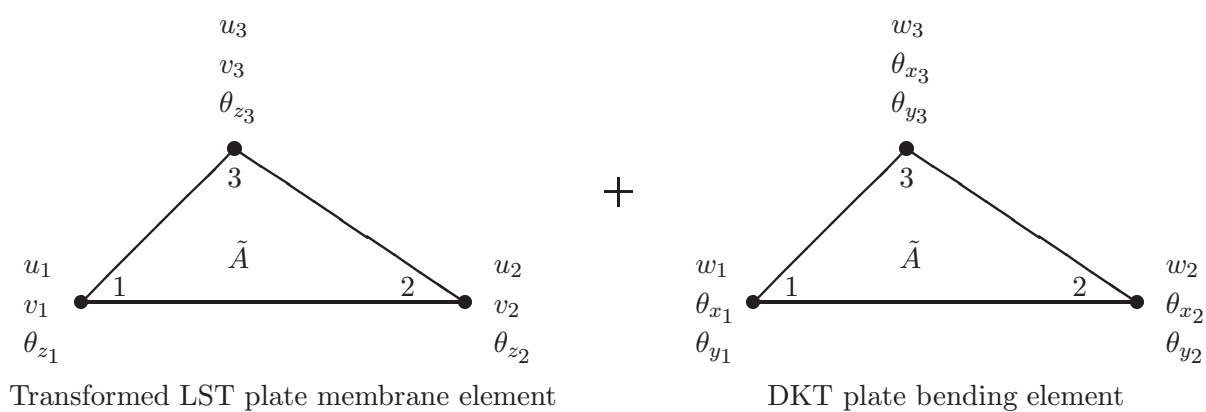
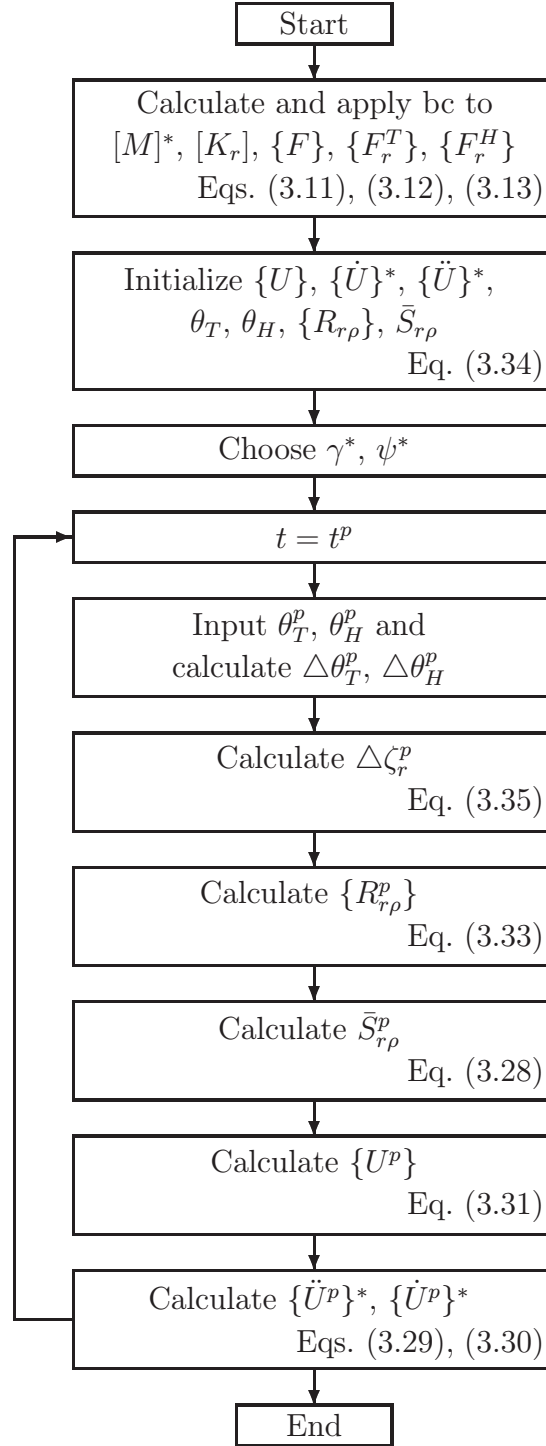


Figure 3.1: Plate membrane and bending elements comprising the triangular flat shell element.



*if inertia is to be included

Figure 3.2: TVATDKT-L solution procedure for calculation of nodal displacements, velocities, and accelerations.

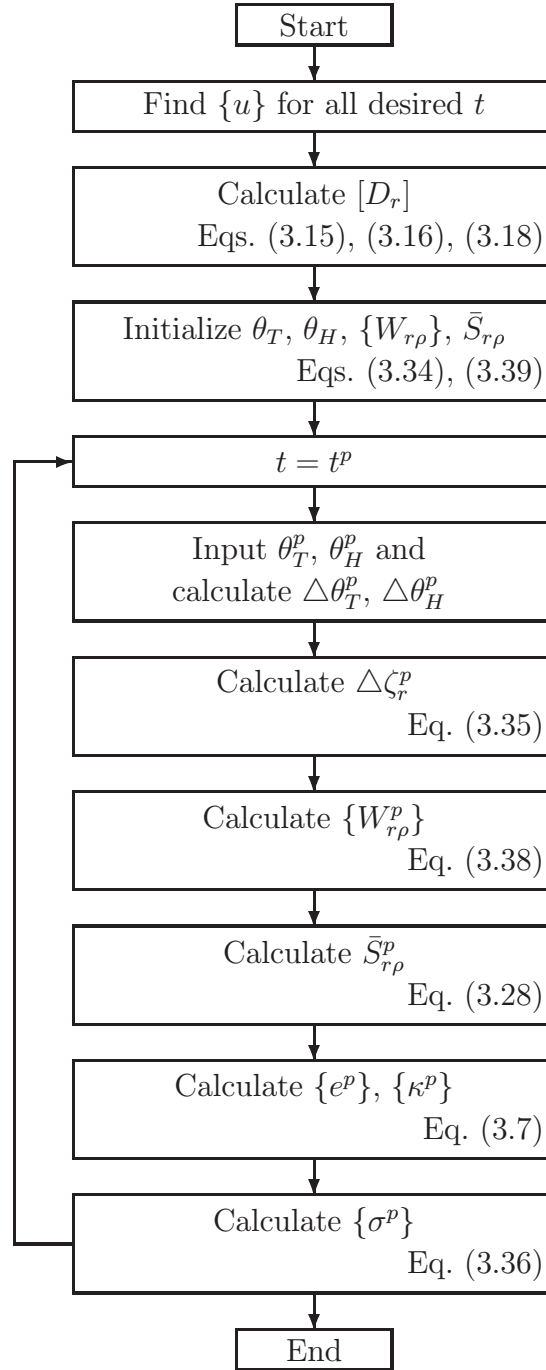


Figure 3.3: TVATDKT-L solution procedure for calculation of elemental stress and strain at a specified point.

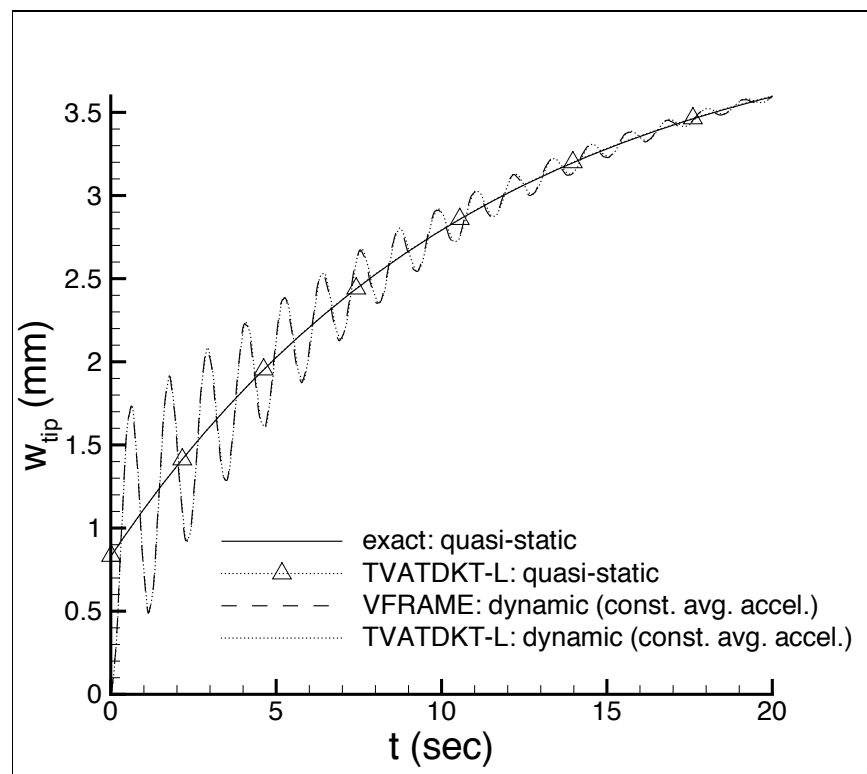


Figure 3.4: Tip deflection of a viscoelastic cantilever beam under a step tip force.

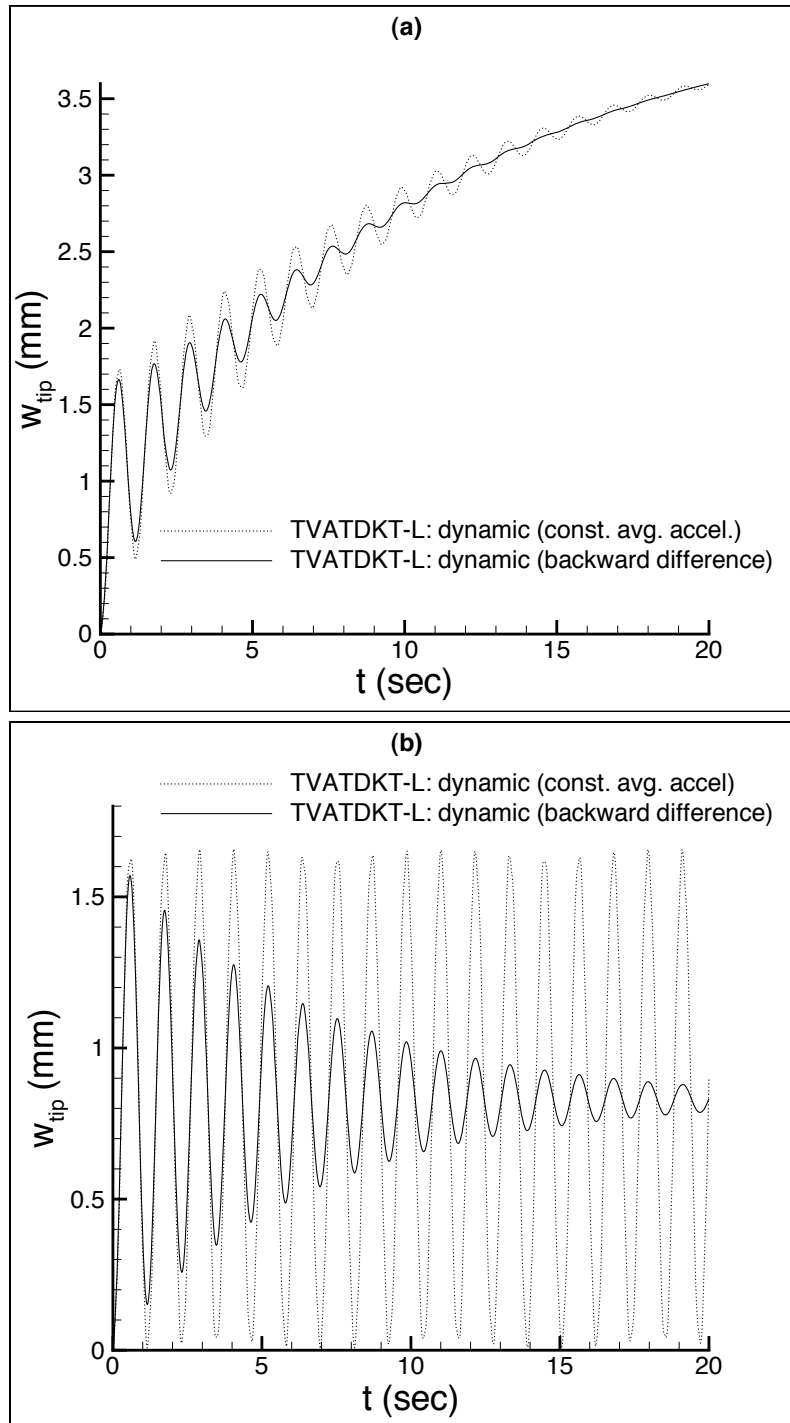


Figure 3.5: Comparison of constant average acceleration and backward difference Newmark methods for the tip deflection of a cantilever beam under a step tip force: (a) viscoelastic results; (b) elastic results using initial viscoelastic properties.

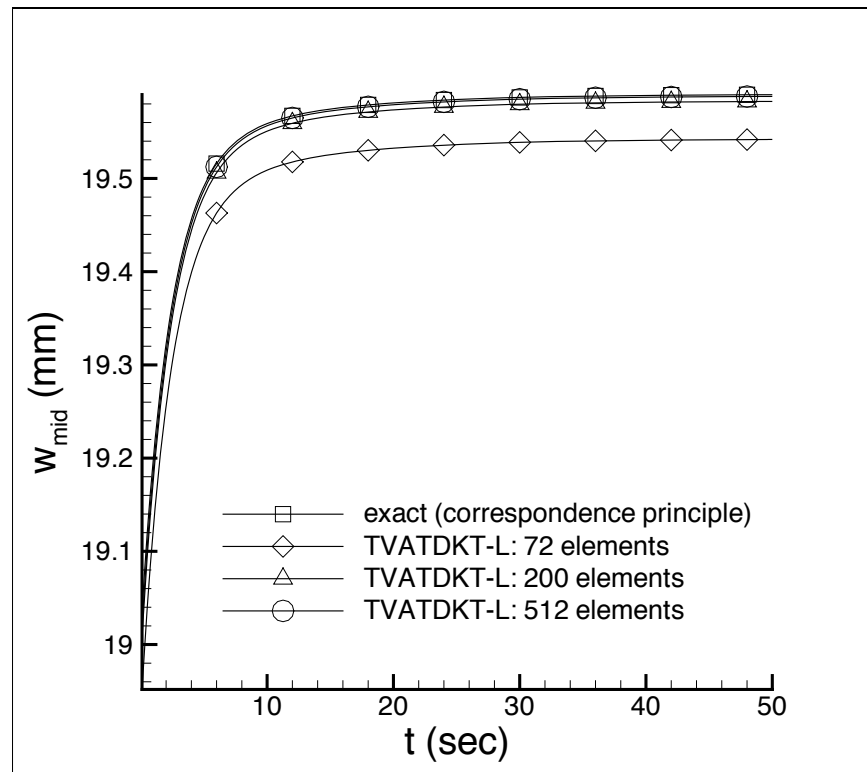


Figure 3.6: Midpoint transverse deflection of a simply supported square viscoelastic $[0/90]_s$ composite plate under uniform pressure load.

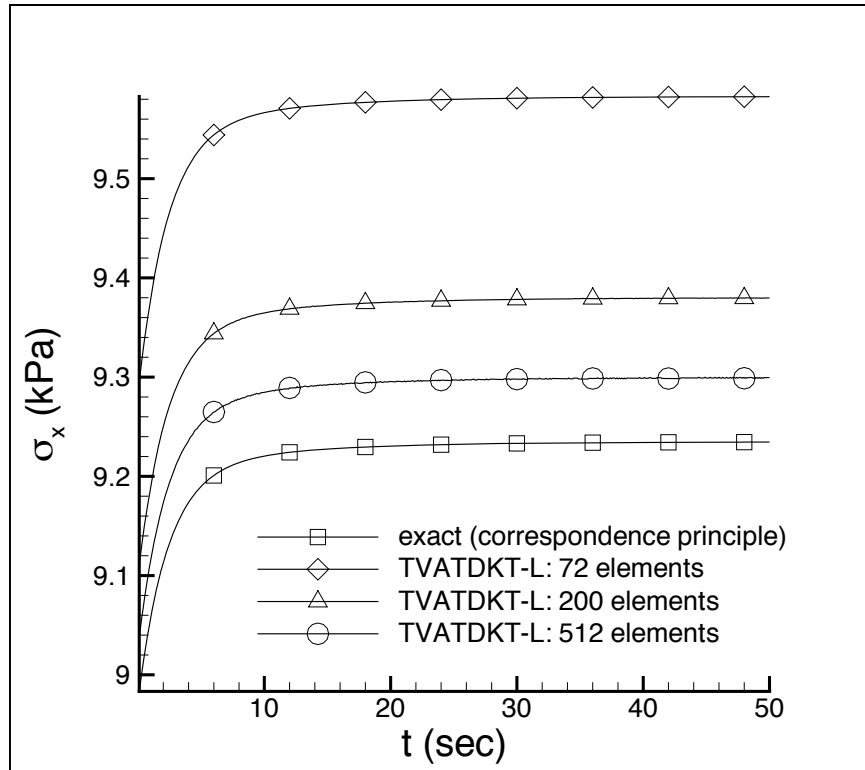


Figure 3.7: Midpoint σ_x at $z = z_{max}$ of a simply supported square viscoelastic $[0/90]_s$ composite plate under uniform pressure load.

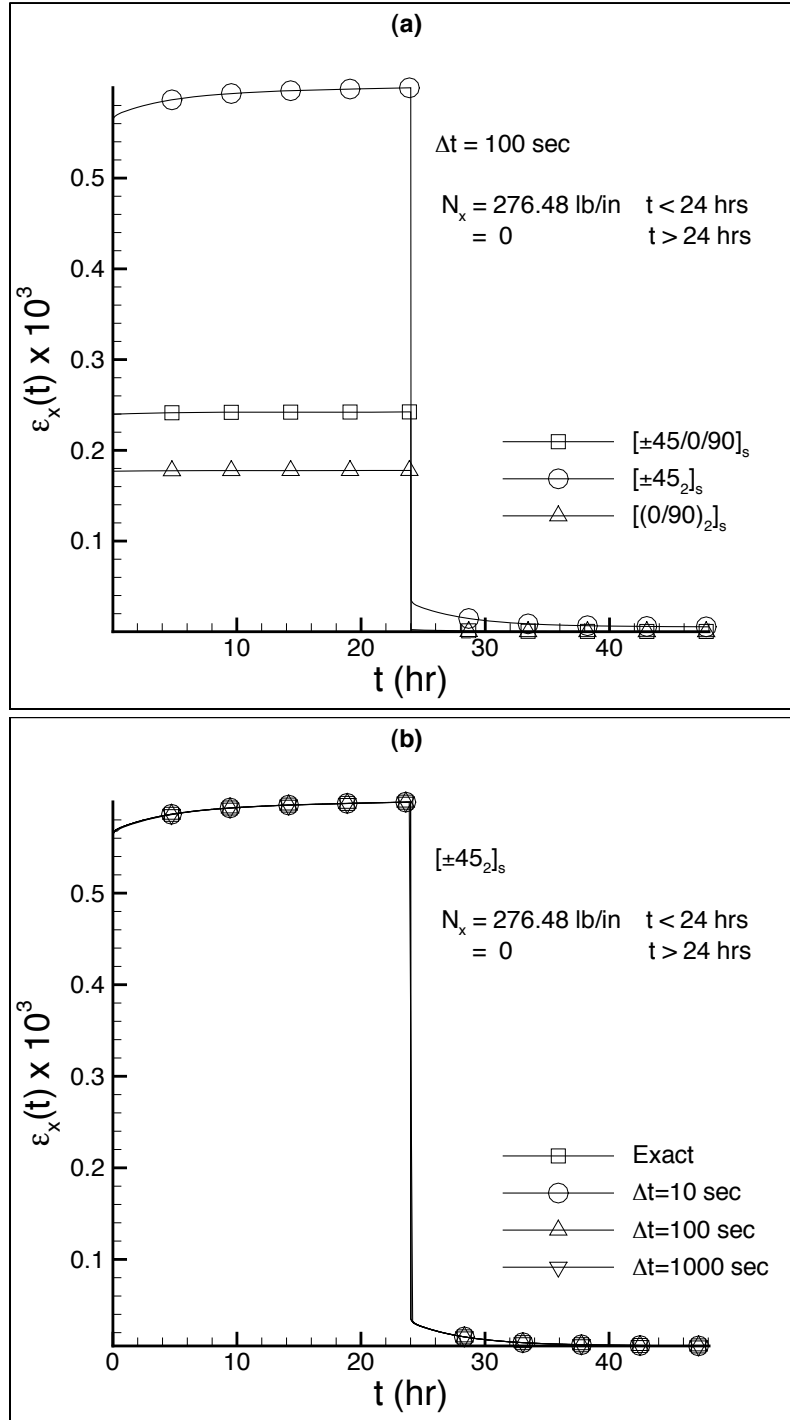


Figure 3.8: Free viscoelastic composite plates subjected to sudden loading and unloading analyzed using TVATDKT-L: (a) $[\pm 45/0/90]_s$, $[\pm 45]_{2s}$ and $[(0/90)_2]_s$ with $\Delta t = 100 \text{ sec}$; (b) $[\pm 45]_{2s}$ with $\Delta t = 10, 100, 1000 \text{ sec}$.

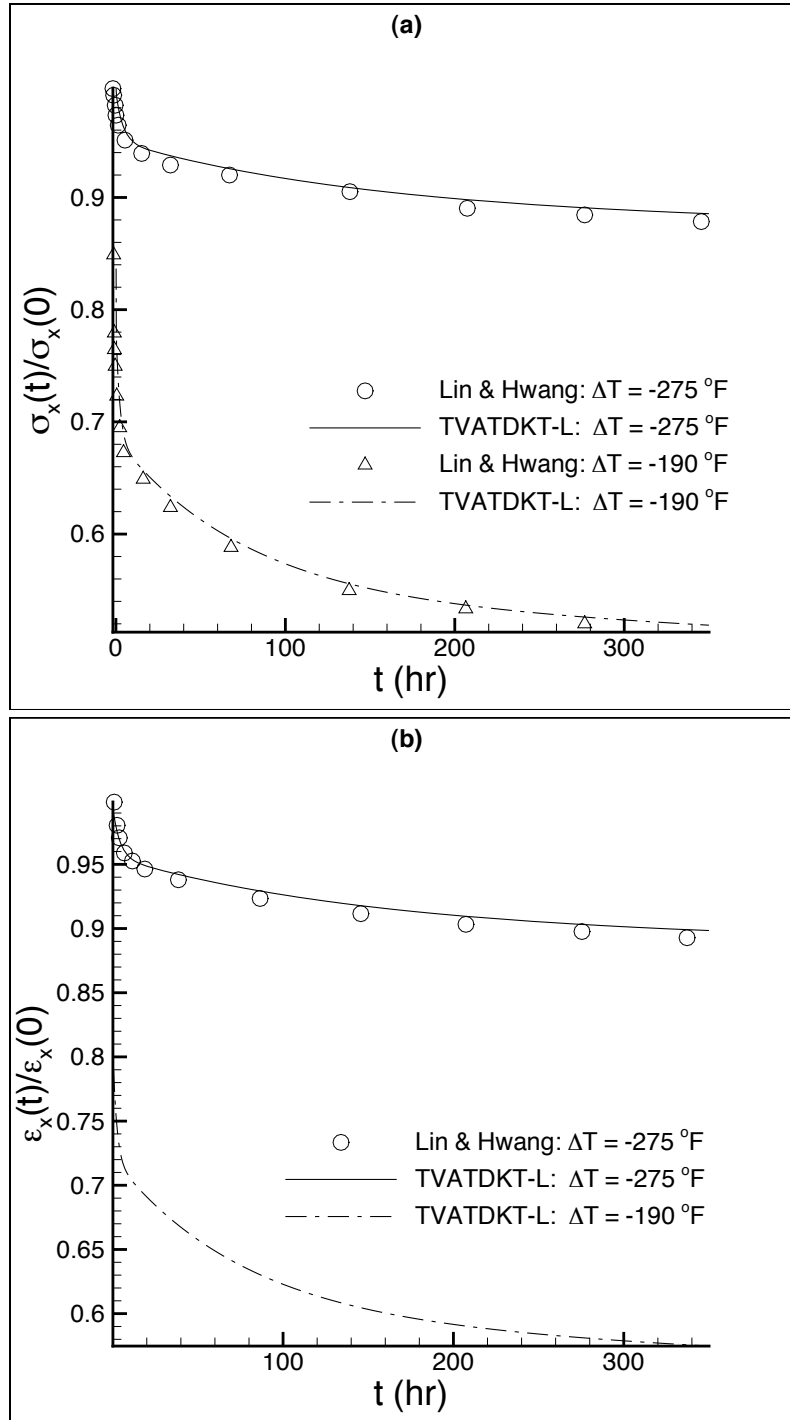


Figure 3.9: Free viscoelastic $[\pm 45/0/90]_s$ plate subjected to uniform ΔT applied at $t = 0$ and held constant thereafter: (a) normalized σ_x in 0° layer; (b) normalized ϵ_x .

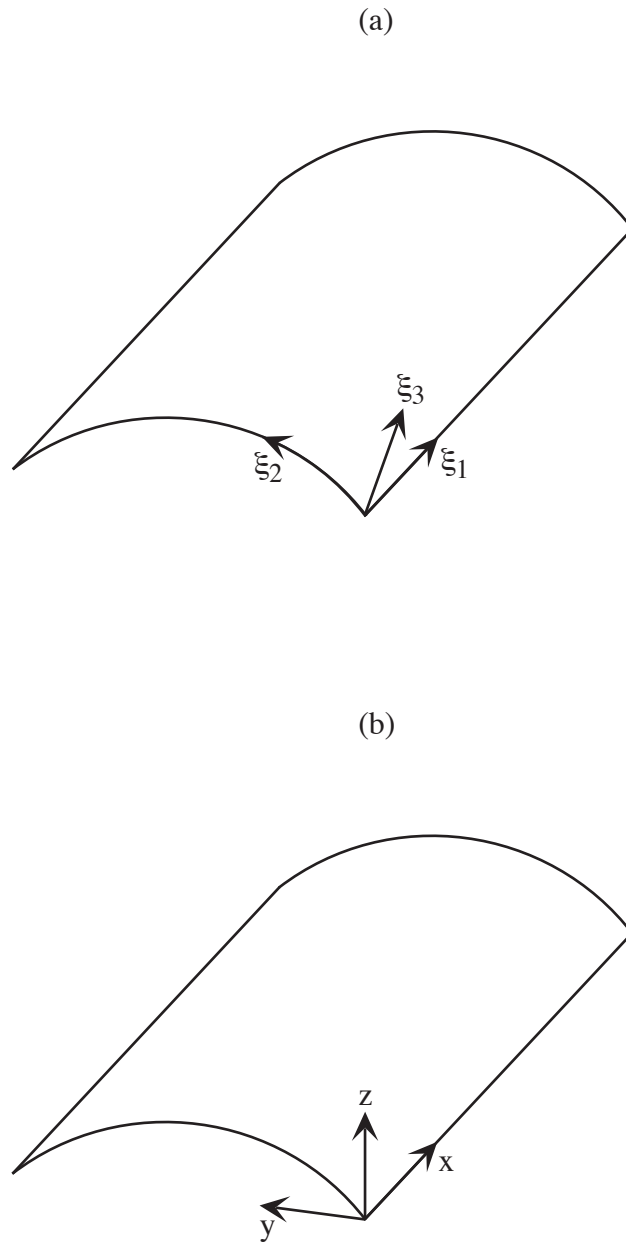


Figure 3.10: Coordinate systems for cylindrical panels under uniform pressure: (a) shell coordinate system; (b) global x - y - z coordinate system.

