**NONLINEAR FLEXURAL-FLEXURAL-TORSIONAL DYNAMICS OF
METALLIC AND COMPOSITE BEAMS**

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(ABSTRACT)

A combination of Newton's second law, a transformation using three consecutive Euler angles, and Taylor expansions is used to develop three nonlinear integro-differential equations describing the flexural-flexural-torsional vibration of metallic and composite beams. The twisting curvature is used to define a physical twisting variable which makes the equations of motion unique and independent of the rotation sequence of the Euler angles.

A numerical-perturbation approach is used to analyze the response of metallic and composite beams to parametric and external excitations. First, the linear eigenfunctions and natural frequencies are calculated using a combination of the state-space concept and the fundamental-matrix method. Then, the method of multiple scales is used to construct a set of nonlinear autonomous first-order ordinary-differential equations describing the slow-time modulation of the amplitudes and phases of the interacting modes in the presence of one-to-one and/or two-to-one internal resonances. The inversion symmetry, D_4 symmetry, and $O(2)$ symmetry of the system are studied using the modulation equations. The solutions of the modulation equations may be fixed points, limit cycles, or chaotic solutions. The frequency and amplitude of excitation are used as bifurcation parameters in the

study. The stability of a fixed point is determined by the eigenvalues of the Jacobian matrix of the modulation equations. A fixed point is found to lose stability in either a saddle-node bifurcation or a Hopf bifurcation. Limit cycles and their periods are located using a shooting technique coupled with a Newton-Raphson iteration method. The stability of a limit cycle is studied using Floquet theory. Chaotic attractors are identified by their frequency spectra and Poincare maps.

The inertia nonlinearity is of the softening type and dominates any single-mode planar response of a cantilever beam except that of the first mode, whereas the geometric nonlinearity is of the hardening type and dominates nonplanar motions of all modes and the planar motion of the first mode. The bending-twisting coupling breaks the inversion symmetry of the chordwise vibration of a composite beam and makes the vibration nonplanar when the excitation is planar. Nonplanar motions of a cantilever beam can be (a) steady whirling motions with constant amplitudes and phases, which correspond to fixed points of the modulation equations, (b) whirling motions of the beating type with the amplitudes and phases being periodically modulated, which correspond to limit-cycle solutions of the modulation equations, or (c) chaotic motions with the amplitudes and phases being chaotically modulated. For low frequency modes, the influence of the beam's weight and rotary inertia are negligible, but the torsional inertia is important for beams with rectangular cross sections due to their small torsional rigidities. The natural frequency bias, modal damping, and frequency and amplitude of excitation are important bifurcation parameters which control nonplanar vibrations.

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CHAPTER I

INTRODUCTION

1.1 *Statement of Problem*

The dynamic behavior of a long slender beam is of interest in connection with helicopter rotor blades, lengthy robot manipulators, spacecraft antennae, flexible satellites, wind-turbine blades, and other systems that perform large and/or complex motions.

A linear formulation shows that vibrations of an isotropic beam in the two principal planes are independent of each other; consequently, a forced motion in one principal plane is always stable in that plane. But when the vibration amplitude is large, various secondary effects (e.g., rotary inertia, warping, gravity, etc.) become significant, and several nonlinear effects (e.g., axial inertia forces, nonlinear curvatures, mid-plane stretching, lateral static loads, material and damping

nonlinearities, and buckling) come into play. Some of the secondary effects will introduce extra nonlinear terms in the equations of motion and boundary conditions. Owing to the participation of these nonlinear terms, many nonlinear phenomena are predicted and/or observed experimentally.

Because of their high strength-to-weight ratio, long fatigue life, resistance to corrosion, high damping, structural simplicity, and their possible use for aeroelastic tailoring, advanced laminated structures made of fiber-reinforced composite materials, such as boron-epoxy, graphite-epoxy, and boron-aluminum, have emerged as primary materials for rotor blades of a new generation of helicopters and for other advanced aerospace vehicles and are showing great promise for improved performance. One important elastic behavior of composite structures is the elastic coupling among extensional, bending, and torsional stiffnesses, which is desirable in some cases. For example, the extension-twisting coupling produces different twist distributions when a two-speed rotor of a helicopter is rotating at different speeds. Moreover, the bending-twisting coupling produces a pitch-flap stability of helicopter rotor blades. But, for a composite structural element, elastic couplings may result in a complicated dynamic behavior even if the vibration amplitude is small, and some secondary effects may become significant.

For a long slender beam with a compact cross section, when there is a finite flexural displacement in the low flexural rigidity direction, it is easy to induce some flexural displacements in the high flexural rigidity direction due to low torsional rigidity and elastic and nonlinear couplings. Because the beam is long, when the torsional angle and flexural displacements are large, the warping and shear strains are still negligible. Hence, for a lengthy beam, Euler-Bernoulli theory is commonly used,

which is a one-dimensional theory. If shear deformation is significant, it can also be taken into account in an averaged sense in a one-dimensional model, which is known as Timoshenko's beam theory (i.e., first-order shear theory). Because of the presence of two flexural displacements and torsion, equations describing the flexural-flexural-torsional vibration of a beam in three dimensions are needed. However, the nonlinear kinematics involved in three-dimensional analyses are complex due to the presence of large rotations, which are not vector quantities.

In this thesis, we develop mathematically consistent nonlinear equations describing the flexural-flexural-torsional vibration of composite beams, investigate the influence of nonlinearities, structural couplings, and secondary effects on the dynamic behavior of a long flexible beam, and analyze the response of a flexible beam to periodic loads.

1.2 *Literature Review*

Although the nonlinear dynamics of beams or beam-like structures has been studied for a long time, many nonlinear dynamic behaviors still need to be studied, identified, and controlled. Here we mention only representative publications that permit us to direct attention to issues of primary concern.

1.2.1 Beam Theories

The most common beam theories deal with linear, in-plane deflection of isotropic beams. They are (a) the exact elasticity equations, (b) Euler-Bernoulli (or engineering) beam theory, and (c) Timoshenko's beam theory. Chree [15] derived the exact equations in cylindrical coordinates from the general equations of the theory of elasticity, and Cowper [18] gave a plane stress elasticity solution for the fundamental frequency of a hinged-hinged beam with a rectangular cross section. Some static, plane-stress beam problems can be treated by Saint Venant's semi-inverse method [100], but only a few problems can be solved by using the theory of elasticity. In the Euler-Bernoulli beam theory [91], it is assumed that a plane cross section perpendicular to the tension axis before deformation remains plane and perpendicular to the deformed tension axis after deformation. In other words, the shear strain is neglected. From the plane stress solution for a simply supported beam carrying a uniform load [91], it is found that the in-plane shear stress is the order h/L , where h and L are the thickness and length of the beam, respectively, and the transverse normal stress is the order $(h/L)^2$. Hence, the transverse normal stress is neglected in most of beam analyses, but the shear stress is not negligible for problems with large aspect ratios h/L , short wavelengths, or high frequencies. The shear strain is assumed to be the same at all points over a given cross section in Timoshenko's beam theory [97-99], which is known as the first-order shear theory. For the first-order shear theory, a shear correction coefficient [17,98], the value of which depends on the shape of the cross section, is introduced to account for the fact that the shear stress and shear strain are not uniformly distributed over the cross section. Krishna Murty [57] included shear deformation in his three linear

second-order approximate equations for transverse oscillations of uniform short beams. Heyliger and Reddy [41] presented a higher-order shear deformation theory, in which some quadratic terms are included and no shear correction coefficient is needed. Bauchau and Hong [7,8], Stemple and Lee [95], and Kane et al. [52] accounted for the warpings induced by torsion and/or bendings in their equations of motion of composite beams. For the Euler-Bernoulli theory, only one displacement variable is used. For Timoshenko's beam theory, two variables are introduced. For high-order shear deformation theories, three [41,57] or more [92] variables are needed. Definitely, if in-plane shear deformations, warping induced by out-of-plane shear deformations, and torsion-related warpings need to be considered, more variables are required, and the finite-element method is commonly used in these analyses [8,95]. However, these considerations about shear deformation and warping may not apply to the study of nonlinear vibrations, where the fundamental modes exhibiting low natural frequencies are the quantities of interest [41].

When the deflections and rotations are large, several nonlinearities must be considered. To derive and/or solve the equations of motion including all of the beam effects and nonlinearities is very difficult, especially for a beam posed in three-dimensional space. In the existing literature, one often finds that a particular effect is introduced in an ad hoc fashion and all other effects are neglected based on the authors' judgement or on the results of former work. To investigate large amplitude whirling motions of a simply supported beam, Ho, Scott, and Eisley [43] accounted for large deformations through the use of Green's strain measure in the longitudinal direction. Heyliger and Reddy [41] and Sheinman and Adan [92] used the Von Karman strain components to account for large deflections and rotations. Although Bolotin [10] and Moody [67] showed that nonlinear-inertia effects are far

more significant than nonlinear elasticity effects on the vibrations of thin rods, Crespo da Silva and Glynn [19,20] used the extended Hamilton's principle and included the nonlinear inertia and the nonlinear bending and twisting curvatures in their derivation of the equations of motion describing the nonplanar, nonlinear dynamics of an inextensional beam. They showed that the generally neglected nonlinear terms arising from the curvature are the same order as the nonlinear terms due to inertia and they might have a significant influence on the response of the system. An experimental study by Evan-Iwanowski et al. [30] also shows that the nonlinear geometric terms have definite effects on the nonstationary parametric response of a straight column. Friedmann [33] indicated that rotary-wing aeroelasticity is inherently nonlinear and that the nonlinear terms in the equations of motion are important in the study of aeroelastic stability. Crespo da Silva and Hodges [23,24] derived the differential equations of motion for a rotating beam, such as a helicopter rotor blade, with the objective of retaining all possible contributions due to higher-order nonlinearities and investigated the influence of these terms on the motion of a helicopter rotor blade. They found that the most significant cubic nonlinear terms are those associated with structural geometric nonlinearities in the torsion equation. Busby and Weingarten [12] used a combination of the finite-element method and the method of averaging to study the influence of induced stretching on the steady-state response of a beam under periodic excitation. Sathyamoorthy [86] gave a survey of the studies that investigate the effects of geometric nonlinearities due to stretching and curvature and material nonlinearities of the Ramberg-Osgood and Ludwick types in beams.

In the development of nonlinear equations of motion for a flexible beam posed in the three-dimensional space, three successive Euler-like rotations are commonly used

to relate the deformed and undeformed states [4,7,19,21,26,45-47,52,59,65,83,84]. Alkire [4] showed that the transformation matrix and curvatures obtained by using Euler-like rotations are unique, but the form varies according to the choice of the twist variable. Hence, different sequences of Euler rotations will result in different equations of motion [4], and the equations are asymmetric because there are implicit asymmetric terms included in the twist variable [45,46,84]. Hodges and Dowell [46] used Hamilton's principle and Newton's second law to develop a comprehensive set of equations with quadratic nonlinearities that describe the dynamics of beams. They considered the offsets between the mass centroid and elastic and area centroid axes and identified several nonlinear terms. Dowell, Traybar, and Hodges [26] devised a simple, non-rotating beam with a tip weight to evaluate the theory of Hodges and Dowell [46]. The results show that there are systematic differences between theory and experiment when tip deflections become large. In a follow-on work of [46], Hodges et al. [45] pointed out that the nonlinear equations of motion governing flexural deformations cannot be consistent unless they include all nonlinear terms up through third order. Rosen et al. [82] derived a more accurate set of equations than those of Hodges and Dowell [46] by including some nonlinear terms with order higher than two. Their numerical results are in good agreement with the experimental data of Ref.[26]. Rosen et al. [83,84] added cubic terms to the curvatures used in Ref. [82] to improve the modeling.

Hinnant and Hodges [42] combined multibody and finite-element technology to analyze the nonlinear vibrations of a cantilever beam. Their results are in good agreement with the experimental data in [26]. In the development of nonlinear equations of motion of twisted rotor blades, Houbolt and Brooks [48] identified many coupling-type terms which are associated with centrifugal forces. Hodges and Peters

[47] showed that the twisting angle due to flexural displacements must be considered in the derivation of the lateral buckling equation of beams and in the calculation of buckling loads. Peters [78] pointed out that the classical buckling load formulas that neglect the effect of bending in the plane of greatest flexural rigidity prior to buckling are not always conservative for infinitely slender beams. Rosen et al. [85] argued that the equations of motion derived by using three Euler-like rotations seem to contain the basis for coupled bending-torsion buckling because the first flapwise frequency tends to zero as the static tip load increases.

A review of recent developments in the analysis of laminated beams and plates is given by Kapania and Raciti [53,54]. Sathyamoorthy [87] reviewed the use of finite-element methods in the analysis of beams.

1.2.2 Secondary Effects

Because the ratio E/G of Young's modulus to shear modulus is between 20 to 50 in modern composites but between 2.5 to 3.0 in isotropic materials [2], shear effects are significant for composite materials. Nayfeh [70] investigated the effects of transverse shear and rotary inertia on the nonlinear transverse vibrations of inhomogeneous beams with finite axial restraints. The results show that the frequency increases with amplitude and axial restraint, and the transverse shear and rotary inertia decrease the frequency. Huang [50] derived the frequency and normal mode equations of uniform Timoshenko beams with simple boundary conditions. It was shown that shear deformation and rotary inertia have definite influence on the natural frequencies and mode shapes of beams, especially stubby beams. In the case of

very large static bending of a cantilever beam, Sinclair [93] found that both the contributions of the shear and longitudinal deformations to the beam response are of the same order. Rosen et al. [85] indicated that the influence of rotary inertia on the frequencies of the lowest modes is insignificant. Kapania and Raciti [55] indicated that, for thin composite plates whose length/thickness is greater than 50, transverse shear effects can be neglected in the large deflection theory. Rao et al. [79] stated that when $L/r > 100$, where r denotes the radius of gyration, the shear effect and rotary inertia are negligible. Adams et al. [2] stated that shear deformation can be neglected for isotropic beams if the aspect ratio $L/h > 20$; but for unidirectional carbon-fiber-reinforced plastics, an aspect ratio the order of 100 is necessary if shear effects are to be ignored.

It is well known that the damping of structures has a large influence on the amplitude and phase angle in the frequency region near resonance and that the damping ratio of composite materials, especially of nonmetal composites, is much higher than that of structural steels. Furthermore, the damping ratio of structures exhibits both frequency and amplitude dependence. All damping ratios in the experimental results of Schultz and Tsai's [89] and Ray and Bert's [80] are in the range of 0.02% to 2.8%. Using the modified Kennedy-Pancu method [77], Siu and Bert [94] obtained analytically the damping ratios of laminated boron-epoxy plates with all edges free and various angles of orientation. The damping ratios they obtained are in the range of 0.09% to 3.31%. For isotropic materials, experimental results [9,89] usually show that modal dampings decrease as the modal number increases. But for a composite beam, the damping ratio may increase with mode number [89]. Adams et al. [2] found that the damping ratio of composite materials can be as high as 5.5 %. The highest damping ratio of boron-epoxy plates used in Clary's [16] experiment is 5.3 %. Adams

et al. [3] indicated that the damping capacity of composites under torsion is higher than that under flexure. A comprehensive mathematical technique has been developed by Ni et al. [76] for predicting the damping of laminated composite beams. They showed that the torsional motion induced by bending-twisting coupling may result in high modal damping ratios for flexural vibrations. Air damping is due to both air viscosity and inertia effects. The effect of air damping was found to be significant for beams of low damping and high modulus at high amplitudes [2]. Experimental results obtained by Adams et al. [2] also show that the damping ratios of composite materials increase with temperature.

Because the change in the gravitational potential of a nonlinear dynamic system cannot be exactly balanced by changes in the elastic potential resulting from static deflections, we cannot use the static equilibrium position as a reference state for a displacement variable and then delete the gravitational potential from the equations [72]. Sato et al. [88] treated parametric vibrations of a simply-supported horizontal beam carrying a concentrated mass under the influence of gravity. Their results show that, in addition to parametric resonances, forced resonances occur due to initial static deflections. Zavodney [104] studied a two-degree-of-freedom system and indicated that the static deflection results in a quadratic term.

For a compact beam, warping effect is usually neglected, but its influence on the torsional rigidity can be accounted for by using the theory of elasticity [100]. Evensen [31] showed that the influence of the boundary conditions decreases as the mode number increases.

1.2.3 Nonlinear Dynamic Behaviors

Frequency entrainment, multiple solutions, jumps, subharmonic resonances, superharmonic resonances, supersubharmonic resonances, combination resonances, internal resonances, saturation, quenching, Hopf bifurcation, limit cycles, symmetry-breaking and period-multiplying bifurcation, chaos, etc. are nonlinear phenomena, which cannot be studied or predicted by linear theory.

For a nonlinear system, the predominant output response to an input sinusoid may be at a different frequency, either a subharmonic or superharmonic of the input frequency [72].

In the absence of an input, a system may possess a self-excited (limit-cycle) response [68,72], e.g., the van der Pol oscillator. The response of a self-excited system to a harmonic input may be a limit-cycle, a quasiperiodic, or a chaotic response.

For some nonlinear dynamic systems, although the equations describing their temporal evolution are entirely deterministic and without noise, their future developments are chaotic and unpredictable, and the motions sustained are irregular aperiodic oscillations. Because these systems are extremely sensitive to initial conditions, they are said to possess memory.

A dynamical system is said to be fluttering if it has a stable closed orbit. Often flutter is suggested if a system linearized about a fixed point has two complex conjugate eigenvalues with a positive real part. However, a general proclamation of this sort

is false because results from local stability analysis cannot predict the global behavior of the system. Tousi and Bajaj [101] used the method of averaging and a combination of a shooting technique and a scheme proposed by Chau and Lin [14] to study in detail the cascades of period-doubling bifurcations of a two degree-of-freedom mechanical system. They found that some of the sequences of period-doubling bifurcations are compatible with the Feigenbaum universality constant. For a single-degree-of-freedom, second-order, nonlinear dynamic system subject to a periodic force, the steady-state response depends on the starting time as well as two initial conditions, and the system is called a third-order system. Symmetry-breaking and period-doubling bifurcations and chaotic motions can only occur in nonlinear systems whose order is no less than three. For a system with two or more degrees of freedom, Hopf bifurcations of a limit cycle (or secondary Hopf bifurcation) are possible.

1.2.4 External Resonances, Internal Resonances, and Structural Couplings

To cause chaotic motions in a one-degree-of-freedom system, one needs a large level of excitation, whereas chaos can occur in a two-degree-of-freedom system at low levels of excitation in the presence of internal resonances.

Henry and Tobias [40] investigated theoretically and experimentally the one-to-one modal coupling of a two-degree-of-freedom system. Crespo da Silva and Glynn [21] used a perturbation method to investigate the planar and out-of-plane response of a clamped-clamped/sliding beam subject to a planar distributed harmonic excitation.

In the analysis of large amplitude out-of-plane oscillations of a thin circular ring, Maganty and Bickford [59] studied the case of a one-to-one internal resonance and found amplitude-modulated motions. Crespo da Silva and Glynn [22] studied the effect of support asymmetry on the free, nonplanar resonant motions of a fixed-free beam with a one-to-one internal resonance. They showed that the torsional-flexural coupling is important when support conditions are asymmetric. Crespo da Silva and Zaretsky [25] investigated the in-plane, three-to-one modal coupling between the third and second modes of a cantilever beam. Modal coupling and the beating phenomenon were observed in the experiments of Ray and Bert [80], which deal with large-amplitude vibrations of a simply supported flexible beam.

For a slender, compact, uniform, initially straight, linearly elastic rod subject to a lateral harmonic excitation in one principal direction, Haight and King [38] investigated the effect of nonlinear inertia terms on the response of the first three modes by using the Ritz averaging method. They identified unstable regions in the planar response curves which correspond to parametrically excited nonplanar motions, without obtaining quantitative results of the resulting nonplanar motions. From their experimental work, they concluded that the nonplanar motions are steady whirling motions with the point on the neutral axis tracing an elliptical path. Using stability analyses, they showed that damping can cause the disappearance of the planar instability phenomenon. Hyer [49] used Haight and King's equations to analyze quantitatively the whirling motions of the same problem. He concluded that all nonplanar vibrations are steady whirling motions. But he indicated that the nonlinear curvature, damping, shear deformation, and rotary inertia might affect the whirling response characteristics. For the free motions of compact beams with fixed

ends, Ho et al. [44] showed that in resonant cases whirling motions are of the beating type, whereas in non-resonant situations they have a steady-state behavior.

Taking into account the nonlinear inertia terms and considering linear curvature in the differential equations of motion, Haight and King [37] obtained the planar frequency-response curves of a parametrically excited rod by means of an averaging method. They also identified unstable regions in the planar response curves, which correspond to stable motions in the other principal plane. They did not find nonplanar motions. Ho, Scott and Easley [43] showed that the nonlinear terms due to stretching are cubic and cause whirling motions. They concluded that the larger the forcing amplitude is the more likely whirling motions occur. They found that, in some parameter range, both planar and nonplanar motions are possible. They indicated that the regions in which both planar and nonplanar motions are unstable might correspond to beating motions or to steady-state motions with a more complex modal structure. Nayfeh and Mook [72] included the motion-induced stretching effect in the governing equations of beams and investigated many cases of primary and secondary resonances. They showed that, in case of modal interaction, if the high-frequency mode is directly excited, the response of the low-frequency mode could be very large due to an energy transfer from the high-frequency mode. But, if the low-frequency mode is directly excited, the response of the high-frequency mode is usually small.

Tezak et al. [96] studied nonlinear transverse oscillations of parametrically excited columns, including the effects of mid-plane stretching. The results exhibit the jump phenomenon and show that two rather different excitations can produce practically the same response when there is an internal resonance. Nayfeh et al. [73]

investigated forced nonlinear oscillations of prismatic beams with mid-plane stretching, they found that the response to a harmonic excitation can involve strong modal interactions when the natural frequencies are commensurable. In their study of the response of a hinged-clamped beam to a simple harmonic excitation, Nayfeh et al. [74] considered the effects of geometric (stretching) and material nonlinearities (Ramberg-Osgood material). They showed that multi-mode nonlinear analyses predict complicated responses in case of modal coupling, which cannot be predicted by single-mode nonlinear analyses or by multi-mode linear analyses. Their results also show that the response may experience period-doubling bifurcations culminating in a chaotic motion.

Extension-twisting and bending-twisting couplings of a composite beam have been extensively studied [1,7,8,55,65,103]. Krenk [56] showed that coupling between extension and torsion exists even for pretwisted isotropic beams. Kapania and Raciti [55] showed that unsymmetrically laminated beams are characterized by a bending-stretching coupling, which induces a quadratic nonlinearity.

Nayfeh [68] and Nayfeh and Mook [72] studied primary, subharmonic, superharmonic, supersubharmonic, principle parametric, fundamental parametric, and combination resonances of many nonlinear systems. Zavodney [104] gave a review of the parametric vibration of nonlinear systems. A small parametric excitation can produce a large response when the excitation frequency is not close to any of the natural frequencies of the system [72]. Furthermore, owing to certain nonlinear coupling terms and the presence of commensurability among the natural frequencies, a parametrically excited mode may in turn indirectly excite other modes through an autoparametric resonance. These phenomena are not disclosed by a linear

approximation to these systems. In the case of primary resonance, an external force of order ε^2 can result in a response of order ε , where ε denotes a small parameter. Moreover, modes with different frequencies can be excited simultaneously by a single-frequency harmonic forcing function through internal resonance. Miles [64] studied the primary resonant motion of the free surface of a liquid in a vertical circular cylinder which is subject to a simple harmonic lateral excitation and showed that when damping is sufficiently small, limit cycles and chaotic motions are possible. Ginsberg [35] studied the response of a two-degree-of-freedom system to a primary resonant excitation in the presence of a one-to-one internal resonance. Ginsberg's results show that increasing the damping coefficient beyond a limiting value causes the elimination of bifurcation points and two-mode responses.

Tso [102] studied parametrically induced torsional vibrations due to shortening effects in an elastic cantilever of rectangular cross section under a dynamic axial loading. It was shown that when the applied frequency is close to twice one of the longitudinal natural frequencies, the corresponding torsional modes may be excited parametrically. Burton and Kolowith [11] presented experimental and analytical results for the response of a flexible, parametrically excited, vertically mounted beam. They added a nonlinear damping term to the equations of Ref. [19]. They concluded that the steady-state amplitudes of the periodic motions of the fourth mode are well predicted by theory. But they observed chaotic motions at high excitation levels. For the same system, Haddow and Hasan [36] also observed chaotic motions, nonplanar motions, and in-plane modal interactions in their experimental work.

Nayfeh and Balachandran [71] reviewed theoretical and experimental studies of the influence of modal interactions due to one-to-one, two-to-one, three-to-one, and

combination internal resonances on the nonlinear response of harmonically excited structural and dynamical systems.

1.3 Objectives and Scope of the Dissertation

In this thesis, we employ a transformation using three consecutive Euler angles and Newton's second law to derive mathematically consistent nonlinear equations of motion and nonlinear boundary conditions describing the flexural-flexural-torsional vibration of flexible metallic and composite beams. The method of multiple scales [68,69] is then used to determine approximate solutions of these equations and boundary conditions for the case of cantilever metallic and composite beams. We study the effects of nonlinear curvatures, nonlinear inertias, gravity, and structural couplings and describe qualitatively and quantitatively the planar and nonplanar motions. Some of the main characteristics of the nonplanar motions are determined, and some of the possible bifurcations of the responses when the excitation frequency or amplitude is varied are investigated. We also demonstrate the existence of Hopf, symmetry-breaking, and period-doubling bifurcations, and chaotically modulated motions. Some control related characteristics of slewing metallic and composite beams are also studied.

Next, we present a brief summary of each of the following chapters.

In Chapter II, we derive the nonlinear equations and boundary conditions governing the motion of metallic and composite beams, discuss the characteristics of the

obtained equations, and use D'Alembert principle to derive the forcing terms induced by the motion of the boundaries.

In Chapter III, we study the flexural-flexural response of a cantilever isotropic beam to a principal parametric resonance in the presence of a one-to-one internal resonance between the two flexural modes.

In Chapter IV, we study the flexural-flexural-torsional response of a cantilever isotropic beam to a principal parametric resonance in the presence of a one-to-one internal resonance between the two flexural modes and a two-to-one internal resonance between the torsional mode and the two flexural modes.

In Chapter V, we study planar and nonplanar flexural-flexural responses of a cantilever isotropic beam to a primary resonant excitation of an in-plane flexural mode in the presence of a one-to-one internal resonance between the in-plane and out-of-plane flexural modes.

In Chapter VI, we obtain the exact linear eigensolutions of a slewing isotropic beam, which includes the effects of rotary inertia and the hub's inertia. For a composite beam made of three prismatic segments and exhibiting bending-twisting coupling, a powerful technique based on the state-space concept and the fundamental-matrix method is used to obtain the exact eigensolutions. A finite-element method is also used to study the same problem. The results obtained by using the fundamental-matrix and finite-element methods are compared with each other. Some dynamic characteristics and control related properties of slewing beams are studied.

In Chapter VII, we study the flexural-flexural-torsional response of a cantilever, symmetrically laminated composite beam to a flapwise resonant excitation in the presence of a two-to-one internal resonance between the out-of-plane flexural mode and the directly excited bending-twisting mode.

In Chapter VIII, we study the flexural-flexural-torsional response of a cantilever, symmetrically laminated composite beam to a chordwise resonant excitation in the presence of a two-to-one internal resonance between the directly excited flexural mode and the out-of-plane bending-twisting mode.

In Chapter IX, we draw some conclusions and give few recommendations for further study.

CHAPTER II

NONLINEAR EQUATIONS OF MOTION

A vector approach using Newton's second law is used to develop the nonlinear equations describing the flexural-flexural-torsional vibrations of metallic and composite beams. Alternatively, one can use a variational approach using the extended Hamilton's principle. Three consecutive Euler angles are used to relate the deformed and undeformed states. Because the twisting-related Euler angle ϕ is not an independent Lagrangian coordinate, twisting curvature is used to define the twist angle, and the resulting equations of motion are symmetric and independent of the rotation sequence of the Euler angles. The equations contain structural coupling terms and quadratic and cubic nonlinearities due to curvature and inertia. Some comparisons with other derivations are made, and the characteristics of the modeling are addressed.

2.1 The Complete Equations of Motion of Slewing

Composite Beams

For a long slender beam with small strains and moderate deflections and rotations, the following assumptions are usually true: (a) warping, shear deformation, and Poisson effects are negligible; and (b) bending and twisting moments at any arbitrary position along the beam are proportional, respectively, to the nonlinear expressions of the local bending curvature and twist rate. Although warping of the cross section is neglected, its influence on the torsional rigidity can be accounted for by using the theory of elasticity [100]. To simplify the analysis, we assume that the beam is initially straight, has closed cross sections and no pretwist angle, and that the mass centroid, area centroid, and elastic axes coincide.

