DEVELOPMENT AND APPLICATIONS OF FINITE ELEMENTS IN TIME DOMAIN

by

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Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY
IN
AEROSPACE AND OCEAN ENGINEERING

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December 4, 1996

Blacksburg, Virginia

Keywords: Time finite element, Sensitivity, Parametric identification, Optimal control

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(ABSTRACT)

A bilinear formulation is used for developing the time finite element method (TFM) to obtain transient responses of both linear, nonlinear, damped and undamped systems. Also the formulation, used in the *h*-, *p*- and *hp*-versions, is extended and found to be readily amenable to multi-degree-of-freedom systems. The resulting linear and nonlinear algebraic equations for the transient response are differentiated to obtain the sensitivity of the response with respect to various design parameters. The present developments were tested on a series of linear and nonlinear examples and were found to yield, when compared with other methods, excellent results for both the transient response and its sensitivity to system parameters. Mostly, the results were obtained using the Legendre polynomials as basis functions, though, in some cases other orthogonal polynomials namely, Hermite, Chebyshev, and integrated Legendre polynomials were also employed (but to no great advantage). A key advantage of TFM, and the one often overlooked in its past applications, is the ease in which the sensitivity of the transient response with respect to various design parameters can be obtained.

Since a considerable effort is spent in determining the sensitivity of the response with respect to system parameters in many algorithms for parametric identification, an identification procedure based on the TFM is developed and tested for a number of nonlinear single- and two-degree-of-freedom system problems. An advantage of the TFM is the easy calculation of the sensitivity of the transient response with respect to various design parameters, a key requirement for gradient-based parameter identification schemes. The method is simple, since one obtains the sensitivity of the response to system parameters by differentiating the algebraic equations, not original differential equations. These sensitivities are used in Levenberg-Marquardt iterative direct method to identify parameters for nonlinear

single- and two-degree-of-freedom systems. The measured response was simulated by integrating the example nonlinear systems using the given values of the system parameters. To study the influence of the measurement noise on parameter identification, random noise is added to the simulated response. The accuracy and the efficiency of the present method is compared to a previously available approach that employs a multistep method to integrate nonlinear differential equations. It is seen, for the same accuracy, the present approach requires fewer data points.

Finally, the TFM for optimal control problems based on Hamiltonian weak formulation is proposed by adopting the p- and hp-versions as a finite element discretization process. The p-version can be used to improve the accuracy of the solution by adding more unknowns to each element without refining the mesh. The usage of hierarchical type of shape functions can lead to a significant saving in computational effort for a given accuracy. A set of Legendre polynomials are chosen as higher order shape functions and applied to two simple minimization problems for optimal control. The proposed formulation provides very accurate results for these problems.

ACKNOWLEDGMENTS

Though all of the people who have made some contributions to the thoughts and ideas which culminated in this work are too numerous to name singularly. However, there are a special few to whom I am particularly indebted that I would like to mention individually.

First of all, I am extremely grateful to my advisor Dr. Rakesh K. Kapania for his considerable assistance and helpful discussions not only as a professor but also a friend. This dissertation would not have been completed without his help.

I would like to express my sincere thanks to Dr. E. R. Johnson, Dr. E. Nikolaidis, and Dr. M. P. Singh, for serving on the supervising committee. Also deserving special mention is Dr. Eugene M. Cliff for many helpful discussions concerning calculus of variations and optimal control.

Thanks are also due to Prof. S. N. Atluri for bringing his work on development and application of time finite element method for multi-rigid body dynamics to my attention.

I would also like to thank Dr. B. Grossman, Department Head of the Aerospace and Ocean engineering for providing generous computer resources for this research.

Finally, I dedicate this work to my parents who have encouraged me to continue my study and provided financial support so that I may pursue my study without any difficulties.

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LIST OF SYMBOLS

Chapter 2

$\mathbf{a}$, a_i	= linear term in the variational formulation
$\mathbf{B}$, B_{ij}	= bilinear term in the variational formulation
C	= damping coefficient
$\mathbf{C}$	= 2×2 damping matrix
$\mathbf{d}$, d_k	= system parameters
δ	= variation operator
$\delta(t)$	= Dirac's delta function
$F(t)$	= externally applied dynamic load
$\mathbf{F}(t)$	= load vector
$\tilde{\mathbf{g}}$	= bilinear form
K	= stiffness coefficient
$\mathbf{K}$	= 2×2 stiffness matrix
M	= mass coefficient
$\mathbf{M}$	= 2×2 mass matrix
$\mathbf{p}$, $\mathbf{q}$	= generalized coordinates
Q_i	= nonconservative generalized forces
s_i, $\dot{s}_i$	= generalized coordinates: displacement, velocity
$\mathbf{T}$	= kinetic energy of the system
T_0	= initial time
T_f	= final time
$u(t)$	= transient response
$\mathbf{u}(t)$	= transient response vector
$v(t)$	= test function
$\mathbf{v}(t)$	= test function vector
$\mathbf{V}$	= potential energy of the system
λ	= Lagrange multiplier
$\phi_j(t)$	= basis functions
$\psi_i(t)$	= admissible functions
$\{ \}$	= column vector
$< >$	= row vector

Chapter 3

$\overline{\mathbf{B}}$, $\overline{\mathbf{B}}^*$	= augmented stiffness matrix

Chapter 4

$L(\mathbf{d})$	= objective function of the system
T	= record length of the measured response
u_a	= time series of the analytical response
u_m	= time series of the measured response
$\mathbf{H}$	= Hessian of the objective function
$\mathbf{g}$	= gradient of the objective function

Chapter 5

$\mathbf{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$	= Lagrangian of the system
$\mathbf{p}$	= generalized momenta
$\mathbf{q}$	= generalized coordinates
$\mathbf{Q}$	= nonconservative generalized forces applied to the system
J	= performance index
$\triangleq$	= equals by definition; denotes
R^n	= set of all ordered n-tuples of real numbers
$L(x, u, t)$	= integrand of the performance index
x	= state
u	= control
$N(t)$, $M(t)$	= Legendre polynomial functions
α, ν	= discrete Lagrange multipliers
$\lambda(t)$	= Lagrange multiplier functions
$\phi(x, t)$	= discrete functions defined at T_0 and T_f
$\psi(x, t)$	= constraints imposed on the states at T_0 and T_f
$\hat{x}$	= nodal values of x

1. INTRODUCTION

1.1 Background

When confronted with a dynamic problem, usually the structure is discretized by using the finite element method or a modal superposition approach, for instance, hence reducing the problem to a set of ordinary differential equations in time that can be solved with the help of many time stepping approaches [1, 2, 3, 4, 5, 6, 7, 8]. This kind of procedure is widely used in practice and fairly well understood.

The other possible approach dealing with that problem is the use of the time finite element method (TFM) based on Hamilton's Law of Varying Action (as will be detailed later). There are several potential advantages to the time finite element formulation. First, the TFM can be applied to both energy equation, will be seen in Chapter 5, and directly to the equations of motion. The second advantage is the straightforward derivation of higher order approximations in time. Usually, the TFM approximation yields an accuracy superior to that of more conventional time stepping schemes at same computational cost. Furthermore, the formulation is easy and convenient for computer implementation.

A formulation based on Ritz method in conjunction with an extension of Hamilton's Law of Varying Action will be applied to obtain numerical solutions of any time dependent problems for mechanical systems. Also, the study for the manner in which its performance might be affected by the choice of basis functions and length of time interval will be attempted.

1.2 Literature Review

In recent years, the TFM has found increasing popularity, especially with researchers involved with studying transient response and dynamic stability of periodic systems, such as the aeroelastic stability of helicopter rotor blades (Borri *et al.* [9]; Peters and Izadpanah [10]; Achar and Gaonkar [11]) and the multi-rigid body dynamics (Borri *et al.* [12, 13]; Mello *et al.* [14]; Atluri and Cazzani [15]). Based on these and many other efforts (to name a few: Pian and O'Brien [16]; Gurtin [17, 18]; Sandhu and Pister [19]; Tonti [20]; Atluri [21]; Herrera and Bielak [22]; Bailey [23, 24, 25, 26]; Reddy [27, 28]; Levinson [29]; Hitzl and Levinson [30]; Hitzl [31]; Tong and Pian [32]; Reissner [33]) dedicated to develop appropriate variational framework for generating approximate solutions in space and time domain, it appears that the TFM has emerged as a viable approach for studying the transient response of systems (*i.e.* for solving initial value problems) also. A brief review of the key development leading to the current state of the finite element method as applied to initial value problems is given below.

The TFM was first introduced by Argyris and Scharpf [34] who employed Hermite cubic interpolation polynomials (akin to the beam finite element) to express the response over each time finite element. The response and the velocity at any given instant were thus obtained in terms of the displacement and the velocity at two "pivotal points", namely the start and the end of the time element. The method, based on the Hamilton's principle, was applied to a single-degree-of-freedom system but no numerical examples were considered. A generalization of the formulation allowing for an arbitrary number (p) of pivotal points and each pivotal point having an arbitrary number of time derivatives (a) of the response, was also given (but, once again, without any numerical results). Fried [35], using an 'Extended Hamilton's Principle', applied this approach to study the transient response of a damped system (nonconservative systems) and transient heat conduction in a slab whose surface was subjected to a harmonically varying temperature. Fried used a step by step approach in which the final conditions (for displacement and velocity) for one time element can be considered as initial conditions for the next time finite element, as in any time-marching technique. This was done to avoid storing and working with large matrices. Zienkiewicz and Parekh [36], also working with problems in heat conduction, used a time finite element approach that was based on Galerkin procedure of obtaining weighted residual equations over a time interval. On integration, the resulting equation was given in terms of the initial and final states.

An important development in the evolution of time finite element method was a series of papers by Bailey [23, 24, 25, 26] in which the author pointed out the need for applying Hamilton's law of varying action, briefly, Hamilton's law, *not* Hamilton's principle, in solving problems in elastodynamics. Using Hamilton's law, Bailey used the classical Ritz method, with simple polynomials as the basis functions, to study elastodynamic response of beams. Baruch and Riff [37], Riff and Baruch [38] and Borri *et al.* [9] developed time finite element using Hamilton's law. Baruch and Riff [37] noted that one can use six different formulations (each having different combinations of initial and final constraints) of the Hamilton's law for each degree-of-freedom. Borri *et al.* [12, 13], Mello *et al.* [14] and Atluri and Cazzani [15] applied the primal and mixed form of Hamilton's law to develop piecewise Lagrange-type time finite elements to solve nonlinear equations of multi-rigid body dynamics.

Wu [39] recognizing the limitations of constrained variational principles like the extended Hamilton's principle to nonconservative forces, presented an unconstrained variational principle in which the constraints were applied using the well known technique of Lagrange multipliers. Simkins [40] presented a procedure, consisting of introducing all boundary and essential conditions into the 'variational statement' as natural boundary conditions, making the variational statement suitable for obtaining approximate solutions for initial and boundary value problems. It was pointed out that the proposed procedure when applied to the Euler equations leads to a modified Hamilton's principle as was given by Tiersten [41]. In a subsequent paper, Simkins [42] employed this variational statement to develop finite element in time.

Commenting on the paper by Simkins [40], Smith [43] noted that what the former calls a variational statement for solving initial value problems is really an application of the weighted residual method to those problems. Building on this observation, Peters and Izadpanah [10] offered a bilinear formulation of elastodynamics as an alternative source to develop approximate methods to solve these problems. As was earlier done by Wu [39], Peters and Izadpanah employed Lagrange multiplier method to account for various end constraints. More specific choice and use of Lagrange multiplier to satisfy the various constraint conditions can be seen in the works by Atluri [21], Borri *et al.* [12, 13], Mello *et al.* [14] and Atluri and Cazzani [15]. A distinct advantage of this augmented, bilinear formulation is the natural convergence of the end conditions. To achieve this, the natural convergence of the end constraints is very important as the end conditions of one segment are used as initial conditions for the next segment. Using the proposed augmented bilinear formulation and the h-, p-, and the hp-versions, Peters and Izadpanah solved a number of examples related to dynamic response of linear systems.

The p-version finite element method [44], initiated by Szabó and Babuška in 1970, is similar to the Ritz method, but there is one important difference. In the p-version of the finite element method, the domain of interest is divided into convex subdomain and the polynomial approximates are piecewise smooth only over individual convex subdomains. In the Ritz method, on the other hand, the solution over the entire domain is approximated by smooth functions. This difference accounts for the greater versatility and higher rate of convergence of the p-version of the finite element method over both the Ritz method and the h-version of the finite element method. It has been implemented to study various aspects of stress analysis and has shown very good results, particularly in connection with problems which have singularities.

In addition to the transient response analysis of the dynamic systems, the study of the transient response sensitivity of the dynamic systems are important for the optimum design of structures. Both parametric identification and evaluation of the effect of design changes on the dynamic response of the system are based on the knowledge of sensitivity. Additionally it is easier to determine the character of the motion of the dynamic system from the critical stability viewpoint by observing the sensitivity coefficients. A number of approaches are currently being used for obtaining sensitivity of the transient response, namely: direct differentiation of the governing differential equation, adjoint variable method, finite difference method, Green's function method, and Fourier amplitude sensitivity method. A review of the existing methods to obtain sensitivity of the transient response is given by Adelman and Haftka [45] and Haftka and Adelman [46]. Tomović [47] mentioned the basic concept of sensitivity and the importance of the sensitivity analysis of modern automatic control system. Greene and Haftka [48] developed a semi-analytical method which is the combination of analytical differentiation of the transient equation with finite difference matirx derivatives. The method directly differentiated the equation of motion, but the derivatives of coefficients were calculated by finite difference approximations. Hsieh and Arora [49] proposed an expanded global matrix formulation for the design sensitivity analysis of dynamic response constraints. Recently Zhang *et al.* [50] proposed an alternative formulation which can eliminate the computation of eigenvector derivatives for the design sensitivity analysis based on the reduced system. Wang and Lu [51] have presented a procedure that uses discrete Fourier transform to obtain sensitivity of the transient response of the linear systems.

Among the current methods the direct differentiation method is straight forward and is quite efficient when the number of design variables is small. The proposed time finite element method based approach can be considered to be similar to the direct differentiation approach. The key difference being that in the proposed approach differentiation of the algebraic equations and not that of the original differential equations is performed. The present approach is thus simpler than the direct differentiation approach.

At present, there is a considerable interest in the control of transient response of structures under external disturbances. The design of the control system is made difficult by the fact that simple analytical models that are often used in the design phase are not adequate for the controller design. Identification techniques are used to determine the system parameters to accurately determine the system model that can be used for controller design. While a number of studies are available for the identification of linear systems, only a few such studies are available for nonlinear systems. These include methods based on the method of multiple scales, iterative and noniterative direct methods, and state space mappings. A review of the nonlinear system identification used in the structural, mechanical and control engineering is given by Natke *et al.* [52]. Their focus was on the detection of nonlinearities, the formation of mathematical models and techniques for parameter identification. Some of the identification techniques for nonlinear systems currently being used are linear system realization theory [53, 54, 55, 56], statistical linearization and the use of extended Kalman filter.

Lee and Chang [57] introduced an orthogonal-function approximation technique (defined over the complete domain), which was developed for solving nonlinear systems, to the system identification. They used Jacobi, Ultraspherical, Chebyshev and Legendre polynomials for the approximations but no results were presented for identification of nonlinear systems. Batill and Bacarro [58] identified the parameters in a highly nonlinear differential equation governing the motion of an aircraft landing gear. The identification process involved relating the variation of the equations in the state variables to the corresponding equations dealing with the initial conditions. The variation of the error function is then made to vanish via changes in the parameters, which are treated as state variables, along with the displacements and velocity of the system. Mook [59] developed a technique for processing noisy state-observable, discrete time domain measurements of a nonlinear dynamic system to estimate both the state trajectory and model error through satisfaction of a covariance constraint. Using a number of examples, he showed that the method is capable of identifying unknown model parameters based on a least-squares formulation. Normann and Kapania [60] presented a method that was based on single and multiple step methods of integrating nonlinear differential equations. The system parameters for a number of examples were determined using the iterative direct method. Most recently Hamel and Jategaonkar [61] presented a review of the the successful applications of various system identification techniques to identify parametric models for flight vehicles. The present aircraft parameter estimation is mainly categorized into three parts, namely: instrumentation and filters, flight test techniques and analysis of flight data. The methods of data analysis used for the aircraft parameter estimation, include: the equation error method, output error method, filter error method and neural network-based methods. They demonstrated the successful application of system identification methodology to a broad range of flight vehicle modeling problems using the selected examples.

Most of the system identification methods are based on minimization of the square of the error between the measured response and that of the identified model. This is the classical least squared approach in which the error is minimized by treating the problem as an unconstrained optimization problem. Solving for unconstrained optimization problems may require sensitivity of the response with respect to various system parameters. These sensitivities are often obtained using either finite difference or by solving a large set of differential equations.

1.3 Objective and Scope of the Dissertation

In search for an alternative formulation for finding the response of transient problem, Kapania and Park [62], in a recent study, extended the bilinear formulation suggested by Peters and Izadpanah for linear undamped systems to linear and nonlinear, damped and undamped systems. The bilinear formulation in the time domain benefits from the large foundation of mathematical theorems [63] and knowledge already developed for the p-version finite element [64] in structural problems. An advantage of the present method over the finite difference approach, the most common way to find the sensitivity, is that one does not need to perform a convergence study to select an appropriate step size for obtaining the sensitivity. Also, note that the method yields an excellent approximation by using a relatively large time step size. The method can be computationally excellent when accurate results are needed. Moreover, the same technique can be used for solving multi-degree-of-freedom problems, and mixed initial value and boundary value problems.

One of the objectives of this study is to evaluate the performance of the augmented bilinear formulation of elastodynamics in determining the transient response of damped (nonconservative) linear and nonlinear systems. Whenever possible, the numerical results are compared with existing results or those obtained using Runge-Kutta fourth-order method. For all the cases studied, the results are obtained using Legendre polynomials as basis functions [65, 66, 67]. The results were also obtained using other polynomials, namely: Chebyshev, Hermite and integrated Legendre polynomials as basis functions, but without any significant improvement in either accuracy or efficiency. The results obtained by using these polynomials are thus not being presented.

In addition to the transient response calculations, the TFM approach is natural for obtaining the sensitivity of the transient response of linear and nonlinear and damped and undamped systems, as the sensitivity can be easily obtained by performing direct differentiation of the nonlinear algebraic equations resulting from the application of the proposed finite element method.

The second objective of this study is to evaluate the performance of the TFM in obtaining the sensitivity of the transient response of various linear and nonlinear and damped and undamped systems. The results obtained from the present approach are compared with those obtained with the central finite difference approach, using step sizes obtained from a convergence study. No such convergence study is needed when the present approach, employing direct differentiation of the algebraic equations resulting from the TFM, is employed.

The third objective of this study is to exploit the TFM for performing the parameter identification. The TFM along with the iterative direct method [68] are applied to a number of nonlinear single and two-degree-of-freedom systems. At first, in parameter identification, an objective function is formulated as a quadratic functional between the measured response of the given system and the analytic response of the mathematical model. Then the minimization of the objective function is performed to determine the system parameters. Here we adopt the Levenberg [71] and Marquardt [72] method which involves only first partial derivatives of the response with respect to various system parameters for the minimization process. The noise is added to a simulated response for studying the effects of measurement errors on the identification procedures. The numerical results are compared with those available from a previous study by Normann and Kapania [60] and Kapania and Normann [69].

Finally, the TFM based on a mixed form of Hamilton's weak formulation, developed by Hodges and Bless [77, 78], is applied to a boundary value problem [80, 82, 83]. In the finite element formulation, unlike Hodges and Bless who used linear type shape functions, higher order shape functions in p-version are applied and tested for solving two simple optimal control problems.

In summary, the possible use of the TFM in many transient problems is tested. We found that there must be a high potential benefit when the method is used for obtaining transient response of linear and nonlinear systems.

2. TRANSIENT RESPONSE AND ITS SENSITIVITY USING FINITE ELEMENTS IN TIME

2.1 Overview

In its application to the solution of engineering problems, the finite element discretization has been implemented almost to the spatial problems. For dynamic or time dependent problems whose solutions as functions of time are of interest, a step by step procedure of finite difference is usually employed.

In recent years, the time finite element method, in which the time is discretized in a number of finite elements and the response history over each element is expressed in terms of basis functions in the time co-ordinate, has had great success and popularity, especially with researchers studying transient response and dynamic stability of periodic systems.

In seeking for an alternative variational formulation which treats initial value problems, the bilinear formulation by using the Lagrange multipliers as suggested by Peters and Izadpanah [10] is extended to obtain the transient response and the response sensitivity of non-conservative damped linear and nonlinear systems. The numerical results are compared with existing results or those obtained using Runge-Kutta and Newmark $\beta = 1/4$ methods. The results were obtained using one of Chebyshev, Hermite, integrated Legendre and Legendre polynomials as basis functions, but without any significant improvement in either accuracy or efficiency.

Also the time finite element formulations are given here for obtaining the sensitivity of the transient response of various linear, nonlinear and damped, undamped systems. The numerical results obtained from the proposed approach are compared with those obtained with the central finite difference approximation by using step sizes obtained from a convergence study. The technique which makes this extension possible is described below.

2.2 Mathematical Formulation for Single-Degree-of-Freedom Systems

2.2.1 Bilinear Form for Damped Linear Systems

Consider a simple one-degree-of-freedom spring-mass-damper system given as

$$\begin{aligned} M\ddot{u}(t) + C\dot{u}(t) + Ku(t) = F(t) \qquad T_0 < t \leq T_f \\ u(0) = u_0 \qquad\qquad \dot{u}(0) = u_1 \end{aligned} \tag{2.1}$$

Here M, C, and K are respectively the mass, damping and stiffness coefficients, and $F(t)$ is the externally applied dynamic load.

Multiplying Eq. (2.1) with a test function (or weight function) $v(t)$ and integrating with respect to time gives

$$\int_{T_0}^{T_f} v \cdot (M\ddot{u} + C\dot{u} + Ku - F)\, dt = 0 \tag{2.2}$$

Integrating the first term in Eq. (2.2) by parts, we get

$$\Big[Mv\dot{u}\Big]_{T_0}^{T_f} + \int_{T_0}^{T_f} (Kvu + Cv\dot{u} - M\dot{v}\dot{u} - vF)\, dt = 0 \tag{2.3}$$

For the correct formulation of Eq. (2.3), we need to investigate the possible constraints on the initial and end conditions of the system. Many previous works offered various ideas. Especially Baruch and Riff [37] suggested six different formulations for each degree of freedom. Peters and Izadpanah [10] suggested that $v(T_f)$ in Eq. (2.3) has to be zero to eliminate the unknown final momentum $M\dot{u}(T_f)$ but $v(t)$ must not be allowed to vanish at T_0 to insure a natural convergence of the initial momentum $M\dot{u}(T_0)$. The variational formulation based on Hamilton's law gives the following form

$$\delta \int_{T_0}^{T_f} (\mathbf{T} - \mathbf{V})\, dt + \int_{T_0}^{T_f} Q_i \delta s_i dt - \frac{\partial \mathbf{T}}{\partial \dot{s}_i} \delta s_i \Big|_{T_0}^{T_f} = 0 \tag{2.4}$$

and included the displacement variations at T_0 and T_f. Baruch and Riff [37] demonstrated that Hamilton's law with a constraint of $\delta s_i(T_f) = 0$ showed the best convergence. In this thesis, we follow the formulation suggested by Peters and Izadpanah [10]. The trial function $u(t)$ can be expressed as

$$u(t) = \sum_{j=1}^{N} q_j \phi_j(t) \tag{2.5}$$

where ϕ_j are the basis functions in the form of Legendre polynomials, though same results were also obtained using Hermite, Chebyshev, and integrated Legendre polynomials.

Substituting Eq. (2.5) into Eq. (2.3), we get a equation of the form (Zienkiewicz and wood [84]):

$$\sum_{j=1}^{N} q_j \Big\{ \int_{T_0}^{T_f} \Big(Kv\phi_j + Cv\dot{\phi}_j - M\dot{v}\dot{\phi}_j \Big)\, dt \Big\} = \int_{T_0}^{T_f} Fvdt - Mv(T_f)\dot{u}(T_f) + Mv(T_0)\dot{u}(T_0) \tag{2.6}$$

Let $v(t) = \delta q_i \psi_i(t)$, $1 \leq i \leq N$, where $\psi_i(t)$ are the admissible functions chosen from the same set of polynomials for trial functions. The test functions may or may not be same as the trial functions. Then for each $\psi_i(t)$, Eq. (2.6) becomes

$$\mathbf{Bq} = \mathbf{a} \tag{2.7}$$

where

$$\begin{aligned} B_{ij} &= \int_{T_0}^{T_f} \left(K\psi_i\phi_j + C\psi_i\dot{\phi}_j - M\dot{\psi}_i\dot{\phi}_j \right) dt \\ a_i &= M\psi_i(T_0)\dot{u}(T_0) - M\psi_i(T_f)\dot{u}(T_f) + \int_{T_0}^{T_f} F\psi_i dt \end{aligned}$$

We impose the initial condition, $u(T_0) = u_0$ in the form of a constraint. This is done by augmenting Eq. (2.7) with an additional Eq. (2.8).

$$u_0 = \sum_{j=1}^{N} q_j\phi_j(T_0) \tag{2.8}$$

$$v(T_f) = \delta q_i\psi_i(T_f) = 0 \tag{2.9}$$

The second constraint Eq. (2.9) was suggested by Peters and Izadpanah [10] and can be included by using the method of Lagrange multiplier, by multiplying an arbitrary Lagrange multiplier λ to Eq. (2.9) and adding the product to the left hand side of Eq. (2.7).

$$\sum_{j=1}^{N} B_{ij}q_j + \lambda M\psi_i(T_f) = {a_i}^* \tag{2.10}$$

where

$${a_i}^* = M\psi_i(T_0)\dot{u}(T_0) + \int_{T_0}^{T_f} F\psi_i dt$$

The $M\psi_i(T_f)\dot{u}(T_f)$ term in a_i has been eliminated due to the constraint Eq. (2.9). Equations (2.7) and (2.10) can be written in the matrix form:

$$\begin{bmatrix} \mathbf{B} & \{M\psi_i(T_f)\} \\ \langle\phi_j(T_0)\rangle & 0 \end{bmatrix} \begin{Bmatrix} q_j \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \mathbf{a}^* \\ [u(t)]_{t=T_0} \end{Bmatrix} \tag{2.11}$$

where

$$\begin{aligned} \mathbf{B} &= \int_{T_0}^{T_f} \left(K\psi_i\phi_j + C\psi_i\dot{\phi}_j - M\dot{\psi}_i\dot{\phi}_j \right) dt \\ \mathbf{a}^* &= M\psi_i(T_0)\dot{u}(T_0) + \int_{T_0}^{T_f} F\psi_i dt \\ \lambda &= [\dot{u}(t)]_{t=T_f} \end{aligned}$$

and i, j are row and column index, respectively. In the case of multiple elements, $u(t)$ and λ at end point for a particular element should be used as the initial conditions for the following element.

2.2.2 Transient Response Sensitivity of Linear Systems

The transient sensitivity calculation is equivalent to the mathematical problem of obtaining the derivatives of the solutions with respect to the independent variables. The straightforward differentiation of Eq. (2.11) with respect to the design parameter d_k, allows us to write the following sensitivity equation.

$$\begin{bmatrix} \mathbf{B} & \{M\psi_i(T_f)\} \\ \langle\phi_j(T_0)\rangle & 0 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial q_j}{\partial d_k} \\ \dfrac{\partial \lambda}{\partial d_k} \end{Bmatrix} + \begin{bmatrix} \dfrac{\partial \mathbf{B}}{\partial d_k} & \left\{\dfrac{\partial M}{\partial d_k}\psi_i(T_f)\right\} \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} q_j \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \mathbf{a}^*}{\partial d_k} \\ \dfrac{\partial u(T_0)}{\partial d_k} \end{Bmatrix} \tag{2.12}$$

where

$$\frac{\partial \mathbf{B}}{\partial d_k} = \int_{T_0}^{T_f} \left(\frac{\partial K}{\partial d_k}\psi_i\phi_j + \frac{\partial C}{\partial d_k}\psi_i\dot{\phi}_j - \frac{\partial M}{\partial d_k}\dot{\psi}_i\dot{\phi}_j \right) dt$$

$$\frac{\partial \mathbf{a}^*}{\partial d_k} = \frac{\partial M}{\partial d_k}\psi_i(T_0)\left[\dot{u}(t)\right]_{t=T_0} + M\psi_i(T_0)\left[\frac{\partial \dot{u}(t)}{\partial d_k}\right]_{t=T_0}$$

For the first element, we have

$$\frac{\partial \mathbf{a}^*}{\partial d_k} = \frac{\partial M}{\partial d_k}\psi_i(T_0)\left[\dot{u}(t)\right]_{t=T_0}$$

2.2.3 Transient Response of Nonlinear Systems

While the solution of a linear equation system can be accomplished without difficulty in a manner described above, this is not possible for nonlinear systems. Analytical procedures for the treatment of nonlinear differential equations are difficult and require extensive mathematical study. The differential equation describing a nonlinear oscillatory system over a given length of time, $T_0 < t \leq T_f$, may have a general form:

$$\mathbf{g}(u, \dot{u}, \ddot{u}, t, \mathbf{d}) = 0 \tag{2.13}$$

where $\mathbf{g}$ may be nonlinear functions of u and $\dot{u}$.
The bilinear formulation of Eq. (2.13) gives us a general form:

$$\tilde{\mathbf{g}}(\mathbf{q}, \mathbf{d}) = 0 \tag{2.14}$$

where d_k, $k = 1, 2, ..., K$ are the K design parameters and the vector $\mathbf{q}$ denotes the generalized coordinates.

Equation (2.14), at times, may also be written as

$$\tilde{\mathbf{g}} = \mathbf{a} - \mathbf{B}\mathbf{q} = 0 \tag{2.15}$$

where $\mathbf{a}$ is the load vector and $\mathbf{B}$ is the nonlinear "stiffness" matrix and a function of generalized coordinates $\mathbf{q}$. The most obvious and direct way to solve Eq. (2.14) is by an iterative method (Burden and Faires [85]). The iteration is terminated when an 'error', i.e.,

$$\mathbf{e} = \mathbf{q}^{(n)} - \mathbf{q}^{(n-1)} \tag{2.16}$$

becomes sufficiently small. Usually some norm of the error is determined and iteration continues until this norm is sufficiently small. For this research, the stopping criterion is to iterate until

$$\frac{\|\mathbf{e}^{(k)} - \mathbf{e}^{(k-1)}\|_\infty}{\|\mathbf{e}^{(k)}\|_\infty} \leq \varepsilon (= 5.0 \times 10^{-5}) \tag{2.17}$$

2.2.4 Transient Response Sensitivity of Nonlinear Systems

The sensitivity of the transient response of a nonlinear system can be obtained by taking the derivative of both sides of Eq. (2.14) with respect to d_k. This gives

$$\frac{\partial \tilde{g}_i}{\partial d_k} + \sum_{j=1}^{N} \frac{\partial \tilde{g}_i}{\partial q_j} \frac{\partial q_j}{\partial d_k} = 0 \tag{2.18}$$

Note that from Eq. (2.18) it is clear that the design sensitivity equation is linear even though the analysis problem is nonlinear. $\partial \tilde{g}_i / \partial q_j$ is called Jacobian or the "tangent" stiffness matrix. Since the vector of generalized coordinates q_j is already available from the transient response analysis, the first derivatives of the generalized coordinates $\partial q_j / \partial d_k$ can be easily calculated by solving Eq. (2.18).

$$\frac{\partial q_j}{\partial d_k} = - \left[\frac{\partial \tilde{g}_i}{\partial q_j} \right]^{-1} \frac{\partial \tilde{g}_i}{\partial d_k} \tag{2.19}$$

In matrix form, the sensitivity equation can be obtained by taking the derivatives of Eq. (2.15) with respect to d_k.

$$\frac{\partial a_i}{\partial d_k} - \sum_{j=1}^{N} \frac{\partial B_{ij}}{\partial d_k} q_j - \sum_{m=1}^{N} \left(\sum_{j=1}^{N} B_{ij} \frac{\partial q_j}{\partial q_m} + \sum_{j=1}^{N} \frac{\partial B_{ij}}{\partial q_m} q_j \right) \frac{\partial q_m}{\partial d_k} = 0 \tag{2.20}$$

This equation may be written symbolically as

$$\left[\; B_{ij} \; \right] \left\{ \; \frac{\partial q_j}{\partial d_k} \; \right\} + \left[\; \frac{\partial B_{ij}}{\partial d_k} \; \right] \left\{ \; q_j \; \right\} = \left\{ \; \frac{\partial a_i}{\partial d_k} \; \right\} - \left\{ \; \sum_{j=1}^{N} \left(\sum_{m=1}^{N} \frac{\partial B_{im}}{\partial q_j} q_m \right) \frac{\partial q_j}{\partial d_k} \; \right\} \tag{2.21}$$

This reduces to

$$\left[\; B^*_{ij} \; \right] \left\{ \; \frac{\partial q_j}{\partial d_k} \; \right\} + \left[\; \frac{\partial B_{ij}}{\partial d_k} \; \right] \left\{ \; q_j \; \right\} = \left\{ \; \frac{\partial a_i}{\partial d_k} \; \right\} \tag{2.22}$$

where

$$B_{ij}^{*} = B_{ij} + \sum_{m=1}^{N} \frac{\partial B_{im}}{\partial q_j} q_m \tag{2.23}$$

For illustration purpose, consider a damped oscillation of a mass and a nonlinear spring, over a given length of time $T_0 < t \leq T_f$. The governing equation is given by

$$\begin{gathered} M\ddot{u}(t) + C\dot{u}(t) + Ku(t) + \mu u(t)^3 = F(t) \\ u(0) = u_0 \qquad\qquad \dot{u}(0) = u_1 \end{gathered} \tag{2.24}$$

where $F(t)$ is an applied external load and M, C, K are mass, damping and stiffness coefficients, respectively. When $\mu = 0$, the frequency of oscillation is that of the linear system. So the frequency of the nonlinear oscillation will depend on the amplitude of oscillation as well as on μ. The bilinear formulation of Eq. (2.24) in a matrix form yields

$$\begin{bmatrix} \mathbf{B} & \{M\psi_i(T_f)\} \\ \langle \phi_j(T_0) \rangle & 0 \end{bmatrix} \begin{Bmatrix} q_j \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \mathbf{a}^{*} \\ [u(t)]_{t=T_0} \end{Bmatrix} \tag{2.25}$$

where

$$\begin{aligned} \mathbf{B} &= \int_{T_0}^{T_f} \left(K\psi_i\phi_j + C\psi_i\dot{\phi}_j - M\dot{\psi}_i\dot{\phi}_j \right) dt \\ \mathbf{a}^{*} &= M\psi_i(T_0)\dot{u}(T_0) + \int_{T_0}^{T_f} \left\{ F\psi_i - \mu\psi_i \left(\sum_{l=1}^{N} q_l\phi_l \right)^3 \right\} dt \end{aligned}$$

It is noted here that both sides include the unknown generalized coordinates q_l in the above formulation. By differentiating Eq. (2.25) with respect to the design parameter d_k, we obtain

$$\begin{bmatrix} \mathbf{B}^{*} & \{M\psi_i(T_f)\} \\ \langle \phi_j(T_0) \rangle & 0 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial q_j}{\partial d_k} \\ \dfrac{\partial \lambda}{\partial d_k} \end{Bmatrix} + \begin{bmatrix} \dfrac{\partial \mathbf{B}}{\partial d_k} & \left\{ \dfrac{\partial M}{\partial d_k}\psi_i(T_f) \right\} \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} q_j \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \mathbf{a}^{**}}{\partial d_k} \\ \dfrac{\partial u(T_0)}{\partial d_k} \end{Bmatrix} \tag{2.26}$$

where

$$\begin{aligned} \mathbf{B}^{*} &= \int_{T_0}^{T_f} \left\{ K\psi_i\phi_j + C\psi_i\dot{\phi}_j - M\dot{\psi}_i\dot{\phi}_j + 3\mu\psi_i\phi_j \left(\sum_{l=1}^{N} q_l\phi_l \right)^2 \right\} dt \\ \frac{\partial \mathbf{B}}{\partial d_k} &= \int_{T_0}^{T_f} \left(\frac{\partial K}{\partial d_k}\psi_i\phi_j + \frac{\partial C}{\partial d_k}\psi_i\dot{\phi}_j - \frac{\partial M}{\partial d_k}\dot{\psi}_i\dot{\phi}_j \right) dt \\ \frac{\partial \mathbf{a}^{**}}{\partial d_k} &= \frac{\partial M}{\partial d_k}\psi_i(T_0)\left[\dot{u}(t)\right]_{t=T_0} + M\psi_i(T_0)\left[\frac{\partial \dot{u}(t)}{\partial d_k}\right]_{t=T_0} - \int_{T_0}^{T_f} \frac{\partial \mu}{\partial d_k}\psi_i \left(\sum_{l=1}^{N} q_l\phi_l \right)^3 dt \end{aligned}$$

Here q_l are known values since they were calculated in the transient response analysis.

2.3 Mathematical Formulation for Multidegree-of-Freedom Systems

2.3.1 Transient Response of Linear Two-Degree-of-Freedom Systems

Consider a linear two-degree-of-freedom system given by

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{F}(t) \tag{2.27}$$

where

$$\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}, \mathbf{K} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

with initial conditions

$$u_1(0) = u_{10} \qquad u_2(0) = u_{20} \qquad \dot{u}_1(0) = v_{10} \qquad \dot{u}_2(0) = v_{20}$$

Here m_{ij}, c_{ij} and k_{ij} are respectively mass, damping and stiffness coefficients of the system.

Multiplying Eq. (2.27) with test functions (or weight functions) $\mathbf{v}(t)$ and integrating with respect to time for the bilinear formulation results in

$$\int_{T_0}^{T_f} v_1\Big(m_{11}\ddot{u}_1 + m_{12}\ddot{u}_2 + c_{11}\dot{u}_1 + c_{12}\dot{u}_2 + k_{11}u_1 + k_{12}u_2 - f_1\Big)dt = 0 \tag{2.28}$$

$$\int_{T_0}^{T_f} v_2\Big(m_{21}\ddot{u}_1 + m_{22}\ddot{u}_2 + c_{21}\dot{u}_1 + c_{22}\dot{u}_2 + k_{21}u_1 + k_{22}u_2 - f_2\Big)dt = 0 \tag{2.29}$$

Integrating Eqs. (2.28) and (2.29) by parts yields

$$\begin{aligned}&\Big[m_{11}v_1\dot{u}_1\Big]_{T_0}^{T_f} + \Big[m_{12}v_1\dot{u}_2\Big]_{T_0}^{T_f} + \\ &\int_{T_0}^{T_f}\Big(k_{11}v_1u_1 + k_{12}v_1u_2 + c_{11}v_1\dot{u}_1 + c_{12}v_1\dot{u}_2 - m_{11}\dot{v}_1\dot{u}_1 - m_{12}\dot{v}_1\dot{u}_2 - v_1f_1\Big)dt \\ &= 0\end{aligned} \tag{2.30}$$

$$\begin{aligned}&\Big[m_{21}v_2\dot{u}_1\Big]_{T_0}^{T_f} + \Big[m_{22}v_2\dot{u}_2\Big]_{T_0}^{T_f} + \\ &\int_{T_0}^{T_f}\Big(k_{21}v_2u_1 + k_{22}v_2u_2 + c_{21}v_2\dot{u}_1 + c_{22}v_2\dot{u}_2 - m_{21}\dot{v}_2\dot{u}_1 - m_{22}\dot{v}_2\dot{u}_2 - v_2f_2\Big)dt \\ &= 0\end{aligned} \tag{2.31}$$

Let trial functions be

$$u_1(t) = \sum_{j=1}^{N} p_j\phi_j(t) \tag{2.32}$$

$$u_2(t) = \sum_{j=1}^{N} q_j\phi_j(t) \tag{2.33}$$

where p_j and q_j are generalized coordinates.