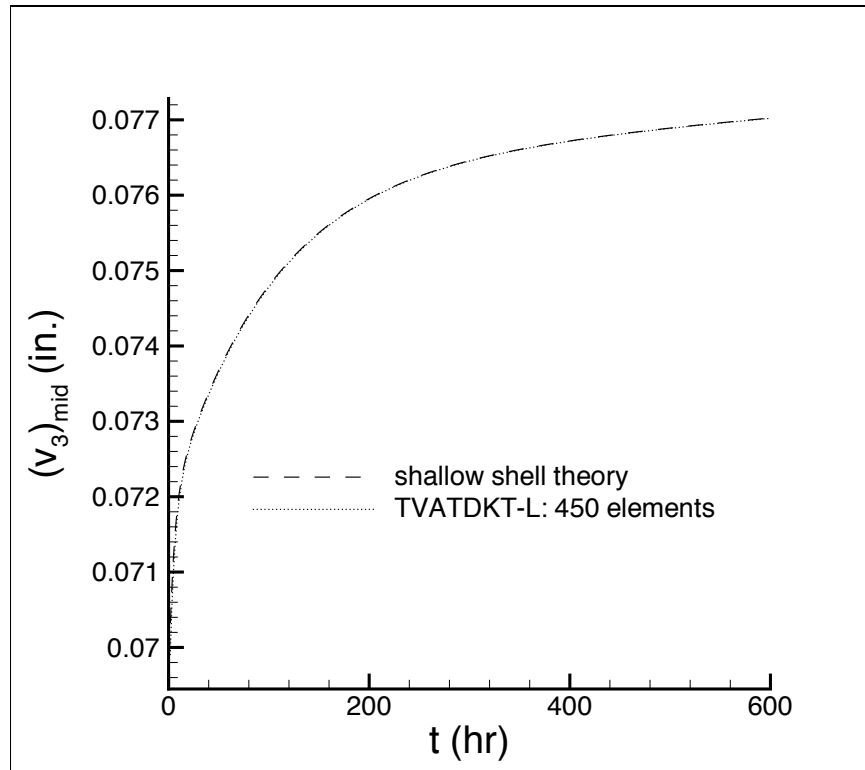


Figure 3.11: Midpoint transverse deflection, $(v_3)_{mid}$, of a viscoelastic 12° half-angle $[0/90]_s$ cylindrical panel subjected to a step uniform pressure load.

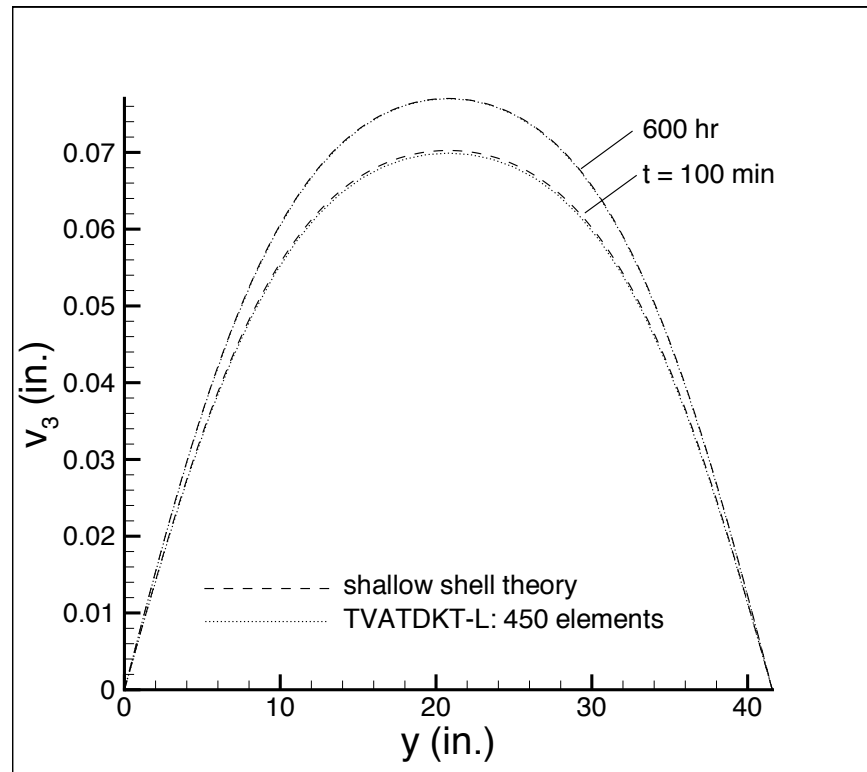


Figure 3.12: Transverse deflection, v_3 , at panel mid-section ($\xi_1 = 1/2 l_1$) of a viscoelastic $[0/90]_s$ cylindrical panel with 12° half-angle subjected to a step uniform pressure load. The results are shown for $t = 100$ min and $t = 600$ hr.

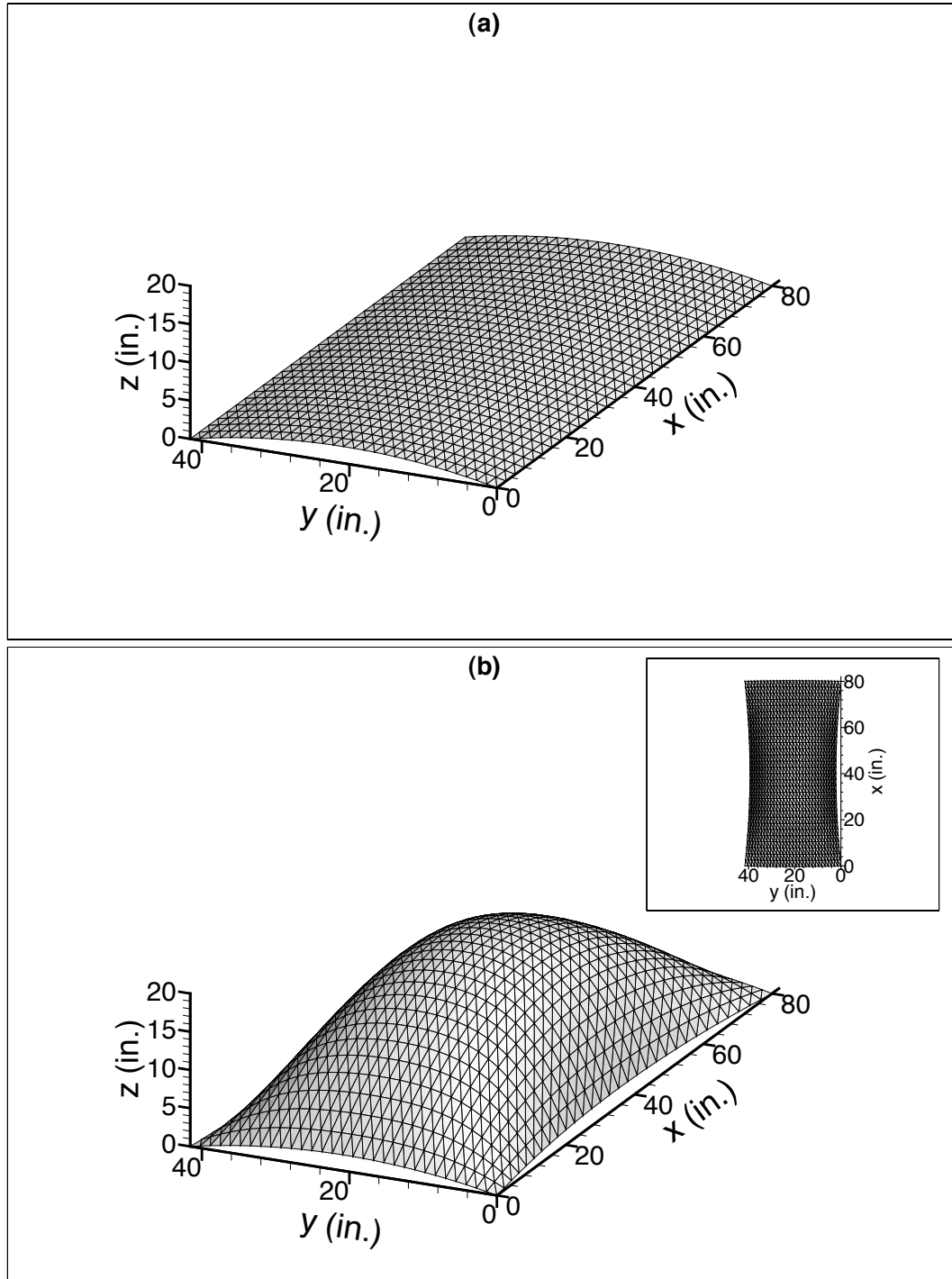


Figure 3.13: Viscoelastic $[0/90]_s$ cylindrical panel with 12° half-angle subjected to step uniform pressure: (a) undeformed mesh; (b) deformed mesh at $t = 600$ hr with displacements multiplied by 200.

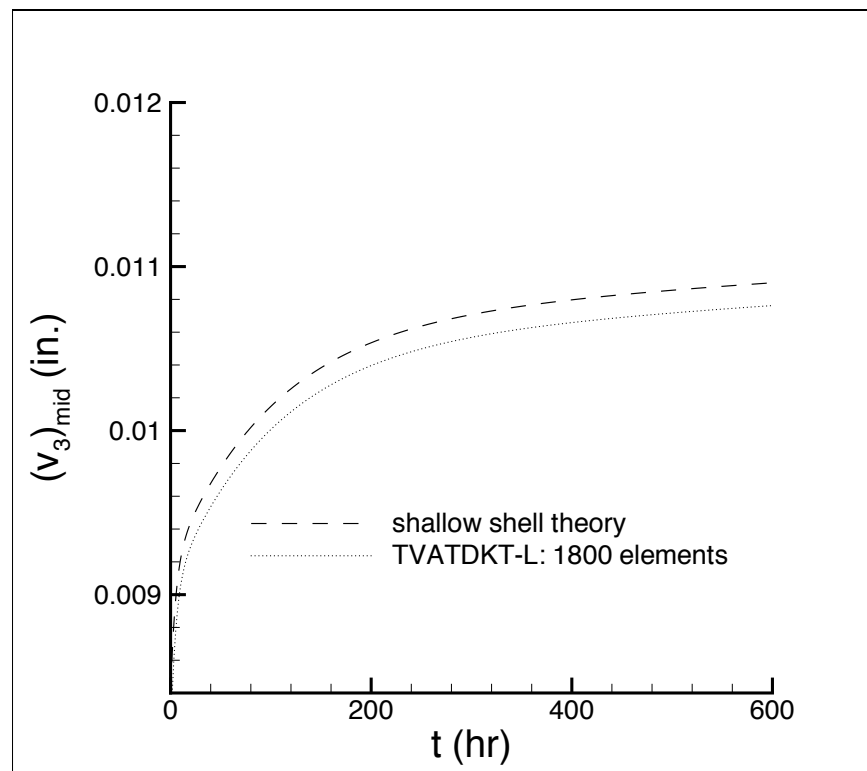


Figure 3.14: Midpoint transverse deflection, $(v_3)_{mid}$, of a viscoelastic 18° half-angle $[0/90]_s$ cylindrical panel subjected to a step uniform pressure load.

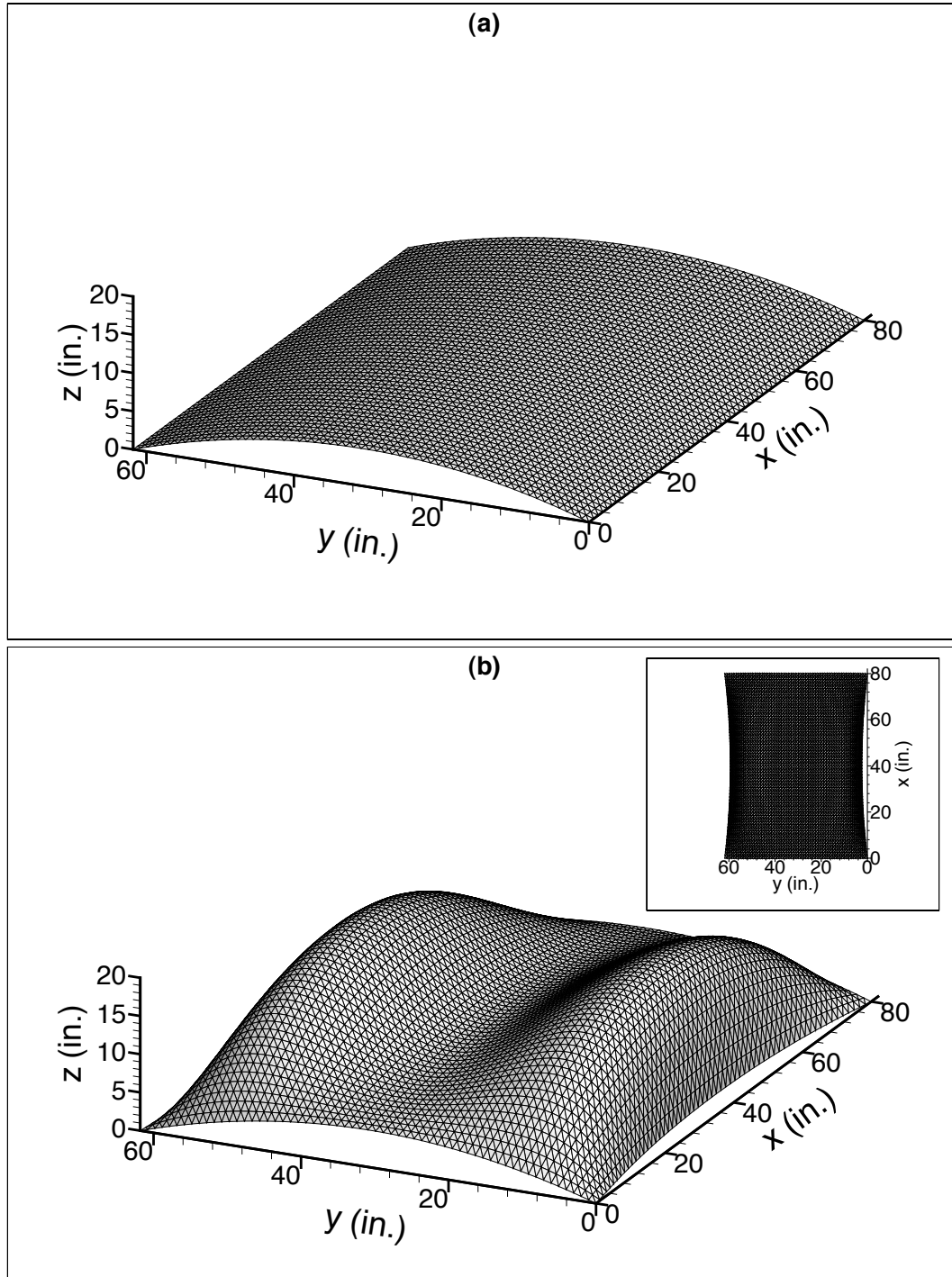


Figure 3.15: Viscoelastic $[0/90]_s$ cylindrical panel with 18° half-angle subjected to step uniform pressure: (a) undeformed mesh; (b) deformed mesh at $t = 600$ hr with displacements multiplied by 300.

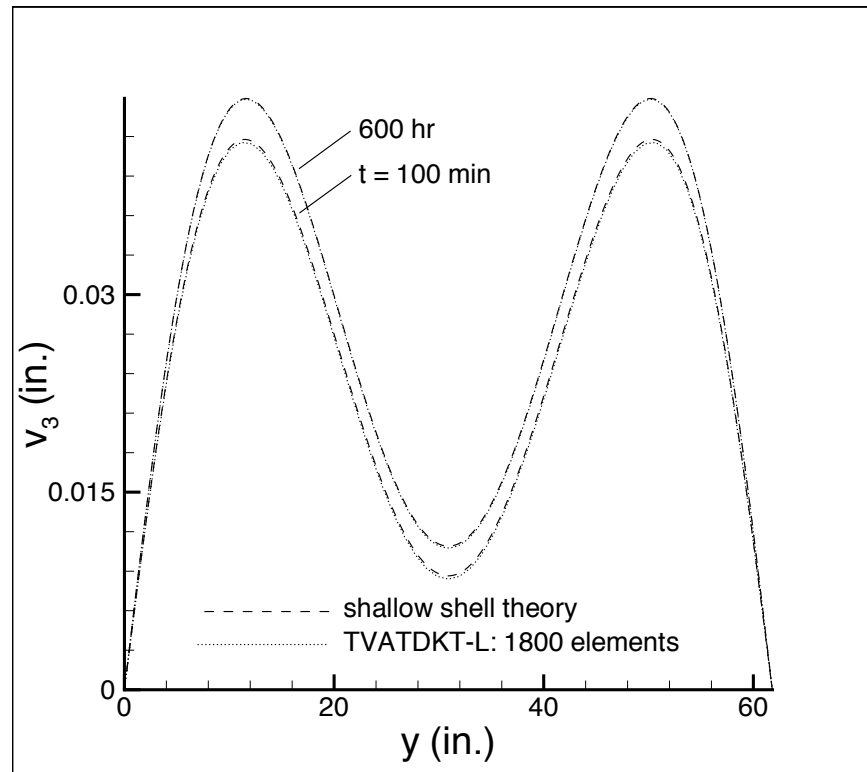


Figure 3.16: Transverse deflection, v_3 , at panel mid-section ($\xi_1 = 1/2 l_1$) of a viscoelastic $[0/90]_s$ cylindrical panel with 18° half-angle subjected to a step uniform pressure load. Results are shown for $t = 100$ min and $t = 600$ hr.

Chapter 4

Geometrically-Nonlinear Analysis of Linear Viscoelastic Composites

In this chapter, the triangular flat shell element formulation will be extended to the large deformation range. The incremental/iterative solution technique is described in detail. The elemental internal force vector is derived from the expression for the virtual work of the internal stresses, while the expression for the variation in the internal force vector corresponding to arbitrary virtual displacements is used to develop the elemental tangent stiffness matrix. A range of numerical examples are solved, demonstrating the accuracy and capability of the formulation. Creep buckling or snap-through instabilities occur in some of the presented examples.

The geometrically-nonlinear formulation for the linear elastic case has been presented by Mohan and Kapania.⁴⁹ Henceforth, the geometrically-nonlinear, linear elastic element will be referred to as ATDKT-NL (Allman Triangle, Discrete-Kirchhoff Theory triangle, geometrically Non-Linear), while the present linear viscoelastic extension will be termed TVATDKT-NL. Most of this chapter is presented in Refs. 51–53.

4.1 Formulation

4.1.1 Incremental Finite Element Analysis

For the regime of large deflections/rotations, the static elastic response for load levels up to the maximum load is of interest, as is the entire time history of response in the dynamic elastic case and the quasi-static and dynamic viscoelastic cases. Hence, an incremental approach is usually used in the finite element analysis of large deformation processes. In the static elastic case, the increments correspond to steps in the loading, while for the dynamic elastic case and the quasi-static and dynamic viscoelastic cases, the increments correspond to steps in the physical time.

For a geometrically-nonlinear formulation, appropriate work-conjugate stress and strain measures must be used. One such choice is the second Piola Kirchhoff (PK2) stress tensor and the Green–Saint-Venant strain tensor. These two tensors can be measured in any known configuration, leading to two alternative methods referred to as the total Lagrangian method and the updated Lagrangian method. In the total Lagrangian method, the original configuration is used as the reference state for the PK2 stress and Green–Saint-Venant strain tensors throughout the entire deformation process, whereas for the updated Lagrangian method the reference state is updated throughout the deformation process. For both methods, an iterative technique such as the Newton-Raphson method is used to converge the response in each increment. In the updated Lagrangian approach, the reference state may be updated after each iteration, or only at the end of each increment when the converged quasi-static or dynamic equilibrium state has been found. Both variations have been implemented by Fafard *et al.*⁵⁴ respectively as ULF2 and ULF1. ULF1 is equivalent to using a total Lagrangian approach within each increment and an updated Lagrangian approach from increment to increment. Here, the ULF1 method will be used.

Throughout this chapter, left superscripts will be used to denote the configuration in which a quantity occurs, whereas left subscripts will be used to denote the configuration in which the quantity is measured. For instance, the global internal force vector, $\{F_{int}\}$,

occurring at time t^p but measured in the configuration corresponding to time t^{p-1} , is denoted as $\{^p_{p-1}F_{int}\}$.

4.1.2 Newton-Raphson Method

Assuming that the equilibrium state at time t^{p-1} is known, the incremental equations needed for the determination of the equilibrium state at time t^p are developed as follows. The current global residual vector $\{^p_{p-1}\Psi\}$ is determined in terms of the current global internal force vector $\{^p_{p-1}F_{int}\}$ and the current global external force vector $\{^p_{p-1}F_{ext}\}$ as

$$\{^p_{p-1}\Psi(\{U\})\} = \{^p_{p-1}F_{int}(\{U\})\} - \{^p_{p-1}F_{ext}(\{U\})\} \quad (4.1)$$

where $\{U\}$ is the global vector of incremental displacements for the current time increment. The goal is to obtain $\{U\}$ giving the residual as $\{0\}$. Iterative equations for $\{U\}$ are obtained by linearizing $\{^p_{p-1}\Psi(\{U^{(n+1)}\})\}$ about $\{U^{(n)}\}$ and driving $\{^p_{p-1}\Psi(\{U^{(n+1)}\})\}$ to $\{0\}$ where $(n+1)$ denotes the current iteration. That is, $\{U^{(n+1)}\}$ is determined from

$$\{^p_{p-1}\Psi(\{U^{(n+1)}\})\} \approx \{^p_{p-1}\Psi(\{U^{(n)}\})\} + \left. \frac{\partial\{\Psi\}}{\partial\{U\}} \right|_{\{U^{(n)}\}} \{U^{(n+1)} - U^{(n)}\} = \{0\} \quad (4.2)$$

In the present formulation, the only external load which is taken to be deformation-dependent is the pressure load which is denoted as $\{F_P\}$. Then $\{\Delta U^{(n+1)}\}$, the change in $\{U\}$ from the previous iteration, is obtained by solving

$$[K_T^{(n)} - K_P^{(n)}] \{\Delta U^{(n+1)}\} = \{^p_{p-1}F_{ext}(\{U^{(n)}\})\} - \{^p_{p-1}F_{int}(\{U^{(n)}\})\} \quad (4.3)$$

where $[K_T]$ is the global tangent stiffness matrix given by

$$[K_T] = \frac{\partial\{F_{int}\}}{\partial\{U\}} \quad (4.4)$$

and $[K_P]$ is the global pressure stiffness matrix given by

$$[K_P] = \frac{\partial\{F_P\}}{\partial\{U\}} \quad (4.5)$$

The derivation of the deformation-dependent pressure load $\{F_p\}$ and the pressure stiffness matrix $[K_p]$ is presented in Refs. 52 and 55. Because neither $\{F_p\}$ nor $[K_p]$ involves material properties, no modifications are necessary to use these quantities when modeling linear

viscoelastic composite structures. Quantities corresponding to $[K_T]$, $[K_P]$, $\{U\}$, $\{F_{ext}\}$, and $\{F_{int}\}$ for an element will be denoted with the corresponding lower case letters.

4.1.3 Derivation of the Internal Force Vector

The internal force vector $\{f_{int}\}$ for an element is derived from the principle of virtual work as follows:

$$\delta W_{int} = \{\delta u\}^T \{f_{int}\} = \int_{A^{p-1}} \int_{z_{min}}^{z_{max}} \{\delta \epsilon_{p-1}^p\}^T \{S_{p-1}^p\} dz dA^{p-1} \quad (4.6)$$

where $\{\epsilon_{p-1}^p\}$ is the vector of total Green–Saint-Venant strains, $\{S_{p-1}^p\}$ is the vector of second Piola-Kirchhoff (PK2) stresses, and A^{p-1} refers to the area of the element at time t^{p-1} . Note that $\{\epsilon_{p-1}^p\}$ and $\{S_{p-1}^p\}$ can be expressed in component form for any defined coordinate system. In the present formulation, for each element they are expressed in a local coordinate system defined by the element geometry at time t^{p-1} . The reason for this choice will become clear later in the formulation. At any given time, an element has the following local coordinate system: the local x -axis is aligned with side 1-2 of the element, while the local y -axis lies in the plane of the element, and the local z -axis points in the direction of the normal to the element. An element undergoing deformation from t^{p-1} to t^p is shown in Fig. 4.1. The local coordinate systems for the element at times t^{p-1} and t^p are also indicated.

Assuming thin shells undergoing moderate incremental rotations, the total Green–Saint-Venant strains are written as

$$\{\epsilon_{p-1}^p\} = \{e_{p-1}^p\} + z \{\kappa_{p-1}^p\} \quad (4.7)$$

where $\{e_{p-1}^p\}$ and $\{\kappa_{p-1}^p\}$ respectively denote the total mid-surface inplane strains and bending curvatures. The variation in the vector of Green–Saint-Venant strains is then

$$\{\delta \epsilon_{p-1}^p\} = \{\delta e_{p-1}^p\} + z \{\delta \kappa_{p-1}^p\} \quad (4.8)$$

The incremental midplane strains $\{\Delta_{p-1}^p e\}$ and bending curvatures $\{\Delta_{p-1}^p \kappa\}$ are given in

terms of the incremental deformations as

$$\{\Delta_{p-1}^p e\} = \{p_{p-1} e\} - \{p_{p-1}^{-1} e\} = \left\{ \begin{array}{c} u_{,x} + 1/2 (u_{,x}^2 + v_{,x}^2 + w_{,x}^2) \\ v_{,y} + 1/2 (u_{,y}^2 + v_{,y}^2 + w_{,y}^2) \\ u_{,y} + v_{,x} + u_{,x} u_{,y} + v_{,x} v_{,y} + w_{,x} w_{,y} \end{array} \right\} \quad (4.9)$$

and

$$\{\Delta_{p-1}^p \kappa\} = \{p_{p-1} \kappa\} - \{p_{p-1}^{-1} \kappa\} = \{\beta_{x,x} \quad \beta_{y,y} \quad (\beta_{x,y} + \beta_{y,x})\}^T \quad (4.10)$$

where u, v , and w are the incremental midplane translations, β_x and β_y are the incremental rotations of the midplane normal about the positive local y - and negative x -axes, respectively, and $(\cdot)_{,x}$ and $(\cdot)_{,y}$ respectively denote $\partial(\cdot)/\partial x$ and $\partial(\cdot)/\partial y$.

In terms of the incremental nodal displacements, the spatial derivatives of the incremental displacements for an element are written as⁴⁹

$$\{u_{,x} \quad u_{,y} \quad v_{,x} \quad v_{,y} \quad w_{,x} \quad w_{,y}\}^T = [G_2]\{u\} \quad (4.11)$$

The incremental inplane strains are then easily determined using Eq. (4.9). The incremental bending curvatures are written in terms of the incremental nodal displacements as⁴⁹

$$\{\Delta_{p-1}^p \kappa\} = [B_{dkt}]\{u\} \quad (4.12)$$

At a given location within an element, $[G_2]$ and $[B_{dkt}]$ depend only upon the nodal coordinates of the element at the end of the last fully converged time step. The variations of the in-plane strains and bending curvatures can be written in terms of the variations of the incremental nodal displacements as

$$\{\delta_{p-1}^p e\} = [G_1(\{u\})][G_2]\{\delta u\} \quad (4.13)$$

$$\{\delta_{p-1}^p \kappa\} = [B_{dkt}]\{\delta u\} \quad (4.14)$$

where $[G_1]$ is given by

$$[G_1] = \begin{bmatrix} 1 + u_{,x} & 0 & v_{,x} & 0 & w_{,x} & 0 \\ 0 & u_{,y} & 0 & 1 + v_{,y} & 0 & w_{,y} \\ u_{,y} & 1 + u_{,x} & 1 + v_{,y} & v_{,x} & w_{,y} & w_{,x} \end{bmatrix} \quad (4.15)$$

The current force resultant $\{^p_{p-1}N\}$ and force-couple resultant $\{^p_{p-1}M\}$ are defined as

$$\{^p_{p-1}N\} = \int_{z_{min}}^{z_{max}} \{^p_{p-1}S\} dz \quad (4.16)$$

$$\{^p_{p-1}M\} = \int_{z_{min}}^{z_{max}} z \{^p_{p-1}S\} dz \quad (4.17)$$

Although $\{N\}$ and $\{M\}$ are here termed force resultant and force-couple resultant, respectively, it should be noted that $\{N\}$ and $\{M\}$ have little physical meaning, because they are based upon integrating PK2 pseudo-stresses through the thickness.