Two coordinate systems are introduced, as shown in Figure 2.1. The x - y - z coordinate system is fixed on the hub, and we will call it the reference coordinate system. The $\xi - \eta - \zeta$ system, which will be called the local coordinate system, coincides with the principal axes of the observed differential element. Because assumption (a) implies that the Euler-Bernoulli hypothesis can be applied, the shape of the cross-section remains same under loads and the ξ axis is tangent to the neutral axis of the beam. Consequently, a differential beam element can be considered a rigid body and we can develop a one-dimensional model. Hence, the motion can be described by three translational displacements u , v and w of the mass center and three rotational displacements. Here we use three consecutive Euler angles ψ , θ , and ϕ to describe the rotation from the undeformed position to the deformed one. The rotations are executed in the order shown in Figure 2.2 : first, ψ around the z axis; second, θ

around the y_1 axis; and last, ϕ around the ξ axis. In other words, we assume deformations in the following sequence : u, v, w, and ϕ . This sequence is only tied to the mathematical modeling.

It follows from Figure 2.2 that the transformation that relates the undeformed coordinate system x-y-z to the deformed coordinate system $\xi - \eta - \zeta$ is

$$\begin{Bmatrix} \vec{i}_\xi \\ \vec{i}_\eta \\ \vec{i}_\zeta \end{Bmatrix} = [T] \begin{Bmatrix} \vec{i}_x \\ \vec{i}_y \\ \vec{i}_z \end{Bmatrix}, \quad (2.1)$$

where $\vec{i}_\xi, \vec{i}_\eta, \vec{i}_\zeta$ are base vectors of the local coordinate system,

$\vec{i}_x, \vec{i}_y, \vec{i}_z$ are base vectors of the reference coordinate system,

and the transformation matrix $[T]$ is given by

$$\begin{aligned} [T] &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta \cos \psi & \cos \theta \sin \psi & -\sin \theta \\ -\cos \phi \sin \psi + \sin \phi \sin \theta \cos \psi & \cos \phi \cos \psi + \sin \phi \sin \theta \sin \psi & \sin \phi \cos \theta \\ \sin \phi \sin \psi + \cos \phi \sin \theta \cos \psi & -\sin \phi \cos \psi + \cos \phi \sin \theta \sin \psi & \cos \phi \cos \theta \end{bmatrix}. \end{aligned} \quad (2.2)$$

According to transformation theory,

$$[T]^{-1} = [T]^T. \quad (2.3)$$

We can represent the displacements in terms of the Euler coordinate $\tilde{s}$ as $u(\tilde{s}, t)$, $v(\tilde{s}, t)$, $w(\tilde{s}, t)$, $\phi(\tilde{s}, t)$, $\theta(\tilde{s}, t)$, and $\psi(\tilde{s}, t)$, where $\tilde{s}$ is the deformed arc length measured from the origin of the x-y-z coordinate system to the deformed position of the observed element. This is a one-dimensional model. Alternatively, we can represent the displacements in terms of the Euler coordinates x, y, and z as $u(x,y,z,t)$, $v(x,y,z,t)$, $w(x,y,z,t)$, $\phi(x,y,z,t)$, $\theta(x,y,z,t)$, and $\psi(x,y,z,t)$, where (x,y,z) denotes the instantaneous location of the mass centroid of the observed element. This is also a one-dimensional model because $v = y$ and $w = z$. We can also use the Lagrangian coordinate s to represent the displacements as $u(s,t)$, $v(s,t)$, $w(s,t)$, $\phi(s,t)$, $\theta(s,t)$ and $\psi(s,t)$, where s is the distance from the origin of the x-y-z coordinate to the undeformed position of the observed element. To avoid the calculation of comoving derivatives and troubles in evaluating Euler strains in the derivation, we use the Lagrangian coordinate s .

To obtain the relation between the curvatures and the Euler angles, we use Kirchoff's kinetic analogy [58]. It follows from Figures 2.1 and 2.2 that the angular velocity of the element is

$$\vec{\omega}(s,t) = \dot{\psi} \vec{i}_z + \dot{\theta} \vec{i}_{y_1} + \dot{\phi} \vec{i}_\xi + \dot{\Theta} \vec{i}_y ,$$

where the dot stands for $\partial/\partial t$. Using (2.1) and knowing that

$$\vec{i}_{y_1} = \cos \phi \vec{i}_\eta - \sin \phi \vec{i}_\zeta ,$$

we obtain

$$\vec{\omega}(s,t) = \omega_\xi \vec{i}_\xi + \omega_\eta \vec{i}_\eta + \omega_\zeta \vec{i}_\zeta ,$$

where

$$\omega_{\xi} = \dot{\phi} - \dot{\psi} \sin \theta + \dot{\Theta} \sin \psi \cos \theta , \quad (2.4)$$

$$\omega_{\eta} = \dot{\psi} \cos \theta \sin \phi + \dot{\theta} \cos \phi + \dot{\Theta} \cos \psi \cos \phi + \dot{\Theta} \sin \psi \sin \theta \sin \phi , \quad (2.5)$$

$$\omega_{\zeta} = \dot{\psi} \cos \theta \cos \phi - \dot{\theta} \sin \phi - \dot{\Theta} \cos \psi \sin \phi + \dot{\Theta} \sin \psi \sin \theta \cos \phi . \quad (2.6)$$

According to Kirchoff's kinetic analogy [58], the curvatures can be obtained from (2.4)-(2.6) by replacing $\partial/\partial t$ with $\partial/\partial \tilde{s}$ and putting $\Theta = 0$ (because $\Theta \neq 0$ corresponds to a rigid-body motion). Hence the twisting curvature $\tilde{\rho}_{\xi}$ and bending curvatures $\tilde{\rho}_{\eta}$ and $\tilde{\rho}_{\zeta}$ are given by

$$\tilde{\rho}_{\xi} = \phi^{+} - \psi^{+} \sin \theta , \quad (2.7)$$

$$\tilde{\rho}_{\eta} = \psi^{+} \cos \theta \sin \phi + \theta^{+} \cos \phi , \quad (2.8)$$

$$\tilde{\rho}_{\zeta} = \psi^{+} \cos \theta \cos \phi - \theta^{+} \sin \phi , \quad (2.9)$$

where the plus stands for $\partial/\partial \tilde{s}$. We note that (2.7)-(2.9) can also be obtained by using the definition of curvatures, as shown in Appendix A. If e denotes the normal strain along the ξ axis, we have

$$d\tilde{s} = (1 + e)ds . \quad (2.10)$$

Moreover, letting

$$\rho_i = \tilde{\rho}_i(1 + e) \quad (2.11)$$

and substituting (2.10) and (2.11) into (2.7)-(2.9), we obtain

$$\rho_{\xi} = \phi' - \psi' \sin \theta , \quad (2.12)$$

$$\rho_{\eta} = \psi' \cos \theta \sin \phi + \theta' \cos \phi , \quad (2.13)$$

$$\rho_{\zeta} = \psi' \cos \theta \cos \phi - \theta' \sin \phi , \quad (2.14)$$

where the prime denotes $\partial/\partial s$. Here we must state that the ρ_i are "normalized" and not real curvatures unless the beam is inextensional.

Using the transport theorem, Kirchoff's kinetic analogy, and (2.1), we obtain

$$\frac{\partial}{\partial \tilde{s}} \begin{Bmatrix} \vec{i}_{\xi} \\ \vec{i}_{\eta} \\ \vec{i}_{\zeta} \end{Bmatrix} = [\tilde{K}] \begin{Bmatrix} \vec{i}_{\xi} \\ \vec{i}_{\eta} \\ \vec{i}_{\zeta} \end{Bmatrix} = [\tilde{K}][\mathcal{T}] \begin{Bmatrix} \vec{i}_x \\ \vec{i}_y \\ \vec{i}_z \end{Bmatrix} , \quad (2.15)$$

where the curvature matrix $[\tilde{K}]$ is given by

$$[\tilde{K}] \equiv \begin{bmatrix} 0 & \tilde{\rho}_{\zeta} & -\tilde{\rho}_{\eta} \\ -\tilde{\rho}_{\zeta} & 0 & \tilde{\rho}_{\xi} \\ \tilde{\rho}_{\eta} & -\tilde{\rho}_{\xi} & 0 \end{bmatrix} . \quad (2.16)$$

Using (2.15) and (2.1) and the fact that the base vectors along the coordinates x , y and z are not functions of $\tilde{s}$, we obtain

$$\frac{\partial[\mathcal{T}]}{\partial \tilde{s}} = [\tilde{K}][\mathcal{T}] . \quad (2.17)$$

Substituting (2.10) and (2.11) into (2.17) yields

$$\frac{\partial[\mathcal{T}]}{\partial s} = [K][\mathcal{T}] \quad (2.18)$$

In the derivation that follows, the following nomenclature will be used:

m : mass per unit of undeformed length

$\{F\}$: internal resultant forces defined in local frame , $\{F\} = \{F_1 F_2 F_3\}^T$

$\{Q^L\}$: load per unit of undeformed length, measured in local frame
(e.g., aerodynamic loads)

$\{Q^R\}$: load per unit of undeformed length, measured in reference frame
(e.g., gravity)

$\{a\}$: accelerations defined in reference frame , $\{a\} = \{a_x a_y a_z\}^T$

$\{d\}$: displacements defined in reference frame , $\{d\} = \{u v w\}^T$

$[C_1]$: translational damping matrix of a unit of undeformed length, defined in reference frame

$\{M\}$: internal resultant moments defined in local frame , $\{M\} = \{M_1 M_2 M_3\}^T$

$\{M^L\}$: load per unit of undeformed length, measured in local frame

$\{M^R\}$: load per unit of undeformed length, measured in reference frame

$\{A\}$: rotary inertia per unit of undeformed length, defined in local frame

$$\{A\} = \{A_t A_r A_z\}^T$$

$\{\omega\}$: angular velocity defined in local frame

$[C_2]$: shear damping matrix of a unit of undeformed length, defined in local frame
(i.e., twisting and shear dampings)

It follows from the free-body diagram in Figure 2.3 and Newton's second law that, in the reference frame, the force-equilibrium equation is

$$\begin{aligned}
& -[\mathcal{T}]^T\{F\} + [[\mathcal{T}]^T + \frac{\partial[\mathcal{T}]^T}{\partial\tilde{s}} d\tilde{s}][\{F\} + \frac{\partial\{F\}}{\partial\tilde{s}} d\tilde{s}] \\
& + [\mathcal{T}]^T\{Q^L\}ds + \{Q^R\}ds = m ds \{a\} + [C_1]\{\dot{d}\}ds ,
\end{aligned} \tag{2.19}$$

and the moment-equilibrium equation is

$$\begin{aligned}
& -[\mathcal{T}]^T\{M\} + [[\mathcal{T}]^T + \frac{\partial[\mathcal{T}]^T}{\partial\tilde{s}} d\tilde{s}][\{M\} + \frac{\partial\{M\}}{\partial\tilde{s}} d\tilde{s}] + [\mathcal{T}]^T[d\tilde{s} \vec{i}_x(\vec{F} + \frac{\partial\vec{F}}{\partial\tilde{s}} d\tilde{s})] \\
& + [\mathcal{T}]^T\{M^L\}ds + \{M^R\}ds = [\mathcal{T}]^T\{A\}ds + [\mathcal{T}]^T[C_2]\{\omega\}ds ,
\end{aligned} \tag{2.20}$$

Neglecting higher-order terms and using (2.10), (2.11), and (2.18), we reduce (2.19) and (2.20) into

$$\frac{\partial}{\partial s} \{[\mathcal{T}]^T\{F\}\} + [\mathcal{T}]^T\{Q^L\} + \{Q^R\} = m\{a\} + [C_1]\{\dot{d}\} , \tag{2.21}$$

$$\begin{aligned}
& [\mathcal{T}]^T[\frac{\partial\{M\}}{\partial s} + [K]^T\{M\} + \{(1 + \Theta)\vec{i}_x\vec{F}\}] + [\mathcal{T}]^T\{M^L\} + \{M^R\} = \\
& [\mathcal{T}]^T\{A\} + [\mathcal{T}]^T[C_2]\{\omega\} .
\end{aligned} \tag{2.22}$$

To investigate the torsional motion $\phi(s,t)$, which is defined in the local reference frame, we transform (2.22) into local coordinates by pre-multiplying it with $[T]$. The result is

$$\frac{\partial\{M\}}{\partial s} + [K]^T\{M\} + \{(1 + \Theta)\vec{i}_x\vec{F}\} + \{M^L\} + [T]\{M^R\} = \{A\} + [C_2]\{\omega\} . \tag{2.23}$$

To complete the formulation, we need to express $\{a\}$, $\{A\}$, $\{F\}$, and $\{M\}$ in terms of u , v , w , ψ , θ , and ϕ .

Because the hub can only rotate, the acceleration of the observed element is given by the formula

$$\vec{a} = \frac{d\vec{\omega}_h}{dt} \times \vec{r} + \vec{\omega}_h \times (\vec{\omega}_h \times \vec{r}) + 2\vec{\omega}_h \times \vec{v}_{rel} + \vec{a}_{rel} , \quad (2.24)$$

where

$\vec{\omega}_h$ is the angular velocity of the hub,

$\vec{r}$ is the position vector of the element in the reference frame,

$\vec{v}_{rel}$ is the element's velocity relative to the reference frame, and

$\vec{a}_{rel}$ is the element's acceleration relative to the reference frame. It follows from Figures 2.1 and 2.2 that

$$\vec{r} = (s + u)\vec{i}_x + v\vec{i}_y + w\vec{i}_z , \quad \vec{\omega}_h = \dot{\Theta}\vec{i}_y , \quad \frac{d\vec{\omega}_h}{dt} = \ddot{\Theta}\vec{i}_y ,$$

$$\vec{v}_{rel} = \dot{u}\vec{i}_x + \dot{v}\vec{i}_y + \dot{w}\vec{i}_z , \quad \vec{a}_{rel} = \ddot{u}\vec{i}_x + \ddot{v}\vec{i}_y + \ddot{w}\vec{i}_z .$$

Substituting these expressions into (2.24) yields the components of $\vec{a}$ as

$$a_x = \ddot{\Theta}w - \dot{\Theta}^2(s + u) + 2\dot{\Theta}\dot{w} + \ddot{u} , \quad (2.25)$$

$$a_y = \ddot{v} , \quad (2.26)$$

$$a_z = -\ddot{\Theta}(s + u) - \dot{\Theta}^2w - 2\dot{u}\dot{\Theta} + \ddot{w} . \quad (2.27)$$

To calculate the rotary inertias A_ζ , A_η , and A_ξ , we use the Euler equation

$$\{A\} \equiv [J]\{\dot{\omega}\} + \begin{bmatrix} 0 & -\omega_\zeta & \omega_\eta \\ \omega_\zeta & 0 & -\omega_\xi \\ -\omega_\eta & \omega_\xi & 0 \end{bmatrix} [J]\{\omega\} , \quad (2.28)$$

where $[J]$ is the inertia matrix per unit length of the beam. Because the local coordinates coincide with the principal axes of the element,

$$[J] = \begin{bmatrix} j_\xi & 0 & 0 \\ 0 & j_\eta & 0 \\ 0 & 0 & j_\zeta \end{bmatrix}. \quad (2.29)$$

Hence the Euler equations reduce to

$$A_\xi = j_\xi \dot{\omega}_\xi - (j_\eta - j_\zeta) \omega_\eta \omega_\zeta, \quad (2.30)$$

$$A_\eta = j_\eta \dot{\omega}_\eta - (j_\zeta - j_\xi) \omega_\zeta \omega_\xi, \quad (2.31)$$

$$A_\zeta = j_\zeta \dot{\omega}_\zeta - (j_\xi - j_\eta) \omega_\xi \omega_\eta, \quad (2.32)$$

where $\tilde{\omega}$ is expressed in terms of the Euler angles as shown in (2.4)-(2.6).

Differentiating (2.4)-(2.6) with respect to time, we obtain

$$\dot{\omega}_\xi = \ddot{\phi} - \ddot{\psi} \sin \theta - \dot{\psi} \dot{\theta} \cos \theta + \ddot{\Theta} \sin \psi \cos \theta + \dot{\Theta} \dot{\psi} \cos \psi \cos \theta - \dot{\Theta} \dot{\theta} \sin \psi \sin \theta, \quad (2.33)$$

$$\begin{aligned} \dot{\omega}_\eta = & \ddot{\psi} \cos \theta \sin \phi - \dot{\psi} \dot{\theta} \sin \theta \sin \phi + \dot{\psi} \dot{\phi} \cos \theta \cos \phi + \ddot{\theta} \cos \phi - \dot{\theta} \dot{\phi} \sin \phi \\ & + \ddot{\Theta} \cos \psi \cos \phi - \dot{\Theta} \dot{\psi} \sin \psi \cos \phi - \dot{\Theta} \dot{\phi} \cos \psi \sin \phi + \ddot{\Theta} \sin \psi \sin \theta \sin \phi \\ & + \dot{\Theta} \dot{\psi} \cos \psi \sin \theta \sin \phi + \dot{\Theta} \dot{\theta} \sin \psi \cos \theta \sin \phi + \dot{\Theta} \dot{\phi} \sin \psi \sin \theta \cos \phi, \end{aligned} \quad (2.34)$$

$$\begin{aligned} \dot{\omega}_\zeta = & \ddot{\psi} \cos \theta \cos \phi - \dot{\psi} \dot{\theta} \sin \theta \cos \phi - \dot{\psi} \dot{\phi} \cos \theta \sin \phi - \ddot{\theta} \sin \phi - \dot{\theta} \dot{\phi} \cos \phi \\ & - \ddot{\Theta} \cos \psi \sin \phi + \dot{\Theta} \dot{\psi} \sin \psi \sin \phi - \dot{\Theta} \dot{\phi} \cos \psi \cos \phi + \ddot{\Theta} \sin \psi \sin \theta \cos \phi \\ & + \dot{\Theta} \dot{\psi} \cos \psi \sin \theta \cos \phi + \dot{\Theta} \dot{\theta} \sin \psi \cos \theta \cos \phi - \dot{\Theta} \dot{\phi} \sin \psi \sin \theta \sin \phi. \end{aligned} \quad (2.35)$$

Hence $\{A\}$ can be represented in terms of ψ , θ , and ϕ and their derivatives. The last step to obtain a set of complete equations of motion is to relate $\{F\}$ and $\{M\}$ to the three displacements and three rotations. In other words, we need constitutive equations and strain-displacement relations. The constitutive equations relate the three resultant forces and moments to the six strains — one axial strain, two shear strains, one twisting curvature, and two bending curvatures. And, the strain-displacement relations relate the six strains to the three displacements and three rotations.

For the constitutive equations, we include all possible structural couplings. Hence, we express the constitutive equations in the most general form

$$\{F_1 F_2 F_3 M_1 M_2 M_3\}^T = [S] \{\theta \gamma_{\xi\eta} \gamma_{\xi\xi} \tilde{\rho}_\xi \tilde{\rho}_\eta \tilde{\rho}_\zeta\}^T, \quad (2.36)$$

where the stiffness matrix $[S]$ is given by

$$[S] \equiv \begin{bmatrix} A_{11} & A_{12} & A_{13} & B_{11} & B_{12} & B_{13} \\ A_{12} & A_{22} & A_{23} & B_{21} & B_{22} & B_{23} \\ A_{13} & A_{23} & A_{33} & B_{31} & B_{32} & B_{33} \\ B_{11} & B_{21} & B_{31} & D_{11} & D_{12} & D_{13} \\ B_{12} & B_{22} & B_{32} & D_{12} & D_{22} & D_{23} \\ B_{13} & B_{23} & B_{33} & D_{13} & D_{23} & D_{33} \end{bmatrix}, \quad (2.37)$$

and $\gamma_{\xi\eta}$ and $\gamma_{\xi\xi}$ are the averaged shear strains, which are the same as those used in Timoshenko's beam theory. We note that the matrices $[A]$, $[B]$, and $[D]$ are not the same as those used in the classical laminated theory (CLT) of plates [103]. In CLT,

$\bar{\rho}_\eta$ and $\gamma_{\zeta\eta}$ are assumed to be zeros, and the bending curvature in the $\eta - \zeta$ plane is not zero, which is assumed to be zero for the beam theory we develop here.

The strain-displacement relations relate the curvatures to the Euler angles as in (2.7)-(2.9) and (2.12)-(2.14). It follows from Fig. 2.4(a) and transformation theory that the relationship between the axial strain e and the displacements is

$$1 + e = T_{11}(1 + u') + T_{12}v' + T_{13}w' . \quad (2.38)$$

In the absence of shear strains and warping, the ξ -axis is tangent to the neutral axis of the beam and perpendicular to the cross-section. Hence ψ and θ can be represented (see Fig. 2.4(b)) in terms of the derivatives of the displacements as

$$\cos \psi = \frac{1 + u'}{\sqrt{(1 + u')^2 + v'^2}} , \sin \psi = \frac{v'}{\sqrt{(1 + u')^2 + v'^2}} , \quad (2.39)$$

$$\cos \theta = \sqrt{\frac{(1 + u')^2 + v'^2}{(1 + u')^2 + v'^2 + w'^2}} , \sin \theta = \frac{-w'}{\sqrt{(1 + u')^2 + v'^2 + w'^2}} . \quad (2.40)$$

With these equations, the unknown variables are reduced to four; namely, u , v , w , and ϕ , which means that only four of (2.21) and (2.23) are independent. The other two give expressions for the internal shear forces in terms of these four displacements, as shown later.

Because the shear strains are functions of the local coordinates (η, ζ) on the cross section, the shear effect can only be included in the one-dimensional theory developed here in an averaged sense, as in Timoshenko's beam theory. When the shear strains $\gamma_{\zeta\eta}$ and $\gamma_{\xi\zeta}$ are included, the ξ axis can be chosen as the neutral axis, and a shear coefficient [17] must be introduced, and the shear strains must be included

in the expressions of $\{\omega\}$, $\{A\}$, and $\tilde{\rho}$. Then, we need to solve six second-order partial-differential equations or four highly coupled fourth-order partial-differential equations. However, for a long beam with a solid cross-section, the shear effects are negligible as indicated earlier.

Although general damping can be added to the equations of motion through the damping matrices $[C_1]$ and $[C_2]$, in the following derivation we assume that the dissipation of energy due to internal friction and relative motion between the beam and its support system can be modeled by small, uncoupled, viscous dampers. So we take

$$[C_1] = \begin{bmatrix} \mu_0 & 0 & 0 \\ 0 & \mu_1 & 0 \\ 0 & 0 & \mu_2 \end{bmatrix}. \quad (2.41)$$

If we neglect the shear strains and assume that any possible external damping (e.g., air damping) can be treated as an external load, we do not need to consider the shear deformation related damping in the matrix $[C_2]$. Consequently, we put

$$[C_2] = \begin{bmatrix} \mu_3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \quad (2.42)$$

Here, μ_0 , μ_1 , μ_2 , and μ_3 denote the damping coefficients with respect to u , v , w , and ϕ , respectively.

Because external loads can be added to the equations of motion whenever the need arises, we drop them in the following expressions to simplify the equations. Expanding (2.21) and (2.23) yields the following complete equations of motion:

$$(T_{11}F_1 + T_{21}F_2 + T_{31}F_3)' = ma_x + \mu_0\dot{u} , \quad (2.43)$$

$$(T_{12}F_1 + T_{22}F_2 + T_{32}F_3)' = ma_y + \mu_1\dot{v} , \quad (2.44)$$

$$(T_{13}F_1 + T_{23}F_2 + T_{33}F_3)' = ma_z + \mu_2\dot{w} , \quad (2.45)$$

$$M_1' - \rho_\zeta M_2 + \rho_\eta M_3 = A_\xi + \mu_3\omega_\xi , \quad (2.46)$$

$$M_2' + \rho_\zeta M_1 - \rho_\xi M_3 - (1 + \theta)F_3 = A_\eta , \quad (2.47)$$

$$M_3' - \rho_\eta M_1 + \rho_\xi M_2 + (1 + \theta)F_2 = A_\zeta . \quad (2.48)$$

These are six equations governing the four unknowns u , v , w , and ϕ . Equations (2.47) and (2.48) can be used to eliminate the internal shear forces F_2 and F_3 from (2.43)-(2.46), yielding four nonlinear partial-differential equations for u , v , w , and ϕ . Because u , v , w , and ϕ are the generalized displacements corresponding to the coordinates x , y , z , and ξ , respectively, the natural boundary conditions are

$$F_x \equiv T_{11}F_1 + T_{21}F_2 + T_{31}F_3 = 0 , \quad (2.49)$$

$$F_y \equiv T_{12}F_1 + T_{22}F_2 + T_{32}F_3 = 0 , \quad (2.50)$$

$$F_z \equiv T_{13}F_1 + T_{23}F_2 + T_{33}F_3 = 0 , \quad (2.51)$$

$$M_1 = 0 , \quad (2.52)$$

$$M_y \equiv T_{22}M_2 + T_{32}M_3 = 0 , \quad (2.53)$$

$$M_z \equiv T_{23}M_2 + T_{33}M_3 = 0 , \quad (2.54)$$

at $s=L$. The kinematic boundary conditions are

$$v = \frac{\partial v}{\partial x} = w = \frac{\partial w}{\partial x} = \phi = 0 \quad (2.55)$$

at $s=0$.

2.2 Ordering Scheme and Taylor-Series Expansions

Because the complete nonlinear equations of motion (2.43)-(2.48) are transcendental, closed-form solutions of this nonlinear, two-point-boundary-value-problem are not available. Moreover, direct numerical procedures suffer from instability and convergence problems. Alternatively, one can use a combination of numerical and perturbation methods to solve them. The first step in such an approach is to expand the nonlinear transcendental terms into polynomials.

To obtain the governing equations in polynomial form for small but finite oscillations about the undeformed position, we assume that u , v , w , ϕ , and their derivatives are $O(\varepsilon)$, where ε ($\ll 1$) is a small dimensionless parameter that is used as a bookkeeping device. In this section, we expand all terms in (2.43)-(2.55) in Taylor series and keep nonlinear terms up to $O(\varepsilon^3)$, thereby obtaining third-order nonlinear

equations of motion. Normalization results (shown later) show that the rotary inertias are $O(\varepsilon^2)$, and this fact is used to simplify the equations.