By substituting Eqs. (2.32) and (2.33) into Eqs. (2.30) and (2.31) respectively, we obtain

$$\int_{T_0}^{T_f} \Bigg\{k_{11}v_1\Bigg(\sum_{j=1}^{N} p_j\phi_j\Bigg) + k_{12}v_1\Bigg(\sum_{j=1}^{N} q_j\phi_j\Bigg) + c_{11}v_1\Bigg(\sum_{j=1}^{N} p_j\dot{\phi}_j\Bigg) + c_{12}v_1\Bigg(\sum_{j=1}^{N} q_j\dot{\phi}_j\Bigg) - m_{11}\dot{v}_1\Bigg(\sum_{j=1}^{N} p_j\dot{\phi}_j\Bigg) - m_{12}\dot{v}_1\Bigg(\sum_{j=1}^{N} q_j\dot{\phi}_j\Bigg)\Bigg\}dt$$
$$= m_{11}v_1(T_0)\dot{u}_1(T_0) - m_{11}v_1(T_f)\dot{u}_1(T_f) + m_{12}v_1(T_0)\dot{u}_2(T_0) - m_{12}v_1(T_f)\dot{u}_2(T_f) + \int_{T_0}^{T_f} v_1 f_1 dt \tag{2.34}$$

$$\int_{T_0}^{T_f} \Bigg\{k_{21}v_2\Bigg(\sum_{j=1}^{N} p_j\phi_j\Bigg) + k_{22}v_2\Bigg(\sum_{j=1}^{N} q_j\phi_j\Bigg) + c_{21}v_2\Bigg(\sum_{j=1}^{N} p_j\dot{\phi}_j\Bigg) + c_{22}v_2\Bigg(\sum_{j=1}^{N} q_j\dot{\phi}_j\Bigg) - m_{21}\dot{v}_2\Bigg(\sum_{j=1}^{N} p_j\dot{\phi}_j\Bigg) - m_{22}\dot{v}_2\Bigg(\sum_{j=1}^{N} q_j\dot{\phi}_j\Bigg)\Bigg\}dt$$
$$= m_{21}v_2(T_0)\dot{u}_1(T_0) - m_{21}v_2(T_f)\dot{u}_1(T_f) + m_{22}v_2(T_0)\dot{u}_2(T_0) - m_{22}v_2(T_f)\dot{u}_2(T_f) + \int_{T_0}^{T_f} v_2 f_2 dt \tag{2.35}$$

Let test functions be

$$v_1(t) = \delta p_i\psi_i(t), \qquad 1 \le i \le M \tag{2.36}$$
$$v_2(t) = \delta q_i\psi_i(t), \qquad 1 \le i \le M \tag{2.37}$$

Then for each δp_j and δq_j, equations (2.34) and (2.35) become

$$\int_{T_0}^{T_f} \Bigg\{k_{11}\psi_i\Bigg(\sum_{j=1}^{N} p_j\phi_j\Bigg) + k_{12}\psi_i\Bigg(\sum_{j=1}^{N} q_j\phi_j\Bigg) + c_{11}\psi_i\Bigg(\sum_{j=1}^{N} p_j\dot{\phi}_j\Bigg) + c_{12}\psi_i\Bigg(\sum_{j=1}^{N} q_j\dot{\phi}_j\Bigg) - m_{11}\dot{\psi}_i\Bigg(\sum_{j=1}^{N} p_j\dot{\phi}_j\Bigg) - m_{12}\dot{\psi}_i\Bigg(\sum_{j=1}^{N} q_j\dot{\phi}_j\Bigg)\Bigg\}dt$$
$$= m_{11}\psi_i(T_0)\dot{u}_1(T_0) - m_{11}\psi_(T_f)\dot{u}_1(T_f) + m_{12}\psi_i(T_0)\dot{u}_2(T_0) - m_{12}\psi_i(T_f)\dot{u}_2(T_f) + \int_{T_0}^{T_f} \psi_i f_1 dt \tag{2.38}$$

$$\int_{T_0}^{T_f} \Bigg\{k_{21}\psi_i\Bigg(\sum_{j=1}^{N} p_j\phi_j\Bigg) + k_{22}\psi_i\Bigg(\sum_{j=1}^{N} q_j\phi_j\Bigg) + c_{21}\psi_i\Bigg(\sum_{j=1}^{N} p_j\dot{\phi}_j\Bigg) + c_{22}\psi_i\Bigg(\sum_{j=1}^{N} q_j\dot{\phi}_j\Bigg) - m_{21}\dot{\psi}_i\Bigg(\sum_{j=1}^{N} p_j\dot{\phi}_j\Bigg) - m_{22}\dot{\psi}_i\Bigg(\sum_{j=1}^{N} q_j\dot{\phi}_j\Bigg)\Bigg\}dt$$

$$= m_{21}\psi_i(T_0)\dot{u}_1(T_0) - m_{21}\psi_i(T_f)\dot{u}_1(T_f) + m_{22}\psi_i(T_0)\dot{u}_2(T_0) -$$
$$m_{22}\psi_i(T_f)\dot{u}_2(T_f) + \int_{T_0}^{T_f} \psi_i f_2 dt \tag{2.39}$$

Rearranging Eqs. (2.38) and (2.39) yields

$$\sum_{j=1}^{N} \left(B_{ij}{}^{(11)} p_j + B_{ij}{}^{(12)} q_j \right) = a_i{}^{(1)} \tag{2.40}$$
$$\sum_{j=1}^{N} \left(B_{ij}{}^{(21)} p_j + B_{ij}{}^{(22)} q_j \right) = a_i{}^{(2)} \tag{2.41}$$

where

$$\begin{aligned}
B_{ij}{}^{(11)} &= \int_{T_0}^{T_f} \left(k_{11}\psi_i\phi_j + c_{11}\psi_i\dot{\phi}_j - m_{11}\dot{\psi}_i\dot{\phi}_j \right) dt \\
B_{ij}{}^{(12)} &= \int_{T_0}^{T_f} \left(k_{12}\psi_i\phi_j + c_{12}\psi_i\dot{\phi}_j - m_{12}\dot{\psi}_i\dot{\phi}_j \right) dt \\
B_{ij}{}^{(21)} &= \int_{T_0}^{T_f} \left(k_{21}\psi_i\phi_j + c_{21}\psi_i\dot{\phi}_j - m_{21}\dot{\psi}_i\dot{\phi}_j \right) dt \\
B_{ij}{}^{(22)} &= \int_{T_0}^{T_f} \left(k_{22}\psi_i\phi_j + c_{22}\psi_i\dot{\phi}_j - m_{22}\dot{\psi}_i\dot{\phi}_j \right) dt
\end{aligned}$$

and

$$\begin{aligned}
a_i{}^{(1)} &= m_{11}\psi_i(T_0)\dot{u}_1(T_0) - m_{11}\psi_i(T_f)\dot{u}_1(T_f) + m_{12}\psi_i(T_0)\dot{u}_2(T_0) - \\
&\quad m_{12}\psi_i(T_f)\dot{u}_2(T_f) + \int_{T_0}^{T_f} \psi_i f_1 dt \\
a_i{}^{(2)} &= m_{21}\psi_i(T_0)\dot{u}_1(T_0) - m_{21}\psi_i(T_f)\dot{u}_1(T_f) + m_{22}\psi_i(T_0)\dot{u}_2(T_0) - \\
&\quad m_{22}\psi_i(T_f)\dot{u}_2(T_f) + \int_{T_0}^{T_f} \psi_i f_2 dt
\end{aligned}$$

Imposition of initial conditions, $u_i(T_0)$, $i = 1, 2$, can be done by augmenting Eqs. (2.40) and (2.41) with additional equations. Thus,

$$\sum_{j=1}^{N} \left\{ \left(B_{ij}{}^{(11)} p_j + B_{ij}{}^{(12)} q_j \right) + \phi_j(T_0) p_j \right\} = a_i{}^{(1)} \tag{2.42}$$
$$\sum_{j=1}^{N} \left\{ \left(B_{ij}{}^{(21)} p_j + B_{ij}{}^{(22)} q_j \right) + \phi_j(T_0) q_j \right\} = a_i{}^{(2)} \tag{2.43}$$

For the natural convergence of the end condition, a constraint $\psi_i(T_f) = 0$ can be included by using Lagrange multipliers. Now,

$$\sum_{j=1}^{N} \left\{ \left(B_{ij}{}^{(11)} p_j + B_{ij}{}^{(12)} q_j \right) + \phi_j(T_0) p_j \right\} +$$

$$\lambda_1 m_{11}\psi_i(T_f) + \lambda_2 m_{12}\psi_i(T_f) = a_i^{*(1)} \tag{2.44}$$

$$\sum_{j=1}^{N}\left\{\left(B_{ij}{}^{(21)}p_j + B_{ij}{}^{(22)}q_j\right) + \phi_j(T_0)q_j\right\} +$$
$$\lambda_1 m_{21}\psi_i(T_f) + \lambda_2 m_{22}\psi_i(T_f) = a_i^{*(2)} \tag{2.45}$$

where

$$\begin{aligned} a_i^{*(1)} &= m_{11}\psi_i(T_0)\dot{u}_1(T_0) + m_{12}\psi_i(T_0)\dot{u}_2(T_0) + \int_{T_0}^{T_f}\psi_i f_1 dt \\ a_i^{*(2)} &= m_{21}\psi_i(T_0)\dot{u}_1(T_0) + m_{22}\psi_i(T_0)\dot{u}_2(T_0) + \int_{T_0}^{T_f}\psi_i f_2 dt \end{aligned}$$

Combinig Eqs. (2.44) and (2.45) and putting into a matrix form, the transient response equation for a linear two-degree-of-freedom system becomes

$$\begin{bmatrix} \mathbf{B}^{(11)} & \mathbf{B}^{(12)} & \{m_{11}\psi_i(T_f)\} & \{m_{12}\psi_i(T_f)\} \\ \mathbf{B}^{(21)} & \mathbf{B}^{(22)} & \{m_{21}\psi_i(T_f)\} & \{m_{22}\psi_i(T_f)\} \\ <\phi_j(T_0)> & 0 & 0 & 0 \\ 0 & <\phi_j(T_0)> & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{p} \\ \mathbf{q} \\ \lambda_1 \\ \lambda_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{a}^{*(1)} \\ \mathbf{a}^{*(2)} \\ u_1(0) \\ u_2(0) \end{Bmatrix} \tag{2.46}$$

where i and j are row and column index, respectively. The final displacements, $u_k(T_f)$ and the final velocities, λ_k for one segment should be used as initial conditions for the following segment in multiple elements.

2.3.2 Transient response sensitivity with respect to design parameter d_k

The differentiation of Eq. (2.46) with respect to design parameter d_k gives the following sensitivity equation, similar to those of single-degree-of-freedom cases.

$$\begin{bmatrix} \mathbf{B}^{(11)} & \mathbf{B}^{(12)} & \{m_{11}\psi_i(T_f)\} & \{m_{12}\psi_i(T_f)\} \\ \mathbf{B}^{(21)} & \mathbf{B}^{(22)} & \{m_{21}\psi_i(T_f)\} & \{m_{22}\psi_i(T_f)\} \\ <\phi_j(T_0)> & 0 & 0 & 0 \\ 0 & <\phi_j(T_0)> & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial\mathbf{p}}{\partial d_k} \\ \dfrac{\partial\mathbf{q}}{\partial d_k} \\ \dfrac{\partial\lambda_1}{\partial d_k} \\ \dfrac{\partial\lambda_2}{\partial d_k} \end{Bmatrix} +$$

$$\begin{bmatrix} \dfrac{\partial\mathbf{B}^{(11)}}{\partial d_k} & \dfrac{\partial\mathbf{B}^{(12)}}{\partial d_k} & \left\{\dfrac{\partial m_{11}}{\partial d_k}\psi_i(T_f)\right\} & \left\{\dfrac{\partial m_{12}}{\partial d_k}\psi_i(T_f)\right\} \\ \dfrac{\partial\mathbf{B}^{(21)}}{\partial d_k} & \dfrac{\partial\mathbf{B}^{(22)}}{\partial d_k} & \left\{\dfrac{\partial m_{21}}{\partial d_k}\psi_i(T_f)\right\} & \left\{\dfrac{\partial m_{22}}{\partial d_k}\psi_i(T_f)\right\} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{p} \\ \mathbf{q} \\ \lambda_1 \\ \lambda_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial\mathbf{a}^{*(1)}}{\partial d_k} \\ \dfrac{\partial\mathbf{a}^{*(2)}}{\partial d_k} \\ \dfrac{\partial u_1(T_0)}{\partial d_k} \\ \dfrac{\partial u_2(T_0)}{\partial d_k} \end{Bmatrix} \tag{2.47}$$

where

$$\begin{aligned}
\frac{\partial B_{ij}{}^{(11)}}{\partial d_k} &= \int_{T_0}^{T_f} \left(\frac{\partial k_{11}}{\partial d_k} \psi_i \phi_j + \frac{\partial c_{11}}{\partial d_k} \psi_i \dot{\phi}_j - \frac{\partial m_{11}}{\partial d_k} \dot{\psi}_i \dot{\phi}_j \right) dt \\
\frac{\partial B_{ij}{}^{(12)}}{\partial d_k} &= \int_{T_0}^{T_f} \left(\frac{\partial k_{12}}{\partial d_k} \psi_i \phi_j + \frac{\partial c_{12}}{\partial d_k} \psi_i \dot{\phi}_j - \frac{\partial m_{12}}{\partial d_k} \dot{\psi}_i \dot{\phi}_j \right) dt \\
\frac{\partial B_{ij}{}^{(21)}}{\partial d_k} &= \int_{T_0}^{T_f} \left(\frac{\partial k_{21}}{\partial d_k} \psi_i \phi_j + \frac{\partial c_{21}}{\partial d_k} \psi_i \dot{\phi}_j - \frac{\partial m_{21}}{\partial d_k} \dot{\psi}_i \dot{\phi}_j \right) dt \\
\frac{\partial B_{ij}{}^{(22)}}{\partial d_k} &= \int_{T_0}^{T_f} \left(\frac{\partial k_{22}}{\partial d_k} \psi_i \phi_j + \frac{\partial c_{22}}{\partial d_k} \psi_i \dot{\phi}_j - \frac{\partial m_{22}}{\partial d_k} \dot{\psi}_i \dot{\phi}_j \right) dt
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial a_i^{*(1)}}{\partial d_k} &= \frac{\partial m_{11}}{\partial d_k} \psi_i(T_0) \dot{u}_1(T_0) + m_{11} \psi_i(T_0) \frac{\partial \dot{u}_1(T_0)}{\partial d_k} + \\
&\quad \frac{\partial m_{12}}{\partial d_k} \psi_i(T_0) \dot{u}_2(T_0) + m_{12} \psi_i(T_0) \frac{\partial \dot{u}_2(T_0)}{\partial d_k}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial a_i^{*(2)}}{\partial d_k} &= \frac{\partial m_{21}}{\partial d_k} \psi_i(T_0) \dot{u}_1(T_0) + m_{21} \psi_i(T_0) \frac{\partial \dot{u}_1(T_0)}{\partial d_k} + \\
&\quad \frac{\partial m_{22}}{\partial d_k} \psi_i(T_0) \dot{u}_2(T_0) + m_{22} \psi_i(T_0) \frac{\partial \dot{u}_2(T_0)}{\partial d_k}
\end{aligned}$$

The numerical examples in references (Wang and Lu [51]; Thomson [86]; Humar and Xia [87]) was chosen to show the exploitation of the present method. The results were compared with those obtained using the Newmark $\beta = 1/4$ method and the central difference approximation.

2.3.3 Newmark $\beta = \frac{1}{4}$ Method

The Newmark time stepping algorithm (Newmark [7, 88]; Craig [89]) was introduced in 1959, a step-by-step solution using time steps Δt_i, can be now considered for the two degrees-of-freedom system.

The equations of motion to be integrated are

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{Bmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix} \tag{2.48}$$

with initial conditions

$$u_1(0) = u_{10} \qquad u_2(0) = u_{20} \qquad \dot{u}_1(0) = v_{10} \qquad \dot{u}_2(0) = v_{20}$$

The initial accelerations, $\ddot{u}_1(0)$ and $\ddot{u}_2(0)$, can now be obtained from Eq. (2.48). Those are

$$\begin{aligned} \ddot{u}_1(0) &= \frac{1}{m_1}\Big(f_1(0) - c_{11}\dot{u}_1(0) - c_{12}\dot{u}_2(0) - k_{11}u_1(0) - k_{12}u_2(0)\Big) \\ \ddot{u}_2(0) &= \frac{1}{m_2}\Big(f_2(0) - c_{21}\dot{u}_1(0) - c_{22}\dot{u}_2(0) - k_{21}u_1(0) - k_{22}u_2(0)\Big) \end{aligned} \tag{2.49}$$

The acceleration in the time interval t_i to t_{i+1} is taken to be the average of the initial and final values of acceleration. This idea is embodied in the following equations.

$$\begin{aligned} \dot{u}_{i+1} &= \dot{u}_i + \frac{\Delta t_i}{2}\left(\ddot{u}_i + \ddot{u}_{i+1}\right) \\ u_{i+1} &= u_i + \dot{u}_i \Delta t_i + \left(\frac{1}{2} - \beta\right)\ddot{u}_i \left(\Delta t_i\right)^2 + \beta \ddot{u}_{i+1}\left(\Delta t_i\right)^2 \end{aligned} \tag{2.50}$$

Putting $\beta = \frac{1}{4}$ into Eq. (2.50), the incremental quantities, $\Delta\ddot{u}_i$ and $\Delta\dot{u}_i$, become

$$\begin{aligned} \Delta\ddot{u}_i &= \ddot{u}_{i+1} - \ddot{u}_i = \frac{4}{\Delta {t_i}^2}\left(\Delta u_i - \dot{u}_i \Delta t_i\right) - 2\ddot{u}_i \\ \Delta\dot{u}_i &= \dot{u}_{i+1} - \dot{u}_i = \frac{2}{\Delta t_i}\Delta u_i - 2\dot{u}_i \end{aligned} \tag{2.51}$$

Since Eq. (2.48) is satisfied at both t_i and t_{i+1}, we can rewrite Eq. (2.48) as

$$\begin{aligned} &\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \Delta\ddot{u}_{1i}(t) \\ \Delta\ddot{u}_{2i}(t) \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{Bmatrix} \Delta\dot{u}_{1i}(t) \\ \Delta\dot{u}_{2i}(t) \end{Bmatrix} + \\ &\qquad \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} \Delta u_{1i}(t) \\ \Delta u_{2i}(t) \end{Bmatrix} = \begin{Bmatrix} \Delta f_{1i}(t) \\ \Delta f_{2i}(t) \end{Bmatrix} \end{aligned} \tag{2.52}$$

Substituting Eq. (2.51) into Eq. (2.52) results in

$$\begin{aligned} &m_1\left\{\frac{4}{\Delta {t_i}^2}\left(\Delta u_{1i} - \dot{u}_{1i}\Delta t_i\right) - 2\ddot{u}_{1i}\right\} + c_{11}\left\{\frac{2}{\Delta t_i}\Delta u_{1i} - 2\dot{u}_{1i}\right\} + \\ &\qquad c_{12}\left\{\frac{2}{\Delta t_i}\Delta u_{2i} - 2\dot{u}_{2i}\right\} + k_{11}\Delta u_{1i} + k_{12}\Delta u_{2i} = \Delta f_{1i} \end{aligned} \tag{2.53}$$

$$\begin{aligned} &m_2\left\{\frac{4}{\Delta {t_i}^2}\left(\Delta u_{2i} - \dot{u}_{2i}\Delta t_i\right) - 2\ddot{u}_{2i}\right\} + c_{21}\left\{\frac{2}{\Delta t_i}\Delta u_{1i} - 2\dot{u}_{1i}\right\} + \\ &\qquad c_{22}\left\{\frac{2}{\Delta t_i}\Delta u_{2i} - 2\dot{u}_{2i}\right\} + k_{21}\Delta u_{1i} + k_{22}\Delta u_{2i} = \Delta f_{2i} \end{aligned} \tag{2.54}$$

Rearranging Eqs. (2.53) and (2.54) and putting those into a matrix form yields

$$
\begin{bmatrix} \dfrac{4m_1}{\Delta t_i{}^2} + \dfrac{2c_{11}}{\Delta t_i} + k_{11} & \dfrac{2c_{12}}{\Delta t_i} + k_{12} \\ \dfrac{2c_{21}}{\Delta t_i} + k_{21} & \dfrac{4m_2}{\Delta t_i{}^2} + \dfrac{2c_{22}}{\Delta t_i} + k_{22} \end{bmatrix} \begin{Bmatrix} \Delta u_{1i} \\ \Delta u_{2i} \end{Bmatrix}
= \begin{Bmatrix} \Delta f_{1i} + \dfrac{4m_1}{\Delta t_i} + 2c_{11}\dot{u}_{1i} + 2c_{12}\dot{u}_{2i} + 2m_1\ddot{u}_{1i} \\ \Delta f_{2i} + \dfrac{4m_2}{\Delta t_i} + 2c_{21}\dot{u}_{1i} + 2c_{22}\dot{u}_{2i} + 2m_2\ddot{u}_{2i} \end{Bmatrix} \tag{2.55}
$$

Solving for Δu_{1i}, Δu_{2i} and the updated values of u_{1i} and u_{2i} determined from

$$
\begin{aligned} u_{1(i+1)} &= u_{1i} + \Delta u_{1i} \\ u_{2(i+1)} &= u_{2i} + \Delta u_{2i} \end{aligned} \tag{2.56}
$$

Equation (2.55) can be solved by the Newton-Raphson method (Tillerson *et al.* [91]) which is one of the most popular methods for the solution of systems of nonlinear algebraic equations.

2.3.4 Transient Response and Sensitivity of Nonlinear Two-Degree-of-Freedom Systems

As we have seen in the case of single-degree-of-system, obtaining the response and sensitivity with respect to design parameter of the nonlinear system depends strongly on the problem under consideration. The validity and accuracy of the proposed method is demonstrated with a nonlinear two-degree-of-freedom system having cubic nonlinearities problem.

$$
\begin{aligned} &\ddot{u}_1 + \omega_1{}^2 u_1 + 2\mu_1\dot{u}_1 + \alpha_1 u_1{}^3 + \alpha_2 u_1{}^2 u_2 + \alpha_3 u_1 u_2{}^2 + \alpha_4 u_2{}^3 = 0 \\ &\ddot{u}_2 + \omega_2{}^2 u_2 + 2\mu_2\dot{u}_2 + \alpha_5 u_1{}^3 + \alpha_6 u_1{}^2 u_2 + \alpha_7 u_1 u_2{}^2 + \alpha_8 u_2{}^3 = 0 \end{aligned} \tag{2.57}
$$

with initial conditions

$$
u_1(0) = 1.5, \qquad \dot{u}_1(0) = 0.0, \qquad u_2(0) = -1.0, \qquad \dot{u}_2(0) = 0.0
$$

Based on the formulation of the linear two-degree-of-freedom system, Appendix A.1 and A.2 describes the detailed derivation for the transient response and the response sensitivity with respect to design parameters.

2.4 Numerical Examples

2.4.1 Linear System

First consider a single degree-of-freedom spring-mass system (Wang and Lu [51]) with viscous damping:

$$\ddot{u}(t) + 4\dot{u}(t) + 400u(t) = \delta(t), \qquad 0 < t \leq 5 \tag{2.58}$$

where δ is Dirac's delta function.

The response for the undamped case shown in Fig. 2.1 is obtained by the proposed method using Legendre polynomials of the third degree and fifty line elements of equal time step of $\Delta t = 0.1$. The response obtained using the present approach along with the exact response is presented in Fig. 2.1. Figure 2.2 represents the sensitivity with respect to the mass and the results are compared with the exact solution. Figure 2.3 shows the sensitivity of the response with respect to the stiffness parameter as obtained using the proposed method and the results are compared with the exact solution. Since damping is not considered, it is seen in Figs. 2.2 and 2.3 that the sensitivities are diverging with time. On the basis of observing the two sensitivity curves, we can conclude that "the original system will be unstable, or at best will have oscillations of steady amplitude" (Tomović [47]). The time history of response obtained using the proposed method and that from the exact solution are shown in Fig. 2.4 for the damped system. The sensitivities with respect to mass, damping, and stiffness are presented in Figs. 2.5-2.7, respectively. The exact results are also plotted in these figures and it can be seen that, for this linear damped system, the sensitivity results obtained from the proposed method are in good agreement with those obtained from the exact solutions.

2.4.2 Nonlinear Softening System

Consider a softening spring-mass system (Chen *et al.* [92]) without viscous damping:

$$\begin{gathered} \ddot{u}(t) + 100\tanh(u(t)) = 0 \qquad 0 < t \leq 5 \\ u(0) = 0.0 \qquad \dot{u}(0) = 25.0 \end{gathered} \tag{2.59}$$

The domain with a range of time $0 < t \leq 5$ is divided into twenty five elements of equal time steps of $\Delta t = 0.2$ and, unless mentioned otherwise, Legendre polynomials of the sixth degree are used as basis functions in the calculation. Figure 2.8 represents the responses of the undamped case. For comparison, the Runge-Kutta fourth-order method (Burden and Faires [85]) for the second-order system is used to approximate the solutions using $\Delta t =$

0.1 and 0.01. To verify the results for the response sensitivities, the second-order central difference approximation is used to calculate the response sensitivities with respect to the design parameters.

Given a function $u(d_k)$, the central difference approximations $\frac{\Delta u}{\Delta d_k}$ to the sensitivity $\frac{du}{d_k}$ of u with respect to a design parameter d_k is given as

$$\frac{\Delta u}{\Delta d_k} \cong \frac{u\left(d_k + \Delta d_k\right) - u\left(d_k - \Delta d_k\right)}{2\Delta d_k} \tag{2.60}$$

It is possible to employ higher-order finite difference approximations, but they are rarely used because of the high computational cost. Figure 2.9a shows the sensitivities with respect to mass as obtained using the present approach and the central difference approach. The values of the central difference approximation with $\Delta\mathrm{M} = 0.01$ and 0.005 are presented along with the result of the proposed method. Figure 2.9b presents the effect of the step size, Δd_k, on the response sensitivity. As can be seen from this figure, as expected (Haftka and Gürdal [93]), step size plays an important role in the calculations of the response sensitivity. A step size of 0.01 appears to yield good results. The advantage of the proposed method is that there is no need to perform a convergence study as is the case in the finite difference method. Figure 2.10 shows the sensitivity of the transient response with respect to the stiffness parameter for the undamped softening system. The approximate sensitivities as obtained from the central difference method using, ΔK=1.0 and 0.1, are also presented in that figure. The step size appears to have limited effect on the sensitivity of the response with respect to the stiffness parameter. Figure 2.11 shows the comparison of the transient responses for the damped nonlinear system as obtained using the Runge-Kutta fourth-order method and the proposed method. Twenty five time finite elements of equal length and Legendre polynomials of the third degree are used in the present calculation. Figures 2.12-2.14 present the response sensitivities with respect to mass, damping and stiffness, respectively.

2.4.3 Nonlinear Van der Pol Equation

As a second example, consider a nonlinear Van der Pol equation (Shampine [94]; Jordan and Smith [95]) of the form:

$$\begin{gathered} \ddot{u}(t) + \epsilon\left(u(t)^2 - 1\right)\dot{u}(t) + u(t) = 0 \qquad 0 < t \leq 20 \\ u(0) = 2.0 \qquad \dot{u}(0) = 0.0 \end{gathered} \tag{2.61}$$

For $\epsilon > 0$, all non-trivial solutions converge to a limit cycle, a periodic solution. So the exact solution must oscillate between -2.0 and 2.0 in this example. Figure 2.15 shows the approximate solution in the case of ϵ=5.0 with time steps of 0.025, 0.0167 and 0.005 in the interval (0,20). The Legendre polynomials of the fourth degree are used as basis functions. The accuracy of the solutions can be improved significantly by reducing the step size from Δt=0.025 to Δt=0.005. Results from the Runge-Kutta fourth-order method with a step size of 0.005 is also presented for comparison purpose. Figure 2.16a shows the response sensitivity with respect to ϵ along with the system response. In Fig. 2.16b, the sensitivity results obtained from the present analytic approach are compared with those obtained using the central difference scheme. The result indicates that the peak values of the response sensitivity increases with time.

2.4.4 Nonlinear Oscillator

As a third example, we studied nonlinear oscillators of the form:

$$\ddot{u}(t) + \alpha u(t) + \mu u(t)^3 = 0 \tag{2.62}$$

where α and μ are given constants. Here we consider a special case by setting the values of α and μ as 0 and 1, respectively. The given initial conditions are

$$u(0) = \beta \qquad\qquad \dot{u}(0) = 0.0 \tag{2.63}$$

where β is a constant. The transient responses are found for the system with β=1.0, 2.0 and 3.0. They are shown and compared in Fig. 2.17. The domain with a range of time $0 < t \leq 15$ is divided equally with a time step of $\Delta t = 0.2$. The Legendre polynomials of the third degree are used as basis functions for the calculation of the transient response. For comparison, corresponding results are also obtained by using the Runge-Kutta fourth-order method with $\Delta t = 0.05$ and an excellent agreement is seen. Figure 2.18 shows the response sensitivities with respect to parameter μ. An initial displacement β=2.0 is used in the sensitivity calculations. The results obtained from the proposed method are in a good agreement with those obtained from using a central difference approximation with $\Delta\mu$=0.01.

2.4.5 Nonlinear Hardening System

A cubic spring-mass model with viscous damping is represented by the following nonlinear equation:

$$\begin{gathered} \ddot{u}(t) + 4\dot{u}(t) + 400u(t) + u(t)^3 = 0 \qquad 0 < t \leq 5 \\ u(0) = 1.0, \qquad \dot{u}(0) = 0.0 \end{gathered} \tag{2.64}$$

The domain with a range of time $0 < t \leq 5$ is divided into twenty five elements of equal time steps of $\Delta t = 0.2$ and Legendre polynomials of the fifth degree are used as basis functions in the calculation. Figure 2.19 represents the transient response. The sensitivities with respect to mass, damping and stiffness are calculated and shown in Figs. 2.20-2.22, respectively.

2.4.6 Linear Two-Degree-of-Freedom Systems having simple forces

Consider the solution of the equation of motion of a two-degree-of-freedom system (Thomson [86]). Let $m_1 = 100Kg$, $m_2 = 25Kg$, $k_1 = 36KN/m$, $k_2 = 36KN/m$ and the system is subjected to a single force, $f_2 = 4000N$.

The equations of motion of the system become

$$\begin{bmatrix} 100 & 0 \\ 0 & 25 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \end{Bmatrix} + \begin{bmatrix} 72000 & -36000 \\ -36000 & 36000 \end{bmatrix} \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 4000 \end{Bmatrix} \tag{2.65}$$

with initial conditions

$$u_1(0) = 0 \qquad u_2(0) = 0 \qquad \dot{u}_1(0) = 0 \qquad \dot{u}_2(0) = 0$$

The domain with a range of time $0 < t \leq 1$ is divided into twenty elements of equal time steps of $\Delta t = 0.05$ and Legendre polynomials of the second degree are used as basis functions in the calculation. Figure 2.23 shows the transient responses, u_1 and u_2, and compared with those obtained by using the Newmark $\beta = 1/4$ method with $\Delta t = 0.01$.

2.4.7 Linear Two-Degree-of-Freedom System with the Ground Acceleration

Consider a linear damped two-degree-of-freedom system subjected to a base acceleration input (Humar and Xia [87]). Let $m_1 = 2$, $m_2 = 1$, $k_1 = 128$, $k_2 = 64$, $c_1 = 3.3941$, $c_2 = 1.1314$ and the ground acceleration is a rectangular pulse of magnitude -10 and duration 2 seconds.

The equations of motion of the system can be expressed as

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1(t) \\ \ddot{u}_2(t) \end{Bmatrix} + \begin{bmatrix} 4.5255 & -1.1314 \\ -1.1314 & 1.1314 \end{bmatrix} \begin{Bmatrix} \dot{u}_1(t) \\ \dot{u}_2(t) \end{Bmatrix} + \begin{bmatrix} 192 & -64 \\ -64 & 64 \end{bmatrix} \begin{Bmatrix} u_1(t) \\ u_2(t) \end{Bmatrix} = \begin{Bmatrix} -m_1\ddot{u}_g \\ -m_2\ddot{u}_g \end{Bmatrix} \tag{2.66}$$

with initial conditions

$$u_1(0) = 0 \qquad u_2(0) = 0 \qquad \dot{u}_1(0) = 0 \qquad \dot{u}_2(0) = 0$$

Here $\ddot{u}_g$ denotes the ground acceleration expressed as the rectangular pulse. The transient responses, u_1 and u_2, and the response sensitivities with respect to k_1 are presented respectively in Figs. 2.24 and 2.25. The results are compared with those of the Newmark $\beta = 1/4$ method with a time step of $\Delta t = 0.05$ and the central difference approximations. The domain with a range of time $0 < t \leq 3$ is divided into sixty elements of equal time steps of $\Delta t = 0.05$ and Legendre polynomials of the second degree are used as basis functions in the calculation.

2.4.8 Nonlinear Two-Degree-of-Freedom System having cubic Nonlinearities

Governing equations with cubic nonlinearities (Nayfeh and Mook [76]), associated with the vibration of strings, beams and plates, are considered

$$\begin{aligned} \ddot{u}_1 + {\omega_1}^2 u_1 + 2\mu_1 \dot{u}_1 + \alpha_1 {u_1}^3 + \alpha_2 {u_1}^2 u_2 + \alpha_3 u_1 {u_2}^2 + \alpha_4 {u_2}^3 = 0 \\ \ddot{u}_2 + {\omega_2}^2 u_2 + 2\mu_2 \dot{u}_2 + \alpha_5 {u_1}^3 + \alpha_6 {u_1}^2 u_2 + \alpha_7 u_1 {u_2}^2 + \alpha_8 {u_2}^3 = 0 \end{aligned} \tag{2.67}$$

with initial conditions

$$u_1(0) = 1.5, \qquad \dot{u}_1(0) = 0.0, \qquad u_2(0) = -1.0, \qquad \dot{u}_2(0) = 0.0$$

for the following values of the system parameters

$$\begin{aligned} {\omega_1}^2 &= 25.0, \quad \mu_1 = 0.35, \quad \alpha_1 = 5.0, \quad \alpha_2 = 0.5, \quad \alpha_3 = 0.25, \quad \alpha_4 = 3.0 \\ {\omega_2}^2 &= 17.0, \quad \mu_2 = 0.25, \quad \alpha_5 = 2.5, \quad \alpha_6 = 0.75, \quad \alpha_7 = 0.2, \quad \alpha_8 = 5.0 \end{aligned}$$

The domain with a range of time, $0 < t \leq 10$, is divided into one hundred elements of equal time steps. Figures 2.26, 2.27 and 2.28 show respectively the transient response of the system and the response sensitivity with respect to design parameter μ_1 and μ_2. The results obtained using $\Delta t = 0.1$ are compared with those of Newmark $\beta = 1/4$ method with $\Delta t = 0.01$ for the case of transient response. The response sensitivities are compared with those from central difference approximations.