Using Eqs. (4.6), (4.8), (4.13), (4.14), (4.16), and (4.17), the internal force vector for an element is determined to be

$$\{^p_{p-1}f_{int}\} = \int_{A^{p-1}} \left([G_2]^T [G_1]^T \{^p_{p-1}N\} + [B_{dkt}]^T \{^p_{p-1}M\} \right) dA^{p-1} \quad (4.18)$$

4.1.4 Derivation of Force and Force-Couple Resultants

The current PK2 stresses are written in the following incremental form:

$$\{^p_{p-1}S\} = \{^{p-1}_{p-1}S\} + \{\Delta^p_{p-1}S\} \quad (4.19)$$

Note that $\{^{p-1}_{p-1}S\}$ is actually a vector of Cauchy stresses. Obviously, after an increment has been converged, it will be necessary to transform the PK2 stress $\{^p_{p-1}S\}$ into the Cauchy stress $\{^p_pS\}$ in order to be used for the next increment. This transformation is performed using the deformation gradient tensor.³⁴ However, for the case of small incremental strains, it can be shown that the current PK2 stress ($\{^p_{p-1}S\}$) expressed in the local coordinate system corresponding to the element at time t^{p-1} is approximately equal to the current Cauchy stress ($\{^p_pS\}$) expressed in the local coordinate system corresponding to the element at time t^p .⁵⁴ This approximation was successfully employed by Mohan and Kapania⁴⁹ and will be used here also.

For the case of a hygrothermo-rheologically simple linear viscoelastic composite material, the stress and strain are related as follows:

$$\{^p_{p-1}S\} = \int_0^{t^p} [\bar{Q}(\zeta^p - \zeta')] \left\{ \frac{\partial_{p-1}^{\tau} \bar{\epsilon}}{\partial \tau} \right\} d\tau \quad (4.20)$$

$$\{^{p-1}_{p-1}S\} = \int_0^{t^{p-1}} [\bar{Q}(\zeta^{p-1} - \zeta')] \left\{ \frac{\partial_{p-1}^{\tau} \bar{\epsilon}}{\partial \tau} \right\} d\tau \quad (4.21)$$

where $\{\bar{\epsilon}\}_{p-1}^\tau$ represents the vector of total mechanical strains up to time τ and is given in terms of the corresponding total strains as

$$\begin{aligned}\{\bar{\epsilon}\}_{p-1}^\tau &= \{\epsilon\}_{p-1}^\tau - \theta_T^\tau \{\alpha\} - \theta_H^\tau \{\bar{\beta}\} \\ &= \{e\}_{p-1}^\tau + z \{\kappa\}_{p-1}^\tau - \theta_T^\tau \{\alpha\} - \theta_H^\tau \{\bar{\beta}\}\end{aligned}\quad (4.22)$$

where θ_T^τ and θ_H^τ are the deviation of the temperature and moisture from the thermal and hygroscopic strain free states at time τ , respectively, and $\{\alpha\}$ and $\{\bar{\beta}\}$ respectively are the transformed coefficients of thermal and hygroscopic expansion. Both the temperature and moisture are assumed to be uniform through the thickness of the laminate, while the coefficients of thermal and hygroscopic expansion depend upon the fiber orientation of the layer being considered. In Eqs. (4.20) and (4.21), the reduced times ζ^p and ζ^{p-1} correspond to t^p and t^{p-1} , respectively, and the reduced time ζ' corresponds to τ . Also note that in Eqs. (4.20) and (4.21), the strains for all past history are referred to the known configuration corresponding to time t^{p-1} . This choice will simplify the resulting expressions for the incremental PK2 stresses.

The mathematical description of the kernel $[Q(\zeta - \zeta')]$ in the material law is the same as given in Chapter 3. Some of the key details are repeated here for completeness and ease of reference. Each layer of the laminate is assumed to be orthotropic and made of the same material so that four relaxation moduli (Q_{11} , Q_{12} , Q_{22} , and Q_{66}) describe the laminate material behavior. The following contracted notation is used for the relaxation moduli:

$$Q_1 = Q_{11}, \quad Q_2 = Q_{12}, \quad Q_3 = Q_{22}, \quad Q_4 = Q_{66} \quad (3.17)$$

Each relaxation modulus is expressed in terms of a Prony series as follows, resulting in a linear viscoelastic material representation:

$$Q_r(t) = Q_r^\infty + \sum_{\rho=1}^{N_r} Q_{r\rho} e^{-\frac{t}{\lambda_{r\rho}}} \quad \text{for } r = 1, 2, 3, 4 \quad (3.19)$$

where the $\lambda_{r\rho}$'s denote relaxation times governing the material response characteristics. Each reduced stiffness is allowed to have its own reduced time scale denoted by ζ_r in order to

model the possibility that it may be affected differently by the hygrothermal environment. The matrix of relaxation moduli in the material law can then be written as

$$[\bar{Q}(\zeta - \zeta')] = \sum_{r=1}^4 Q_r(\zeta_r - \zeta'_r) [D_r] = \sum_{r=1}^4 \left[Q_r^\infty + \sum_{\rho=1}^{N_r} Q_{r\rho} e^{-\frac{\zeta_r - \zeta'_r}{\lambda_{r\rho}}} \right] [D_r] \quad (3.20)$$

where

$$\zeta_r = \int_0^t \frac{dt'}{A_r[T(t'), H(t')]} \quad \text{and} \quad \zeta'_r = \int_0^\tau \frac{dt'}{A_r[T(t'), H(t')]} \quad (3.22)$$

Recall that $[D_i]$ is the transformed reduced stiffness matrix for the elastic case with $Q_i = 1$ and the other three Q_r 's equal to zero.

The current PK2 stresses are determined as

$$\{^p_{p-1}S\} = \sum_{r=1}^4 Q_r^\infty [D_r] \{^p_{p-1}\bar{\epsilon}\} + \sum_{r=1}^4 \sum_{\rho=1}^{N_r} \{^p_{p-1}V_{r\rho}\} \quad (4.23)$$

where

$$\{^p_{p-1}V_{r\rho}\} = \int_0^{t^p} Q_{r\rho} e^{-\frac{\zeta_r^p - \zeta'_r}{\lambda_{r\rho}}} [D_r] \left\{ \frac{\partial_{p-1}^\tau \bar{\epsilon}}{\partial \tau} \right\} d\tau \quad (4.24)$$

Likewise, the PK2 stresses at the end of the last time increment are

$$\{^{p-1}_{p-1}S\} = \sum_{r=1}^4 Q_r^\infty [D_r] \{^{p-1}_{p-1}\bar{\epsilon}\} + \sum_{r=1}^4 \sum_{\rho=1}^{N_r} \{^{p-1}_{p-1}V_{r\rho}\} \quad (4.25)$$

Writing the expression for $\{^{p-1}_{p-1}V_{r\rho}\}$ and evaluating the hereditary integral over t^{p-1} to t^p by approximating $\{\partial_{p-1}^\tau \bar{\epsilon} / \partial \tau\}$ to be constant over the time step, the following recurrence relation is obtained for $\{^p_{p-1}V_{r\rho}\}$:

$$\{^p_{p-1}V_{r\rho}\} = e^{-\frac{\Delta \zeta_r^p}{\lambda_{r\rho}}} \{^{p-1}_{p-1}V_{r\rho}\} + \bar{S}_{r\rho}^p Q_{r\rho} [D_r] \left\{ \Delta_{p-1}^p \bar{\epsilon} \right\} \quad (4.26)$$

where $\bar{S}_{r\rho}^p$ is defined by Eq. (3.27) which is repeated here for convenience:

$$\bar{S}_{r\rho}^p = \frac{1}{\Delta t^p} \int_{t^{p-1}}^{t^p} e^{-\frac{\zeta_r^p - \zeta'_r}{\lambda_{r\rho}}} d\tau \quad (3.27)$$

Once again, the horizontal shift factor will be approximated to be constant over a time step so that $\bar{S}_{r\rho}^p$ is evaluated using Eq. (3.28) which is also repeated here:

$$\bar{S}_{r\rho}^p = \frac{1}{\Delta \zeta_r^p} \lambda_{r\rho} \left[1 - e^{-\frac{\Delta \zeta_r^p}{\lambda_{r\rho}}} \right] \quad (3.28)$$

Using Eqs. (4.23), (4.25), and (4.26), the incremental PK2 stresses $\{\Delta_{p-1}^p S\}$ are given by

$$\begin{aligned} \{\Delta_{p-1}^p S\} &= \sum_{r=1}^4 \left[Q_r^\infty + \sum_{\rho=1}^{N_r} \bar{S}_{r\rho}^p Q_{r\rho} \right] [D_r] \{\Delta_{p-1}^p \bar{e}\} \\ &\quad - \sum_{r=1}^4 \sum_{\rho=1}^{N_r} \left[1 - e^{-\frac{\Delta \zeta_r^p}{\lambda_{r\rho}}} \right] \{V_{r\rho}^{p-1}\} \end{aligned} \quad (4.27)$$

The last term gives the change in the PK2 stresses resulting from the relaxation over the current time step of the stresses corresponding to the strains that occurred up to the end of the previous time step.

The current force and force-couple resultants are determined by using Eq. (4.27) in

$$\{N_{p-1}^p; M_{p-1}^p\} = \int_{z_{min}}^{z_{max}} (1; z) \left(\{V_{p-1}^{p-1} S\} + \{\Delta_{p-1}^p S\} \right) dz \quad (4.28)$$

Using the decomposition of the mechanical Green–Saint-Venant strains given by Eq. (4.22), the current force resultant is thus

$$\begin{aligned} \{N_{p-1}^p\} &= \{V_{p-1}^{p-1} N\} - \sum_{r=1}^4 \sum_{\rho=1}^{N_r} \left[1 - e^{-\frac{\Delta \zeta_r^p}{\lambda_{r\rho}}} \right] \{V_{r\rho}^{p-1} N_{r\rho}\} \\ &\quad + [\tilde{A}^p] \{\Delta_{p-1}^p e\} + [\tilde{B}^p] \{\Delta_{p-1}^p \kappa\} \\ &\quad - \sum_{r=1}^4 \left[Q_r^\infty + \sum_{\rho=1}^{N_r} \bar{S}_{r\rho}^p Q_{r\rho} \right] \left(\Delta \theta_T^p \{N_r^T\} + \Delta \theta_H^p \{N_r^H\} \right) \end{aligned} \quad (4.29)$$

where

$$[\tilde{A}^p; \tilde{B}^p; \tilde{D}^p] = \sum_{r=1}^4 \left[Q_r^\infty + \sum_{\rho=1}^{N_r} \bar{S}_{r\rho}^p Q_{r\rho} \right] [\bar{A}_r; \bar{B}_r; \bar{D}_r] \quad (4.30)$$

and

$$[\bar{A}_r; \bar{B}_r; \bar{D}_r] = \int_{z_{min}}^{z_{max}} (1; z; z^2) [D_r] dz \quad (4.31)$$

and

$$\{V_{p-1}^{p-1} N_{r\rho}\} = \int_{z_{min}}^{z_{max}} \{V_{p-1}^{p-1} V_{r\rho}\} dz \quad (4.32)$$

$$\{N_r^T\} = \int_{z_{min}}^{z_{max}} [D_r] \{\alpha\} dz \quad (4.33)$$

$$\{N_r^H\} = \int_{z_{min}}^{z_{max}} [D_r] \{\bar{\beta}\} dz \quad (4.34)$$

Note that for the case where all $Q_{r\rho}$ are equal to zero, $[\tilde{A}^p; \tilde{B}^p; \tilde{D}^p]$ reduce to the usual matrices $[A; B; D]$ corresponding to an elastic laminate with reduced stiffnesses Q_1^∞ - Q_4^∞ .

Likewise, the current force-couple resultant is

$$\begin{aligned} \{^p_{p-1}M\} &= \{^{p-1}_{p-1}M\} - \sum_{r=1}^4 \sum_{\rho=1}^{N_r} \left[1 - e^{-\frac{\Delta\zeta_r^p}{\lambda_{r\rho}}} \right] \{^{p-1}_{p-1}M_{r\rho}\} \\ &\quad + [\tilde{B}^p] \{\Delta^p_{p-1}e\} + [\tilde{D}^p] \{\Delta^p_{p-1}\kappa\} \\ &\quad - \sum_{r=1}^4 \left[Q_r^\infty + \sum_{\rho=1}^{N_r} \bar{S}_{r\rho}^p Q_{r\rho} \right] \left(\Delta\theta_T^p \{M_r^T\} + \Delta\theta_H^p \{M_r^H\} \right) \end{aligned} \quad (4.35)$$

where

$$\{^{p-1}_{p-1}M_{r\rho}\} = \int_{z_{min}}^{z_{max}} z \{^{p-1}_{p-1}V_{r\rho}\} dz \quad (4.36)$$

$$\{M_r^T\} = \int_{z_{min}}^{z_{max}} z [D_r] \{\alpha\} dz \quad (4.37)$$

$$\{M_r^H\} = \int_{z_{min}}^{z_{max}} z [D_r] \{\bar{\beta}\} dz \quad (4.38)$$

Hence, the current force and force-couple resultants are equal to the summation of their previous values, some viscoelastic memory loads, and terms corresponding to the current incremental strains. The viscoelastic memory loads account for the relaxation over the current time step of the force and force-couple resultants corresponding to the strains that occurred up to the end of the previous time step. The factor $\bar{S}_{r\rho}^p$ which appears in the $[\tilde{A}^p; \tilde{B}^p; \tilde{D}^p]$ matrices and the hygrothermal terms accounts for the relaxation over the current time step of the force and force-couple resultants corresponding to the current incremental strains. Recall that the factor $\bar{S}_{r\rho}^p$ as defined by Eq. (3.27) resulted from assuming the current incremental strains to vary linearly over the current time step.

Based upon Eq. (4.26), recursion relations are written for $\{^p_{p-1}N_{r\rho}\}$ and $\{^p_{p-1}M_{r\rho}\}$ as follows:

$$\begin{aligned} \{^p_{p-1}N_{r\rho}\} &= e^{-\frac{\Delta\zeta_r^p}{\lambda_{r\rho}}} \{^{p-1}_{p-1}N_{r\rho}\} + \bar{S}_{r\rho}^p Q_{r\rho} \left([\bar{A}_r] \{\Delta^p_{p-1}e\} + [\bar{B}_r] \{\Delta^p_{p-1}\kappa\} \right. \\ &\quad \left. - \Delta\theta_T^p \{N_r^T\} - \Delta\theta_H^p \{N_r^H\} \right) \end{aligned} \quad (4.39)$$

$$\begin{aligned} \{^p_{p-1}M_{r\rho}\} &= e^{-\frac{\Delta\zeta_r^p}{\lambda_{r\rho}}} \{^{p-1}_{p-1}M_{r\rho}\} + \bar{S}_{r\rho}^p Q_{r\rho} \left([\bar{B}_r] \{\Delta^p_{p-1}e\} + [\bar{D}_r] \{\Delta^p_{p-1}\kappa\} \right. \\ &\quad \left. - \Delta\theta_T^p \{M_r^T\} - \Delta\theta_H^p \{M_r^H\} \right) \end{aligned} \quad (4.40)$$

4.1.5 Derivation of the Tangent Stiffness Matrix

The elemental tangent stiffness matrix $[k_T]$ is determined by taking the variation of the elemental internal force vector and recognizing that

$$\{\delta_{p-1}^p f_{int}\} = \frac{\partial \{\delta_{p-1}^p f_{int}\}}{\partial \{u\}} \{\delta u\} = [k_T] \{\delta u\} \quad (4.41)$$

Based upon Eq. (4.18), the variation in $\{\delta_{p-1}^p f_{int}\}$ is

$$\begin{aligned} \{\delta_{p-1}^p f_{int}\} &= \int_{A^{p-1}} \left([G_2]^T [G_1]^T \{\delta_{p-1}^p N\} + [B_{dkt}]^T \{\delta_{p-1}^p M\} \right. \\ &\quad \left. + [G_2]^T [\delta G_1]^T \{\delta_{p-1}^p N\} \right) dA^{p-1} \end{aligned} \quad (4.42)$$

The last term occurs because $[G_1]$ depends on the incremental displacements. This term can be shown to be

$$[G_2]^T [\delta G_1]^T \{\delta_{p-1}^p N\} = [G_2]^T \begin{bmatrix} \mathcal{N} & 0 & 0 \\ 0 & \mathcal{N} & 0 \\ 0 & 0 & \mathcal{N} \end{bmatrix} [G_2] \{\delta u\} \quad (4.43)$$

where

$$[\mathcal{N}] = \begin{bmatrix} \delta_{p-1}^p N_x & \delta_{p-1}^p N_{xy} \\ \delta_{p-1}^p N_{xy} & \delta_{p-1}^p N_y \end{bmatrix} \quad (4.44)$$

and

$$\{\delta_{p-1}^p N\} = \begin{Bmatrix} \delta_{p-1}^p N_x \\ \delta_{p-1}^p N_y \\ \delta_{p-1}^p N_{xy} \end{Bmatrix} \quad (4.45)$$

Recalling that the temperature and moisture histories are taken to be prescribed, the variations in $\{\delta_{p-1}^p N\}$ and $\{\delta_{p-1}^p M\}$ are simply

$$\{\delta_{p-1}^p N\} = [\tilde{A}^p][G_1][G_2]\{\delta u\} + [\tilde{B}^p][B_{dkt}]\{\delta u\} \quad (4.46)$$

$$\{\delta_{p-1}^p M\} = [\tilde{B}^p][G_1][G_2]\{\delta u\} + [\tilde{D}^p][B_{dkt}]\{\delta u\} \quad (4.47)$$

The elemental tangent stiffness matrix is finally determined as

$$\begin{aligned} [k_T] &= \int_{A^{p-1}} \left([G_2]^T [G_1]^T [\tilde{A}^p][G_1][G_2] + [G_2]^T [G_1]^T [\tilde{B}^p][B_{dkt}] \right. \\ &\quad \left. + [B_{dkt}]^T [\tilde{B}^p][G_1][G_2] + [B_{dkt}]^T [\tilde{D}^p][B_{dkt}] \right) dA^{p-1} \end{aligned}$$

$$+[G_2]^T \begin{bmatrix} \mathcal{N} & 0 & 0 \\ 0 & \mathcal{N} & 0 \\ 0 & 0 & \mathcal{N} \end{bmatrix} [G_2] \Big) dA^{p-1} \quad (4.48)$$

If all $Q_{r\rho} = 0$, then the $[\tilde{A}^p; \tilde{B}^p; \tilde{D}^p]$ matrices are equal to the usual $[A; B; D]$ matrices and the tangent stiffness matrix reduces, as it should, to the elastic case as presented by Mohan and Kapania.⁴⁹ For the linear viscoelastic case, the $[\tilde{A}^p; \tilde{B}^p; \tilde{D}^p]$ matrices are used to account for the relaxation over the current time step of the portion of the internal force corresponding to the current incremental strains.

Because the tangent stiffness matrix from each element needs to be calculated in each iteration (full Newton-Raphson method) or every few iterations (modified Newton-Raphson method), incorporating a different set of reduced time scales for each element is not overly cumbersome. Hence, each element is allowed to be at its own temperature and moisture in the present formulation, allowing structures with non-uniform time-dependent hygrothermal fields to be analyzed. For the case of non-uniform hygrothermal fields, an element's temperature and moisture will be approximated as the average of its nodal values. Obviously, the mesh size needed for convergence will depend upon the the spatial distribution of the temperature and moisture throughout the time period of interest.

Note that using a single temperature and moisture for each element at each discrete time considered is consistent with the use of Eq. (4.22), which assumes that neither the temperature nor the moisture varies through the thickness of the structure. In order to model hygrothermal gradients in the thickness direction, the formulation would have to be modified significantly to account for the variation through the thickness of the reduced time scales. If the gradients in the thickness direction are large enough, a single set of reduced time scales for each layer may not be sufficient. For such cases, the individual layers may be divided into sub-layers with independent reduced time scales.

4.1.6 Details of the Iterative Technique

The full Newton-Raphson method will be used such that the change in the incremental displacements for iteration $(n + 1)$ is determined by solving Eq. (4.3). Prior to assembly, the elemental internal force vector, tangent stiffness matrix, pressure load, and pressure stiffness matrix are all transformed to the global coordinate system using the standard transformation techniques. All integrals over the area of an element are evaluated using a three-point rule in area coordinates.⁴⁰

For an element in the first iteration of a time step $(n + 1 = 1)$, the viscoelastic memory and incremental hygrothermal loads are applied so that the internal force vector $\{f_{int}^{(0)}\}_{p-1}^p$ is computed using

$$\begin{aligned} \{f_{p-1}^{(0)}\} &= \{f_{p-1}^{p-1}\} - \sum_{r=1}^4 \sum_{\rho=1}^{N_r} \left[1 - e^{-\frac{\Delta \zeta_r^p}{\lambda_{r\rho}}} \right] \{f_{p-1}^{p-1} N_{r\rho}\} \\ &\quad - \sum_{r=1}^4 \left[Q_r^\infty + \sum_{\rho=1}^{N_r} \bar{S}_{r\rho}^p Q_{r\rho} \right] \left(\Delta \theta_T^p \{N_r^T\} + \Delta \theta_H^p \{N_r^H\} \right) \end{aligned} \quad (4.49)$$

and

$$\begin{aligned} \{M_{p-1}^{(0)}\} &= \{M_{p-1}^{p-1}\} - \sum_{r=1}^4 \sum_{\rho=1}^{N_r} \left[1 - e^{-\frac{\Delta \zeta_r^p}{\lambda_{r\rho}}} \right] \{M_{p-1}^{p-1} M_{r\rho}\} \\ &\quad - \sum_{r=1}^4 \left[Q_r^\infty + \sum_{\rho=1}^{N_r} \bar{S}_{r\rho}^p Q_{r\rho} \right] \left(\Delta \theta_T^p \{M_r^T\} + \Delta \theta_H^p \{M_r^H\} \right) \end{aligned} \quad (4.50)$$

Note that Eqs. (4.49) and (4.50) result from setting the incremental displacements to zero in Eqs. (4.29) and (4.35), respectively. To be consistent in the evaluation of the tangent stiffness matrix for the first iteration, the $\{N\}$ corresponding to the geometric stiffness term (the last term in Eq. (4.48)) is taken simply as $\{f_{p-1}^{p-1} N\}$. In all subsequent iterations $(n + 1)$, the $\{N\}$ for the geometric stiffness term is taken as $\{f_{p-1}^p N^{(n)}\}$. The incremental displacements are updated as follows:

$$\{U^{(n+1)}\} = \{U^{(n)}\} + \{\Delta U^{(n+1)}\} \quad (4.51)$$

The elemental force and force-couple resultants for $(n + 1) \geq 1$ are updated using

$$\{f_{p-1}^p N^{(n+1)}\} = \{f_{p-1}^p N^{(0)}\} + [\tilde{A}^p] \{\Delta f_{p-1}^p e^{(n+1)}\} + [\tilde{B}^p] \{\Delta f_{p-1}^p \kappa^{(n+1)}\} \quad (4.52)$$

$$\{^p_{p-1}M^{(n+1)}\} = \{^p_{p-1}M^{(0)}\} + [\tilde{B}^p]\{\Delta^p_{p-1}e^{(n+1)}\} + [\tilde{D}^p]\{\Delta^p_{p-1}\kappa^{(n+1)}\} \quad (4.53)$$

where $\{\Delta^p_{p-1}e^{(n+1)}\}$ and $\{\Delta^p_{p-1}\kappa^{(n+1)}\}$ respectively are the total incremental midplane in-plane strains and bending curvatures at iteration $(n+1)$.

The solution will be considered to be converged when

$$\frac{\|\{\Delta U_t^{(n+1)}\}\|_\infty}{\|\{U_t^{(n+1)}\}\|_\infty} < tol \quad \text{and} \quad \frac{\|\{\Delta U_r^{(n+1)}\}\|_\infty}{\|\{U_r^{(n+1)}\}\|_\infty} < tol \quad (4.54)$$

and/or

$$\|\{F_{int}^{(n)}\} - \{F_{ext}^{(n)}\}\|_2 < tol \quad (4.55)$$

where $\{U_t\}$ and $\{U_r\}$ refer to the translation and rotation components of the incremental global displacement vector, respectively, and tol is a suitable tolerance. Typically the same tol is used in all three convergence measures. After convergence has been met, $\{N_{r\rho}\}$ and $\{M_{r\rho}\}$ are updated using Eqs. (4.39) and (4.40), respectively.

As previously mentioned, the iterations in a single time increment are performed using a total Lagrangian approach with all integrations carried out over the known configuration at t^{p-1} . Then, after the incremental displacements have been converged, the reference state is updated to the newly converged quasi-static equilibrium state at t^p with the updated global coordinates for node i given by

$$\begin{Bmatrix} X_i(t^p) \\ Y_i(t^p) \\ Z_i(t^p) \end{Bmatrix} = \begin{Bmatrix} X_i(t^{p-1}) \\ Y_i(t^{p-1}) \\ Z_i(t^{p-1}) \end{Bmatrix} + \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} \quad (4.56)$$

Note that the TVATDKT-NL code can still be used for the static analysis of elastic structures under proportional loading by using t as the load proportionality factor. Because the TVATDKT-NL code is intended primarily for the analysis of viscoelastic structures where the load magnitudes are given as functions of the physical time t , the TVATDKT-NL code only uses load control, meaning that t must be increased from increment to increment. Hence, the TVATDKT-NL code is unable to trace the entire load-deflection behavior of elastic structures exhibiting snap-through behavior. In order to accomplish this, a special

method such as the Riks method⁵⁶ or the modified Riks method⁵⁷ must be used. In both of these methods, the load parameter is allowed to vary within each increment and the additional equation needed to completely determine all of the unknowns at each iteration comes from an imposed equality constraint. In Riks method,⁵⁶ the constraint is placed on the arc length of the path followed in the load-deflection hyperspace, whereas in the modified Riks method,⁵⁷ the square of the L2-norm of the incremental displacements is specified.

In general, finite rotations do not add vectorially (the final position of a body subjected to finite rotations about arbitrary axes depends upon the order of the rotations).⁵⁸ Hence, in general, it is not accurate to just add the incremental rotations together to determine the total rotations, even though all the rotations are represented in component form in the same stationary global coordinate system. Although techniques exist for the parameterization of large rotations,^{58–60} they are not employed here, because the total accumulated rotations are not needed in the updated Lagrangian approach that is used from increment to increment. However, because a total Lagrangian approach is used within each increment and the iterative changes in the rotations are added vectorially, the present formulation is limited to moderate values of incremental rotations. This simplifies the formulation somewhat, but at the cost of using more increments in determining the large-rotation response of structures. Given the wide availability of computational resources at the present time, this cost is not prohibitive.

Because no relevant terms have been neglected in the strain-displacement relations for thin shells, the present formulation is valid for thin shells undergoing small or large strains under the action of the applied loads, provided that the use of the constitutive law given in Eq. (4.20) is still valid and the incremental rotations are moderately small. For thick shells, transverse shear may be important and may need to be included in the formulation. Note that Eq. (4.20) corresponds to a linear relation between the PK2 stress tensor and the history of the Green–Saint-Venant strain tensor. It was found by Batra and Yu⁶¹ that such a relationship may give qualitatively incorrect predictions for isotropic solids undergoing finite strains. Batra and Yu⁶¹ showed that a linear relationship between the Cauchy stress tensor and the history of the relative Green–Saint-Venant strain tensor provided results

which qualitatively agreed with experimental observations.

4.2 Numerical Examples

Several numerical examples are presented to demonstrate the accuracy of the TVATDKT-NL formulation and to present some key characteristics of the geometrically-nonlinear response of viscoelastic structures. The first is a cantilever under a tip moment with two different time variations. Next, a ring under non-uniform external pressure is analyzed with the follower effects of the pressure load taken into account. Then, the creep buckling of a column under an axial load is examined. Following this, a cylindrical panel exhibiting snap-through behavior under the action of a centrally applied point force is studied. Next, a composite cylindrical panel experiences snap-through under a uniform deformation-dependent pressure load. In the final example, the thermal postbuckling of a composite plate is analyzed. Although the commercial code ABAQUS is used in validating some of the results, it should be noted that the current version of ABAQUS (ABAQUS/Standard Version 5.8⁶²) is unable to handle the small or large deformations of viscoelastic composites.

