# Chapter 4

# Dispersion Analysis

As with any numerical method, the discretization of space and time introduces undesirable dispersion behavior in the numerical results. The other main source of numerical dispersion in a TLM mesh is the use of stubs which usually arise in trying to model anisotropic and inhomogeneous media and sometimes to guarantee synchronization in the three coordinate directions in a TLM mesh [21 ].

The dispersion relation is an implicit function of the mesh propagation constant in the three coordinate directions, the cell dimensions, the operating frequency and the constitutive parameters of the medium. The derivation of the dispersion relation of a generally graded TLM mesh with stubs will follow the general approach in [22-23].

Let's consider two nodes a and b. a is some arbitrary node in a generally graded TLM mesh which in general has twelve transmission lines and six stub ports. b is some hypothetical node which is formed by the transmission lines that can be reached at time $t+\Delta t$ by the pulses scattered from node a at time t. The scattered voltages at node a can be written in terms of the incident voltages as

$$V_a^r = SV_a^i \tag{4.1}$$

where S is the scattering matrix. If the propagation constant along the link lines is $K_0$, then the incident voltage at a can be written in terms of the scattered voltage at B as

$$V_a^i = TV_b^r \tag{4.2}$$

where T is the connection matrix which is an 18 x 18 diagonal matrix whose nonzero elements are $T_{ii} = e^{-jK_0 \Delta l/}$ for i =1 to i =15 and $T_{ii} = -e^{-jK_0 \Delta l/}$ for i=16, 17 and 18. The negative sign comes from the fact that these entries in the T matrix correspond to the short circuited stubs which have negative reflection coefficients. From Floquet's theorem or the monochromatic wave condition

$$V_b^r = PV_a^r \tag{4.3}$$

where P is a nondiagonal matrix whose nonzero coefficients are

$$\begin{aligned} &P_{1,12} = P_{5,7} = e^{jK_y \Delta y} && P_{2,9} = P_{4,8} = e^{jK_z \Delta z} \\ &P_{3,11} = P_{6,10} = e^{jK_x \Delta x} && P_{7,5} = P_{12,1} = e^{-jK_y \Delta y} \\ &P_{8,4} = P_{9,2} = e^{-jK_z \Delta z} && P_{10,6} = P_{11,3} = e^{-jK_x \Delta x} \end{aligned} \tag{4.4}$$

Substituting from (4.1) and (4.2) into (4.3) gives

$$V_b^r = PSTV_b^r \quad \text{which implies} \quad \det[I - PST] = 0 \tag{4.5}$$

The above condition is the dispersion relation. In this work, simplified versions of equation (4.5) are usually derived and solved numerically. Simplification is usually done by assuming propagation in a 2-D space and considering only either modes comprising $H_z$ , $E_x$ , and $E_y$ or those comprising $E_z$ , $H_x$ , and $H_y$. Only the nodes that are responsible for the field components of interest are used in solving (4.5). In the following discussion, the dispersion error will refer to the error between the simulated or mesh propagation constant obtained by solving equation (4.5) and the actual propagation constant of the medium.

### *4.1 The Dispersion behavior of the SCN, HSCN and SSCN*

In this section, results are given for the dispersion properties of the SCN as compared to the SSCN and the type II HSCN. The dispersion behavior is derived from the eigenvalue equation given in equation (4.5) for the modes comprising $H_z$ , $E_x$, and $E_y$ assuming 2-D propagation in the x-y plane.

Figure (4.1) shows the resulting family of dispersion curves for the SCN, the HSCN and the SSCN. The TLM mesh is assumed to be nonuniform having dx = 0.2 cm, dy = dz =0.1 cm modeling a medium of $\varepsilon_r = \mu_r = 1$. The dispersion error is plotted versus the angle from the x axis ($\theta$). The maximum time step $\Delta t$ is given relative to the time step in a uniform mesh with size 0.1 cm referred to as $\Delta t_0$. The maximum allowable time step for the SCN and the HSCN are the same and equal to $\Delta t_0$, whereas that provided by the SSCN is about 1.2 $\Delta t_0$. When the dispersion behavior was evaluated at the maximum allowable time step for each node, the HSCN was found to have the best dispersion properties as compared to the SCN and the SSCN, while the SCN has the worst dispersion behavior and the SSCN lies between the two. However, the SSCN has the advantage of being able to operate at a larger time step without the loss of stability which implies fewer number of iterations. When the dispersion properties of the SSCN were evaluated with a time step equal to $\Delta t_0$, the dispersion error was tremendously improved even over the HSCN. Figure (4.2) shows a comparison of the dispersion characteristic of the three nodes in a uniform mesh modeling a medium of $\varepsilon_r = 5$, $\mu_r$=1. For this case, the HSCN has the most superior performance over the SCN and the SSCN. It is worth mentioning, however, that the SSCN as reported in [10] has the advantage of having a unique and unilateral dispersion irrespective of the propagating mode. All other stub loaded nodes will experience different dispersion errors for different propagating modes [24]. This phenomenon may be attributed to the fact that open circuited stubs would directly affect the corresponding electric field components whereas short circuited stubs would affect the corresponding magnetic field components. As a result, depending on the values of these stubs in different coordinate directions, different modes associated with different field components will be treated differently. The fact that the SSCN has a unique dispersion relation makes it easier to correct for the dispersion error.