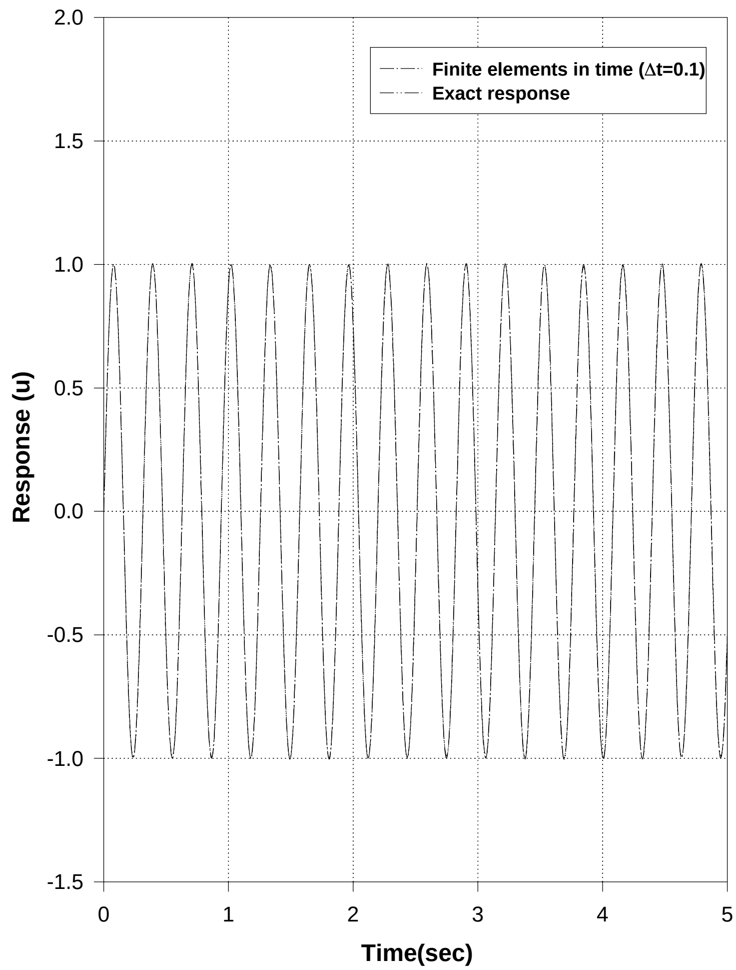

Fig. 2.1 Transient response of a linear undamped system ($\ddot{u} + 400u = \delta(t)$)

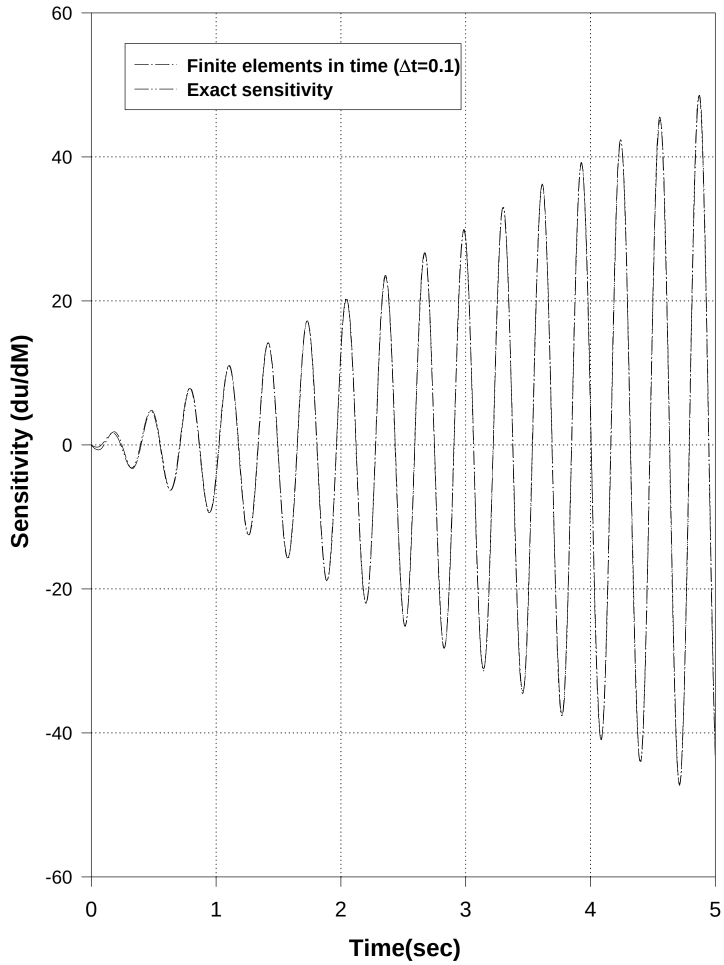

Fig. 2.2 Sensitivity of the response with respect to mass ($\ddot{u} + 400u = \delta(t)$)

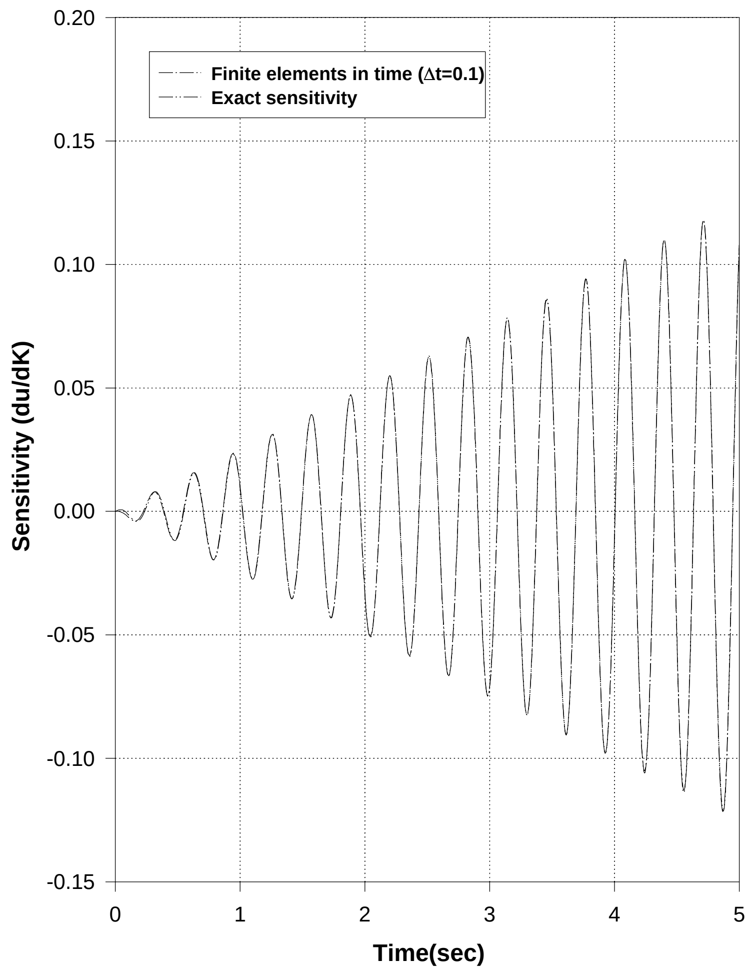

Fig. 2.3 Sensitivity of response with respect to stiffness ($\ddot{u} + 400u = \delta(t)$)

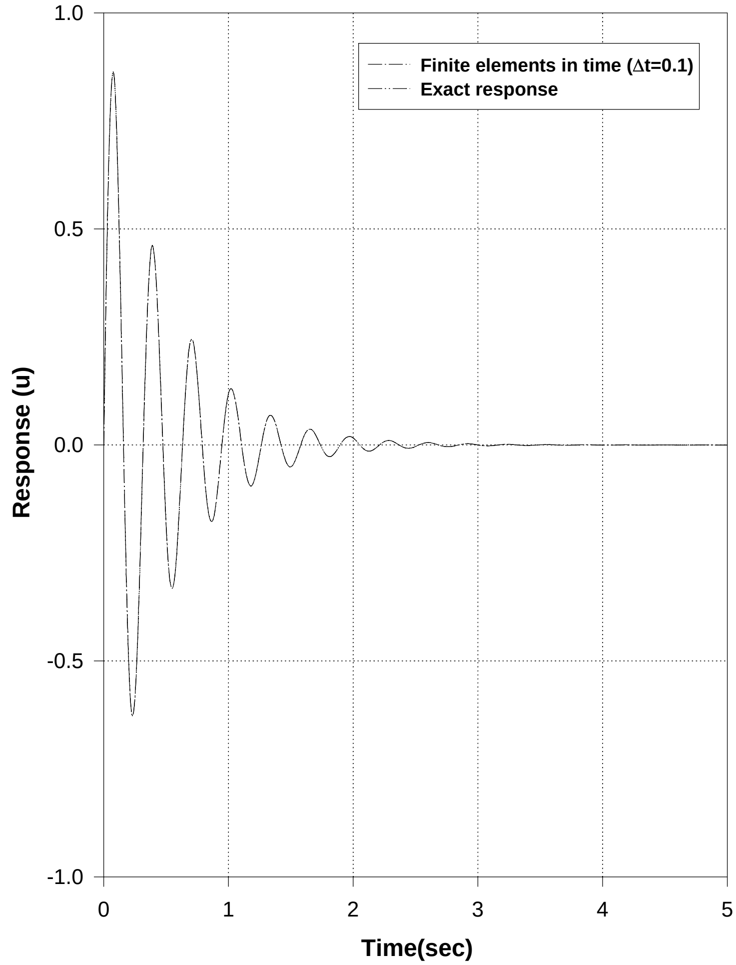

Fig. 2.4 Transient response of a linear damped system ($\ddot{u} + 4\dot{u} + 400u = \delta(t)$)

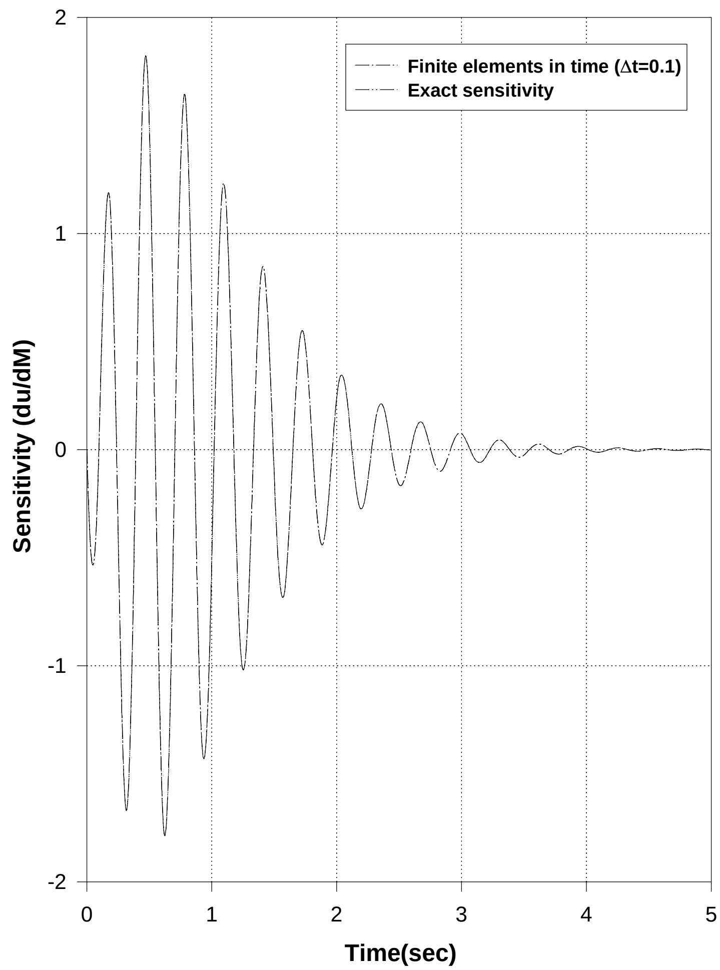

Fig. 2.5 Sensitivity of the response with respect to mass ($\ddot{u} + 4\dot{u} + 400u = \delta(t)$)

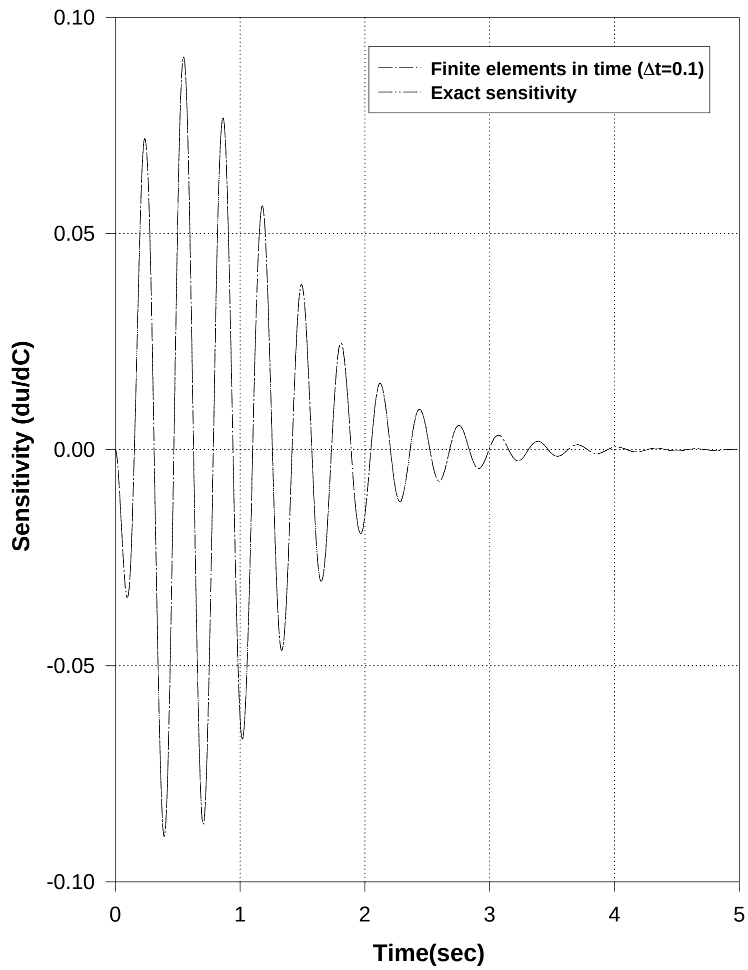

Fig. 2.6 Sensitivity of the response with respect to damping ($\ddot{u} + 4\dot{u} + 400u = \delta(t)$)

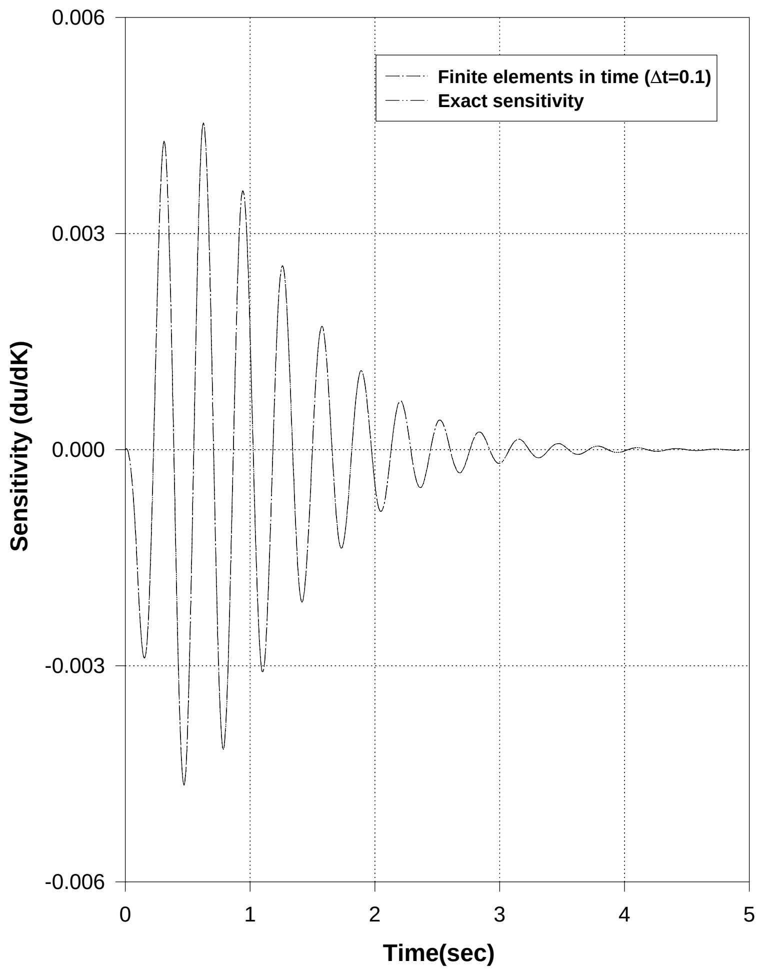

Fig. 2.7 Sensitivity of the response with respect to stiffness ($\ddot{u} + 4\dot{u} + 400u = \delta(t)$)

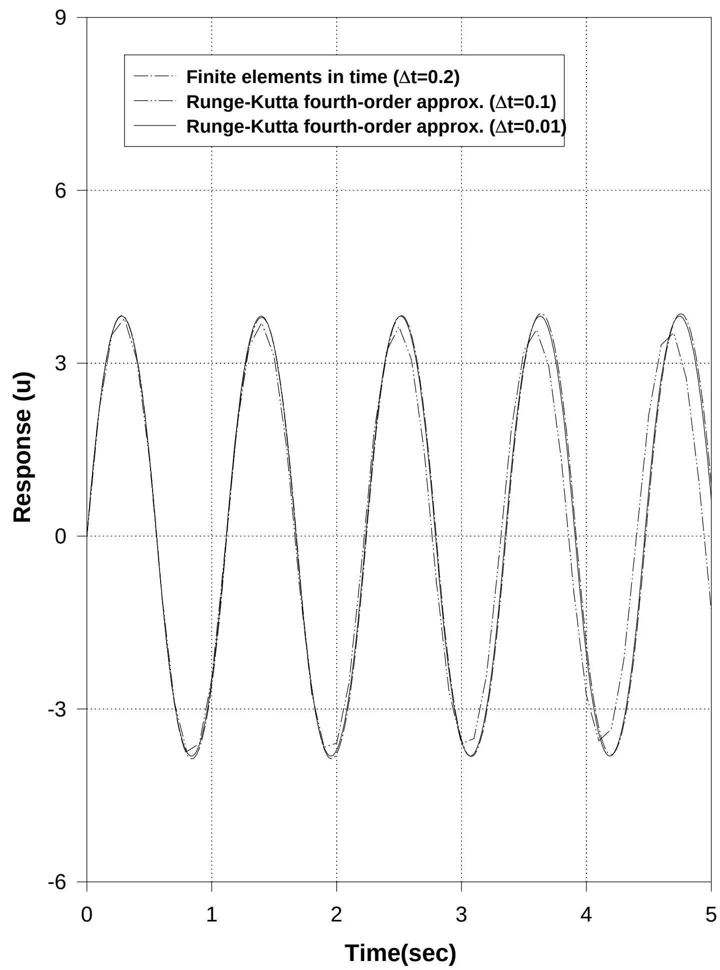

Fig. 2.8 Transient response of a nonlinear system without damping ($\ddot{u} + 100 tanh(u) = 0$)

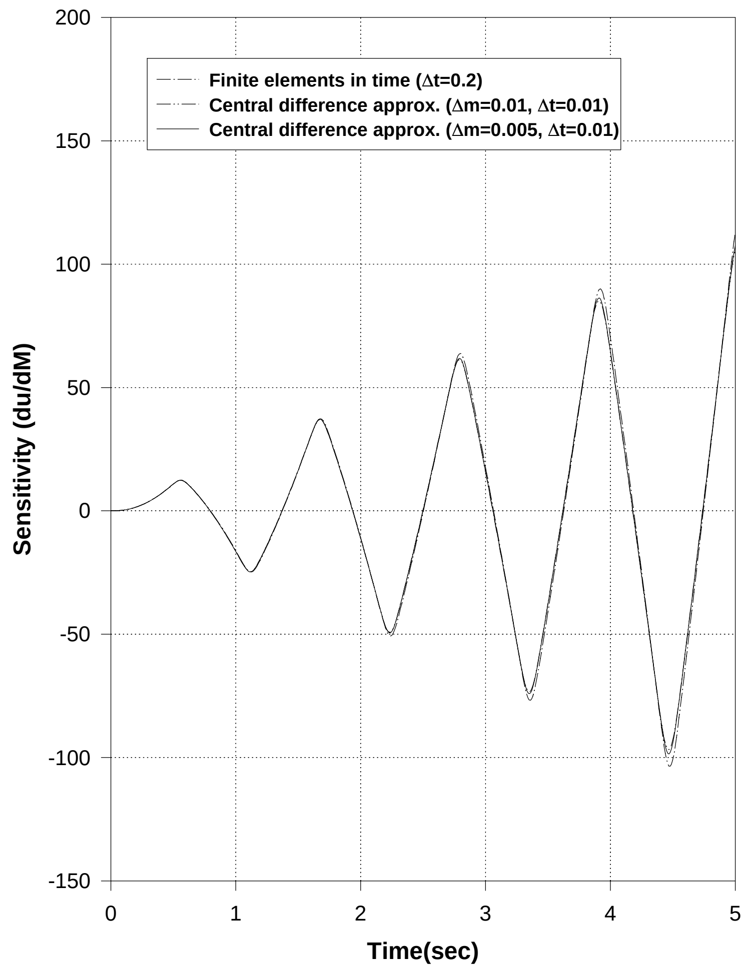

Fig. 2.9a Sensitivity of the response with respect to mass
$(\ddot{u} + 100tanh(u) = 0)$

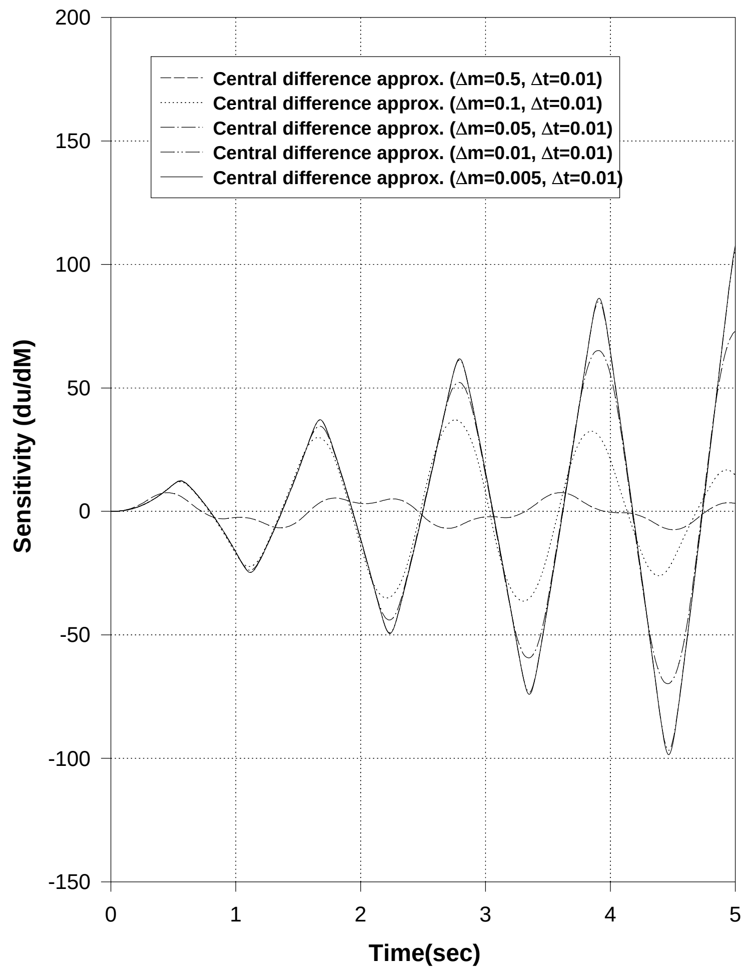

Fig. 2.9b Effect of step size on the response sensitivity with respect to mass ($\ddot{u} + 100tanh(u) = 0$)

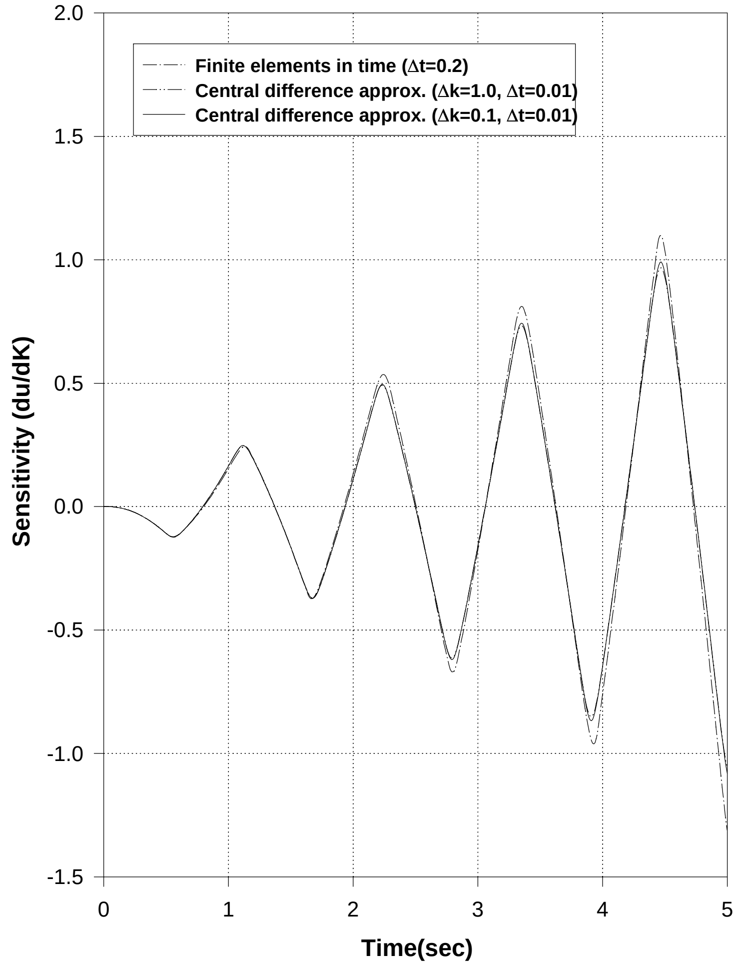

Fig. 2.10 Sensitivity of the response with respect to stiffness ($\ddot{u} + 100tanh(u) = 0$)

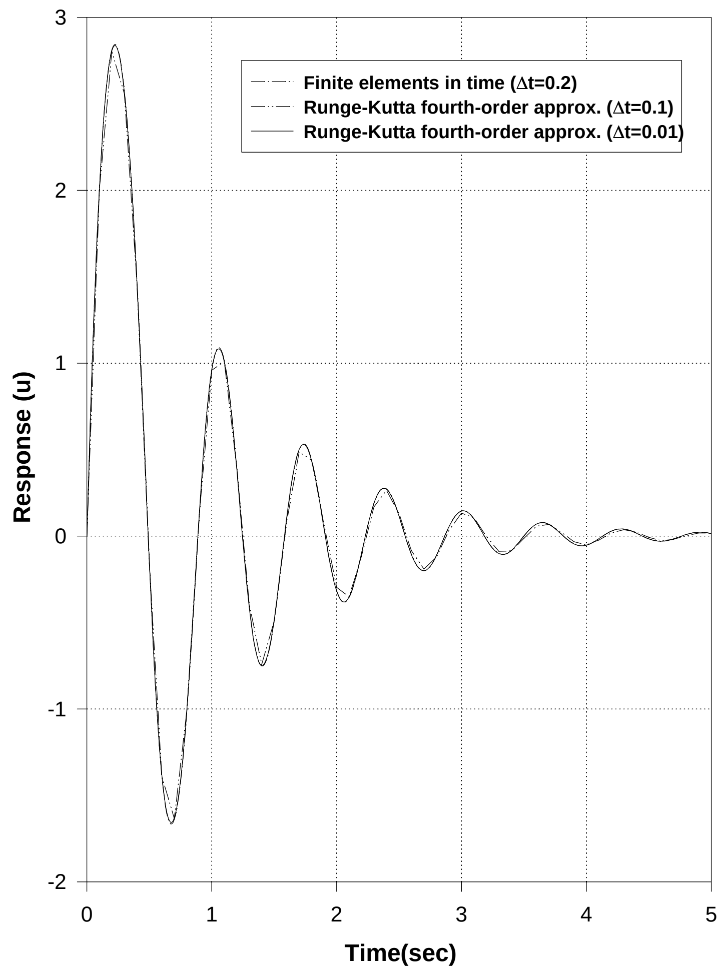

Fig. 2.11 Transient response of a nonlinear system with damping
$(\ddot{u} + 2\dot{u} + 100tanh(u) = 0)$

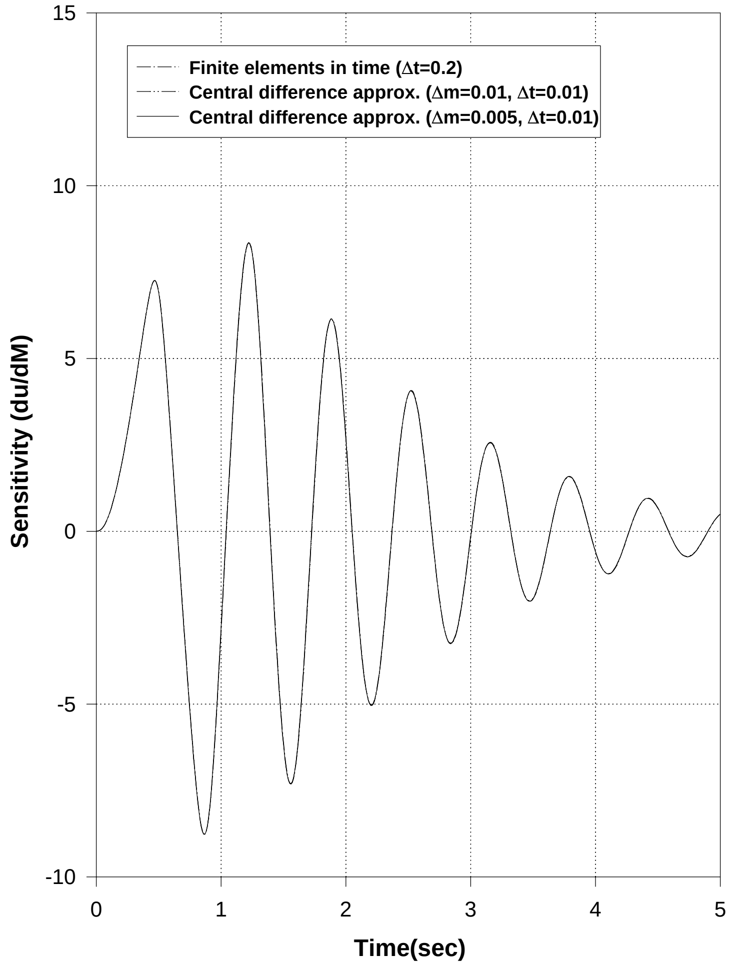

Fig. 2.12 Sensitivity of the response with respect to mass ($\ddot{u} + 2\dot{u} + 100tanh(u) = 0$)

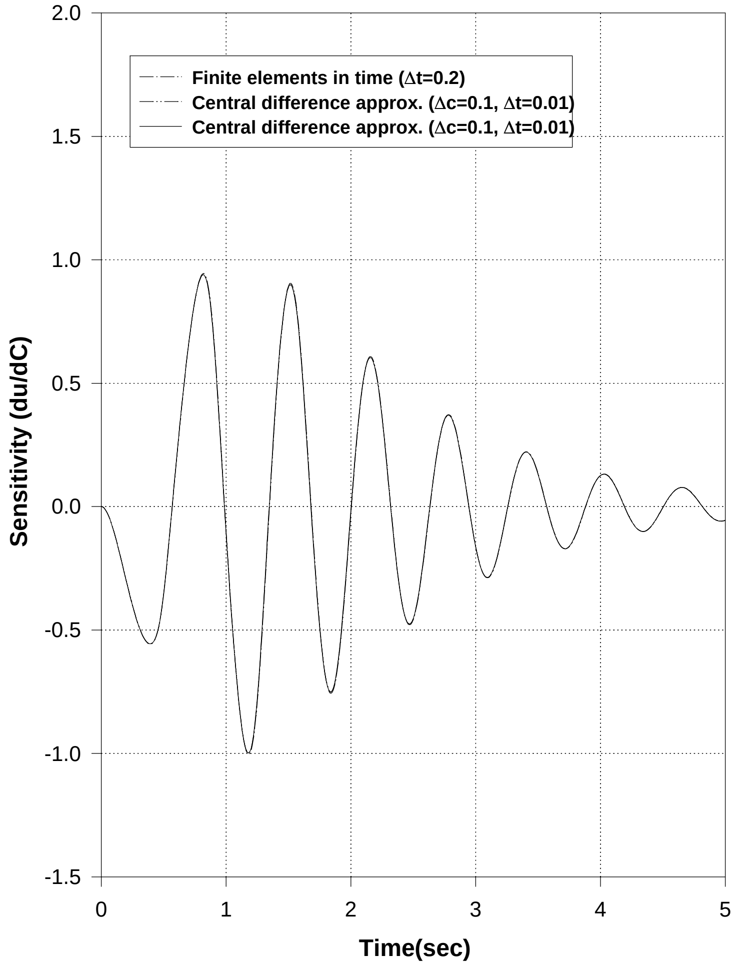

Fig. 2.13 Sensitivity of the response with respect to damping ($\ddot{u} + 2\dot{u} + 100tanh(u) = 0$)

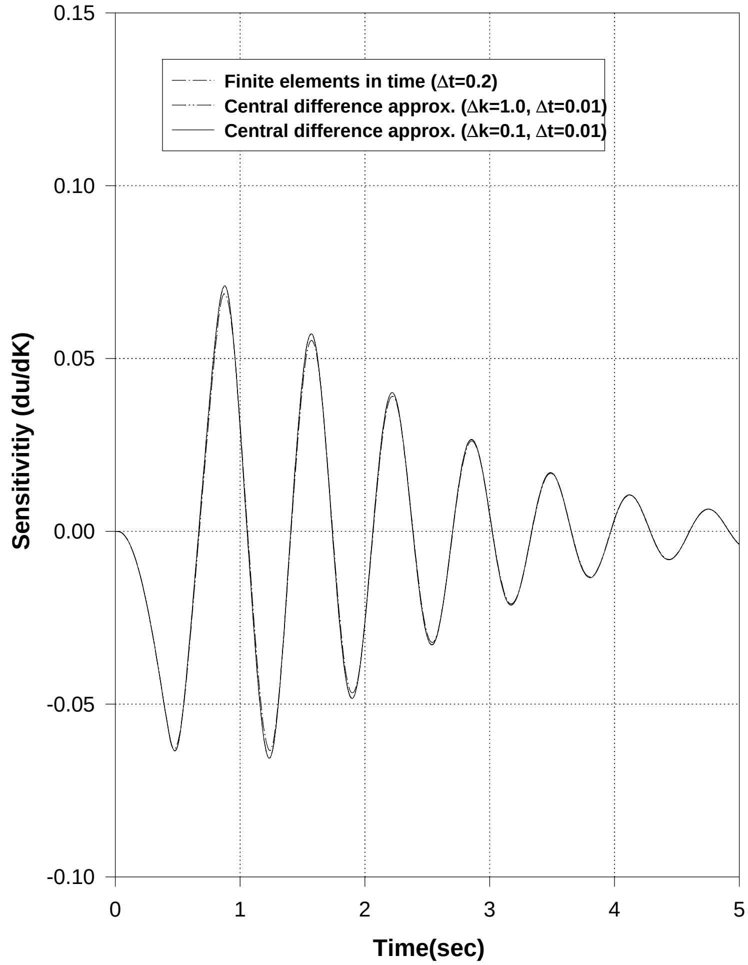

Fig. 2.14 Sensitivity of the response with respect to stiffness ($\ddot{u} + 2\dot{u} + 100tanh(u) = 0$)

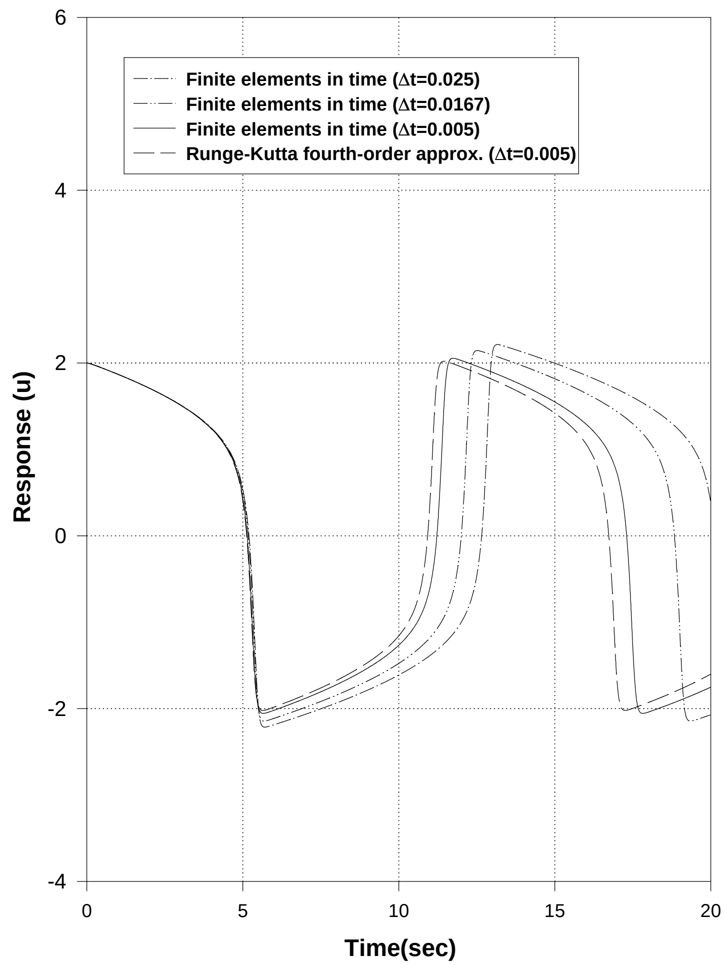

Fig. 2.15 Transient responses of the Van der Pol equation
$(\ddot{u} + \epsilon(u^2 - 1.0)\dot{u} + u = 0, \quad \epsilon = 5.0)$

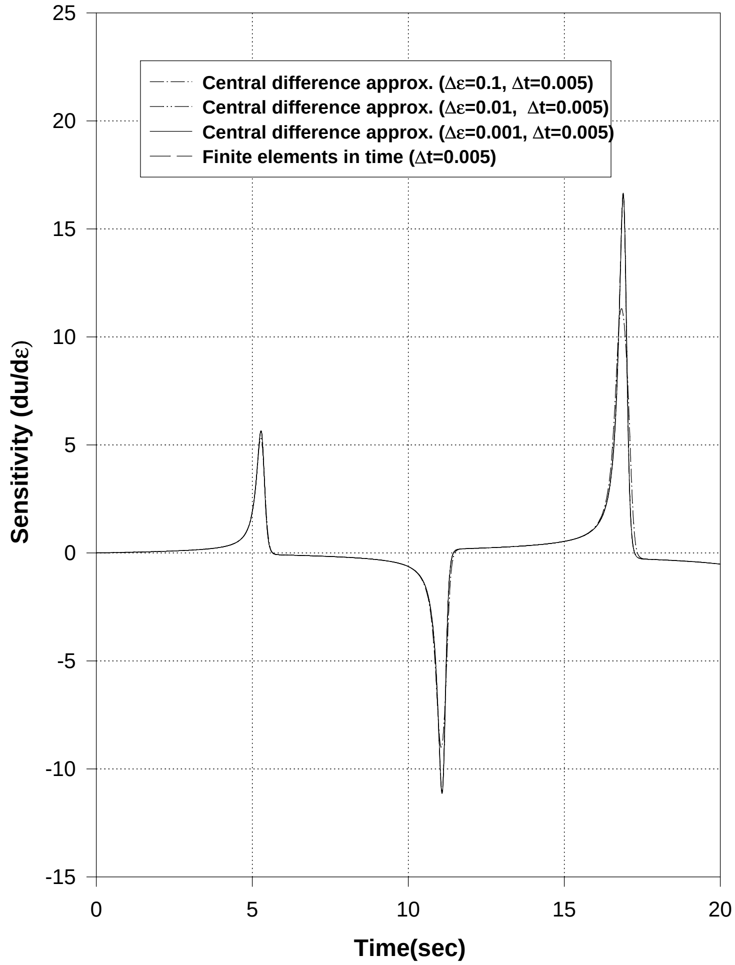

Fig. 2.16a Sensitivity of the response with respect to ϵ ($\ddot{u} + \epsilon(u^2 - 1.0)\dot{u} + u = 0, \quad \epsilon = 5.0$)