4.2.1 Cantilever Under Tip Moment

The beam is 10.0 m long with a 1.0×0.1 m rectangular cross-section. The relaxation modulus for the isotropic cantilever is taken to be

$$E(t) = E(0) [0.5 + 0.5 \exp(-t/6)] \quad (4.57)$$

where t is in minutes. The initial modulus $E(0)$ is taken to be 1.2×10^8 Pa. In order to model the structure as a beam using a shell element, Poisson's ratio is set to zero.

Two time histories of tip moment with the following time variations are applied:

$$\text{Case I: } M(t) = 1000\pi \sin(t/3)[1 - u(t - 3\pi)] \text{ N}\cdot\text{m} \quad (4.58)$$

$$\text{Case II: } M(t) = 1000\pi \left\{ \sin(t/3)[1 - u(t - 1.5\pi)] + u(t - 1.5\pi) \right\} \text{ N}\cdot\text{m} \quad (4.59)$$

where t is in minutes and $u(t)$ is the unit step function. The time histories of M for Cases I and II are shown in an inset in Fig. 4.2.

The exact solution for this example can be developed as follows. For an elastic beam, the radius of curvature R is given by

$$\frac{1}{R} = \frac{\bar{M}}{EI} = \frac{v_{,xx}}{[1 + (v_{,x})^2]^{3/2}} \quad (4.60)$$

where $\bar{M}$ is the internal bending moment, E is the elastic modulus, I is the area moment of inertia, v is the transverse deflection, and $(\cdot)_{,x}$ represents $d(\cdot)/dx$. For a cantilever under a tip moment, the bending moment is constant throughout the beam and equal to the applied tip moment. Thus, the cantilever has a constant curvature. Also, in standard Euler-Bernoulli beam theory, the neutral axis does not exhibit strain. Hence, under any applied tip moment, the cantilever being studied here has a constant midplane length. This means that the angle θ subtended by the midplane of the elastic cantilever under tip moment M is given by

$$\theta = \frac{L}{R} = \frac{ML}{EI} \quad (4.61)$$

Equivalently, θ is the angle of rotation at the tip of the cantilever. The linear viscoelastic solution is developed using the correspondence principle³⁷ which gives θ for the linear viscoelastic cantilever as

$$\theta = \mathcal{L}^{-1} \left\{ \frac{\hat{M}L}{s\hat{E}I} \right\} \quad (4.62)$$

where $\mathcal{L}$ is the Laplace operator, $(\hat{\cdot})$ denotes the Laplace transform of $(\cdot)$, and s is the Laplace variable.

An exception to the rule that finite rotations do not add vectorially occurs when the axes of rotation are parallel.⁵⁸ Thus, the total rotation θ for this problem from the TVATDKT-NL analysis can be computed by simply adding the incremental rotation values together. Shown in Fig. 4.2 are the time histories of θ for Cases I and II computed using 40 TVATDKT-NL elements and the correspondence principle. A time step of 0.1 min was used in marching the TVATDKT-NL finite element solution. Figure 4.3 depicts a side view of the deflected meshes for both cases of applied tip moment at various times. For Case I, the cantilever has non-zero deflection when the tip moment returns to zero at $t = 3\pi$ min, due to the creep that has occurred under the applied load history. For Case I at large t , the cantilever has essentially

returned to its initial configuration. For Case II after $t = 1.5\pi$ min, the applied tip moment is held constant at 1000π N · m. Hence, for large t in Case II, the cantilever has deformed into a complete circle. The differences between the finite element and correspondence principle results cannot be distinguished.

4.2.2 Circular Ring Under External Pressure

The ring radius R is 6.0 in., while the width b and thickness h are 1.0 in. and 0.05 in., respectively. The ring is composed of acetal resin (an engineering thermoplastic) with an initial elastic modulus $E(0) = 410,000$ psi which corresponds to a temperature of 73°F .⁶³ Rogers and Lee⁶⁴ fit experimental creep data for the first 600 hr presented by Warriner⁶³ to a creep compliance $D(t)$ of

$$\frac{D(t)}{D(0)} = 1.0 + 0.00076 t + 1.12 [1 - \exp(-0.055 t)] \quad (4.63)$$

where t is in hr. Hence, the creep compliance over the first 600 hr is represented using a 4-parameter fluid model. The corresponding relaxation modulus, $E(t)$, can be found using Laplace transforms and was given by Yang and Lianis¹⁹ to be

$$\frac{E(t)}{E(0)} = 0.46845 \exp(-0.00035726 t) + 0.53155 \exp(-0.11700 t) \quad (4.64)$$

The ring behaves as a curved beam and Poisson's ratio is set to zero for the finite element analyses.

The external pressure loading P on the ring at any given time t is of the form

$$P(\theta, t) = P_0(t)(1 - q \cos 2\theta) \quad (4.65)$$

where θ is as shown in Fig. 4.4 and always refers to the original configuration (body-attached pressure load). Results will be computed for several non-zero values of q .

Exploiting the symmetry about two planes in both the geometry and the loading, one-fourth of the ring is analyzed using either 40 TVATDKT-NL elements or 20 S4R elements of ABAQUS.⁶² Element S4R is a four-node, doubly-curved, quadrilateral shell element with six

degrees of freedom per node. The following symmetry boundary conditions used at points A and B which are indicated in Fig. 4.4:

$$\text{point } A : u = 0, \quad \theta_y = \theta_z = 0 \quad (4.66)$$

$$\text{point } B : w = 0, \quad \theta_x = \theta_y = 0 \quad (4.67)$$

where u, v , and w are the deflections in the x -, y -, and z -directions and θ_x, θ_y , and θ_z are the corresponding rotations about the x -, y -, and z -axes. In order to prevent rigid body motion in the y -direction, v is restrained at point A . For all finite element analyses, the follower effects of the pressure load are taken into account.

The non-dimensional load-deflection paths for the elastic case are shown in Fig. 4.5. Here the load parameter is taken as P_0 , while the deflection is taken as w_A . The TVATDKT-NL and S4R results match very well. This elastic problem with the follower effects of the pressure load taken into account was analyzed by Seide and Jamjoom⁶⁵ using numerical methods. As indicated in Ref. 65, bifurcation buckling occurs for the case of uniform pressure ($q = 0$) at a non-dimensional load level of $P_0 b R^3 / EI = 3.0$, where I is the area moment of inertia ($I = bh^3/12$). The effect of small non-zero q is to add a small imperfection in the loading so that a path very close to the bifurcation path is traced without the need for a bifurcation analysis to be performed. The original shape and the deformed elastic ring shapes near $P_0 b R^3 / EI = 3.4$ for various values of q are shown in Fig. 4.4.

For viscoelastic analysis, the pressure magnitude $P_0(t)$ is ramped from zero to $\bar{P}_0$ over the first hour, then held constant for another 300 hr, after which it is ramped back to zero over one hour, and thereafter remains zero. A single value of $\bar{P}_0$ which is given as follows is used:

$$\bar{P}_0 = 1.7 \frac{E(0)I}{bR^3} \quad (4.68)$$

The viscoelastic results produced using the TVATDKT-NL and S4R elements are shown in Fig. 4.6. For the TVATDKT-NL finite element analysis, the time step size was 0.025 hr during loading and unloading and 0.25 hr during all other times. Reducing the time step by a factor of two and using 80 elements did not significantly change the TVATDKT-NL results

for the case where $q = 0.001$. For this case, the deflection remains small for approximately the first 130 hrs, after which it increases at a substantially higher rate. Therefore, a delayed buckling due to creep is indicated, with the critical time for buckling estimated as 130 hrs. For the case where $q = 1.0$, the analysis was terminated when $w_A = -R$, which indicates that contact between point A and the point opposite to it has occurred. For the other cases, Fig. 4.6 indicates that the structure recovers a significant portion of its viscoelastic response during and after load removal.

4.2.3 Pinned-Pinned Column Under Axial Load

The column is 3.0 m long with a 0.3×0.03 m cross-section. The relaxation modulus is the same as that used for the cantilever under a tip moment in Subsection 4.2.1. The applied compressive axial load is denoted by P .

For elastic analysis, the stability of a perfect column is determined by examining the evolution of the lowest eigenvalue of the tangent stiffness matrix. Forty TVATDKT-NL elements are used. The critical point using a Young's modulus $E = E(0)$ is determined as $P_{cr} = 88.9268$ N which agrees well with the value of 88.8264 N computed using the classical Euler buckling load formula for a pinned-pinned column given by

$$P_{cr} = \pi^2 \frac{EI}{L^2} \quad (4.69)$$

where I is the area moment of inertia and L is the column length. For an elastic column with $E = E(\infty)$, P_{cr} is determined to be 44.4634 N and 44.4132 N from the finite element analysis and Eq. (4.69), respectively.

The elastic bifurcation problem can be studied without using any special bifurcation procedures incorporated in the TVATDKT-NL code by superimposing an initial imperfection onto the perfect structure. This imperfection is taken to be a small multiplicative constant times the eigenvector corresponding to the zero eigenvalue computed at the critical point of the perfect elastic structure using $E = E(0)$. It is well-known that for a geometrically-perfect, elastic pinned-pinned column, the first buckling mode is simply a half-sine wave, regardless of the value of E . The computed eigenvector shape agreed well with this. Shown in Fig. 4.7

are the load-deflection paths for the elastic case using both the initial and final values of the viscoelastic relaxation modulus. The imperfection is scaled to have a maximum magnitude of 7.5311×10^{-4} m. The column deflection is characterized by the midpoint transverse deflection v_{mid} for the load-deflection curves. Also shown in Fig. 4.7 are the results computed using the elastica solution of Timoshenko and Gere⁶⁶ which assumes an inextensible perfect column. The present results with small initial imperfection approach those of Timoshenko and Gere,⁶⁶ as expected.

The stability of the viscoelastic column is determined by examining whether a small transverse deflection grows large or not with an axial load applied. Instability in such a case is commonly referred to as creep buckling. The small transverse deflection may either be introduced as an initial imperfection in the geometry or by applying a small sinusoidally-varying distributed transverse load. For the present analysis, an initial imperfection is added to the column before the axial load is applied. This imperfection is once again taken to be a small factor times the lowest eigenvector computed at the critical point of the elastic analysis with $E = E(0)$.

The time histories of viscoelastic deflection are computed for a range of axial loads. In each case, the axial load is applied as a creep load at $t = 0$ (i.e., the load is applied suddenly at $t = 0$ and then held constant). However, for clarity only a few representative results are shown in Fig. 4.8. These results, computed using 40 TVATDKT-NL elements and a time step of 1.0 min, are taken to be converged, as using 80 elements with a time step of 0.1 min did not significantly alter the results for $P = 50$ N. Although the axial load is applied as a creep load at $t = 0$ in each case, the time required to reach the final deformation state depends on the applied load level. This arises from the fact that the bending moment throughout the beam caused by the axial load is dependent upon the transverse deflection of the beam. Furthermore, the transverse deformation depends upon the applied loading and the initial imperfection. Hence, the internal bending moment in the column is time-dependent. The final values of v_{mid} from the viscoelastic analyses match those computed using the long-term modulus value in the elastic analysis of the imperfect column.

The creep buckling stability of the column can be characterized by v_{mid} remaining below a given level for a given loading and time duration. Either the loading or time duration of interest is selected and then the safe time t_s or the safe load level P_s is determined. Here, for demonstrative purposes, the viscoelastic deformation will be taken to be acceptable if $v_{mid} < 0.2$ m. The time for v_{mid} to reach 0.2 m is plotted against the axial load in Fig. 4.9. Results are shown for two initial imperfection magnitudes. For an imperfection magnitude of 7.5311×10^{-4} m, the stability curve is computed using 40 data points, while only seven points are used for the 3.7646×10^{-4} m imperfection case. The two stability curves shown in Fig. 4.9 exhibit similar trends. Below an axial load of about 45 N, the structure remains safe for all times. For loads above this, the safe time t_s is determined by examining Fig. 4.9. Note that for loads slightly above the initial elastic buckling load, the viscoelastic column becomes unsafe nearly instantaneously. For relatively large loads, the critical time is not affected significantly by the magnitude of the initial imperfection. However, for loads approaching the final elastic buckling load from above, the safe time t_s computed for a given load P_s is significantly affected by the imperfection magnitude, as the stability curve there becomes horizontal in nature. On the other hand, the safe load P_s for a chosen time period t_s is affected less by the imperfection magnitude, even for loads approaching the final elastic buckling load level. The safe load level for which the viscoelastic deflections will remain safe for all time t for this well-behaved geometrically-nonlinear example could have been determined by examining the elastic load-deflection paths computed using the final value of the viscoelastic relaxation modulus.

4.2.4 Cylindrical Panel Under Point Load

At the panel center, a point load P is applied in the transverse direction pointing towards the center of curvature of the panel cross section. The length of the straight panel edges is 0.508 m. The arc length of the other two edges is also 0.508 m, corresponding to a radius of 2.54 m and a half-angle of 0.1 rad. The panel thickness is taken to be 0.0126 m. The edges having zero curvature are hinged (along those edges, the shell is only allowed to rotate about

the axis corresponding to the edge), while the other two sides are free of any restraints. The hinged edges are parallel to the global x -axis, while the curved edges lie in planes parallel to the global y - z plane. For this problem, the coordinate axes are chosen so that the center point force acts in the negative z -direction.

The panel is taken to be composed of polymethyl methacrylate. Muki and Sternberg⁶⁷ fit the following relation to experimental data for the relaxation modulus:

$$E(t) = 10^{7.85} \left\{ 0.0316 + 63.0(t + 3.98)^{-1/2} - 15.5 \left[7.94(t + 3.98)^{-3/2} - \exp(-1.89t) \right] \right\} \text{ N/m}^2 \quad (4.70)$$

where t is in hours. Using ABAQUS,⁶² the following Prony series is found:

$$\begin{aligned} \frac{E(t)}{E(0)} = & 0.0113497 + 0.41545 \exp(-t/0.47449) + 0.32457 \exp(-t/23.007) + \\ & 0.18726 \exp(-t/214.83) + 0.0613703 \exp(-t/4234.0) \end{aligned} \quad (4.71)$$

where t once again is in hours. This Prony series is accurate for $0 \leq t \leq 1000$ hr. Although more terms could be used to represent $E(t)$ accurately for a larger time period, the chosen Prony series is acceptable for the analyses that will be performed. Poisson's ratio is taken to be constant at a value of 0.35.

Utilizing symmetry boundary conditions on two sides, one-fourth of the panel is modeled under a point load of $0.25P$ using either 128 TVATDKT-NL, 128 ATDKT-NL or 64 S4R elements. Using the initial value of the relaxation modulus, the static elastic behavior is determined. The results are shown in Fig. 4.10, with the total deflection at the panel center in the positive z -direction denoted as w_{mid} . The panel undergoes snap-through at a critical load of approximately 1620 N. Although snap-through is truly a dynamic event, here it is inferred based upon the static load-deflection behavior. In order to completely trace the static load-deflection curve, the modified Riks method⁵⁷ is used in the ATDKT-NL and S4R analyses. Because only one-fourth of the panel is modeled with symmetric boundary conditions, it is not possible to determine whether asymmetric bifurcation buckling takes place or not. Critical points which may be either bifurcation or limit points are indicated by a zero determinant or a zero eigenvalue for the tangent stiffness matrix. Limit points

can be determined using the modified Riks method⁵⁷ without knowing the location of the limit point *a priori*. For bifurcation, special methods can be used to continue on secondary paths.⁵⁶ Checks for bifurcation onto secondary paths corresponding to asymmetrical buckling are not performed here.

As shown in Fig. 4.10, the static elastic load-deflection behavior can be partially traced using the TVATDKT-NL code which uses load-control with the time t taken as the load-proportionality factor. Because the tangent stiffness matrix is singular at the limit point, the limit load can not be both found exactly and bypassed in a single run using load-control. In order to jump from a point just below the limit load to a stable static equilibrium point with higher load value, the incremental values of the load proportionality factor t must be chosen carefully. This jump in the TVATDKT-NL results is indicated in Fig. 4.10 by the nearly horizontal line in the middle of the load-deflection curve.

For the viscoelastic analysis, the loading is ramped from zero to its final value over the first 3.6 sec (10^{-3} hr). Three final values of center point force P are considered: 1500 N, 1000 N, and 500 N. In all cases, a time step of 10^{-4} hr is used during ramp-up. Otherwise, for times sufficiently below and above the critical point, a time step of 0.024 hr is used in the TVATDKT-NL analyses for $P = 1500$ and 1000 N. For these two cases using ABAQUS, an automatic time step selection procedure (based upon the creep strain rates at the beginning and the end of an increment) is used in computing the results for times sufficiently far from the critical point, with the maximum time increment limited to 0.1 hr. For $P = 500$ N, a time step of 0.5 hr is used away from the critical point in both the TVATDKT-NL and S4R analyses. The time histories of the center point deflection for the various cases are shown in Fig. 4.11, using either 128 TVATDKT-NL or 64 S4R elements. Note that the results have been plotted on a $\log_{10}$ time scale. In each case, the deflection magnitude increases at a finite rate until viscoelastic snap-through, which corresponds to a vertical tangent for the deflection versus time curve. After snap-through, the deflection once again continues to increase at a finite rate. The critical times to snap-through are approximately 6.06 min, 2.33 hr, and 82.0 hr for $P = 1500$ N, 1000 N, and 500 N, respectively. Similar to the elastic case

using load-control, the critical point is not found exactly in the single-run results shown in Fig. 4.11, because the analysis is unable to converge precisely at the limit point, regardless of what size time step is used.

Unlike the elastic structure which has a single limit load value for snap-through, the viscoelastic structure will exhibit snapping for a range of loads if those loads are applied for a sufficiently long enough time. Although not computed here, it is likely that a load level exists for which the viscoelastic structure will not snap even for infinite load duration. Also, for sudden application of a load greater than or equal to the elastic limit load value computed using the initial relaxation modulus, the viscoelastic structure would snap instantaneously. Hence, instead of the snapping instability being characterized by a single limit load value as in the elastic case, it is characterized by a range of critical times corresponding to a range of loads in the viscoelastic case.

4.2.5 Cylindrical Panel Under Uniform Pressure

The length of the panel is 80.0 in. The radius associated with the curvature of the panel is 100.0 in. and the panel half-angle is 12° , corresponding to an arc length of 41.89 in. for the two curved panel sides. The panel is taken to be supported on all edges by vertical diaphragms which are rigid in their own planes, but perfectly flexible otherwise.

The composite panel is composed of four layers of GY70/339 graphite-epoxy with a symmetric cross-ply stacking sequence of $[0/90]_s$. Each layer is 0.08 in. thick, giving a total laminate thickness of 0.32 in. The initial elastic properties are taken from Ref. 3 and are as follows:

$$\begin{aligned} E_1 &= 42.0 \text{ Msi}, & G_{12}(0) &= 0.6 \text{ Msi}, \\ E_2(0) &= 0.88 \text{ Msi}, & \nu_{12} &= 0.31 \end{aligned} \tag{4.72}$$

Here E_1 is taken to be fiber-dominated and, therefore, constant in time. The major Poisson's ratio ν_{12} is taken to be constant in computing the reduced stiffnesses Q_1 - Q_4 . Using these assumptions and also treating ν_{21} as constant in time, Q_1 will be constant in time. This is

an acceptable approximation, as Q_1 is fiber-dominated. The viscoelastic time variation for the other three reduced stiffnesses is of the following form:

$$Q_r(t) = Q_r(0) f(t) \quad \text{for } r = 2, 3, 4 \quad (4.73)$$

where

$$f(t) = f_0 + \sum_{i=1}^{10} f_i e^{-\frac{t}{\lambda_i}} \quad (4.74)$$

The components of the normalized viscoelastic time variation $f(t)$ are given in Table 3.1. The values for Q_1 for all t and Q_2 - Q_4 at the initial time are as follows:

$$Q_1 = \frac{E_1}{1 - \nu_{21}\nu_{12}} \quad (4.75a)$$

$$Q_2(0) = \frac{\nu_{12}E_2(0)}{1 - \nu_{21}\nu_{12}} \quad (4.75b)$$

$$Q_3(0) = \frac{E_2(0)}{1 - \nu_{21}\nu_{12}} \quad (4.75c)$$

$$Q_4(0) = G_{12}(0) \quad (4.75d)$$

$$\nu_{21} = \frac{E_2(0)}{E_1} \nu_{12} \quad (4.75e)$$

The panel is taken to be in a hygrothermal environment with a temperature of 151°F and a moisture level of 0.1%. The horizontal shift factor corresponding to these environmental conditions is 10^{-4} as taken from Ref. 2. Furthermore, the panel is assumed to be free of hygrothermal strains in this environment.

Because of the cross-ply stacking sequence and the double symmetry in the geometry, boundary conditions, and loading, the panel response can be analyzed using one-fourth of the panel with proper symmetry conditions enforced on two edges. The deformation-dependent pressure loading of magnitude P_0 is applied pointing towards the center of curvature of the panel (external pressure). The load-deflection curve for the elastic case using the initial material properties given in Eq. (4.72) is shown in Fig. 4.12, where w_{mid} is the midpoint transverse deflection. The present results using 288 ATDKT-NL elements appear to agree fairly well with the results produced using 144 S4R elements. The mesh sizes are sufficient

for accurate calculations, as using either 512 ATDKT-NL or 256 S4R elements did not significantly change either set of results. Note that the modified Riks method⁵⁷ has been used with the ATDKT-NL and S4R analyses in order to trace the entire load-deflection path. Also shown in Fig. 4.12 are the results computed using 288 TVATDKT-NL elements. Once again, the TVATDKT-NL results exhibit snap-through over a single load step. Note that when the panel snaps, it does not snap to an inverted position due to the applied boundary conditions. Once again, even though snap-through is truly a dynamic event, it has been inferred based upon the static load-deflection behavior. The limit load is calculated to be 1.2942 psi. Checks for possible bifurcation buckling are not included in the present analysis.

For viscoelastic analysis, the applied pressure is ramped from 0 to P_0 over the first 10^{-7} sec and held constant thereafter. A time step of $\Delta t = 10^{-8}$ sec ($\Delta \zeta = 10^{-4}$ sec) is used during load ramp-up. After ramp-up, the reduced time step increases as a geometric series as follows:

$$\Delta \zeta^p = \Delta \zeta^0 (r)^p \quad (4.76)$$

The time histories of viscoelastic center point deflections for various final magnitudes P_0 of applied pressure are shown in Fig. 4.13, using $\Delta \zeta^0 = 6000$ sec ($\Delta t^0 = 0.6$ sec) and $r = 1.08$ with 288 TVATDKT-NL elements. Using a time step approximately one order of magnitude smaller for all t ($r = 1.007$) did not significantly alter the viscoelastic results for $P_0 = 1.11$ psi. For the cases exhibiting delayed snap-through, the deflection magnitude increases at a finite rate until viscoelastic snap-through occurs, which is once again indicated by a vertical tangent for the deflection versus time curve. Near viscoelastic snap-through, the time step size is reduced in determining the snap-through point. Once again, the critical point for snap-through is not found exactly in the single-run results shown in Fig. 4.13, because the analysis is unable to converge right at the limit point, regardless of what size time step is used. After snap-through, the viscoelastic deflection then continues to increase.

As can be observed from Fig. 4.13, the viscoelastic structure will exhibit snapping for a range of loads, if those loads are applied for a sufficiently long time. The lower limit on P_0 for which viscoelastic snap-through will occur is found to be approximately 0.86 psi. Note

that for P_0 values below this, the deflection may still grow to be relatively large, even though snap-through does not occur. The time t_s of snap-through is plotted against P_0 in Fig. 4.14. For $P_0 \geq 1.2942$ psi, the viscoelastic panel snaps at the instant the pressure magnitude reaches 1.2942 psi.

A check on the critical time computed for viscoelastic snap-through is performed using the quasi-elastic method.³⁶ Here, the quasi-elastic approximation for the viscoelastic deflection for a given time is computed using a fictitious elastic reduced stiffness matrix $[\bar{Q}_E]$ equal to $[\bar{C}(\zeta^p)]^{-1}$, where $[\bar{C}(\zeta^p)]$ is the matrix of transformed reduced creep compliances. Note that $[\bar{Q}_E(\zeta^p)]$ is not exactly equal to $[\bar{Q}(\zeta^p)]$, because $[\bar{C}(\zeta)]$ and $[\bar{Q}(\zeta)]$ are not inverses of one another except for $\zeta = 0$ and $\zeta = \infty$. Although the quasi-elastic method is more suited to situations where a closed-form solution exists, it can be implemented using a numerical technique such as the finite element method. In order to use a quasi-elastic finite element approach to determine the time history of viscoelastic deformation, a separate finite element analysis must be performed for each discrete time considered. Also, note that in the case of nonlinear deformations where more than one quasi-elastic deformation state corresponds to a given load magnitude, the correct quasi-elastic deformation state can be determined from knowledge of the previous viscoelastic deformations.

Here, the applied pressure loading is ramped-up over an extremely short time such that the pressure loading is applied essentially as a creep load at $t = 0$. For this nonlinear problem, the direction of the pressure loading changes as time evolves. However, this change is very gradual at all times except that corresponding to snap-through. Hence, the computed critical time for viscoelastic snap-through can be checked by determining the elastic load-deflection path with material properties corresponding to $[\bar{Q}_E(\zeta_s)]$, where ζ_s is the reduced time corresponding to t_s . The predicted elastic limit load should then match the applied creep load magnitude used in the viscoelastic analysis.

Shown in Fig. 4.15 are the elastic load-deflection curves corresponding to several values of t_s . Horizontal lines corresponding to the pressure magnitudes P_0 causing viscoelastic snap-through at $t = t_s$ are also shown. The computed elastic limit loads are in good agreement

with the applied loads used in the viscoelastic analysis, indicating that the predicted times t_s for the viscoelastic analyses are accurate. Note that the severity of the snap-through decreases as t_s increases, until a snap-through is no longer indicated. This decrease in snap-through severity is also indicated in Fig. 4.13. For extremely large times, the agreement between the ATDKT-NL and S4R elastic load-deflection curves decreases, likely due to the growing importance of transverse shear effects which are included in the S4R formulation but not in the ATDKT-NL formulation. Hence, the TVATDKT-NL formulation (which reduces to the ATDKT-NL formulation in the elastic case) is likely to be slightly in error for extremely large times.

4.2.6 Thermal Postbuckling of an Antisymmetric Angle-Ply Plate

The plate is square with a side length L equal to 50 in. The plate edges are parallel to the global x - and y -directions. The pinned boundary conditions are as follows:

$$\begin{aligned} x = 0, L : \quad u = w = 0 \quad \text{and} \quad M_x = N_{xy} = 0 \\ y = 0, L : \quad v = w = 0 \quad \text{and} \quad M_y = N_{xy} = 0 \end{aligned} \quad (4.77)$$

The plate has an antisymmetric stacking sequence of $[\pm 45]_a$. Each of the four layers has a thickness of 0.05 in., giving a total laminate thickness of 0.2 in.

The plate is composed of T300/934 graphite-epoxy with the following viscoelastic material properties at $t = 0$ taken from Flaggs and Crossman:³

$$\begin{aligned} E_1 = 21 \text{ Msi}, \quad E_2 = 1.5 \text{ Msi}, \quad \alpha_1 = 0.05 \times 10^{-6}/^\circ\text{F}, \\ G_{12} = 0.7 \text{ Msi}, \quad \nu_{12} = 0.29, \quad \alpha_2 = 16.5 \times 10^{-6}/^\circ\text{F} \end{aligned} \quad (4.78)$$

Because Q_1 is taken to be fiber-dominated, it will be treated as constant in time. The other three Q_r 's have the same normalized viscoelastic time variation and are given by

$$Q_r(t) = Q_r(0)f(t) = Q_r(0) \left[f_0 + \sum_{i=1}^N f_i e^{-\frac{t}{\lambda_i}} \right] \quad \text{for } r = 2, 3, 4 \quad (4.79)$$

Using ABAQUS,⁶² a nine-term ($N = 9$) Prony series is fit to experimental data for T300/934 presented by Crossman *et al.*² The Prony series components are given in Table 4.1. The plate

is assumed to be free of hygrothermal strains in an environment with 0.14% moisture. The viscoelastic matrix is assumed to be hygrothermo-rheologically simple. The horizontal shift factor A_r is taken to be the same for Q_2 - Q_4 and is subsequently denoted by A_{TH} . The values for $\log_{10}[A_{TH}]$ taken from Crossman *et al.*² are presented in Table 4.2. The horizontal shift factors for intermediate temperatures are determined using a linear interpolation for $\log_{10}[A_{TH}]$.

