Appendix A

Derivation of the scattering matrix of a general symmetrical condensed node

In this appendix, the scattering matrix of a general symmetrical condensed node will be derived. The method for deriving the scattering matrix of a general node is based on first obtaining expressions for three nodal voltages V_x, V_y and V_z and three loop currents I_x, I_y and I_z. The scattering equations can then be expressed in terms of these voltages and currents.

The link line impedances are assumed to be different and given by

Z_{jk} (Y_{jk}) is the impedance (admittance) of a j directed line polarized in the k direction.

The x directed equivalent voltage V_x can be obtained from the expression

$$V_x \sum C_i = \sum V_i C_i \tag{A.1}$$

expressing the capacitance in equation (A.1) in terms of the link line admittances, equation (A.1) can be rewritten as

$$V_x \frac{\Delta t}{2}[2Y_{yx} + 2Y_{zx} + Y_{ox}] = \frac{\Delta t}{2}[Y_{yx}(V_{ypx} + V_{ynx}) + Y_{zx}(V_{zpx} + V_{znx}) + Y_{ox}V_{ox}] \tag{A.2}$$

From the conservation of the charge associated with the E_x field component, the following expression can be derived

$$Y_{yx}[V^i_{ynx} - V^r_{ynx} + V^i_{ypx} - V^r_{ypx}] + y_{zx}[V^i_{znx} - V^r_{znx} + V^i_{zpx} - V^r_{zpx}] + Y_{ox}[V^i_{ox} - V^r_{ox}] - V_x C_x = 0 \tag{A.3}$$

Rearranging equation (A.3) gives

$$\begin{aligned} &Y_{yx}[V^r_{ynx} + V^r_{ypx}] + Y_{zx}[V^r_{znx} + V^r_{zpx}] + Y_{ox}V^r_{ox} = \\ &\qquad Y_{yx}[V^i_{ynx} + V^i_{ypx}] + Y_{zx}[V^i_{znx} + V^i_{zpx}] + Y_{ox}V^i_{ox} - V_x G_x \end{aligned} \tag{A.4}$$

expressing the voltage in terms of incident and reflected voltages and writing the reflected voltages in equation (A.4) in terms of the incident voltages gives

$$V_x[2Y_{yx} + 2Y_{zx} + Y_{ox}] = [2Y_{yx}(V^i_{ypx} + V^i_{ynx}) + 2Y_{zx}(V^i_{zpx} + V^i_{znx}) + 2Y_{ox}V^i_{ox} - V_xG_x] \tag{A.5}$$

From equation (A.5), the node voltage V_x can be expressed as

$$V_x = \frac{[2Y_{yx}(V^i_{ypx} + V^i_{ynx}) + 2Y_{zx}(V^i_{zpx} + V^i_{znx}) + Y_{ox}V^i_{ox}]}{[2Y_{yx} + 2Y_{zx} + Y_{ox}]} \tag{A.6}$$

by writing equations similar to (A.1) and (A.3) for V_y and V_z , similar expressions to that in (A.6) can be obtained for V_y and V_z and are given by

$$V_y = \frac{[2Y_{xy}(V^i_{xpy} + V^i_{xny}) + 2Y_{zy}(V^i_{zpy} + V^i_{zny}) + Y_{oy}V^i_{oy}]}{[2Y_{xy} + 2Y_{zy} + Y_{oy}]} \tag{A.7}$$

$$V_z = \frac{[2Y_{yz}(V^i_{ypz} + V^i_{ynz}) + 2Y_{xz}(V^i_{xpz} + V^i_{xnz}) + Y_{oz}V^i_{oz}]}{[2Y_{yz} + 2Y_{xz} + Y_{oz}]} \tag{A.8}$$

To get the loop current I_x, first apply the conservation of magnetic flux principle

$$\sum L_n(I^i_n + I^r_n) = 0 \tag{A.9}$$

considering the current I_x and expressing each link line inductance in terms of the link line impedance and delay

$$\frac{\Delta t}{2}[Z_{yz}(I^i_{ynz} + I^r_{ynz} - I^i_{ypz} - I^r_{ypz}) + Z_{zy}(I^i_{zpy} + I^r_{zpy} - I^i_{zny} - I^r_{zny}) + R_xI_x + Z_{sx}I_x] = 0 \tag{A.10}$$

expressing currents in terms of voltage pulses and rearranging

$$V^r_{ynz} - V^r_{ypz} + V^r_{zpy} - V^r_{zny} + R_xI_x + V^r_{sx} = -(V^i_{ynz} - V^i_{ypz} + V^i_{zpy} - V^i_{zny} + V^i_{sx}) \tag{A.11}$$