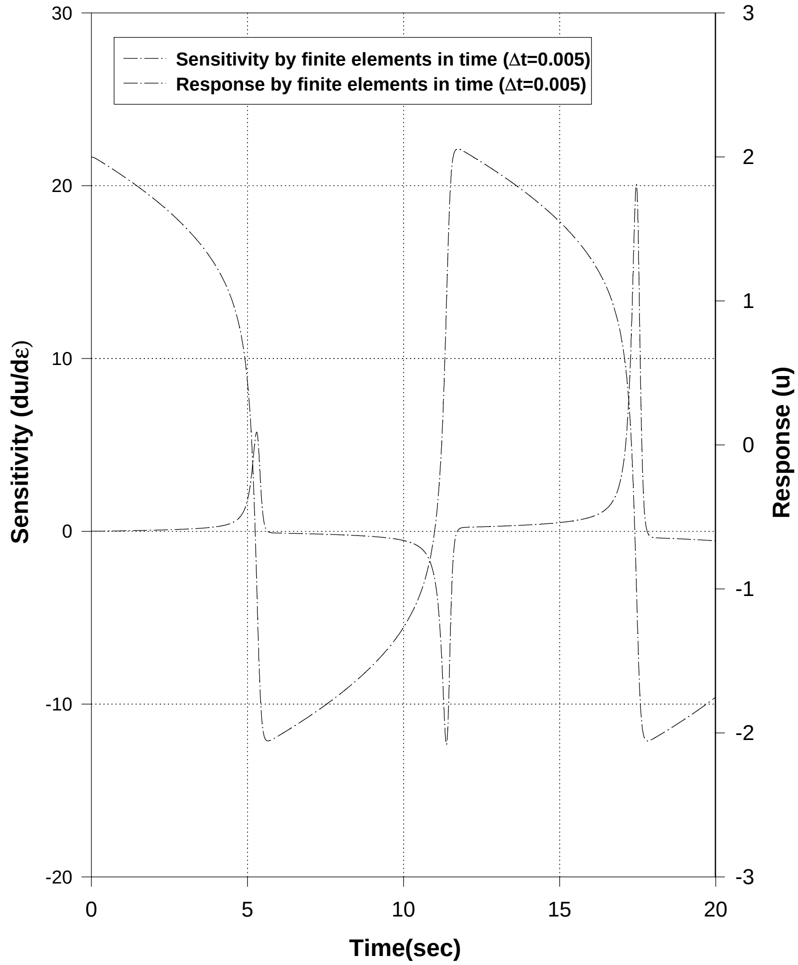

Fig. 2.16b Comparison of the sensitivities with respect to ϵ between time finite elements method and central difference approximations ($\ddot{u} + \epsilon(u^2 - 1.0)\dot{u} + u = 0, \quad \epsilon = 5.0$)

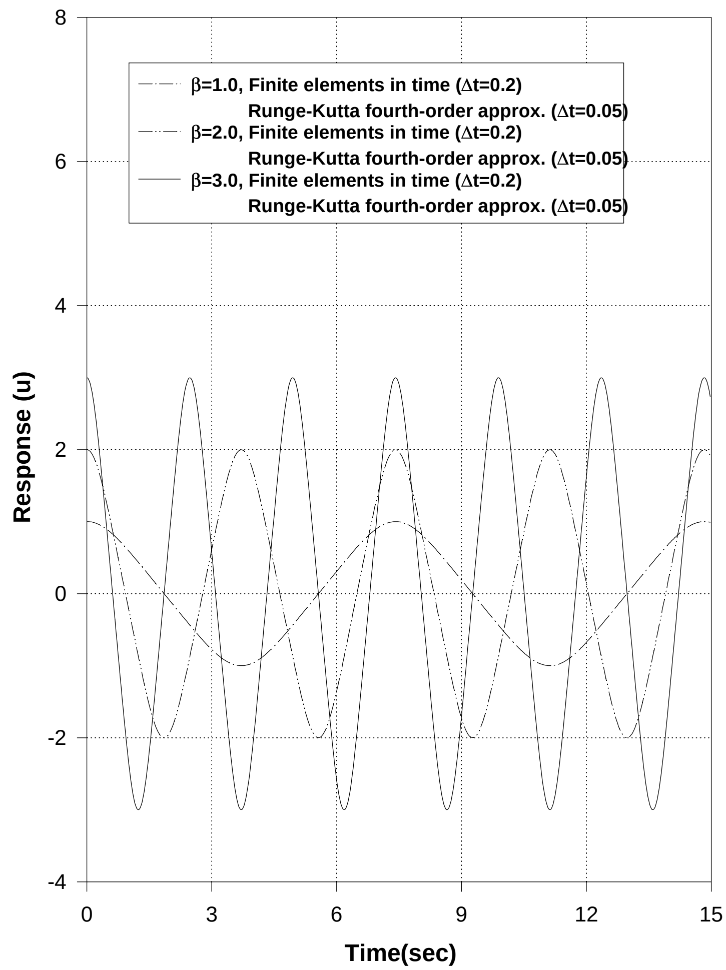

Fig. 2.17 Transient responses of the nonlinear oscillator with various values of β
$(\ddot{u} + \mu u^3 = 0, \quad \mu = 1.0, \quad u(0) = \beta, \quad \dot{u}(0) = 0.0)$

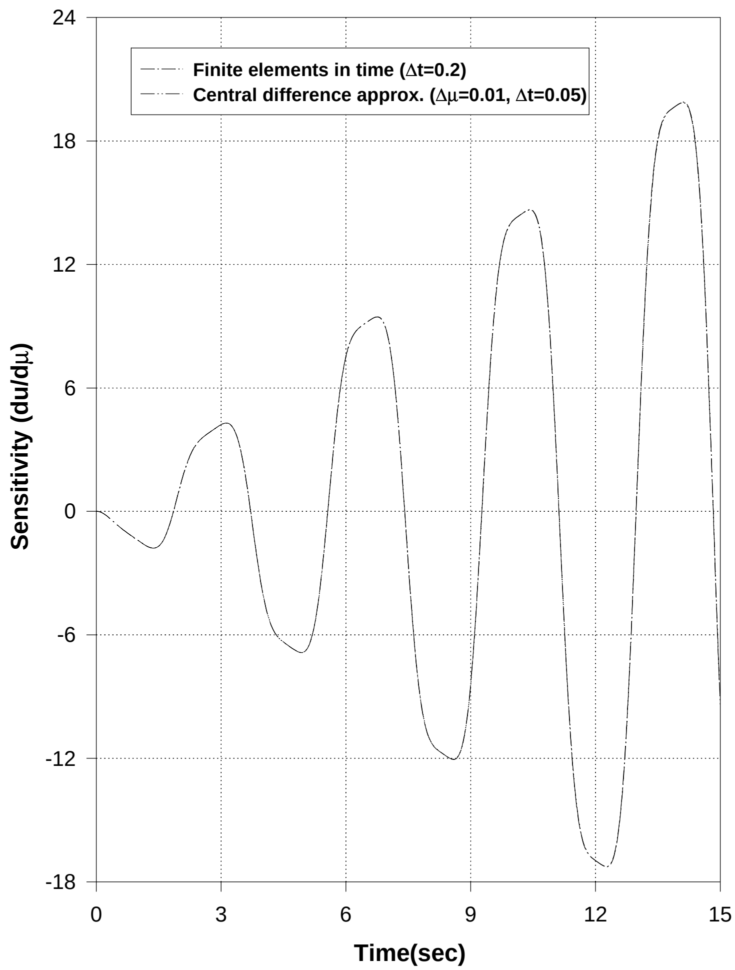

Fig. 2.18 Comparison of the sensitivities with respect to μ between time finite element method and central difference approximations ($\ddot{u}+\mu u^3 = 0, \quad u(0) = 2.0, \dot{u}(0) = 0.0$)

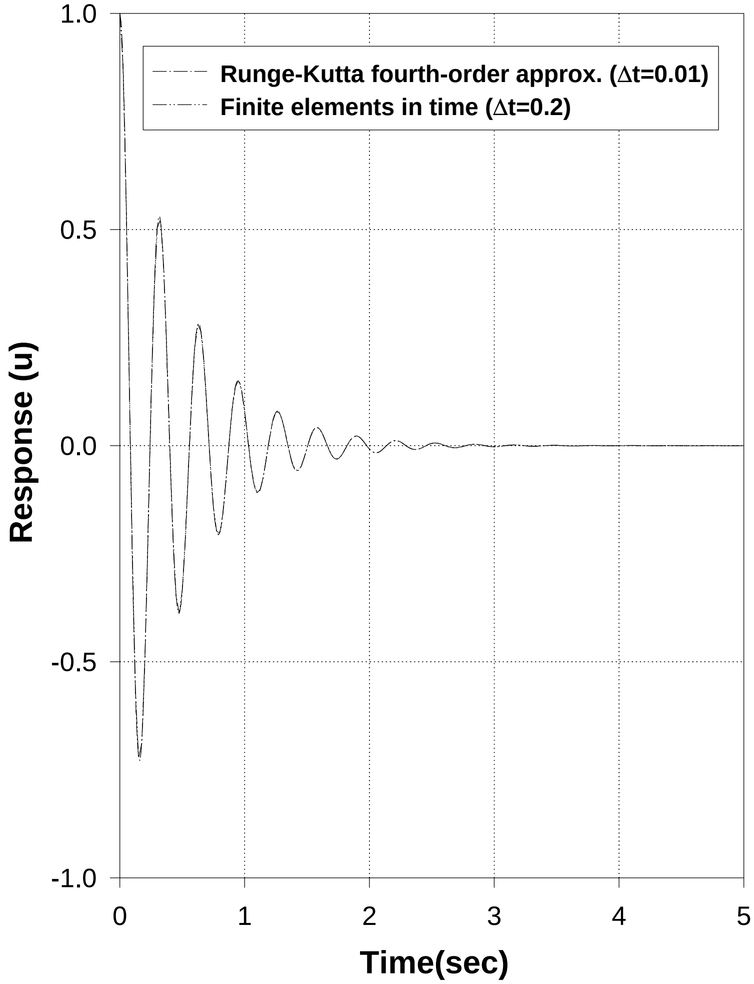

Fig. 2.19 Transient response of a hardening system with damping
$(\ddot{u}(t) + 4\dot{u}(t) + 400u(t) + u(t)^3 = 0)$

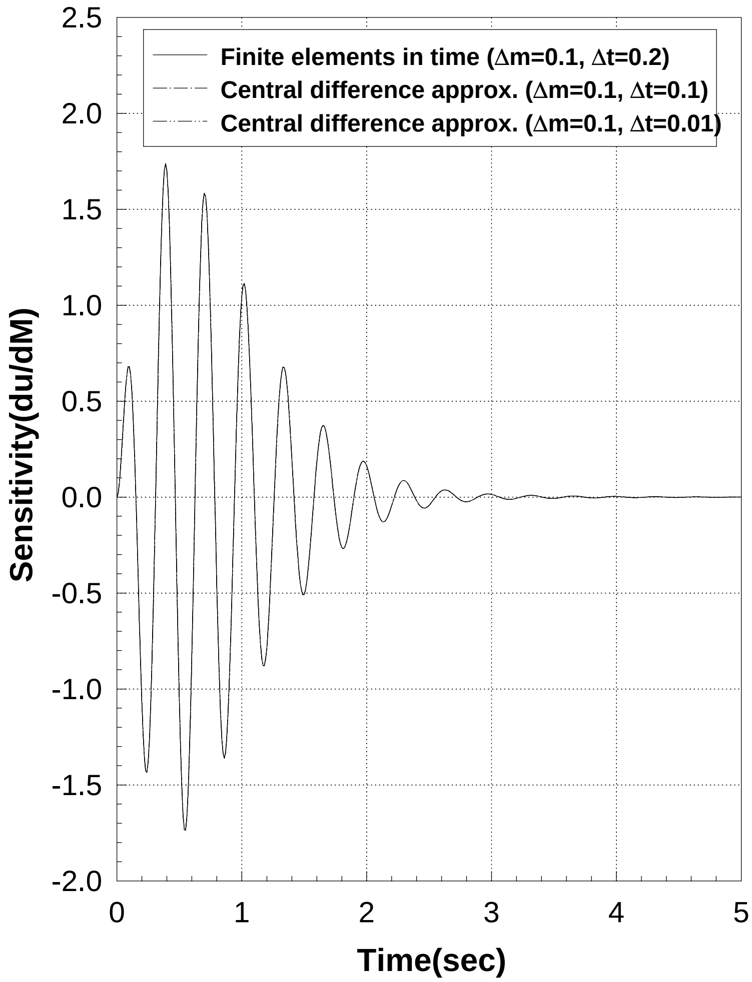

Fig. 2.20 Sensitivity of the response with respect to mass
$(\ddot{u}(t) + 4\dot{u}(t) + 400u(t) + u(t)^3 = 0)$

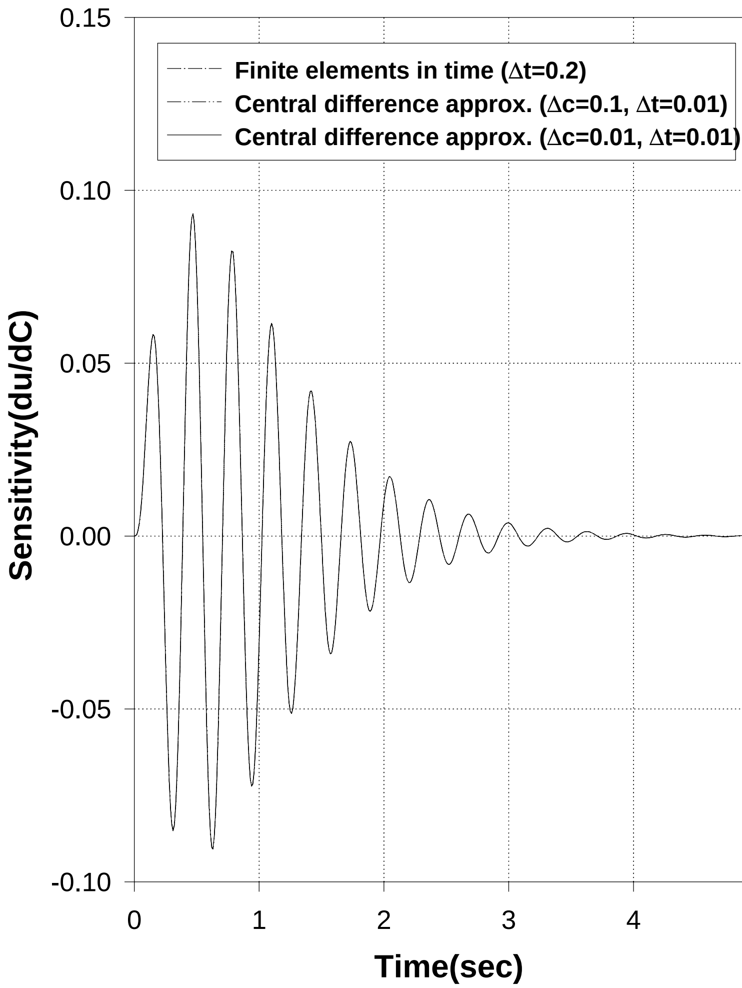

Fig. 2.21 Sensitivity of the response with respect to damping
$(\ddot{u}(t) + 4\dot{u}(t) + 400u(t) + u(t)^3 = 0)$

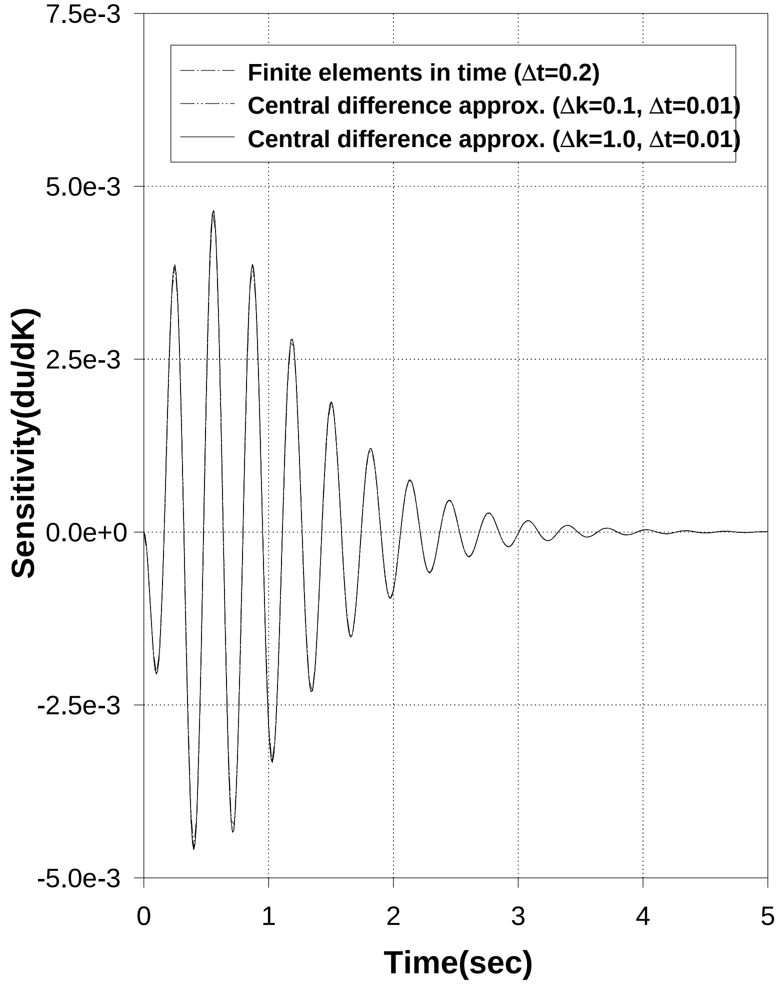

Fig. 2.22 Sensitivity of the response with respect to stiffness
$(\ddot{u}(t) + 4\dot{u}(t) + 400u(t) + u(t)^3 = 0)$

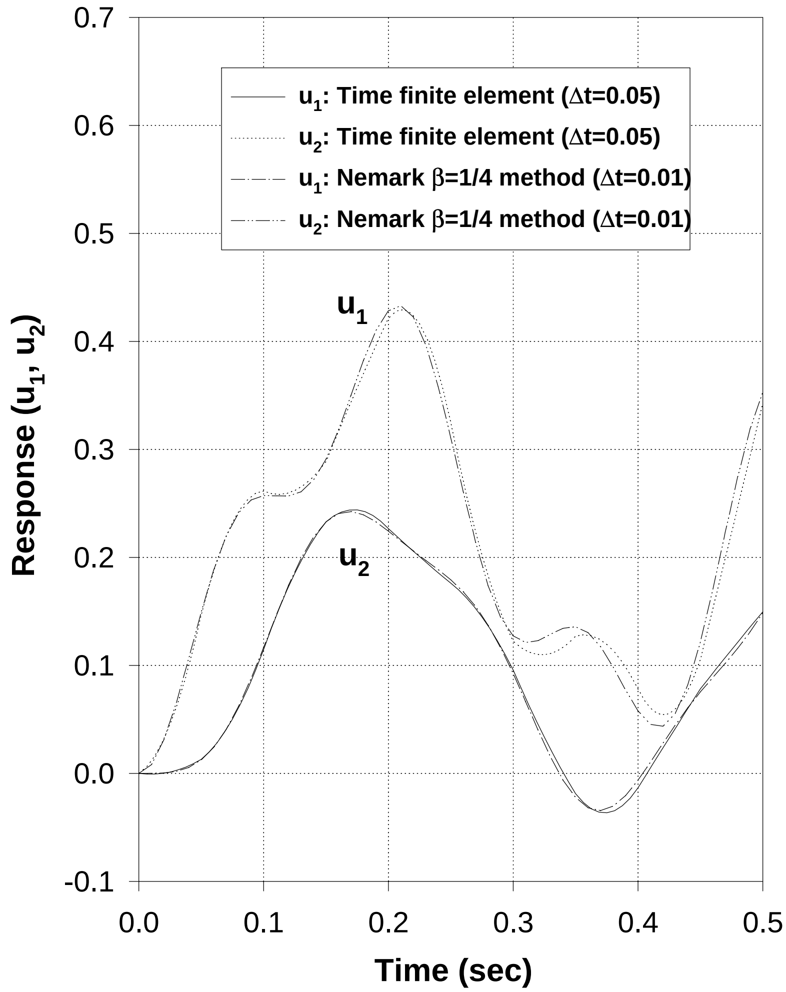

Fig. 2.23 Transient responses of linear 2DOF system having simple forces

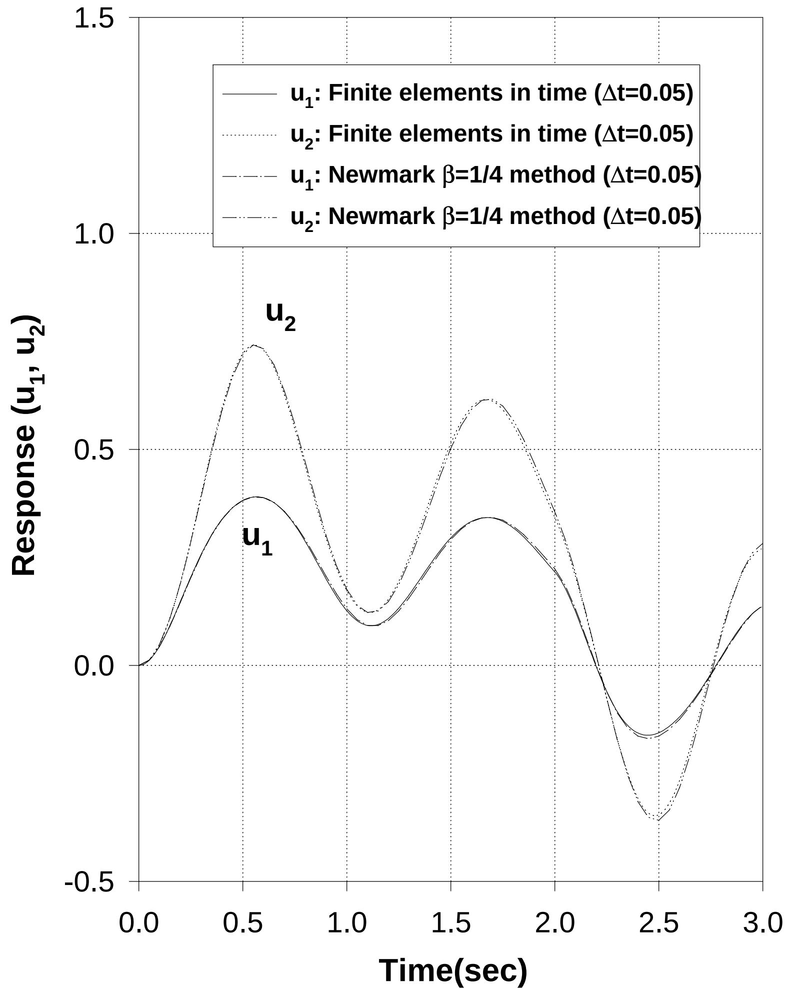

Fig. 2.24 Transient responses of linear 2DOF system with the ground acceleration

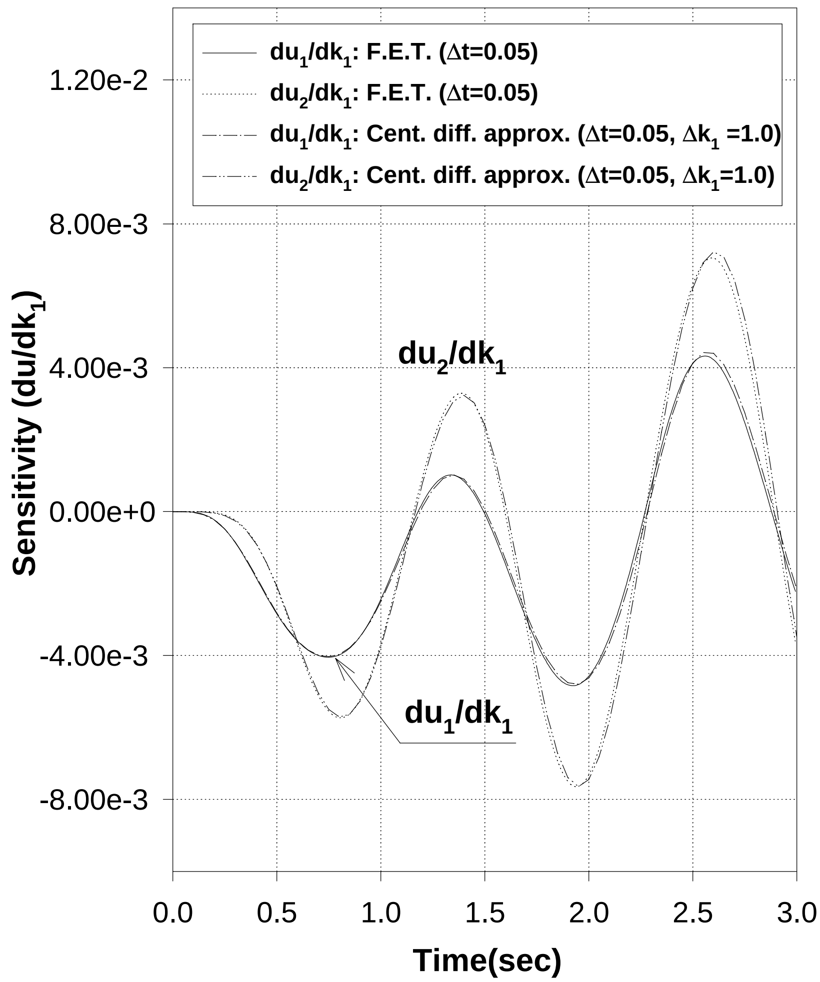

Fig. 2.25 Sensitivity of the responses with respect to k_1 for linear 2DOF system with the ground acceleration

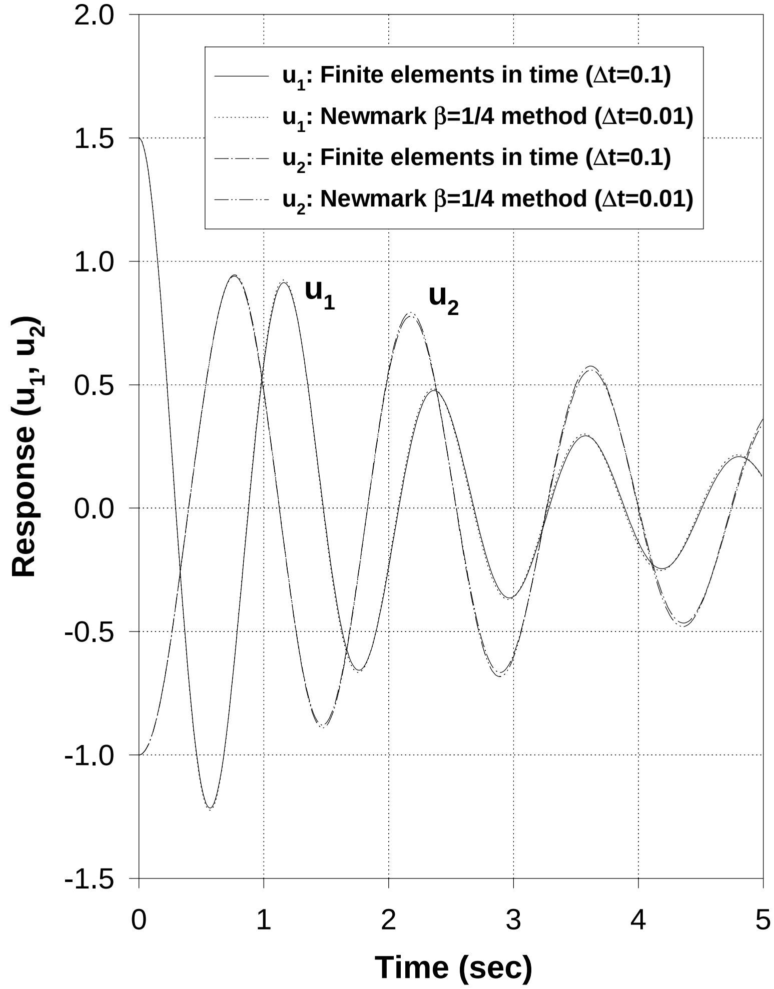

Fig. 2.26 Transient response of nonlinear 2DOF system having cubic nonlinearities

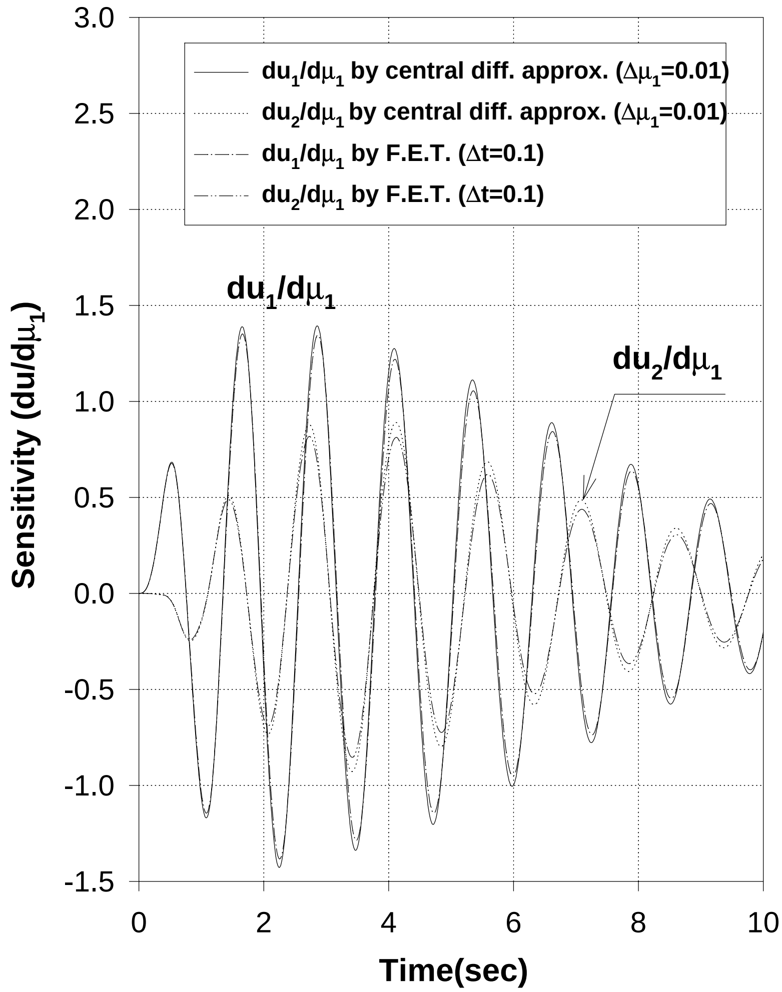

Fig. 2.27 Sensitivity of the response with respect to μ_1 for nonlinear 2DOF system having cubic nonlinearities

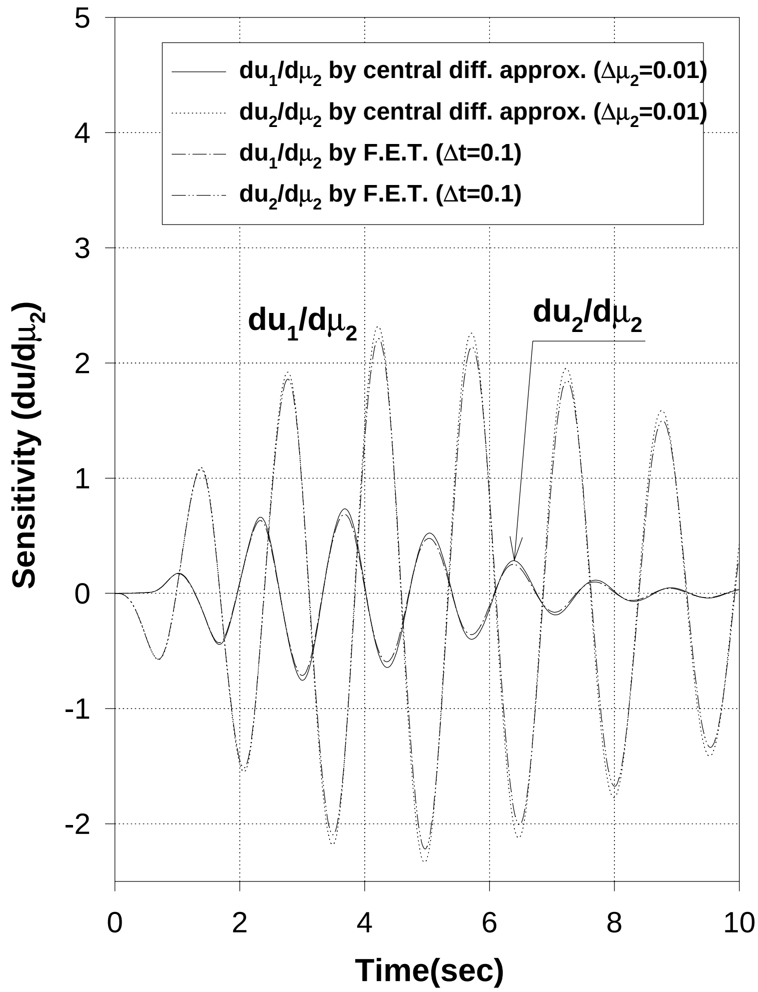

Fig. 2.28 Sensitivity of the response with respect to μ_2 for nonlinear 2DOF system having cubic nonlinearities

3. COMPARISON OF VARIOUS ORTHOGONAL POLYNOMIALS IN THE *hp*-VERSION FINITE ELEMENT METHOD

3.1 Overview

Four sets of orthogonal polynomials, Legendre, Chebyshev, Hermite and Integrated Legendre polynomials are evaluated as basis functions to solve initial value problems governed by second order differential equations using finite elements in time (Kapania and Park [62]). Problems treated are the transient response and the response sensitivity of van der Pol's oscillator, mass on a hardening and a softening spring system and a two-degree-of-freedom system having cubic nonlinearities. The results obtained using four different polynomial basis functions were found to be almost identical. The objective of this study is to find better conditioned systems for the problems under consideration. Condition numbers of the augmented stiffness matrix for the selected problems are estimated by increasing the number of polynomial terms in the expansion. Results for the CPU time and the estimated condition numbers, using IMSL subroutine DLFCRG [96], for each of four basis functions are presented. The present research shows that integrated Legendre polynomials are well suited to be used as the basis function in the time finite element method for solving nonlinear initial value problems.

3.2 Mathematical Formulations

3.2.1 Bilinear Formulation for a Nonlinear System

Consider the following mathematical model of the dynamic system:

$$F(\ddot{u}, \dot{u}, u, t, d_k) = 0 \tag{3.1}$$

where the vector d_k denotes the design parameters of the system.
Applying the bilinear formulation to Eq. (3.1) results in

$$\left[\ B_{ij}\left(q_j\right)\ \right]\left\{\ q_j\ \right\} = \left\{\ A_i\ \right\} \tag{3.2}$$

which is the transient response of a nonlinear system.
Taking the derivative of Eq. (3.2) with respect to design parameter d_k yields

$$\left[\ B^*_{ij}\ \right]\left\{\ \frac{\partial q_j}{\partial d_k}\ \right\} + \left[\ \frac{\partial B_{ij}}{\partial d_k}\ \right]\left\{\ q_j\ \right\} = \left\{\ \frac{\partial a_i}{\partial d_k}\ \right\} \tag{3.3}$$

where

$$B^*_{ij} = B_{ij} + \sum_{m=1}^{N} \frac{\partial B_{im}}{\partial q_j} q_m \tag{3.4}$$

3.2.2 Augmented Stiffness Matrix for Single-Degree-of-Freedom System

As shown section 3.2.1, the nonlinear stiffness matrices $\mathbf{B}$ and $\mathbf{B}^*$ are not generally equal in the nonlinear system. By adding an initial condition and imposing a constraint in the form of Lagrange multipliers, the augmented stiffness matrix $\overline{\mathbf{B}}$ for the transient response may be expressed in the matrix form as

$$\overline{\mathbf{B}} = \begin{bmatrix} \mathbf{B} & \{\psi_i(T_f)\} \\ < \phi_j(T_0) > & 0 \end{bmatrix} \tag{3.5}$$

while the augmented stiffness matrix $\overline{\mathbf{B}}^*$ for the response sensitivity may be expressed as

$$\overline{\mathbf{B}}^* = \begin{bmatrix} \mathbf{B}^* & \{\psi_i(T_f)\} \\ < \phi_j(T_0) > & 0 \end{bmatrix} \tag{3.6}$$

where $\psi_i(t)$ and $\phi_j(t)$ are polynomial basis functions for test and trial functions respectively. In the matrix, { } denotes the column vector and < > denotes the row vector.

3.2.3 Augmented Stiffness Matrix for Two-Degree-of-Freedom System

In the case of two-degree-of-freedom systems, the augmented stiffness matrix $\overline{B}_{ij}$, derived for the first time here, for the transient response may be expressed as

$$\overline{\mathbf{B}} = \begin{bmatrix} \mathbf{B}^{(1)} & 0 & \{\psi_i(T_f)\} & 0 \\ 0 & \mathbf{B}^{(2)} & 0 & \{\psi_i(T_f)\} \\ < \phi_j(T_0) > & 0 & 0 & 0 \\ 0 & < \phi_j(T_0) > & 0 & 0 \end{bmatrix} \tag{3.7}$$

while the augmented stiffness matrix $\overline{B}^*_{ij}$ for the response sensitivity takes the form:

$$\overline{\mathbf{B}}^* = \begin{bmatrix} \mathbf{B}^{*(11)} & \mathbf{B}^{*(12)} & \{\psi_i(T_f)\} & 0 \\ \mathbf{B}^{*(21)} & \mathbf{B}^{*(22)} & 0 & \{\psi_i(T_f)\} \\ < \phi_j(T_0) > & 0 & 0 & 0 \\ 0 & < \phi_j(T_0) > & 0 & 0 \end{bmatrix} \tag{3.8}$$

here i and j respectively denote the row and the column index in the matrix.

3.3 Basis Functions

Since one goal of the computational method is the computational simplicity, one choice for the basis function is polynomials. Basically the orthogonal polynomials (Szegö [65]; Sansonne [66]; Hochstrasser [67]) are selected in order to have simplicity in the computation of the stiffness matrix. The orthogonality has less effect on the computation due to the nature of the formulation. Four sets of orthogonal polynomials, Legendre, Chebyshev, Hermite and Integrated Legendre polynomials are selected as basis functions, each defined over the range $-1 < t \leq 1$. In Integrated Legendre polynomials, the first and second terms are linear combinations of the first two integrals of Legendre polynomials.

$$I_1(t) = \frac{(1-t)}{2} \qquad I_2(t) = \frac{(1+t)}{2} \tag{3.9}$$

The rest of the polynomials $I_n(t)$ are found using the following relationship (Szabó and Babuška [64]):

$$I_n(t) = \frac{1}{\sqrt{2(2n-3)}} \left(P_{n-1}(t) - P_{n-3}(t)\right) \quad n \geq 3 \tag{3.10}$$

where $P_n(t)$ denotes Legendre polynomials.

3.4 Numerical Results

Condition numbers [97] of the augmented stiffness matrix for the selected problems are estimated by increasing the number of polynomial terms in the expansion. Gaussian quadrature [98] is used for the integration process in the calculation of the transient response and the response sensitivity. Results for the CPU time and the estimated condition numbers for each of four basis functions are presented. The CPU time for the selected problems required only one to two seconds (except in the case of van der Pol's oscillator). Those results are not presented here. To estimate the condition numbers, the IMSL subroutine DLFCRG was used. All the computations were performed on an IBM3090-300E mainframe computer. The detailed derivations of the used nonlinear stiffness matrices for the following four examples are given in Chapter 2. For completeness, the final equations are given in Appendix A.3.

3.4.1 van der Pol's Oscillator

Consider a nonlinear van der Pol's equation (Shampine [94]) with a large parameter ϵ given by

$$\ddot{u}(t) + \epsilon\left(u(t)^2 - 1\right)\dot{u}(t) + u(t) = 0, \qquad \epsilon = 5.0, \qquad 0 < t \leq 20 \tag{3.11}$$

with initial conditions

$$u(0) = 2.0, \qquad \dot{u}(0) = 0.0$$

The response of the van der Pol's equation for a large parameter ϵ shows a slow build-up followed by a sudden discharge, repeated periodically. Figures 3.1a and 3.1b show respectively the limit cycle and the corresponding response for the problem using the time finite element method. The results shown in Figs. 3.1a and 3.1b were obtained using Legendre polynomials of degree four as basis functions with varying time steps of $\Delta t =$ 0.005 to 0.000625. Condition numbers of the augmented stiffness matrix were estimated by varying the order of polynomials from 3 to 8 with fixed time steps of $\Delta t = 0.005$.