It follows from (2.39) that

$$\psi = \tan^{-1} \frac{v'}{1 + u'} = v' - v'u' + v'u'^2 - \frac{1}{3} v'^3 + \dots , \quad (2.56)$$

$$\cos \psi = 1 - \frac{1}{2} v'^2 + u'v'^2 + \dots , \quad (2.57)$$

and

$$\sin \psi = v' - u'v' + u'^2v' - \frac{1}{2} v'^3 + \dots . \quad (2.58)$$

Similarly, it follows from (2.40) that

$$\theta = \tan^{-1} \frac{-w'}{\sqrt{(1 + u')^2 + v'^2}} = -w' + u'w' - u'^2w' + \frac{1}{2} w'v'^2 + \frac{1}{3} w'^3 + \dots , \quad (2.59)$$

$$\cos \theta = 1 - \frac{1}{2} w'^2 + w'^2u' + \dots , \quad (2.60)$$

$$\sin \theta = -w' + w'u' - u'^2w' + \frac{1}{2} w'v'^2 + \frac{1}{2} w'^3 + \dots . \quad (2.61)$$

Substituting (2.56)-(2.61) into (2.2), using Taylor-series expansions of $\sin \phi$ and $\cos \phi$, and keeping terms up to $O(\varepsilon^3)$, we obtain

$$T_{11} = 1 - \frac{1}{2} v'^2 - \frac{1}{2} w'^2 + u'v'^2 + u'w'^2 + \dots , \quad (2.62)$$

$$T_{12} = v' - v'u' + v'u'^2 - \frac{1}{2} v'^3 - \frac{1}{2} v'w'^2 + \dots , \quad (2.63)$$

$$T_{13} = w' - u'w' + u'^2w' - \frac{1}{2}v'^2w' - \frac{1}{2}w'^3 + \dots , \quad (2.64)$$

$$T_{21} = -v' + u'v' - w'\phi - u'^2v' + \frac{1}{2}v'^3 + \frac{1}{2}\phi^2v' + \phi u'w' + \dots , \quad (2.65)$$

$$T_{22} = 1 - \frac{1}{2}v'^2 - \frac{1}{2}\phi^2 + u'v'^2 - \phi v'w' + \dots , \quad (2.66)$$

$$T_{23} = \phi - \frac{1}{2}\phi w'^2 - \frac{1}{6}\phi^3 + \dots , \quad (2.67)$$

$$T_{31} = -w' + u'w' + \phi v' - \phi u'v' - w'u'^2 + v'^2w' + \frac{1}{2}w'^3 + \frac{1}{2}\phi^2w' + \dots , \quad (2.68)$$

$$T_{32} = -\phi - v'w' + \frac{1}{2}\phi v'^2 + \frac{1}{6}\phi^3 + 2u'v'w' + \dots , \quad (2.69)$$

$$T_{33} = 1 - \frac{1}{2}w'^2 - \frac{1}{2}\phi^2 + u'w'^2 + \dots . \quad (2.70)$$

Substituting for T_{11} , T_{12} , and T_{13} from (2.62)-(2.64) in (2.38) yields

$$\theta = u' + \frac{1}{2}(1 - u')(v'^2 + w'^2) + \dots . \quad (2.71)$$

Differentiating (2.56) and (2.59) with respect to s , and then substituting the results, (2.60) and (2.61), and the Taylor-series expansions of $\sin \phi$ and $\cos \phi$ into equations (2.12)-(2.14), we obtain the expanded form of the twisting and bending curvatures as

$$\rho_\xi = \phi' + v''w' - 2u'v''w' - u''v'w' + \dots , \quad (2.72)$$

$$\begin{aligned} \rho_\eta = & -w'' + u'w'' + u''w' + v''\phi - \phi u'v'' - \phi u''v' - u'^2w'' - 2u'u''w' \\ & + \frac{1}{2}v'^2w'' + v'v''w' + w''w'^2 + \frac{1}{2}\phi^2w'' + \dots , \end{aligned} \quad (2.73)$$

$$\begin{aligned} \rho_\zeta = & v'' - u'v'' - u''v' + \phi w'' + u'^2v'' + 2u'u''v' - v'^2v'' - \frac{1}{2}v''w'^2 \\ & - \frac{1}{2}\phi^2v'' - \phi u'w'' - \phi u''w' + \dots . \end{aligned} \quad (2.74)$$

Using (2.57), (2.58), (2.60), and (2.61) in (2.4)-(2.6) and (2.33)-(2.35) and knowing that the rotary inertias are second-order parameters, we expand the rotary inertia terms (2.30)-(2.32) as

$$A_\eta = j_\eta(\ddot{\Theta} - \ddot{w}') , \quad (2.75)$$

$$A_\zeta = j_\zeta(\ddot{v}' - (\dot{\Theta}\dot{\phi})) - (j_\zeta - j_\eta)\dot{\Theta}(\dot{\phi} + \dot{\Theta}v') , \quad (2.76)$$

$$\begin{aligned} A_\xi = j_\xi[& \ddot{\phi} + \ddot{v}'w' - 2\ddot{v}'u'w' - 3\ddot{v}'\dot{u}'w' - \ddot{u}'v'w' + \ddot{v}'\dot{w}' - 2u'\ddot{v}'\dot{w}' - \dot{u}'v'\dot{w}' \\ & + \ddot{\Theta}(v' - u'v' + u'^2v' - \frac{1}{2}v'^3 - \frac{1}{2}v'w'^2) \\ & + \dot{\Theta}(\dot{v}' - u'\dot{v}' - \dot{u}'v' + u'^2\dot{v}' + 2\dot{u}'u'v' - \frac{3}{2}\dot{v}'v'^2 - \frac{1}{2}\dot{v}'w'^2 - v'w'\dot{w}') \\ & - (j_\eta - j_\zeta)[\phi\dot{v}'^2 + \dot{u}'\dot{v}'w' - \dot{v}'\dot{w}' + 2u'\dot{v}'\dot{w}' + \dot{u}'v'\dot{w}' - \phi\dot{w}'^2 + \dot{\Theta}(\dot{v}' - u'\dot{v}' - \dot{u}'v' \\ & + u'^2\dot{v}' + 2\dot{u}'u'v' - \frac{3}{2}\dot{v}'v'^2 - \frac{1}{2}\dot{v}'w'^2 - 2\phi^2\dot{v}' + 2\phi\dot{w}' - 2\phi u'\dot{w}' - 2\phi\dot{u}'w' + v'w'\dot{w}') \\ & + \dot{\Theta}^2(-\phi + \phi v'^2 + \frac{2}{3}\phi^3 - v'w' + 2u'v'w')] . \end{aligned} \quad (2.77)$$

Here we keep all nonlinear terms due to the motion of the hub because Θ can be of $O(\varepsilon)$ (e.g., free vibration of a hinged-free beam) or of $O(1)$ (e.g., slewing beam with large $\dot{\Theta}$). Because the linear inertia term $j_\zeta\ddot{\phi}$ in the equation for ϕ is $O(\varepsilon^3)$, we keep quadratic and cubic inertia terms in the expansion of A_ζ . The term $\mu_3\omega_\xi$ is expanded as

$$\mu_3\omega_\xi = \mu_3\dot{\phi} . \quad (2.78)$$

2.3 Equations of Slewing Inextensional Composite Beams

For an inextensional beam, the number of equations can be reduced from four to three if an artificial constraint force is added into the equations. Next, we develop three third-order nonlinear partial-differential equations, which describe the slewing motion of an inextensional composite beam. For an inextensional beam,

$$s = \tilde{s}, \rho_i = \tilde{\rho}_i, \text{ and } e = 0.$$

Because we neglected shear deformation and warping, the constitutive equation (2.36) reduces to

$$F_1 = B_{11}\rho_\xi + B_{12}\rho_\eta + B_{13}\rho_\zeta, \quad (2.79)$$

$$M_1 = D_{11}\rho_\xi + D_{12}\rho_\eta + D_{13}\rho_\zeta, \quad (2.80)$$

$$M_2 = D_{12}\rho_\xi + D_{22}\rho_\eta + D_{23}\rho_\zeta, \quad (2.81)$$

$$M_3 = D_{13}\rho_\xi + D_{23}\rho_\eta + D_{33}\rho_\zeta, \quad (2.82)$$

Putting $e = 0$ in (2.71), solving for u' , and neglecting terms of order higher than three, we obtain

$$u' = -\frac{1}{2}(v'^2 + w'^2), \quad (2.83)$$

Integrating (2.83) and using the geometric boundary condition $u(0,t) = 0$, we obtain

$$u = -\frac{1}{2} \int_0^s (v'^2 + w'^2) ds . \quad (2.84)$$

Equation (2.84) shows that u in this case is a second-order quantity and hence the expansions in Section 2.2 can be simplified further. Following the ordering scheme, substituting (2.83) and (2.84) into (2.72)-(2.74) and (2.62)-(2.70), we obtain the third-order expansions of the curvatures and $[T]$ as

$$\rho_{\xi} = \phi' + v''w' \quad (2.85)$$

$$\rho_{\eta} = -w'' + v''\phi + \frac{1}{2} w''(\phi^2 - w'^2) \quad (2.86)$$

$$\rho_{\zeta} = v'' + w''\phi + \frac{1}{2} v''(v'^2 - \phi^2) + v'w'w'' \quad (2.87)$$

$$[T] = \begin{bmatrix} 1 - \frac{1}{2}(v'^2 + w'^2) & v' & w' \\ \frac{1}{2} v'(\phi^2 - w'^2) - v' - w'\phi & 1 - \frac{1}{2}(v'^2 + \phi^2) - \phi v'w' & \phi - \frac{1}{2} \phi(w'^2 + \frac{1}{3} \phi^2) \\ \frac{1}{2} w'(\phi^2 + v'^2) - w' + v'\phi & \frac{1}{2} \phi(v'^2 + \frac{1}{3} \phi^2) - \phi - v'w' & 1 - \frac{1}{2}(w'^2 + \phi^2) \end{bmatrix} \quad (2.88)$$

Because the beam is not a rigid body, the axial strain can never be zero. One way for implementing the assumption of inextensionality is the addition of an artificial, axial constraint force λ in the equations. Moreover, because there is no axial strain, the axial damping μ_0 vanishes. Therefore, (2.43)-(2.48) can be rewritten as

$$(T_{11}F_1 + T_{21}F_2 + T_{31}F_3)' + (T_{11}\lambda)' = ma_x , \quad (2.89)$$

$$(T_{12}F_1 + T_{22}F_2 + T_{32}F_3)' + (T_{12}\lambda)' = ma_y + \mu_1\dot{v} , \quad (2.90)$$

$$(T_{13}F_1 + T_{23}F_2 + T_{33}F_3)' + (T_{13}\lambda)' = ma_z + \mu_2\dot{w} , \quad (2.91)$$

$$M_1' - \rho_\zeta M_2 + \rho_\eta M_3 = A_\xi + \mu_3\omega_\xi , \quad (2.92)$$

$$M_2' + \rho_\zeta M_1 - \rho_\xi M_3 - F_3 = A_\eta , \quad (2.93)$$

$$M_3' - \rho_\eta M_1 + \rho_\xi M_2 + F_2 = A_\zeta . \quad (2.94)$$

The boundary conditions become

$$F_x = T_{11}(\lambda + F_1) + T_{21}F_2 + T_{31}F_3 = 0 , \quad (2.95)$$

$$F_y = T_{12}(\lambda + F_1) + T_{22}F_2 + T_{32}F_3 = 0 , \quad (2.96)$$

$$F_z = T_{13}(\lambda + F_1) + T_{23}F_2 + T_{33}F_3 = 0 , \quad (2.97)$$

$$M_1 = 0 , \quad (2.98)$$

$$M_y = T_{22}M_2 + T_{32}M_3 = 0 , \quad (2.99)$$

$$M_z = T_{23}M_2 + T_{33}M_3 = 0 , \quad (2.100)$$

at $s = L$ and

$$v = \frac{\partial v}{\partial x} = w = \frac{\partial w}{\partial x} = \phi = 0 , \quad (2.101)$$

$$\tau = I_H \ddot{\Theta} - M_2 , \quad (2.102)$$

at $s = 0$, where (2.102) is a statement of the balance of moments at the hub, τ is the torque applied on the hub, and I_h is the moment of inertia of the hub.

Substituting (2.80)-(2.82), (2.85)-(2.87), (2.75), and (2.76) into (2.93) and (2.94), we obtain the internal shear forces, up to $O(\varepsilon^3)$, as

$$\begin{aligned}
 F_2 &= -M_3' + \rho_\eta M_1 - \rho_\xi M_2 + A_\xi \\
 &= -[D_{33}(v'' + w''\phi - \frac{1}{2}v''\phi^2 + \frac{1}{2}v''v'^2 + v'w'w'') \\
 &\quad + D_{13}(\phi' + v''w') + D_{23}(-w'' + v''\phi + \frac{1}{2}w''\phi^2 - \frac{1}{2}w''w'^2)]' \\
 &\quad + (D_{11} - D_{22})(\phi\phi'v'' - \phi'w'' - v''w'w'') \\
 &\quad + D_{12}(w''^2 - 2\phi v''w'') + D_{13}(-v''w'' - w''^2\phi + v''^2\phi) \\
 &\quad - D_{12}(\phi'^2 + 2\phi'v''w') - D_{23}(\phi'v'' + w''\phi\phi' + v''^2w') \\
 &\quad + j_\xi(\ddot{v}' - (\dot{\Theta}\phi)') - (j_\xi - j_\eta)(\dot{\Theta}\dot{\phi} + \dot{\Theta}^2v') , \tag{2.103}
 \end{aligned}$$

$$\begin{aligned}
 F_3 &= M_2' + \rho_\xi M_1 - \rho_\eta M_3 - A_\eta \\
 &= [D_{22}(-w'' + \phi v'' + \frac{1}{2}w''\phi^2 - \frac{1}{2}w''w'^2) \\
 &\quad + D_{12}(\phi' + v''w') + D_{23}(v'' + w''\phi - \frac{1}{2}v''\phi^2 + \frac{1}{2}v''v'^2 + v'w'w'')]' \\
 &\quad + D_{12}(-v''w'' - w''^2\phi + v''^2\phi) + D_{13}(v''^2 + 2v''\phi w'') \\
 &\quad - D_{13}(\phi'^2 + 2\phi'v''w') - D_{23}(-\phi'w'' + v''\phi\phi' - v''w'w'') \\
 &\quad + (D_{11} - D_{33})(\phi'v'' + \phi\phi'w'' + v''^2w') + j_\eta(\ddot{w}' - \ddot{\Theta}) . \tag{2.104}
 \end{aligned}$$

From (2.89), (2.25), (2.95), (2.79), (2.103), (2.104), and (2.88), λ can be obtained as

$$\begin{aligned}
 \lambda &= \frac{1}{T_{11}} \left[\int_L^s m a_x ds - (T_{11}F_1 + T_{21}F_2 + T_{31}F_3) \right] \\
 &= -D_{11}\phi'(v'w'' - v''w') - [(D_{33} - D_{22})\phi v'']'w' - [(D_{33} - D_{22})\phi w'']'v' \\
 &\quad - (D_{33}v'')'v' - (D_{22}w'')'w' - j_\eta \ddot{\Theta} w'
 \end{aligned}$$

$$\begin{aligned}
& -B_{11}(\phi' + v''w') - B_{12}(-w'' + v''\phi + \frac{1}{2}w''\phi^2 - \frac{1}{2}w''w'^2) \\
& -B_{13}(v'' + w''\phi - \frac{1}{2}v''\phi^2 + \frac{1}{2}v''v'^2 + v'w'w'') - D_{12}'(-w'\phi' - v''w'^2 + \phi'\phi v') \\
& -D_{13}'(v'\phi' + v'v''w' + w'\phi\phi') - D_{23}'(-v'w'' + 2v'v''\phi - 2w''w'\phi - w'v'') \\
& -D_{12}(-v'w''^2 + v'\phi'^2 - w'\phi'' - v'''w'^2 + \phi v'\phi'') \\
& -D_{13}(\phi''v' + 2v'v''w'' + w'\phi\phi'' - v''^2w' + \phi'^2w' + v'v'''w') \\
& -D_{23}(-v'w''' + 2v'v''' \phi - 2w'\phi w''' + 2v'v''\phi' - w'v''' - 2w'w''\phi') \\
& + \int_L^s m(-\frac{1}{2}[\int_0^s (v'^2 + w'^2)ds])'' + \ddot{\Theta}w + 2\dot{\Theta}\dot{w} - \dot{\Theta}^2s + \frac{1}{2}\dot{\Theta}^2\int_0^s (v'^2 + w'^2)ds)ds \\
& + \frac{1}{2}(v'^2 + w'^2)\int_L^s m(\ddot{\Theta}w + 2\dot{\Theta}\dot{w} - \dot{\Theta}^2s)ds .
\end{aligned} \tag{2.105}$$

From (2.105) we see that λ is a second-order variable if $\Theta(t) = 0$ and $B_{ij} = 0$. If $\Theta(t) \neq 0$, λ can be large, especially due to centrifugal forces. If axial or shear strains are large, the structural coupling stiffnesses B_{ij} cannot be neglected.

In view of (2.85), we define the elastic twist angle γ as

$$\gamma \equiv \phi + \int_0^s v''w' ds , \tag{2.106}$$

which satisfies

$$\rho_{\xi} = \gamma' = \phi' + v''w' , \tag{2.107}$$

and the boundary conditions $\gamma(0, t) = \phi(0, t) = 0$. Substituting (2.106) and its derivatives into (2.85)-(2.88), we obtain

$$\rho_{\xi} = \gamma' , \tag{2.108}$$

$$\rho_\eta = -w'' + v''\gamma - v'' \int_0^s v'' w' ds + \frac{1}{2} w''(\gamma^2 - w'^2) , \quad (2.109)$$

$$\rho_\zeta = v'' + w''\gamma + w'' \int_0^s v' w'' ds + \frac{1}{2} v''(v'^2 - \gamma^2) , \quad (2.110)$$

$$[T] = \begin{bmatrix} 1 - \frac{1}{2}(v'^2 + w'^2) & v' & w' \\ \frac{1}{2} v'(\gamma^2 + w'^2 - 2) - w'(\gamma + l_1) & 1 - \frac{1}{2}(v'^2 + \gamma^2 + 2\gamma l_1) & \gamma - l_2 - \frac{1}{2} \gamma(w'^2 + \frac{1}{3} \gamma^2) \\ \frac{1}{2} w'(\gamma^2 + v'^2 - 2) + v'(\gamma - l_2) & \frac{1}{2} \gamma(v'^2 + \frac{1}{3} \gamma^2) - \gamma - l_1 & 1 - \frac{1}{2}(w'^2 + \gamma^2 - 2\gamma l_2) \end{bmatrix} \quad (2.111)$$

where $l_1 \equiv \int_0^s v' w'' ds$ and $l_2 \equiv \int_0^s v'' w' ds$. Substituting (2.106)-(2.110) into (2.79) and (2.103)-(2.105), and then substituting the results, (2.111), (2.26), and (2.27) into (2.90) and (2.91) and expanding (2.92) as we did in obtaining (2.103), we obtain

$$m\ddot{v} + \mu_1 \dot{v} = G_v' + F_v , \quad (2.112)$$

$$m\ddot{w} + \mu_2 \dot{w} = G_w' + F_w , \quad (2.113)$$

$$j_\xi \ddot{\gamma} + \mu_3 \dot{\gamma} = G_\gamma + F_\gamma , \quad (2.114)$$

where

$$\begin{aligned} G_v \equiv & -(D_{33} v'')' - D_{11}(\gamma' w'') - D_{33} v'(v' v'' + w' w'')' \\ & + (D_{22} - D_{33})[(w''\gamma - v''\gamma^2)' - w''' \int_0^s v'' w' ds] \\ & + D_{22}'(w''\gamma - w'' \int_0^s v'' w' ds - v''\gamma^2) - D_{33}'(w''\gamma + w'' \int_0^s v' w'' ds - v''\gamma^2 + v'' v'^2) \end{aligned}$$

$$\begin{aligned}
& + D_{12}(w'^2 - \gamma'^2 - \gamma\gamma'' + \gamma'' \int_0^s v'' w' ds - \gamma v'' w'') \\
& + D_{13}(-\gamma'' - v'' w'' - \frac{1}{2} v'^2 \gamma'' - w''^2 \gamma + \frac{1}{2} \gamma^2 \gamma'' + \gamma \gamma'^2) \\
& + D_{23}(w''' - 2v'' \gamma' + v''^2 w' - 2v''' \gamma + 2v''' \int_0^s v'' w' ds + \frac{1}{2} v'^2 w''') \\
& + \frac{1}{2} w''' w'^2 - 4w'' \gamma \gamma' + w''^2 w' - 2w''' \gamma^2) \\
& + D_{12}'(-\gamma \gamma' + \gamma' \int_0^s v'' w' ds) + D_{13}'(-\gamma' - \frac{1}{2} v'^2 \gamma' + \frac{1}{2} \gamma^2 \gamma') \\
& + D_{23}'(w'' + \frac{1}{2} v'^2 w'' + \frac{1}{2} w'' w'^2 - 2\gamma^2 w'' - 2v'' \gamma + 2v'' \int_0^s v'' w' ds) \\
& + j_\zeta \ddot{v}' - \frac{1}{2} v' \int_L^s m \left[\int_0^s (v'^2 + w'^2) ds \right] ds, \tag{2.115}
\end{aligned}$$

$$\begin{aligned}
G_w \equiv & -(D_{22} w'')' + D_{11}(\gamma' v'') - D_{22} w'(w' w'' + v' v'')' \\
& + (D_{22} - D_{33})[(v'' \gamma + w'' \gamma^2)' + v''' \int_0^s w'' v' ds] \\
& - D_{33}'(v'' \gamma + v'' \int_0^s w'' v' ds + w'' \gamma^2) + D_{22}'(v'' \gamma - v'' \int_0^s w'' v' ds + w'' \gamma^2 - w'' w'^2) \\
& + D_{13}(v'^2 - \gamma'^2 - \gamma \gamma'' - \gamma'' \int_0^s w'' v' ds + \gamma w'' v'') \\
& + D_{12}(\gamma'' - v'' w'' + \frac{1}{2} w'^2 \gamma'' + v''^2 \gamma - \frac{1}{2} \gamma^2 \gamma'' - \gamma \gamma'^2) \\
& + D_{23}(v''' + 2w'' \gamma' + w''^2 v' + 2w''' \gamma + 2w''' \int_0^s w'' v' ds + \frac{1}{2} w'^2 v''') \\
& + \frac{1}{2} v''' v'^2 - 4v'' \gamma \gamma' + v''^2 v' - 2v''' \gamma^2) \\
& + D_{13}'(-\gamma \gamma' - \gamma' \int_0^s w'' v' ds) + D_{12}'(\gamma' + \frac{1}{2} w'^2 \gamma' - \frac{1}{2} \gamma^2 \gamma') \\
& + D_{23}'(v'' + \frac{1}{2} w'^2 v'' + \frac{1}{2} v'' v'^2 - 2\gamma^2 v'' + 2w'' \gamma + 2w'' \int_0^s w'' v' ds)
\end{aligned}$$

$$+ j_\eta \ddot{w}' - \frac{1}{2} w' \int_L^s m \left[\int_0^s (v'^2 + w'^2) ds \right]' ds , \quad (2.116)$$

$$\begin{aligned} G_y \equiv & (D_{11}\gamma')' + (D_{33} - D_{22})[(v''^2 - w''^2)\gamma - v''w''] \\ & + [D_{12}(-w'' + v''\gamma - v'' \int_0^s v''w' ds + \frac{1}{2} w''\gamma^2 - \frac{1}{2} w''w'^2)]' \\ & + [D_{13}(v'' + w''\gamma + w'' \int_0^s w''v' ds - \frac{1}{2} v''\gamma^2 + \frac{1}{2} v''v'^2)]' \\ & - D_{12}(v''\gamma' + w''\gamma\gamma') + D_{13}(-w''\gamma' + v''\gamma\gamma') + D_{23}(w''^2 - 4\gamma v''w'' - v''^2) \\ & + j_\xi \left(\int_0^s v''w' ds \right)' - j_\xi (\dot{v}'w') + (j_\eta - j_\xi)[(\dot{v}'^2 - \dot{w}'^2)\gamma - \dot{w}'\dot{v}'] , \end{aligned} \quad (2.117)$$

and the forcing functions F_v , F_w , and F_y are due to the motion of the hub, which are given by

$$\begin{aligned} F_v = & \{ (j_\eta - j_\xi)(\dot{\Theta}\dot{\gamma} + \dot{\Theta}^2 v') + j_\eta \ddot{\Theta}\gamma - j_\xi(\dot{\Theta}\gamma)' - \frac{1}{2} \dot{\Theta}^2 v'(v'^2 + w'^2) \int_L^s m s ds \\ & + v' \int_L^s m [\ddot{\Theta}w + 2\dot{\Theta}\dot{w} - \dot{\Theta}^2 s + \frac{1}{2} \dot{\Theta}^2 \int_0^s (v'^2 + w'^2) ds] ds \}' , \end{aligned} \quad (2.118)$$

$$\begin{aligned} F_w = & m\dot{\Theta}^2 w - m\ddot{\Theta}(-s + \frac{1}{2} \int_0^s (v'^2 + w'^2) ds) - m\dot{\Theta} \left[\int_0^s (v'^2 + w'^2) ds \right]' + \{ -j_\eta \ddot{\Theta} - \frac{1}{2} \dot{\Theta}^2 w' \\ & (v'^2 + w'^2) \int_L^s m s ds + w' \int_L^s m [\ddot{\Theta}w + 2\dot{\Theta}\dot{w} - \dot{\Theta}^2 s + \frac{1}{2} \dot{\Theta}^2 \int_0^s (v'^2 + w'^2) ds] ds \}' , \end{aligned} \quad (2.119)$$

$$\begin{aligned} F_y = & -j_\xi(\dot{\Theta}v') + (j_\eta - j_\xi)[\dot{\Theta}(\dot{v}' - 2\dot{v}'\gamma^2 + 2\dot{w}'\gamma + 2\dot{w}' \int_0^s v'w'' ds) \\ & + \dot{\Theta}^2(-\gamma - \int_0^s v'w'' ds + \gamma v'^2 + \frac{2}{3} \gamma^3)] . \end{aligned} \quad (2.120)$$

It follows from Fig. 2.4(b) that

$$v_x \equiv \frac{\partial v}{\partial x} = \frac{dv}{ds + du} = \frac{v'}{1 + u'} = v'(1 + \frac{1}{2} v'^2 + \frac{1}{2} w'^2 + \dots) , \quad (2.121)$$

where (2.83) is used. Similarly, we obtain

$$w_x \equiv \frac{\partial w}{\partial x} = w'(1 + \frac{1}{2} v'^2 + \frac{1}{2} w'^2 + \dots) . \quad (2.122)$$

Because $v_x = w_x = 0$ at $s = 0$, we conclude that $v' = w' = 0$ at $s = 0$. Hence, the boundary conditions (2.96)-(2.102) become

$$\begin{aligned} & (D_{22} - D_{33})(w''''\gamma - v''''\gamma^2 - w'''' \int_0^L v''w' ds) - D_{33}v'''' - D_{33}v'(v'v'''' + w'w''''') + j_\zeta \ddot{v}' \\ & + D_{12}(-\gamma\gamma'' + \gamma'' \int_0^L v''w' ds) + D_{13}(-\gamma'' - \frac{1}{2} v'^2\gamma'' + \frac{1}{2} \gamma^2\gamma'') \\ & + D_{23}(w'''' - 2v''''\gamma + 2v'''' \int_0^L v''w' ds + \frac{1}{2} v'^2w'''' + \frac{1}{2} w''''w'^2 - 2w''''\gamma^2) \\ & + (j_\eta - j_\zeta)(\dot{\Theta}\dot{\gamma} + \dot{\Theta}^2v') + j_\eta \ddot{\Theta}\gamma - j_\zeta(\dot{\Theta}\gamma)' = 0 , \end{aligned} \quad (2.123)$$

$$\begin{aligned} & (D_{22} - D_{33})(v''''\gamma + w''''\gamma^2 + v'''' \int_0^L w''v' ds) - D_{22}w'''' - D_{22}w'(w'w'''' + v'v''''') + j_\eta \ddot{w}' \\ & + D_{13}(-\gamma\gamma'' - \gamma'' \int_0^L w''v' ds) + D_{12}(\gamma'' + \frac{1}{2} w'^2\gamma'' - \frac{1}{2} \gamma^2\gamma'') \\ & + D_{23}(v'''' + 2w''''\gamma + 2w'''' \int_0^L w''v' ds + \frac{1}{2} w'^2v'''' + \frac{1}{2} v''''v'^2 - 2v''''\gamma^2) - j_\eta \ddot{\Theta} = 0 , \end{aligned} \quad (2.124)$$

$$\gamma' = 0 , \quad (2.125)$$

$$w'' = 0 , \quad (2.126)$$

$$v'' = 0 \quad (2.127)$$

at $s = L$ and

$$v = v' = w = w' = \gamma = 0 , \quad (2.128)$$

$$\tau = I_H \ddot{\Theta} - D_{12} \gamma' + D_{22} w'' - D_{23} v'' \quad (2.129)$$

at $s = 0$. The obtained equations of motion are valid for arbitrary variations in the stiffnesses and density along the beam's span.

2.3.1 Flexural-Flexural-Torsional Vibrations of Uniform Isotropic Beams

For an isotropic beam, we only need to set $D_{12} = D_{13} = D_{23} = 0$ in (2.108)-(2.110). If the cross section dimensions and material properties of the beam are uniform (i.e., m , D_{11} , D_{22} , D_{33} , j_ζ , j_η , and j_ζ are constant), the equations of motion can be considerably simplified and rewritten in nondimensional form by normalizing the variables using the characteristic length L and the characteristic time $L^2 \sqrt{m/D_{22}}$ as

$$s^* = s/L, v^* = v/L, w^* = w/L, t^* = t \sqrt{D_{22}/mL^4},$$

$$\mu_1^* = \mu_1 L^2 / \sqrt{mD_{22}}, \mu_2^* = \mu_2 L^2 / \sqrt{mD_{22}}, \mu_3^* = \mu_3 / \sqrt{mD_{22}} j_1^*,$$

$$j_1^* = j_\zeta / mL^2, j_2^* = j_\eta / mL^2, j_3^* = j_\zeta / mL^2,$$

For ease of notation in the following analyses, we drop the superscript $*$ and use primes and dots to denote $\frac{\partial}{\partial s^*}$ and $\frac{\partial}{\partial t^*}$, respectively. Because $j_1 = j_2 + j_3 = \frac{1}{12} \left(\frac{b}{L} \right)^2 + \frac{1}{12} \left(\frac{h}{L} \right)^2$ for a rectangular cross section, where b and h denote the width and height of the cross section, we assume that j_1, j_2 and j_3 are second-order quantities. Letting $\beta_y \equiv D_{33}/D_{22}$ and $\beta_z \equiv D_{11}/D_{22}$, we rewrite (2.112)-(2.114) in the nondimensional form

$$\begin{aligned}
\ddot{v} + \mu_1 \dot{v} + \beta_y v^{iv} = & -\beta_y (\gamma' w'')' - \beta_y [v'(v'v'' + w'w'')]'' \\
& - (1 - \beta_y) [(v''\gamma^2 - w''\gamma)' + w''' \int_0^s v'' w' ds]' + j_3 \ddot{v}'' \\
& - \frac{1}{2} \{v' \int_1^s [\int_0^s (v'^2 + w'^2) ds]'' ds\}' + F_v ,
\end{aligned} \tag{2.130}$$

$$\begin{aligned}
\ddot{w} + \mu_2 \dot{w} + w^{iv} = & \beta_y (v''\gamma')' - [w'(v'v'' + w'w'')]'' \\
& + (1 - \beta_y) [(v''\gamma + w''\gamma^2)' + v''' \int_0^s v' w'' ds]' + j_2 \ddot{w}'' \\
& - \frac{1}{2} \{w' \int_1^s [\int_0^s (v'^2 + w'^2) ds]'' ds\}' + F_w ,
\end{aligned} \tag{2.131}$$

$$\begin{aligned}
\ddot{\gamma} + \mu_3 \dot{\gamma} - \frac{\beta_y}{j_1} \gamma'' = & \frac{\beta_y - 1}{j_1} (v''^2 \gamma - w''^2 \gamma - v'' w'') + (\int_0^s v'' w' ds)'' - (\dot{v}' w')' \\
& + \frac{j_2 - j_3}{j_1} (\dot{v}'^2 \gamma - \dot{w}'^2 \gamma - \dot{v}' \dot{w}') + F_\gamma .
\end{aligned} \tag{2.132}$$

Because the rotary inertias are small, the boundary conditions can be simplified.