The viscoelastic plate is subjected to a range of thermal loads. In each case, the plate is initially free of thermal strains at a temperature T_i of 77 °F. The temperature is ramped-up using a linear variation from 77 °F to its final temperature T_f over the first 10^{-7} sec and then held constant. Hence, the thermal load $\Delta T = T_f - T_i$ is applied almost instantaneously. Here, only integer values of ΔT are considered. During ramp-up, the time step Δt is chosen to be $10^{-7}/\Delta T$, giving an increase of 1 °F in each increment. During this period, the change in reduced time ζ is determined as⁶⁸

$$\Delta\zeta^p = \left[\left(\frac{1}{A_{TH}(T^p)} - \frac{1}{A_{TH}(T^{p-1})} \right) / \left(\ln A_{TH}(T^{p-1}) - \ln A_{TH}(T^p) \right) \right] \Delta t^p \quad (4.80)$$

After ramp-up, the time step size is chosen to be

$$\Delta t^p = 100 (r)^p A_{TH}(T_f) \text{ sec} \quad (4.81)$$

where r is the time step size ratio between consecutive increments.

A geometrically-perfect elastic antisymmetric angle-ply plate with the boundary conditions given in Eq. (4.77) undergoes zero deformation under a uniform thermal load until the bifurcation point for thermal buckling is reached.^{69,70} The elastic load-deflection path is determined by first finding the bifurcation point which corresponds to a singular tangent stiffness matrix. Next, the eigenvector corresponding to the zero eigenvalue is determined. A small constant times this eigenvector is then introduced as an initial imperfection in the plate geometry and the analysis is started over. The viscoelastic plates are also analyzed using this initial imperfection.

An approximate viscoelastic solution can be determined using the quasi-elastic approach described by Schapery.³⁶ Both the quasi-elastic and viscoelastic results are computed using a

uniform mesh of 512 TVATDKT-NL elements. For the viscoelastic analysis, $r = 1.08$ is used in Eq. (4.81) to determine the time step. Using a finer mesh of 1152 elements and a time step approximately one order of magnitude smaller for all t ($r = 1.007$) did not significantly change the viscoelastic results for $\Delta T = 100^\circ\text{F}$.

Shown in Fig. 4.16 are the quasi-elastic load-deflection paths using elastic properties corresponding to the viscoelastic relaxation moduli at various values of reduced times. Here w_{mid} denotes the transverse deflection at the plate center. Results computed at other values of reduced time are not shown for clarity in Fig. 4.16. In each case, the critical buckling mode consisted of a single half-sine wave in the x - and y -directions. The initial imperfection used in each case has a maximum magnitude of 2.48877×10^{-4} in. Also shown in Fig. 4.16 are the load-deflection pairs computed using the viscoelastic finite element analysis. Note that “ $\zeta = 0$ ” viscoelastic results are actually the viscoelastic results after load ramp-up (i.e., the ζ for the initial viscoelastic results is not exactly zero, but still relatively very small). Also, the “ $\zeta = \infty$ ” viscoelastic results are actually near-equilibrium results computed at reduced times less than infinity. The mode shape of viscoelastic deformation for each value of ΔT considered remained a single half-sine wave in the x - and y -directions for all times. Note that Fig. 4.16 does not show how the viscoelastic deformation would vary with changes in ΔT applied at a given time. Rather, for the viscoelastic case, the curves shown in Fig. 4.16 can be used together to determine how the viscoelastic deformation varies with time for a range of load levels, where the thermal load in each case is applied nearly instantaneously at $t = 0$ and then held constant. Also, recall that the physical time t corresponding to a given reduced time ζ depends upon the temperature history.

The bifurcation temperature is observed to increase as larger times are used in the quasi-elastic analysis. This counterintuitive result is explained by the fact that the compressive forces caused by the thermal load decrease due to viscoelastic relaxation, which is reflected in the quasi-elastic analysis by the use of degraded Q_2 - Q_4 properties. In each quasi-elastic case, the bifurcation temperature computed using the TVATDKT-NL code agreed well with that computed using the formula neglecting transverse shear effects presented by Tauchert.⁷⁰

The bifurcation temperature at $t = 0$ computed using both Tauchert's closed-form solution⁷⁰ and the finite element code is 15.42 °F, while the final bifurcation temperature is computed to be 68.07 °F and 68.09 °F from the closed-form solution⁷⁰ and the TVATDKT-NL analysis, respectively.

The time histories of deformation computed using the viscoelastic and quasi-elastic finite element analyses are shown in Fig. 4.17. The decrease in viscoelastic deformation as time evolves is also explained by the relaxation of the compressive force resultants caused by the imposed thermal strains and boundary conditions. Note that as T_f increases in magnitude, the time to reach the long-term equilibrium state decreases. This occurs due to the speed-up of the viscoelastic response with increasing T_f as incorporated through the hygrothermo-rheologically simple postulate made for the matrix material. Examining Fig. 4.16, it should be apparent that if the viscoelastic deflections for various values of ΔT were plotted versus reduced time ζ as opposed to physical time t in Fig. 4.17, the various deflection curves would not cross over each other as shown in Fig. 4.17. From Figs. 4.16 and 4.17, it is clear that for thermal loads below the final critical ΔT value of 68.09 °F, the final viscoelastic deformation is essentially zero. The agreement between the quasi-elastic and viscoelastic results appears to be very good, because the stress resultants and deformations change slowly with time and a single mode of viscoelastic deformation occurs for this well-behaved geometrically-nonlinear example.

Table 4.1: Prony series coefficients and time constants for T300/934 graphite-epoxy.

i	f_i	λ_i (sec)
0	1.78398×10^{-1}	
1	4.96164×10^{-2}	5.21100×10^2
2	4.91345×10^{-2}	2.06800×10^4
3	5.29998×10^{-2}	1.07785×10^6
4	5.21760×10^{-2}	2.32300×10^7
5	8.50296×10^{-2}	1.67441×10^9
6	5.97267×10^{-2}	3.24064×10^{10}
7	1.56990×10^{-1}	1.69677×10^{12}
8	9.69593×10^{-2}	5.14749×10^{13}
9	2.18970×10^{-1}	1.49240×10^{15}

Table 4.2: Horizontal shift factors for T300/934 graphite-epoxy in an environment with 0.14% moisture.

T (°F)	$\log_{10}[A_{TH}]$
77	0
122	-2.1176
167	-4.5647
212	-6.3529
257	-8.2353
302	-10.0706
347	-12

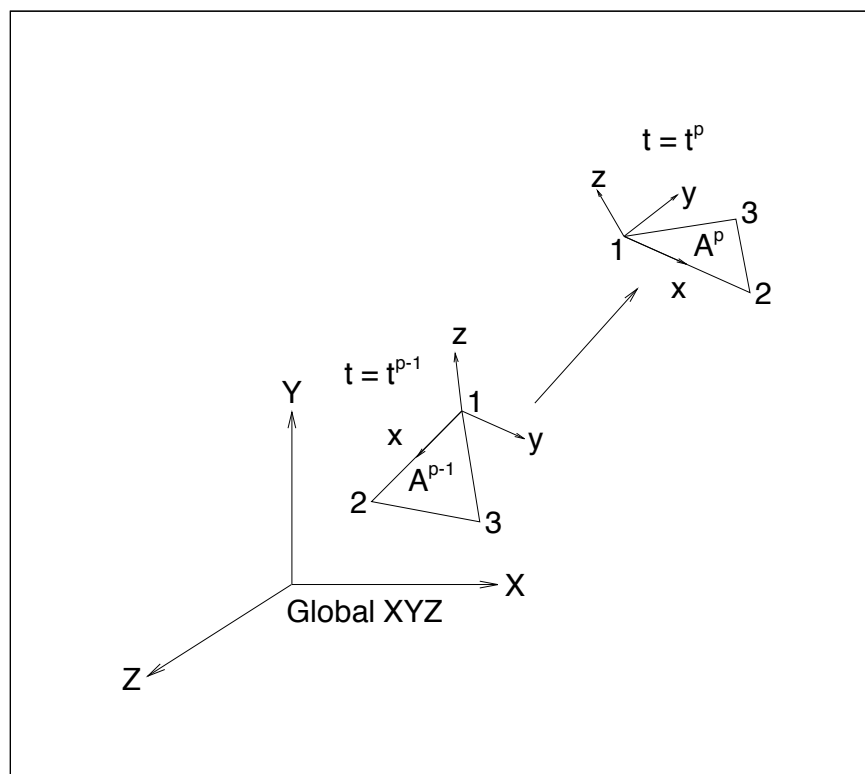


Figure 4.1: Deformation of a triangular flat shell element from time t^{p-1} to time t^p .

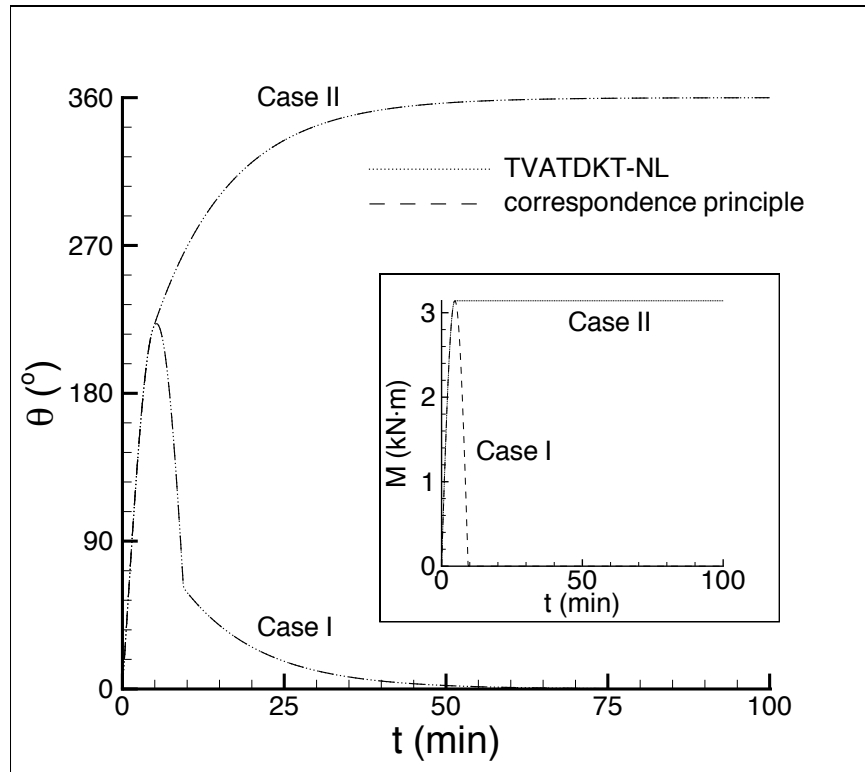


Figure 4.2: Tip rotation of a linear viscoelastic cantilever subjected to a tip moment. The time histories of applied tip moment are shown as an inset.

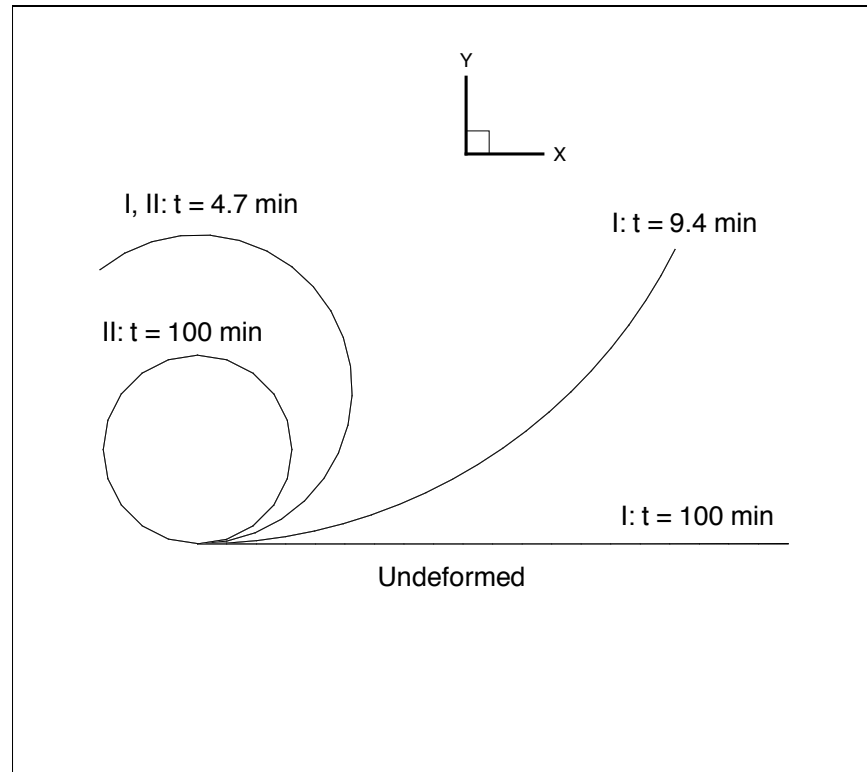


Figure 4.3: Original and deformed TVATDKT-NL meshes at various times for a linear viscoelastic cantilever under two time variations of tip moment.

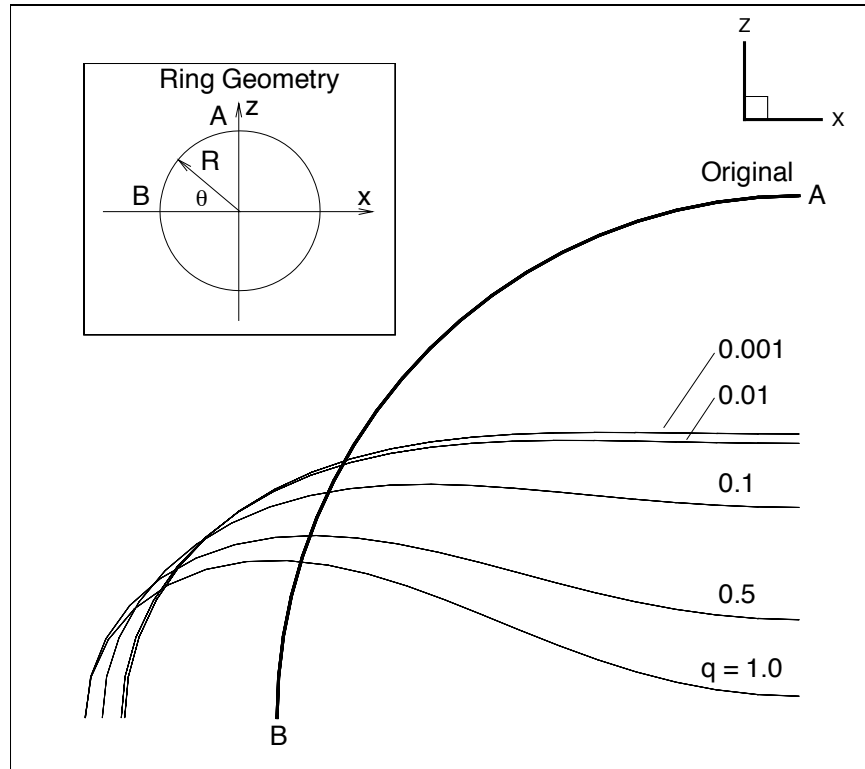


Figure 4.4: Original and deformed shapes for an elastic ring under nonuniform pressure. The deformed ring shapes computed using 40 TVATDKT-NL elements are shown for a non-dimensional load level near $P_0 b R^3 / EI = 3.4$.

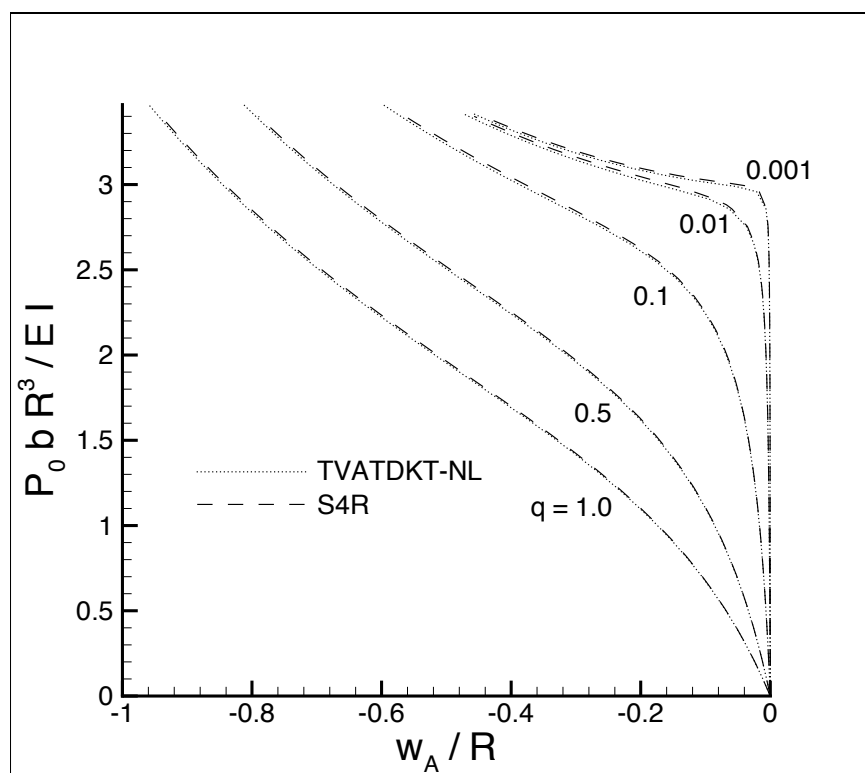


Figure 4.5: Elastic load-deflection behavior of a ring under various nonuniform pressure loads ($P(\theta) = P_0(1 - q \cos 2\theta)$).

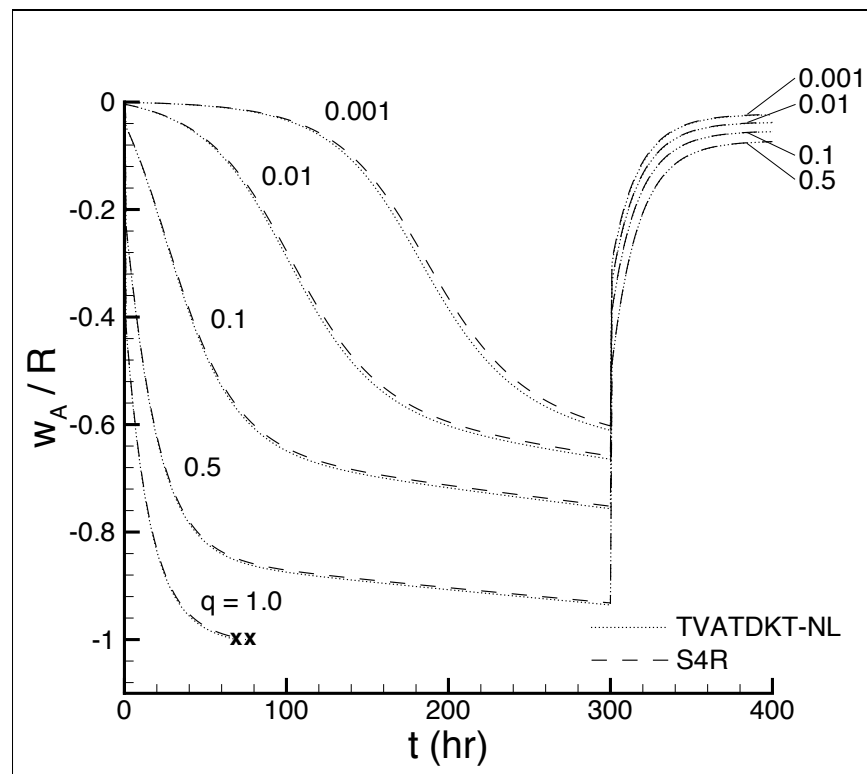


Figure 4.6: Viscoelastic response of a ring under various nonuniform pressure loads.

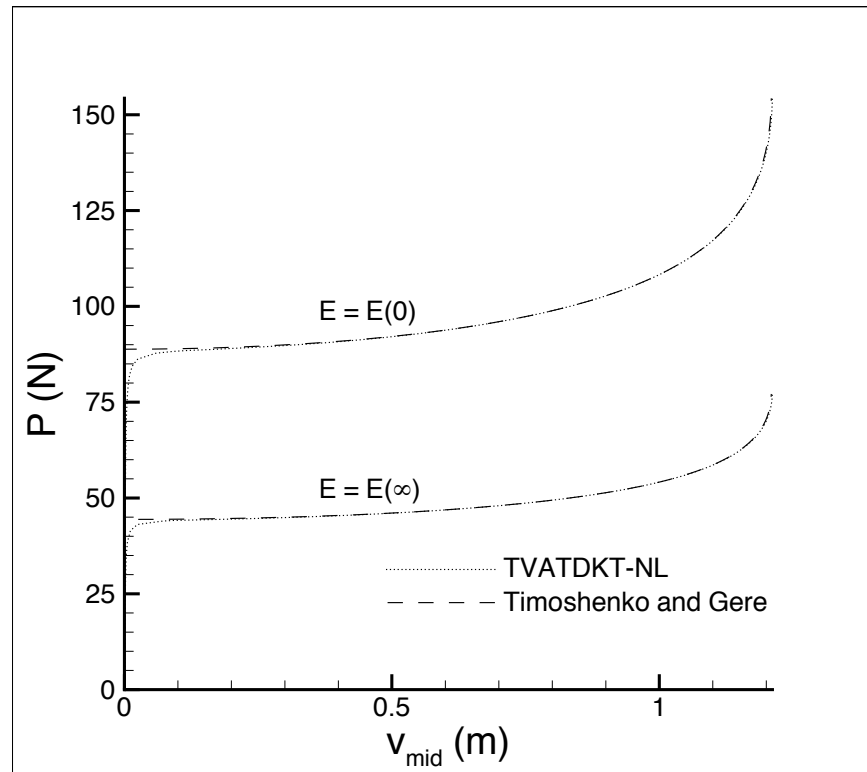


Figure 4.7: Elastic load-deflection paths for a pinned-pinned column under axial load. The TVATDKT-NL results are computed using an initial imperfection amplitude of 7.5311×10^{-4} m. The elastica solution of Timoshenko and Gere⁶⁶ for a geometrically-perfect, inextensible structure is also shown.

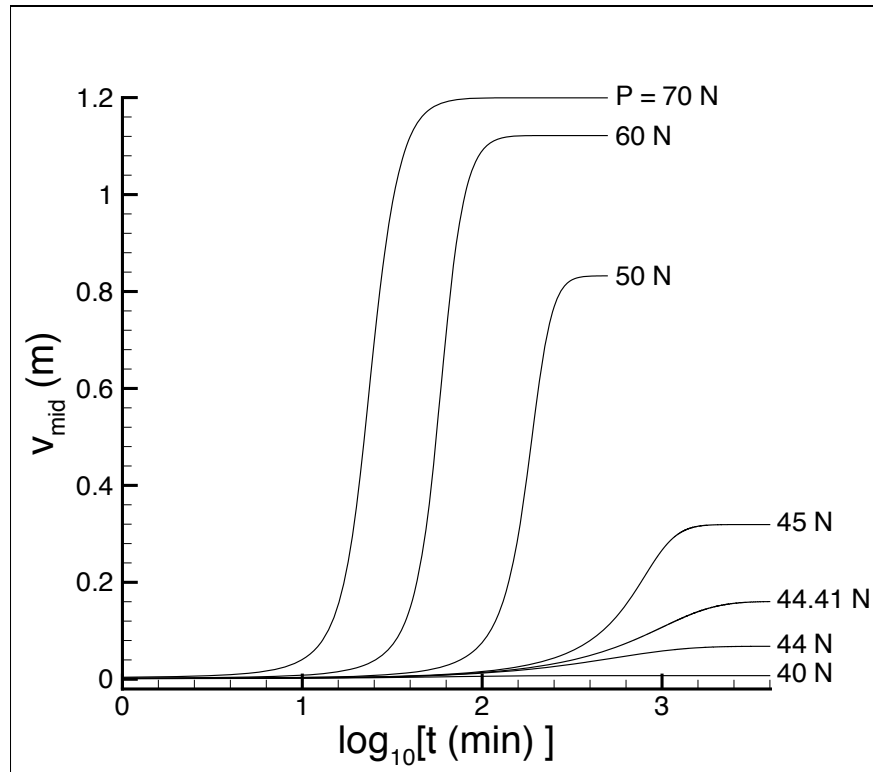


Figure 4.8: Time histories of midpoint transverse deflection, v_{mid} , for various values of axial load P using an initial imperfection amplitude of $7.5311 \times 10^{-4} \text{ m}$.

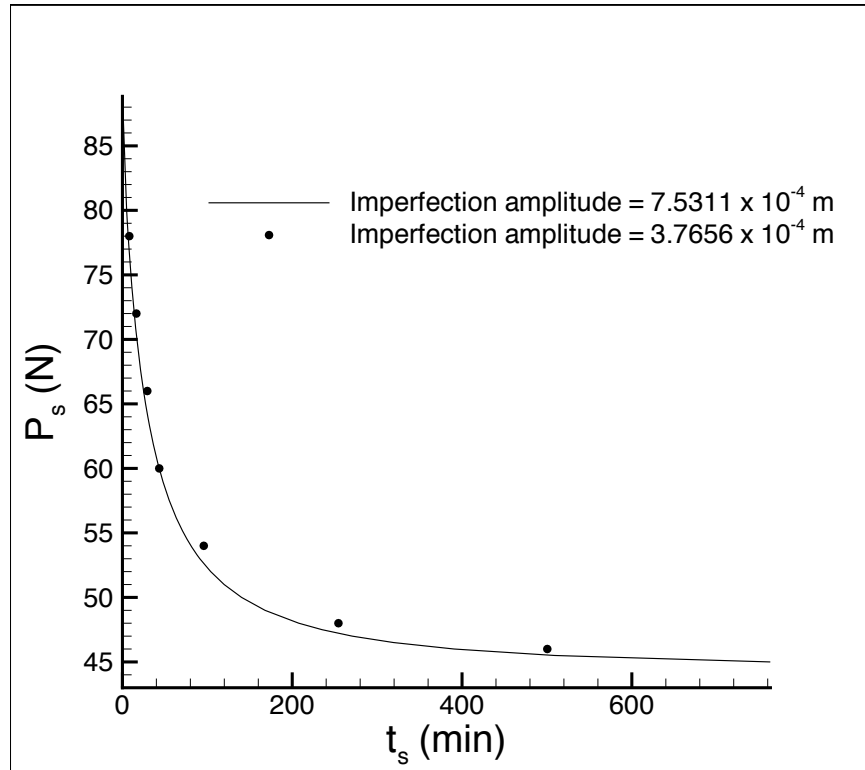


Figure 4.9: Stability curves of a viscoelastic column for two levels of initial imperfection. Here, t_s is the time required to reach $v_{mid} = 0.2$ m under axial load P_s applied as a creep load at $t = 0$.