The equivalent current I_x responsible for the x directed component of the magnetic field can be obtained as

$$I_x \sum I_i = \sum I_i L_i \tag{A.12}$$

expressing the inductances in terms of the link line impedances, equation (A.12) can be written as

$$\frac{\Delta t}{2} I_x[2Z_{yz} + 2Z_{zy} + Z_{sx}] = -\frac{\Delta t}{2}[Z_{yz}(I_{ynz} - I_{ypz}) + Z_{zy}(I_{zpy} - I_{zny}) + Z_{sx} I_x] \qquad (A.13)$$

but $I_{ynz} = \dfrac{V^i_{ynz} - V^r_{ynz}}{Z_{yz}}$ and similarly $I_{zny} = \dfrac{V^i_{zny} - V^r_{zny}}{Z_{zy}}$

substituting in (A.13) and expressing the reflected voltages in terms of the incident voltages from (A.10) gives

$$I_x[2Z_{yz} + 2Z_{zy} + Z_{sx}] = -2[(V^i_{ynz} - V^i_{ypz}) + (V^i_{zpy} - V^i_{zny}) + V_{sx}] - R_x I_x \qquad (A.14)$$

From equation (A.14), the current I_x can be written as

$$I_x = \frac{2[V^i_{ypz} - V^i_{ynz} + V^i_{zny} - V^i_{zpy} - V_{sx}]}{[2Z_{yz} + 2Z_{zy} + Z_{sx} + R_x]} \qquad (A.15)$$

By writing similar expressions to (A.10) and (A.13) for the currents in the y and z directions, similar expressions to (A.15) can be obtained for the currents I_y and I_z and are given by

$$I_y = \frac{2[V^i_{xnz} - V^i_{xpz} + V^i_{zpx} - V^i_{znx} - V_{sy}]}{[2Z_{xz} + 2Z_{zx} + Z_{sy} + R_y]} \qquad (A.16)$$

$$I_z = \frac{2[V^i_{xpy} - V^i_{xny} + V^i_{ypx} - V^i_{ynx} - V_{sz}]}{[2Z_{yx} + 2Z_{xy} + Z_{sz} + R_z]} \qquad (A.17)$$

Writing continuity equations for the electric field in the x direction gives

$$V^i_{xny} + V^r_{xny} + V^i_{xpy} + V^r_{xpy} = V^i_{zny} + V^r_{zny} + V^i_{zpy} + V^r_{zpy} \qquad (A.18)$$

and writing field continuity equation for the H_z gives

$$\frac{(V^i_{ynx} - V^r_{ynx}) - (V^i_{ypx} - V^r_{ypx})}{Z_{yx}} = \frac{(V^i_{xny} - V^r_{xny}) - (V^i_{xpy} + V^r_{xpy})}{Z_{xy}} \qquad (A.19)$$

Equations (A.18) and (A.19) together with conservation of charge equation for E_x and the equation for the conservation of the magnetic flux for H_z, the four equations can be solved simultaneously for V_{ynx} and V_{ypx} to give expression in the form

$$V^r_{ynx} = V_x - I_z Z_{yx} - V^i_{ynx} \qquad (A.20)$$

$$V^r_{ypx} = V_x + I_z Z_{yx} - V^i_{ypx} \qquad (A.21)$$

Similar expressions for other pairs of voltages can be obtained by writing the appropriate charge and flux conservation equations, electric and magnetic continuity equations for the corresponding electric and magnetic fields. The expression for the scattered voltage will in general take the from

$$V_{ipj} = V_j - I_k Z_{ij} - V_{inj} \quad V_{inj} = V_j + I_k Z_{ij} - V_{ipj} \quad \text{if } j \wedge k = i \tag{A.22}$$

$$V_{ipj} = V_j + I_k Z_{ij} - V_{inj} \quad V_{inj} = V_j - I_k Z_{ij} - V_{ipj} \quad \text{if } j \wedge k = -i \tag{A.23}$$