Thus, the boundary conditions become

$$v = w = v' = w' = \gamma = 0 \quad \text{at } s = 0 , \tag{2.133}$$

$$I_h \ddot{\Theta}(t) + w''(0, t) = \tau_0(t) , \tag{2.134}$$

$$\beta_y v'''' - j_3 \ddot{v}' = w'''' - j_2 (\ddot{w}' - \ddot{\Theta}) = v'' = w'' = \gamma' = 0 \quad \text{at } s = 1 , \tag{2.135}$$

$$\text{where } I_h \equiv \frac{I_H}{mL^3} \quad \text{and } \tau_0 \equiv \frac{\tau L}{D_{22}} .$$

2.3.2 Flexural-Flexural Vibrations of Uniform Isotropic Beams

If the torsional rigidity is relatively high (e.g., a beam with a square cross-section and long span), the torsional inertia cannot be excited by low-frequency input forces because the twisting natural frequency is relatively high. Moreover, in the absence of a torque applied along the beam, the twist is only induced by bending deflections. The induced twist angle can be found by setting all terms with time derivatives equal to zero in (2.132). The result is

$$\frac{\beta_y}{j_1} \gamma'' + \frac{1 - \beta_y}{j_1} (v''w'' + w''^2\gamma - v''^2\gamma) = 0 . \quad (2.136)$$

Using the boundary conditions $\gamma(0, t) = \gamma'(1, t) = 0$ in (2.136), we find that the twist angle due to bending is

$$\gamma = \frac{\beta_y - 1}{\beta_y} \int_0^s \int_1^s v''w'' ds ds , \quad (2.137)$$

where the neglected terms are of fourth order. Equation (2.137) shows that the bending induced twisting is a nonlinear phenomenon. Using (2.137) and neglecting terms of order higher than three, we reduce (2.130) and (2.131) to

$$\begin{aligned} \ddot{v} + \mu_1 \dot{v} + \beta_y v^{IV} = & (1 - \beta_y) [w'' \int_1^s v''w'' ds - w''' \int_0^s v''w' ds]' + j_3 \ddot{v}'' \\ & - \frac{(1 - \beta_y)^2}{\beta_y} [w'' \int_0^s \int_1^s v''w'' ds ds]'' - \beta_y [v'(v'v'' + w'w'')]'' \\ & - \frac{1}{2} \{v' \int_1^s [\int_0^s (v'^2 + w'^2) ds]'' ds\}' + F_v , \end{aligned} \quad (2.138)$$

$$\begin{aligned}
\ddot{w} + \mu_2 \dot{w} + w^{iv} = & -(1 - \beta_y) \left[v'' \int_1^s v'' w'' ds - v''' \int_0^s w'' v' ds \right]' + j_2 \ddot{w}'' \\
& - \frac{(1 - \beta_y)^2}{\beta_y} \left[v'' \int_0^s \int_1^s v'' w'' ds ds \right]'' - [w'(w' w'' + v' v'')]'' \\
& - \frac{1}{2} \left\{ w' \int_1^s \left[\int_0^s (v'^2 + w'^2) ds \right]'' ds \right\}' + F_w .
\end{aligned} \tag{2.139}$$

The boundary conditions (2.133)-(2.135) become

$$v = w = v' = w' = 0 \quad \text{at } s = 0 , \tag{2.140}$$

$$I_h \ddot{\Theta}(t) + w''(0, t) = \tau_0(t) , \tag{2.141}$$

$$\beta_y v''' - j_3 \ddot{v}' = w''' - j_2 (\ddot{w}' - \ddot{\Theta}) = v'' = w'' = 0 \quad \text{at } s = 1 , \tag{2.142}$$

2.4 Characteristics of Equations

In this section we discuss the properties of the nonlinear equations of motion and compare them with other derivations.

2.4.1 Comparison with Other Derivations

Except for the forcing terms, structural coupling terms, and some rotary inertia terms with order higher than ε^3 , all other terms in (2.112)-(2.114), when they are expressed

in terms of ϕ , reduce to those obtained by Crespo da Silva and Glynn [19] for the case of isotropic beams.

Comparing (2.86) and (2.87) with the expressions (A13) and (A14) used by Rosen et al. [83] for the nonlinear curvatures, we find that each of the latter is missing a third-order term.

For the case of an in-plane vibration of an inextensional cantilever beam, we substitute $w = \gamma = F_v = 0$ into (2.112) and obtain

$$m\ddot{v} + \mu_1\dot{v} = -D_{33}(v'''' + v''''v'^2 + 4v'v''v'' + v''^3) + j_\zeta\ddot{v}'' - \frac{1}{2}[v'\int_L^s m(\int_0^s v'^2 ds)''ds]' . \quad (2.143)$$

The nonlinear curvature and inertia terms in (2.143) are the same as those derived by Sato et al. [88] and Zavodney [104,105].

2.4.2 An Asymmetric Mathematical Model

In the nonlinear formulation, owing to the use of finite rotations and the contributions from nonlinear bendings, ϕ does not represent the real twist angle with respect to the ξ axis. To show that a nonzero ϕ does not necessarily indicate the presence of torsion along the beam, we consider an inextensional uniform isotropic beam with a square cross section under a load Q acting at the tip of the beam along the direction $\vec{i}_y + \vec{i}_z$. Because $D_{22} = D_{33}$ for this case, any direction parallel to the $\eta - \zeta$ plane is a principal direction; hence, under the load, the beam will bend in the plane defined by the x-axis and $\vec{Q}$, the displacements v and w are equal, and torsion is not

expected. In other words, the following conditions exist: $\rho_\xi = 0$, $-\rho_\eta = \rho_\xi$. Checking (2.85)-(2.87), we see that nontrivial particular values of ϕ are needed in order to satisfy these conditions. This observation is also discussed by Rosen, Loewy, and Mathew [84]. Because a particular value of ϕ is required in order to provide symmetry in the expressions for v and w and because the expression for ρ_η in (2.86) cannot be obtained from (2.87) by replacing w by v and v by -w (i.e., rotate the reference coordinate system by 90° with respect to the x- axis), we conclude that the mathematical model itself is not symmetric. Hence, the equations of motion expressed in terms of ϕ are not symmetric or interchangeable because there are implicit asymmetric terms included in ϕ . Moreover, if the rotation sequence used is θ, ψ, ϕ , which implies that the deformation sequence is u, w, v, ϕ , the twisting curvature becomes [4]

$$\rho_\xi = \phi' - v'w'' .$$

If two sequential rotations (a bending part and a twisting part) are used in the derivation, which implies that the deformation sequence is u, v & w, ϕ , the twisting curvature becomes [4]

$$\rho_\xi = \phi' + \frac{1}{2} (v''w' - v'w'') .$$

From these two different formulations, we can see that nontrivial values of ϕ are produced in a uniform beam having a square cross section when loaded in, for example, the $\bar{i}_y + 2\bar{i}_z$ direction. Therefore, it follows that any twist variable ϕ , defined by using a sequence of three Euler-like rotations or even two sequential rotations, is not a real twisting angle because the deformations u, v, w, and ϕ do not occur in sequence as assumed in the mathematical model that uses Euler angles.

Consequently, the twisting curvature and kinematic boundary conditions must be used to define a real twist angle as done in (2.106).

Using the following transformation, which corresponds to the case in which the reference coordinate system x-y-z is rotated by 90° with respect to the x-axis,

$$\begin{aligned} \gamma \rightarrow \gamma, \quad v \rightarrow -w, \quad w \rightarrow v, \quad j_\xi \rightarrow j_\xi, \quad j_\eta \rightarrow j_\zeta, \quad j_\zeta \rightarrow j_\eta, \quad \mu_1 \rightarrow \mu_1, \quad \mu_2 \rightarrow \mu_3, \quad \mu_3 \rightarrow \mu_2, \\ D_{11} \rightarrow D_{11}, \quad D_{22} \rightarrow D_{33}, \quad D_{33} \rightarrow D_{22}, \quad D_{12} \rightarrow -D_{13}, \quad D_{13} \rightarrow D_{12}, \quad D_{23} \rightarrow -D_{32} (= -D_{23}) \end{aligned}$$

in (2.112)-(2.129), we can interchange the equations of motion and boundary conditions governing v and w with the equation and boundary conditions governing γ being unchanged. Furthermore, (2.108)-(2.110) show that the bending curvatures are also interchangeable. So, the equations of motion (2.112) and (2.113) are symmetric, unique, and independent of the rotation sequence of the three Euler angles used in the mathematical modeling. And v , w , and γ are independent coordinates.

Equation (2.137) shows that, for a lengthy isotropic beam, the frequencies of twist oscillations are linear combinations of the bending frequencies in the x and z directions. If any pretwist angle γ_p exists, it must be considered before using the transformation [4]. Any torque applied to the beam produces a forced torsional motion γ_r . Furthermore, for a composite beam, bending-twisting couplings may induce a torsional motion γ_c . Hence, the total twist angle consists of four parts: γ_p , γ_r , γ_c , and γ . The frequency of γ_c is the same as that of v if the bending motion in the y -direction is coupled with the torsional motion, and the forced torsional motion γ_r may have a different frequency. Hence, we expect the nonlinear torsional motion to be rather complicated.

2.4.3 Lagrangian and Eulerian Coordinates and Curvature Expressions

To show the implications of using an Eulerian coordinate, let us consider the in-plane flexural vibration of an inextensional cantilever isotropic beam. Because $x = s + u$ and the inextensionality condition implies that (see Fig. 2.4(b))

$$dx^2 + dv^2 = ds^2 . \quad (2.144)$$

Putting $ds = dx - du$ in (2.144), expanding, and using the boundary condition $u(0,t) = 0$ yields

$$u_x = -\frac{1}{2} v_x^2 + \dots , \quad u = -\frac{1}{2} \int_0^x v_x^2 dx + \dots . \quad (2.145)$$

Comparing (2.145) and (2.84), we find that, up to order ε^3 , the longitudinal nonlinear inertia term $-m\ddot{u}$ has the same form, irrespective of the coordinate system used. However, when the Lagrangian coordinate s is used, the curvature ρ_ζ can be expressed as

$$\rho_\zeta \equiv \frac{\partial \psi}{\partial s} \equiv \psi' = \frac{\partial \sin^{-1} v'}{\partial s} = \frac{v''}{\sqrt{1 - v'^2}} = v'' \left(1 + \frac{1}{2} v'^2 + \frac{3}{8} v'^4 + \dots \right) . \quad (2.146)$$

On the other hand, when the Eulerian coordinate x is used,

$$\rho_\zeta \equiv \frac{\partial \psi}{\partial s} = \frac{\partial \tan^{-1} v_x}{\partial x} \frac{dx}{ds} = \frac{v_{xx}}{(1 + v_x^2)^{3/2}} = v_{xx} \left(1 - \frac{3}{2} v_x^2 + \frac{15}{8} v_x^4 + \dots \right) , \quad (2.147)$$

where $v_x \equiv \frac{\partial v}{\partial x} = v'(1 + \frac{1}{2} v'^2 + \dots)$. We note that the form of the nonlinear curvature depends on the coordinate being used. Because $|v'| < 1$ for all $|\psi| < 90^\circ$, the series

in (2.146) converges for all $|\psi| < 90^\circ$. However, because $|v_x| < 1$ for $|\psi| < 45^\circ$ and $|v_x| > 1$ for $45^\circ < |\psi| < 90^\circ$, the series in (2.147) converges in the first interval and diverges in the second interval.

If the Eulerian coordinate x is used, the comoving derivative of displacement v (i.e., the velocity in the y -direction of the observed element) is obtained by using (2.145) as

$$\frac{dv}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \frac{dx}{dt} = \dot{v} + v_x \frac{du}{dt} = \dot{v} - \frac{1}{2} v_x \frac{d}{dt} \left(\int_0^x v_x^2 dx \right) + \dots \quad (2.148)$$

Consequently, the nonlinear term is a third-order term. Similarly, the acceleration of the element is given by

$$\begin{aligned} \frac{d^2 v}{dt^2} &= \ddot{v} + 2\dot{v}_x \frac{du}{dt} + v_{xx} \left(\frac{du}{dt} \right)^2 + v_x \frac{d^2 u}{dt^2} + v_x \frac{du_x}{dt} \frac{du}{dt} \\ &= \ddot{v} - \dot{v}_x \frac{d}{dt} \left(\int_0^x v_x^2 dx \right) - \frac{1}{2} v_x \frac{d^2}{dt^2} \left(\int_0^x v_x^2 dx \right) + \dots \end{aligned} \quad (2.149)$$

It follows from (2.148) and (2.149), the Lagrangian coordinate s is more suitable than the Eulerian coordinate x for an inextensional beam because in the latter case the expressions for the velocity and acceleration are more complicated.

For a fully-stretched extensional beam (e.g., a fixed-fixed beam), if we assume that $u = 0$, then $s = x$ and the curvature can be expressed as

$$\begin{aligned} \rho_\zeta &\equiv \frac{\partial \psi}{\partial \tilde{s}} \equiv \psi' = \frac{\partial \tan^{-1} v'}{\partial s} \frac{ds}{d\tilde{s}} = \frac{v''}{1 + v'^2} \frac{ds}{\sqrt{ds^2 + dv^2}} = \frac{v''}{(1 + v'^2)^{3/2}} \\ &= v'' \left(1 - \frac{3}{2} v'^2 + \frac{15}{8} v'^4 + \dots \right) \end{aligned} \quad (2.150)$$

There is no longitudinal inertia force because $u = 0$, but the nonlinear curvature and stretching effect dominate the response and $v_{x \max} = v'_{\max} = \infty$. If we use the Eulerian coordinate $\tilde{s}$, the curvature can be expressed as

$$\rho_{\zeta} \equiv \frac{\partial \psi}{\partial \tilde{s}} = \frac{\partial \sin^{-1} v^{+}}{\partial \tilde{s}} = \frac{v^{++}}{\sqrt{1 - v^{+2}}} = v^{++} \left(1 + \frac{1}{2} v^{+2} + \frac{3}{8} v^{+4} + \dots \right), \quad (2.151)$$

and $v_{\max}^{+} = 1$. However, the comoving derivatives and the stretching length (when boundary conditions are imposed) make the formulation more complicated.

Equation (2.147) is the expression of the curvature found in most of the books of strength of materials. However, it follows from the expressions (2.146), (2.147), and (2.150) that the expression for the nonlinear curvature depends on the coordinate being used and the boundary conditions. Consequently, approximations of the nonlinear curvatures derived from the Von Karman strains, which are commonly used in nonlinear analysis of plates, may be questionable.

2.4.4 Nonlinear Lagrangian Strains and Curvatures

To simplify the discussion, let us consider the in-plane (x-y plane) vibration of an inextensional cantilever beam again. Without shear deformation, the displacements $\tilde{u}$ and $\tilde{v}$ of any point on the cross section are obtained as

$$\tilde{u} = u - \eta \sin \psi, \quad \tilde{v} = v + \eta (\cos \psi - 1). \quad (2.152)$$

Differentiating (2.152) and using (2.83) and (2.56)-(2.58), we obtain, up to third order,

$$\tilde{u}' = -\frac{1}{2} v'^2 - \eta v'' \quad , \quad \tilde{v}' = v' - \eta v' v'' \quad . \quad (2.153)$$

Hence, the nonlinear Lagrangian strain L_{xx} [60] can be expanded as

$$L_{xx} \equiv \tilde{u}' + \frac{1}{2} \tilde{u}'^2 + \frac{1}{2} \tilde{v}'^2 = \eta \left(-v'' - \frac{1}{2} v'' v'^2 \right) + \frac{1}{2} \eta^2 v''^2 + \dots \quad . \quad (2.154)$$

Substituting (2.154) into the identity

$$L_{xx} \equiv \tilde{\theta} + \frac{1}{2} \tilde{\theta}^2$$

yields the $O(\varepsilon^3)$ expansion of the normal strain $\tilde{\theta}$ as

$$\begin{aligned} \tilde{\theta} &= L_{xx} - \frac{1}{2} L_{xx}^2 + \frac{1}{2} L_{xx}^3 + \dots \\ &= -\rho_\zeta \eta + \dots \quad , \end{aligned} \quad (2.155)$$

where (2.146) is used. (2.155) shows that, although the strain may be small, nonlinear strain tensors must be used in the derivation of nonlinear curvatures if the absolute displacements are large. Although the curvature may be small due to small value of v'' , nonlinear curvature expressions are needed if v' is large somewhere in the structure (e.g., a long slender beam). When the curvature is large due to large v'' , a nonlinear curvature expression is definitely needed because v' must be large somewhere in the structure if the span is not very short. From (2.155), we see that, even when the curvature is large, the axial strain is still small if the thickness is small, and linear stress-strain relationship can be assumed.

2.4.5 Using D'Alembert Principle to Derive Forcing Functions

To avoid complex manipulations of the forcing terms due to the motion of the boundaries, one can separately obtain these forcing terms by using D'Alembert principle. We consider three cases : (1) slewing motions, (2) parametric-base excitations, and (3) lateral-base excitations.

Case (1) : slewing motions

Using (2.25)-(2.27), we obtain the following inertia forces due to the motion of the hub (considered as external loads):

$$Q_x^R = -m(a_x - \ddot{u}) = -m[\ddot{\Theta}w - \dot{\Theta}^2(s + u) + 2\dot{\Theta}\dot{w}] , \quad (2.156)$$

$$Q_y^R = -m(a_y - \ddot{v}) = 0 , \quad (2.157)$$

$$Q_z^R = -m(a_z - \ddot{w}) = -m[-\ddot{\Theta}(s + u) - \dot{\Theta}^2w - 2\dot{u}\dot{\Theta}] . \quad (2.158)$$

The extra inertia moments due to Θ can be obtained from (2.75)-(2.77) as

$$M_\xi^L = -j_\xi(\dot{\Theta}v')' + (j_\eta - j_\zeta) [\dot{\Theta}(\dot{v}' + 2\dot{w}'\phi - 2\dot{v}'\phi^2 + 2v'w'\dot{w}') + \dot{\Theta}^2(-\phi - v'w' + \phi v'^2 + \frac{2}{3}\phi^3)] , \quad (2.159)$$

$$M_\eta^L = -j_\eta\ddot{\Theta} , \quad (2.160)$$

$$M_\zeta^L = j_\zeta(\dot{\Theta}\phi)' + (j_\zeta - j_\eta)\dot{\Theta}(\dot{\phi} + \dot{\Theta}v') . \quad (2.161)$$

It follows from (2.89)-(2.94) that

$$(T_{21}\tilde{F}_2 + T_{31}\tilde{F}_3)' + (T_{11}\tilde{\lambda})' + Q_x^R = 0 , \quad (2.162)$$

$$F_y = (T_{22}\tilde{F}_2 + T_{32}\tilde{F}_3)' + (T_{12}\tilde{\lambda})' + Q_y^R , \quad (2.163)$$

$$F_w = (T_{23}\tilde{F}_2 + T_{33}\tilde{F}_3)' + (T_{13}\tilde{\lambda})' + Q_z^R , \quad (2.164)$$

$$F_\eta = M_\xi^L , \quad (2.165)$$

$$-\tilde{F}_3 + M_\eta^L = 0 , \quad (2.166)$$

$$\tilde{F}_2 + M_\zeta^L = 0 , \quad (2.167)$$

where $(\tilde{})$ denotes the change in the internal forces under the inertia forces and moments. The beam is assumed to be inextensional. It follows from (2.166) and (2.167) that

$$\tilde{F}_3 = M_\eta^L , \tilde{F}_2 = -M_\zeta^L . \quad (2.168)$$

Substituting (2.168) and (2.95) into (2.162), we obtain

$$w'j_\eta\ddot{\Theta} + (1 - \frac{1}{2}v'^2 - \frac{1}{2}w'^2)\tilde{\lambda} - \int_L^s m(\ddot{\Theta}w + 2\dot{\Theta}\dot{w} - \dot{\Theta}^2(s+u))ds = 0 . \quad (2.169)$$

Solving (2.169) for $\tilde{\lambda}$ and keeping terms up to order ε^3 , we obtain

$$\tilde{\lambda} = (1 + \frac{1}{2}v'^2 + \frac{1}{2}w'^2)\int_L^s m(\ddot{\Theta}w + 2\dot{\Theta}\dot{w} - \dot{\Theta}^2(s+u))ds - w'j_\eta\ddot{\Theta} . \quad (2.170)$$

Substituting (2.170), (2.168), and (2.106) into (2.163)-(2.165), we obtain (2.118)-(2.120).

Case (2) : parametric-base excitations

If the base motion $F \cos \Omega t$ is assumed to be along the x-direction, then the absolute displacement in the x-direction is

$$\bar{u} = u + F \cos \Omega t .$$

Using (2.25)-(2.27), we obtain the following inertia forces due to the motion of the base:

$$Q_x^R = -m(a_x - \ddot{u}) = mF\Omega^2 \cos \Omega t , \quad (2.171)$$

$$Q_y^R = -m(a_y - \ddot{v}) = 0 , \quad (2.172)$$

$$Q_z^R = -m(a_z - \ddot{w}) = 0 . \quad (2.173)$$

Because there are no extra inertia moments,

$$M_\xi^L = M_\eta^L = M_\zeta^L = 0 . \quad (2.174)$$

Hence,

$$F_y = \tilde{F}_2 = \tilde{F}_3 = 0 , \quad (2.175)$$

$$(T_{11}\tilde{\lambda})' + Q_x^R = 0 . \quad (2.176)$$

Integrating and inverting (2.176) yields

$$\tilde{\lambda} = -\left(1 + \frac{1}{2}v'^2 + \frac{1}{2}w'^2\right) \int_L^s m\Omega^2 F \cos \Omega t ds . \quad (2.177)$$

Using (2.177), we obtain the forcing functions as

$$F_v = (T_{12}\tilde{\lambda})' = -[v''(s-L) + v']m\Omega^2 F \cos \Omega t , \quad (2.178)$$

$$F_w = (T_{13}\tilde{\lambda})' = -[w''(s-L) + w']m\Omega^2 F \cos \Omega t . \quad (2.179)$$

Case (3) : lateral-base excitations

When the base motion $F \cos \Omega t$ is along the z-direction, the absolute displacement in the z-direction is $\bar{w} = w + F \cos \Omega t$. Using (2.25)-(2.27) and taking the gravity g into account, we obtain the following inertia forces due to the base motion:

$$Q_x^R = -m(a_x - \ddot{u}) - mg = -mg , \quad (2.180)$$

$$Q_y^R = -m(a_y - \ddot{v}) = 0 , \quad (2.181)$$

$$Q_z^R = -m(a_z - \ddot{w}) = mF\Omega^2 \cos \Omega t . \quad (2.182)$$

Because there are no extra inertia moments,

$$M_\xi^L = M_\eta^L = M_\zeta^L = 0 . \quad (2.183)$$

Hence,

$$F_y = \tilde{F}_2 = \tilde{F}_3 = 0 , \quad (2.184)$$

$$(T_{11}\tilde{\lambda})' + Q_x^R = 0 . \quad (2.185)$$

Integrating and inverting (2.185) yields

$$\tilde{\lambda} = mg(s - L) .$$

Using this expression for the constraint force and (2.111), we obtain

$$F_v = (T_{12}\tilde{\lambda})' = [v''(s - L) + v']mg , \quad (2.186)$$

$$F_w = (T_{13}\tilde{\lambda})' + Q_z^R = [w''(s - L) + w']mg + mF\Omega^2 \cos \Omega t . \quad (2.187)$$

2.4.6 Stiffness Matrix of a Composite Beam

To obtain analytically the stiffness matrices [A], [B], and [D], we develop a theory similar to the classical laminated plate theory, as shown below. To this end, we need to relate the stress resultants to the shear strains and curvatures. First, the strains and curvatures need to be correlated. Then, the constitutive equations are used. Because the displacements of a material point on the elastic axis are given by

$$u = u(\xi, t) , \quad v = v(\xi, t) , \quad w = w(\xi, t) , \quad (2.188)$$

the displacements of an arbitrary point on the cross section at (ξ, η, ζ) are obtained as

$$\tilde{u} = u - \eta(\theta_\zeta - \gamma_{\xi\eta}) + \zeta(\theta_\eta + \gamma_{\xi\zeta}) , \quad \tilde{v} = v - \zeta\theta_\xi , \quad \tilde{w} = w + \eta\theta_\xi , \quad (2.189)$$

where θ_ξ , θ_η , and θ_ζ denote the rotation angles of the cross section with respect to the ξ , η , and ζ axes, respectively. Accordingly, the strains are obtained as

$$\tilde{\epsilon}_\xi = \frac{\partial \tilde{u}}{\partial \xi} = \epsilon - \eta\rho_\zeta + \zeta\rho_\eta , \quad (2.190)$$

$$\tilde{\gamma}_{\xi\eta} = \frac{\partial \tilde{u}}{\partial \eta} + \frac{\partial \tilde{v}}{\partial \xi} = \gamma_{\xi\eta} - \zeta \rho_{\xi} , \quad (2.191)$$

$$\tilde{\gamma}_{\xi\zeta} = \frac{\partial \tilde{u}}{\partial \zeta} + \frac{\partial \tilde{w}}{\partial \xi} = \gamma_{\xi\zeta} + \eta \rho_{\xi} , \quad (2.192)$$

$$\text{where } e = \frac{\partial u}{\partial \xi} , \rho_{\xi} = \frac{\partial \theta_{\xi}}{\partial \xi} , \rho_{\eta} = \frac{\partial \theta_{\eta}}{\partial \xi} , \rho_{\zeta} = \frac{\partial \theta_{\zeta}}{\partial \xi} , \theta_{\xi} = \frac{\partial v}{\partial \xi} , \theta_{\eta} = -\frac{\partial w}{\partial \xi}$$

and $\partial \gamma_{\xi\eta}/\partial \xi$ and $\partial \gamma_{\xi\zeta}/\partial \xi$ are neglected. Equations (2.190)-(2.192) are the same as those derived in [8], where a tensor analysis was used and some extra terms due to warping deformation were included.

Next, we consider a symmetrically laminated composite beam with each lamina being parallel to the $\xi - \zeta$ plane. Using coordinate transformations, we obtain the constitutive equations

$$\begin{Bmatrix} \sigma_{\xi} \\ \sigma_{\eta} \\ \sigma_{\zeta} \\ \tau_{\eta\zeta} \\ \tau_{\xi\zeta} \\ \tau_{\xi\eta} \end{Bmatrix} = \begin{bmatrix} Q'_{11} & Q'_{12} & Q'_{13} & 0 & Q'_{15}/2 & 0 \\ Q'_{12} & Q'_{22} & Q'_{23} & 0 & Q'_{25}/2 & 0 \\ Q'_{13} & Q'_{23} & Q'_{33} & 0 & Q'_{35}/2 & 0 \\ 0 & 0 & 0 & Q'_{44}/2 & 0 & Q'_{46}/2 \\ Q'_{15}/2 & Q'_{25}/2 & Q'_{35}/2 & 0 & Q'_{55}/2 & 0 \\ 0 & 0 & 0 & Q'_{46}/2 & 0 & Q'_{66}/2 \end{bmatrix} \begin{Bmatrix} \tilde{\epsilon}_{\xi} \\ \tilde{\epsilon}_{\eta} \\ \tilde{\epsilon}_{\zeta} \\ \tilde{\gamma}_{\eta\zeta} \\ \tilde{\gamma}_{\xi\zeta} \\ \tilde{\gamma}_{\xi\eta} \end{Bmatrix} , \quad (2.193)$$

where

$$Q'_{11} = m^4 Q_{11} + 2m^2 n^2 Q_{13} + n^4 Q_{33} + 4m^2 n^2 Q_{55} ,$$

$$Q'_{12} = m^2 Q_{12} + n^2 Q_{23} ,$$

$$Q'_{13} = m^2 n^2 (Q_{11} + Q_{33} - 4Q_{55}) + (m^4 + n^4) Q_{13} ,$$

$$Q'_{15} = 2mn^3(Q_{33} - Q_{13}) + 2m^3n(Q_{13} - Q_{11}) + 4mn(m^2 - n^2)Q_{55} ,$$

$$Q'_{22} = Q_{22} ,$$

$$Q'_{23} = n^2Q_{12} + m^2Q_{23} ,$$

$$Q'_{25} = 2mn(Q_{23} - Q_{12}) ,$$

$$Q'_{33} = n^4Q_{11} + m^4Q_{33} + 2m^2n^2(Q_{13} + 2Q_{55}) ,$$

$$Q'_{35} = 2mn^3(Q_{13} - Q_{11}) + 2m^3n(Q_{33} - Q_{13}) + 4mn(n^2 - m^2)Q_{55} ,$$

$$Q'_{44} = 2m^2Q_{44} + 2n^2Q_{66} ,$$

$$Q'_{46} = 2mn(Q_{44} - Q_{66}) ,$$

$$Q'_{55} = 2m^2n^2(Q_{11} - 2Q_{13} + Q_{33}) + 2(m^2 - n^2)^2Q_{55} ,$$

$$Q'_{66} = 2n^2Q_{44} + 2m^2Q_{66} ,$$

$$m \equiv \cos \alpha , \quad n \equiv \sin \alpha ,$$

and the Q_{ij} are represented in terms of nine material constants [103].