## 4.2 Dispersion in TFDTLM

The dispersion behavior of the TFDTLM scheme can be analyzed in the same way as in the time domain TLM, where the dispersion characteristic of a general TLM mesh can be derived by solving an eigenvalue equation in (4.5)

The matrix P has a similar form to that in time domain TLM except for the fact that propagation constants or wavenumbers along the three coordinate directions can be complex for a lossy medium. These propagation constants are written as $\gamma_x$, $\gamma_y$ and $\gamma_z$ for the x, y, and z directions, respectively. The matrix T is a diagonal matrix with nonzero elements equal to $e^{-\gamma\Delta l}$ where $\gamma$ is the approximated propagation constant of the medium. To simplify the analysis, equation (4.5) is solved numerically for a 2-D propagation case in the xy plane and only for the modes associated with the three field components $H_z$, $E_x$, and $E_y$ .

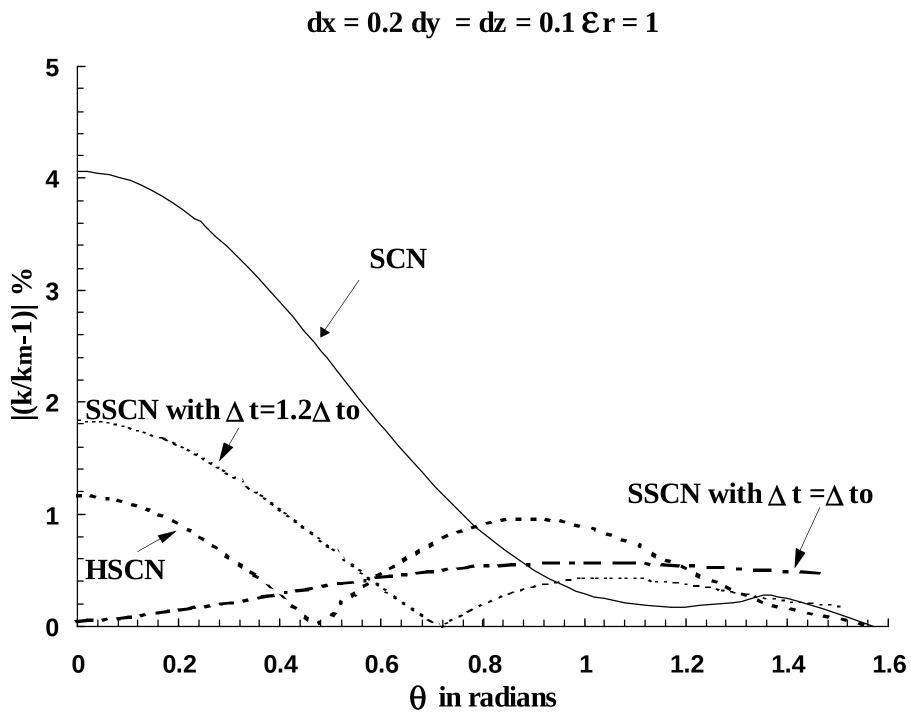


Figure 4.1 Comparison between the dispersion properties of the SCN, the HSCN and the SSCN for a nonuniform cell