Table 3.1a CPU time for calculating transient response and response sensitivity: van der Pol's oscillator

N^*	Legendre	Chebyshev	Hermite	Int. Legendre
3	6.0(9.0)	5.0(8.0)	6.0(8.0)	7.0(12.0)
4	6.0(12.0)	6.0(11.0)	6.0(11.0)	8.0(17.0)
5	7.0(17.0)	7.0(14.0)	7.0(15.0)	9.0(24.0)
6	9.0(23.0)	8.0(19.0)	8.0(21.0)	12.0(33.0)
7	11.0(30.0)	9.0(25.0)	10.0(27.0)	14.0(44.0)
8	13.0(39.0)	11.0(32.0)	12.0(36.0)	17.0(57.0)

* order of polynomials

Table 3.1a presents the required CPU time for calculating the transient response and the response sensitivity (in parentheses) of the given system. Note that the computation time is increasing as the order of polynomials increases. The Integrated Legendre polynomials require more computation time as compared to the other polynomials. This is expected since an integration process is needed for generating Integrated Legendre polynomials while others are generated by appropriate recurrence procedures.

Results of the estimated condition numbers are presented in Table 3.1b (also in Fig. 3.2). The columns in Table 3.1b present condition numbers of the augmented stiffness matrix for the transient response and the response sensitivity (in parentheses) cases. In both cases, condition numbers are generally increasing along with the order of polynomials except for the case of Integrated Legendre polynomials. For the latter case, they are independent of the polynomial order.

Condition numbers obtained using Hermite polynomials of degree 6, 7, and 8 show extremely high values compared with the condition numbers obtained for other polynomials. This is due to the fact that the numerical values of diagonal terms are getting real big whenever a polynomial term is added in the expansion for the basis function (see Appendix A.3). The condition numbers for the case of Integrated Legendre polynomials consistently have far smaller values. This is due to the fact that the values in the "stiffness matrix" remained same even after additional polynomial terms are added to the basis function expansion (see

Appendix A.3). Condition numbers obtained using Legendre and Chebyshev polynomials show a similar behavior but with a little higher values for the Legendre polynomials. Note that the augmented stiffness matrix using Chebyshev polynomials is not sparse as is the case for Legendre polynomials. This is due to nonuniform weights being used for Chebyshev polynomials.

3.4.2 Mass on a Nonlinear Hardening Spring

Consider a damped oscillation of a mass on a nonlinear hardening spring over a given length of time, $0 < t \leq 5$, which is defined by

$$M\ddot{u}(t) + C\dot{u}(t) + Ku(t) + \mu u(t)^3 = 0.0 \tag{3.12}$$

with initial conditions

$$u(0) = 1.0 \qquad \dot{u}(0) = 0.0$$

for the following values of system parameters:

$$M = 1.0, \qquad C = 4.0, \qquad K = 400.0, \qquad \mu = 1.0$$

The domain, $0 < t \leq 5$, is divided into fifty elements of equal time steps. Condition numbers were estimated by varying the order of polynomials from 3 to 8. Results for the estimated condition numbers for transient response and the response sensitivity (in parentheses) are given in Table 3.2 (also in Fig. 3.3).

We failed to estimate condition numbers for Hermite polynomials of degree 6, 7 and 8. Again the Integrated Legendre case shows smallest condition numbers and the condition number is independent of the order of the polynomial being used.

3.4.3 Mass on a Nonlinear Softening Spring

A damped oscillation of a mass on a nonlinear softening spring model with a time length, $0 < t \leq 5$, is represented by

$$M\ddot{u}(t) + C\dot{u}(t) + \alpha \tanh(u(t)) = 0 \tag{3.13}$$

with initial conditions

$$u(0) = 0, \qquad \dot{u}(0) = 25.0$$

for the following values of system parameters:

$$M = 1.0, \qquad C = 1.0, \qquad \alpha = 100.0$$

In order to estimate condition numbers, the domain was divided into fifty elements of equal time steps and the order of polynomials were again increased from 3 to 8. Table 3.3 (also in Fig. 3.4) presents condition numbers of the augmented stiffness matrix for the transient response and the response sensitivity (in parentheses).

We noted that the Integrated Legendre case again shows smallest values for the condition number and the condition number remains unchanged as the number of terms is increased. The Hermite case shows extremely high condition numbers as the order of the polynomial is increased.

3.4.4 Two-Degree-of-Freedom System having Cubic Nonlinearities

As a fourth example, a two-degree-of-freedom system governed by differential equations having cubic nonlinearities of Chapter 2 will be examined

$$
\begin{aligned}
\ddot{u}_1 + {\omega_1}^2 u_1 + 2\mu_1 \dot{u}_1 + \alpha_1 {u_1}^3 + \alpha_2 {u_1}^2 u_2 + \alpha_3 u_1 {u_2}^2 + \alpha_4 {u_2}^3 = 0 \\
\ddot{u}_2 + {\omega_2}^2 u_2 + 2\mu_2 \dot{u}_2 + \alpha_5 {u_1}^3 + \alpha_6 {u_1}^2 u_2 + \alpha_7 u_1 {u_2}^2 + \alpha_8 {u_2}^3 = 0
\end{aligned}
\tag{3.14}
$$

with initial conditions

$$
u_1(0) = 1.5, \qquad \dot{u}_1(0) = 0.0, \qquad u_2(0) = -1.0, \qquad \dot{u}_2(0) = 0.0
$$

for the following values of the system parameters

$$
\begin{aligned}
{\omega_1}^2 &= 25.0, \quad \mu_1 = 0.35, \quad \alpha_1 = 5.0, \quad \alpha_2 = 0.5, \quad \alpha_3 = 0.25, \quad \alpha_4 = 3.0 \\
{\omega_2}^2 &= 17.0, \quad \mu_2 = 0.25, \quad \alpha_5 = 2.5, \quad \alpha_6 = 0.75, \quad \alpha_7 = 0.2, \quad \alpha_8 = 5.0
\end{aligned}
$$

The estimated condition numbers of the augmented stiffness matrix for the transient response and the response sensitivity (in parentheses) is given in Table 3.4 (also in Fig. 3.5). Condition numbers for the response sensitivity are almost same as those for the transient response case. The results show very similar trend as those for the case of single-degree-of-freedom systems.

In the case of Hermite polynomials, again we failed to estimate the condition numbers when higher order polynomials (5, 6, 7 and 8) were used. The Integrated Legendre case shows the lowest numbers among all the polynomials.

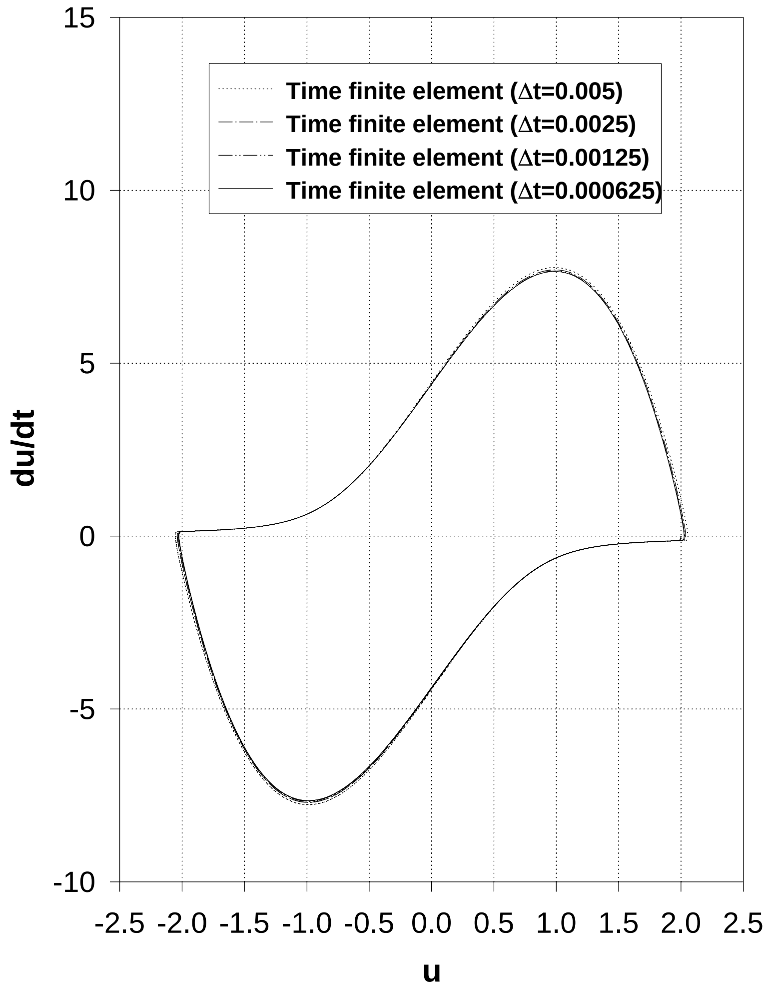

Fig. 3.1a Limit cycles of van der Pol's equation with $\epsilon = 5.0$

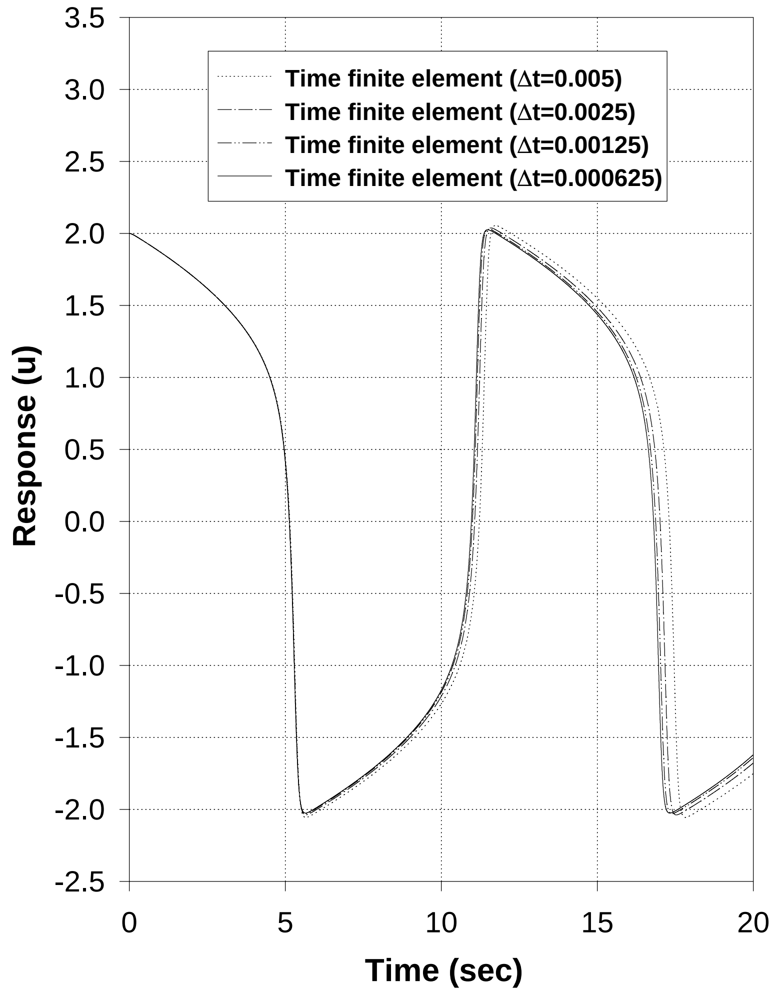

Fig. 3.1b Response curves corresponding to limit cycles with various time step sizes

Table 3.1b Condition numbers of augmented stiffness matrix: van der Pol's oscillator

N^*	Legendre	Chebyshev	Hermite	Int. Legendre
3	5.74E+3(5.65E+3)	1.21E+4(1.20E+4)	6.15E+4 (6.08E+4)	823(822)
4	1.07E+4(1.05E+4)	2.55E+4(2.51E+4)	1.11E+6 (1.09E+6)	823(823)
5	1.80E+4(1.78E+4)	4.58E+4(4.51E+4)	9.29E+6 (9.14E+6)	823(823)
6	2.79E+4(2.74E+4)	7.43E+4(7.32E+4)	1.13E+8 (1.13E+8)	823(823)
7	4.10E+4(4.03E+4)	1.12E+5(1.11E+5)	3.52E+9 (3.53E+9)	823(823)
8	5.74E+4(5.65E+4)	1.61E+5(1.59E+5)	5.61E+11(5.62E+11)	823(823)

* order of polynomials

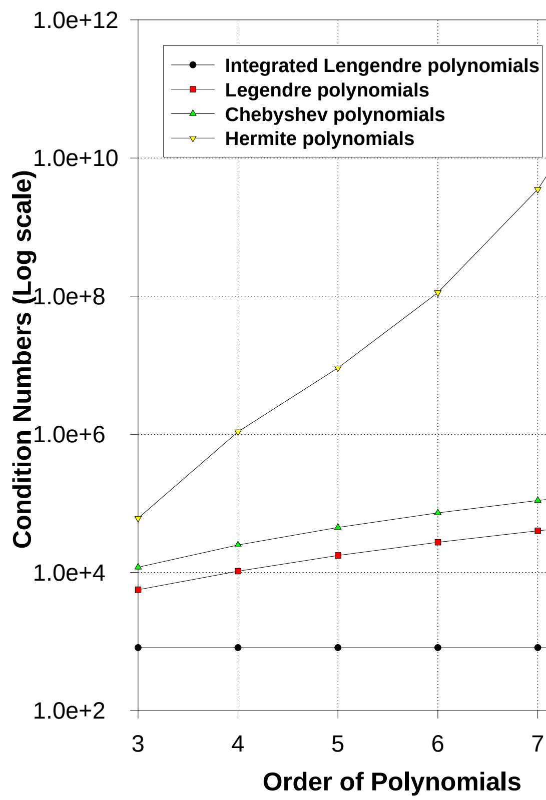

Fig. 3.2 Condition numbers of augmented stiffness matrix: van der Pol's oscillator

Table 3.2 Condition numbers of augmented stiffness matrix: hardening spring model

N^*	Legendre	Chebyshev	Hermite	Int. Legendre
3	2.26E+3(5.40E+3)	4.65E+3(1.13E+4)	6.10E+3(3.92E+4)	117(630)
4	4.34E+3(1.01E+4)	1.01E+4(2.40E+4)	2.75E+5(9.39E+5)	116(630)
5	7.54E+3(1.72E+4)	1.85E+4(4.33E+4)	2.43E+6(7.95E+6)	118(630)
6	1.17E+4(2.66E+4)	3.02E+4(7.05E+4)	**	118(630)
7	1.73E+4(3.92E+4)	4.59E+4(1.06E+5)	**	118(630)
8	2.43E+4(5.49E+4)	6.60E+4(1.53E+5)	**	118(630)

* order of polynomials ** not available

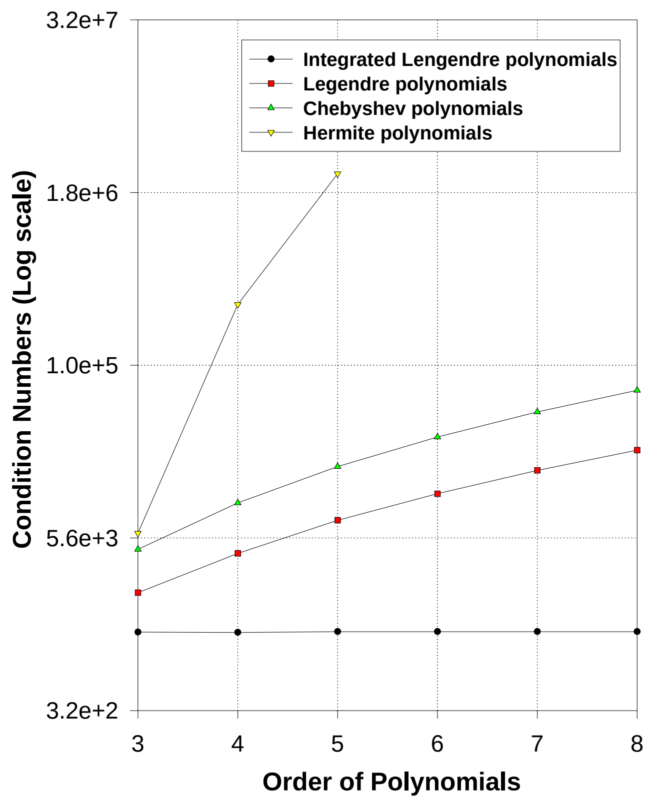

Fig. 3.3 Condition numbers of augmented stiffness matrix: hardening spring model

Table 3.3 Condition numbers of the augmented stiffness matrix: softening spring model

N^*	Legendre	Chebyshev	Hermite	Int. Legendre
3	1.55E+2(2.82E+3)	3.27E+2(5.95E+3)	1.61E+3(3.02E+4)	22(402)
4	2.87E+2(5.23E+3)	6.85E+2(1.25E+3)	2.89E+4(5.43E+4)	22(402)
5	4.85E+2(8.84E+3)	1.23E+3(2.24E+4)	2.43E+5(4.57E+6)	22(402)
6	7.49E+2(1.37E+4)	2.00E+3(3.64E+4)	1.34E+7(5.98E+7)	22(402)
7	1.10E+3(2.01E+4)	3.02E+3(5.51E+4)	3.52E+9(3.52E+10)	22(402)
8	1.54E+3(2.81E+4)	4.33E+3(7.91E+4)	5.60E+11(5.60E+12)	22(402)

* order of polynomials

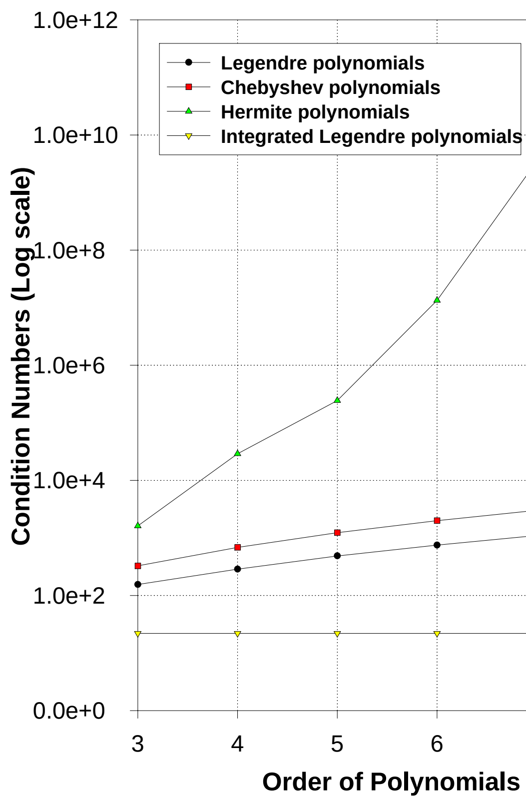

Fig. 3.4 Condition numbers of augmented stiffness matrix: softening spring model

Table 3.4 Condition numbers of augmented stiffness matrix for transient response: two-degree-of-freedom system with cubic nonlinearities

N^*	Legendre	Chebyshev	Hermite	Int. Legendre
2	4.04E+2(4.19E+2)	7.17E+2(7.43E+2)	2.81E+3(2.91E+3)	88(91)
3	9.41E+2(9.75E+2)	1.98E+3(2.06E+3)	9.84E+3(1.02E+4)	88(91)
4	1.75E+3(1.81E+3)	4.17E+3(4.32E+3)	1.78E+5(1.84E+5)	88(91)
5	2.95E+3(3.06E+3)	7.48E+3(7.75E+3)	**	88(91)
6	4.56E+3(4.73E+3)	1.21E+4(1.26E+4)	**	88(91)
7	6.71E+3(6.95E+3)	1.84E+4(1.90E+4)	**	88(91)
8	9.39E+3(9.73E+3)	2.63E+4(2.73E+4)	**	88(91)

* order of polynomials ** not available

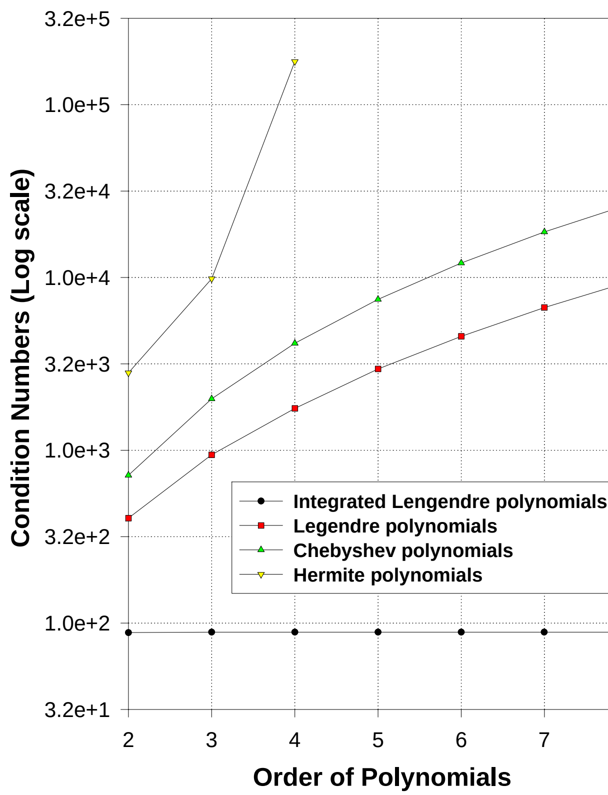

Fig. 3.5 Condition numbers of augmented stiffness matrix for transient response: two-degree-of-freedom system with cubic nonlinearities

4. PARAMETRIC IDENTIFICATION OF NONLINEAR STRUCTURAL DYNAMIC SYSTEMS

4.1 Overview

At present, most of the system identification methods are based on minimization of the square of the error between the measured response and that of the identified model. This is the classical least squared approach in which the error is minimized by treating the problem as an unconstrained optimization problem. Most of the algorithms for solving unconstrained optimization problems require sensitivity of the response with respect to various system parameters. These sensitivities are often obtained using either finite difference or by solving a large set of differential equations. In this thesis, an alternative approach, based on the TFM is employed to identify a series of single-degree-of-freedom and a two-degree-of-freedom nonlinear systems.

An advantage of the present method over the finite difference approach, the most common way to find the sensitivity, is that one does not need to perform a convergence study to select an appropriate step size for obtaining the sensitivities. Also, the method can be applied as a step by step procedure, thereby avoiding the need for dealing with large matrices.

The TFM along with the iterative direct method [60, 68, 69] are applied to a number of damped single- and two-degree-of-freedom nonlinear systems. Considering all the advantages and the numerical results, it is clear that the TFM is very much suitable for system identification.

4.2 Transient Response and Its Sensitivity of a Nonlinear System

The sensitivity of the transient response of a nonlinear system can be obtained by taking the derivative of Eq. (2.11) with respect to d_k in Chapter 2. Applications of the aforementioned equations for determining sensitivity of transient responses of a large number of linear and nonlinear problems are given in chapter 2.

4.3 Parameter Identification

4.3.1 Iterative Direct Method

As a first step, in parameter identification, an objective function is formulated as a quadratic functional between the measured response of the given system and an analytic response of the mathematical model. Then the system parameters can be determined using the method presented by Levenberg [71] and Marquardt [72] for minimizing the objective function. To avoid large computational costs involved with obtaining second order partial derivatives with respect to the design parameters, the method involves only first partial derivatives of the response with respect to various system parameters .

4.3.2 Objective Function Formulation

The objective function for a single-degree-of-freedom system is given as

$$L(\mathbf{d}) = \int_0^T (u_a - u_m)^2 \, dt \tag{4.1}$$

and the same for a two-degree-of-freedom-system is given as

$$L(\mathbf{d}) = \int_0^T \left\{ (u_{1a} - u_{1m})^2 + (u_{2a} - u_{2m})^2 \right\} dt \tag{4.2}$$

where u_a and u_m, respectively, denote the time series of the analytical and measured displacement, velocity or acceleration response, T is the record length of the measured response and $\mathbf{d}$ is the parameter vector to be determined. The time series u_a is an analytic solution of the assumed model for the given system. The measured data u_m was simulated by obtaining the response of the system using given parameters. Effect of random noise on the identification of the parameters is also studied by corrupting the analytical solution with random noise with varying rms values. The objective function is minimized by setting the partial derivatives of L with respect to various system parameters equal to zero.

The vector of first derivatives of $L(\mathbf{d})$ with respect to design parameter d_k, for the single-degree-of-freedom system, yields

$$\frac{\partial \left(L(\mathbf{d}) \right)}{\partial d_k} = \int_0^T 2 \left(u_a - u_m \right) \left(\frac{\partial u_a}{\partial d_k} \right) dt \tag{4.3}$$

Similarly the vector of first derivatives of $L(\mathbf{d})$ with respect to d_k, for the two-degree-of-freedom-system, can be expressed as

$$\frac{\partial \left(L(\mathbf{d}) \right)}{\partial d_k} = \int_0^T 2 \left\{ \left(u_{1a} - u_{1m} \right) \left(\frac{\partial u_{1a}}{\partial d_k} \right) + \left(u_{2a} - u_{2m} \right) \left(\frac{\partial u_{2a}}{\partial d_k} \right) \right\} dt \tag{4.4}$$

Hence $\partial u_a/\partial d_k$ for all k are the first-order sensitivity of u_a with respect to kth system parameter. Note that using the time finite element method, these sensitivities are obtained by simply solving the set of linear algebraic equations and not by solving a large set of ordinary differential equations as was done in the past studies. These sensitivities are obtained by solving the sensitivity equations formulated in Eq. (2.22) in Chapter 2.

The time series for u_a and $\partial u_a/\partial d_k$ are then used to minimize the objective function $L(\mathbf{d})$.

4.3.3 Objective Function Minimization by Levenberg-Marquardt Method

Minimization of the objective function $L(\mathbf{d})$ is accomplished by Newton's method [70, 85]. If $\mathbf{d}^{(i)}$ denotes the trial values of $\mathbf{d}$ after ith iteration, then $\mathbf{d}^{(i+1)}$ is obtained as

$$\mathbf{d}^{(i+1)} = \mathbf{d}^{(i)} + h^{(i)}\boldsymbol{\Delta}\mathbf{d} \tag{4.5}$$

where $\boldsymbol{\Delta}\mathbf{d}$ is the correction vector and $h^{(i)}$ is the step size which is set equal to one. In order to calculate the correction vector such that, at each iteration, the value of objective function $L(\mathbf{d})$ will decrease most rapidly, a steepest-descent type procedure is adopted. In general, the steepest-descent direction is the negative gradient of the function with respect to the design parameters d_k and takes the form:

$$\mathbf{g} = -\left\{\frac{\partial L}{\partial d_1}, \frac{\partial L}{\partial d_2}, \cdots \frac{\partial L}{\partial d_k}\right\}^t \tag{4.6}$$

where $\partial L/\partial d_k$ is the rate of change of L with respect to the design parameters d_k. The Hessian $\mathbf{H}$ of the objective function is of the form:

$$\mathbf{H} = \frac{\partial}{\partial d_k}\left(\frac{\partial L}{\partial d_l}\right) \tag{4.7}$$

Therefore, the equation for correction vector $\boldsymbol{\Delta}\mathbf{d}$ takes the form:

$$\mathbf{H}\boldsymbol{\Delta}\mathbf{d} = -\mathbf{g} \tag{4.8}$$

At each iteration, the gradient and Hessian are calculated, then a new vector $\mathbf{d}^{(i+1)}$ is found. The iteration is terminated when a predefined convergence criterion is met. The gradient of the objective function L for the single-degree-of-freedom system takes the form:

$$\mathbf{g} = 2\int_0^T \left(\frac{\partial u_a}{\partial d_k}\right)(u_a - u_m)\,dt \tag{4.9}$$

and the gradient of the two-degree-of-freedom system can be written as

$$\mathbf{g} = 2\int_0^T \left\{\left(\frac{\partial u_{1a}}{\partial d_k}\right)(u_{1a} - u_{1m}) + \left(\frac{\partial u_{2a}}{\partial d_k}\right)(u_{2a} - u_{2m})\right\}dt \tag{4.10}$$

Also the Hessian for the single-degree-of-freedom system is given by

$$\mathbf{H} = 2\int_0^T \left[\left(\frac{\partial u_a}{\partial d_k}\right)\left(\frac{\partial u_a}{\partial d_l}\right) + \frac{\partial}{\partial d_k}\left(\frac{\partial u_a}{\partial d_l}\right)(u_a - u_m)\right] dt \tag{4.11}$$

while the Hessian for the two-degree-of-freedom system is expressed as

$$\begin{aligned} \mathbf{H} \;=\; & 2\int_0^T \left[\left(\frac{\partial u_{1a}}{\partial d_k}\right)\left(\frac{\partial u_{1a}}{\partial d_l}\right) + \frac{\partial}{\partial d_k}\left(\frac{\partial u_{1a}}{\partial d_l}\right)(u_{1a} - u_{1m}) \right. \\ & \left. + \left(\frac{\partial u_{2a}}{\partial d_k}\right)\left(\frac{\partial u_{2a}}{\partial d_l}\right) + \frac{\partial}{\partial d_k}\left(\frac{\partial u_{2a}}{\partial d_l}\right)(u_{2a} - u_{2m})\right] dt \end{aligned} \tag{4.12}$$

where k, l are row and column indices, respectively. Equations (4.11) and (4.12) require second derivatives for the calculation of Hessian. Usually the calculation of second derivatives requires large computational costs. In order to avoid this second derivatives calculations, the Levenberg-Marquardt method is used. In this method, the correction vector is obtained by solving the following set of algebraic equations. Specifically, the equation at the ith iteration has the form:

$$\left[\mathbf{N}^{*(i)} + \lambda^{(i)}\mathbf{I}\right] \boldsymbol{\Delta}\mathbf{d}^{*(i)} = \mathbf{g}^{*(i)} \tag{4.13}$$

where

$$g_k{}^{*(i)} = \frac{g_k{}^{(i)}}{\sqrt{N_{kk}{}^{(i)}}}, \qquad N_{kl}{}^{*(i)} = \frac{N_{kl}{}^{(i)}}{\sqrt{N_{kk}{}^{(i)} N_{ll}{}^{(i)}}}$$

$$\Delta d_k{}^{*(i)} = \Delta d_k \sqrt{N_{kk}{}^{(i)}} \tag{4.14}$$

and for the single-degree-of-freedom system:

$$\begin{aligned} g_k{}^{(i)} &= 2\int_0^T \left(\frac{\partial u_a{}^{(i)}}{\partial d_k}\right)(u_a{}^{(i)} - u_m) dt \\ N_{kl}{}^{(i)} &= 2\int_0^T \left(\frac{\partial u_a{}^{(i)}}{\partial d_k}\right)\left(\frac{\partial u_a{}^{(i)}}{\partial d_l}\right) dt \end{aligned} \tag{4.15}$$

and for the two-degree-of-freedom system:

$$\begin{aligned} g_k{}^{(i)} &= 2\int_0^T \left\{ \left(\frac{\partial u_{1a}{}^{(i)}}{\partial d_k}\right)(u_{1a}{}^{(i)} - u_{1m}) + \left(\frac{\partial u_{2a}{}^{(i)}}{\partial d_k}\right)(u_{2a}{}^{(i)} - u_{2m})\right\} dt \\ N_{kl}{}^{(i)} &= 2\int_0^T \left[\left(\frac{\partial u_{1a}{}^{(i)}}{\partial d_k}\right)\left(\frac{\partial u_{1a}{}^{(i)}}{\partial d_l}\right) + \left(\frac{\partial u_{2a}{}^{(i)}}{\partial d_k}\right)\left(\frac{\partial u_{2a}{}^{(i)}}{\partial d_l}\right)\right] dt \end{aligned} \tag{4.16}$$

where λ is a scaling factor, chosen to increase the size of the correction vector components if the objective function value has been found to decrease in the preceding step. The value of λ is set as 0.01 initially and changed by a factor of ten during successive iterations according to the objective function values. To obtain $\lambda^{(i+1)}$, $\lambda^{(i)}$ is multiplied by ten if the objective function has increased and divided by ten if the objective function has decreased. After the system is solved for $\mathbf{\Delta d}^*$,

$$\Delta d_k = \frac{\Delta {d_k}^{*(i)}}{\sqrt{{N_{kk}}^{(i)}}} \tag{4.17}$$

each component of the scaled version of the correction vector has to be scaled back using Eq. (4.17).

For one of the examples studied here, the conjugate gradient method [73, 74], was also applied to minimize the objective function.

4.3.4 Objective Function Minimization by Conjugate Gradient Method

The conjugate gradient method [73, 74], combining the advantages of both first- and second-order gradient algorithms, is attractive to try to associate conjugacy properties with the steepest descent method in an attempt to achieve both efficiency and reliability. Basically the method is built around two key ideas. First, a sequence of directions, $s_1, s_2, \ldots, s_n$ is generated which has the orthogonality or conjugacy property with respect to L_{dd} $(= \partial^2 L/\partial \mathbf{d}^2)$; that is,

$$\left[s_i^t L_{dd} s_j\right] = 0, \quad i \neq j \tag{4.18}$$

Second, a sequence of one-dimensional searches is made along each of the conjugate directions to find the optimum in that direction.

$$\mathbf{s}^{(i+1)} = -\mathbf{r}^{(i+1)} + \alpha^{(i)}\mathbf{s}^{(i)} \tag{4.19}$$

where

$$\begin{aligned} \mathbf{r}^{(i+1)} &= \mathbf{r}(\mathbf{p}^{(i+1)}) \\ \alpha^{(i)} &= \frac{\mathbf{r}^{(i+1)t}\mathbf{r}^{(i+1)}}{\mathbf{r}^{(i)t}\mathbf{r}^{(i)}} \end{aligned} \tag{4.20}$$

Initially, the algorithm behaves like a first-order gradient method but, as iteration proceeds, it behaves more like a second-order gradient method. In this paper, the minimization technique developed by Fletcher and Reeves [74] was used for the calculation.

4.4 NUMERICAL RESULTS

The performance of the proposed time finite element method (TFM) based approach was evaluated by identifying parameters for a number of single and two-degree-of-freedom systems. The parameters were identified from the impulse response (to simulate the measured response) of the example systems, obtained by using the TFM to integrate the nonlinear system equations with given system parameters. For computational ease, the impulse excitation was simulated by imposing initial velocity conditions. For all the examples studied, all the data points were used to determine the system parameters. However, for one example, results were also obtained using fewer (50) data points than those available (100 and 200). This (the use of two different meshes, a finer one for response generation and a coarser one for parameter identification) is often done in inverse problems to simulate the fact that the results from a finer mesh will be more representative (will contain higher-frequency components) of the physical system and using fewer data points for identification one can evaluate the robustness of the algorithm.

To simulate noise measurement, random noises with different r.m.s. (5%, 10% and 20%) values were generated and added to the simulated response. The corresponding simulated corrupted data $\overline{u}_m$ [75] is given by

$$\overline{u}_m = u_m(1 + r) \tag{4.21}$$

where r is a uniformly distributed random number, generated by IMSL subroutine DRNUN, that is scaled and shifted to a range of $(-\alpha, \alpha)$ by using subroutines DSCAL and DADD. In this study, α was chosen as 0.05 and 0.1 for 10% and 20% noises, respectively. All computations were performed on IBM 3090-300E mainframe computer.

4.4.1 System with Nonlinear Damping and Cubic Nonlinearity

Consider a nonlinear single-degree-of-freedom [75] system given by

$$\begin{gathered} \ddot{u}(t) + a_1 u(t) + a_2 u(t)^3 + a_3 \dot{u}(t) + a_4 \dot{u}(t)^3 = 0, \ 0 < t \leq 5 \\ u(0^+) = 0.0 \qquad \dot{u}(0^+) = 5.0 \end{gathered} \tag{4.22}$$

with the following given values of the system parameters:

$$a_1 = 25.0, \qquad a_2 = 2.5, \qquad a_3 = 1.0, \qquad a_4 = 0.1 \tag{4.23}$$

Parameters to be identified are: a_1, a_2, a_3, a_4 and a_5, the initial velocity $\dot{u}(0^+)$. In the simulation, the record length of 5 seconds was divided into 25 and 50 data points. Initial values of the parameters were chosen arbitrary, $a_1 = 22.5$, $a_2 = 1.0$, $a_3 = 0.5$, $a_4 = 0.3$ and $a_5 = 3.5$.

Table 4.1 presents the results of the identified system using 25 points and 50 points of the simulated response. The results using noise-free measured data converged to the exact (given) values of parameters in both cases. The results have been rounded to two decimal points. For data with 20% noise, parameter a_2 is 50.8% in error when 25 data points were used. The error, however, decreases dramatically (to 4.8%) when 50 data points were used. Figure 4.1 shows identified response and the simulated uncorrupted data. Figures 4.2-4.3 present results obtained using 50 data points and simulated data corrupted with 10% and 20% noise respectively. Figures 4.4-4.5 show the identified response sensitivities with respect to parameters a_2 and a_4, respectively. Note that the convergence rate of the parameter a_4 is much faster than that of the parameter a_2. This may be due to the fact that the transient response is significantly more sensitive to a_4 (see Fig. 4.5) than to a_2 (see Fig. 4.4).

4.4.2 System with combined Quadratic and Cubic Nonlinearities

As a second example, a single-degree-of-freedom system with combined quadratic and cubic nonlinearity and with viscous damping [60] was considered

$$\begin{gathered} \ddot{u}(t) + a_1\dot{u}(t) + a_2u(t) + a_3u(t)^2 + a_4u(t)^3 = 0, \;\; 0 < t \leq 5 \\ u(0^+) = 0.0 \qquad \dot{u}(0^+) = 5.0 \end{gathered} \tag{4.24}$$

for the following two cases of numerical values of the system parameters:

Case (1)

$$a_1 = 1.0, \qquad a_2 = 25.0, \qquad a_3 = 0.1, \qquad a_4 = 0.5 \tag{4.25}$$

Case (2)

$$a_1 = 1.0, \qquad a_2 = 25.0, \qquad a_3 = 2.5, \qquad a_4 = 5.0 \tag{4.26}$$

For Case (1), initial values were chosen arbitrarily as, $a_1 = 0.7$, $a_2 = 22.5$, $a_3 = 0.5$, $a_4 = 1.0$ and $a_5 = 3.5$, and for Case (2), the same were chosen as $a_1 = 0.7$, $a_2 = 22.5$, $a_3 = 1.0$, $a_4 = 2.5$ and $a_5 = 3.5$.

Tables 4.2-4.3 present the numerical results for both cases as compared with those given by Normann and Kapania [60]. Record length was taken to be 5 seconds and was divided into 25 and 50 data points. In Case (1), the identified parameter a_4 shows the worst accuracy when 25 data points of the 20% simulated corrupted data were used. The accuracy for a_4 improved by 70% when 50 data points were taken. The results for Case (2) show a similar trend. For Case (1), Figs. 4.6-4.8 present, respectively, the identified responses and the simulated data corrupted with 0 %, 10% and 20% r.m.s. values. Figures 4.9-4.10 show the sensitivities of the identified response with respect to system parameters a_1 and a_3, respectively. It was observed that the parameter a_3 converges much slower than the other parameters. This is, again, due to the fact that the response is relatively insensitive to a_3 as compared to other parameters (see Figs. 4.9 and 4.10). Note that Normann and Kapania used the record length as 5 seconds and 335 data points for representing the measured data.