For beams with no external loads acting on the lateral surfaces, we can reasonably assume that

$$\sigma_\eta = \sigma_\zeta = \tau_{\eta\zeta} = 0 . \quad (2.194)$$

Hence, (2.193) reduces to

$$\begin{Bmatrix} \sigma_\xi \\ \tau_{\xi\eta} \\ \tau_{\xi\zeta} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{16} & \bar{Q}_{15} \\ \bar{Q}_{16} & \bar{Q}_{66} & \bar{Q}_{65} \\ \bar{Q}_{15} & \bar{Q}_{65} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \tilde{\varepsilon}_\xi \\ \tilde{\gamma}_{\xi\eta} \\ \tilde{\gamma}_{\xi\zeta} \end{Bmatrix} , \quad (2.195)$$

where

$$\bar{Q}_{11} = Q'_{11} + Q'_{12}C_3 + Q'_{13}C_1 ,$$

$$\bar{Q}_{15} = \frac{1}{2} Q'_{15} + Q'_{13}C_2 + Q'_{12}C_4 ,$$

$$\bar{Q}_{55} = \frac{1}{2} Q'_{55} + \frac{1}{2} Q'_{25} C_4 + \frac{1}{2} Q'_{35} C_2 ,$$

$$\bar{Q}_{66} = -\frac{1}{2} \frac{Q'^2_{46}}{Q'_{44}} + \frac{1}{2} Q'_{66} , \quad \bar{Q}_{16} = \bar{Q}_{65} = 0 ,$$

$$C_1 = \frac{Q'_{12} Q'_{23} - Q'_{13} Q'_{22}}{Q'_{22} Q'_{33} - Q'^2_{23}} , \quad C_2 = \frac{Q'_{25} Q'_{23} - Q'_{35} Q'_{22}}{2(Q'_{22} Q'_{33} - Q'^2_{23})} ,$$

$$C_3 = \frac{Q'_{12} Q'_{33} - Q'_{13} Q'_{23}}{Q'^2_{23} - Q'_{22} Q'_{33}} , \quad C_4 = \frac{Q'_{25} Q'_{33} - Q'_{35} Q'_{23}}{2(Q'^2_{23} - Q'_{22} Q'_{33})} .$$

It must be pointed out that for a laminate with the fibers being arbitrary tailored, the matrix $[\bar{Q}]$ can be a full matrix.

Because internal forces and moments can be expressed in terms of the stresses as

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \int_{-b/2}^{b/2} \sum_{k=1}^N \int_{\eta_{k-1}}^{\eta_k} \begin{Bmatrix} \sigma_\xi \\ \tau_{\xi\eta} \\ \tau_{\xi\zeta} \end{Bmatrix} d\eta d\zeta , \quad \begin{Bmatrix} M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \int_{-b/2}^{b/2} \sum_{k=1}^N \int_{\eta_{k-1}}^{\eta_k} \begin{Bmatrix} -\zeta \tau_{\xi\eta} + \eta \tau_{\xi\zeta} \\ \zeta \sigma_\xi \\ -\eta \sigma_\xi \end{Bmatrix} d\eta d\zeta , \quad (2.196)$$

we obtain

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B]^T & [D] \end{bmatrix} \begin{Bmatrix} \gamma_{\xi\eta} \\ \gamma_{\xi\zeta} \\ \rho_\xi \\ \rho_\eta \\ \rho_\zeta \end{Bmatrix} , \quad (2.197)$$

where the stiffness matrices are given by

$$[A] = \sum_{k=1}^N b t_k \begin{bmatrix} \bar{Q}_{11} & 0 & \bar{Q}_{15} \\ 0 & \bar{Q}_{66} & 0 \\ \bar{Q}_{15} & 0 & \bar{Q}_{55} \end{bmatrix}, \quad [B] = \sum_{k=1}^N \frac{1}{2} b (\eta_k^2 - \eta_{k-1}^2) \begin{bmatrix} \bar{Q}_{15} & 0 & -\bar{Q}_{11} \\ 0 & 0 & 0 \\ \bar{Q}_{55} & 0 & -\bar{Q}_{15} \end{bmatrix},$$

$$[D] = \sum_{k=1}^N \begin{bmatrix} \bar{Q}_{66} C_5 + \bar{Q}_{55} C_6 & 0 & -\bar{Q}_{15} C_6 \\ 0 & \bar{Q}_{11} C_5 & 0 \\ -\bar{Q}_{15} C_6 & 0 & \bar{Q}_{11} C_6 \end{bmatrix}.$$

Here, $t_k (= \eta_k - \eta_{k-1})$ is the thickness of the k th lamina, N is the total number of laminae, $C_5 = \frac{1}{12} t_k b^3$, and $C_6 = \frac{1}{3} b (\eta_k^3 - \eta_{k-1}^3)$.

2.4.7 Procedures for Deriving Equations of Motion for Extensional Beams

For unsymmetrically laminated composite beams and beams with both ends constrained (e.g., a clamped-pinned beam), the bending-extension coupling and the mid-plane stretching are not negligible. Following a procedure similar to that used in the extensional case and knowing that $\epsilon \neq 0$, one can derive third-order equations describing the extensional-flexural-flexural-torsional vibrations of composite beams. Because of extensionality,

$$d\tilde{s} = (1 + \epsilon) ds, \quad \rho_l = \tilde{\rho}_l (1 + \epsilon). \quad (2.198)$$

Because the shear strains are neglected, the constitutive equations (2.36) reduce to

$$F_1 = A_{11} \epsilon + B_{11} \tilde{\rho}_\xi + B_{12} \tilde{\rho}_\eta + B_{13} \tilde{\rho}_\zeta, \quad (2.199)$$

$$M_1 = B_{11}\theta + D_{11}\tilde{\rho}_\xi + D_{12}\tilde{\rho}_\eta + D_{13}\tilde{\rho}_\zeta \quad , \quad (2.200)$$

$$M_2 = B_{12}\theta + D_{12}\tilde{\rho}_\xi + D_{22}\tilde{\rho}_\eta + D_{23}\tilde{\rho}_\zeta \quad , \quad (2.201)$$

$$M_3 = B_{13}\theta + D_{13}\tilde{\rho}_\xi + D_{23}\tilde{\rho}_\eta + D_{33}\tilde{\rho}_\zeta \quad , \quad (2.202)$$

Using (2.47) and (2.48), we obtain the internal shear forces as

$$F_2 = \frac{1}{1 + \Theta} (-M_3' + \rho_\eta M_1 - \rho_\xi M_2 + A_\zeta) \quad , \quad (2.203)$$

$$F_3 = \frac{1}{1 + \Theta} (M_2' + \rho_\zeta M_1 - \rho_\xi M_3 - A_\eta) \quad . \quad (2.204)$$

The four governing equations are

$$(T_{11}F_1 + T_{21}F_2 + T_{31}F_3)' = ma_x + \mu_0\dot{u} \quad , \quad (2.205)$$

$$(T_{12}F_1 + T_{22}F_2 + T_{32}F_3)' = ma_y + \mu_1\dot{v} \quad , \quad (2.206)$$

$$(T_{13}F_1 + T_{23}F_2 + T_{33}F_3)' = ma_z + \mu_2\dot{w} \quad , \quad (2.207)$$

$$M_1' - \rho_\zeta M_2 + \rho_\eta M_3 = A_\xi + \mu_3\omega_\xi \quad . \quad (2.208)$$

By using (2.62)-(2.78), (2.25)-(2.27), (2.198), and the expanded forms of F_2 and F_3 , one can obtain four equations governing the four unknowns u , v , w , and ϕ .

Figure 2.1. Coordinate systems: $x-y-z$ = reference frame, which is fixed on the hub; $\xi-\eta-\zeta$ coincides with the principal axes of the cross section, which is a local frame fixed on the cross section.

Figure 2.2. Three successive Euler angle rotations of a differential beam element.

Figure 2.3. Free-body diagram of a differential beam element.

Figure 2.4. (a) Transformation geometry, (b) relationship between displacements and Euler angles.

CHAPTER III

PARAMETRICALLY EXCITED FLEXURAL-FLEXURAL VIBRATIONS OF AN ISOTROPIC BEAM

In this chapter, we use the developed nonlinear integro-differential equations of motion to investigate planar and nonplanar responses of isotropic cantilevered beams to a principal parametric excitation, as shown in Fig. 3.1. The beam is assumed to undergo flexure about two principal axes and torsion. Two uniform beams with rectangular cross sections are considered: one has an aspect ratio near unity, and the other has an aspect ratio near 6.27. In both cases, the beam possesses a one-to-one internal resonance with one of the natural flexural frequencies in one plane being approximately equal to one of the natural flexural frequencies in the second plane.

3.1 Equations of Motion

We assume that the beam is uniform and long, the torsional motion cannot be excited by low-frequency input forces because the torsional frequency is higher than any flexural frequency, and the rotary inertias are negligible. Hence, the equations of motion are obtained from (2.138), (2.139), (2.178), and (2.179) as

$$\begin{aligned}\ddot{v} + \mu_1 \dot{v} + \beta_y v^{iv} = & (1 - \beta_y) \left[w'' \int_1^s v'' w'' ds - w''' \int_0^s v'' w' ds \right]' \\ & - \frac{(1 - \beta_y)^2}{\beta_y} \left[w'' \int_0^s \int_1^s v'' w'' ds ds \right]'' - \beta_y [v'(v'v'' + w'w'')]'' \\ & - \frac{1}{2} \{ v' \int_1^s \left[\int_0^s (v'^2 + w'^2) ds \right]'' ds \}' - [v''(s-1) + v'] F \Omega^2 \cos(\Omega t) ,\end{aligned}\quad (3.1)$$

$$\begin{aligned}\ddot{w} + \mu_2 \dot{w} + w^{iv} = & -(1 - \beta_y) \left[v'' \int_1^s v'' w'' ds - v''' \int_0^s w'' v' ds \right]' \\ & - \frac{(1 - \beta_y)^2}{\beta_y} \left[v'' \int_0^s \int_1^s v'' w'' ds ds \right]'' - [w'(v'v'' + w'w'')]'' \\ & - \frac{1}{2} \{ w' \int_1^s \left[\int_0^s (v'^2 + w'^2) ds \right]'' ds \}' - [w''(s-1) + w'] F \Omega^2 \cos(\Omega t) ,\end{aligned}\quad (3.2)$$

where the forcing terms (2.178) and (2.179) are nondimensionalized using the characteristic length L and the characteristic time $L^2 \sqrt{m/D_{22}}$. From (2.140) and (2.142), we obtain the boundary conditions

$$v = w = v' = w' = 0 \text{ at } s = 0 , \quad (3.3a)$$

$$v''' = w''' = v'' = w'' = 0 \text{ at } s = 1 . \quad (3.3b)$$

In (3.1) and (3.2), the terms in the first three square brackets are nonlinear geometric terms, those in the curly bracket are nonlinear inertia terms, and those in the last square bracket are parametric excitation forces due to the base motion. We note that all the nonlinear terms are cubic. And if $\beta_y = 1$ (e.g., a beam with square cross section), the nonlinear geometric terms in the first two square brackets disappear and (3.1) becomes the same as (3.2) because any axis on the cross section is a principal axis.

3.2 Multiple Scale Analysis

To analyze the coupled nonlinear equations (3.1) and (3.2), we employ the method of multiple scales [68,69]. Toward this end a small parameter ε is introduced as a bookkeeping device in the following perturbation analysis. We seek a first-order uniform expansion and consider weakly nonlinear vibrations. Thus, we assume that an approximation to the solutions of $v(s,t)$ and $w(s,t)$ can be expressed as

$$v(s, T_0, T_2; \varepsilon) = \varepsilon v_1(s, T_0, T_2, \dots) + \varepsilon^3 v_3(s, T_0, T_2, \dots) + \dots, \quad (3.4a)$$

$$w(s, T_0, T_2; \varepsilon) = \varepsilon w_1(s, T_0, T_2, \dots) + \varepsilon^3 w_3(s, T_0, T_2, \dots) + \dots. \quad (3.4b)$$

Here, $T_0 = t$, a fast scale characterizing motions with the natural frequencies ω_{mn} and Ω ; and $T_2 = \varepsilon^2 t$, a slow scale characterizing the modulations of the amplitudes and phases. Moreover, we let $\mu_1 = \mu_2 = \varepsilon^2 \mu$ and $F = \varepsilon^2 f$. For the case of a one-to-one internal resonance, we expand β_y as

$$\beta_y = 1 + \delta_0 + \varepsilon^2 \delta_2 \quad (3.5)$$

so that when $\delta_2 = 0$ the internal resonance is perfectly tuned. Substituting (3.4) into (3.1) and (3.2) and equating coefficients of like powers of ε , we obtain

Order ε :

$$D_0^2 v_1 + (1 + \delta_0) v_1^{iv} = 0 , \quad (3.6)$$

$$D_0^2 w_1 + w_1^{iv} = 0 , \quad (3.7)$$

$$v_1 = w_1 = v_1' = w_1' = 0 \text{ at } s = 0 , \quad (3.8a)$$

$$v_1'' = w_1'' = v_1''' = w_1''' = 0 \text{ at } s = 1 , \quad (3.8b)$$

Order ε^3 :

$$\begin{aligned} D_0^2 v_3 + (1 + \delta_0) v_3^{iv} = & -2D_0 D_2 v_1 - \mu D_0 v_1 - \frac{(1 - \beta_y)^2}{\beta_y} [w_1'' \int_0^s \int_1^s v_1'' w_1'' ds ds]'' \\ & - \delta_2 v_1^{iv} + (1 - \beta_y) [w_1'' \int_1^s v_1'' w_1'' ds - w_1''' \int_0^s v_1'' w_1' ds]' \\ & - \beta_y [v_1' (v_1' v_1'' + w_1' w_1'')]'' - \frac{1}{2} \{v_1' \int_1^s [\int_0^s (v_1'^2 + w_1'^2) ds]'' ds\}' \\ & - [(s - 1)v_1'' + v_1'] f \Omega^2 \cos(\Omega t) , \end{aligned} \quad (3.9)$$

$$\begin{aligned} D_0^2 w_3 + w_3^{iv} = & -2D_0 D_2 w_1 - \mu D_0 w_1 - (1 - \beta_y) [v_1'' \int_1^s v_1'' w_1'' ds - v_1''' \int_0^s w_1'' v_1' ds]' \\ & - \frac{(1 - \beta_y)^2}{\beta_y} [v_1'' \int_0^s \int_1^s v_1'' w_1'' ds ds]'' - \frac{1}{2} \{w_1' \int_1^s [\int_0^s (v_1'^2 + w_1'^2) ds]'' ds\}' \\ & - [w_1' (v_1' v_1'' + w_1' w_1'')]'' - [(s - 1)w_1'' + w_1'] f \Omega^2 \cos(\Omega t) , \end{aligned} \quad (3.10)$$

$$v_3 = w_3 = v_3' = w_3' = 0 \text{ at } s = 0 , \quad (3.11a)$$

$$v_3'' = w_3'' = v_3''' = w_3''' = 0 \text{ at } s = 1 , \quad (3.11b)$$

where $D_n \equiv \partial/\partial T_n$.

The general solutions of (3.6)-(3.8) consist of two infinite sequences of modes corresponding to two infinite sequences of frequencies. Any mode that is not directly excited or indirectly excited will decay to zero with time due to the presence of damping [72]. In this chapter, we consider the principal parametric excitation of one of the modes in the z direction, say the n th mode, which in turn excites one of the modes in the y direction, say the m th mode, through a one-to-one internal or autoparametric resonance. Thus, the former mode is directly excited by the parametric resonance and the latter mode is indirectly excited through the autoparametric resonance. We assume that none of the other modes in the y and z directions is involved in any autoparametric resonance with these two modes. Consequently, the nondecaying solutions of (3.6)-(3.8) can be expressed as

$$v_1(s, T_0, T_2, \dots) = V_m(s)A_1(T_2)e^{i\omega_{1m}T_0} + cc, \quad (3.12a)$$

$$w_1(s, T_0, T_2, \dots) = W_n(s)A_2(T_2)e^{i\omega_{2n}T_0} + cc, \quad (3.12b)$$

where cc stands for the complex conjugate of the preceding terms,

$$\omega_{1m} = z_m^2 \sqrt{1 + \delta_0}, \quad \omega_{2n} = z_n^2,$$

$$V_i(s) = W_i(s) = \cosh z_i s - \cos z_i s + \frac{\cos z_i + \cosh z_i}{\sin z_i + \sinh z_i} (\sin z_i s - \sinh z_i s),$$

and z_i is a root of the characteristic equation $1 + \cos z \cosh z = 0$. The lowest three roots are 1.8751, 4.6941, 7.8548. Using the characteristic equation, it can be proved that $\int_0^1 V_i^2(s) ds = \int_0^1 W_i^2(s) ds = 1$. We note that ω_{2n} is the linear natural frequency of the n th mode of vibration in the z direction and ω_{1m} is approximately equal to the linear

natural frequency of the m th mode of vibration in the y direction. The latter actual frequency is $z_m^2 \sqrt{1 + \delta_0 + \varepsilon^2 \delta_2}$.

To determine the equations describing the functions $A_i(T_2)$, we use the Galerkin procedure to reduce (3.9) and (3.10) into ordinary-differential equations. To this end, we let

$$v_{1t}(T_0, T_2) = A_1(T_2)e^{i\omega_{1m}T_0} + cc, \quad w_{1t}(T_0, T_2) = A_2(T_2)e^{i\omega_{2n}T_0} + cc, \quad (3.13a)$$

$$v_{3t}(T_0, T_2) = \int_0^1 V_m(s)v_3(s, T_0, T_2)ds, \quad w_{3t}(T_0, T_2) = \int_0^1 W_n(s)w_3(s, T_0, T_2)ds. \quad (3.13b)$$

Multiplying (3.9) and (3.10) by $V_m(s)$ and $W_n(s)$, respectively, and then integrating over the domain of s , we obtain

$$\begin{aligned} D_0^2 v_{3t} + \omega_{1m}^2 v_{3t} = & -2D_0 D_2 v_{1t} - \mu D_0 v_{1t} - \delta_2 z^4 v_{1t} \\ & + (1 - \beta_y) \alpha_1 v_{1t} w_{1t}^2 - \frac{(1 - \beta_y)^2}{\beta_y} \alpha_2 v_{1t} w_{1t}^2 - \beta_y \alpha_3 v_{1t} w_{1t}^2 - \beta_y \alpha_4 v_{1t}^3 \\ & - \frac{1}{2} \alpha_5 v_{1t} D_0^2 v_{1t}^2 - \frac{1}{2} \alpha_6 v_{1t} D_0^2 w_{1t}^2 - \alpha_7 f \Omega^2 v_{1t} \cos(\Omega t), \end{aligned} \quad (3.14)$$

$$\begin{aligned} D_0^2 w_{3t} + \omega_{2n}^2 w_{3t} = & -2D_0 D_2 w_{1t} - \mu D_0 w_{1t} \\ & - (1 - \beta_y) \beta_1 v_{1t}^2 w_{1t} - \frac{(1 - \beta_y)^2}{\beta_y} \beta_2 v_{1t}^2 w_{1t} - \beta_3 v_{1t}^2 w_{1t} - \beta_4 w_{1t}^3 \\ & - \frac{1}{2} \beta_5 w_{1t} D_0^2 w_{1t}^2 - \frac{1}{2} \beta_6 w_{1t} D_0^2 v_{1t}^2 - \beta_7 f \Omega^2 w_{1t} \cos(\Omega t), \end{aligned} \quad (3.15)$$

where the α_i and β_i are defined in Appendix B and can be obtained by numerical integration.

To investigate the behavior of the beam to a principal parametric excitation in the presence of a one-to-one autoparametric resonance, we introduce the detuning parameter σ defined by

$$\Omega = 2\omega_{2n}(1 + \varepsilon^2 \sigma) \quad (3.16)$$

and let $\omega_{1m} = \omega_{2n}$. Substituting (3.13a) and (3.16) into (3.14) and (3.15) and eliminating the terms that produce secular terms, we obtain the solvability conditions

$$\begin{aligned} & -i(2\omega_{1m}A_1' + \mu\omega_{1m}A_1) - \delta_2 z^4 A_1 - [\delta_0 \alpha_1 + \frac{\delta_0^2}{\beta_y} \alpha_2 + (1 + \delta_0)\alpha_3](2A_1 A_2 \bar{A}_2 + \bar{A}_1 A_2^2) \\ & - 3(1 + \delta_0)\alpha_4 A_1^2 \bar{A}_1 + 2\alpha_5 \omega_{1m}^2 \bar{A}_1 A_1^2 + 2\alpha_6 \omega_{2n}^2 \bar{A}_1 A_2^2 - 2\alpha_7 f g \omega_{2n}^2 \bar{A}_1 e^{2i\omega_{2n}\sigma T_2} = 0, \end{aligned} \quad (3.17a)$$

$$\begin{aligned} & -i(2\omega_{2n}A_2' + \mu\omega_{2n}A_2) + (\delta_0 \beta_1 - \frac{\delta_0^2}{\beta_y} \beta_2 - \beta_3)(2A_2 A_1 \bar{A}_1 + \bar{A}_2 A_1^2) \\ & - 3\beta_4 A_2^2 \bar{A}_2 + 2\beta_5 \omega_{2n}^2 \bar{A}_2 A_2^2 + 2\beta_6 \omega_{1m}^2 \bar{A}_2 A_1^2 - 2\beta_7 f g \omega_{2n}^2 \bar{A}_2 e^{2i\omega_{2n}\sigma T_2} = 0. \end{aligned} \quad (3.17b)$$

Substituting the polar form

$$A_1 = \frac{1}{2} a_1(T_2) e^{i\theta_1(T_2)} \text{ and } A_2 = \frac{1}{2} a_2(T_2) e^{i\theta_2(T_2)}$$

into (3.17) and separating real and imaginary parts, we obtain

$$2\omega_{1m}a_1' + [R_1 + R_2 a_2^2 \sin 2(v_1 - v_2) + R_6 \sin 2v_1]a_1 = 0, \quad (3.18a)$$

$$[2\omega_{1m}v_1' - R_3 - R_4 a_1^2 + R_5 a_2^2 + R_2 a_2^2 \cos 2(v_1 - v_2) + R_6 \cos 2v_1]a_1 = 0, \quad (3.18b)$$

$$2\omega_{2n}a_2' + [E_1 + E_2 a_1^2 \sin 2(v_1 - v_2) + E_6 \sin 2v_2]a_2 = 0, \quad (3.18c)$$

$$[2\omega_{2n}v_2' - E_3 - E_4 a_2^2 - E_5 a_1^2 - E_2 a_1^2 \cos 2(v_1 - v_2) + E_6 \cos 2v_2]a_2 = 0, \quad (3.18d)$$

where the R_i and E_i are defined in Appendix B and

$$v_1 = \sigma\omega_{2n}T_2 - \theta_1, \quad v_2 = \sigma\omega_{2n}T_2 - \theta_2. \quad (3.19)$$

Periodic solutions of the beam correspond to the fixed points (i.e., constant solutions) of (3.18), which in turn correspond to $a_1' = a_2' = v_1' = v_2' = 0$. And to the first-order approximation the response of the beam is given by

$$v(s,t) = \varepsilon V_m(s) a_1(\varepsilon^2 t) \cos\left(\frac{1}{2} \Omega t - v_1\right) + \dots, \quad (3.20a)$$

$$w(s,t) = \varepsilon W_n(s) a_2(\varepsilon^2 t) \cos\left(\frac{1}{2} \Omega t - v_2\right) + \dots. \quad (3.20b)$$

To determine the stability of the non-trivial and trivial fixed points and hence the periodic solutions given by (3.20), we find it convenient to introduce the Cartesian coordinates p_i and q_i as

$$p_i = a_i \cos v_i, \quad q_i = a_i \sin v_i. \quad (3.21)$$

Differentiating (3.19) and (3.21) with respect to T_2 , we obtain

$$\theta_i' = \sigma\omega_{2n} - v_i', \quad (3.22a)$$

$$p_i' = a_i' \cos v_i - a_i v_i' \sin v_i, \quad (3.22b)$$

$$q_i' = a_i' \sin v_i + a_i v_i' \cos v_i. \quad (3.22c)$$

It follows from (3.18), (3.19), (3.21), and (3.22) that

$$-2\omega_{1m}p_1' = R_3q_1 + R_4(\rho_1^2 + q_1^2)q_1 - R_5(\rho_2^2 + q_2^2)q_1 + R_1p_1$$

$$+ R_2[-2q_2p_2p_1 + q_1(p_2^2 - q_2^2)] + R_6q_1 , \quad (3.23a)$$

$$\begin{aligned} -2\omega_{1m}q_1' = & -R_3p_1 - R_4(p_1^2 + q_1^2)p_1 + R_5(p_2^2 + q_2^2)p_1 + R_1q_1 \\ & + R_2[(p_2^2 - q_2^2)p_1 + 2q_1p_2q_2] + R_6p_1 , \end{aligned} \quad (3.23b)$$

$$\begin{aligned} -2\omega_{2n}p_2' = & E_3q_2 + E_4(p_2^2 + q_2^2)q_2 + E_5(p_1^2 + q_1^2)q_2 + E_1p_2 \\ & + E_2[q_2(q_1^2 - p_1^2) + 2p_2p_1q_1] + E_6q_2 , \end{aligned} \quad (3.23c)$$

$$\begin{aligned} -2\omega_{2n}q_2' = & -E_3p_2 - E_4(p_2^2 + q_2^2)p_2 - E_5(p_1^2 + q_1^2)p_2 + E_1q_2 \\ & - E_2[p_2(p_1^2 - q_1^2) + 2q_2p_1q_1] + E_6p_2 . \end{aligned} \quad (3.23d)$$

Because these are first-order autonomous ordinary-differential equations, the stability of a particular fixed point with respect to an infinitesimal disturbance proportional to $\exp(\lambda T_2)$ is determined by the eigenvalues of the Jacobian matrix of the right-hand sides of (3.23). A given fixed point is stable if and only if the real parts of all eigenvalues are less than or equal to zero. If there is a pair of complex conjugate eigenvalues having positive real parts, amplitude- and phase-modulated motions are expected.

Because the most natural way systems have frequencies that are nearly equal is due to some symmetry of the system, we want to discuss some symmetry properties before closing this section. If the solvability conditions (3.17a) and (3.17b) are presented in terms of variables Z_1 and Z_2 , where

$$Z_1 \equiv A_1 e^{i\omega_{1m}t} = (p_1 - iq_1) \frac{1}{2} e^{i\Omega t/2} , \quad Z_2 \equiv A_2 e^{i\omega_{2n}t} = (p_2 - iq_2) \frac{1}{2} e^{i\Omega t/2} ,$$

they are called normal form equations. When the modulation equations (3.23a)-(3.23d) are invariant under the transformation $(p_i, q_i) \rightarrow (-p_i, -q_i)$, the system

is said to possess Z_2 (inversion) symmetry. Because (3.23a)-(3.23d) are invariant under the transformations $(p_1, q_1, p_2, q_2) \rightarrow (-p_1, -q_1, p_2, q_2)$ and $(p_1, q_1, p_2, q_2) \rightarrow (p_1, q_1, -p_2, -q_2)$, the system exhibits $Z_2 - Z_2$ symmetry [90]. This $Z_2 - Z_2$ symmetry is also confirmed by the fact that (3.18a)-(3.18d) are invariant under the transformations $(a_1, v_1, a_2, v_2) \rightarrow (-a_1, v_1, a_2, v_2)$ and $(a_1, v_1, a_2, v_2) \rightarrow (a_1, v_1, -a_2, v_2)$. Because of the $Z_2 - Z_2$ symmetry, there are four possible solutions coexisting, which are

$(\pm p_1, \pm q_1, p_2, q_2)$ and $(\pm p_1, \pm q_1, -p_2, -q_2)$,
or $(\pm a_1, v_1, a_2, v_2)$ and $(\pm a_1, v_1, -a_2, v_2)$,
or (a_1, v_1, a_2, v_2) , $(a_1, v_1 + \pi, a_2, v_2)$, $(a_1, v_1, a_2, v_2 + \pi)$, and $(a_1, v_1 + \pi, a_2, v_2 + \pi)$.

For a fixed-point solution, these four possible solutions reduce to two because (3.20a) and (3.20b) are symmetric solutions; that is,

$$v(s, t) = -v(s, t + 2\pi/\Omega), \quad w(s, t) = -w(s, t + 2\pi/\Omega).$$

If furthermore the cross section is square (i.e., $\delta_0 = \delta_2 = 0$), then

$$R_1 = E_1, R_2 = -E_2, R_3 = E_3, R_4 = E_4, R_5 = E_5, R_6 = -E_6.$$

Hence, the system exhibits D_4 symmetry [90] because it is invariant under the transformation $(p_1, q_1, p_2, q_2) \rightarrow (p_2, q_2, p_1, q_1)$. Moreover, because it is an isotropic beam and any axis on the cross section is a principal axis, $O(2)$ symmetry [90] exists in this system because the modulation equations are invariant under any rotation of the coordinate x - y - z with respect to the x -axis; that is, the transformation $(y, z) \rightarrow (y \cos \theta + z \sin \theta, -y \sin \theta + z \cos \theta)$ for any $\theta \in [0, 2\pi)$. Hence, when there is D_4 symmetry in the system, (3.17a) and (3.17b) reduce to

$$i\omega_{1m}(2A_1' + \mu A_1) = \Pi_1 A_1^2 \bar{A}_1 + \Pi_2 A_1 A_2 \bar{A}_2 + \Pi_3 \bar{A}_1 A_2^2 - 2\alpha_7 f \omega_{2n}^2 \bar{A}_1 e^{2i\omega_{2n}\sigma T_2} ,$$

$$i\omega_{2n}(2A_2' + \mu A_2) = \Pi_1 A_2^2 \bar{A}_2 + \Pi_2 A_1 \bar{A}_1 A_2 + \Pi_3 A_1^2 \bar{A}_2 - 2\alpha_7 f \omega_{2n}^2 \bar{A}_2 e^{2i\omega_{2n}\sigma T_2} ,$$

where

$$\Pi_1 = -3\alpha_3 + 2\alpha_6 \omega_{2n}^2 , \quad \Pi_2 = -2\alpha_3 , \quad \Pi_3 = -\alpha_3 + 2\alpha_6 \omega_{2n}^2 .$$

They confirm the identity

$$\Pi_1 = \Pi_2 + \Pi_3$$

of a Hamiltonian system with $O(2)$ symmetry [90].