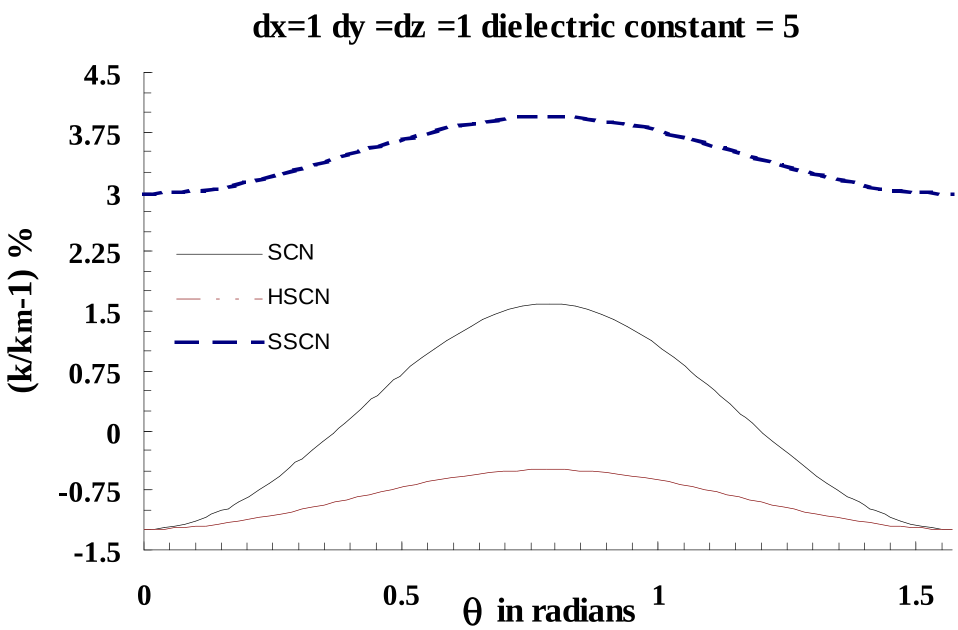


Figure 4.2 Comparison between the dispersion properties of the SCN, the HSCN and the SSCN for a uniform cell with $\varepsilon_r = 5$

## 4.3 Comparison with Dispersion in a TD HSCN

The dispersion relation of a stub loaded HSCN can be derived from the general condition in equation (4.5). The dispersion equation is solved numerically for both the proposed TFDTLM and the HSCN with the simplifying assumptions in section 4.1 and the error in the propagation vector is compared. A uniform mesh is considered with $\varepsilon_r \neq 1$ for a lossless medium in one case and then a lossy medium in another case. The dispersion error is calculated at a frequency where the cell dimension is 0.15 times the corresponding wavelength. The dispersion error refers to the error between the mesh propagation constant k and the actual propagation constant of the medium $k_m$.

### *4.3.1 Dispersion in lossless inhomogeneous medium*

In this section, the dispersion error of the TFDTLM in a lossless inhomogeneous medium is compared to that in HSCN II. The cell is assumed to be uniform having dx = dy = dz = $2\Delta\ell$ = 0.5 cm. In the TFDTLM, a first order approximation filter is used. The filter coefficients are optimized in a frequency range where the maximum cell dimension is less than or equal to 0.125 times the corresponding wavelength. The filter has the form $F_1$ given by

$$F_1 = \quad {}^{-\gamma_1 \Delta l_1} \left( \frac{a_0 + a_1 e^{-\gamma_1 \Delta l_1}}{b_0 + b_1 e^{-\gamma_1 \Delta l_1}} \right) \tag{4.6}$$

Figure (4.3) compares the percentage dispersion error calculated for a type II HSCN and the TFDTLM in a lossless medium with $\varepsilon_r$ = 5. From Figure (4.3), it appears that dispersion error of the TFDTLM is always positive with a minimum of 0 and a maximum of 0.88% and an average of 0.4%, whereas that of the type II HSCN is always negative with a minimum magnitude of 0.5 % and maximum magnitude of 1.2% and an average of 0.9 %. Hence it appears that even for a lossless medium, the TFDTLM scheme still showed some improvement over type II HSCN. It is worth mentioning that in the results predicted in Figure (4.3), the bilateral dispersion has not been taken into consideration. The bilateral dispersion reported in [24] results from the fact that for stub loaded TLM nodes, there exist both negative and positive dispersion errors, or stated otherwise, different modes of propagation will be associated with different dispersion error. For the case considered, we have looked into the three field components $E_x$, $E_y$, and $H_z$ and because type II HSCN employs short circuited stubs which directly affect the H fields, it is expected, that for the three field components chosen, the error associated with this node should be at a minimum. On the other hand, for the modes associated with $E_z$, $H_y$ and $H_x$, the error should be at a maximum. Hence the average error should be higher than that shown in Figure (4.3). On the other hand, because the TFDTLM in the above analysis didn't employ any stubs, it did not experience any bilateral dispersion and the dispersion error is unique irrespective of the mode of propagation. This property as in the SSCN would make it easier to correct for the dispersion error [10].