For comparison purposes, the parameters for this example (Case (2)) were also identified using a conjugate gradient method. Table 4.4a presents results from this comparison using 50 data points. These 50 points were selected from population sizes of 50, 100 and 200 data points. The results show that Levenberg-Marquardt method is more accurate than the conjugate gradient method in this case. For noise-free case, the results were not influenced by the population size. But the results for 10% and 20 % simulated corrupted data were. The inaccuracy of identified parameters for nonlinear terms was increased with an increase in the population size. Table 4.4b presents the results using 100 data points in the simulation. The 100 data points in the simulation were chosen from population sizes of 100 and 200 data points. The results show a similar trend as the 50 data points case.

4.4.3 System with Linear and Quadratic Damping

Consider a single-degree-of-freedom system with linear and quadratic damping [60]:

$$\begin{gathered} \ddot{u}(t) + a_1\dot{u}(t) + a_2 u(t) + a_3\dot{u}(t)\,|\dot{u}(t)| = 0,\ \ 0 < t \le 5 \\ u(0^+) = 0.0 \qquad \dot{u}(0^+) = 5.0 \end{gathered} \tag{4.27}$$

for the following two cases of numerical values of the system parameters:

Case (1)

$$a_1 = 1.0, \qquad a_2 = 25.0, \qquad a_3 = 0.5 \tag{4.28}$$

Case (2)

$$a_1 = 1.0, \qquad a_2 = 25.0, \qquad a_3 = 2.5 \tag{4.29}$$

As initial values, $a_1 = 0.7$, $a_2 = 22.5$, $a_3 = 1.0$ and $a_4 = 3.5$ were chosen for Case (1), and $a_1 = 0.7$, $a_2 = 22.5$, $a_3 = 1.5$ and $a_4 = 3.5$ for Case (2).

Tables 4.5-4.6 present the numerical results for both cases as compared to those given by Normann and Kapania. For Case (1), 25 and 50 data points were used and for Case (2), 50 and 100 data points were used. Also we used Integrated Legendre polynomials as basis functions which gave a better conditioned system of equations. For Case (1), the present approach using 50 points gives results which are more accurate than those given by Normann and Kapania. In Case (2), the results using 100 data points show good accuracy. Figures 4.11-4.12 present respectively the identified responses and the simulated data corrupted with 10% and 20% noise respectively for Case (2).

4.4.4 Two-Degree-of-Freedom System having Cubic Nonlinearities

Consider a two-degree-of-freedom system having cubic nonlinearities [69, 76]:

$$\begin{aligned} \ddot{u}_1 &= -{\omega_1}^2 u_1 - 2\mu_1 \dot{u}_1 - \alpha_1 {u_1}^3 - \alpha_2 {u_1}^2 u_2 - \alpha_3 u_1 {u_2}^2 - \alpha_4 {u_2}^3 \\ \ddot{u}_2 &= -{\omega_2}^2 u_2 - 2\mu_2 \dot{u}_2 - \alpha_5 {u_1}^3 - \alpha_6 {u_1}^2 u_2 - \alpha_7 u_1 {u_2}^2 - \alpha_8 {u_2}^3 \end{aligned} \tag{4.30}$$

with initial conditions:

$$u_1(0) = 1.5, \qquad \dot{u}_1(0) = 0.0, \qquad u_2(0) = -1.0, \qquad \dot{u}_2(0) = 0.0 \tag{4.31}$$

for the following values of system parameters:

$$\begin{aligned} \omega_1^2 &= 25.0, \quad \mu_1 = 0.35, \quad \alpha_1 = 5.0, \quad \alpha_2 = 0.5, \quad \alpha_3 = 0.25, \quad \alpha_4 = 3.0 \\ \omega_2^2 &= 17.0, \quad \mu_2 = 0.25, \quad \alpha_5 = 2.5, \quad \alpha_6 = 0.75, \quad \alpha_7 = 0.2, \quad \alpha_8 = 5.0 \end{aligned} \tag{4.32}$$

This type of systems are associated with many physical systems such as the vibration of strings, beams and plates.

One-Step Identification Procedure: The domain ($0 < t \leq 10$) is divided into 100 elements of equal time steps. Legendre polynomials of the second degree are used as basis functions for the time finite element method. For simplicity, initial velocities have not been treated as unknown parameters. The parameters to be identified are $\{\omega_1^2, \mu_1, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \omega_2^2, \mu_2, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$. The initial trial values of system parameters were chosen as

$$\begin{aligned} \omega_1^2 &= 20.0, \quad \mu_1 = 0.50, \quad \alpha_1 = 3.0, \quad \alpha_2 = 0.20, \quad \alpha_3 = 0.15, \quad \alpha_4 = 4.0 \\ \omega_2^2 &= 14.0, \quad \mu_2 = 0.55, \quad \alpha_5 = 4.5, \quad \alpha_6 = 0.35, \quad \alpha_7 = 0.35, \quad \alpha_8 = 2.0 \end{aligned} \tag{4.33}$$

The results, Table 4.7, show that the method after 1000 iterations did not converge to any values, the parameters for nonlinear terms, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$, were particularly unreasonable.

Two-Step Identification Procedure: A two step procedure was then adopted for the direct iterative method. First the parameters for the linear terms, $\{\omega_1^2, \mu_1, \omega_2^2, \mu_2\}$, were identified, then those for the nonlinear terms. For linear terms, the simulated data were generated using very small initial displacements ($u_1(0) = 0.05, u_2(0) = -0.03$). Since the original system, Eq. (4.33), behaves almost linearly when subjected to very small initial conditions, identification was performed assuming the system to be linear:

$$\begin{aligned} \ddot{u}_1 &= -{\omega_1}^2 u_1 - 2\mu_1 \dot{u}_1 \qquad u_1(0) = 0.05, \quad \dot{u}_1(0) = 0.0 \\ \ddot{u}_2 &= -{\omega_2}^2 u_2 - 2\mu_2 \dot{u}_2 \qquad u_2(0) = -0.03, \quad \dot{u}_2(0) = 0.0 \end{aligned} \tag{4.34}$$

The simulated data, however, was obtained by integrating the actual nonlinear system. Next, the parameters for nonlinear terms were identified by keeping the parameters corresponding to linear terms $\{{\omega_1}^2, \mu_1, {\omega_2}^2, \mu_2\}$ as fixed. Identification was performed on the following system.

$$\begin{aligned} \ddot{u}_1 &= -{\omega_1}^2 u_1 - 2\mu_1 \dot{u}_1 - \alpha_1 {u_1}^3 - \alpha_2 {u_1}^2 u_2 - \alpha_3 u_1 {u_2}^2 - \alpha_4 {u_2}^3 \\ \ddot{u}_2 &= -{\omega_2}^2 u_2 - 2\mu_2 \dot{u}_2 - \alpha_5 {u_1}^3 - \alpha_6 {u_1}^2 u_2 - \alpha_7 u_1 {u_2}^2 - \alpha_8 {u_2}^3 \end{aligned} \tag{4.35}$$

with initial conditions:

$$u_1(0) = 1.5, \qquad \dot{u}_1(0) = 0.0, \qquad u_2(0) = -1.0, \qquad \dot{u}_2(0) = 0.0 \tag{4.36}$$

Table 4.8 presents the numerical results of the identified system using 100 points of the simulated data. The results using noise-free simulated data easily converged to the given values of system parameters. Also results using data corrupted by 5%, 10% and 20% random noise show reasonably accurate values of system parameters. Figures 4.13-4.14 present respectively the identified responses for u_1 and u_2 with the simulated data corrupted by 20% random noise.

Table 4.1 Numerical results for the system with nonlinear damping and cubic nonlinearity

Parameters	a_1	a_2	a_3	a_4	a_5
Initial values	**22.5**	**1.0**	**0.5**	**0.3**	**3.5**
25 data points					
Identified (0%)	25.00	2.50	1.00	0.10	5.00
Identified (10%)	24.87	3.17	1.00	0.10	5.09
Identified (20%)	24.74	3.77	1.00	0.11	5.21
55 data points					
Identified (0%)	25.00	2.50	1.00	0.10	5.00
Identified (10%)	24.98	2.57	1.00	0.10	5.01
Identified (20%)	24.96	2.62	1.00	0.10	5.03
Exact values	**25.0**	**2.5**	**1.0**	**0.1**	**5.0**

Table 4.2 Numerical results for the system with combined quadratic and cubic nonlinearities, Case (1)

Parameters	a_1	a_2	a_3	a_4	a_5
Initial values	**0.7**	**22.5**	**0.5**	**1.0**	**3.5**
25 data points					
Identified (0%)	1.00	25.00	0.10	0.50	5.00
Identified (10%)	1.01	24.87	0.25	0.90	5.04
Identified (20%)	1.03	24.74	0.41	1.29	5.08
50 data points					
Identified (0%)	1.00	25.0	0.10	0.50	5.00
Identified (10%)	1.00	24.99	0.22	0.47	5.00
Identified (20%)	1.00	24.97	0.34	0.45	5.00
335 data points[a]					
Identified (0%)	1.00	25.0	0.10	0.50	5.00
Identified (10%)	0.98	25.05	0.12	0.31	4.97
Identified (20%)	0.98	25.10	0.13	0.12	4.94
Exact values	**1.0**	**25.0**	**0.1**	**0.5**	**5.0**

[a]Results from Normann and Kapania, 1990

Table 4.3 Numerical results for the system with combined quadratic and cubic nonlinearities, Case (2)

Parameters	a_1	a_2	a_3	a_4	a_5
Initial values	**0.7**	**22.5**	**1.0**	**2.5**	**3.5**
25 data points					
Identified (0%)	1.00	25.00	2.50	5.00	5.00
Identified (10%)	0.99	24.99	2.32	5.28	4.91
Identified (20%)	0.97	24.96	2.11	5.61	4.81
50 data points					
Identified (0%)	1.00	25.00	2.50	5.00	5.00
Identified (10%)	1.00	25.01	2.61	4.92	5.00
Identified (20%)	1.01	25.03	2.73	4.80	4.99
335 data points[a]					
Identified (0%)	1.00	25.00	2.50	5.00	5.00
Identified (10%)	0.98	25.04	2.56	4.79	4.96
Identified (20%)	0.98	25.10	2.62	4.58	4.93
Exact values	**1.0**	**25.0**	**2.5**	**5.0**	**5.0**

[a]Results from Normann and Kapania, 1990

Table 4.4a Comparison of numerical results obtained from Levenberg-Marquardt and Conjugate Gradient methods for the system with combined quadratic and cubic nonlinearities, Case (2), 50 data points

Noise								
	0%			10%			20%	
50 data points selected from population sizes of:								
50	100	200	50	100	200	50	100	200
Levenberg-Marquardt Method								
1.00	1.00	1.00	1.00	0.99	1.00	1.00	0.98	1.00
25.00	25.00	25.00	24.96	24.98	25.01	24.92	24.96	25.02
2.50	2.50	2.50	2.56	2.17	2.96	2.61	1.82	3.42
5.00	5.00	5.00	5.06	5.23	4.90	5.12	5.48	4.80
5.00	5.00	5.00	5.00	4.94	5.02	5.01	4.88	5.05
Conjugate Gradient Method								
1.00	1.00	1.00	1.00	0.99	1.00	1.00	0.98	1.01
25.00	25.00	25.00	24.95	24.97	25.00	24.92	24.95	25.01
2.50	2.53	2.53	2.60	2.19	3.01	2.67	1.82	3.47
5.00	5.28	5.28	5.34	5.53	5.16	5.34	5.79	5.06
5.00	5.01	5.01	5.01	4.95	5.03	5.01	4.89	5.05

Table 4.4b Comparison of numerical results obtained from Levenberg-Marquardt and Conjugate Gradient methods for the system with combined quadratic and cubic nonlinearities, Case (2), 100 data points

Noise					
0%		10%		20%	
100 data points selected from population sizes of:					
100	200	100	200	100	200
Levenberg-Marquardt Method					
1.00	1.00	0.99	0.99	0.99	0.99
25.00	25.00	25.00	24.99	25.01	24.99
2.50	2.50	2.18	2.89	1.85	3.28
5.00	5.00	5.03	4.92	5.07	4.84
5.00	5.00	4.97	4.99	4.94	4.98
Conjugate Gradient Method					
1.00	1.00	0.99	0.99	0.99	0.99
25.00	25.00	25.00	24.99	25.01	24.99
2.51	2.50	2.18	2.90	1.85	3.30
5.07	5.07	5.10	4.98	5.15	4.90
5.00	5.00	4.97	4.99	4.94	4.98

Table 4.5 Numerical results for the system with quadratic damping, Case (1)

Parameters	a_1	a_2	a_3	a_4
Initial values	**0.7**	**22.5**	**0.5**	**3.5**
25 data points				
Identified (0%)	1.00	25.00	0.50	5.00
Identified (10%)	0.99	25.04	0.51	5.04
Identified (20%)	0.97	25.09	0.53	5.08
50 data points				
Identified (0%)	1.00	25.00	0.50	5.00
Identified (10%)	1.00	25.00	0.50	5.00
Identified (20%)	1.00	25.01	0.50	5.00
335 data points[a]				
Identified (0%)	1.00	25.00	0.50	5.00
Identified (10%)	1.00	24.99	0.48	4.92
Identified (20%)	1.00	24.97	0.46	4.84
Exact	**1.00**	**25.00**	**0.50**	**5.0**

[a]Results from Normann and Kapania, 1990

Table 4.6 Numerical results for the system with quadratic damping, Case (2)

Parameters	a_1	a_2	a_3	a_4
Initial values	**0.7**	**22.5**	**1.5**	**3.5**
25 data points				
Identified (0%)	1.00	25.00	2.50	5.00
Identified (10%)	1.05	25.00	2.40	4.86
Identified (20%)	1.08	24.99	2.34	4.78
100 data points				
Identified (0%)	1.00	25.00	2.50	5.00
Identified (10%)	1.00	24.97	2.49	5.00
Identified (20%)	0.99	24.94	2.52	5.04
335 data points[a]				
Identified (0%)	1.00	25.00	2.50	5.00
Identified (10%)	0.96	24.96	2.49	4.92
Identified (20%)	0.94	24.92	2.49	4.84
Exact values	**1.00**	**25.00**	**2.50**	**5.0**

[a]Results from Normann and Kapania, 1990

Table 4.7 Numerical results for the two-degree-of-freedom system using one-step procedure and no noise

Parameters	Initial	Identified 0% noise	Exact
ω_1^2	**20.0**	26.07	**25.00**
μ_1	**0.5**	0.48	**0.35**
α_1	**3.0**	3.60	**5.00**
α_2	**0.2**	2.32	**0.50**
α_3	**0.15**	1.54	**0.25**
α_4	**4.0**	-0.96	**3.00**
ω_2^2	**14.0**	18.38	**17.00**
μ_2	**0.55**	0.31	**0.25**
α_5	**4.5**	1.85	**2.50**
α_6	**0.35**	-4.08	**0.75**
α_7	**0.35**	-3.95	**0.20**
α_8	**2.0**	3.63	**5.00**

Table 4.8 Numerical results for the two-degree-of-freedom system using two-step procedure

Parameters	Initial	Noise				Exact
		0%	5%	10%	20%	
ω_1^2	**20.0**	25.00(25.00)	25.00(24.58)	25.01(24.23)	25.01(23.72)	**25.00**
μ_1	**0.5**	0.35(0.35)	0.35(0.34)	0.35(0.34)	0.35(0.35)	**0.35**
α_1	**3.0**	5.00(5.00)	5.24(5.97)	5.49(6.76)	6.01(7.77)	**5.00**
α_2	**0.2**	0.49(0.50)	0.86(1.72)	1.25(2.72)	2.04(4.07)	**0.50**
α_3	**0.15**	0.25(0.25)	0.10(0.77)	-0.07(1.23)	-0.42(1.99)	**0.25**
α_4	**4.0**	3.00(3.00)	2.85(2.92)	2.68(2.79)	2.35(2.44)	**3.00**
ω_2^2	**14.0**	17.00(17.00)	17.00(17.11)	17.00(17.27)	16.99(17.75)	**17.00**
μ_2	**0.55**	0.25(0.25)	0.25(0.26)	0.25(0.26)	0.25(0.27)	**0.25**
α_5	**4.5**	2.50(2.50)	2.53(2.73)	2.56(2.93)	2.64(3.21)	**2.50**
α_6	**0.35**	0.75(0.75)	0.91(0.61)	1.10(0.34)	1.48(-0.56)	**0.75**
α_7	**0.35**	0.20(0.20)	0.32(-0.30)	0.45(-0.91)	0.69(-2.40)	**0.20**
α_8	**2.0**	5.00(5.00)	4.95(4.75)	4.89(4.40)	4.76(3.46)	**5.00**

( ); Results from Normann and Kapania, 1990, using 335 data points, unfiltered data, using same initial guess values

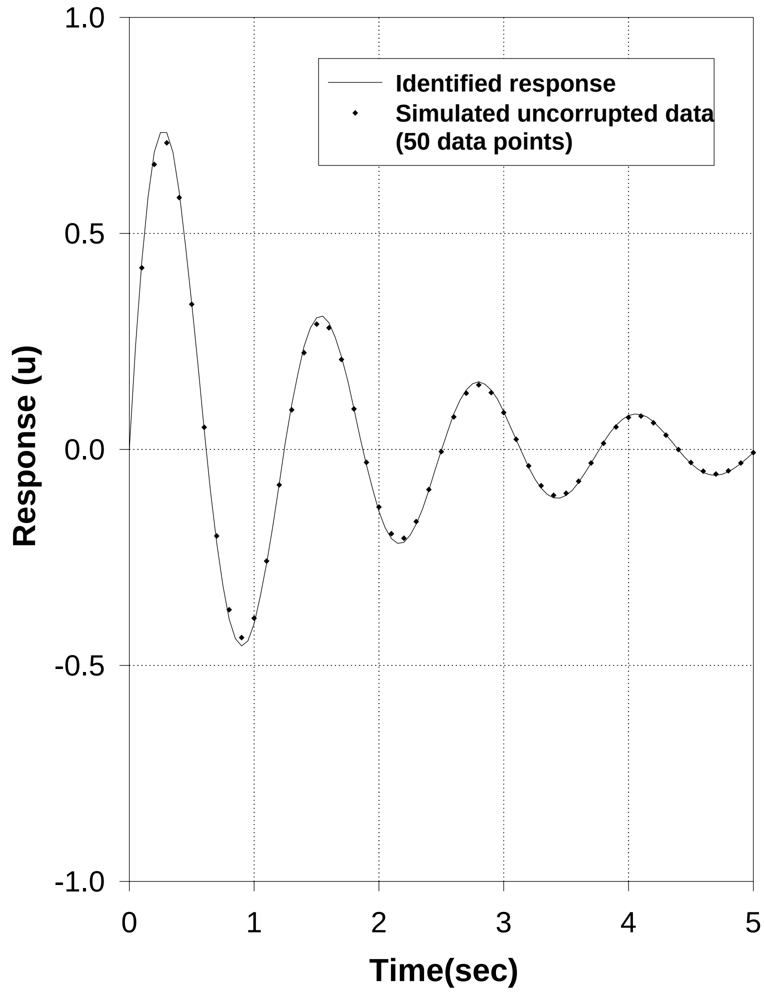

Fig. 4.1 Identified/Simulated response of the system with nonlinear damping and cubic nonlinearity.

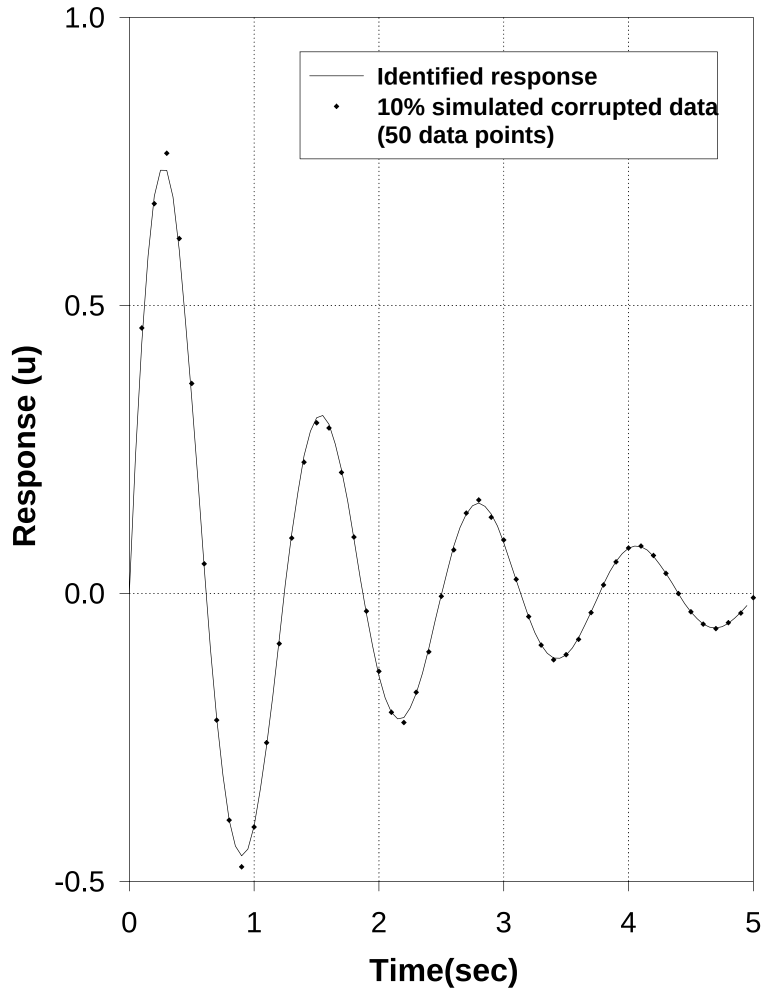

Fig. 4.2 Identified/Simulated response of the system with nonlinear damping and cubic nonlinearity.

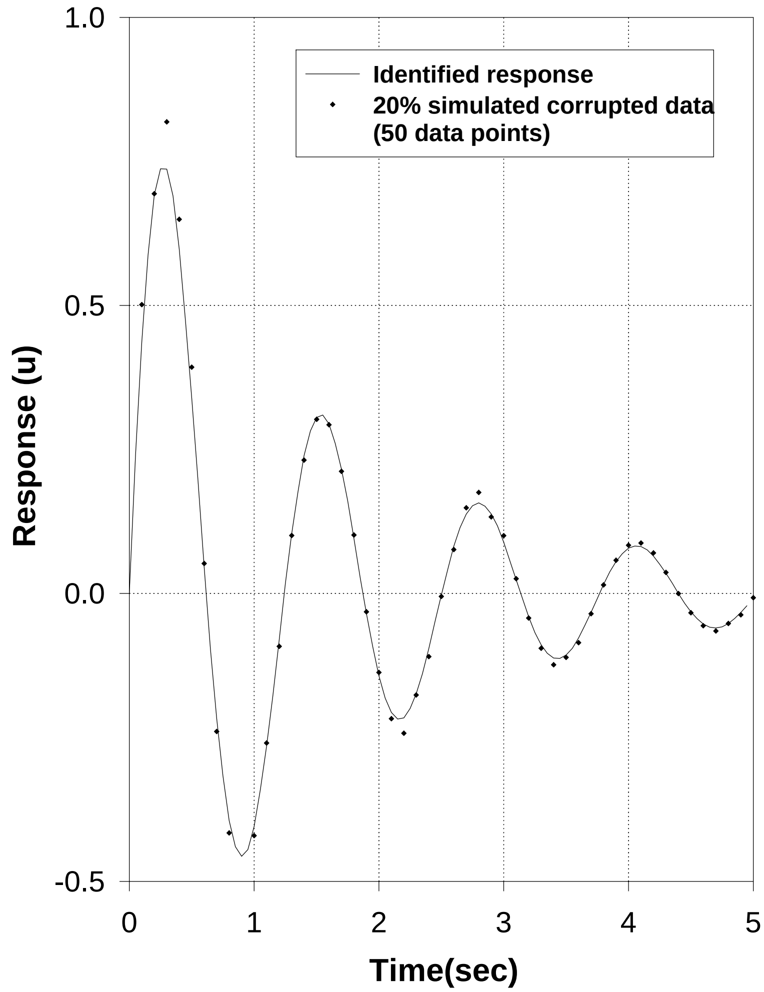

Fig. 4.3 Identified/Simulated response of the system with nonlinear damping and cubic nonlinearity.

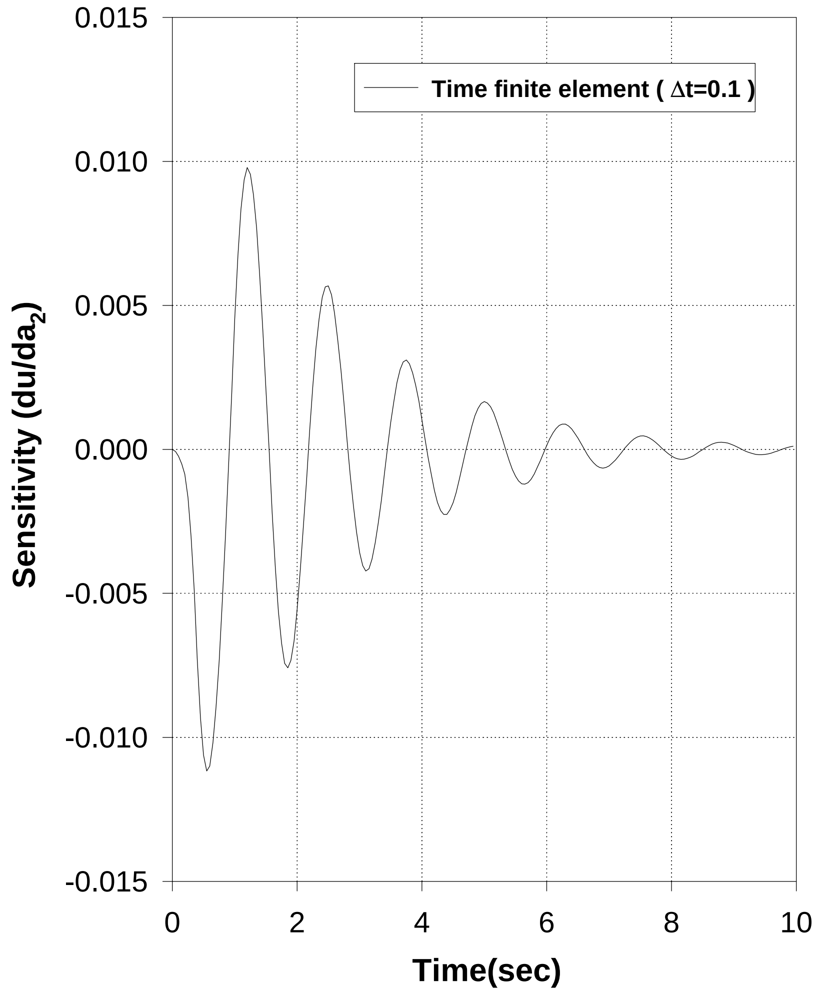

Fig. 4.4 Sensitivity of the identified response with respect to a_2 for the system having nonlinear damping and cubic nonlinearity.

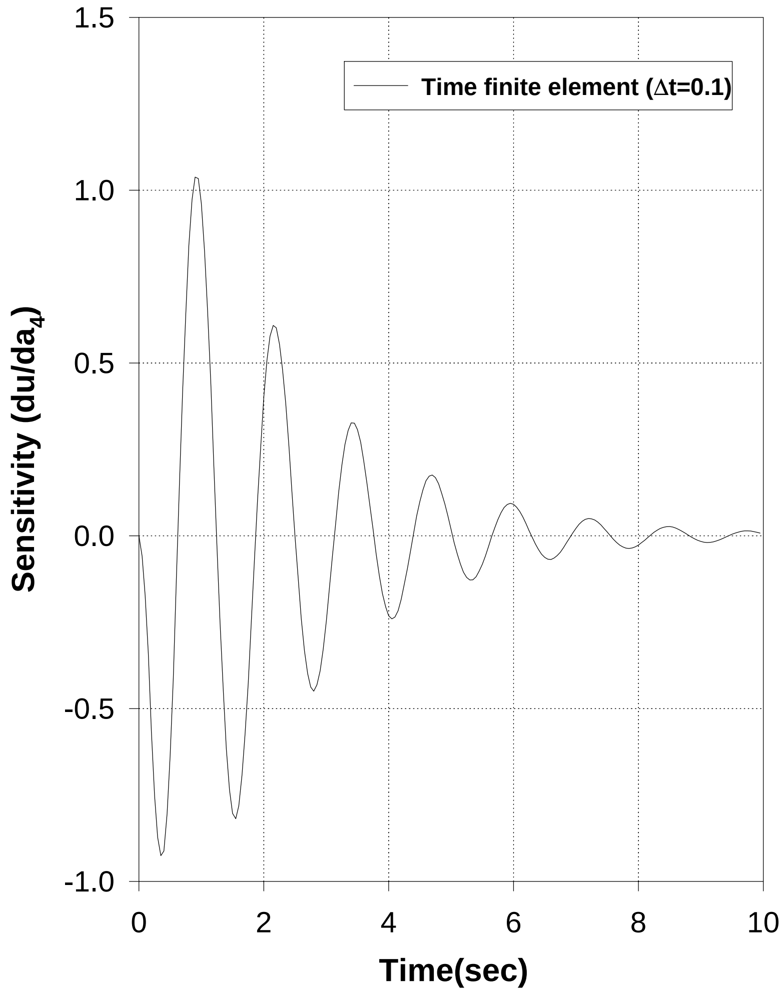

Fig. 4.5 Sensitivity of the identified response with respect to a_4 for the system having nonlinear damping and cubic nonlinearity.

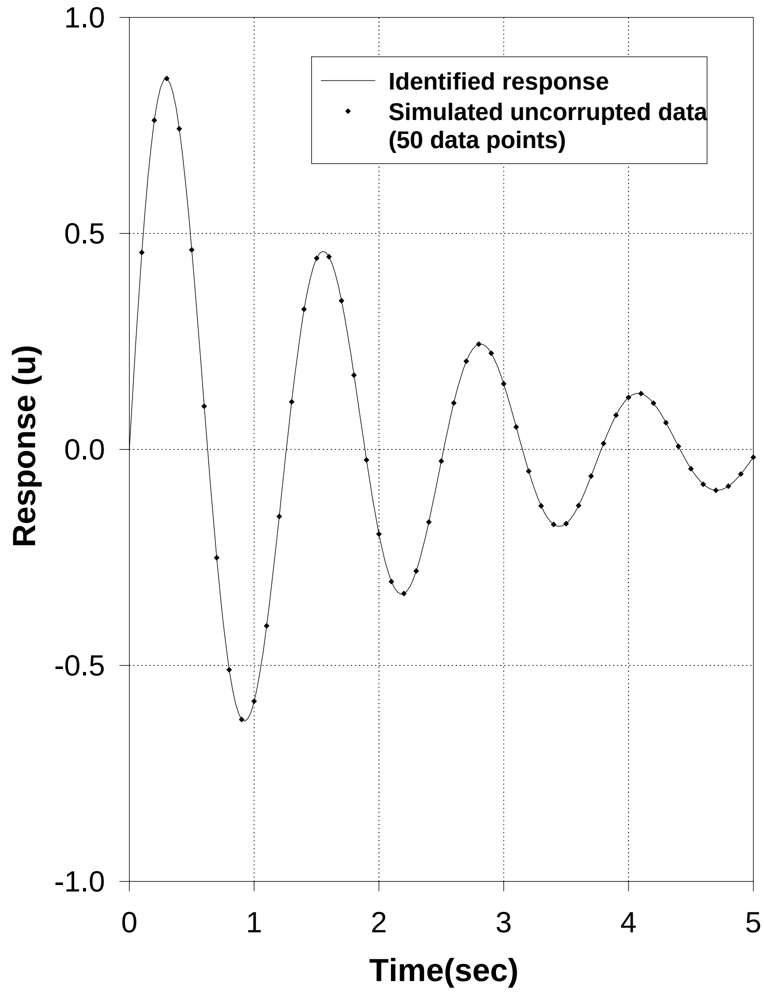

Fig. 4.6 Identified/Simulated response of the system with combined quadratic and cubic nonlinearity, Case (1).

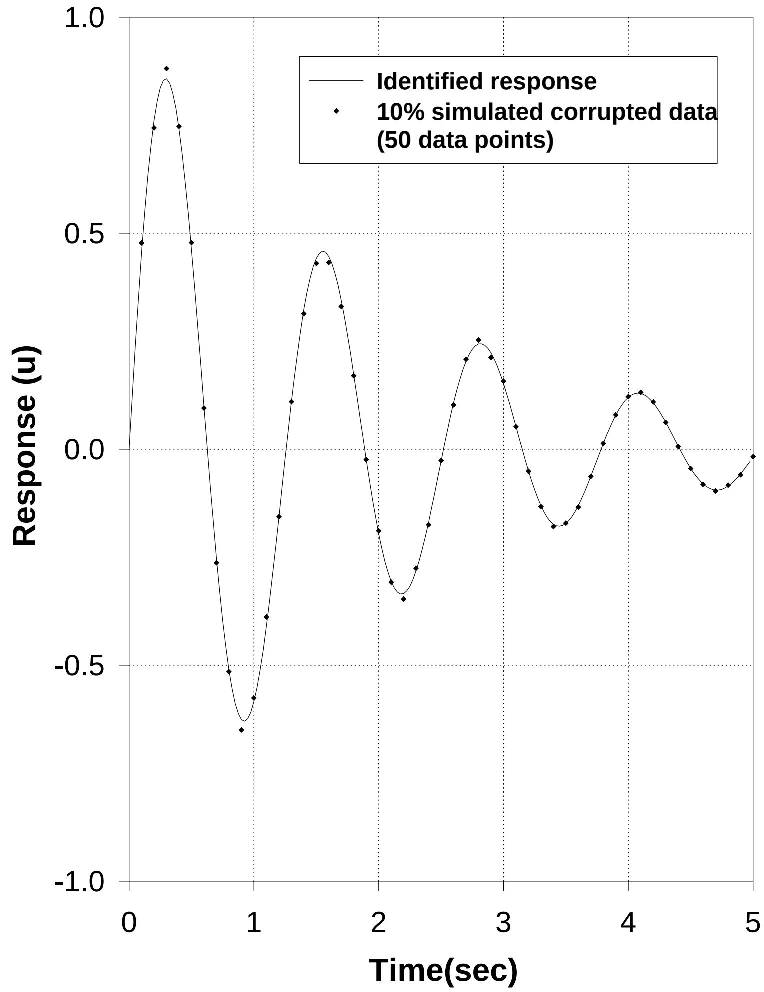

Fig. 4.7 Identified/Simulated response of the system with combined quadratic and cubic nonlinearity, Case (1).

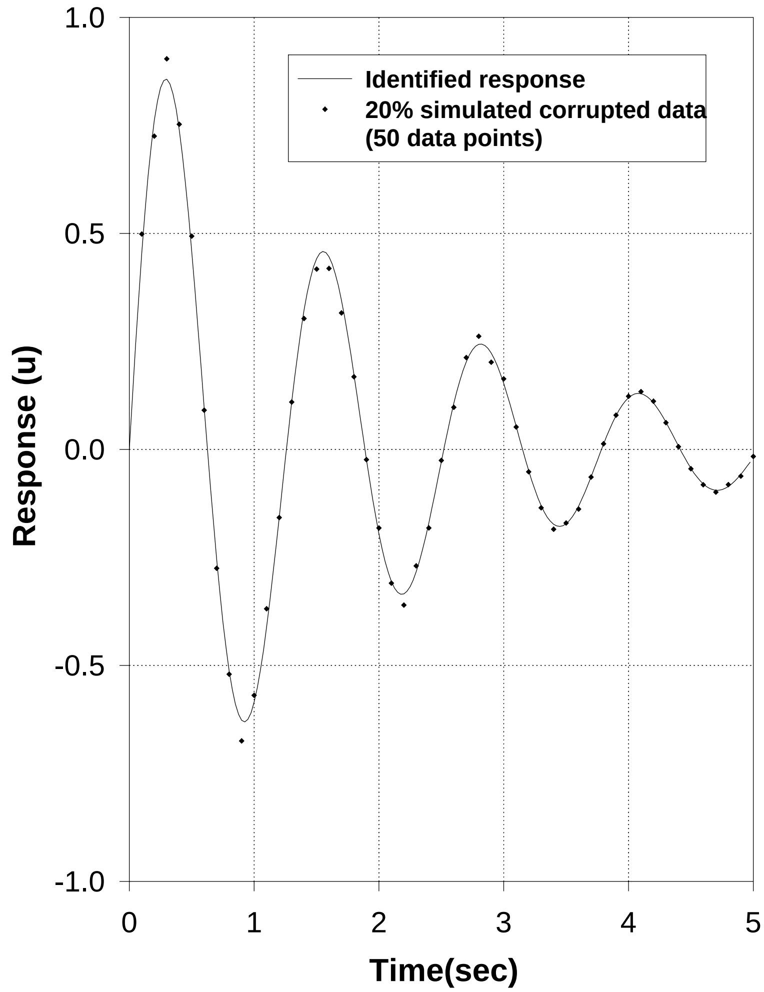

Fig. 4.8 Identified/Simulated response of the system with combined quadratic and cubic nonlinearity, Case (1).

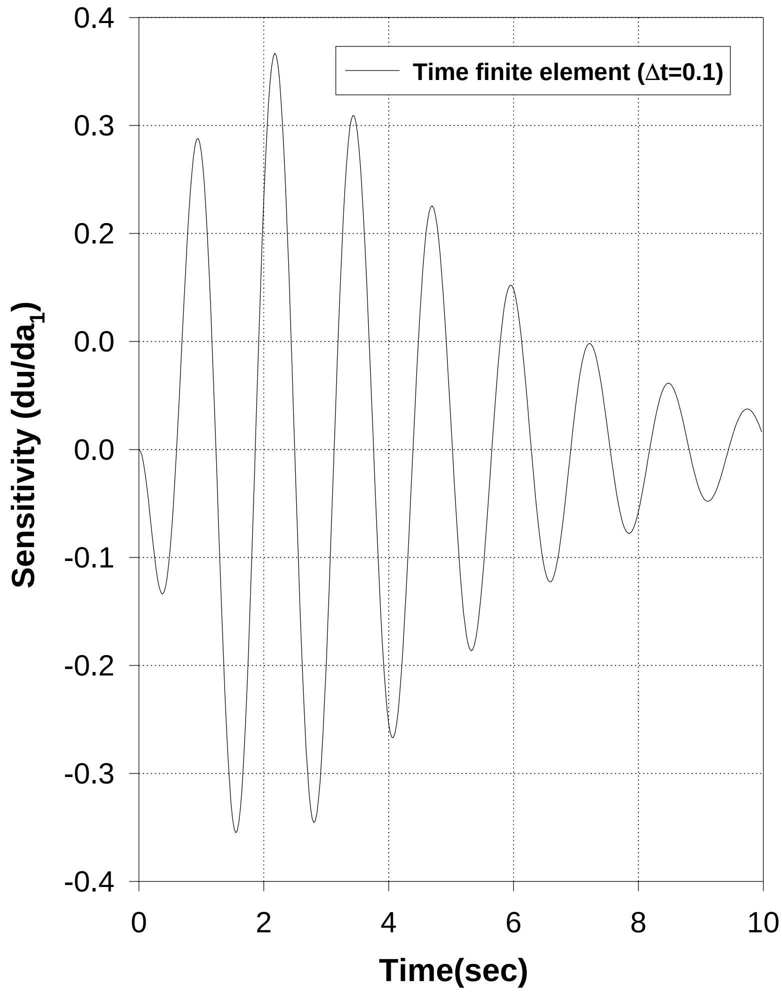

Fig. 4.9 Sensitivity of the identified response with respect to a_1 for the system having combined quadratic and cubic nonlinearity, Case (1).