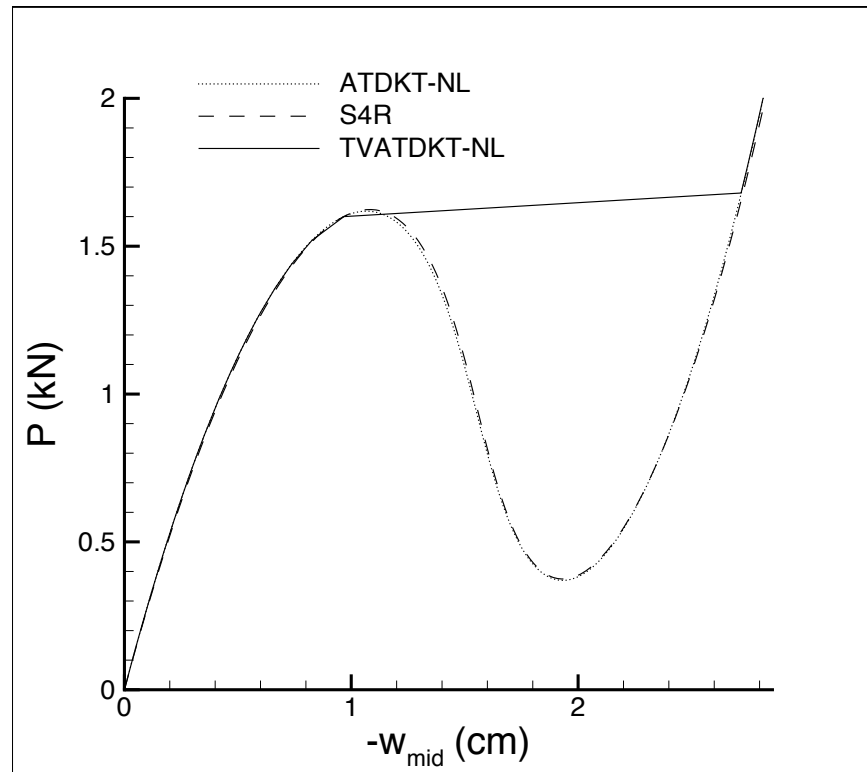


Figure 4.10: Static load-deflection behavior of a hinged cylindrical panel under centrally applied point load. These elastic results are computed using the initial value of the viscoelastic relaxation modulus.

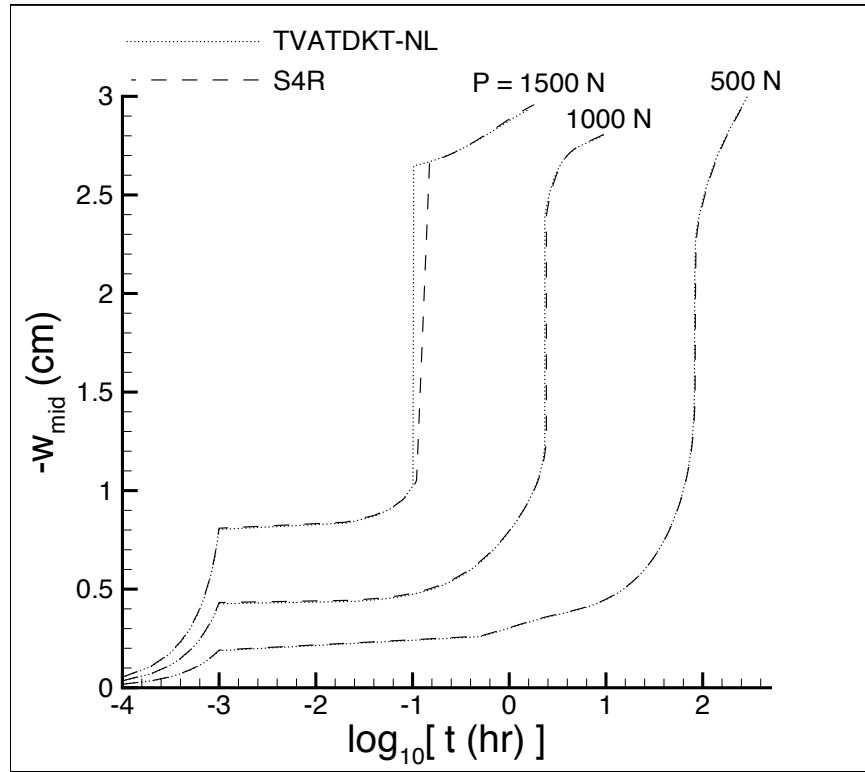


Figure 4.11: Time histories of viscoelastic deflection for various final values of center point force applied to a hinged cylindrical panel. The center force P is ramped from zero to its final value over the first 3.6 sec.

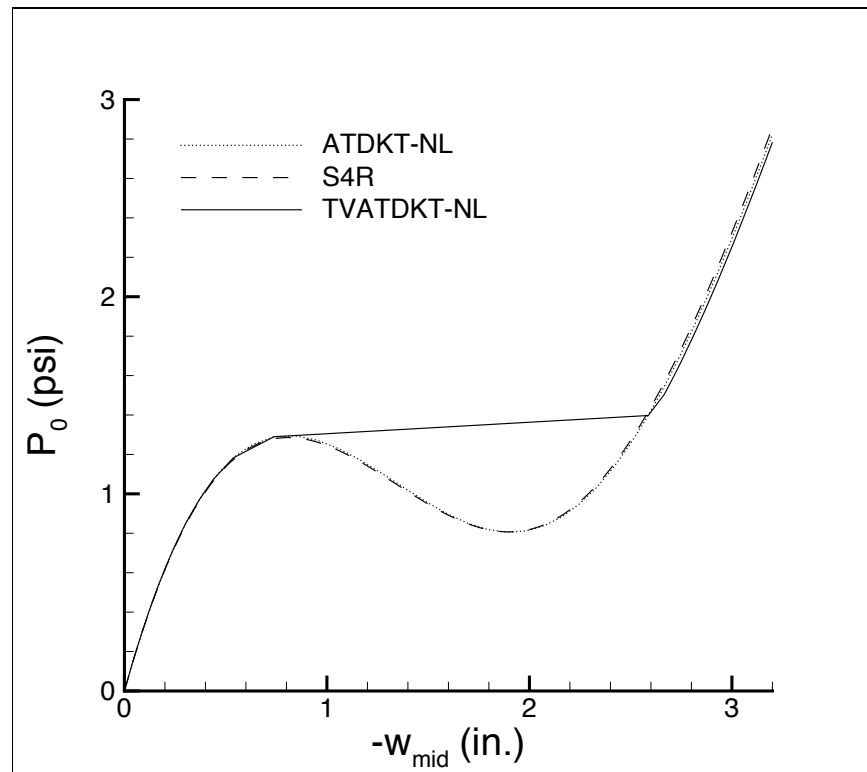


Figure 4.12: Elastic load-deflection path (using the initial viscoelastic material properties) for a composite cylindrical panel under uniform pressure.

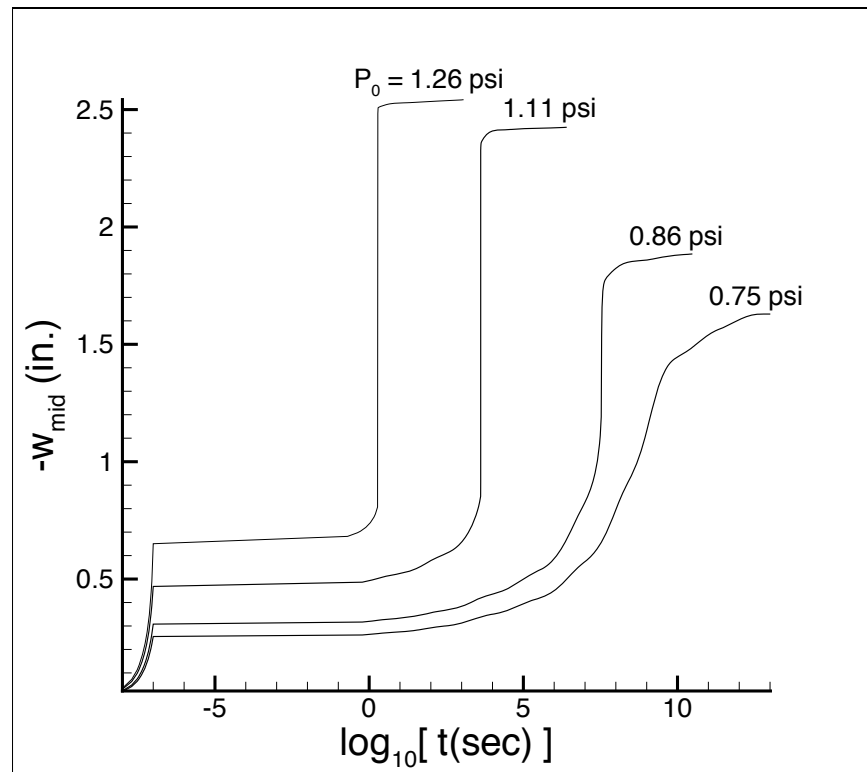


Figure 4.13: Time histories of center point deflection for a viscoelastic composite cylindrical panel under various values of step pressure load.

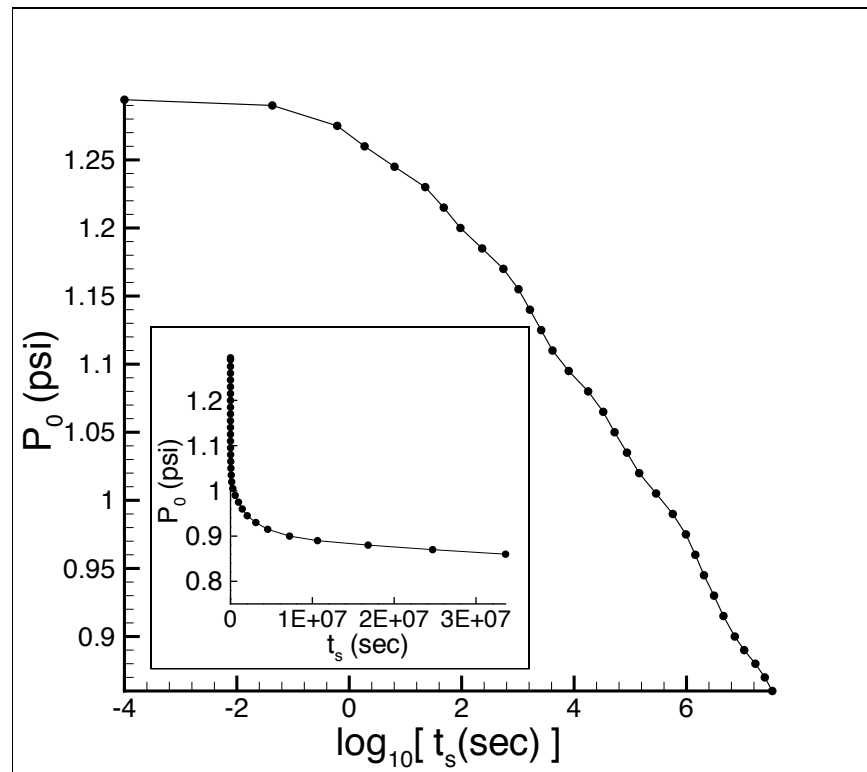


Figure 4.14: Step pressure magnitude P_0 versus critical time t_s to viscoelastic snap-through for a composite cylindrical panel.

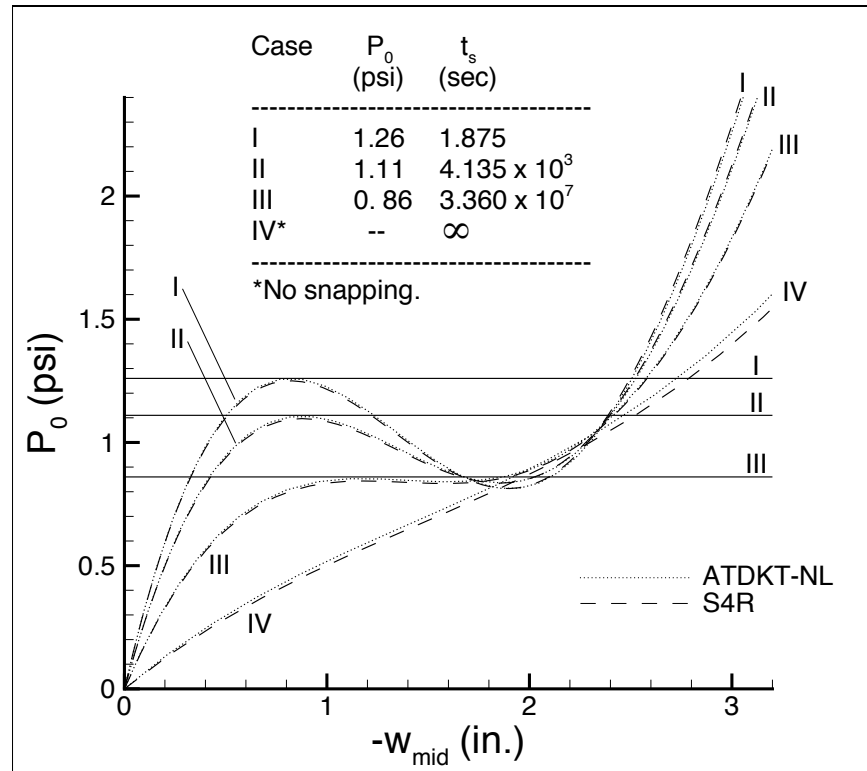


Figure 4.15: Elastic load-deflection paths using creep compliance properties corresponding to time t_s . The horizontal lines correspond to the load-levels P_0 necessary to cause snapping at t_s as predicted by the viscoelastic TVATDKT-NL analyses.

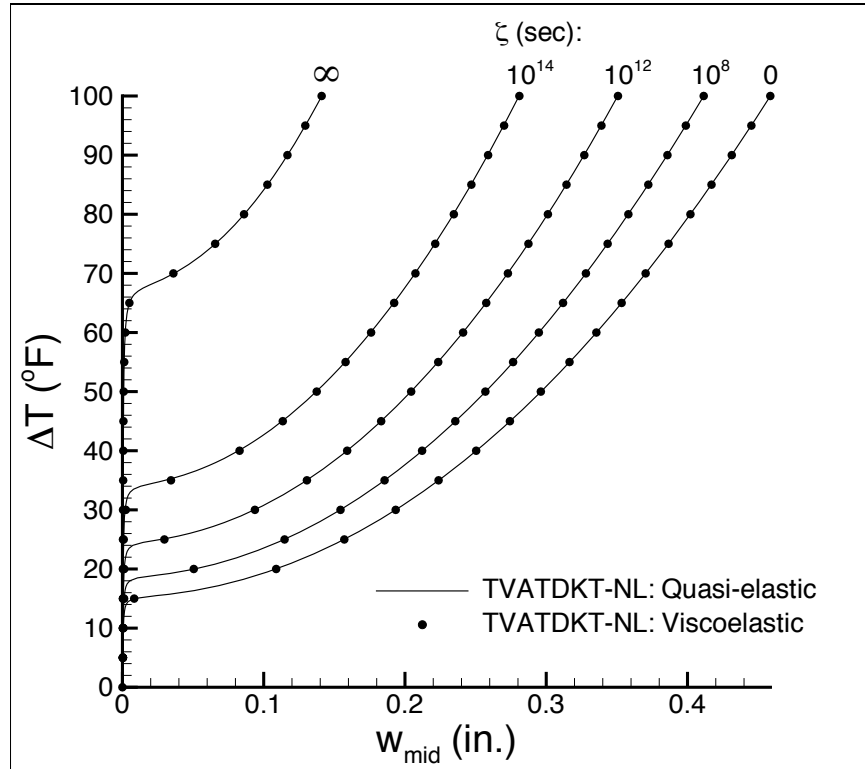


Figure 4.16: Thin viscoelastic $[\pm 45]_a$ plate under uniform thermal load applied nearly instantaneously at $t = 0$ and then held constant. A small initial imperfection is used in both the quasi-elastic and viscoelastic analyses. ($\Delta T = T_f - T_i$, $T_i = 77^\circ\text{F}$)

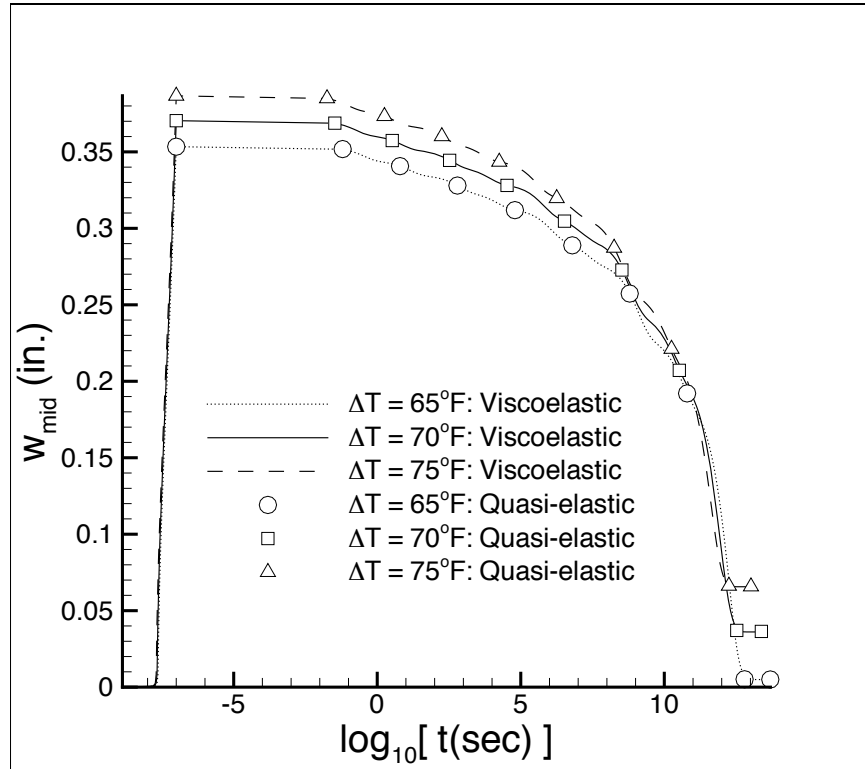


Figure 4.17: Time history of viscoelastic midpoint transverse deflection of thin antisymmetric panel under various values of thermal load computed using viscoelastic and quasi-elastic TVATDKT-NL analyses. A small initial imperfection is used in both the quasi-elastic and viscoelastic analyses. ($\Delta T = T_f - T_i$, $T_i = 77^\circ\text{F}$)

Chapter 5

Summary and Conclusions

Numerical analysis of linear viscoelastic isotropic beams and plane frames and laminated plates and shells has been formulated and conducted. The work on the isotropic beams and plane frames was performed to gain a better understanding of numerical analysis of linear viscoelastic structures, before the more difficult analysis of laminated plates and shells was begun. The solution of isotropic beam problems was first carried out using Galerkin's method for the spatial domain. The Galerkin shape functions were chosen to be orthonormal so that the resulting time domain equations for the shape function amplitudes are independent of one another. Although several forms of the governing equations for the shape function amplitudes were derived, a numerical time integration scheme was presented only for one form of the governing equations. In this time integration scheme, the shape function amplitudes are determined using a composite numerical integration scheme for the hereditary integral term as proposed by White.⁶ Using Prony series for the relaxation modulus, the hereditary integral at each time step is broken into incremental and memory parts. The number of viscoelastic memory terms is equal to the number of decaying exponential terms used in representing the relaxation modulus. These memory loads are evaluated using recursion relations. For dynamic analysis, the Newmark method is used to approximate the inertia term by a finite difference expression. Only homogeneous structures under mechanical loads were considered.

This numerical time integration scheme was then used with a plane frame element. The linear viscoelastic plane frame element equations for homogeneous structures were developed

from the linear elastic finite element equations by analogy to the relationship between the material laws for one-dimensional linear viscoelastic and linear elastic structures. The resulting linear viscoelastic equations are decoupled using the modes for an elastic structure with a Young's modulus equal to unity. The developed VFRAME code allows a limited number of modes to be used in representing the structural response. The number of modes needed for accuracy is, of course, problem dependent. The solution technique employed in the VFRAME formulation can be used for any homogeneous, isotropic, linear viscoelastic structure discretized using the finite element method. The Galerkin and VFRAME formulations were shown to be valid through a numerical example for which the exact quasi-static and dynamic results could be obtained. The VFRAME formulation was also shown to be valid for the analysis of viscoelastic arches.

Next, a triangular flat shell element for linear elastic composite plates and shells was extended to the linear viscoelastic case. The flat shell element is the combination of the Discrete Kirchhoff Theory (DKT) plate bending element and an element similar to the Allman Triangle, but derived by transforming the midside translations of the Linear Strain Triangle (LST) into corner translations and rotations. The element has three translations and three rotations at each node for a total of 18 degrees of freedom. Geometrically-linear and nonlinear formulations were developed and are termed TVATDKT-L and TVATDKT-NL, respectively.

Laminated composite plates and shells under mechanical and hygrothermal loads are modeled. The structural material is assumed to be linear viscoelastic and hygrothermo-rheologically simple. For fiber-reinforced polymer-matrix composites, the polymer matrix is typically viscoelastic, while the fibers are elastic. The present shell formulations include the capability to model the effect of elastic or viscoelastic fibers. Each layer is assumed to be orthotropic in a local coordinate system, resulting in the stress and strain for a layer being related through four independent relaxation moduli. The effect of a change in the temperature and/or moisture level upon a hygrothermo-rheologically simple material is to simply shift horizontally the curve of the relaxation modulus versus the log of time. This shifting

of the relaxation modulus curve can be avoided by using a single master relaxation modulus curve and a reduced time scale which depends upon the hygrothermal environment. For the TVATDKT-L and TVATDKT-NL formulations, the hygrothermo-rheologically simple postulate is incorporated by using four reduced time scales corresponding to four master relaxation curves.

The TVATDKT-L formulation takes advantage of the fact that for the regime of small deformations, the structural stiffness and the internal and external load distributions are independent of the deformations. Using Prony series for the relaxation moduli allows each hereditary integral appearing in the assembled finite element equations to be divided into incremental and memory parts. The viscoelastic memory terms are evaluated using recurrence relations involving only quantities from the previous two time steps. Because each layer is assumed to be made of the same material and the material behavior is characterized by four independent relaxation moduli, only four assembled stiffness matrices and thermal and hygroscopic load vectors are needed to evaluate the incremental hereditary integral terms and to update the viscoelastic memory terms, regardless of the number of terms used in the Prony series for the relaxation moduli. However, the total number of viscoelastic memory terms is equal to the number of exponential terms used to represent the relaxation moduli. Dynamic analysis is performed by including the inertia term, which is handled using the Newmark method.

Several numerical examples were performed to validate the TVATDKT-L formulation for the quasi-static and dynamic analysis of hygrothermo-rheologically simple linear viscoelastic laminated plates and shells. The results produced using the TVATDKT-L formulation agree well with solutions determined using the correspondence principle and results available in the existing literature for composite structures under mechanical loads, thermal loads, and load removal. For the case of an isotropic cantilever under a tip load, the dynamic results from the TVATDKT-L code matched the results obtained from the VFRAME code. It was shown that a poor choice of the Newmark parameters can lead to algorithmic damping.

Although the formulation for the small-deformation analysis of linear viscoelastic compos-

ite plates and shells was given for a triangular flat shell element, it could be easily employed with any known element which has been formulated for linear elastic composites. Recall that the four assembled stiffness matrices and thermal and hygroscopic load vectors needed for linear viscoelastic analysis are each what result for a linear elastic structure with one of the four reduced stiffnesses (Q_{11} , Q_{12} , Q_{22} , or Q_{66}) equal to unity and the other three reduced stiffnesses equal to zero. Furthermore, the time integration technique is independent of the element used in generating the assembled finite element equations.

The nonlinear equations for the TVATDKT-NL formulation are solved in an incremental/iterative fashion, with the full Newton-Raphson method used to fully converge each time increment. The TVATDKT-NL formulation is based upon Green–Saint-Venant strains and second Piola-Kirchhoff (PK2) stresses. The reference state used for these stress and strain measures in the TVATDKT-NL formulation is that at the end of the last fully-converged time step so that a total Lagrangian approach is used within each time step and an updated Lagrangian approach is used from time step to time step. The Green–Saint-Venant strains for all past history in the hereditary integral form of the linear viscoelastic material law are also referred to the configuration at the start of the current time increment. This results in a relatively simple form for the elemental incremental stresses in terms of the incremental strains and viscoelastic memory loads. The quantities needed to evaluate the memory loads for each element are updated at the end of an increment using recursion relations involving their previous values and terms corresponding to the incremental strains that were just converged. The viscoelastic tangent stiffness matrix is similar to that occurring in the elastic case, but with the usual $[A; B; D]$ matrices replaced by matrices which account for the relaxation over the current time step of the viscoelastic stresses corresponding to the current incremental strains. Deformation-independent mechanical loads, deformation-dependent pressure loads, and hygrothermal loads can be applied simultaneously. Although the TVATDKT-NL formulation and its implementation can be used for the regime of small deformations, it is computationally much more efficient to use the TVATDKT-L formulation for such problems.

The large deformations and stability of linear elastic and viscoelastic structures were examined. Deformation-dependent pressure loads were applied in some of the examples. For situations where a geometrically-perfect elastic structure would undergo bifurcation buckling with no prebuckling deformation, it was necessary to add an imperfection either in the loading or geometry in order to study the stability of a similar viscoelastic structure using a full viscoelastic analysis. Creep buckling instability for such situations was defined as a characteristic deflection growing beyond a chosen limit. The critical time for instability of viscoelastic structures is of course affected by the size of the imperfection. Hence, for the analysis of non-academic problems, the imposed imperfection must be chosen carefully, based upon the expected imperfections that would occur in the geometry and loading of the actual structure. Viscoelastic buckling or snap-through was observed to occur for a range of applied loads over a range of critical times, as opposed to occurring at a single load value as in the elastic case. The critical time to viscoelastic buckling or snap-through increased as the applied load decreased.

The quasi-elastic approach provides a convenient way to assess the accuracy of a full viscoelastic analysis when other results (analytical, numerical, or experimental) are unavailable. Although not shown here, it is well-known that the quasi-elastic method provides sufficient accuracy only in situations where the viscoelastic relaxation moduli (or creep compliances), the stress resultants, and the deformations all vary slowly with time. It should be noted that to determine the deflection history of complex viscoelastic structures, a numerical approach such as the finite element method must be used. For finite element analysis, it is much more cumbersome to use a quasi-elastic approach than a full viscoelastic approach to determine the complete time history of viscoelastic deflection, especially in the regime of large deformations.

The present work detailed the analysis of structures composed of linear viscoelastic materials for which Prony series could be used to represent the relaxation moduli and the creep compliances. Although many other material laws exist for the description of viscoelastic damping, the numerical efficiency of using Prony series should be apparent, since recursion

relations can be developed and used in computing the needed hereditary integral terms. Even though a large number of Prony series terms may be needed to adequately describe the material response characteristics, the evaluation of the viscoelastic memory terms and the incremental stiffness matrices is straightforward and computationally inexpensive.

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Appendix A

Brief Introduction to Linear Viscoelasticity

A.1 Introduction

This appendix is intended only to provide a brief introduction to some of the basics of linear viscoelasticity. For a more comprehensive study of linear viscoelasticity, the reader is directed to the texts by Flügge,³⁸ Findley *et al.*,⁷¹ and Creus.⁷² This appendix follows the text by Flügge³⁸ to a limited extent. Furthermore, only the case of uni-axial stress and strain will be discussed.

A.2 Basic Rheological Models for Uni-axial Stress

The elastic behavior of viscoelastic materials is modeled using springs, while dashpots are used to model the viscous behavior of such materials. By using various combinations of spring and dashpot elements in series and/or parallel connections, a wide range of linear viscoelastic materials can be modeled. The four basic elements which are used to build up more complex material models will be discussed subsequently.

A.2.1 Elastic Solid

A linear elastic solid is a material for which the stress σ is linearly proportional to the strain ϵ through Young's modulus E as follows:

$$\sigma = E\epsilon \quad (\text{A.1})$$

Obviously this represents a solid, as a constant non-zero stress results in a constant non-zero strain. A wide range of engineering materials can be represented as being linear elastic. The mechanical analog for an elastic solid is a spring, as shown in Fig. A.1.

A.2.2 Viscous Fluid

In a linear viscous fluid, the stress is linearly proportional to the strain rate. The constitutive law is simply

$$\sigma = \eta \dot{\epsilon} \quad (\text{A.2})$$

where $(\dot{})$ denotes $d()/dt$. Obviously this represents a fluid, as a constant non-zero stress causes continual deformation. Also note that it is not possible to deform the viscous fluid model instantly, as a step change in ϵ would require that an infinite stress be applied. A dashpot is the mechanical analog of the viscous fluid model, as shown in Fig. A.2.