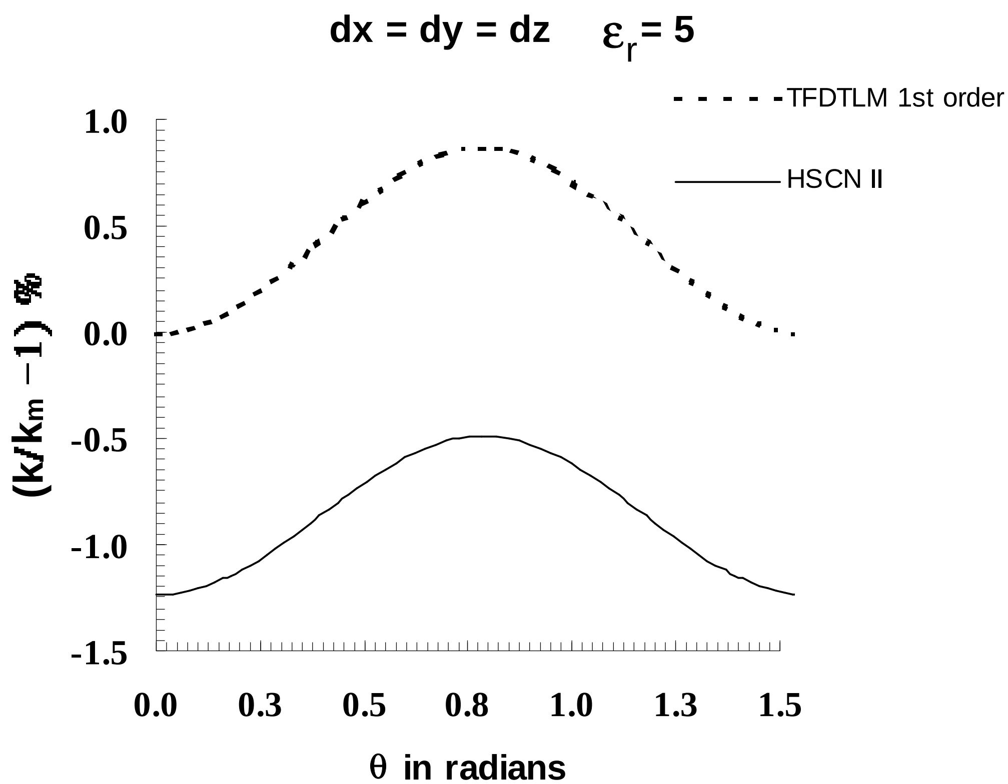


Figure 4.3 Comparison between the dispersion properties of the TFDTLM and HSCN II for uniform cell with $\varepsilon_r = 5$

**Table 4.1 Comparison between the dispersion properties the TFDTLM and HSCN II for uniform cell with $\varepsilon_r = 5$**

| $\varepsilon_r$=5 | minimum error | maximum error | average \| error \| |
|---|---|---|---|
| HSCN II | -0.5 % | -1.2 % | 0.9% |
| 1st order TFDTLM | 0 | 0.88 % | 0.4 % |

Figure (4.4) shows similar results to those in (4.3) but with $\varepsilon_r = 20$. The same first order filter approximation is used with the TFDTLM The Figure shows that the behavior of the HSCN continues to degrade by increasing the relative dielectric constant. The TFDTLM on the other hand, almost maintains the same order of accuracy. It is worth noting that even for such a relatively high relative dielectric constant, a first order approximation filter in the TFDTLM can still provide almost the same order of accuracy as with lower relative dielectric constants. This conclusion can have a significant effect on improving the computational efficiency of the TFDTLM

### *4.3.2 Dispersion in lossy inhomogeneous medium*

In this section, the dispersion behavior of the TFDTLM in a lossy inhomogeneous medium is analyzed. First and second order approximation filters are used. The filters have are denoted $F_1$ and $F_2$, respectively

$$F_1 = {}^{-\gamma_1 \Delta l_1}\left(\frac{a_0 + a_1 e^{-\gamma_1 \Delta l_1}}{b_0 + b_1 e^{-\gamma_1 \Delta l_1}}\right) \qquad F_2 = {}^{-\gamma_1 \Delta l_1}\left(\frac{a_0 + a_1 e^{-\gamma_1 \Delta l_1} + a_2 e^{-2\gamma_1 \Delta l_1}}{b_0 + b_1 e^{-2\gamma_1 \Delta l_1} + b_2 e^{-2\gamma_1 \Delta l_1}}\right) \tag{4.7}$$

The reference medium is considered free space with $\gamma_1 = j\omega\sqrt{\mu_0 \varepsilon_0}$. The filter coefficients are optimized in a frequency range where the maximum cell dimension is less than 0.125 times the corresponding wavelength. The cell is assumed uniform with dx = dy = dz = 0.5 cm. The dispersion characteristics of the TFDTLM will be compared to both the HSCN and the SSCN.

Figure (4.5) shows a comparison of the dispersion characteristics of the SSCN the HSCN II and first and second order TFDTLM in a lossy medium with $\varepsilon_r = 5$ and $\sigma = 0.1$ s/m ( a loss tangent of about 0.1 at the considered frequency). The Figure shows that the SSCN has a relatively poor dispersion characteristics for a lossy inhomogeneous medium, a behavior already observed in Figure (4.2). The first order TFDTLM provides significant improvement over the SSCN, however the HSCN II is still better. A second order TFDTLM on the other hand has a superior performance over both the SSCN and the HSCN II. Table 4.2 compares the minimum, maximum, and average magnitude error in the SSCN, the HSCN II and first and second order TFDTLM.