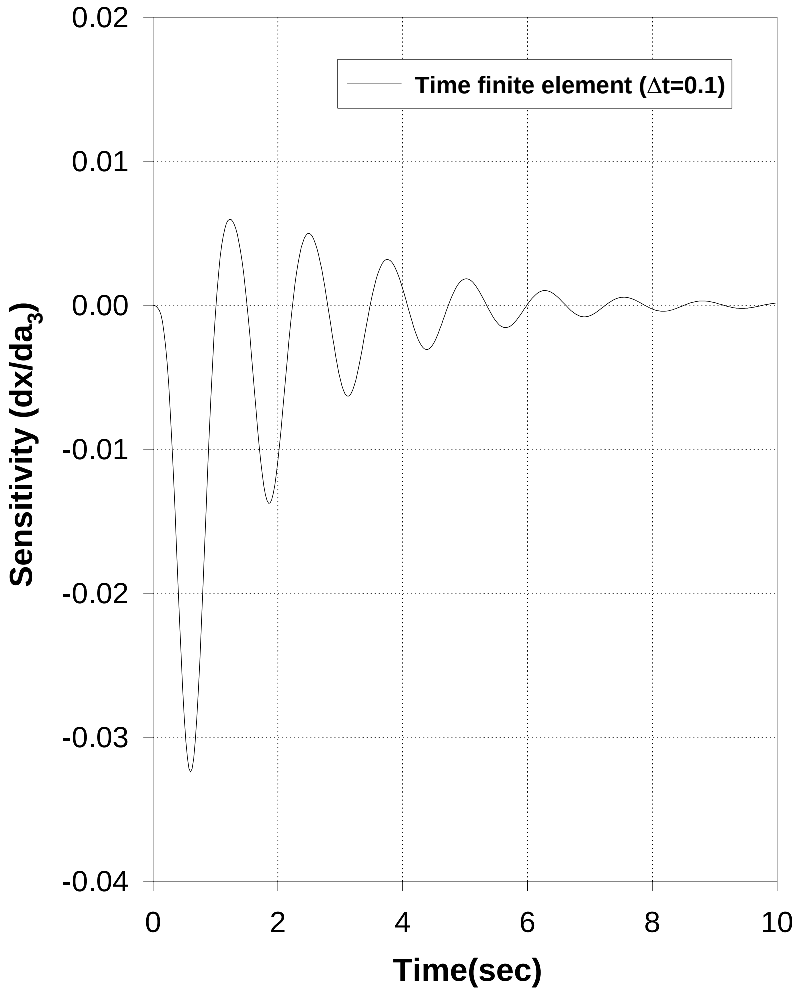

Fig. 4.10 Sensitivity of the identified response with respect to a_3 for the system having combined quadratic and cubic nonlinearity, Case (1).

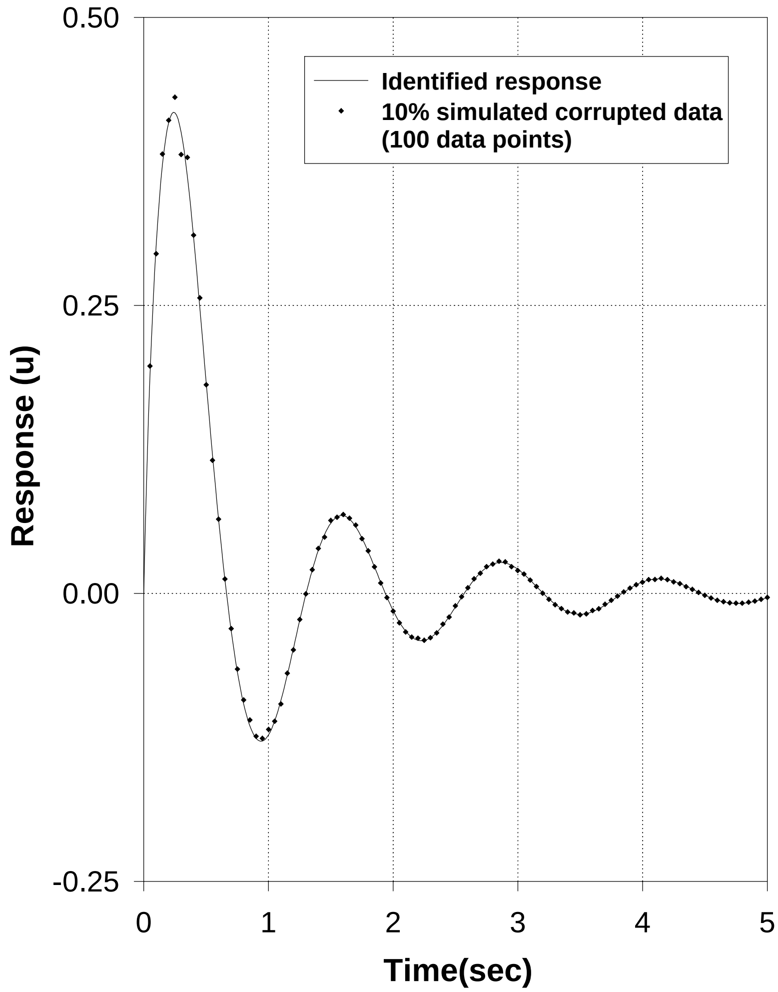

Fig. 4.11 Identified/Simulated response of the system with linear and quadratic damping, Case (2).

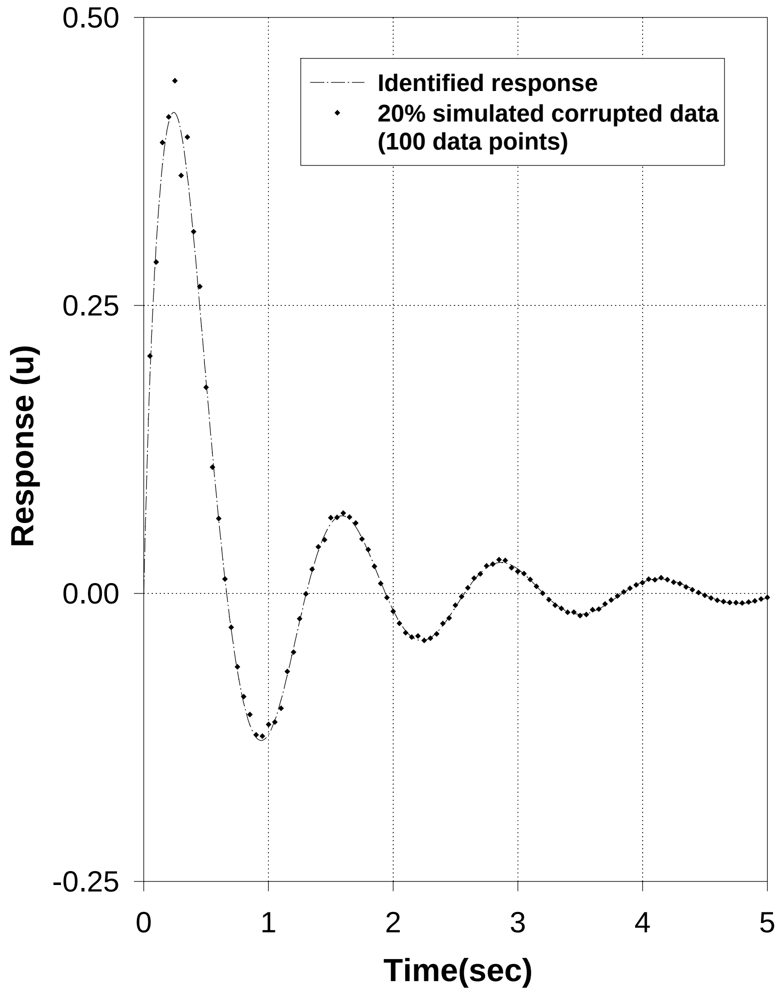

Fig. 4.12 Identified/Simulated response of the system with linear and quadratic damping, Case (2).

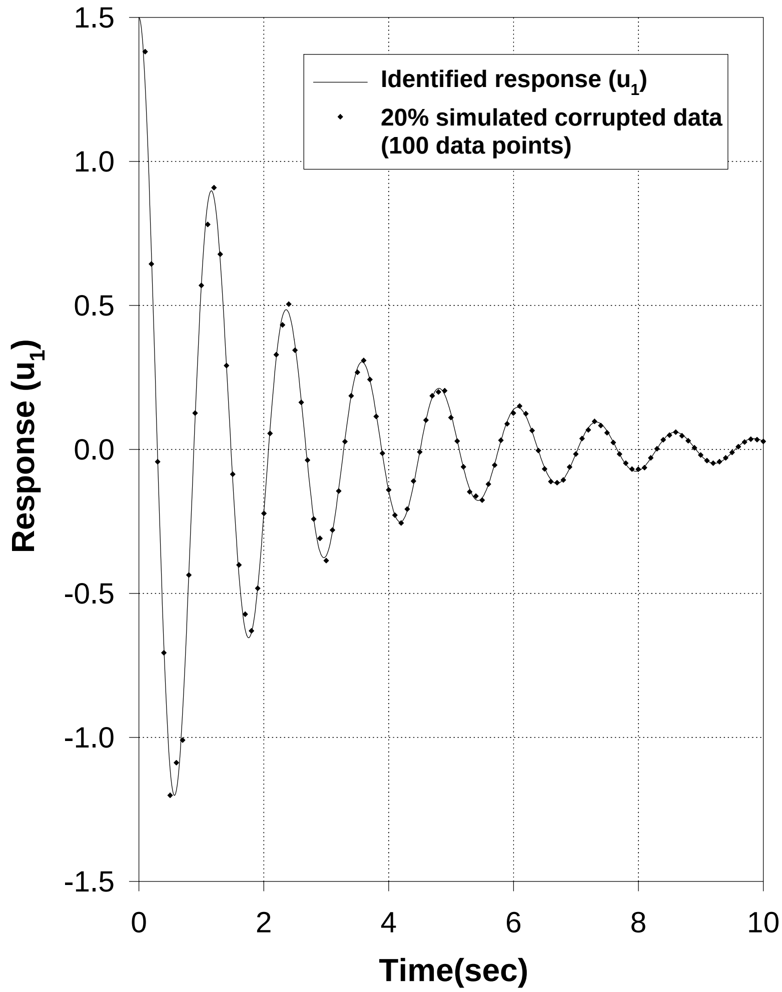

Fig. 4.13 Identified/Simulated response of two-degree-of-freedom system having cubic nonlinearities, u_1.

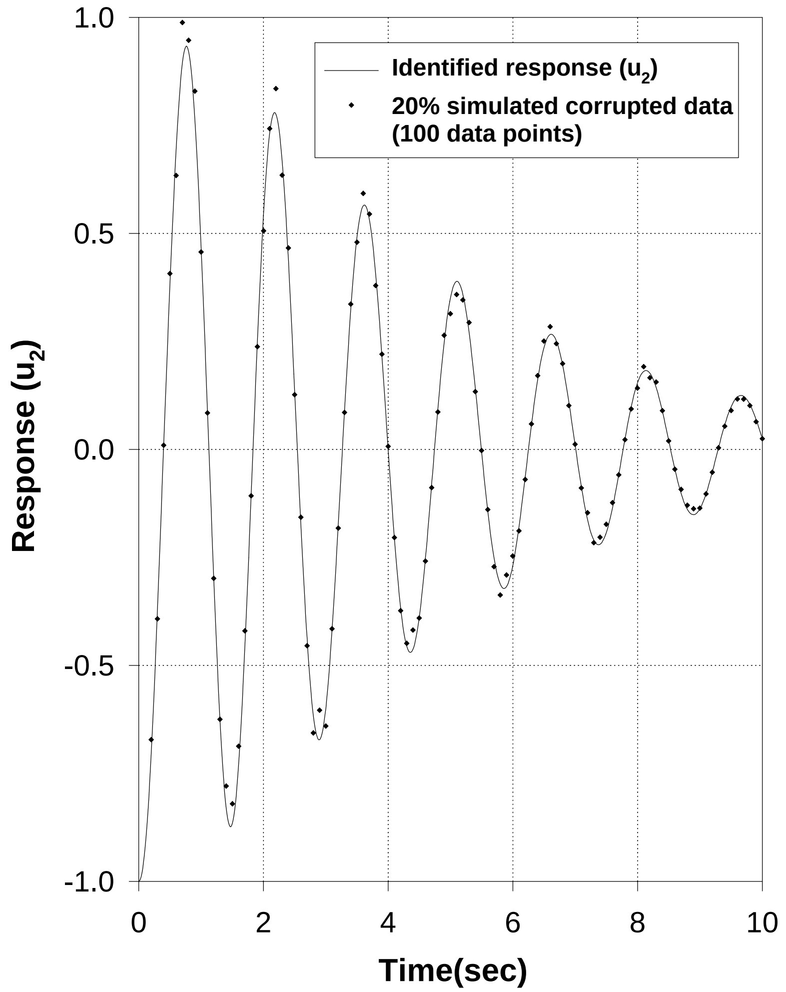

Fig. 4.14 Identified/Simulated response of two-degree-of-freedom system having cubic nonlinearities, u_2.

5. TIME FINITE ELEMENT FOR OPTIMAL CONTROL

5.1 Overview

Many dynamic systems - such as aerospace systems - are nonlinear and/or time-varying, and the techniques for analysis and design of linear, time-invariant systems are, in general, not applicable to these more complicated systems. In many instances, it became clear that a more systematic approach was desirable. This led to a renewed interest in the application of calculus of variations to study those systems. The weak variational formulation for optimal control problems by Hodges and Bless [77, 78, 79] is reviewed and possibility was checked for the development of the finite element method using higher order polynomial functions. To evaluate the proposed method, the approach is applied to two simple optimal problems. Solutions for state, costate and control are obtained using both one and multiple elements with various order of polynomials. The results show the accuracy is excellent with a fewer number of elements than those used by other researchers.

5.2 Weak Hamiltonian Formulation

5.2.1 Hamilton's Weak Principle

In recent years, the use of Hamilton's classical principle [23, 24, 25, 26, 37, 38] to obtain numerical solutions of initial and boundary value problems for mechanical systems has seen a renewed interest. The variational formulation, known as Hamilton's Weak Principle (HWP) [9], described the real motion at any time between T_0 and T_f.

$$\int_{T_0}^{T_f} \delta\mathbf{L}dt + \int_{T_0}^{T_f} \delta\mathbf{q}^t\mathbf{Q}dt = \delta\mathbf{q}^t\mathbf{p}\Big|_{T_0}^{T_f} \tag{5.1}$$

where $\mathbf{L}$ and $\mathbf{Q}$, respectively denote the Lagrangian of the system and the nonconservative generalized forces applied to the system, $\mathbf{p}$ is the generalized momenta, $\mathbf{q}$ is the generalized coordinates and T_0 and T_f are initial and final time respectively.

Equation (5.1) can be derived by combining D'Alembert's principle for the inertia force and the principle of virtual work for the static equilibrium of a system. The term "*weak*" is used since the given system differential equation is represented in an equivalent integral form of a functional with dependent variables. This formulation can be a basis for the TFM to develop consistent and efficient approximations for the determination of the response of mechanical systems. Research based on HWP is very active in the areas of periodic solutions for autonomous systems [99] and multibody dynamics [13, 14]. The following section explains the weak form by Hodges and Bless [77] based on the variation of the performance index for the optimal control.

5.2.2 Weak Formulation for Optimal Control

Consider a system defined over a time interval from T_0 to T_f by a set of n states x and a set of m controls u. Here x is piecewise smooth, u is piecewise continuous, and f is continuous on $R^n \times R^m \times R^1$. The states of the system are governed by a set of differential equations. Consider a performance index J of the form

$$J = \int_{T_0}^{T_f} L(x, u, t)dt + \phi(x(T_f), T_f) \tag{5.2}$$

subject to the system equations

$$\dot{x}(t) = f(x(t), u(t), t), \quad x(T_0) \text{ given}, \quad T_0 \leq t \leq T_f \tag{5.3}$$

with the terminal constraints

$$\psi[x(T_f), T_f] = 0 \tag{5.4}$$

where ψ are q vector functions of the state variables and time specified at unspecified terminal time.

To derive an optimality system for this problem, the terminal constraints $\psi(x, t)$ are adjoined to J with Lagrange multipliers ν as follows

$$J = \int_{T_0}^{T_f} L(x, u, t)dt + \Big[\phi(x, t) + \nu^t \psi(x, t)\Big]_{T_0}^{T_f} \tag{5.5}$$

The new performance index J adjoined by the system differential equations with Lagrange multiplier functions $\lambda(t)$ can be written as

$$J = \int_{T_0}^{T_f} \Big[L(x, u, t) + \lambda^t(t)(f - \dot{x})\Big]dt + \Phi\Big|_{T_0}^{T_f} \tag{5.6}$$

where $\Phi = \phi(x, t) + \nu^t \psi(x, t)$. Here $\lambda(t)$, also called as influence functions, will be referred to as costates.

In order to transform all strong boundary conditions into natural boundary conditions, a constraint equation for continuity at T_0 and T_f is adjoined to the performance index. Introducing

$$x\Big|_{T_0} \triangleq \lim_{t \to t_0^+} x(t) \qquad x\Big|_{T_f} \triangleq \lim_{t \to t_f^-} x(t) \tag{5.7}$$

$$\hat{x}\Big|_{T_0} \triangleq x(T_0) \qquad \hat{x}\Big|_{T_f} \triangleq x(T_f) \tag{5.8}$$

The continuity at T_0 and T_f are enforced weakly to the performance index by using a set of discrete undetermined multipliers, α, defined at T_0 and T_f as follows

$$\bar{J} = \int_{T_0}^{T_f} \Big[L(x,u,t) + \lambda^t(f - \dot{x})\Big]\, dt + \Phi\Big|_{T_0}^{T_f} + \alpha^t\,(x - \hat{x})\Big|_{T_0}^{T_f} \tag{5.9}$$

This allows the final weak formulation to have only natural(weak) boundary conditions and the shape functions to be chosen from a less restricted class of functions.

As a next step for the weak formulation, the first variation of $\bar{J}$, the necessary condition for finding an extremal of $\bar{J}$, has to be zero. After performing several integrations by parts and enforcing the continuity conditions (For more detailed formulation, see Ref.[77].), the final form of the weak formulation for optimal control by Hodges and Bless [77] can be written as

$$\begin{aligned}
\delta\bar{J} \;=\; & \int_{T_0}^{T_f} \Bigg\{ \delta\dot{\lambda}^t x - \delta\dot{x}^t\lambda + \delta x^t \Bigg[\left(\frac{\partial L}{\partial x}\right)^t + \left(\frac{\partial f}{\partial x}\right)^t \lambda \Bigg] + \delta\lambda^t f \\
& + \delta u^t \Bigg[\left(\frac{\partial L}{\partial u}\right)^t + \left(\frac{\partial f}{\partial u}\right)^t \lambda \Bigg] \Bigg\} dt + \delta T_f \Bigg[L + \lambda^t(f - \dot{x}) + \frac{\partial \Phi}{\partial x} \Bigg]_{T_f} \\
& + \delta\nu^t\psi\Big|_{T_0}^{T_f} + \delta x^t\hat{\lambda}\Big|_{T_0}^{T_f} - \delta\lambda^t\hat{x}\Big|_{T_0}^{T_f} = 0
\end{aligned} \tag{5.10}$$

5.3 Finite Element Discretization using Higher Order Shape Functions

For the finite element discretization process, we may choose linear shape functions [77] for the variables in Eq. (5.10). But we now turn our attention to obtain p-version shape functions using hierarchical polynomials [100] such as Legendre, Chebyshev, Hermite polynomials. These shape functions are suitable for obtaining highly accurate solutions.

5.3.1 Coordinate Transformation

First consider the transformation [101] from the global (or problem) coordinate system t to a local coordinate system ξ which has the origin at the center of the element and is scaled such that $\xi = -1$ at the left end node and $\xi = 1$ at the right end node. The transformation is achieved by the linear "stretch" transformation given by

$$\xi = \frac{2t}{\Delta T} - \frac{(T_0 + T_f)}{\Delta T} \tag{5.11}$$

where ΔT equals to $T_f - T_0$. The transformation Eq. (5.11) transforms the coordinate t ($T_0 \le t \le T_f$) to a nondimensional time ξ ($-1 \le \xi \le 1$). Rewriting Eq. (5.10) using this transformation, one then obtains

$$\delta\bar{J} \;=\; \int_{-1}^{1} \Bigg\langle \delta\lambda'^t x - \delta x'^t\lambda + \frac{\Delta T}{2} \Bigg\{ \delta x^t \Bigg[\left(\frac{\partial L}{\partial x}\right)^t + \left(\frac{\partial f}{\partial x}\right)^t \lambda \Bigg] + \delta\lambda^t f$$

$$+\,\delta u^t\Big[\Big(\frac{\partial L}{\partial u}\Big)^t+\Big(\frac{\partial f}{\partial u}\Big)^t\lambda\Big]\Big\}\Big\rangle d\xi+\delta\xi\Big[L+\lambda^t(f-\dot{x})+\frac{\partial\Phi}{\partial x}\Big]_{\xi=1}$$
$$+\,\delta\nu^t\psi\Big|_{-1}^{1}+\delta x^t\hat{\lambda}\Big|_{-1}^{1}-\delta\lambda^t\hat{x}\Big|_{-1}^{1} \tag{5.12}$$

where prime denotes the differentiation with respect to ξ. C^0 type shape functions [102, 103] may be used to represent δx, $\delta\lambda$ and δu. These functions are chosen to be hierarchical "bubble" functions.

$$\begin{aligned}
\delta x &= \frac{(1-\xi)}{2}\delta x_0+\frac{(1+\xi)}{2}\delta x_1+\sum_{i=1}^{n-2}\delta x_i{}^* N_{i+1}(\xi) \qquad -1\le\xi\le 1\\
\delta\lambda &= \frac{(1-\xi)}{2}\delta\lambda_0+\frac{(1+\xi)}{2}\delta\lambda_1+\sum_{i=1}^{n-2}\delta\lambda_i^* N_{i+1}(\xi) \qquad -1\le\xi\le 1\\
\delta u &= \frac{(1-\xi)}{2}\delta u_0+\frac{(1+\xi)}{2}\delta u_1+\sum_{i=1}^{n-2}\delta u_i{}^* N_{i+1}(\xi) \qquad -1\le\xi\le 1
\end{aligned} \tag{5.13}$$

Then the derivatives of δx and $\delta\lambda$ with respect to ξ can be written as

$$\begin{aligned}
\delta x' &= \frac{1}{2}(\delta x_1-\delta x_0)+\sum_{i=1}^{n-2}\delta x_i{}^* N'{}_{i+1}(\xi) \qquad -1\le\xi\le 1\\
\delta\lambda' &= \frac{1}{2}(\delta\lambda_1-\delta\lambda_0)+\sum_{i=1}^{n-2}\delta\lambda_i^* N'{}_{i+1}(\xi) \qquad -1\le\xi\le 1
\end{aligned} \tag{5.14}$$

Also the approximate values of x and λ are taken as continuous functions on the element interior while for distinct, discrete values on the element boundaries. Thus,

$$\begin{aligned}
x &= \sum_{j=1}^{n-1}\bar{x}_j M_j(\xi) \qquad \lambda=\sum_{j=1}^{n-1}\bar{\lambda}_j M_j(\xi) \qquad -1<\xi<1\\
&\quad x=\hat{x}_0 \qquad\qquad \lambda=\hat{\lambda}_0 \qquad\qquad \xi=-1\\
&\quad x=\hat{x}_1 \qquad\qquad \lambda=\hat{\lambda}_1 \qquad\qquad \xi=1
\end{aligned} \tag{5.15}$$

Here $\hat{x}_0$ and $\hat{\lambda}_0$ denote discrete values of x and λ at the left node of the element, $\hat{x}_1$ and $\hat{\lambda}_1$ denote discrete values of $\mathbf{x}$ and λ at the right node of the element. Also the approximation for u can be written as

$$u=\sum_{j=1}^{n-1}\bar{u}_j M_j(\xi) \qquad -1\le\xi\le 1 \tag{5.16}$$

5.4 Numerical Examples

5.4.1 A Simple Optimal Control Problem I using One Element

Consider the following optimal control problem [78] where x, λ and u are scalars.

$$\begin{gathered} J = \frac{1}{2}x(1)^2 + \int_0^1 \frac{1}{2}u^2 dt \qquad 0 < t < 1 \\ \dot{x} = tu \qquad x(0) = 4 \end{gathered} \tag{5.17}$$

With $L = \frac{1}{2}u^2$, $f = tu$ and a given initial condition, the final form of the weak formulation of Eq. (5.12) for this problem takes form of

$$\begin{aligned} \delta\bar{J} &= \int_{T_0}^{T_f} \left\{ \delta\dot{\lambda}^t x - \delta\dot{x}^t\lambda + \delta\lambda^t f + \delta u^t \left[\left(\frac{\partial L}{\partial u}\right)^t + \left(\frac{\partial f}{\partial u}\right)^t \lambda \right] \right\} dt \\ &\quad + \delta x^t \hat{\lambda}\Big|_{T_0}^{T_f} - \delta\lambda^t \hat{x}\Big|_{T_0}^{T_f} \end{aligned} \tag{5.18}$$

and boundary conditions are $\hat{x}(0) = 4$ and $\hat{x}(T_f) = \hat{\lambda}(T_f)$. Rewriting Eq. (5.18) by use of transformation Eq. (5.11) yields

$$\begin{aligned} \delta\bar{J} &= \int_{-1}^{1} \left\langle \delta\lambda'^t x - \delta x'^t\lambda + \frac{\Delta T}{2}\left\{ \delta\lambda^t f + \delta u^t \left[\left(\frac{\partial L}{\partial u}\right)^t + \left(\frac{\partial f}{\partial u}\right)^t \lambda \right] \right\} \right\rangle d\xi \\ &\quad + \delta x^t \hat{\lambda}\Big|_{-1}^{1} - \delta\lambda^t \hat{x}\Big|_{-1}^{1} \end{aligned} \tag{5.19}$$

Substituting Eq. (5.13) into the first two terms in the integrand and the trailing terms in Eq. (5.19) results in

$$\begin{aligned} \int_{-1}^{1} \delta\lambda'^t x d\xi - \delta\lambda^t \hat{x}\Big|_{-1}^{1} &= \delta\boldsymbol{\lambda}\mathbf{A}\mathbf{x} \\ -\int_{-1}^{1} \delta x'^t \lambda d\xi + \delta x^t \hat{\lambda}\Big|_{-1}^{1} &= -\delta\mathbf{x}\mathbf{A}\boldsymbol{\lambda} \end{aligned} \tag{5.20}$$

where

$$\begin{aligned} \delta\boldsymbol{\lambda} &= \left\{ \delta\lambda_0, \quad \delta\lambda_1, \quad \delta\lambda_1^*, \quad \delta\lambda_2^* \right\} \\ \delta\mathbf{x} &= \left\{ \delta x_0, \quad \delta x_1, \quad \delta x_1^*, \quad \delta x_2^* \right\} \end{aligned} \tag{5.21}$$

and

$$
\mathbf{A} = \begin{bmatrix} 1 & 0 & -\frac{1}{2}\int_{-1}^{1} M_1(\xi)d\xi & -\frac{1}{2}\int_{-1}^{1} M_2(\xi)d\xi & -\frac{1}{2}\int_{-1}^{1} M_3(\xi)d\xi \\ 0 & -1 & \frac{1}{2}\int_{-1}^{1} M_1(\xi)d\xi & \frac{1}{2}\int_{-1}^{1} M_2(\xi)d\xi & \frac{1}{2}\int_{-1}^{1} M_3(\xi)d\xi \\ N_2(-1) & -N_2(1) & \int_{-1}^{1} N'_2(\xi)M_1(\xi)d\xi & \int_{-1}^{1} N'_2(\xi)M_2(\xi)d\xi & \int_{-1}^{1} N'_2(\xi)M_3(\xi)d\xi \\ N_3(-1) & -N_3(1) & \int_{-1}^{1} N'_3(\xi)M_1(\xi)d\xi & \int_{-1}^{1} N'_3(\xi)M_2(\xi)d\xi & \int_{-1}^{1} N'_3(\xi)M_3(\xi)d\xi \end{bmatrix} \tag{5.22}
$$

where $\mathbf{A}$ is $n \times (n+1)$ matrix and

$$
\begin{aligned} \mathbf{x} &= \left\{ \; \hat{x}_0, \quad \hat{x}_1, \quad \bar{x}_1, \quad \bar{x}_2, \quad \bar{x}_3 \; \right\}^t \\ \boldsymbol{\lambda} &= \left\{ \; \hat{\lambda}_0, \quad \hat{\lambda}_1, \quad \bar{\lambda}_1, \quad \bar{\lambda}_2, \quad \bar{\lambda}_3 \; \right\}^t \end{aligned} \tag{5.23}
$$

Substituting Eq. (5.13) into the third term in the integrand in Eq. (5.19) yields

$$
\frac{\Delta T}{2}\int_{-1}^{1} \delta\lambda^t f d\xi = \delta\boldsymbol{\lambda}\mathbf{D}\mathbf{u} \tag{5.24}
$$

where

$$
\mathbf{u} = \left\{ \; \hat{u}_0, \quad \hat{u}_1, \quad \bar{u}_1, \quad \bar{u}_2, \quad \bar{u}_3 \; \right\}^t \tag{5.25}
$$

and

$$
\mathbf{D} = \begin{bmatrix} 0 & 0 & \frac{\Delta T}{2}\int_{-1}^{1} t\frac{(1-\xi)}{2}M_1(\xi)d\xi & \frac{\Delta T}{2}\int_{-1}^{1} t\frac{(1-\xi)}{2}M_2(\xi)d\xi & \frac{\Delta T}{2}\int_{-1}^{1} t\frac{(1-\xi)}{2}M_3(\xi)d\xi \\ 0 & 0 & \frac{\Delta T}{2}\int_{-1}^{1} t\frac{(1+\xi)}{2}M_1(\xi)d\xi & \frac{\Delta T}{2}\int_{-1}^{1} t\frac{(1+\xi)}{2}M_2(\xi)d\xi & \frac{\Delta T}{2}\int_{-1}^{1} t\frac{(1+\xi)}{2}M_3(\xi)d\xi \\ 0 & 0 & \frac{\Delta T}{2}\int_{-1}^{1} tN_2(\xi)M_1(\xi)d\xi & \frac{\Delta T}{2}\int_{-1}^{1} tN_2(\xi)M_2(\xi)d\xi & \frac{\Delta T}{2}\int_{-1}^{1} tN_2(\xi)M_3(\xi)d\xi \\ 0 & 0 & \frac{\Delta T}{2}\int_{-1}^{1} tN_3(\xi)M_1(\xi)d\xi & \frac{\Delta T}{2}\int_{-1}^{1} tN_3(\xi)M_2(\xi)d\xi & \frac{\Delta T}{2}\int_{-1}^{1} tN_3(\xi)M_3(\xi)d\xi \end{bmatrix} \tag{5.26}
$$

where $\mathbf{D}$ is $n \times (n+1)$ matrix and $t = \frac{\Delta T}{2}(1+\xi)$ in the matrix.
Substituting Eq. (5.13) into the fourth term in the integrand in Eq. (5.19) yields

$$
\frac{\Delta T}{2}\int_{-1}^{1} \delta u^t \Big\{ \left(\frac{\partial L}{\partial u}\right)^t + \left(\frac{\partial f}{\partial u}\right)^t \lambda \Big\} d\xi = \delta\mathbf{u}\mathbf{C}\mathbf{u} + \delta\mathbf{u}\mathbf{D}\boldsymbol{\lambda} \tag{5.27}
$$

where

$$
\delta\mathbf{u} = \left\{ \; \delta u_0, \quad \delta u_1, \quad \delta u_1^*, \quad \delta u_2^* \; \right\} \tag{5.28}
$$

and

$$\mathbf{C} = \begin{bmatrix} 0 & 0 & \frac{\Delta T}{2}\int_{-1}^{1}\frac{(1-\xi)}{2}M_1(\xi)d\xi & \frac{\Delta T}{2}\int_{-1}^{1}\frac{(1-\xi)}{2}M_2(\xi)d\xi & \frac{\Delta T}{2}\int_{-1}^{1}\frac{(1-\xi)}{2}M_3(\xi)d\xi \\ 0 & 0 & \frac{\Delta T}{2}\int_{-1}^{1}\frac{(1+\xi)}{2}M_1(\xi)d\xi & \frac{\Delta T}{2}\int_{-1}^{1}\frac{(1+\xi)}{2}M_2(\xi)d\xi & \frac{\Delta T}{2}\int_{-1}^{1}\frac{(1+\xi)}{2}M_3(\xi)d\xi \\ 0 & 0 & \frac{\Delta T}{2}\int_{-1}^{1}N_2(\xi)M_1(\xi)d\xi & \frac{\Delta T}{2}\int_{-1}^{1}N_2(\xi)M_2(\xi)d\xi & \frac{\Delta T}{2}\int_{-1}^{1}N_2(\xi)M_3(\xi)d\xi \\ 0 & 0 & \frac{\Delta T}{2}\int_{-1}^{1}N_3(\xi)M_1(\xi)d\xi & \frac{\Delta T}{2}\int_{-1}^{1}N_3(\xi)M_2(\xi)d\xi & \frac{\Delta T}{2}\int_{-1}^{1}N_3(\xi)M_3(\xi)d\xi \end{bmatrix} \tag{5.29}$$

where $\mathbf{C}$ is $n \times (n+1)$ matirx. Combining all four matrix equations leads to a matrix form of equations,

$$\begin{Bmatrix} \delta\boldsymbol{\lambda} \\ \delta\mathbf{x} \\ \delta\mathbf{u} \end{Bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{D} \\ \mathbf{0} & \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} & \mathbf{C} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \\ \mathbf{u} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{5.30}$$

Since $\delta\boldsymbol{\lambda}$, $\delta\mathbf{x}$ and $\delta\mathbf{u}$ are arbitrary, Eq. (5.30) leads directly to the final matrix equation to be solved,

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{D} \\ \mathbf{0} & \mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} & \mathbf{C} \end{bmatrix} \begin{Bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \\ \mathbf{u} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{Bmatrix} \tag{5.31}$$

Figures 5.1-5.3 respectively show the state, costate and control for the system. The results were obtained using one element with various order of Legendre polynomials. In the case of states (see Fig. 1), the exact solution was obtained using the third order of Legendre polynomials. The exact solution was obtained in the case of control (see Fig. 3) using the first order Legendre polynomials.

5.4.2 Optimal Control Problem II using Multiple Elements

The problem [80, 81] is to minimize

$$\begin{aligned} J &= \frac{1}{2}x(T)^2 + \int_0^1 \frac{1}{2}u^2 dt \qquad 0 < t < T \\ \dot{x} &= h(t)u, \quad h(t) = 1 + t - \frac{3}{17}t^2 \end{aligned} \tag{5.32}$$

where $T = 3$sec, x and u are scalars, and the initial condition is $x(0) = -5355692/268515$.

The difference between this problem and the problem I is the system equation. So Eq. (5.18) can be also used as a governing equation for this problem. In the formulation, we may put $L = \frac{1}{2}u^2$ and $f = (1 + t - 3/17t^2)u$.

In this example, the derivation of the assembly of element equations in matrix forms is shown. For illustration purpose, the derivation is restricted to the matrix **A** which is a part of the element equation in a matrix form (see Eq. (5.31)). Suppose that the domain of the problem is divided into two elements of equal time lengths with the value of $n = 5$. Then the assembled global matrix has the form of

$$\begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} & 0 & A_{13}^{(1)} & A_{14}^{(1)} & 0 & 0 & \cdots \\ A_{21}^{(1)} & A_{22}^{(1)} + A_{11}^{(2)} & A_{12}^{(2)} & A_{23}^{(1)} & A_{24}^{(1)} & A_{13}^{(2)} & A_{14}^{(2)} & \cdots \\ 0 & A_{21}^{(2)} & A_{22}^{(2)} & 0 & 0 & A_{23}^{(2)} & A_{24}^{(2)} & \cdots \\ A_{31}^{(1)} & A_{32}^{(1)} & 0 & A_{33}^{(1)} & A_{34}^{(1)} & 0 & 0 & \cdots \\ 0 & A_{31}^{(2)} & A_{32}^{(2)} & 0 & 0 & A_{33}^{(2)} & A_{34}^{(2)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{Bmatrix} \hat{X}_1 \\ \hat{X}_2 \\ \hat{X}_3 \\ \bar{x}_1^{(1)} \\ \bar{x}_2^{(1)} \\ \bar{x}_1^{(2)} \\ \bar{x}_2^{(2)} \\ \vdots \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{Bmatrix} \tag{5.33}$$

The assembly of the rest element matrices, **C** and **D**, is same as above.

We label the values of x at the global nodes with $\hat{X}_i (i = 1, 2, ..., N)$ where N is the total number of global nodes. Then we have the following correspondence, so called interelement continuity conditions, between the local nodal values and the global nodal values.

$$\hat{x}_0^{(1)} = \hat{X}_1, \qquad \hat{x}_1^{(1)} = \hat{X}_2 = \hat{x}_0^{(2)}, \qquad \hat{x}_1^{(2)} = \hat{X}_3 \tag{5.34}$$

Figures 5.4-5.5 respectively show state and control of the system. The results were obtained using three element with various order of Legendre polynomials.