A.2.3 Maxwell Fluid

In the Maxwell fluid model, a dashpot is connected to a spring in series, as shown in Fig. A.3. The constitutive law for such a material is derived as follows. When a stress σ is applied to both ends of such a model, the total strain ϵ of the element is

$$\epsilon = \epsilon_s + \epsilon_d \quad (\text{A.3})$$

where ϵ_s and ϵ_d are the strains across the spring and dashpot elements, respectively. The stress carried by each element is the same and equal to σ . Hence, the strains for the spring and dashpot are determined according to the following equations:

$$\epsilon_s = \frac{\sigma}{E} \quad \text{and} \quad \dot{\epsilon}_d = \frac{\sigma}{\eta} \quad (\text{A.4})$$

Thus,

$$\sigma + \frac{\eta}{E} \dot{\sigma} = \eta \dot{\epsilon} \quad (\text{A.5})$$

It should be obvious that this represents a fluid, as a non-zero stress can only be carried through a non-zero strain rate.

The Maxwell fluid model will now be subjected to creep loading in which a stress σ_0 is applied suddenly at $t = 0$ and then held constant (i.e., $\sigma = \sigma_0 u(t)$, where $u(t)$ is the unit step function). Taking the Laplace Transform of Eq. (A.5) with all initial conditions equal to zero and rearranging yields:

$$\hat{\epsilon} = \left(\frac{1}{\eta s} + \frac{1}{E} \right) \hat{\sigma} \quad (\text{A.6})$$

where $(\hat{\cdot})$ denotes the Laplace transform of $(\cdot)$ and s is the Laplace variable. Note that here $\hat{\sigma} = \sigma_0/s$. Taking the inverse Laplace transform of Eq. (A.6) gives

$$\epsilon = \sigma_0 \left(\frac{1}{E} + \frac{t}{\eta} \right) \quad (\text{A.7})$$

A plot of this creep response is shown as the dashed line in Fig. A.4. As can be seen from the plot, immediately after the load is applied, the response is $\epsilon_0 = \sigma_0/E$ which is the same as for an elastic material with Young's modulus of E , which here is referred to as either the impact or initial modulus.

Now suppose that the model is unloaded suddenly at time t_1 . That is, an additional stress $\tilde{\sigma} = -\sigma_0 u(t - t_1)$ is applied. The strain $\tilde{\epsilon}$ due to $\tilde{\sigma}$ is zero for $t < t_1$ and is as follows for $t > t_1$:

$$\tilde{\epsilon} = -\sigma_0 \left(\frac{1}{E} + \frac{t - t_1}{\eta} \right) \quad (\text{A.8})$$

Hence, the total strain response for $t > t_1$ is the superposition of the responses given in Eqs. (A.7) and (A.8) and is given by

$$\epsilon = \epsilon_1 = \frac{\sigma_0 t_1}{\eta} \quad (\text{A.9})$$

As indicated by dotted line in Fig. A.4, the material experiences instantaneous partial recovery. This is because the strain in the spring is instantly recovered, while the strain in the dashpot is never recovered, since a dashpot does not have any strain energy associated with

it. Alternatively, a dashpot dissipates all the energy that was used to cause its deformation, resulting in a nonconservative deformation process.

Now consider the case where a strain ϵ_0 is applied suddenly at $t = 0$ and then held constant ($\epsilon(t) = \epsilon_0 u(t)$). This is referred to as a relaxation loading. The dashpot does not deform instantly. Hence, all of the strain ϵ_0 initially results from the spring being stretched. Because the stresses carried by the spring and dashpot are the same, the initial stress can be determined as the initial stress in the spring as follows:

$$\sigma(0) = E\epsilon_0 = \sigma_0 \quad (\text{A.10})$$

Using the initial condition given in Eq. (A.10), Eq. (A.5) can be integrated to give

$$\sigma(t) = E\epsilon_0 e^{-\frac{E}{\eta}t} = \sigma_0 e^{-\frac{t}{\lambda}} \quad (\text{A.11})$$

where $\lambda = \eta/E$. A plot of this response is shown in Fig. A.5. Note that λ is a characteristic measure of the material response rate to relaxation loading and here is termed the relaxation time. By the time $t = \lambda$, the exponential term in Eq. (A.11) has decayed by 63%. For this model, λ is equal to the time required for the stress to reach zero following a path which is initially tangent to the stress-time curve for relaxation loading.

A.2.4 Kelvin Solid

In the Kelvin solid model, a spring is connected to a dashpot in parallel as shown in Fig. A.6. The total stress σ is the sum of the stresses in the spring and dashpot elements, while the strain ϵ is the same in both elements. That is

$$\sigma = E\epsilon + \eta\dot{\epsilon} \quad (\text{A.12})$$

Consider the case where the Kelvin model is subjected to a creep test ($\sigma(t) = \sigma_0 u(t)$). The initial strain is zero, because it is impossible to deform a dashpot instantaneously. Hence, the strain response can be shown to be

$$\epsilon(t) = \frac{\sigma_0}{E}(1 - e^{-\frac{t}{\lambda}}) \quad (\text{A.13})$$

where λ is equal to η/E . This response is shown as the dashed line in Fig. A.7. For large t , the strain approaches an asymptotic value $\epsilon_\infty = \sigma_0/E$. Hence, in this case, the asymptotic modulus, E_∞ , is equal to the stiffness E of the spring. This process is referred to as delayed elasticity, as the strain ultimately approaches a constant value under a constant load, but does not do so immediately. Since immediately after load application the spring is unstretched, all of σ_0 is initially carried by the dashpot through the nonzero strain rate for the model. As time evolves, the spring is stretched at the expense of a decreasing strain rate which lowers the stress carried by the dashpot. Ultimately, all of the stress is carried by the spring.

For this model, λ characterizes the creep response rate and is here termed the retardation time. The term $(1 - \exp(-t/\lambda))$ in Eq. (A.13) reaches 63% of its final value by the time $t = \lambda$. Here, λ is equal to the time required for the strain to reach its asymptotic value following a path initially tangent to the strain versus time curve for creep loading.

If after an elapsed time of t_1 under $\sigma(t) = \sigma_0 u(t)$, the strain is held constant at $\epsilon_1 = \epsilon(t_1)$, the stress response for $t > t_1$ is found from Eqs. (A.12) and (A.13) to be

$$\sigma(t) = \sigma_1 = E\epsilon_1 = \sigma_0 \left(1 - e^{-\frac{t_1}{\lambda}}\right) \quad (\text{A.14})$$

Hence, the stress immediately relaxes to this value, but it relaxes no further as shown by the dotted line in Fig. A.7.

Now suppose the Kelvin solid model is once again subjected to a creep test, but the model is unloaded at time t_1 (i.e., a total load of $[\sigma_0 u(t) - \sigma_0 u(t - t_1)]$ is applied). The strain for $t > 0$ due to $\sigma_0 u(t)$ is given by Eq. (A.13). The strain resulting from $-\sigma_0 u(t - t_1)$ is zero for $t < t_1$ and for $t > t_1$ it is determined by multiplying Eq. (A.13) by -1 and replacing t by $t - t_1$. Hence, the total strain response for $t > t_1$ is given by

$$\epsilon = \frac{\sigma_0}{E} \left(e^{\frac{t_1}{\lambda}} - 1 \right) e^{-\frac{t}{\lambda}} \quad (\text{A.15})$$

Ultimately, this leads to full recovery as shown in Fig. A.8.

Neither the Maxwell fluid nor Kelvin solid model is sufficient to accurately describe the response of real linear viscoelastic fluids and solids. For instance, the Maxwell fluid model

does not display a decreasing strain rate under a constant stress and does not exhibit time-dependent recovery after being unloaded. On the other hand, the Kelvin solid model does not model time-dependent stress relaxation and does not deform instantly under a suddenly applied stress. Rather, these models are presented to examine some of the basic features of linear viscoelasticity and to be used with isolated springs and dashpots in building up more complex and accurate material models.

A.3 Differential Form of Constitutive Law

In general, a linear viscoelastic material law can be expressed in the following form:

$$\sigma + p_1\dot{\sigma} + p_2\ddot{\sigma} + \cdots + p_m\sigma^{(m)} = q_0\epsilon + q_1\dot{\epsilon} + q_2\ddot{\epsilon} + \cdots + q_n\epsilon^{(n)} \quad (\text{A.16})$$

It should be noted that p_0 has been chosen to be unity in writing Eq. (A.16). Equation (A.16) can be rewritten in operator form as follows:

$$\underline{P}(\sigma) = \underline{Q}(\epsilon) \quad (\text{A.17})$$

where

$$\underline{P}(\cdot) = \sum_{k=0}^m p_k \frac{d^k(\cdot)}{dt^k} \quad \text{and} \quad \underline{Q}(\cdot) = \sum_{k=0}^n q_k \frac{d^k(\cdot)}{dt^k} \quad (\text{A.18})$$

For Eq. (A.16) to correspond to a real linear viscoelastic material, the p 's and q 's must satisfy certain mathematical relations. If Eq. (A.16) is determined from a mechanical analog, it is only necessary to ensure that the spring and dashpot constants are real and positive. If $q_0 = 0$, the material law represents a fluid, since a constant load gives a nonzero strain rate at all times. If however $q_0 \neq 0$, the constitutive law is for a solid, as at large times under a constant load, a constant strain equal to σ/q_0 will result. Hence, the long term modulus for a linear viscoelastic solid is simply q_0 . For Eq. (A.16) to result from the combination of springs and dashpots connected in series and/or parallel, either $n = m$ or $n = m + 1$. The initial material response to the sudden application of a stress or strain is governed by the highest order derivatives appearing in Eq. (A.16). If $n = m$, the material has instantaneous response with an initial modulus equal to q_m/p_m . If however, $n = m + 1$, the material will

not have instantaneous response, because to impose a strain suddenly would require infinite stress.

A.4 Mechanical Analogs for General Linear Viscoelastic Materials

For a particular material which has its constitutive behavior governed by Eq. (A.16), a number of mechanical analogs can be developed. However, such a material law will have unique generalized Maxwell ladder and Kelvin chain representations. These generalized models will be discussed subsequently.

A.4.1 Generalized Maxwell Ladder

In the generalized Maxwell ladder, N Maxwell elements are connected in parallel as shown in Fig. A.9. Also as indicated in Fig. A.9, an isolated spring and/or dashpot may be included in the model, if needed. Instantaneous elasticity is demonstrated by generalized Maxwell ladders lacking an isolated damper. A generalized Maxwell ladder lacking an isolated spring behaves as a fluid, whereas if an isolated spring is included, the model responds as a solid.

A.4.2 Generalized Kelvin Chain

The generalized Kelvin chain consists of N Kelvin elements connected in series as shown in Fig. A.10. An isolated spring is included to model instantaneous response, while the presence or absence of an isolated dashpot will result in fluid or solid behavior, respectively.

Both the generalized Maxwell ladder and Kelvin chain can model instantaneous response, delayed elasticity with a range of retardation times, stress relaxation with various relaxation times, and fluid flow.

A.4.3 Relation Between Generalized Model Forms and Differential Form of the Constitutive Law

Writing the material law as shown in Eq. (A.16) allows an easy determination of the number of springs and dashpots needed in representing the material by a Maxwell ladder or Kelvin chain. The number of springs and dashpots needed is simply equal to the number of nonzero p 's and q 's appearing in Eq. (A.16) with p_0 not counted. The presence/absence of an isolated spring and/or dashpot can be determined for the generalized models based upon whether the material experiences instantaneous response or not and whether the material behaves as a solid or fluid. Likewise, if a generalized model is known, the differential form of the constitutive law is easily determined based upon the number of springs and dashpots, solid/fluid behavior, and whether the model has instantaneous response or not. It should be noted that the only linear viscoelastic materials with unique mechanical analogs are the elastic solid, viscous fluid, Maxwell fluid, and Kelvin solid.

A.5 Creep Compliance and Relaxation Modulus

A.5.1 Creep Compliance

The creep compliance $D(t)$ of a viscoelastic material is defined to be the strain response to a unit creep load applied at $t = 0$. That is, if $\sigma(t) = u(t)$, then $\epsilon(t) = D(t)$. For linear viscoelastic materials, the strain response to $\sigma_0 u(t)$ is simply $\sigma_0 D(t)$. Since it is physically impossible for a tension bar to get shorter under a constant applied tensile stress, $D(t)$ is a monotonically increasing function or at least constant. For $t < 0$, $D(t)$ is defined to be zero. Specific creep compliances can be determined using the standard differential form for the constitutive law given in Eq. (A.17). Taking the Laplace transform of Eq. (A.17) for the case where all initial conditions are zero gives

$$\hat{P}(s)\hat{\sigma}(s) = \hat{Q}(s)\hat{\epsilon}(s) \quad (\text{A.19})$$

where

$$\hat{P}(s) = \sum_{k=0}^m p_k s^k \quad \text{and} \quad \hat{Q}(s) = \sum_{k=0}^n q_k s^k \quad (\text{A.20})$$

Using $\sigma = u(t)$ ($\hat{\sigma} = 1/s$) and $\hat{\epsilon} = \hat{D}$, Eq. (A.19) can be rearranged to give

$$\hat{D}(s) = \frac{\hat{P}(s)}{s\hat{Q}(s)} \quad (\text{A.21})$$

Ultimately, $D(t)$ can be determined by splitting Eq. (A.21) into a partial fraction expansion and then taking the inverse Laplace transform.

A.5.2 Relaxation Modulus

The relaxation modulus $E(t)$ is defined to be the stress response to a unit strain applied at $t = 0$ and then held constant ($\epsilon(t) = u(t)$). The relaxation modulus, which is also known as the relaxation compliance or the specific relaxation function, is a material property and is either monotonically decreasing or at most constant for all real viscoelastic materials. Similar to $D(t)$, $E(t)$ is zero for all $t < 0$. The equation corresponding to Eq. (A.21) for $\hat{E}(s)$ is

$$\hat{E}(s) = \frac{\hat{Q}(s)}{s\hat{P}(s)} \quad (\text{A.22})$$

A.5.3 Relations Between Creep Compliance and Relaxation Modulus

Using Eqs. (A.21) and (A.22), it can be shown that

$$\hat{D}\hat{E} = \frac{1}{s^2} \quad (\text{A.23})$$

Taking the inverse Laplace transform of Eq. (A.23) and using the convolution theorem gives

$$\int_0^t D(t-\tau)E(\tau) d\tau = t \quad (\text{A.24})$$

or

$$\int_0^t E(t-\tau)D(\tau) d\tau = t \quad (\text{A.25})$$

Two other general relations between $D(t)$ and $E(t)$ can be determined using the initial and final value theorems, which respectively state

$$\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} s \hat{f}(s) \quad (\text{A.26})$$

and

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s \hat{f}(s) \quad (\text{A.27})$$

Applying these two theorems to Eq. (A.23) and using the fact that the limit of a product is the product of the limits gives

$$D(0)E(0) = 1 \quad (\text{A.28})$$

$$D(\infty)E(\infty) = 1 \quad (\text{A.29})$$

Noting Eqs. (A.21), (A.22), (A.28), and (A.29), it should be apparent that $E(t)$ and $D(t)$ are reciprocals of each other only at $t = 0$ and $t = \infty$, except for the special case of an elastic solid which has $D = 1/E$.

A.5.4 Examples

If the Kelvin chain parameters are known for a general linear viscoelastic material law, it is fairly easy to write the creep compliance in terms of the Kelvin chain spring and dashpot constants. Also, if the Maxwell ladder parameters are known, the relaxation modulus can be written directly in terms of those parameters. Hence, $D(t)$ for the Kelvin chain and $E(t)$ for the Maxwell ladder will be considered subsequently. In the final example, $D(t)$ and $E(t)$ will be determined specifically for the three-parameter solid in terms of both the generalized model parameters and the constants appearing in the differential form of the material law.

Generalized Maxwell Ladder

Because the strain is the same in each “rung” of the generalized Maxwell ladder and the total stress is simply the sum of the stresses from each rung, the relaxation modulus of the

generalized Maxwell ladder is easily derived. The stress σ_i carried by the i^{th} Maxwell fluid element is governed by the following equation:

$$\sigma_i + \frac{\eta_i}{E_i} \dot{\sigma}_i = \eta_i \dot{\epsilon} \quad (\text{A.30})$$

where the strain for the Maxwell ladder is ϵ . The stress in the isolated dashpot is $\eta_0 \dot{\epsilon}$, whereas the stress in the isolated spring is $E_\infty \epsilon$. Hence, the Laplace transform of the total stress is as follows:

$$\hat{\sigma} = \left[E_\infty + \sum_{i=1}^N \left(\frac{s E_i}{s + \frac{E_i}{\eta_i}} \right) + s \eta_0 \right] \hat{\epsilon} \quad (\text{A.31})$$

For a relaxation test, $\hat{\epsilon} = 1/s$ and $\hat{\sigma} = \hat{E}(s)$. Hence, the relaxation modulus is determined to be

$$E(t) = E_\infty + \sum_{i=1}^N E_i e^{-\frac{t}{\lambda_i}} + \eta_0 \delta(t) \quad (\text{A.32})$$

where $\lambda_i = \eta_i/E_i$ and $\delta(t)$ is the Dirac delta generalized function.

Generalized Kelvin Chain

For the generalized Kelvin model, the creep compliance function is more easily derived than the relaxation modulus, as the stress σ carried by each Kelvin unit and the isolated spring and dashpot units (if present) is the same and the total strain is the sum of the strains from each unit. The strain ϵ_i in each Kelvin unit is governed by

$$\sigma = E_i \epsilon_i + \eta_i \dot{\epsilon}_i \quad i = 1, \dots, N \quad (\text{A.33})$$

The strain in the isolated spring is σ/E_0 , while the strain rate in the isolated dashpot is σ/η_∞ . Hence, the Laplace transform of the total strain ϵ is given by

$$\hat{\epsilon} = \left[\frac{1}{E_0} + \sum_{i=1}^N \frac{1}{\eta_i} \left(\frac{1}{s + \frac{E_i}{\eta_i}} \right) + \frac{1}{\eta_\infty s} \right] \hat{\sigma} \quad (\text{A.34})$$

For a creep test, $\hat{\sigma} = 1/s$ and $\hat{\epsilon} = \hat{D}(s)$. Thus, the creep compliance for the generalized Kelvin chain is determined as

$$D(t) = \frac{1}{E_0} + \sum_{i=1}^N \frac{1}{E_i} \left(1 - e^{-\frac{t}{\lambda_i}} \right) + \frac{t}{\eta_\infty} \quad (\text{A.35})$$

where $\lambda_i = \eta_i/E_i$.

Because any material with constitutive behavior governed by Eq. (A.16) can be represented using a generalized Maxwell ladder or Kelvin chain, it should be apparent that $E(t)$ and $D(t)$ for such a linear viscoelastic material will always be of the form given in Eqs. (A.32) and (A.35), respectively.

Three-Parameter Solid

The three-parameter solid or the standard solid consists of a spring in series with a Kelvin unit or a spring in parallel with a Maxwell unit as shown in Fig. A.11. The constitutive law for this model may be shown to be

$$\sigma + p_1 \dot{\sigma} = q_0 \epsilon + q_1 \dot{\epsilon} \quad (\text{A.36})$$

In terms of the Kelvin chain parameters, the constants in the constitutive law are

$$p_1 = \frac{\eta_a}{E_a + E_0} \quad (\text{A.37})$$

$$q_0 = \frac{E_0 E_a}{E_a + E_0} \quad (\text{A.38})$$

$$q_1 = \frac{E_0 \eta_a}{E_a + E_0} \quad (\text{A.39})$$

On the other hand, the constants are given in terms of the Maxwell ladder parameters as

$$p_1 = \frac{\eta_b}{E_b} \quad (\text{A.40})$$

$$q_0 = E_\infty \quad (\text{A.41})$$

$$q_1 = \left(1 + \frac{E_\infty}{E_b}\right) \eta_b \quad (\text{A.42})$$

Note that for a given three-parameter solid, the spring and dashpot constants for the two equivalent mechanical analogs are not equal, although they can be related to each other.

Using Eq. (A.21), $\hat{D}$ is found to be

$$\hat{D} = \frac{1 + p_1 s}{s(q_0 + q_1 s)} = \frac{1}{q_0} \left(\frac{1}{s}\right) + \left(\frac{p_1}{q_1} - \frac{1}{q_0}\right) \left(\frac{1}{s + q_0/q_1}\right) \quad (\text{A.43})$$

Taking the inverse Laplace Transform of Eq. (A.43) gives

$$D(t) = \frac{1}{q_0} + \left(\frac{p_1}{q_1} - \frac{1}{q_0} \right) e^{-\frac{q_0}{q_1}t} \quad (\text{A.44})$$

Using Eqs. (A.37), (A.38), and (A.39), the creep compliance is simplified to be

$$D(t) = \frac{1}{E_0} + \frac{1}{E_a} \left(1 - e^{-\frac{t}{\lambda_a}} \right) \quad (\text{A.45})$$

where $\lambda_a = \eta_a/E_a$.

Performing similar steps using Eq. (A.22), the relaxation modulus for the three-parameter solid is determined to be

$$E(t) = q_0 + \left(\frac{q_1}{p_1} - q_0 \right) e^{-\frac{t}{p_1}} \quad (\text{A.46})$$

Substituting in the expressions for p_1 , q_0 , and q_1 respectively given by Eqs. (A.40), (A.41), and (A.42), $E(t)$ can be rewritten as

$$E(t) = E_\infty + E_b e^{-\frac{t}{\lambda_b}} \quad (\text{A.47})$$

where $\lambda_b = \eta_b/E_b$.

Equations (A.45) and (A.47) could have been determined directly using Eqs. (A.35) and (A.32), respectively. Hence, rather than using Eqs. (A.21) and (A.22), it should be apparent that it is much easier to determine $D(t)$ and $E(t)$ in terms of the Kelvin chain parameters and Maxwell ladder parameters, respectively, provided that those parameters are known. Also, if a Prony series for $E(t)$ is known, it is fairly easy to determine the Maxwell ladder parameters for the material. Likewise, the spring and dashpot constants for the Kelvin chain representation of the material can be found directly when a Prony series for $D(t)$ is known.

A.6 Hereditary Integrals

A.6.1 Derivation

An alternative to solving Eq. (A.16) when either complicated stress or strain histories are prescribed is to evaluate an appropriate hereditary integral. These integrals are easily derived from basic calculus concepts and the creep compliance and relaxation modulus functions.

Consider a linear viscoelastic, one-dimensional body to be undisturbed (free of stress and strain) for $t < 0$. If a stress $\sigma(t) = \sigma(0) u(t)$ is applied, the strain response would simply be $\epsilon(t) = \sigma(0) D(t)$. If instead a stress $\Delta\sigma_1 u(t - t_1)$ is applied, $\epsilon(t)$ would be $\Delta\sigma_1 D(t - t_1)$. Recall that $D(t)$ is zero for negative t arguments, giving zero strain for $t < t_1$ under such a stress. If both of these stresses are applied to a linear viscoelastic material, the response is the superposition of the individual responses (i.e., $\epsilon(t)$ is $\sigma(0)D(t) + \Delta\sigma_1 D(t - t_1)$).

A general stress history can be represented as

$$\sigma(t) = \sigma(0) u(t) + \sum_{i=1}^N \Delta\sigma_i u(t - t_i) \quad (\text{A.48})$$

The strain response to such a stress history is simply

$$\epsilon(t) = \sigma(0) D(t) + \sum_{i=1}^N \Delta\sigma_i D(t - t_i) \quad (\text{A.49})$$

Now consider the case where the stress is written as

$$\sigma(t) = \sigma(0) u(t) + \int_{0+}^t \frac{d\sigma(\tau)}{d\tau} d\tau \quad (\text{A.50})$$

Here, $\sigma(0^-) = 0$ and $\sigma(0^+) = \sigma(0)$. The strain response is then

$$\epsilon(t) = D(t) \sigma(0) + \int_{0+}^t D(t - \tau) \frac{d\sigma(\tau)}{d\tau} d\tau \quad (\text{A.51})$$

Another form of Eq. (A.51) can be found by applying integration by parts. That is,

$$\epsilon(t) = D(t) \sigma(0) + D(t - \tau) \sigma(\tau) \Big|_{0+}^t - \int_{0+}^t \frac{dD(t - \tau)}{d\tau} \sigma(\tau) d\tau \quad (\text{A.52})$$

Simplifying gives

$$\epsilon(t) = D(0) \sigma(t) - \int_{0+}^t \frac{dD(t - \tau)}{d\tau} \sigma(\tau) d\tau \quad (\text{A.53})$$

Using the chain rule, the derivative appearing in the hereditary integral in Eq. (A.53) can be rewritten as follows:

$$\frac{d}{d\tau} [D(t - \tau)] = \frac{d}{d(t - \tau)} [D(t - \tau)] \frac{d(t - \tau)}{d\tau} = - \left(\frac{dD(t - \tau)}{d(t - \tau)} \right) \quad (\text{A.54})$$

Hence, another form for the hereditary integral equation for ϵ is

$$\epsilon(t) = D(0) \sigma(t) + \int_{0+}^t \frac{dD(t - \tau)}{d(t - \tau)} \sigma(\tau) d\tau \quad (\text{A.55})$$

As noted in Ref. 38, the first term in Eq. (A.51) represents the strain due to the initial stress, while the hereditary integral in Eq. (A.51) gives the additional strain due to any changes in the stress which occurred. As also explained in Ref. 38, the first term in Eqs. (A.53) and (A.55) represents the strain that would occur if the current stress had been applied instantly at t and the second term in those equations gives the additional strain due to the creep that occurred under the actual load history used to arrive at the current state of stress.

Finally, consider the case where the initial stress is not separated out in Eq. (A.50). That is, the stress is written as

$$\sigma(t) = \int_{0^-}^t \frac{d\sigma(\tau)}{d\tau} d\tau \quad (\text{A.56})$$

The strain is then given as

$$\epsilon(t) = \int_{0^-}^t D(t - \tau) \frac{d\sigma(\tau)}{d\tau} d\tau \quad (\text{A.57})$$

Equation (A.51) can be derived from Eq. (A.57) for the case when a step change occurs in the stress at $t = 0$, as the derivative of the stress at $t = 0$ in such a case is equal to the Dirac delta generalized function.

Similar derivations can be given when the strain history is prescribed. In such a case, σ is replaced by ϵ and vice versa. Instead of the creep compliance, $D(t)$, the relaxation modulus, $E(t)$, is used. The resulting hereditary integral forms for the constitutive law are

$$\sigma(t) = E(t) \epsilon(0) + \int_{0^+}^t E(t - \tau) \frac{d\epsilon(\tau)}{d\tau} d\tau \quad (\text{A.58})$$

$$\sigma(t) = E(0) \epsilon(t) - \int_{0^+}^t \frac{dE(t - \tau)}{d\tau} \epsilon(\tau) d\tau \quad (\text{A.59})$$

$$\sigma(t) = E(0) \epsilon(t) + \int_{0^+}^t \frac{dE(t - \tau)}{d(t - \tau)} \epsilon(\tau) d\tau \quad (\text{A.60})$$

$$\sigma(t) = \int_{0^-}^t E(t - \tau) \frac{d\epsilon(\tau)}{d\tau} d\tau \quad (\text{A.61})$$

Often, in practice, the lower limit on the hereditary integral will be written as 0 with either the value 0^+ or 0^- implied.

If the viscoelastic body is not undisturbed for $t < 0$, then either Eq. (A.57) and (A.61) should be used with the lower integration limit changed accordingly to the time when the

stress or strain is first applied or simply written as $-\infty$. Of course, the origin of the time scale could always be changed to the point where stress or strain is first applied so that the hereditary integral forms given explicitly in this subsection could be used.