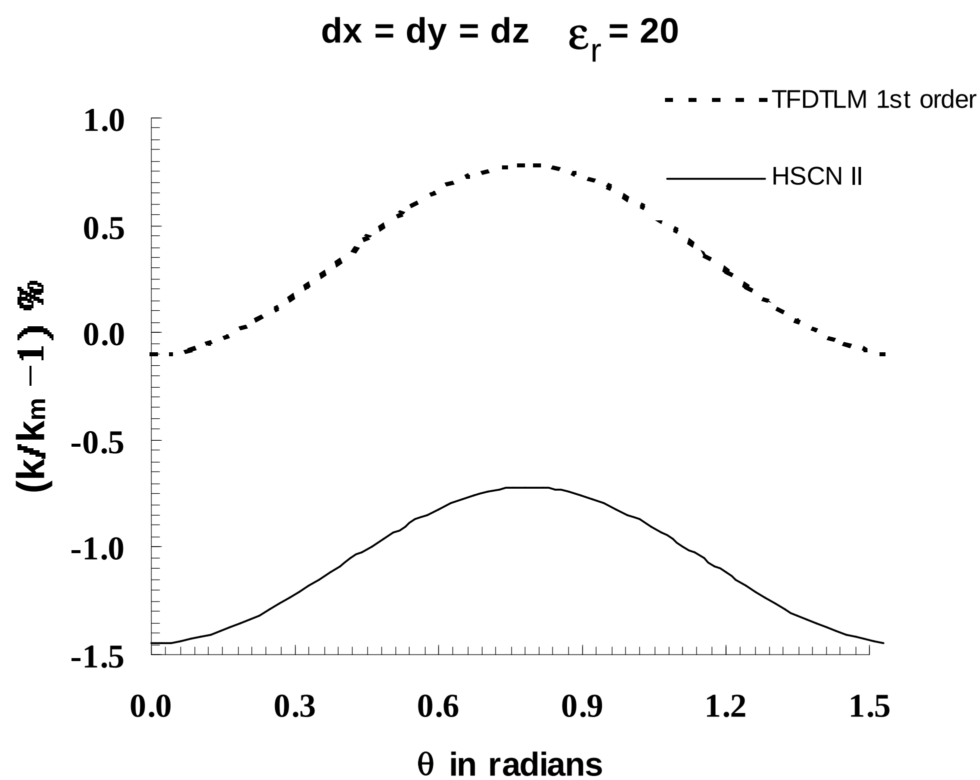


Figure 4.4 Comparison between the dispersion properties of the TFDTLM and HSCN II for uniform cell with $\varepsilon_r = 20$

**Table 4.2 Comparison between the dispersion properties of the TFDTLM, HSCN and the SSCN for in a lossy inhomogeneous medium with $\varepsilon_r = 5$ and $\sigma = 0.1$ s/m for a uniform cell**

| $\varepsilon_r = 5, \sigma = 0.1$ | minimum \| error \| | maximum \|error\| | average \| error\| |
|---|---|---|---|
| SSCN | 2.98 % | 3.94 % | 3.45 % |
| HSCN II | 0.54 % | 1.35 % | 0.98 % |
| 1st order TFDTLM | 1.27 | 1.49 % | 1.4 % |
| 2nd order TFDTLM | 0 % | 0.89 % | 0.43 % |

Figure (4.6) shows a comparison of the dispersion behavior of the HSCN II and a second order TFDTLM for higher loss tangent. The relative dielectric constant is chosen to be 8 with a conductivity $\sigma$ of 1 s/m, a loss tangent of about 0.2. The Figure shows that the behavior of the HSCN is significantly degraded for such a relatively high loss tangent. The TFDTLM on the other hand almost maintains the same order of accuracy as in a lossless homogeneous medium with a very slight degradation. Table 4.3 summarizes the minimum, maximum and average dispersion error in both the HSCN II and the second order TFDTLM.

**Table 4.3 Comparison between the dispersion properties of a second order TFDTLM and the HSCN II in a lossy inhomogeneous medium with $\varepsilon_r = 8$ and $\sigma = 1$ s/m for uniform cell**

| $\varepsilon_r = 8, \sigma = 1$ | minimum \| error \| | maximum \|error\| | average \| error\| |
|---|---|---|---|
| HSCN II | 2.13 % | 4.55 % | 3.44 % |
| 2nd order TFDTLM | 0.1 % | 0.85 % | 0.43 % |

Another important conclusion can also be derived from the results in Figure (4.6). It has been shown that a second order approximation filter can provide an acceptable order of accuracy even for a lossy inhomogeneous medium with a relatively high loss tangent. This conclusion can help improve the computational efficiency of the TFDTLM significantly.