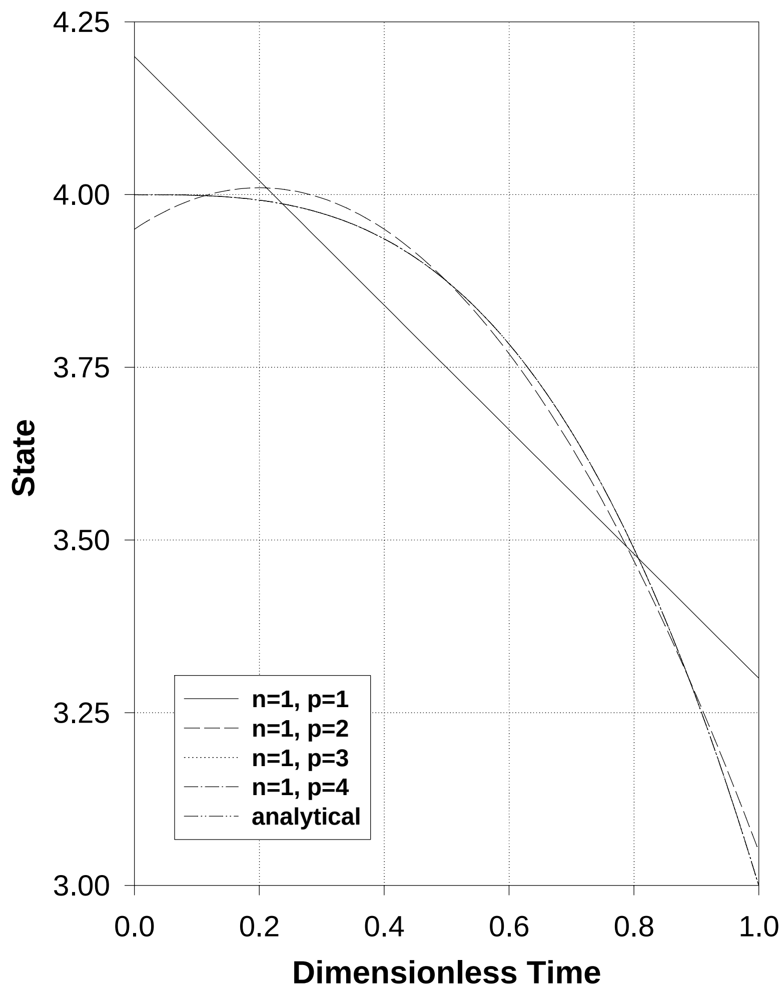

Fig. 5.1 State vs time for example problem I using one element.

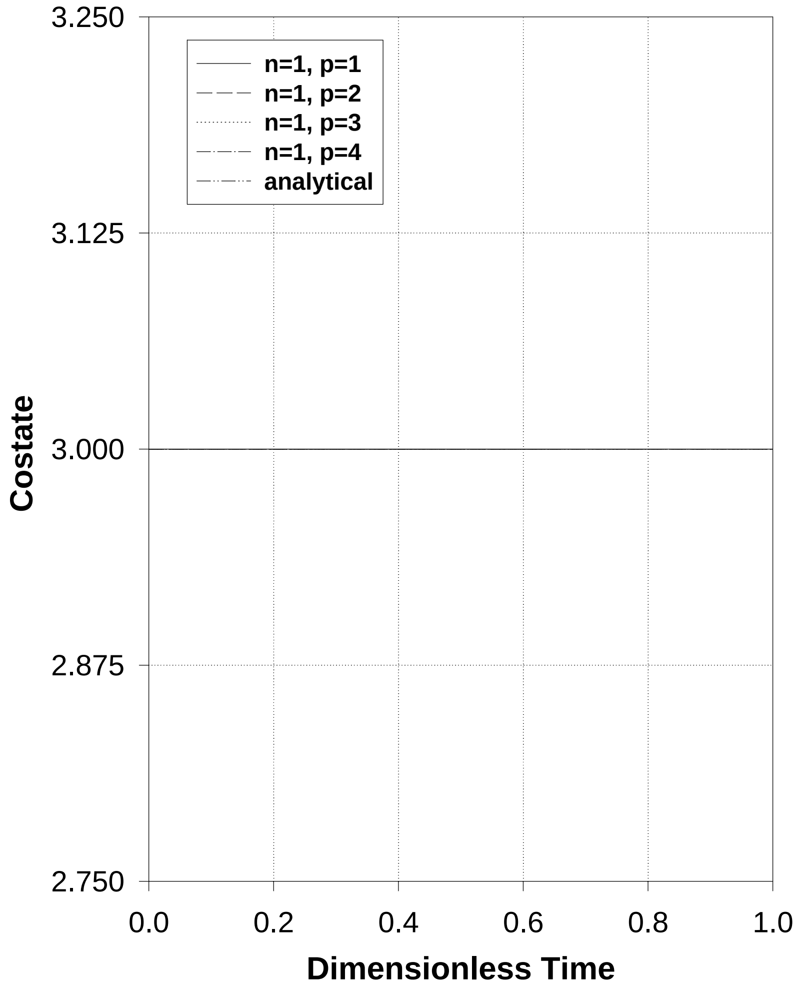

Fig. 5.2 Costate vs time for example problem I using one element.

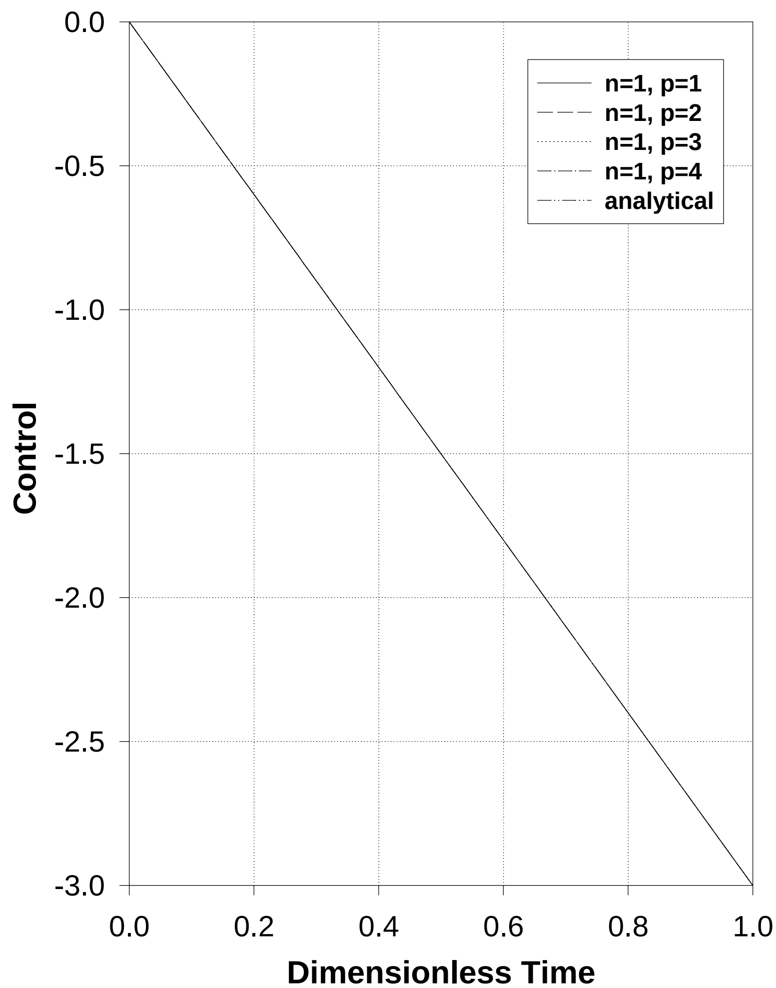

Fig. 5.3 Control vs time for example problem I using one element.

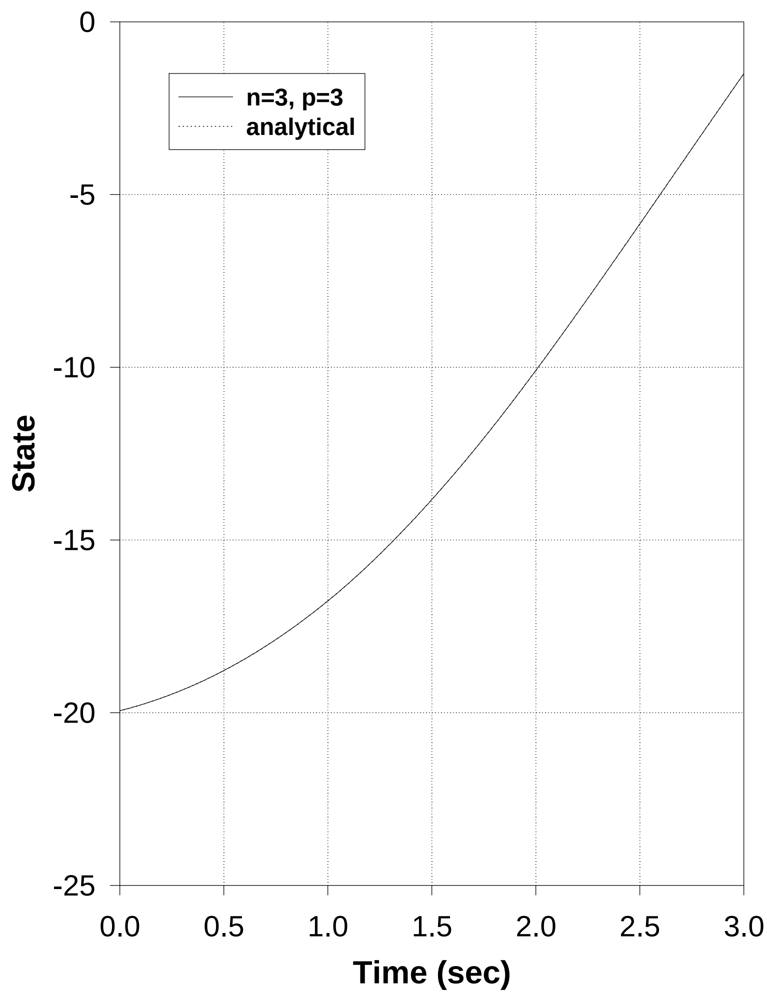

Fig. 5.4 State vs time for example problem II using three elements.

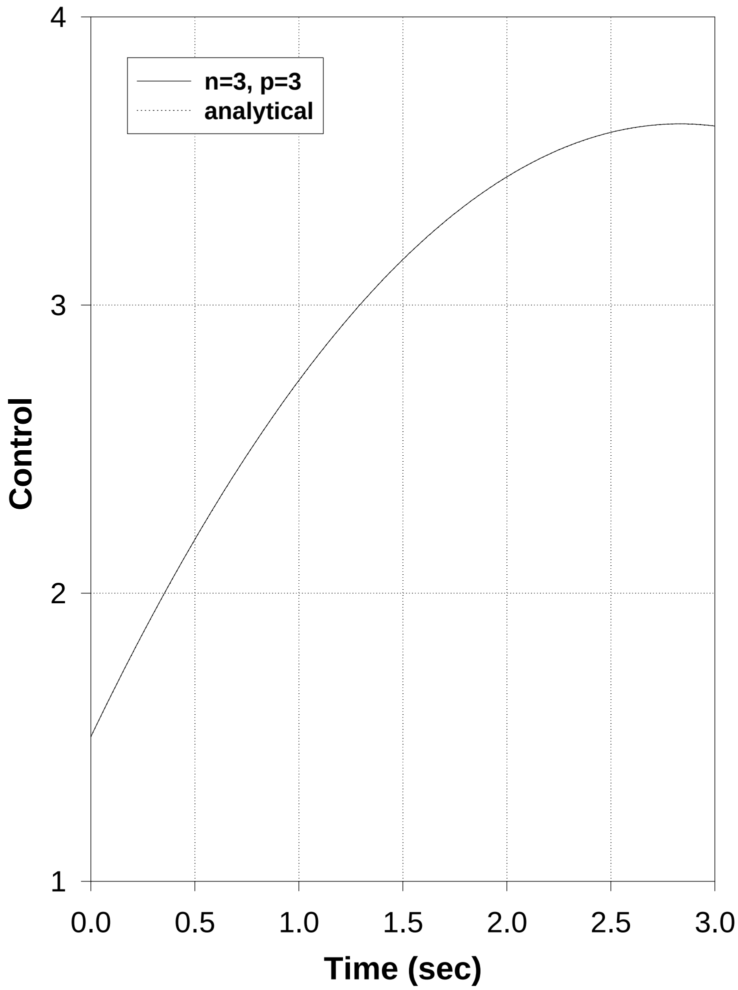

Fig. 5.5 Control vs time for example problem II using three elements.

6. CONCLUSION AND FUTURE WORK

6.1 Summary and Conclusion

In Chapter 2, the bilinear formulation, proposed earlier by Peters and Izadpanah for linear systems, is extended to solve nonlinear transient problems. This method is easily extended to the time finite element formulation for the initial value problems by adopting a constraint on the test functions, $v(T_0) \neq 0$ and $v(T_f) = 0$. The bilinear formulation can be proved to be a convergent method, provided that the formulation has appropriate constraint on the test functions. The use of a Lagrange multiplier method for applying the constraint $v(T_f) = 0$ eliminates various numerical difficulties faced in earlier implementation of the method. Throughout this paper, Legendre polynomials are used as basis functions. The results, in some cases, were also obtained using other polynomials such as Chebyshev, Hermite, and Integral form of Legendre polynomials and were found to be almost identical to those obtained using Legendre polynomials. An advantage of the use of the orthogonal functions as basis functions is that the resulting "stiffness matrix" is numerically well behaved. Also, it is convenient to use hierarchical form of basis functions since it allows additional higher order basis functions within elements without changing the mesh and without removing basis functions that are already in use. As a result, one need not calculate the entire matrix anew when higher-order basis functions are added to improve the accuracy of the approximation. The present approach is thus ideal for adaptive schemes. By using a time finite element formulation, not only the transient responses but also sensitivities of the transient response are calculated easily (by performing a direct differentiation of the resulting algebraic equations). An advantage of the present approach over the central difference approach is that one does not need to perform a convergence study to select an appropriate step size for obtaining the sensitivities. The numerical results for the presented examples show very good agreement between the results obtained using the present approach and those available either exactly or obtained from central difference approximations. Based on the results presented here, the proposed method appears to be a good choice for calculating both the transient response and its sensitivity with respect to various system parameters for linear and nonlinear, damped and undamped systems.

In Chapter 3, in order to find the best suitable polynomials for the proposed method, the basis function in *hp*-version of the TFM for solving initial value problems were examined by using four cases of nonlinear second order systems. Four different types of orthogonal polynomials, namely: Chebyshev, Hermite, Legendre and Integrated Legendre are evaluated. Condition numbers of the resulting augmented stiffness matrices are obtained as a function of number of terms used in the expansion. For all the four examples, the Integrated Legendre polynomials show the best performance. The choice of Hermite polynomials is not recommended based on the results presented here. Compared to the other polynomials, Integrated Legendre polynomials require large CPU times.

In Chapter 4, the time finite element based iterative direct method is found to be very effective for the identification of parameters in the nonlinear examples studied. Especially for the two-degree-of-freedom nonlinear system, a two-step approach in which parameters corresponding to linear and nonlinear terms were identified separately was necessary for the convergence of the process. Comparing with the results from Normann and Kapania's study, the proposed method uses fewer number of data points for getting the same accuracy. Good results were obtained for both single- and two-degree-of-freedom system examples. Based on the results presented here, the proposed method appears to be a good choice for performing parametric identification of nonlinear systems.

In chapter 5, *p*- and *hp*- version TFM is proposed based on the weak formulation for optimal control by Hodges and Bless. The proposed formulation showed the time finite element method can be used to the solution of the optimal control problems. The numerical results of two simple optimal control problems are compared with the analytic solutions. The accuracy along with the order of polynomials is of particular interest in the study. When highly accurate solutions are required, the method will be a good choice to use.

6.2 Future Work

Finally, as a future work, it is proposed that the present approach be used in:

- Extending the formulation in time domain to space-time domain.
- Deriving formulation for the second order sensitivity.
- Deriving for the *p*- and *hp*- version TFM:
  - with state-control inequality constraints.
  - for discontinuities in the states and/or discontinuities in the system equations.

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APPENDIX A

A.1 Transient Response of a Two-Degree-of-Freedom System having Cubic Nonlinearities

The bilinear formulation is derived for a two-degree-of-freedom system having cubic nonlinearities (Nayfeh and Mook [76]) governed by

$$\ddot{u}_1 + {\omega_1}^2 u_1 + 2\mu_1 \dot{u}_1 + \alpha_1 {u_1}^3 + \alpha_2 {u_1}^2 u_2 + \alpha_3 u_1 {u_2}^2 + \alpha_4 {u_2}^3 = 0 \tag{A.1}$$

$$\ddot{u}_2 + {\omega_2}^2 u_2 + 2\mu_2 \dot{u}_2 + \alpha_5 {u_1}^3 + \alpha_6 {u_1}^2 u_2 + \alpha_7 u_1 {u_2}^2 + \alpha_8 {u_2}^3 = 0 \tag{A.2}$$

with initial conditions

$$u_1(0) = u_{10} \qquad u_2(0) = u_{20} \qquad \dot{u}_1(0) = v_{10} \qquad \dot{u}_2(0) = v_{20}$$

where ω_1, ω_2 are natural frequencies, and μ_1, μ_2 are linear viscous damping of the systems. Also α_i, $i = 1, 2..8$, are arbitrary constants.

Multiplying governing Eqs. (A.1) and (A.2) with test functions (or weight functions) $\mathbf{v}(t)$ and integrating with respect to time, we obtain

$$\int_{T_0}^{T_f} v_1 \cdot \left(\ddot{u}_1 + {\omega_1}^2 u_1 + 2\mu_1 \dot{u}_1 + \alpha_1 {u_1}^3 + \alpha_2 {u_1}^2 u_2 + \alpha_3 u_1 {u_2}^2 + \alpha_4 {u_2}^3\right) dt = 0 \tag{A.3}$$

$$\int_{T_0}^{T_f} v_2 \cdot \left(\ddot{u}_2 + {\omega_2}^2 u_2 + 2\mu_2 \dot{u}_2 + \alpha_5 {u_1}^3 + \alpha_6 {u_1}^2 u_2 + \alpha_7 u_1 {u_2}^2 + \alpha_8 {u_2}^3\right) dt = 0 \tag{A.4}$$

Integrating Eqs. (A.3) and (A.4) by parts results in

$$\begin{aligned} [v_1 \dot{u}_1]_{T_0}^{T_f} &+ \int_{T_0}^{T_f} \Big({\omega_1}^2 v_1 u_1 + 2\mu_1 v_1 \dot{u}_1 + \alpha_1 v_1 {u_1}^3 + \alpha_2 v_1 {u_1}^2 u_2 \\ &+ \alpha_3 v_1 u_1 {u_2}^2 + \alpha_4 v_1 {u_2}^3 - \dot{v}_1 \dot{u}_1\Big) dt = 0 \end{aligned} \tag{A.5}$$

$$\begin{aligned} [v_2 \dot{u}_2]_{T_0}^{T_f} &+ \int_{T_0}^{T_f} \Big({\omega_2}^2 v_2 u_2 + 2\mu_2 v_2 \dot{u}_2 + \alpha_5 v_2 {u_1}^3 + \alpha_6 v_2 {u_1}^2 u_2 \\ &+ \alpha_7 v_2 u_1 {u_2}^2 + \alpha_8 v_2 {u_2}^3 - \dot{v}_2 \dot{u}_2\Big) dt = 0 \end{aligned} \tag{A.6}$$

Let trial functions be

$$u_1(t) = \sum_{j=1}^{N} p_j \phi_j(t) \tag{A.7}$$

$$u_2(t) = \sum_{j=1}^{N} q_j \phi_j(t) \tag{A.8}$$

where p_j and q_j are generalized coordinates and ϕ_j are basis functions. Substituting Eqs. (A.7) and (A.8) into Eqs. (A.5) and (A.6) respectively yields

$$\int_{T_0}^{T_f} \left\{ {\omega_1}^2 v_1 \left(\sum_{j=1}^{N} p_j \phi_j \right) + 2\mu_1 v_1 \left(\sum_{j=1}^{N} p_j \dot{\phi}_j \right) - \dot{v}_1 \left(\sum_{j=1}^{N} p_j \dot{\phi}_j \right) \right\} dt$$
$$= v_1(T_0)\dot{u}_1(T_0) - v_1(T_f)\dot{u}_1(T_f) - \int_{T_0}^{T_f} \left\{ \alpha_1 v_1 \left(\sum_{j=1}^{N} p_j \phi_j \right)^3 \right.$$
$$+ \alpha_2 v_1 \left(\sum_{j=1}^{N} p_j \phi_j \right)^2 \left(\sum_{j=1}^{N} q_j \phi_j \right) + \alpha_3 v_1 \left(\sum_{j=1}^{N} p_j \phi_j \right) \left(\sum_{j=1}^{N} q_j \phi_j \right)^2$$
$$\left. + \alpha_4 v_1 \left(\sum_{j=1}^{N} q_j \phi_j \right)^3 \right\} dt \tag{A.9}$$

$$\int_{T_0}^{T_f} \left\{ {\omega_2}^2 v_2 \left(\sum_{j=1}^{N} q_j \phi_j \right) + 2\mu_2 v_2 \left(\sum_{j=1}^{N} q_j \dot{\phi}_j \right) - \dot{v}_2 \left(\sum_{j=1}^{N} q_j \dot{\phi}_j \right) \right\} dt$$
$$= v_2(T_0)\dot{u}_2(T_0) - v_2(T_f)\dot{u}_2(T_f) - \int_{T_0}^{T_f} \left\{ \alpha_5 v_2 \left(\sum_{j=1}^{N} p_j \phi_j \right)^3 \right.$$
$$+ \alpha_6 v_2 \left(\sum_{j=1}^{N} p_j \phi_j \right)^2 \left(\sum_{j=1}^{N} q_j \phi_j \right) + \alpha_7 v_2 \left(\sum_{j=1}^{N} p_j \phi_j \right) \left(\sum_{j=1}^{N} q_j \phi_j \right)^2$$
$$\left. + \alpha_8 v_2 \left(\sum_{j=1}^{N} q_j \phi_j \right)^3 \right\} dt \tag{A.10}$$

Assuming test functions, $\mathbf{v}(t)$, as variations of trial functions yields

$$v_1(t) = \delta p_i \psi_i(t), \qquad 1 \le i \le M \tag{A.11}$$
$$v_2(t) = \delta q_i \psi_i(t), \qquad 1 \le i \le M \tag{A.12}$$

Then for each δp_i and δq_i, equations (A.9) and (A.10) become

$$\sum_{j=1}^{N} p_j \left\{ \int_{T_0}^{T_f} \left({\omega_1}^2 \psi_i \phi_j + 2\mu_1 \psi_i \dot{\phi}_j - \dot{\psi}_i \dot{\phi}_j \right) dt \right\}$$
$$= \psi_i(T_0)\dot{u}_1(T_0) - \psi_i(T_f)\dot{u}_1(T_f) - \int_{T_0}^{T_f} \psi_i \left\{ \alpha_1 \left(\sum_{l=1}^{N} p_l \phi_l \right)^3 \right.$$
$$+ \alpha_2 \left(\sum_{l=1}^{N} p_l \phi_l \right)^2 \left(\sum_{l=1}^{N} q_l \phi_l \right) + \alpha_3 \left(\sum_{l=1}^{N} p_l \phi_l \right) \left(\sum_{l=1}^{N} q_l \phi_l \right)^2$$

$$+ \alpha_4 \left(\sum_{l=1}^{N} q_l \phi_l \right)^3 \Bigg\} dt \tag{A.13}$$

$$\begin{aligned}
&\sum_{j=1}^{N} q_j \left\{ \int_{T_0}^{T_f} \left({\omega_2}^2 \psi_i \phi_j + 2\mu_2 \psi_i \dot{\phi}_j - \dot{\psi}_i \dot{\phi}_j \right) dt \right\} \\
&= \psi_i(T_0)\dot{u}_2(T_0) - \psi_i(T_f)\dot{u}_2(T_f) - \int_{T_0}^{T_f} \psi_i \left\{ \alpha_5 \left(\sum_{l=1}^{N} p_l \phi_l \right)^3 \right. \\
&\quad + \alpha_6 \left(\sum_{l=1}^{N} p_l \phi_l \right)^2 \left(\sum_{l=1}^{N} q_l \phi_l \right) + \alpha_7 \left(\sum_{l=1}^{N} p_l \phi_l \right) \left(\sum_{l=1}^{N} q_l \phi_l \right)^2 \\
&\quad \left. + \alpha_8 \left(\sum_{l=1}^{N} q_l \phi_l \right)^3 \right\} dt
\end{aligned} \tag{A.14}$$

Rearranging and simplifying Eqs. (A.13) and (A.14) yields

$$\sum_{j=1}^{N} {B_{ij}}^{(1)} p_j = {A_i}^{(1)} \tag{A.15}$$

$$\sum_{j=1}^{N} {B_{ij}}^{(2)} q_j = {A_i}^{(2)} \tag{A.16}$$

where

$$\begin{aligned}
{B_{ij}}^{(1)} &= \int_{T_0}^{T_f} \left({\omega_1}^2 \psi_i \phi_j + 2\mu_1 \psi_i \dot{\phi}_j - \dot{\psi}_i \dot{\phi}_j \right) dt \\
{B_{ij}}^{(2)} &= \int_{T_0}^{T_f} \left({\omega_2}^2 \psi_i \phi_j + 2\mu_2 \psi_i \dot{\phi}_j - \dot{\psi}_i \dot{\phi}_j \right) dt
\end{aligned}$$

and

$$\begin{aligned}
{A_i}^{(1)} &= \psi_i(T_0)\dot{u}_1(T_0) - \psi_i(T_f)\dot{u}_1(T_f) - \int_{T_0}^{T_f} \psi_i \left\{ \alpha_1 \left(\sum_{l=1}^{N} p_l \phi_l \right)^3 \right. \\
&\quad + \alpha_2 \left(\sum_{l=1}^{N} p_l \phi_l \right)^2 \left(\sum_{l=1}^{N} q_l \phi_l \right) + \alpha_3 \left(\sum_{l=1}^{N} p_l \phi_l \right) \left(\sum_{l=1}^{N} q_l \phi_l \right)^2 \\
&\quad \left. + \alpha_4 \left(\sum_{l=1}^{N} q_l \phi_l \right)^3 \right\} dt
\end{aligned}$$

$${A_i}^{(2)} = \psi_i(T_0)\dot{u}_2(T_0) - \psi_i(T_f)\dot{u}_2(T_f) - \int_{T_0}^{T_f} \psi_i \left\{ \alpha_5 \left(\sum_{l=1}^{N} p_l \phi_l \right)^3 \right.$$

$$+ \alpha_6 \left(\sum_{l=1}^{N} p_l \phi_l \right)^2 \left(\sum_{l=1}^{N} q_l \phi_l \right) + \alpha_7 \left(\sum_{l=1}^{N} p_l \phi_l \right) \left(\sum_{l=1}^{N} q_l \phi_l \right)^2$$

$$\left. + \alpha_8 \left(\sum_{l=1}^{N} q_l \phi_l \right)^3 \right\} dt$$

Imposition of initial conditions $u_i(T_0) = u_i(0)$, $i = 1, 2$, can be done by augmenting Eqs. (A.15) and (A.16) with additional equations. Thus,

$$\sum_{j=1}^{N} \left(B_{ij}{}^{(1)} + \phi_j(T_0) \right) p_j = A_i{}^{(1)} \quad \text{(A.17)}$$

$$\sum_{j=1}^{N} \left(B_{ij}{}^{(2)} + \phi_j(T_0) \right) q_j = A_i{}^{(2)} \quad \text{(A.18)}$$

For the natural convergence of the end condition, a constraint $\psi_i(T_f) = 0$, which was suggested by Peters and Izadpanah, can be included by using Lagrange multipliers.

$$\sum_{j=1}^{N} \left(B_{ij}{}^{(1)} + \psi_j(T_0) \right) p_j + \lambda_1 \psi_i(T_f) = A_i{}^{*(1)} \quad \text{(A.19)}$$

$$\sum_{j=1}^{N} \left(B_{ij}{}^{(2)} + \psi_j(T_0) \right) q_j + \lambda_2 \psi_i(T_f) = A_i{}^{*(2)} \quad \text{(A.20)}$$

where

$$A_i{}^{*(1)} = \psi_i(T_0)\dot{u}_1(T_0) - \int_{T_0}^{T_f} \psi_i \left\{ \alpha_1 \left(\sum_{l=1}^{N} p_l \phi_l \right)^3 \right.$$

$$+ \alpha_2 \left(\sum_{l=1}^{N} p_l \phi_l \right)^2 \left(\sum_{l=1}^{N} q_l \phi_l \right) + \alpha_3 \left(\sum_{l=1}^{N} p_l \phi_l \right) \left(\sum_{l=1}^{N} q_l \phi_l \right)^2$$

$$\left. + \alpha_4 \left(\sum_{l=1}^{N} q_l \phi_l \right)^3 \right\} dt$$

$$A_i{}^{*(2)} = \psi_i(T_0)\dot{u}_2(T_0) - \int_{T_0}^{T_f} \psi_i \left\{ \alpha_5 \left(\sum_{l=1}^{N} p_l \phi_l \right)^3 \right.$$

$$+ \alpha_6 \left(\sum_{l=1}^{N} p_l \phi_l \right)^2 \left(\sum_{l=1}^{N} q_l \phi_l \right) + \alpha_7 \left(\sum_{l=1}^{N} p_l \phi_l \right) \left(\sum_{l=1}^{N} q_l \phi_l \right)^2$$

$$\left. + \alpha_8 \left(\sum_{l=1}^{N} q_l \phi_l \right)^3 \right\} dt$$

For the expression of $A_i{}^{*(1)}$ and $A_i{}^{*(2)}$, the second terms in equations $A_i{}^{(1)}$ and $A_i{}^{(2)}$ are eliminated due to the use of Lagrange multipliers.

Combining Eqs. (A.19) and (A.20) and putting into a matrix form, the transient response equation can be expressed as

$$\begin{bmatrix} \mathbf{B}^{(1)} & 0 & \{\psi_i(T_f)\} & 0 \\ 0 & \mathbf{B}^{(2)} & 0 & \{\psi_i(T_f)\} \\ <\phi_j(T_0)> & 0 & 0 & 0 \\ 0 & <\phi_j(T_0)> & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{p} \\ \mathbf{q} \\ \lambda_1 \\ \lambda_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{A}^{*(1)} \\ \mathbf{A}^{*(2)} \\ u_1(T_0) \\ u_2(T_0) \end{Bmatrix} \tag{A.21}$$

A.2 Response Sensitivity with respect to various Design Parameters d_k

The straightforward differentiation of Eq. (A.21) with respect to d_k gives the following sensitivity equation. Thus,

$$\begin{bmatrix} \mathbf{B}^{*(11)} & \mathbf{B}^{*(12)} & \{\psi_i(T_f)\} & 0 \\ \mathbf{B}^{*(21)} & \mathbf{B}^{*(22)} & 0 & \{\psi_i(T_f)\} \\ <\phi_j(T_0)> & 0 & 0 & 0 \\ 0 & <\phi_j(T_0)> & 0 & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial \mathbf{p}}{\partial d_k} \\ \frac{\partial \mathbf{q}}{\partial d_k} \\ \frac{\partial \lambda_1}{\partial d_k} \\ \frac{\partial \lambda_2}{\partial d_k} \end{Bmatrix}
+ \begin{bmatrix} \frac{\partial \mathbf{B}^{(1)}}{\partial d_k} & 0 & 0 & 0 \\ 0 & \frac{\partial \mathbf{B}^{(2)}}{\partial d_k} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{p} \\ \mathbf{q} \\ \lambda_1 \\ \lambda_2 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \mathbf{A}^{**(1)}}{\partial d_k} \\ \frac{\partial \mathbf{A}^{**(2)}}{\partial d_k} \\ \frac{\partial u_1(T_0)}{\partial d_k} \\ \frac{\partial u_2(T_0)}{\partial d_k} \end{Bmatrix} \tag{A.22}$$

where

$$\begin{aligned}
B_{ij}{}^{*(11)} &= B_{ij}{}^{(1)} + \int_{T_0}^{T_f} \psi_i \phi_j \left\{ 2\alpha_2 \left(\sum_{l=1}^{N} p_l \phi_l\right) \left(\sum_{l=1}^{N} q_l \phi_l\right) + \alpha_3 \left(\sum_{l=1}^{N} q_l \phi_l\right)^2 \right. \\
&\quad \left. + 3\alpha_1 \left(\sum_{l=1}^{N} p_l \phi_l\right)^2 \right\} dt \\
B_{ij}{}^{*(12)} &= \int_{T_0}^{T_f} \psi_i \phi_j \left\{ \alpha_2 \left(\sum_{l=1}^{N} p_l \phi_l\right)^2 + 2\alpha_3 \left(\sum_{l=1}^{N} p_l \phi_l\right) \left(\sum_{l=1}^{N} q_l \phi_l\right) + 3\alpha_4 \left(\sum_{l=1}^{N} q_l \phi_l\right)^2 \right\} dt \\
B_{ij}{}^{*(21)} &= \int_{T_0}^{T_f} \psi_i \phi_j \left\{ 3\alpha_5 \left(\sum_{l=1}^{N} p_l \phi_l\right)^2 + 2\alpha_6 \left(\sum_{l=1}^{N} p_l \phi_l\right) \left(\sum_{l=1}^{N} q_l \phi_l\right) + \alpha_7 \left(\sum_{l=1}^{N} q_l \phi_l\right)^2 \right\} dt \\
B_{ij}{}^{*(22)} &= B_{ij}{}^{(2)} + \int_{T_0}^{T_f} \psi_i \phi_j \left\{ 2\alpha_7 \left(\sum_{l=1}^{N} p_l \phi_l\right) \left(\sum_{l=1}^{N} q_l \phi_l\right) + \alpha_6 \left(\sum_{l=1}^{N} p_l \phi_l\right)^2 \right. \\
&\quad \left. + 3\alpha_8 \left(\sum_{l=1}^{N} q_l \phi_l\right)^2 \right\} dt
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial {B_{ij}}^{(1)}}{\partial d_k} &= \int_{T_0}^{T_f} \left(\frac{\partial {\omega_1}^2}{\partial d_k} \psi_i \phi_j + 2 \frac{\partial \mu_1}{\partial d_k} \psi_i \dot{\phi}_j \right) dt \\
\frac{\partial {B_{ij}}^{(2)}}{\partial d_k} &= \int_{T_0}^{T_f} \left(\frac{\partial {\omega_2}^2}{\partial d_k} \psi_i \phi_j + 2 \frac{\partial \mu_2}{\partial d_k} \psi_i \dot{\phi}_j \right) dt \\
\frac{\partial A_i^{**(1)}}{\partial d_k} &= \psi_i(T_0) \frac{\partial \dot{u}_1(T_0)}{\partial d_k} \\
\frac{\partial A_i^{**(2)}}{\partial d_k} &= \psi_i(T_0) \frac{\partial \dot{u}_2(T_0)}{\partial d_k}
\end{aligned}$$

A.3 Nonlinear Stiffness Matrices

A.3.1 van der Pol's Oscillator

The nonlinear stiffness matrices, $\mathbf{B}$ and $\mathbf{B}^*$, for the van der Pol's oscillator take the following forms

$$B_{ij} = \int_{T_0}^{T_f} \left\{ \psi_i \phi_j + \epsilon \psi_i \dot{\phi}_j \left\{ \left(\sum_{l=1}^{N} q_l \phi_l \right)^2 - 1 \right\} - \dot{\psi}_i \dot{\phi}_j \right\} dt \quad \text{(A.23)}$$

$$B_{ij}^* = B_{ij} + \int_{T_0}^{T_f} 2\epsilon \psi_i \phi_j \left(\sum_{l=1}^{N} q_l \phi_l \right) \left(\sum_{m=1}^{N} q_m \dot{\phi}_m \right) dt \quad \text{(A.24)}$$

A.3.2 Mass on a Nonlinear Hardening Spring

The stiffness matrix B_{ij} for the transient response has the following form

$$B_{ij} = \int_{T_0}^{T_f} \left(K \psi_i \phi_j + C \psi_i \dot{\phi}_j - M \dot{\psi}_i \dot{\phi}_j \right) dt \quad \text{(A.25)}$$

while the nonlinear stiffness matrix B_{ij}^* for the response sensitivity takes the following form

$$B_{ij}^* = B_{ij} + \int_{T_0}^{T_f} 3\mu \psi_i \phi_j \left(\sum_{l=1}^{N} q_l \phi_l \right)^2 dt \quad \text{(A.26)}$$

A.3.3 Mass on a Nonlinear Softening Spring

The stiffness matrix B_{ij} for the transient response takes the form

$$B_{ij} = \int_{T_0}^{T_f} \left(C \psi_i \dot{\phi}_j - M \dot{\psi}_i \dot{\phi}_j \right) dt \quad \text{(A.27)}$$

while the nonlinear stiffness matrix B_{ij}^* for the response sensitivity takes the following form

$$B_{ij}^* = B_{ij} + \int_{T_0}^{T_f} \frac{K \psi_i \phi_j}{\cosh^2 \left(\sum_{l=1}^{N} q_l \phi_l \right)} dt \quad \text{(A.28)}$$

A.4 Augmented Stiffness Matrix for van der Pol's Oscillator

Numerical values of the augmented matrix for the transient response case in the van der Pol's oscillator are presented. The values were generated using Hermite and Integrated Legendre polynomials of degree seven as basis functions. The values for the response sensitivity case are similar.

Two-digit rounding values of the matrix **B** using Hermite polynomials are reported as

$$\begin{bmatrix} -\mathbf{.50e-2} & .31e+2 & -.33e-2 & -.63e+2 & -.40e-2 & -.13e+3 & .10e+1 \\ .00e+0 & -\mathbf{.32e+4} & .84e+2 & .64e+4 & -.60e+3 & .13e+5 & .20e+1 \\ -.33e-2 & -.21e+2 & -\mathbf{.17e+5} & .18e+3 & .12e+6 & -.18e+4 & .20e+1 \\ .00e+0 & .64e+4 & -.30e+3 & -\mathbf{.54e+5} & .24e+4 & .56e+6 & -.40e+1 \\ -.40e-2 & -.25e+2 & .12e+6 & -.11e+4 & -\mathbf{.96e+6} & .17e+5 & -.20e+2 \\ .00e+0 & .13e+5 & .16e+4 & .56e+6 & -.14e+5 & -\mathbf{.85e+7} & -.80e+1 \\ .10e+1 & -.20e+1 & .20e+1 & .40e+1 & -.20e+2 & .80e+1 & .00e+0 \end{bmatrix}$$

Two-digit rounding values of the matrix **B** using Integrated Legendre polynomials are reported as

$$\begin{bmatrix} -\mathbf{.20e+3} & .20e+3 & .10e+1 & .33e+0 & -.16e+0 & .00e+0 & .00e+0 \\ .20e+3 & -\mathbf{.20e+3} & -.11e+1 & .31e+0 & .16e+0 & .00e+0 & .10e+1 \\ -.13e+1 & .13e+1 & -\mathbf{.40e+3} & -.96e+0 & -.49e-2 & .10e+1 & .00e+0 \\ -.36e-2 & .36e-2 & .74e+0 & -\mathbf{.40e+3} & -.56e+0 & -.32e-2 & .00e+0 \\ .54e-1 & -.54e-1 & .31e-2 & .51e+0 & -\mathbf{.40e+3} & -.41e+0 & .00e+0 \\ .00e+0 & .00e+0 & -.50e-1 & .23e-2 & .38e+0 & -\mathbf{.40e+3} & .00e+0 \\ .10e+1 & .00e+0 & .00e+0 & .00e+0 & .00e+0 & .00e+0 & .00e+0 \end{bmatrix}$$

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