3.3 Numerical Results

Here we use the false position method [34] to solve for fixed-point solutions. Then, we determine the eigenvalues of the Jacobian matrix using the numerical routine EIGZF from the IMSL library. We investigate the responses of two different beams to principal parametric excitations. One beam has an almost square cross section, which implies that the natural frequencies of $v(s,t)$ and $w(s,t)$ of the same mode are very close. The other beam has a cross section with $b/h \simeq 6.2673$; in this case, the natural frequency of the second mode of $v(s,t)$ is close to that of the first mode of $w(s,t)$. For all the following cases, we assume that the material is isotropic, Poisson's ratio $\nu = 0.3$, and the cross section is rectangular with width b and height h . From material mechanics and the theory of elasticity, we know that

$$\beta_y = D_{33}/D_{22} = (h/b)^2 \simeq 1 + \delta_0 \quad \text{and} \quad \beta_z = D_{11}/D_{22} = 6k_1 \frac{(h/b)^2}{(1 + \nu)},$$

where $D_{11} = k_1 b h^3 E / (2 + 2\nu)$, k_1 can be calculated using Eqs.(168) and (171) in Ref.[100], and E is Young's modulus.

When solving for the planar response, we let either $w(s,t) = 0$ or $v(s,t) = 0$. Unless specified, we investigate the stability by assuming disturbances in both the y and z directions. To discuss the stability of trivial solutions, we let $p_i = q_i = 0$ in the Jacobian matrix.

3.3.1 Near Square Cross Section

In this case, $\delta_0 = 0.0$ and $\beta_z = 0.6489$. We let $\mu = 0.05$ and $f = 0.03$. Figure 3.2(a) shows the response curves of the first mode. It is well known that, for planar responses, if only in-plane disturbances are considered then the upper branch of a_2 will be stable [72]. But due to the disturbances in the y -direction, this branch is unstable. It follows from (3.18a) and (3.18b) that

$$\sin 2v_1 = \frac{-\mu}{2\omega_{1m}\alpha_7 f} \quad \text{and} \quad \sin 2v_2 = \frac{-\mu}{2\omega_{2n}\beta_7 f}$$

So, if $\mu = 0$, then the phase angles $v_1 = v_2 = -90^\circ$ for the upper branches, and $v_1 = v_2 = 0^\circ$ for the lower branches. But the damping changes the phase angles slightly. If we increase μ , the upper and lower branches will move closer to each other. If we increase f , the separation between these two branches will increase. We note from Fig. 3.2(a) that the planar solution in the stiff direction (i.e., z direction)

is always unstable and hence the planar response is always in the weak direction (i.e., y direction).

It follows from (3.17) that the effective nonlinear spring coefficients α_e and β_e in the y and z directions are

$$\alpha_e = (1 + \delta_0)\alpha_4 - \frac{2}{3}\alpha_5\omega_{1m}^2, \quad \beta_e = \beta_4 - \frac{2}{3}\beta_5\omega_{2n}^2.$$

Since numerical integration shows that α_4 , α_5 , β_4 , and β_5 are positive, the nonlinear geometric terms $(1 + \delta_0)\alpha_4$ and β_4 are of the hardening type, whereas the nonlinear inertia terms $\frac{2}{3}\alpha_5\omega_{1m}^2$ and $\frac{2}{3}\beta_5\omega_{2n}^2$ are of the softening type. Hence, the overall type of the effective nonlinearity in the y(z) direction depends on the relative magnitudes of $\alpha_4(\beta_4)$ and $\alpha_5(\beta_5)$ and the mode shape. Because s is the undeformed length, (3.20) implies that the nodal points are moving back and forth when the beam is vibrating with large amplitude, especially for high frequency modes, but the number of nodes is constant. Furthermore, we expect that the nonlinear inertia terms due to the shortening effect become important for high frequency modes because the inertia force is proportional to ω_{im}^2 .

For the first mode, Fig. 3.2(a) shows that the planar response curves¹ are bent to the right, which implies that the nonlinear geometric terms dominate the response because they have a hardening effect. Neglecting the nonlinear geometric terms (i.e., letting $\alpha_i = \beta_i = 0$ for $i = 1, 2, 3$, and 4), we obtain the response curves shown in Fig. 3.2(b). Comparing Figures 3.2(a) and 3.2(b) shows that neglecting the geometric

¹ For all the response curves in this dissertation, we use solid lines to denote stable solutions, dashed lines to denote unstable solutions with at least one eigenvalue being positive, and dotted lines to denote unstable fixed-point solutions with a pair of complex conjugate eigenvalues having positive real parts.

nonlinearity yields frequency curves that are even qualitatively wrong. The nonlinearity changes from a hardening to a softening type. Moreover, nonplanar solutions cannot be predicted without including the geometric nonlinearity.

Figure 3.3(a) shows the response curves of the second mode. We note that for the upper branches of planar motions $v_1 = v_2 \simeq 0^\circ$, and for the lower branches $v_1 = v_2 \simeq -90^\circ$. These phase angles are different from those found in the response of the first mode (Figure 3.2(a)). Owing to disturbances in the z-direction, the planar motion in the y-direction is unstable along the branch AB. Similarly, the planar motion in the z-direction is unstable along the branch CD due to disturbances in the y-direction. Another point is that the planar response curves are bent to the left, which means that the overall effective nonlinearity is of the softening type and hence the nonlinear inertia terms dominate the response. Neglecting the geometric nonlinearity, we obtain the frequency-response curves shown in Figure 3.3(b). Although the planar frequency-curves are qualitatively similar, neglecting the geometric nonlinearity underpredicts the response amplitudes. Moreover, as in the case of the lower mode, nonplanar solutions cannot be predicted if the geometric nonlinearity is neglected. Comparing Figures 3.2(a) and 3.3(a), we note that for a fixed value of f , the amplitudes of the planar motion of the second mode are definitely smaller than those of the first mode.

For a beam having a square cross section (i.e., $\delta_0 = \delta_2 = 0$), there are no nonplanar response curves found in the first and second mode vibrations and the planar response curves of vibrations in the y and z directions are the same. Moreover, the upper branches of the planar response curves are marginally stable because one of the four eigenvalues of the Jacobian matrix of the modulation equations is zero and

all others have negative real parts. And, because of the $O(2)$ symmetry, a planar motion along any arbitrary direction is always a solution but it is marginally stable.

The response curves for the third (see Fig. 3.4) or higher modes are similar to those in Fig. 3.3. Since the relative importance of the geometric nonlinearity decreases as the natural frequency increases, the error introduced by neglecting the geometric nonlinearity decreases as the mode number increases.

Next, we return to Figures 3.2(a) and 3.3(a) to discuss the nonplanar response curves of the first two modes. Along the branch AB of Fig. 3.2(a), the nontrivial solutions are unstable with two of the eigenvalues of the Jacobian matrix being complex conjugate having a positive real part. As σ decreases from a value larger than that corresponding to the point B, the nonplanar fixed point loses stability at B with a complex conjugate pair of eigenvalues moving into the right-half plane. This corresponds to the extensively studied Hopf bifurcation [61]. Based on the Hopf bifurcation theorem [39], one expects amplitude- and phase-modulated motions for values of σ near B. We note that the phase angles of nonplanar responses are not constant, but the difference between v_1 and v_2 is close to 90° . Furthermore, all the nonplanar response curves are bent to the right even for the second mode. This is not unexpected because as discussed earlier the nonlinear geometric terms control the nonplanar motion. For the first mode, the nonlinear geometric terms have a hardening effect, and hence the amplitudes of nonplanar motion are smaller than those of planar motion. On the other hand, for the second and third modes, the nonlinear geometric terms overcome the inertia terms and produce nonplanar motions that are larger than the planar motions.

If we increase the absolute value of δ_2 (i.e., increase the deviation of the cross section from a square), the planar response curves of the y-direction motion move away from those of the z-direction motion. Figures 3.5 and 3.6 show the response curves for the first two modes. Because of out-of-plane disturbances, the planar response is unstable along the branch AB of Fig. 3.5, which also shows the nonplanar response of the first mode. Here we have a Hopf bifurcation at point C. Using a fifth- and sixth-order Runge-Kutta routine to integrate (3.23) for $\sigma = -0.0353$ along the branch AC for a long period of time, we obtain the amplitude-modulation behavior shown in Fig. 3.7(a). The phase difference is not equal to 180° as in the case of internal resonance of free vibrations [22]. And Fig. 3.7(b) shows the spectrum of a_{1n} , which is obtained by using the fast Fourier transform algorithm. Since the amplitudes and phases are not constant but periodic with a period that is larger than that corresponding to free oscillations, the resulting motion is quasi-periodic having two periods (i.e., motion on a torus). This motion can be better visualized by plotting the motion of the tip-end of the beam, as shown in Fig. 3.8(a). This figure shows that the elliptical route keeps changing the lengths of axes and direction, and it also shows the twisting motion. Because of the nonlinear terms, the inertia force and the elastic restoring force in the y-direction are not proportional to $v(s,t)$ and the inertia force and the elastic restoring force in the z-direction are not proportional to $w(s,t)$, and hence the resultant force is not parallel to the total displacement in the y-z plane and it induces a twisting moment on the beam. This is a whirling motion of the beating type. Because the modulation frequency of the amplitudes is commensurable with the input frequency Ω for this special σ value, the route repeats itself after a period. If we denote the solution shown in Fig. 3.8(a) by (i), then due to the $Z_2 - Z_2$ symmetry, there are three other possible solutions (ii), (iii), and (iv). Solutions (ii), (iii), and (iv) can be obtained by rotating the solution (i) by 180° with respect to the y-axis, the

z-axis, and the y-axis and then the z-axis, respectively. For the case of steady whirling motion, such as that shown in Fig. 3.8(b), there are only two possible solutions: (i) and (ii). The other two (iv) and (iii) can be obtained from (i) and (ii) using the above transformation.

Increasing σ to 0.0753 and integrating (3.23), we obtain an attractor whose projection onto the $a_2 - a_1$ plane is shown in Fig. 3.9(a). And from the spectrum of a_{1n} shown in Fig. 3.9(b), we find that the basic frequency does not change much. But, increasing σ further to 0.07627 produces a period-doubling bifurcation of the attractor as shown in Fig. 3.10. Increasing σ further produces a bifurcation of this attractor to a fixed point, yielding a periodic rather than an aperiodic nonplanar motion. Thus, the response is a steady whirling motion. In performing the Fourier harmonic analysis, we did not use any window (or taper) functions, but we took 4096 sampling points extending over about fifty times the period of the wave form. So the leakage is expected to be negligible. The zero-frequency components for all the figures were eliminated.

Figure 3.6 shows also the nonplanar response of the second mode. We note that the Hopf bifurcation points E and F are adjacent to stable and unstable branches. Checking the variation of the behavior of the attractor with σ between points E and F, we find that all long-time histories of the amplitudes are not periodic. Figure 3.11 shows that the motion is chaotic as evident in the spectrum having a broad-band component. In Figs. 3.5 and 3.6, the nonplanar response curves always bifurcate from the planar response curves at frequencies higher than the natural frequency. The reason is that the motion in the y-direction has the same mode shape as that in

the z-direction motion but it has a smaller linear natural frequency; that is, it belongs to a lower-energy motion.

Next we combine the planar, nonplanar, and trivial solutions to explain all possible motions. From Fig. 3.2(a), we note that if we decrease or increase σ slowly and smoothly and consider different initial conditions, the possible routes of a_1 are $1-2-\underline{3-4}-5-6-7$, $1-2-8-5-6-7$, $8-5-6-7$, $7-6-5-8$, and $\underline{11-3-4}-5-6-7$, where the underlined numbers represent nonplanar motions and the primed numbers represent the corresponding route of a_2 if it is nontrivial. It should be noted that if the changes in σ are not slow then the nonstationary excitation will result in the penetration phenomenon [29,30,66] and the actual path of a_1 will be different from that described above.

From Fig. 3.3(a), we note that the possible routes of a_1 are $1-2-3-\underline{4-5}-6$, $\underline{7-4-5}-6$, and $10-8-9-\underline{5-4-7}$. For the route $1-2-3-\underline{4-5}-6$, we note that the branches 1-2-3 and 5-6 correspond to planar motions in the z- and y-directions, respectively. Thus, the motion changes from one plane to the other. If $a_1 = a_2 = 0$, the local stability analysis shows that the branch 2-3 of a_1 is unstable. But here $a_2 \neq 0$, so the branch 2-3 is stable and it corresponds to a planar motion in the z-direction. For Fig. 3.5, the possible routes of a_1 are $1-2-\underline{3-4}-5-6-7-8-9$, $\underline{11-3-4}-5-6-7-8-9$, and $9-8-7-10$. For Fig. 3.6, the possible routes of a_1 are $1-2-3-\underline{4-5}-6-7-8$, $\underline{9-4-5}-6-7-8$, and $14-13-12-7-10-\underline{11-4-9}$.

3.3.2 Rectangular Cross Section

In this case, $\delta_0 = -0.9745$ (i.e., $b/h \simeq 6.2673$) and $\beta_y = 0.0352$, and we let $\mu = 0.05$ and $f = 0.03$. For this beam, $\omega_{21} = \omega_{12}$. We investigate the possibility of nonplanar vibrations comprised of the first mode in the z-direction and the second mode in the y-direction.

Figure 3.12 shows the response curves of v and w when $\delta_2 = -0.001$. Because β_y is not equal to unity, the nonlinear terms due to coupling between torsion and lateral vibrations come into play and the stable regions become more complicated. We note that there are two branches of nonplanar response curves: one bifurcates from the unstable branch of the planar response curves of $v(s,t)$ and the other bifurcates from the planar response curves of $w(s,t)$. The upper part of the branch which bifurcates from A and B is unstable except in a small region close to point A, and the solutions correspond to amplitude- and phase-modulated motions. However, the lower part of this branch has a long stable segment, which begins from point E and ends at $\sigma = -130000$ where it merges with the unstable segment below it. This is due to the low torsional rigidity of rectangular compact beams. When $v \neq 0$, small disturbances in the z-direction can induce large-amplitude vibrations in the z-direction via torsional motion due to the low torsional rigidity. We can see that the planar response amplitude a_1 is larger than the nonplanar response amplitude a_{1n} . It is due to energy transfer from planar vibrations in the y-direction to out-of-plane vibrations.

Checking the long-time history of all the fixed-point solutions having a pair of complex conjugate eigenvalues with positive real parts, we find that they correspond to chaotic motions.

As in the case of near square cross sections, multiple attractors coexist in the response of rectangular cross sections. When σ is smaller than a value corresponding to point 1 in Fig. 3.12, three attractors are possible: the trivial attractor, a nontrivial attractor in which $a_2 = 0$ and $a_1 \neq 0$, and a nonplanar attractor. If the input frequency corresponds to point 2, there are two attractors : a planar attractor in the y-direction and a planar attractor in the z-direction. For a value of σ corresponding to point 3, there are four possibilities: a trivial attractor, a planar attractor in the y-direction, a periodic nonplanar attractor, and an amplitude- and phase-modulated nonplanar attractor. When more than one attractor exist, the initial conditions determine the response.

For $\delta_2 = -0.05$, the perfect resonant point of v is far below $\sigma = 0$ and the planar response curves of v are separated from those of w as shown in Fig. 3.13. We note that for $\sigma \simeq 0$ the only possible motion is a planar motion consisting of the first mode in the z-direction. Again there are two nonplanar response curves that bifurcate from the planar response curves of v . As in the case $\delta_2 = -0.001$ (Fig.3.12), the nonplanar response curve that bifurcates from points A and D is unstable except in a small region close to A, which corresponds to amplitude- and phase-modulated motions. Again, below the perfect resonant point of v , there is a large stable nonplanar segment along the lower branch.

Figure 3.14 shows the responses when $\delta_2 = 0.05$. Both of the nonplanar response curves bifurcate from the planar response curves of w . Moreover, around $\sigma = 0$ there are some amplitude- and phase-modulated motions, and the lower branch of nonplanar response curves is stable in a large frequency range.

Checking the steady motion of the tip-end of the beam, we find that the path is a straight line inclined 45° to the y -axis. Because the flexural rigidity of bending in the z -direction is much larger than that of bending in the y -direction, when $v = 0$ the large flexural rigidity resists bending in the z -direction and there is no displacement in the z -direction resulting from the torsional motion. But when $v \neq 0$, w can be easily excited by disturbances in the z -direction because the torsional rigidity of a compact rectangular beam is very low. This explains why the route of the tip-end is a straight line.

Neglecting the geometric nonlinearity for the rectangular beam, we obtain the response curves in Fig. 3.15. Comparing Fig. 3.12 with Fig. 3.15, we observe that the nonlinear geometric terms increase the planar response amplitude a_1 , stabilize the planar response amplitude a_2 , and more importantly produce nonplanar responses. Consequently, neglecting the geometric nonlinearity leads to erroneous results for all modes and cross sections.

In this chapter, we investigated the dynamic behavior of out-of-plane, one-to-one modal interactions. However, a three-to-one in-plane modal coupling between the second and third modes may occur in these beams because $\omega_{13}/\omega_{12} \simeq 3$ and the governing equations contain cubic nonlinearities [72]. To investigate this problem with in-plane and out-of-plane modal interactions, one must take into consideration at least four modes for near square beams (second and third modes of v and w) and at least three modes for rectangular beams with $b/h \simeq 6.2673$ (first mode of w and second and third modes of v). When the second mode is directly excited (i.e. $\Omega \simeq 2\omega_{12}$), the response of the third mode may not be large [72]. On the other hand,

if the third mode is directly excited, the response of the second mode can be very large due to energy transfer from the third mode to the second mode.

Figure 3.2. Response curves of the first mode for a beam with an aspect ratio $b/h \approx 1.0$: mode (1,1), $\omega_{11} = \omega_{21}$, $\delta_0 = 0.0$, $\delta_2 = -0.05$, $\mu = 0.05$, $f = 0.03$, $\beta_1 = 0.6489$; a_1, a_2 = planar response amplitudes; a_{1n}, a_{2n} = nonplanar response amplitudes; (a) with (b) without nonlinear geometric terms.

Figure 3.3. Response curves of the second mode for a beam with an aspect ratio $b/h \approx 1.0$: mode (2,2) , $\omega_{12} = \omega_{22}$, $\delta_0 = 0.0$, $\delta_2 = -0.05$, $\mu = 0.05$, $f = 0.03$, $\beta_y = 0.6489$; a_1, a_2 = planar response amplitudes; a_{1n}, a_{2n} = nonplanar response amplitudes; (a) with (b) without nonlinear geometric terms.

Figure 3.4. Response curves of the third mode for a beam with an aspect ratio $b/h \approx 1.0$: mode (3,3) , $\omega_{13} = \omega_{23}$, $\delta_0 = 0.0$, $\delta_2 = -0.3$, $\mu = 0.05$, $f = 0.03$, $\beta_1 = 0.6489$; a_1, a_2 = planar response amplitudes; a_{1n}, a_{2n} = nonplanar response amplitudes; (a) with (b) without nonlinear geometric terms.

Figure 3.6. Response curves of the second mode for a beam with an aspect ratio $b/h \approx 1.0$: mode (2,2) , $\omega_{12} = \omega_{22}$, $\delta_0 = 0.0$, $\delta_2 = -1.2$, $\mu = 0.05$, $f = 0.03$, $\beta_1 = 0.6489$; a_1 , a_2 = planar response amplitudes; a_{1n} , a_{2n} = nonplanar response amplitudes.

Figure 3.7. (a) A projection of the trajectory onto the $a_1 - a_2$ plane, (b) the Fourier harmonic analysis of a_{1n} : $b/h \simeq 1.0$, mode (1,1), $\omega_{11} = \omega_{21}$, $\delta_0 = 0.0$, $\delta_2 = -0.5$, $\mu = 0.05$, $f = 0.03$, $\beta_y = 0.6489$, $\sigma = -0.0353$.

Figure 3.8. The path of the tip-end of the beam for the case of an amplitude- and phase-modulated motion : $b/h \simeq 1.0$, mode (1,1) , $\omega_{11} = \omega_{21}$, $\delta_0 = 0.0$, $\delta_2 = -0.5$, $\mu = 0.05$, $f = 0.03$, $\beta_1 = 0.6489$, $\varepsilon = 0.5$, (a) $\sigma = -0.0353$, (b) $\sigma = 0.1067$ (steady whirling motion).

Figure 3.9. (a) A projection of the trajectory onto the $a_1 - a_2$ plane, (b) the Fourier harmonic analysis of a_{1n} : $b/h \simeq 1.0$, mode (1,1), $\omega_{11} = \omega_{21}$, $\delta_0 = 0.0$, $\delta_2 = -0.5$, $\mu = 0.05$, $f = 0.03$, $\beta_1 = 0.6489$, $\sigma = 0.0753$.

Figure 3.10. (a) A projection of the trajectory onto the $a_1 - a_2$ plane, (b) the Fourier harmonic analysis of a_{1n} : $b/h \simeq 1.0$, mode (1,1), $\omega_{11} = \omega_{21}$, $\delta_0 = 0.0$, $\delta_2 = -0.5$, $\mu = 0.05$, $f = 0.03$, $\beta_1 = 0.6489$, $\sigma = 0.07627$.

Figure 3.11. The Fourier harmonic analysis of a_{1n} : $b/h \simeq 1.0$, mode (2,2) , $\omega_{12} = \omega_{22}$, $\delta_0 = 0.0$, $\delta_2 = -1.2$, $\mu = 0.05$, $f = 0.03$, $\beta_y = 0.6489$, $\sigma = -0.2624$.

Figure 3.12. Response curves of a beam with an aspect ratio $b/h \approx 6.2673$: mode (2,1) , $\omega_{12} = \omega_{21}$, $\delta_0 = -0.9745$, $\delta_2 = -0.001$, $\mu = 0.05$, $f = 0.03$, $\beta_1 = 0.0352$; a_1, a_2 = planar response amplitudes; a_{1n}, a_{2n} = nonplanar response amplitudes.

Figure 3.13. Response curves of a beam with an aspect ratio $b/h \approx 6.2673$: mode (2,1) , $\omega_{12} = \omega_{21}$, $\delta_0 = -0.9745$, $\delta_2 = -0.05$, $\mu = 0.05$, $f = 0.03$, $\beta_1 = 0.0352$; a_1, a_2 = planar response amplitudes; a_{1n}, a_{2n} = nonplanar response amplitudes.

Figure 3.14. Response curves of a beam with an aspect ratio $b/h \approx 6.2673$: mode (2,1) , $\omega_{12} = \omega_{21}$, $\delta_0 = -0.9745$, $\delta_2 = 0.05$, $\mu = 0.05$, $f = 0.03$, $\beta_r = 0.0352$; a_1, a_2 = planar response amplitudes; a_{1n}, a_{2n} = nonplanar response amplitudes.

Figure 3.15. Response curves of $v(s,t)$ and $w(s,t)$ obtained by neglecting the geometric nonlinearity for a beam with an aspect ratio $b/h \simeq 6.2673$: mode (2,1) , $\omega_{12} = \omega_{21}$, $\delta_0 = -0.9745$, $\delta_2 = -0.001$, $\mu = 0.05$, $f = 0.03$, $\beta_r = 0.0352$.

CHAPTER IV

PARAMETRICALLY EXCITED

FLEXURAL-FLEXURAL-TORSIONAL VIBRATIONS

OF AN ISOTROPIC BEAM

Three nonlinear integro-differential equations of motion for an inextensional beam are used to investigate planar and nonplanar responses of a fixed-free beam to a principal parametric excitation (see Fig. 3.1). The beam has an almost square cross section, is assumed to undergo flexure about two principal axes and torsion, and possesses a one-to-one internal resonance between two flexural modes in two principal planes and a two-to-one internal resonance between the first torsional mode and the two flexural modes. The method of multiple scales is used to construct a first-order uniform expansion for the interaction of the three resonant (two flexural and one torsional) modes, yielding six first-order nonlinear ordinary-differential equations governing the amplitudes and phases of the modes of vibration.

4.1 Equations of Motion

We assume that the beam is uniform. The equations of motion are obtained from (2.130)-(2.132), (2.178), and (2.179) as

$$\begin{aligned} \ddot{v} + \beta_y v^{iv} = \varepsilon \{ & -\mu_1 \dot{v} - \beta_y (\gamma' w'')' - \beta_y [v'(v'v'' + w'w'')] \}' \\ & - (1 - \beta_y) [(v''\gamma^2 - w''\gamma)'' + (w'' \int_0^s v'' w' ds)'] + j_3 \ddot{v}'' \\ & - \frac{1}{2} [v' \int_1^s [\int_0^s (v'^2 + w'^2) ds]'' ds]' - [v''(s-1) + v'] f \Omega^2 \cos \Omega t \} , \end{aligned} \quad (4.1)$$

$$\begin{aligned} \ddot{w} + w^{iv} = \varepsilon \{ & -\mu_2 \dot{w} + \beta_y (v''\gamma')' - [w'(v'v'' + w'w'')] \}' \\ & + (1 - \beta_y) [(v''\gamma + w''\gamma^2)'' + (v'' \int_0^s v' w'' ds)'] + j_2 \ddot{w}'' \\ & - \frac{1}{2} [w' \int_1^s [\int_0^s (v'^2 + w'^2) ds]'' ds]' - [w''(s-1) + w'] f \Omega^2 \cos \Omega t \} , \end{aligned} \quad (4.2)$$

$$\begin{aligned} \ddot{\gamma} - \frac{\beta_y}{j_1} \gamma'' = \varepsilon \{ & -\mu_3 \dot{\gamma} + (\int_0^s v'' w' ds)'' - (\dot{v}' w')' - \frac{1 - \beta_y}{j_1} (v''^2 \gamma - w''^2 \gamma - v' w'') \\ & + \frac{j_2 - j_3}{j_1} (\dot{v}'^2 \gamma - \dot{w}'^2 \gamma - \dot{v}' \dot{w}') \} , \end{aligned} \quad (4.3)$$

where ε is a nondimensional parameter that is used as a bookkeeping device and will be set equal to unity in the final solutions. The boundary conditions are

$$v = w = v' = w' = \gamma = 0 \text{ at } s = 0, \quad (4.4)$$

$$v'' = w'' = v''' = w''' = \gamma' = 0 \text{ at } s = 1. \quad (4.5)$$

When $\ddot{\gamma}$ and $\dot{\gamma}$ are negligible, one can solve for γ from (4.3) in terms of v and w . The result is $\gamma = 0(vw)$. Then, substituting for γ into (4.1) and (4.2), one finds that the

nonlinear terms are cubic. However, when $\ddot{\gamma}$ is not negligible, which may be the case for short or composite beams, γ may be $O(v)$ or $O(w)$ and hence the nonlinear terms in (4.1)-(4.3) are quadratic. In this paper, we investigate the influence of these quadratic terms.