### *4.3.3 Dispersion in lossy inhomogeneous medium with a nonuniform cell*

This section will discuss dispersion of the TFDTLM for a lossy inhomogeneous medium and a nonuniform cell. The dispersion behavior of the TFDTLM will be compared to that of the SSCN and the HSCN. The TFDTLM can handle the situation of a nonuniform cell in a simple and direct way. The set of equations in (3.15) are function of the normalized link line impedances (normalized by the complex intrinsic impedance of the medium) and the equivalent cell dimension $\Delta\ell$. In order to account for nonuniform cells, the set of equations are solved simultaneously as in the SSCN for the normalized link line impedances and the equivalent cell dimension $\Delta\ell$. The reason they are chosen to be solved the same way as in the SSCN is that

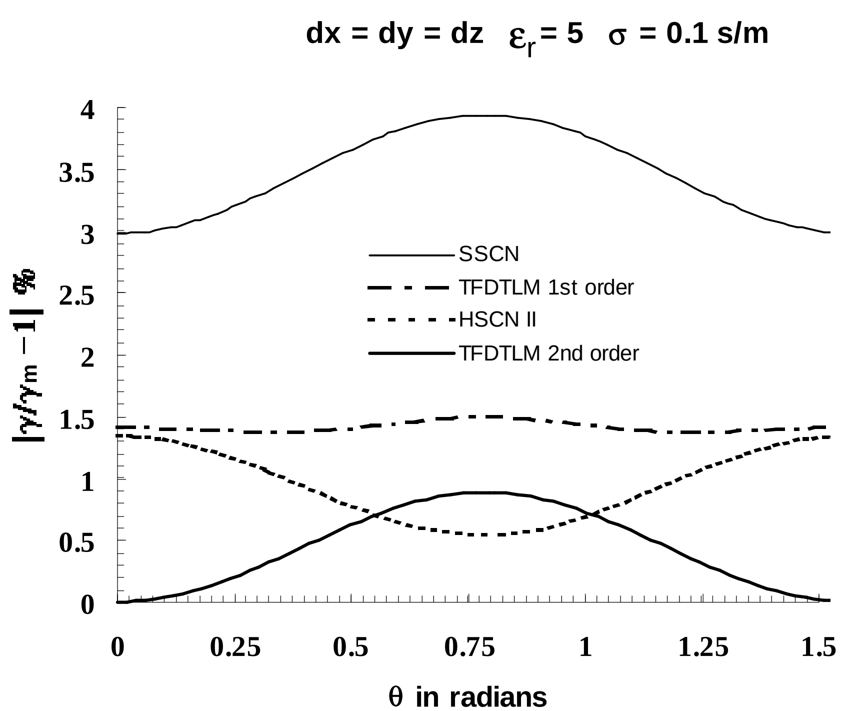


Figure 4.5 Comparison between the dispersion properties of the TFDTLM and HSCN II in a lossy inhomogeneous medium $\varepsilon_r = 5$, $\sigma = 0.1$ s/m for a uniform cell

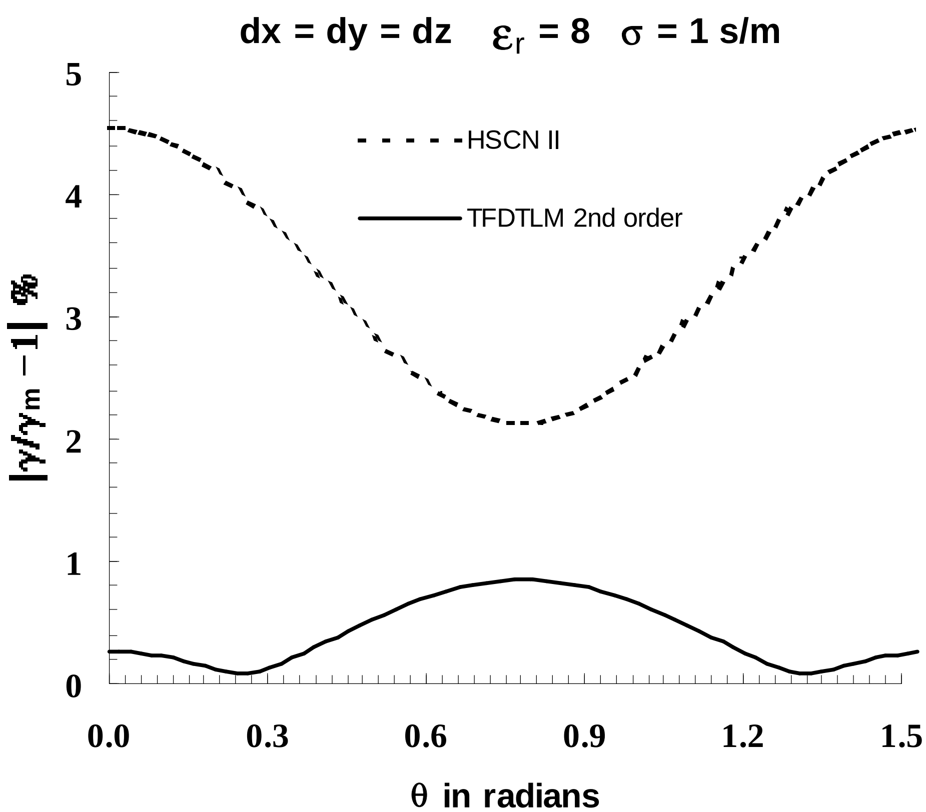


Figure 4.6 Comparison between the dispersion properties of the TFDTLM and HSCN II in a lossy inhomogeneous medium $\varepsilon_r = 8$, $\sigma = 1$ s/m for a uniform cell

the SSCN proved to have superior dispersion characteristics in a homogenous lossless medium with nonuniform cell [10]. The set of equations can in general be solved as in a general symmetrical condensed node (GSCN) as illustrated in [25] for optimum dispersion behavior in a nonuniform cell. The solution of equation (3.15) implicitly assumes that the propagation delay along any cell dimension is equal to the propagation delay in the medium. The role of the TFDTLM becomes significant in approximating the propagation delay in different media with different frequency dependent material parameters in terms of the propagation constant of some reference medium.