A.6.2 Example

Consider three loading histories which are presented in Fig. A.12. For the Maxwell material, the strain response to the Case I loading can be obtained by using Eq. (A.57). Note that from Eq. (A.7), $D(t)$ is $(1/E + t/\eta)$. The mathematics is as follows for $t > t_1$:

$$\begin{aligned}\epsilon_I(t) &= \int_0^{t_1} \left(\frac{\sigma_1}{t_1} \right) \left(\frac{1}{E} + \frac{(t - \tau)}{\eta} \right) d\tau \\ &= \frac{\sigma_1}{t_1} \left[\left(\frac{1}{E} + \frac{t}{\eta} \right) \tau - \frac{1}{2\eta} (\tau)^2 \right]_0^{t_1} \\ &= \frac{\sigma_1}{E} - \frac{\sigma_1 t_1}{2\eta} + \frac{\sigma_1}{\eta} t\end{aligned}\tag{A.62}$$

If σ_1 is applied as a step input at t_1 , as in loading Case II, the resulting strain for $t > t_1$ is

$$\begin{aligned}\epsilon_{II}(t) &= \sigma_1 D(t - t_1) \\ &= \frac{\sigma_1}{E} + \frac{\sigma_1}{\eta} (t - t_1) \\ &= \frac{\sigma_1}{E} - \frac{\sigma_1 t_1}{\eta} + \frac{\sigma_1}{\eta} t\end{aligned}\tag{A.63}$$

This is observed to be less than the strain in Case I by $\sigma_1 t_1 / 2\eta$ for all $t > t_1$. Subjecting the Maxwell model to the Case III loading, the strain for $t > t_1$ is easily found to be

$$\epsilon_{III}(t) = \sigma_1 D(t) = \frac{\sigma_1}{E} + \frac{\sigma_1}{\eta} t\tag{A.64}$$

For all times $t > t_1$, this strain is larger than the strains in Cases I and II by $\sigma_1 t_1 / 2\eta$ and $\sigma_1 t_1 / \eta$, respectively. This simple example illustrates an important property of linear viscoelastic fluids: the differences in material response to several loading histories which result in the same steady-state stress state remain, regardless of how much time elapses.

Subjecting the three-parameter solid to the same three loading histories that were applied to the Maxwell model will yield a different fundamental result. The creep compliance for

this model is given by equation Eq. (A.44). Using Eq. (A.57) for Case I with $t > t_1$ gives

$$\begin{aligned}\epsilon_I(t) &= \int_0^{t_1} \left(\frac{\sigma_1}{t_1} \right) \left[\frac{1}{q_0} + \left(\frac{p_1}{q_1} - \frac{1}{q_0} \right) e^{-\frac{q_0}{q_1}(t-\tau)} \right] d\tau \\ &= \frac{\sigma_1}{q_0} + \frac{\sigma_1}{t_1} \left(\frac{p_1}{q_0} - \frac{q_1}{q_0^2} \right) \left(e^{\frac{q_0}{q_1}t_1} - 1 \right) e^{-\frac{q_0}{q_1}t}\end{aligned}\quad (\text{A.65})$$

Case II and III load histories respectively give the following results for $t > t_1$:

$$\epsilon_{II}(t) = \sigma_1 D(t - t_1) = \frac{\sigma_1}{q_0} + \sigma_1 \left(\frac{p_1}{q_1} - \frac{1}{q_0} \right) e^{\frac{q_0}{q_1}t_1} e^{-\frac{q_0}{q_1}t} \quad (\text{A.66})$$

and

$$\epsilon_{III}(t) = \sigma_1 D(t) = \frac{\sigma_1}{q_0} + \sigma_1 \left(\frac{p_1}{q_1} - \frac{1}{q_0} \right) e^{-\frac{q_0}{q_1}t} \quad (\text{A.67})$$

As t grows very large, ϵ_I , ϵ_{II} , and ϵ_{III} all approach σ_1/q_0 . This is referred to as the fading memory characteristic of linear viscoelastic solids, as the material “forgets” how the final load was initially applied.

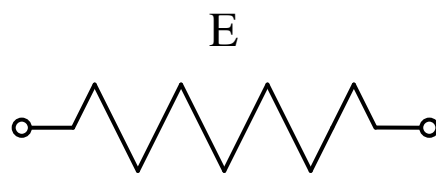


Figure A.1: Mechanical analog for the linear elastic solid material model.

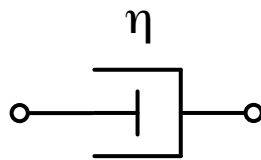


Figure A.2: Mechanical analog for the linear viscous fluid material model.

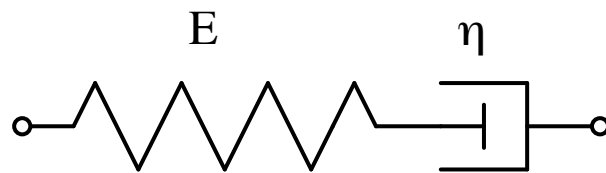


Figure A.3: Mechanical analog for the Maxwell fluid material model.

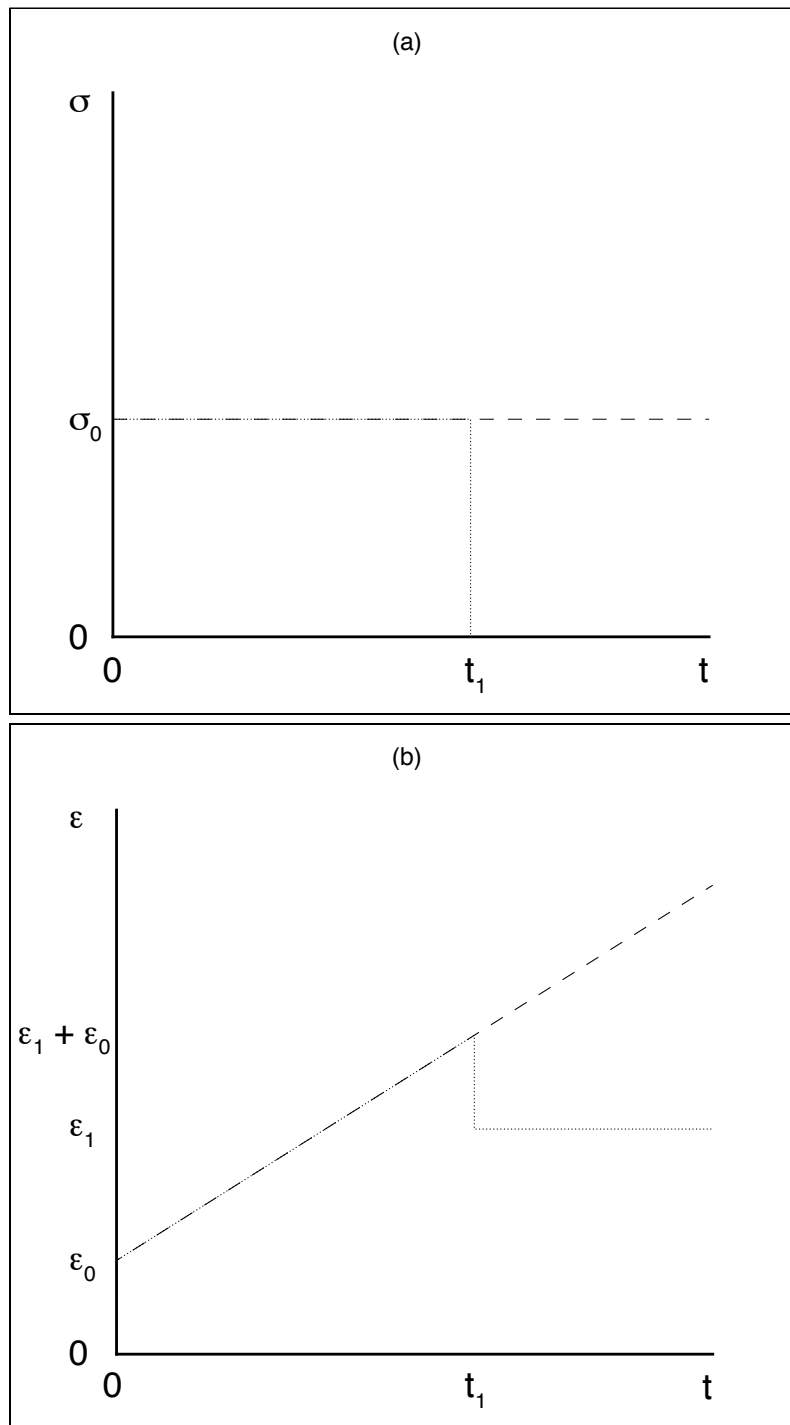


Figure A.4: Uni-axial stress applied to the Maxwell fluid model: (a) stress history; (b) strain history.

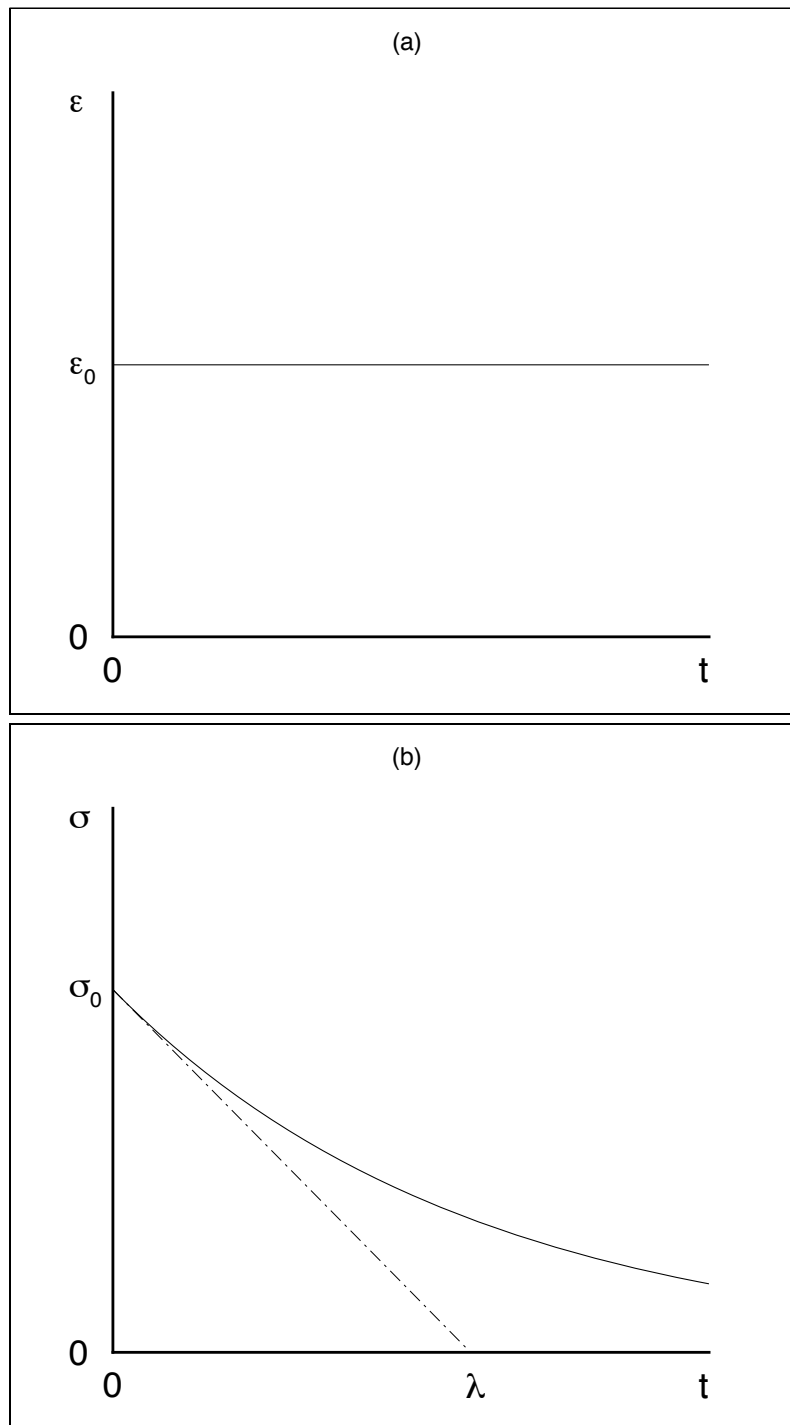


Figure A.5: Uni-axial strain applied to the Maxwell fluid model: (a) strain history; (b) stress history.

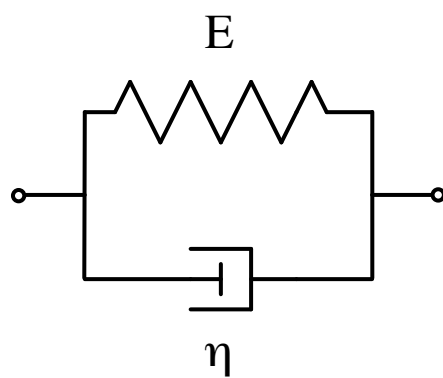


Figure A.6: Mechanical analog for the Kelvin solid material model.

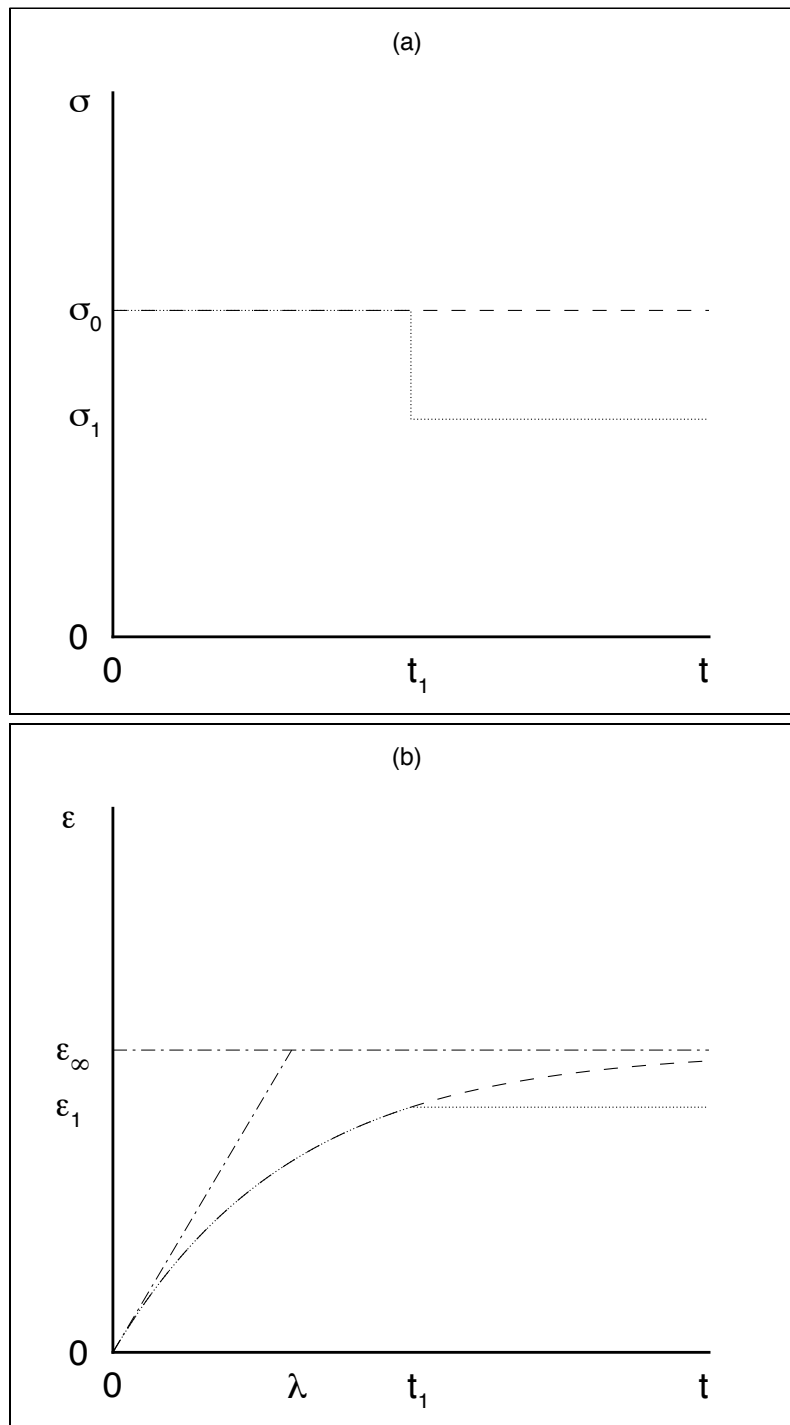


Figure A.7: Uni-axial loading applied to the Kelvin solid model: (a) stress history; (b) strain history.

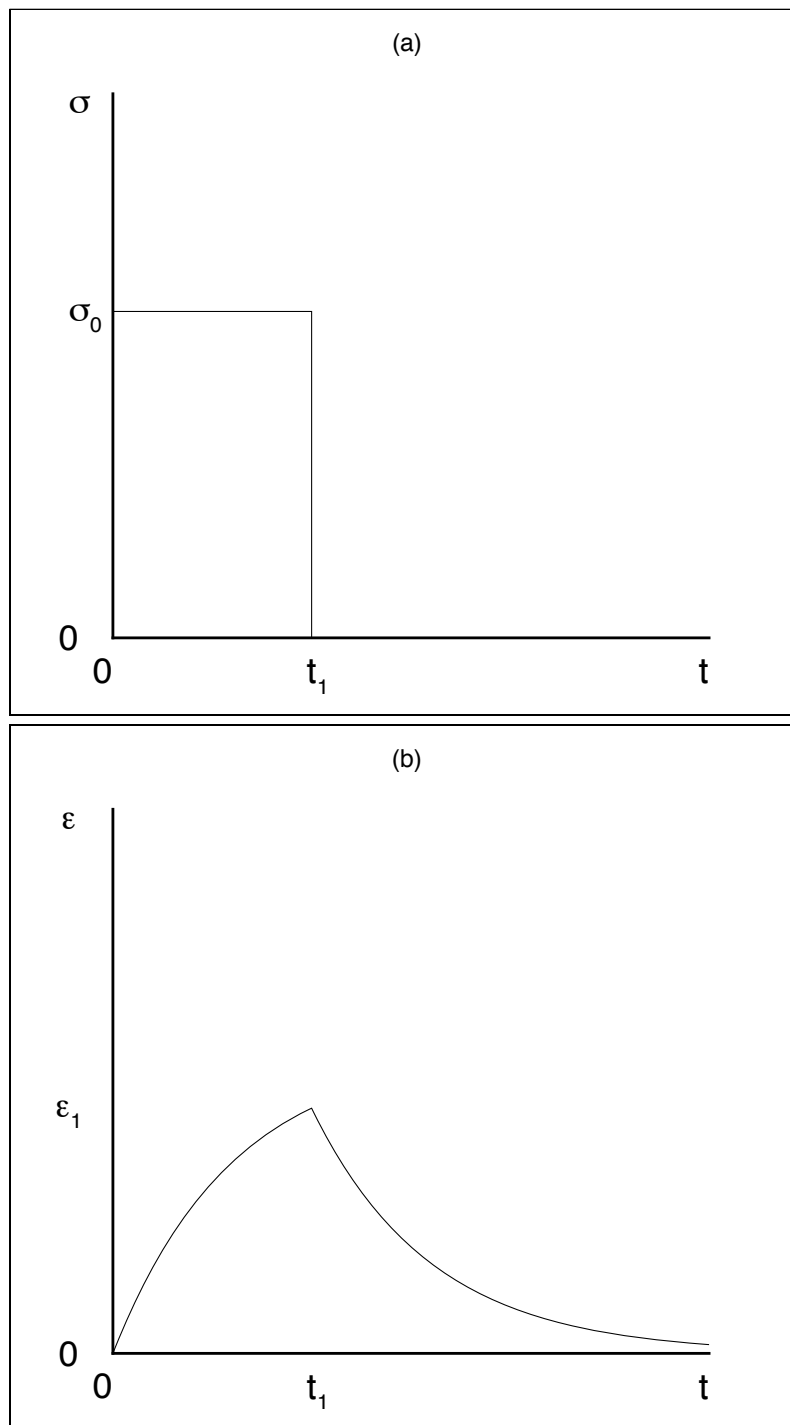


Figure A.8: Uni-axial stress applied to the Kelvin solid model with complete load removal: (a) stress history; (b) strain history.

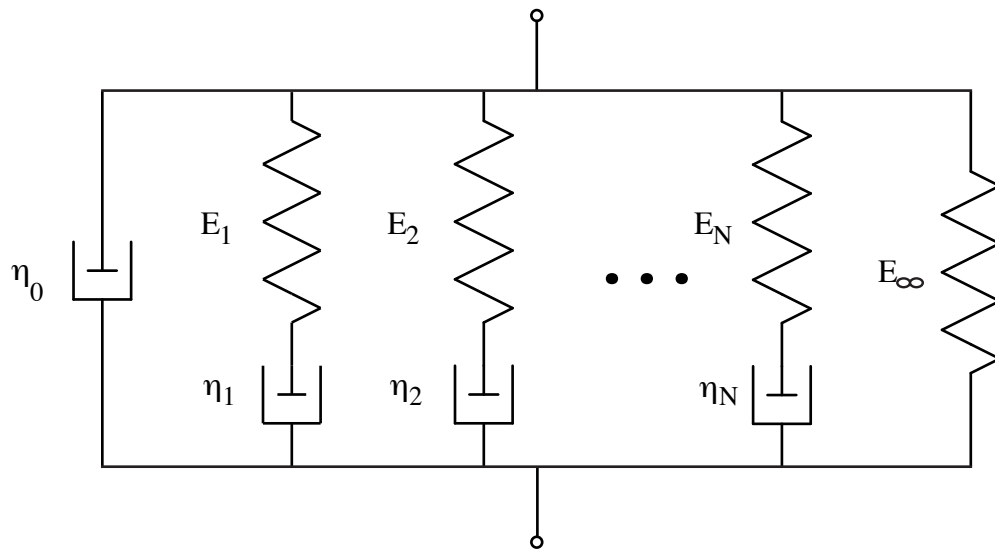


Figure A.9: Generalized Maxwell ladder mechanical analog for linear viscoelastic materials.

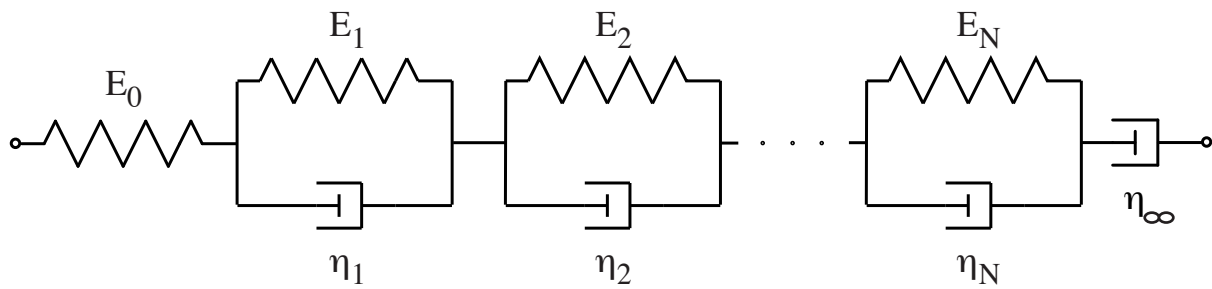


Figure A.10: Generalized Kelvin chain mechanical analog for linear viscoelastic materials.

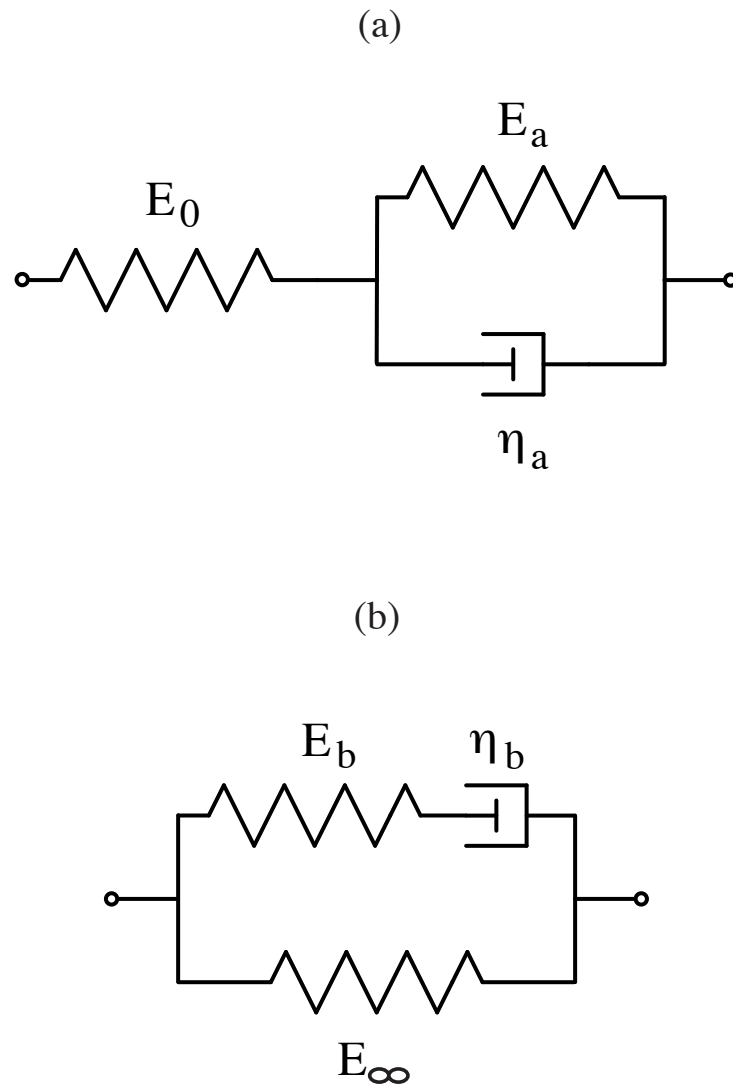


Figure A.11: Mechanical analogs for the three-parameter solid model: (a) Kelvin chain; (b) Maxwell ladder.

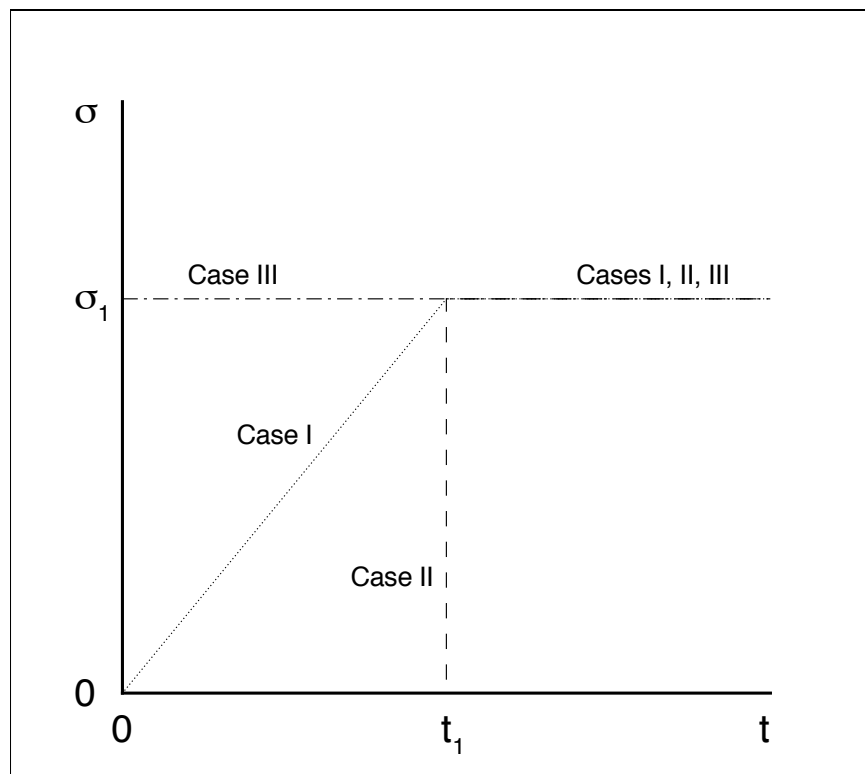


Figure A.12: Three time histories of applied stress.

Vita

Daniel C. Hammerand was born December 8, 1970 in Dubuque, Iowa. He graduated from Northside High School in Roanoke, Virginia in 1989. He then enrolled at Virginia Tech and received his bachelor's and master's degrees in Aerospace Engineering in 1994 and 1996, respectively. After completing his Ph.D., he will begin a post-doctoral position at Sandia National Laboratories in Albuquerque, New Mexico.