4.2 Multiple Scale Analysis

To analyze the solutions of the coupled nonlinear equations (4.1) through (4.3) for small but finite amplitudes, we employ the method of multiple scales [68,69]. We seek a first-order uniform expansion in the form

$$v(s, T_0, T_1; \varepsilon) = v_0(s, T_0, T_1) + \varepsilon v_1(s, T_0, T_1) + \dots, \quad (4.6)$$

$$w(s, T_0, T_1; \varepsilon) = w_0(s, T_0, T_1) + \varepsilon w_1(s, T_0, T_1) + \dots, \quad (4.7)$$

$$\gamma(s, T_0, T_1; \varepsilon) = \gamma_0(s, T_0, T_1) + \varepsilon \gamma_1(s, T_0, T_1) + \dots. \quad (4.8)$$

where $T_0 = t$, a fast scale characterizing motions with the natural frequencies ω_i and Ω ; and $T_1 = \varepsilon t$, a slow scale characterizing the modulations of the amplitudes and phases. In terms of the new scales, the time derivatives become

$$\frac{\partial}{\partial t} = D_0 + \varepsilon D_1 + \dots, \quad \frac{\partial^2}{\partial t^2} = D_0^2 + 2\varepsilon D_0 D_1 + \dots, \quad (4.9)$$

where $D_n \equiv \partial/\partial T_n$. To investigate autoparametric resonances, we expand β_y and β_z as

$$\beta_y = \left(\frac{h}{b}\right)^2 \equiv 1 + \delta_0 + \varepsilon \delta_2, \quad \frac{\beta_y}{j_1} = \frac{6}{1+\nu} k_1 \left(\frac{h}{b}\right)^2 \frac{12L^2}{b^2 + h^2} \equiv K_y + \varepsilon \delta_1, \quad (4.10)$$

where the coefficient k_1 is tabulated in Ref. [100] and ν is Poisson's ratio. The influence of the axial displacement due to warping is indirectly included through k_1 , which is calculated by using elasticity theory. When $\delta_2 = 0$ the internal resonance between w and v is perfectly tuned and when $\delta_1 = 0$ the internal resonance between w and γ is perfectly tuned. Substituting (4.6)-(4.10) into (4.1)-(4.5) and equating coefficients of like powers of ε , we obtain

Order ε^0 :

$$D_0^2 v_0 + (1 + \delta_0) v_0^{iv} = 0, \quad (4.11)$$

$$D_0^2 w_0 + w_0^{iv} = 0, \quad (4.12)$$

$$D_0^2 \gamma_0 - K_y \gamma_0'' = 0, \quad (4.13)$$

$$v_0 = w_0 = v_0' = w_0' = \gamma_0 = 0 \text{ at } s = 0, \quad (4.14)$$

$$v_0'' = w_0'' = v_0''' = w_0''' = \gamma_0' = 0 \text{ at } s = 1, \quad (4.15)$$

Order ε :

$$\begin{aligned} D_0^2 v_1 + (1 + \delta_0) v_1^{iv} = & -2D_0 D_1 v_0 - \mu_1 D_0 v_0 - \delta_2 v_0^{iv} - \beta_y (w_0'' \gamma_0')' \\ & - (1 - \beta_y) [(v_0'' \gamma_0^2 - w_0'' \gamma_0)'' + (w_0''' \int_0^s v'' w' ds)'] + j_3 D_0^2 v_0'' \\ & - \beta_y [v_0' (v_0' v_0'' + w_0' w_0'')] - \frac{1}{2} [v_0' \int_1^s D_0^2 \int_0^s (v_0'^2 + w_0'^2) ds ds]' \\ & - [(s-1) v_0'' + v_0'] \Omega^2 \cos(\Omega T_0), \end{aligned} \quad (4.16)$$

$$\begin{aligned}
D_0^2 w_1 + w_1^{iv} = & -2D_0 D_1 w_0 - \mu_2 D_0 w_0 + \beta_y (v_0'' \gamma_0')' \\
& + (1 - \beta_y) [(v_0'' \gamma_0 + w_0'' \gamma_0^2)'' + (v_0''' \int_0^s v_0' w_0'' ds)'] + j_2 D_0^2 w_0'' \\
& - [w_0' (w_0' w_0'' + v_0' v_0'')] - \frac{1}{2} [w_0' \int_1^s D_0^2 \int_0^s (v_0'^2 + w_0'^2) ds ds]' \\
& - [(s-1)w_0'' + w_0'] f \Omega^2 \cos(\Omega T_0) , \tag{4.17}
\end{aligned}$$

$$\begin{aligned}
D_0^2 \gamma_1 - K_y \gamma_1'' = & -2D_0 D_1 \gamma_0 - \mu_3 D_0 \gamma_0 + \delta_1 \gamma_0'' + D_0^2 \int_0^s v_0'' w_0' ds \\
& - \frac{1 - \beta_y}{j_1} [v_0''^2 \gamma_0 - w_0''^2 \gamma_0 - v_0'' w_0''] - D_0 (w_0' D_0 v_0') \\
& + \frac{j_2 - j_3}{j_1} [(D_0 v_0')^2 \gamma_0 - (D_0 w_0')^2 \gamma_0 - D_0 v_0' D_0 w_0'] , \tag{4.18}
\end{aligned}$$

$$v_1 = w_1 = v_1' = w_1' = \gamma_1 = 0 \text{ at } s = 0 , \tag{4.19}$$

$$v_1'' = w_1'' = v_1''' = w_1''' = \gamma_1' = 0 \text{ at } s = 1 . \tag{4.20}$$

The general solutions of (4.11)-(4.15) consist of three infinite sequences of modes corresponding to three infinite sequences of frequencies. Consequently, the solutions of (4.11)-(4.15) can be expressed as

$$v_0(s, T_0, T_1) = \sum_{m=1}^{\infty} V_m(s) A_{1m}(T_1) e^{i\omega_{1m} T_0} + cc , \tag{4.21}$$

$$w_0(s, T_0, T_1) = \sum_{n=1}^{\infty} W_n(s) A_{2n}(T_1) e^{i\omega_{2n} T_0} + cc , \tag{4.22}$$

$$\gamma_0(s, T_0, T_1) = \sum_{k=1}^{\infty} \Gamma_k(s) A_{3k}(T_1) e^{i\omega_{3k}T_0} + cc, \quad (4.23)$$

where cc stands for the complex conjugate of the preceding terms,

$$\omega_{1k} = z_k^2 \sqrt{1 + \delta_0}, \quad \omega_{2k} = z_k^2, \quad \omega_{3k} = (k - \frac{1}{2})\pi\sqrt{K_y}, \quad (4.24)$$

$$V_k(s) = W_k(s) = \cosh z_k s - \cos z_k s + \frac{\cos z_k + \cosh z_k}{\sin z_k + \sinh z_k} (\sin z_k s - \sinh z_k s), \quad (4.25)$$

$$\Gamma_k(s) = \sin[(k - \frac{1}{2})\pi s], \quad (4.26)$$

and z_k is a root of

$$1 + \cos z \cosh z = 0. \quad (4.27)$$

The lowest ten roots of (4.27) are 1.8751, 4.6941, 7.8548, 10.996, 14.137, 17.279, 20.420, 23.562, 26.704 and 29.845. The shape functions V_k , W_k , and Γ_k satisfy

$$\int_0^1 V_k^2(s) ds = \int_0^1 W_k^2(s) ds = 2 \int_0^1 \Gamma_k^2(s) ds = 1. \quad (4.28)$$

We note that ω_{2n} is the linear natural frequency of the n th mode of vibration in the z direction and ω_{1m} is approximately equal to the linear natural frequency of the m th mode of vibration in the y direction. The latter actual frequency is $z_m^2 \sqrt{1 + \delta_0 + \varepsilon \delta_2}$. And the linear natural frequency of the k th torsional mode is $(k - 1/2)\pi\sqrt{K_y + \varepsilon \delta_1}$.

In dimensional form, the linear natural frequencies are

$$v(s,t) : \omega_{1m} = \frac{z_m^2}{L} \sqrt{\frac{h^2}{12}} \sqrt{\frac{EA}{mL^2}} \quad \text{when } \delta_2 = 0 , \quad (4.29)$$

$$w(s,t) : \omega_{2n} = \frac{z_n^2}{L} \sqrt{\frac{b^2}{12}} \sqrt{\frac{EA}{mL^2}} , \quad (4.30)$$

$$\gamma(s,t) : \omega_{3k} = \left(k - \frac{1}{2}\right) \pi \sqrt{\frac{12k_1 h^2}{(2+2\nu)(b^2 + h^2)}} \sqrt{\frac{EA}{mL^2}} \quad \text{when } \delta_1 = 0 , \quad (4.31)$$

$$u(s,t) : \omega_k = \left(k - \frac{1}{2}\right) \pi \sqrt{\frac{EA}{mL^2}} , \quad (4.32)$$

where A is the cross sectional area and E is Young's modulus.

It turns out that in the presence of damping all modes that are not directly excited by the load or indirectly excited through an autoparametric (internal) resonance will decay with time [72]. Hence, to proceed further, we need to specify the modes that are directly excited and the modes that are internally coupled with them. It follows from (4.29)-(4.32) that the state of autoparametric resonance depends on b/h and L/b . For example, when $b \approx h$, it follows from (4.29) and (4.30) that $\omega_{1m} \approx \omega_{2m}$ (one-to-one resonance) for any m . As a second example, if $b/h = 34.3882$ and $\nu = 0.3$, then $k_1 = 0.32722$ and

$$\omega_{1m} \frac{L}{b} \sqrt{\frac{mL^2}{EA}} = 0.029516, 0.18497, 0.51794, 1.0150, 1.6777, \dots ,$$

$$\omega_{2n} \frac{L}{b} \sqrt{\frac{mL^2}{EA}} = 1.0150, 6.3608, 17.811, 34.904, 57.693, \dots ,$$

$$\omega_{3k} \sqrt{\frac{mL^2}{EA}} = 0.056111, 0.16833, 0.28056, 0.39278, 0.50500, \dots ,$$

$$\omega_k \sqrt{\frac{mL^2}{EA}} = 1.5708, 4.7124, 7.8540, \dots$$

If furthermore we choose a beam with $L/b = 36.1774$, then $\omega_{31} = 2\omega_{14} = 2\omega_{21}$. Thus, the fourth mode in the y direction is involved in a one-to-one autoparametric resonance with the first mode in the z direction and both of them are involved in a two-to-one autoparametric resonance with the first torsional mode. In this paper, we consider the case in which the nth mode in the z direction is excited by a principal parametric resonance (i.e., $\Omega \approx 2\omega_{2n}$) when

$$\omega_{3k} = 2\omega_{1m} = 2\omega_{2n} . \quad (4.33)$$

To quantitatively describe the nearness of the principal parametric resonance, we introduce the detuning parameter σ defined by

$$\Omega = 2\omega_{2n}(1 + \varepsilon\sigma) . \quad (4.34)$$

We assume that none of the kth torsional mode, mth flexural mode in the y direction, and nth flexural mode in the z direction is internally coupled with any other mode. Consequently, owing to the fact that all modes that are not directly or indirectly excited decay with time, we express the solution of the first-order problem as in (4.21)-(4.23) without the summation signs.

Substituting (4.33), (4.34), and (4.21)-(4.23) without the summations into (4.16)-(4.18) yields

$$D_0^2 v_1 + (1 + \delta_0) v_1^{IV} = H_1(s, T_1) e^{i\omega_{1m} T_0} + cc + NST \quad (4.35)$$

$$D_0^2 w_1 + w_1^{IV} = H_2(s, T_1) e^{i\omega_{2n} T_0} + cc + NST \quad (4.36)$$

$$D_0^2 \gamma_1 - K_y \gamma_1'' = H_3(s, T_1) e^{i\omega_{3k} T_0} + cc + NST \quad (4.37)$$

where NST stands for terms that do not produce secular terms. Because the homogenous parts of (4.35)-(4.37), (4.19), and (4.20) are the same as (4.11)-(4.15) and because the latter have nontrivial solutions, the inhomogeneous problems (4.35)-(4.37), (4.19) and (4.20) have solutions only if solvability conditions are satisfied [68]. In this case, the solvability conditions demand that the right-hand sides of (4.35)-(4.37) be orthogonal to the solutions of the adjoint problems. It turns out that these problems are self-adjoint and hence the solutions of their adjoints are $V_m(s)$, $W_n(s)$, and $\Gamma_k(s)$. Consequently, the solvability conditions become

$$\int_0^1 H_1(s, T_1) V_m(s) ds = 0 \quad (4.38)$$

$$\int_0^1 H_2(s, T_1) W_n(s) ds = 0 \quad (4.39)$$

$$\int_0^1 H_3(s, T_1) \Gamma_k(s) ds = 0 \quad (4.40)$$

Substituting (4.33), (4.34), and (4.21)-(4.23) into (4.16)-(4.18), collecting the coefficients of $\exp(i\omega_{1m} T_0)$, $\exp(i\omega_{2n} T_0)$, and $\exp(i\omega_{3k} T_0)$ to determine H_1 , H_2 , and H_3 , respectively, substituting the results into (4.38)-(4.40), and rearranging, we obtain

$$\begin{aligned} & -i\omega_{1m}(2A_{1m}' + \mu_1 A_{1m}) - (R_5 + 2\sigma\omega_{1m}^2)A_{1m} - R_2 \bar{A}_{1m} e^{2i\sigma\omega_{1m} T_1} + 2R_3 \bar{A}_{2n} A_{3k} \\ & + 4R_6 A_{1m}^2 \bar{A}_{1m} + 4R_4 \bar{A}_{1m} A_{2n}^2 + 4R_7 A_{1m} A_{2n} \bar{A}_{2n} - 4R_8 A_{1m} A_{3k} \bar{A}_{3k} = 0, \end{aligned} \quad (4.41)$$

$$-i\omega_{2n}(2A_{2n}' + \mu_2 A_{2n}) - (E_5 + 2\sigma\omega_{2n}^2)A_{2n} - R_2 \bar{A}_{2n} e^{2i\sigma\omega_{2n}T_1} + 2E_3 \bar{A}_{1m} A_{3k} + 4E_6 A_{2n}^2 \bar{A}_{2n} + 4E_4 \bar{A}_{2n} A_{1m}^2 + 4E_7 A_{2n} A_{1m} \bar{A}_{1m} + 4E_8 A_{2n} A_{3k} \bar{A}_{3k} = 0 , \quad (4.42)$$

$$-\frac{1}{2} i\omega_{3k}(2A_{3k}' + \mu_3 A_{3k}) + \delta_1 \alpha_{42} A_{3k} + 4L_2 A_{1m} A_{2n} + 4L_4 A_{3k} A_{1m} \bar{A}_{1m} + 4L_5 A_{3k} A_{2n} \bar{A}_{2n} = 0 , \quad (4.43)$$

where the R_i , E_i , and L_i are defined in Appendix C.

Substituting the polar form

$$A_{1m} = \frac{1}{2} a_1(T_1) e^{i\theta_1(T_1)}, A_{2n} = \frac{1}{2} a_2(T_1) e^{i\theta_2(T_1)} \text{ and } A_{3k} = \frac{1}{2} a_3(T_1) e^{i\theta_3(T_1)} \quad (4.44)$$

into (4.41)-(4.43) and separating real and imaginary parts, we obtain

$$2\omega_{1m}(a_1' + \frac{1}{2} \mu_1 a_1) + a_1 R_2 \sin 2v_1 - a_3 a_2 R_3 \sin v_3 - a_1 a_2^2 R_4 \sin(2v_1 - 2v_2) = 0 , \quad (4.45)$$

$$2\omega_{1m} a_1 v_1' + a_1 (R_5 + R_2 \cos 2v_1) - a_3 a_2 R_3 \cos v_3 - a_1 a_2^2 R_4 \cos(2v_1 - 2v_2) - a_1^3 R_6 - a_1 a_2^2 R_7 + a_1 a_3^2 R_8 = 0 , \quad (4.46)$$

$$2\omega_{2n}(a_2' + \frac{1}{2} \mu_2 a_2) + a_2 E_2 \sin 2v_2 - a_3 a_1 E_3 \sin v_3 - a_2 a_1^2 E_4 \sin(2v_2 - 2v_1) = 0 , \quad (4.47)$$

$$2\omega_{2n} a_2 v_2' + a_2 (E_5 + E_2 \cos 2v_2) - a_3 a_1 E_3 \cos v_3 - a_2 a_1^2 E_4 \cos(2v_2 - 2v_1) - a_2^3 E_6 - a_2 a_1^2 E_7 - a_2 a_3^2 E_8 = 0 , \quad (4.48)$$

$$\omega_{3k}(a_3' + \frac{1}{2} \mu_3 a_3) + a_1 a_2 L_2 \sin v_3 = 0 , \quad (4.49)$$

$$\omega_{3k} a_3 (v_3' - v_1' - v_2') + a_3 L_3 + a_1 a_2 L_2 \cos v_3 + a_3 a_1^2 L_4 + a_3 a_2^2 L_5 = 0 , \quad (4.50)$$

where

$$v_1 = \sigma\omega_{1m}T_1 - \theta_1, \quad v_2 = \sigma\omega_{2n}T_1 - \theta_2, \quad v_3 = \theta_3 - \theta_1 - \theta_2. \quad (4.51)$$

To the first-order approximation,

$$v(s,t) = V_m(s)a_1(\varepsilon t) \cos\left(\frac{1}{2}\Omega t - v_1\right) + \dots, \quad (4.52)$$

$$w(s,t) = W_n(s)a_2(\varepsilon t) \cos\left(\frac{1}{2}\Omega t - v_2\right) + \dots, \quad (4.53)$$

$$\gamma(s,t) = \Gamma_k(s)a_3(\varepsilon t) \cos(\Omega t + v_3 - v_1 - v_2) + \dots, \quad (4.54)$$

where the a_i and v_i are given by (4.45)-(4.50).

Periodic solutions of the beam correspond to the fixed points (i.e., constant solutions) of (4.45)-(4.50), which in turn correspond to $a_1' = a_2' = a_3' = v_1' = v_2' = v_3' = 0$. To determine the stability of the fixed points and hence the periodic solutions, we introduce the Cartesian coordinates p_i and q_i defined by

$$p_1 = a_1 \cos v_1, \quad q_1 = a_1 \sin v_1, \quad (4.55)$$

$$p_2 = a_2 \cos v_2, \quad q_2 = a_2 \sin v_2, \quad (4.56)$$

$$p_3 = a_3 \cos(-v_3 + v_1 + v_2), \quad q_3 = a_3 \sin(-v_3 + v_1 + v_2). \quad (4.57)$$

Substituting (4.55)-(4.57) into (4.45)-(4.50) yields

$$\begin{aligned} 2\omega_{1m}p_1' = & -\mu_1\omega_{1m}p_1 + (R_5 - R_2)q_1 + R_3(-q_3p_2 + p_3q_2) - R_4(2p_2q_2p_1 - p_2^2q_1 + q_2^2q_1) \\ & - R_6q_1(p_1^2 + q_1^2) - R_7q_1(p_2^2 + q_2^2) + R_8q_1(p_3^2 + q_3^2), \end{aligned} \quad (4.58)$$

$$2\omega_{1m}q_1' = -\mu_1\omega_{1m}q_1 - (R_5 + R_2)p_1 + R_3(p_3p_2 + q_3q_2) - R_4(-2q_1p_2q_2 + p_1q_2^2 - p_1p_2^2)$$

$$+ R_6 p_1(p_1^2 + q_1^2) + R_7 p_1(p_2^2 + q_2^2) - R_8 p_1(p_3^2 + q_3^2) , \quad (4.59)$$

$$\begin{aligned} 2\omega_{2n} p_2' = & -\mu_2 \omega_{2n} p_2 + (E_5 - E_2) q_2 + E_3(-q_3 p_1 + p_3 q_1) - E_4(2p_1 q_1 p_2 - p_1^2 q_2 + q_1^2 q_2) \\ & - E_6 q_2(p_2^2 + q_2^2) - E_7 q_2(p_1^2 + q_1^2) - E_8 q_2(p_3^2 + q_3^2) , \end{aligned} \quad (4.60)$$

$$\begin{aligned} 2\omega_{2n} q_2' = & -\mu_2 \omega_{2n} q_2 - (E_5 + E_2) p_2 + E_3(p_3 p_1 + q_3 q_1) - E_4(-2q_2 p_1 q_1 + p_2 q_1^2 - p_2 p_1^2) \\ & + E_6 p_2(p_2^2 + q_2^2) + E_7 p_2(p_1^2 + q_1^2) + E_8 p_2(p_3^2 + q_3^2) , \end{aligned} \quad (4.61)$$

$$\omega_{3k} p_3' = -\frac{1}{2} \mu_3 \omega_{3k} p_3 - L_3 q_3 - L_2(q_1 p_2 + q_2 p_1) - L_4 q_3(p_1^2 + q_1^2) - L_5 q_3(p_2^2 + q_2^2) , \quad (4.62)$$

$$\omega_{3k} q_3' = -\frac{1}{2} \mu_3 \omega_{3k} q_3 + L_3 p_3 - L_2(-p_1 p_2 + q_1 q_2) + L_4 p_3(p_1^2 + q_1^2) + L_5 p_3(p_2^2 + q_2^2) . \quad (4.63)$$

Then, the stability of a particular fixed point with respect to an infinitesimal disturbance proportional to $\exp(\lambda T_1)$ is determined by the eigenvalues of the Jacobian matrix of the right-hand sides of (4.58)-(4.63). A given fixed point is stable if and only if all the eigenvalues are in the left-half of the complex plane. Varying a control parameter, such as the amplitude or frequency of excitation, one finds that a fixed point may lose its stability in one of two ways. First, a real eigenvalue moves into the right-half of the complex plane along the real axis. Second, a pair of complex eigenvalues move transversely into the right-half of the complex plane. The latter corresponds to the Hopf bifurcation and the resulting response is either a periodically or a chaotically modulated motion.

We note that (4.58)-(4.63) are invariant under the transformations

$$(p_1, q_1, p_2, q_2, p_3, q_3) \rightarrow (-p_1, -q_1, p_2, q_2, -p_3, -q_3) ,$$

$$(p_1, q_1, p_2, q_2, p_3, q_3) \rightarrow (p_1, q_1, -p_2, -q_2, -p_3, -q_3) ,$$

$$(p_1, q_1, p_2, q_2, p_3, q_3) \rightarrow (-p_1, -q_1, -p_2, -q_2, p_3, q_3) .$$

For a square cross section, $\delta_0 = \delta_2 = 0$ and

$$R_1 = E_1, R_2 = E_2, R_3 = -E_3, R_4 = E_4, R_5 = E_5, R_6 = E_6, R_7 = E_7,$$

$$R_8 = E_8 = L_2 = L_4 = L_5 = 0 .$$

In obtaining $L_2 = 0$, we used the fact that $\alpha_{49} = 2\alpha_{47}$, which can be proved by using integration by parts. Then, it follows from (4.49) that $a_3 \rightarrow 0$ as $T_1 \rightarrow \infty$ and hence it follows from (4.44) that $A_{3k} \rightarrow 0$ as $T_1 \rightarrow \infty$. Consequently, (4.41) and (4.42) reduce to

$$\begin{aligned} i\omega_{1m}(2A_{1m}' + \mu_1 A_{1m}) + j_3 \omega_{1m}^2 \alpha_3 A_{1m} = \\ 4R_6 A_{1m}^2 \bar{A}_{1m} + 4R_7 A_{1m} A_{2n} \bar{A}_{2n} + 4R_4 \bar{A}_{1m} A_{2n}^2 - R_2 \bar{A}_{1m} e^{2i\sigma\omega_{1m}T_1}, \end{aligned} \quad (4.64)$$

$$\begin{aligned} i\omega_{2n}(2A_{2n}' + \mu_2 A_{2n}) + j_2 \omega_{2n}^2 \alpha_{23} A_{2n} = \\ 4R_6 A_{2n}^2 \bar{A}_{2n} + 4R_7 A_{1m} \bar{A}_{1m} A_{2n} + 4R_4 A_{1m}^2 \bar{A}_{2n} - R_2 \bar{A}_{2n} e^{2i\sigma\omega_{2n}T_1}, \end{aligned} \quad (4.65)$$

where the third terms on the left-hand sides of (4.64) and (4.65) are due to the fact that we neglected the rotary inertias j_2 and j_3 in obtaining the linear eigenfunctions. Consequently, torsional oscillations cannot be excited for a beam with a near square cross section.

The system has D_4 symmetry [90] because it is invariant under the transformation

$$(p_1, q_1, p_2, q_2) \rightarrow (p_2, q_2, p_1, q_1) .$$

Moreover, because the beam is isotropic and any axis on the square cross section is a principal axis, the modulation equations are invariant under any rotation of the

coordinate x - y - z with respect to the x -axis; that is, the transformation $(y, z) \rightarrow (y \cos \theta + z \sin \theta, -y \sin \theta + z \cos \theta)$ for any $\theta \in [0, 2\pi)$. Thus, the system possesses an $O(2)$ symmetry [90]. It follows from Appendix C that

$$R_6 = R_4 + R_7 ,$$

which is a condition of a Hamiltonian system with $O(2)$ symmetry [90].

4.3 Numerical Results

In this chapter, we focus on the first two flexural modes of an isotropic beam with an almost square cross section. The responses of high frequency modes are similar to those of the second modes. For a beam with a near square cross section, it follows from Appendix C that $L_2 = 0$ and hence it follows from (4.49) that $a_3 \rightarrow 0$ as $t \rightarrow \infty$ and consequently torsional oscillations cannot be excited by longitudinal base-excitations. Therefore, the only difference in the results presented in this chapter and those presented in Chapter III is the influence of the rotary inertias $j_3 \ddot{v}''$ and $j_2 \ddot{w}''$. If the beam has a rectangular cross section, torsional motions can be excited by the two flexural modes through a two-to-one internal resonance and the torsional inertia will come into play. For all cases, we take $\mu_1 = \mu_2 = 0.05$, $f = 0.03$ and $\nu = 0.3$.

To determine the stability of the trivial solutions, we let $p_i = q_i = 0$ in the Jacobian matrix. Inspecting the eigenvalues, we identify the following four transition values for σ :

$$\sigma_{11} = \frac{R_5 - \sqrt{R_2^2 - R_1^2}}{2\omega_{1m}^2} , \quad \sigma_{12} = \frac{R_5 + \sqrt{R_2^2 - R_1^2}}{2\omega_{1m}^2} ,$$

$$\sigma_{21} = \frac{E_5 - \sqrt{E_2^2 - E_1^2}}{2\omega_{2n}^2} , \quad \sigma_{22} = \frac{E_5 + \sqrt{E_2^2 - E_1^2}}{2\omega_{2n}^2} .$$

The trivial solutions are unstable when $\sigma_{11} < \sigma < \sigma_{12}$ and $\sigma_{21} < \sigma < \sigma_{22}$.

First Mode:

For this case, $b/h \simeq 1$ and $L/b \simeq 2.2689$. Figures 4.1(a) and (b) show the frequency-response curves for $\delta_2 = -0.5$ and $\delta_2 = -0.05$, respectively. The stable trivial solutions are denoted by solid curves whereas the unstable solutions are denoted by broken curves in Fig. 4.1. The values σ_{11} and σ_{21} are pitchfork bifurcation points, whereas the values σ_{12} and σ_{22} are subcritical pitchfork bifurcation points. Because this is a very short beam, warping and shear deformations cannot be neglected. Therefore, the results shown in Figs. 4.1 (a) and (b) cannot be used for isotropic beams, but they can be used for some other beams with special cross sections or special material properties (e.g., composite materials) where the torsional rigidity is low and the beam is so long so that the warping and shear effects can be neglected. Comparing Fig.4.1(a) with Fig.3.5 and Fig.4.1(b) with Fig.3.2(a), we note that the rotary inertias j_2 and j_3 do not have much influence on the magnitudes of the responses, but they slightly shift the four transition values σ_{11} , σ_{12} , σ_{21} , and σ_{22} to the right. The rotary inertia increases the first natural frequencies of v and w . It follows from (4.1) and (4.2) that the rotary inertias $\bar{v}''j_3$ and $\bar{w}''j_2$ decreases the natural

frequencies of $v(s,t)$ and $w(s,t)$, respectively, only when $\int_0^1 V_m V_m'' ds$ and $\int_0^1 W_n W_n'' ds$ are negative.

To study the solutions of the modulation equations between the Hopf bifurcation points A ($\sigma = -0.0361$) and C ($\sigma = 0.0761$) in Fig.4.1(a), we use a Runge-Kutta routine to integrate (4.58)-(4.61) for a long period of time and obtain the limit-cycle attractors shown in Fig.4.2 where each plot is for a different value of σ . We note that a small limit cycle is born when σ increases beyond the Hopf bifurcation point A. As σ increases further, the limit cycle becomes large and asymmetric with respect to the unstable fixed point (indicated by a plus), as shown in Fig. 4.2(a). Figures 4.2(b) and (c) show the limit cycles at $\sigma = -0.011128$ and 0.0607 , respectively. We can see that the limit cycle increases in size and becomes complicated when σ increases far beyond point A. When σ increases from 0.0607 to 0.061 , the attractor experiences its first period-tripling bifurcation, which is revealed by Fig. 4.2(d). Figures 4.2(e) and (f) show that the attractor undergoes two period-doubling bifurcations as σ increases from 0.061 to 0.06494 . With further increase in σ , the attractor jumps onto an attractor having a period close to that of the attractor shown in Fig. 4.2(c). The size of the limit cycle continues to increase with σ even when σ increases beyond point C ($\sigma = 0.0761$). The large-amplitude attractor at $\sigma = 0.078$, which is shown in Fig. 4.2(h), coexists with the stable fixed point (indicated by a plus); thus point C is a subcritical Hopf-bifurcation point. This large-amplitude attractor jumps onto the fixed-point attractor at $\sigma = 0.079$. Starting with different initial conditions, we find that a very small limit cycle coexists with the large-amplitude limit cycle for $0.073 < \sigma < 0.0761$. Comparing Fig.4.2 with Figs. 3.7, 3.9, and 3.10, we note that the rotary inertias significantly change the frequency range where the limit-cycle attractor exists and the period, amplitude, and period-multiplying sequence of the limit-cycle attractor.

Figure 4.3 shows the long-time history of the attractors shown in Figs. 4.2(c)-(f). They show clearly the amplitude modulation behavior of the two coupled modes and the period-multiplying sequence.

Second Mode:

For this case $b/h \simeq 1$ and $L/b \simeq 14.2188$. For $\delta_2 = -0.05$ and $\delta_2 = 0.3$, the frequency-response curves are shown in Fig.4.4(a) and Fig.4.4(b), respectively. Comparing Figs. 4.4(a) and 3.3(a), we note again that the rotary inertias do not have a significant effect on the nonplanar periodic motions, but the rotary inertias slightly decrease the natural frequencies of v and w .