Figure (4.7) compares the dispersion behavior of a second order TFDTLM, the SSCN and the HSCN in a lossy inhomogeneous medium having $\varepsilon_r = 5$ and $\sigma = 0.1$ s/m. The cell is assumed to be nonuniform having dx = 2dy = 2dz = 1cm. In the TFDTLM, the link line impedances are chosen to account for the nonuniform cell dimension by solving the set of equations in (3.15). A second order approximation filter is then used to approximate the medium propagation factor in terms of that of the reference medium. The reference medium is taken to be free space. The filter coefficients are optimized in a frequency range where the maximum cell dimension is less than 0.125 times the corresponding wavelength. Figure (4.7) shows that the maximum dispersion is along the maximum cell dimension. In the SSCN, the link line impedances are chosen to account for the nonuniform cell and satisfy the medium dielectric constant. Lossy stubs are added to account for losses. The SSCN, as shown in Figure (4.7) has the worst dispersion behavior. The TFDTLM and the HSCN have almost the same amount of dispersion when both are operating at the maximum permissible equivalent cell dimension, the maximum equivalent cell dimension that would guarantee all normalized link line impedances are positive. The maximum permissible equivalent cell dimension in the HSCN is simply the maximum permissible time step multiplied by the speed of light in air. Although the dispersion behavior of both the HSCN and the TFDTLM are very close for the maximum permissible equivalent cell dimension, it is worth mentioning that the maximum permissible equivalent cell dimension in the TFDTLM is equal to almost 1.2 times that required by the HSCN, i.e. the number of iterations required by the TFDTLM is less than 85% that required by the HSCN. When the equivalent cell dimension is dropped to 0.25 dx which is equal to that required by the HSCN, the Figure shows that the dispersion of the TFDTLM is significantly improved over the HSCN.

Figure (4.8) shows results similar to those in Figure (4.7) but with higher relative dielectric constant, larger loss tangent and larger cell ratio. $\varepsilon_r$ is taken to be 8 and $\sigma = 1$ s/m with dx = 2dy =2 dz = 2 cm. The Figure shows that the behavior of the HSCN is significantly degraded and so is the behavior of the SSCN. The TFDTLM on the other hand operating at equivalent cell dimension equal to the maximum permissible equivalent cell dimension required by the HSCN is able to provide significantly less amount of error.

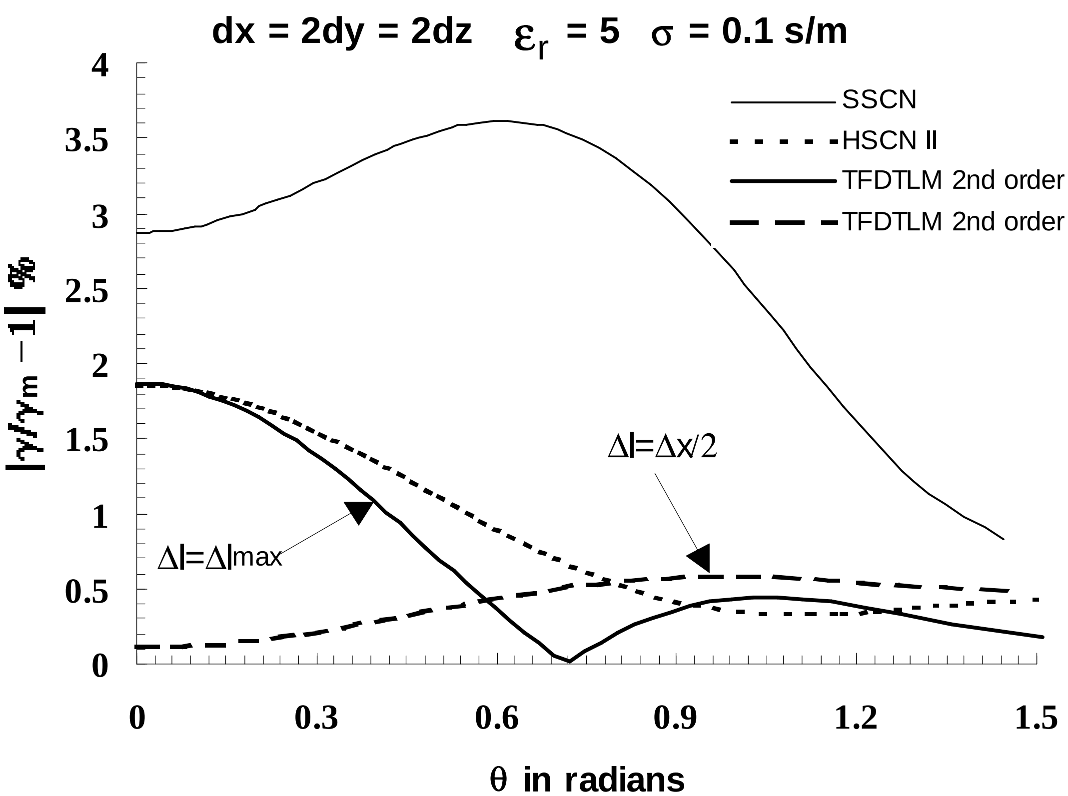


Figure 4.7 Comparison between the dispersion properties of the TFDTLM the SSCN and HSCN II in a lossy inhomogeneous medium $\varepsilon_r = 5$, $\sigma = 0.1$ s/m for a non uniform cell

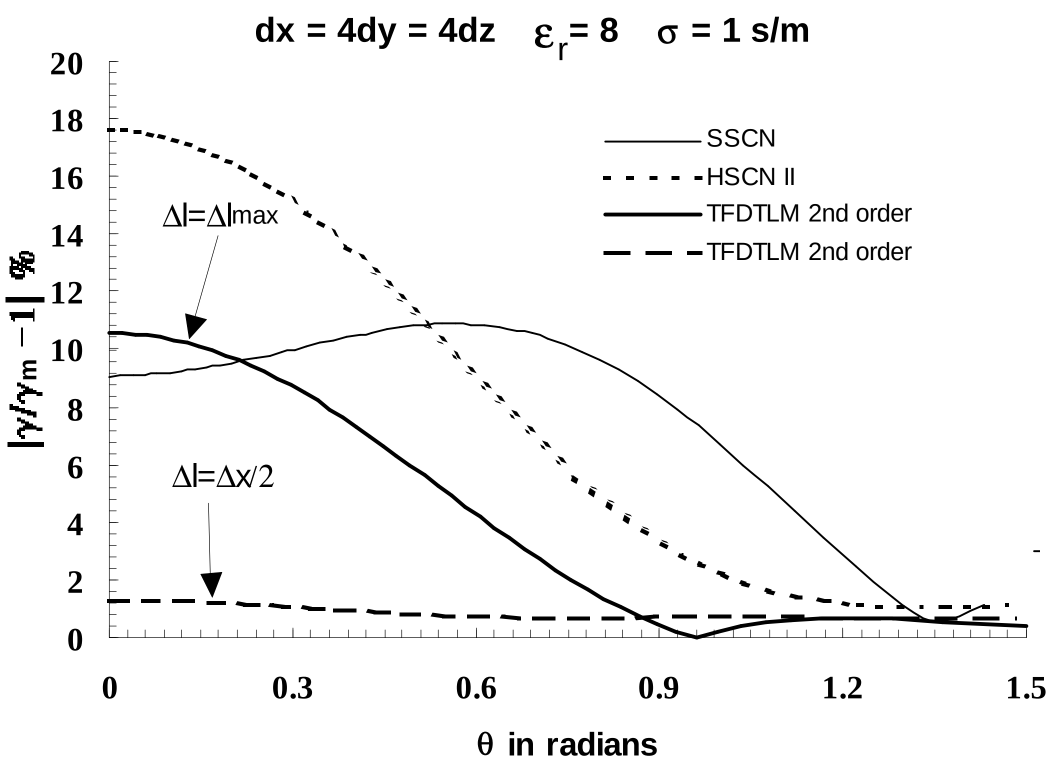


Figure 4.8 Comparison between the dispersion properties of the TFDTLM, the SSCN and HSCN II in a lossy inhomogeneous medium $\varepsilon_r = 8$, $\sigma = 1$ s/m for a nonuniform cell

## 4.4 Summary

In this chapter, the dispersion behavior of the SCN, the HSCN, and the SSCN were analyzed. The dispersion characteristics of the TFDTLM was also derived. The dispersion behavior of the TFDTLM was then compared to that of the HSCN and the SSCN. It was found that the TFDTLM has less dispersion error than the HSCN and the SSCN in modeling a lossless inhomogeneous medium. Furthermore, the dispersion error of the HSCN and the SSCN were significantly degraded as the relative dielectric constant increased. For the TFDTLM on the other hand, and even for a relatively high relative dielectric constant, a first order approximation filter was able to provide almost the same order of accuracy as with lower relative dielectric constants. The TFDTLM also proved to have superior dispersion characteristics as compared to the HSCN and the SSCN in modeling lossy inhomogeneous medium with uniform cells as well as with nonuniform cells. It has been shown that a second order approximation filter can provide an acceptable order of accuracy even for a lossy inhomogeneous medium with a relatively high loss tangent