If we express the equations of motion in terms of v , w , and ϕ (see (2.106)), use the expression $\Gamma_k(s)$ in (4.26) for the k th 'mode shape' of ϕ , and carry out same perturbation analysis as before, we obtain Figs. 4.5 and 4.6. Comparing Figs. 4.1(b) and 4.5, we note that there are some differences, especially in the stability and magnitude of nonplanar responses. Comparing Figs.4.4(b) and 4.6, we find that very different nonplanar responses are predicted by these two formulations. The torsional amplitude from the ϕ -formulation is non-trivial because $L_2 \neq 0$ but there is no torsional motion in the γ -formulation. Most of the nonplanar responses obtained using the ϕ -formulation are unstable or amplitude- and phase-modulated motions and the response curve is broken into two. However, most of the nonplanar responses from the γ -formulation are stable and the response curve is continuous. Because ϕ is not an independent variable and does not represent the real twist angle, the results from the γ -formulation should be used.

For a beam with a near square cross section, one can use the false position method [34] to search for $2v_1 - 2v_2$ between 0 and 2π and solve (4.45)-(4.48) for the fixed points. To obtain the fixed-point solutions for a beam with a rectangular cross-section, one needs to solve (4.45)-(4.50). Because at least two of the six unknowns in the modulation equations need to be guessed in solving this problem, one needs to use optimization search techniques or homotopy methods.

Figure 4.1. Frequency-response curves of the first mode: $b/h \approx 1.0$, $L/b \approx 2.2689$, $\omega_{11} = \omega_{21}$, $\delta_1 = 0.01$, $\mu_1 = \mu_2 = 0.05$, $f = 0.03$; a_1, a_2 = planar response amplitudes; a_{1n}, a_{2n} = nonplanar response amplitudes; (a) $\delta_2 = -0.5$ and (b) $\delta_2 = -0.05$.

Figure 4.2. A projection of the trajectory onto the $a_1 - a_2$ plane: $b/h \simeq 1.0$, $L/b \simeq 2.2689$, $\omega_{11} = \omega_{21}$, $\delta_1 = 0.01$, $\delta_2 = -0.5$, $\mu_1 = \mu_2 = 0.05$, $f = 0.03$, (a) $\sigma = -0.036$, (b) $\sigma = -0.011128$, (c) $\sigma = 0.0607$, (d) $\sigma = 0.061$.

Figure 4.2. continued, (e) $\sigma = 0.0645$, (f) $\sigma = 0.06494$, (g) $\sigma = 0.065$, (h) $\sigma = 0.078$.

Figure 4.3. Long time history of attractors at (a) $\sigma = 0.0607$, (b) $\sigma = 0.061$, (c) $\sigma = 0.0645$, and (d) $\sigma = 0.06494$: $b/h \simeq 1.0$, $L/b \simeq 2.2689$, $\omega_{11} = \omega_{21}$, $\delta_1 = 0.01$, $\delta_2 = -0.5$, $\mu_1 = \mu_2 = 0.05$, $f = 0.03$.

Figure 4.4. Frequency-response curves of the second mode: $b/h \approx 1.0$, $L/b \approx 14.2188$, $\omega_{12} = \omega_{22}$, $\delta_1 = 0.01$, $\mu_1 = \mu_2 = 0.05$, $f = 0.03$; a_1, a_2 = planar response amplitudes; a_{1n}, a_{2n} = nonplanar response amplitudes; (a) $\delta_2 = -0.05$ and (b) $\delta_2 = 0.3$.

Figure 4.5. Frequency-response curves of the first mode: $b/h \approx 1.0$, $L/b \approx 2.2689$, $2\omega_{11} = 2\omega_{21} = \omega_{31}$, $\delta_1 = 0.01$, $\delta_2 = -0.05$, $\mu_1 = \mu_2 = \mu_3 = 0.05$, $f = 0.03$; a_1, a_2 = planar response amplitudes; a_{1n}, a_{2n} = nonplanar response amplitudes; using the equations of motion in terms of ϕ .

Figure 4.6. Frequency-response curves of the second mode: $b/h \simeq 1.0$, $L/b \simeq 14.2188$, $2\omega_{12} = 2\omega_{22} = \omega_{31}$, $\delta_1 = 0.01$, $\delta_2 = 0.3$, $\mu_1 = \mu_2 = \mu_3 = 0.05$, $f = 0.03$; a_1, a_2 = planar response amplitudes; a_{1n}, a_{2n} = nonplanar response amplitudes; using the equations of motion in terms of ϕ .

CHAPTER V

FORCED FLEXURAL-FLEXURAL VIBRATIONS OF AN ISOTROPIC BEAM

The forced planar and nonplanar responses of an isotropic cantilevered beam subject to a lateral harmonic base-excitation along the chordwise direction (see Fig. 5.1) are investigated using two nonlinear coupled integral-differential equations of motion. The equations contain cubic nonlinearities due to curvature and inertia. Two uniform beams with rectangular cross sections are considered: the aspect ratios of their cross sections are near unity and 6.27. In both cases, the beam possesses a one-to-one internal resonance with one of the natural flexural frequencies in one plane being approximately equal to one of the natural flexural frequencies in the second plane.

5.1 Equations of Motion

We assume that the beam is uniform and long, the torsional motion cannot be excited by low-frequency input forces because the torsional frequency is higher than any flexural frequency, and the rotary inertias are negligible. Hence, the equations of motion are obtained from (2.138), (2.139), (2.186), and (2.187) as

$$\begin{aligned}\ddot{v} + \mu_1 \dot{v} + \beta_y v^{iv} = & (1 - \beta_y) [w'' \int_1^s v'' w'' ds - w''' \int_0^s v'' w' ds]' \\ & - \frac{(1 - \beta_y)^2}{\beta_y} [w'' \int_0^s \int_1^s v'' w'' ds ds]'' - \beta_y [v'(v' v'' + w' w'')] ' \\ & - \frac{1}{2} \{v' \int_1^s [\int_0^s (v'^2 + w'^2) ds] \ddot{} ds\}' + [v''(s - 1) + v'] \frac{L^3}{D_{22}} mg, \quad (5.1)\end{aligned}$$

$$\begin{aligned}\ddot{w} + \mu_2 \dot{w} + w^{iv} = & -(1 - \beta_y) [v'' \int_1^s v'' w'' ds - v''' \int_0^s w'' v' ds]' \\ & - \frac{(1 - \beta_y)^2}{\beta_y} [v'' \int_0^s \int_1^s v'' w'' ds ds]'' - [w'(w' w'' + v' v'')] ' \\ & - \frac{1}{2} \{w' \int_1^s [\int_0^s (v'^2 + w'^2) ds] \ddot{} ds\}' + [w''(s - 1) + w'] \frac{L^3}{D_{22}} mg \\ & + F \Omega^2 \cos \Omega t, \quad (5.2)\end{aligned}$$

where the forcing terms (2.186) and (2.187) are nondimensionalized using the characteristic length L and the characteristic time $L^2 \sqrt{m/D_{22}}$. From (2.140) and (2.142), we obtain the boundary conditions:

$$v = w = v' = w' = 0 \text{ at } s = 0, \quad (5.3a)$$

$$v''' = w''' = v'' = w'' = 0 \text{ at } s = 1. \quad (5.3b)$$

In (5.1) and (5.2), the terms in the first three square brackets are nonlinear geometric terms, those in the curly bracket are the nonlinear inertia terms, the terms in the last square bracket are due to the gravitational load and the use of the inextensibility constraint, and $F\Omega^2 \cos \Omega t$ is the normalized forcing function due to the base motion. Apparently, all the nonlinear terms are cubic. And if $\beta_y = 1$ (i.e., a beam with square cross section), the nonlinear geometric terms in the first two square brackets disappear and (5.1) and (5.2) are interchangeable except for the forcing term. The terms $[w''(s-1) + w'] \frac{L^3}{D_{22}} mg$ and $[v''(s-1) + v'] \frac{L^3}{D_{22}} mg$ are found to be small, as shown later.

5.2 Multiple Scale Analysis

To analyze the coupled nonlinear equations (5.1) and (5.2), we employ the method of multiple scales [68]. Toward this end a small parameter ε is introduced as a bookkeeping device in the following perturbation analysis. Because all nonlinear terms are cubic and we seek a first-order uniform expansion, we assume that an approximation to $v(s,t)$ and $w(s,t)$ can be expressed as

$$v(s, T_0, T_2; \varepsilon) = \varepsilon v_1(s, T_0, T_2, \dots) + \varepsilon^3 v_3(s, T_0, T_2, \dots) + \dots, \quad (5.4a)$$

$$w(s, T_0, T_2; \varepsilon) = \varepsilon w_1(s, T_0, T_2, \dots) + \varepsilon^3 w_3(s, T_0, T_2, \dots) + \dots. \quad (5.4b)$$

Here, $T_0 = t$ is a fast scale, characterizing motions with the natural frequencies ω_{mn} and Ω , and $T_2 = \varepsilon^2 t$ is a slow scale, characterizing the modulations of the amplitudes

and phases. Moreover, we let $\mu_1 = \mu_2 = \varepsilon^2 \mu$, $F = \varepsilon^3 f$, and $\beta_y = 1 + \delta_0 + \varepsilon^2 \delta_2$. Substituting (5.4) into (5.1) and (5.2) and equating coefficients of like powers of ε , we obtain

Order ε :

$$D_0^2 v_1 + (1 + \delta_0) v_1^{iv} = 0 , \quad (5.5)$$

$$D_0^2 w_1 + w_1^{iv} = 0 , \quad (5.6)$$

$$v_1 = w_1 = v_1' = w_1' = 0 \text{ at } s = 0 , \quad (5.7a)$$

$$v_1'' = w_1'' = v_1''' = w_1''' = 0 \text{ at } s = 1 , \quad (5.7b)$$

Order ε^3 :

$$\begin{aligned} D_0^2 v_3 + (1 + \delta_0) v_3^{iv} = & -2D_0 D_2 v_1 - \mu D_0 v_1 - \frac{(1 - \beta_y)^2}{\beta_y} [w_1'' \int_0^s \int_1^s v_1'' w_1'' ds ds]'' \\ & - \delta_2 v_1^{iv} + (1 - \beta_y) [w_1'' \int_1^s v_1'' w_1'' ds - w_1''' \int_0^s v_1'' w_1' ds]' \\ & - \beta_y [v_1' (v_1' v_1'' + w_1' w_1'')]'' - \left\{ \frac{v_1'}{2} \int_1^s \left[\int_0^s (v_1'^2 + w_1'^2) ds \right]'' ds \right\}' \\ & + [v_1''(s - 1) + v_1'] g_0 , \end{aligned} \quad (5.8)$$

$$\begin{aligned} D_0^2 w_3 + w_3^{iv} = & -2D_0 D_2 w_1 - \mu D_0 w_1 - (1 - \beta_y) [v_1'' \int_1^s v_1'' w_1'' ds - v_1''' \int_0^s w_1'' v_1' ds]' \\ & - \frac{(1 - \beta_y)^2}{\beta_y} [v_1'' \int_0^s \int_1^s v_1'' w_1'' ds ds]'' - \left\{ \frac{w_1'}{2} \int_1^s \left[\int_0^s (v_1'^2 + w_1'^2) ds \right]'' ds \right\}' \\ & - [w_1' (v_1' v_1'' + w_1' w_1'')]'' + [w_1''(s - 1) + w_1'] g_0 \\ & + \Omega^2 \cos(\Omega T_0) , \end{aligned} \quad (5.9)$$

$$v_3 = w_3 = v_3' = w_3' = 0 \text{ at } s = 0 , \quad (5.10a)$$

$$v_3'' = w_3'' = v_3''' = w_3''' = 0 \text{ at } s = 1, \quad (5.10b)$$

where $D_n \equiv \partial/\partial T_n$ and $g_0 \equiv L^3 mg/(D_{22} \varepsilon^2)$.

The general solutions of (5.5)-(5.7) consist of two infinite sequences of modes corresponding to two infinite sequences of frequencies. Any mode that is not directly excited or indirectly excited will decay to zero with time due to the presence of damping [72]. In this paper, we consider the primary-resonant excitation of one of the modes in the z direction, say the n th mode, and which in turn excites one of the modes in the y direction, say the m th mode, through a one-to-one internal or autoparametric resonance. Thus, the former mode is directly excited by the primary resonance and the latter mode is indirectly excited through the autoparametric resonance. We assume that none of the other modes in the y and z directions is involved in any autoparametric resonance with either the m th mode in the y direction or the n th mode in the z direction. Consequently, the nondecaying solutions of (5.5)-(5.7) can be expressed as

$$v_1(s, T_0, T_2, \dots) = V_m(s)A_1(T_2)e^{i\omega_{1m}T_0} + cc, \quad (5.11a)$$

$$w_1(s, T_0, T_2, \dots) = W_n(s)A_2(T_2)e^{i\omega_{2n}T_0} + cc, \quad (5.11b)$$

where cc stands for the complex conjugate of the preceding terms,

$$\omega_{1m} = z_m^2 \sqrt{1 + \delta_0}, \quad \omega_{2n} = z_n^2,$$

$$V_m(s) = W_n(s) = \cosh z_j s - \cos z_j s + \frac{\cos z_j + \cosh z_j}{\sin z_j + \sinh z_j} (\sin z_j s - \sinh z_j s). \quad (5.12)$$

Here, z_l is a root of $1 + \cos z \cosh z = 0$. Its lowest three roots are 1.8751, 4.6941, and 7.8548. The eigenfunctions satisfy $\int_0^1 V_m^2(s) ds = \int_0^1 W_n^2(s) ds = 1$. We note that ω_{2n} is the linear natural frequency of the n th mode of vibration in the z direction and ω_{1m} is approximately equal to the linear natural frequency of the m th mode of vibration in the y direction. The latter actual frequency is $z_m^2 \sqrt{1 + \delta_0 + \varepsilon^2 \delta_2}$.

To determine the equations describing the functions $A_i(T_2)$, we use the Galerkin procedure to reduce (5.8) and (5.9) into ordinary-differential equations. To this end, we let

$$v_{1t}(T_0, T_2) = A_1(T_2) e^{i\omega_{1m}T_0} + cc, \quad w_{1t}(T_0, T_2) = A_2(T_2) e^{i\omega_{2n}T_0} + cc, \quad (5.13a)$$

$$v_{3t}(T_0, T_2) = \int_0^1 V_m(s) v_3(s, T_0, T_2) ds, \quad w_{3t}(T_0, T_2) = \int_0^1 W_n(s) w_3(s, T_0, T_2) ds. \quad (5.13b)$$

Multiplying (5.8) and (5.9) by $V_m(s)$ and $W_n(s)$, respectively, and integrating over the domain of s , we obtain

$$\begin{aligned} D_0^2 v_{3t} + \omega_{1m}^2 v_{3t} = & -2D_0 D_2 v_{1t} - \mu D_0 v_{1t} - \delta_2 z^4 v_{1t} \\ & + (1 - \beta_y) \alpha_1 v_{1t} w_{1t}^2 - \frac{(1 - \beta_y)^2}{\beta_y} \alpha_2 v_{1t} w_{1t}^2 - \beta_y \alpha_3 v_{1t} w_{1t}^2 - \beta_y \alpha_4 v_{1t}^3 \\ & - \frac{1}{2} \alpha_5 v_{1t} D_0^2 v_{1t}^2 - \frac{1}{2} \alpha_6 v_{1t} D_0^2 w_{1t}^2 + \alpha_7 g_0 v_{1t}, \end{aligned} \quad (5.14)$$

$$\begin{aligned} D_0^2 w_{3t} + \omega_{2n}^2 w_{3t} = & -2D_0 D_2 w_{1t} - \mu D_0 w_{1t} \\ & - (1 - \beta_y) \beta_1 v_{1t}^2 w_{1t} - \frac{(1 - \beta_y)^2}{\beta_y} \beta_2 v_{1t}^2 w_{1t} - \beta_3 v_{1t}^2 w_{1t} - \beta_4 w_{1t}^3 \\ & - \frac{1}{2} \beta_5 w_{1t} D_0^2 w_{1t}^2 - \frac{1}{2} \beta_6 w_{1t} D_0^2 v_{1t}^2 + \beta_7 g_0 w_{1t} + \beta_8 \Omega^2 \cos \Omega T_0, \end{aligned} \quad (5.15)$$

where the α_i and β_i are defined in Appendix B and can be obtained by numerical integration.

To investigate the system response under a primary-resonant excitation, we introduce the detuning parameter $\hat{\sigma}$ defined by

$$\Omega = \omega_{2n}(1 + \varepsilon^2 \hat{\sigma}) \quad (5.16)$$

and we let $\omega_{1m} = \omega_{2n}$, which represents a one-to-one internal resonance. We assume that none of the other modes is involved in an autoparametric resonance with these modes. Substituting (5.13a) into (5.14) and (5.15) and eliminating the terms that produce secular terms, we obtain the solvability conditions

$$\begin{aligned} & -i(2\omega_{1m}A_1' + \mu\omega_{1m}A_1) - \delta_2 z^4 A_1 - [\delta_0 \alpha_1 + \frac{\delta_0^2}{\beta_y} \alpha_2 + (1 + \delta_0) \alpha_3](2A_1 A_2 \bar{A}_2 + \bar{A}_1 A_2^2) \\ & - 3(1 + \delta_0) \alpha_4 A_1^2 \bar{A}_1 + 2\alpha_5 \omega_{1m}^2 \bar{A}_1 A_1^2 + 2\alpha_6 \omega_{2n}^2 \bar{A}_1 A_2^2 + \alpha_7 g_0 A_1 = 0 , \end{aligned} \quad (5.17a)$$

$$\begin{aligned} & -i(2\omega_{2n}A_2' + \mu\omega_{2n}A_2) + (\delta_0 \beta_1 - \frac{\delta_0^2}{\beta_y} \beta_2 - \beta_3)(2A_2 A_1 \bar{A}_1 + \bar{A}_2 A_1^2) - 3\beta_4 A_2^2 \bar{A}_2 \\ & + 2\beta_5 \omega_{2n}^2 \bar{A}_2 A_2^2 + 2\beta_6 \omega_{1m}^2 \bar{A}_2 A_1^2 + \beta_7 g_0 A_2 + \frac{1}{2} \beta_8 f \omega_{2n}^2 \theta^{i\omega_{2n} \hat{\sigma} T_2} = 0 . \end{aligned} \quad (5.17b)$$

Substituting the polar form

$$A_1 = \frac{1}{2} a_1(T_2) e^{i\theta_1(T_2)} \quad \text{and} \quad A_2 = \frac{1}{2} a_2(T_2) e^{i\theta_2(T_2)}$$

into (5.17) and separating real and imaginary parts, we obtain the modulation equations

$$2\omega_{1m}a_1' + [R_1 + R_2 a_2^2 \sin 2(\nu_1 - \nu_2)]a_1 = 0 , \quad (5.18a)$$

$$[2\omega_{1m}v_1' - \hat{R}_3 - R_4a_1^2 + R_5a_2^2 + R_2a_2^2 \cos 2(v_1 - v_2)]a_1 = 0 , \quad (5.18b)$$

$$2\omega_{2n}a_2' + [E_1 + E_2a_1^2 \sin 2(v_1 - v_2)]a_2 - \beta_8 f \omega_{2n}^2 \sin v_2 = 0 , \quad (5.18c)$$

$$[2\omega_{2n}v_2' - \hat{E}_3 - E_4a_2^2 - E_5a_1^2 - E_2a_1^2 \cos 2(v_1 - v_2)]a_2 - \beta_8 f \omega_{2n}^2 \cos v_2 = 0 , \quad (5.18d)$$

where the R_i and E_i are defined in Appendix B,

$$\hat{R}_3 = 2\omega_{1m}\omega_{2n}\hat{\sigma} - \delta_2 z^4 + \alpha_7 g_0 ,$$

$$\hat{E}_3 = 2\omega_{2n}^2 \hat{\sigma} + \beta_7 g_0 ,$$

and

$$v_1 = \omega_{2n}\hat{\sigma}T_2 - \theta_1, \quad v_2 = \omega_{2n}\hat{\sigma}T_2 - \theta_2 . \quad (5.19)$$

Periodic solutions of the system correspond to the fixed points (i.e., constant solutions) of (5.18), which in turn correspond to $a_1' = a_2' = v_1' = v_2' = 0$. It follows from (5.19) that $\theta_1' = \omega_{2n}\hat{\sigma}$ and $\theta_2' = \omega_{2n}\hat{\sigma}$. Thus, (5.18) reduce to a set of algebraic equations that can be solved numerically using, for example, the false position method [34]. And to the first-order approximation the response of the beam is given by

$$v(s,t) = \varepsilon V_m(s)a_1(\varepsilon^2 t) \cos(\Omega t - v_1) + \dots , \quad (5.20a)$$

$$w(s,t) = \varepsilon W_n(s)a_2(\varepsilon^2 t) \cos(\Omega t - v_2) + \dots . \quad (5.20b)$$

To determine the stability of the fixed points, we let

$$p_i = a_i \cos v_i \quad \text{and} \quad q_i = a_i \sin v_i . \quad (5.21)$$

Using (5.18), (5.19), and (5.21), we find that

$$\begin{aligned} -2\omega_{1m}p_1' = & \hat{R}_3q_1 + R_4(p_1^2 + q_1^2)q_1 - R_5(p_2^2 + q_2^2)q_1 + R_1p_1 \\ & + R_2[-2q_2p_2p_1 + q_1(p_2^2 - q_2^2)] , \end{aligned} \quad (5.22a)$$

$$\begin{aligned} -2\omega_{1m}q_1' = & -\hat{R}_3p_1 - R_4(p_1^2 + q_1^2)p_1 + R_5(p_2^2 + q_2^2)p_1 + R_1q_1 \\ & + R_2[(p_2^2 - q_2^2)p_1 + 2q_1p_2q_2] , \end{aligned} \quad (5.22b)$$

$$\begin{aligned} -2\omega_{2n}p_2' = & \hat{E}_3q_2 + E_4(p_2^2 + q_2^2)q_2 + E_5(p_1^2 + q_1^2)q_2 + E_1p_2 \\ & + E_2[q_2(q_1^2 - p_1^2) + 2p_2p_1q_1] , \end{aligned} \quad (5.22c)$$

$$\begin{aligned} -2\omega_{2n}q_2' = & -\hat{E}_3p_2 - E_4(p_2^2 + q_2^2)p_2 - E_5(p_1^2 + q_1^2)p_2 + E_1q_2 \\ & - E_2[p_2(p_1^2 - q_1^2) + 2q_2p_1q_1] - \beta_8 f \omega_{2n}^2 . \end{aligned} \quad (5.22d)$$

Because (5.22a)-(5.22d) are first-order autonomous differential equations, the stability of a particular fixed point with respect to an infinitesimal disturbance proportional to $\exp(\lambda T_2)$ is determined by the eigenvalues of the Jacobian matrix of their right-hand sides. A given fixed point is stable if and only if the real parts of all eigenvalues are less than or equal to zero. If there is a pair of complex conjugate eigenvalues having positive real parts, amplitude- and phase-modulated motions are expected.

Because (5.22a)-(5.22d) are invariant under the transformation $(p_1, q_1, p_2, q_2) \rightarrow (-p_1, -q_1, p_2, q_2)$, the system possesses inversion symmetry and there are two possible solutions coexisting, which are

$$(\pm p_1, \pm q_1, p_2, q_2) ,$$

$$\text{or } (\pm a_1, v_1, a_2, v_2) ,$$

$$\text{or } (a_1, v_1, a_2, v_2) \text{ and } (a_1, v_1 + \pi, a_2, v_2) .$$

It follows from the expression of $\hat{R}_3$ and $\hat{E}_3$ that gravity decreases the natural frequencies of $v(s,t)$ and $w(s,t)$ by $\alpha_7 g_0 / (200 \omega_{1m} \omega_{2n})$ % and $\beta_7 g_0 / (200 \omega_{2n}^2)$ % , respectively, which are very small. For example, by checking the dimensions and material properties of the beams used in Refs.[11,36,38], we find that the maximum decrease is 0.96 % , which occurs for the first-mode of Haight and King's beam. Thus, we put

$$\sigma \equiv \hat{\sigma} + \frac{\alpha_7 g_0}{2 \omega_{1m} \omega_{2n}} \text{ for } v(s,t) \quad \text{and} \quad \sigma \equiv \hat{\sigma} + \frac{\beta_7 g_0}{2 \omega_{2n}^2} \text{ for } w(s,t)$$

in the following analysis.

5.3 Numerical Results

We investigate the responses of two different beams to lateral harmonic excitations. One beam has an almost square cross section, which implies that the natural frequency of a given mode of $v(s,t)$ is close to the natural frequency of the same mode of $w(s,t)$. The other beam has a cross section with an aspect ratio b/h close to 6.2673; in this case, the natural frequency of the second mode of $v(s,t)$ is close to that of the first mode of $w(s,t)$. For all the following cases, we assume that the beam is uniform, the material is homogeneous and isotropic and obeys Hooke's law, Poisson's ratio $\nu = 0.3$, and the cross section is rectangular with width b and height h . It follows from material mechanics and the theory of elasticity that

$$\beta_y = D_{33}/D_{22} = (h/b)^2 \simeq 1 + \delta_0 \quad \text{and} \quad \beta_y = D_{11}/D_{22} = 6k_1 \frac{(h/b)^2}{(1 + \nu)}$$

where $D_{11} = k_1 b h^3 E / (2 + 2\nu)$, k_1 can be calculated using equations (168) and (171) of Ref. [100], and E is Young's modulus.

It follows from (5.17) that the effective nonlinear spring coefficients α_e and β_e in the y and z directions are

$$\alpha_e = (1 + \delta_0)\alpha_4 - \frac{2}{3}\alpha_5\omega_{1m}^2 \quad \text{and} \quad \beta_e = \beta_4 - \frac{2}{3}\beta_5\omega_{2n}^2$$

Since α_4 , α_5 , β_4 , and β_5 are positive according to numerical results, the nonlinear geometric terms $(1 + \delta_0)\alpha_4$ and β_4 are of the hardening type, whereas the nonlinear inertia terms $-\frac{2}{3}\alpha_5\omega_{1m}^2$ and $-\frac{2}{3}\beta_5\omega_{2n}^2$ are of the softening type. Hence, the overall type of the effective nonlinearity in the $y(z)$ direction depends on the relative magnitudes of $\alpha_4(\beta_4)$ and $\alpha_5(\beta_5)$ and the mode shapes.

It follows from (5.18b) and (5.18c) that $v(s,t)$ can only be excited indirectly through the autoparametric resonance and that the only directly excited motion is a planar motion in the z -direction. Unless specified, we investigate the stability by assuming disturbances in both the y - and z -directions. We note that $\sigma = 0$ and $\sigma = \delta_z/2(1 + \delta_0)$ correspond to the pure linear resonant points for $w(s,t)$ and $v(s,t)$, respectively. Since $F(= \varepsilon^3 fL)$ is a third-order quantity and $v(s,t)$ and $w(s,t)$ are first-order quantities, we note that, under primary resonance, a small base motion can result in large responses.

5.3.1 Near Square Cross Section

In this case, $\delta_0 = 0.0$ and $\beta_y = 0.6489$. We let $\mu = 0.04$. For a beam with square cross-section, the planar response curves of the first mode with different amplitudes of the base excitation are shown in Fig. 5.2. We note that when the amplitude of the base motion f is small, the response amplitude is small, single-valued, and stable, which corresponds to the linear solution. The phase angles are not constant. If the input frequency is away from the perfect resonant point $\sigma = 0$, v_2 is close to 0° for $\sigma < 0$ and 180° for $\sigma > 0$ as predicted by linear theory. As f increases, the response curves are bent to the right, implying that the nonlinear geometric terms dominate the response and produce an effective hardening spring effect. It is well known that, if there are only in-plane disturbances, the branch AB is unstable but the branches BC and DE are stable [72]. Now owing to out-of-plane disturbances (i.e., disturbances perpendicular to the direction of motion), the branches BC and DE are unstable. Hence, the notion that nonlinearities are only important for large-excitation levels may not be true for a multiple degree-of-freedom system. Figure 5.3 shows the frequency-response curves of the second mode for different values of f . Apparently, the nonlinear inertia terms dominate the response and produce an effective softening spring effect. Again the branch BC is unstable due to out-of-plane disturbances, and the branch AB is unstable due to in-plane disturbances. Furthermore, away from the perfect resonant point $\sigma = 0$, the phase angles of the left and right branches are close to 0° and 180° , respectively. Because the response curves of high-frequency modes are similar to those of the second mode, we concentrate on the first two modes only.

When we decrease the detuning δ_2 from 0.0 to -0.01, the unstable frequency range of the planar response curve, which corresponds to nonplanar motions, is moved from

DE in Fig. 5.2 to $x'y'$ in Fig. 5.4(a). The reason is that, when $\delta_2 < 0.0$, $h < b$ and the beam is more flexible in the y-direction than in the z-direction. So the motion in the y-direction can be easily autoparametrically excited, and the unstable region of the planar response curve shifts to the left and enlarges. When δ_2 is positive, the unstable region xy shifts to the right and narrows down as shown in Fig. 5.4(b) because the beam is stiffer in the y-direction than in the z-direction. It follows from (5.18c) and (5.18d) that when δ_2 changes, the magnitude of the planar response amplitude a_2 does not change because δ_2 appears only in the expression for the coefficient $\hat{R}_3$. It is due to the fact that we have normalized all structure properties with respect to D_{zz} . Figure 5.5 shows the domain of existence of nonplanar responses in the $\delta_2 - \sigma$ plane. As the natural frequency bias δ_2 increases beyond a critical positive value or decreases below a critical negative value, there are no nonplanar motions.

Figure 5.6 shows the effect of damping on the unstable region xy of planar motions. Increasing the damping coefficient μ decreases the amplitude a_2 , but it does not change the region of unstable planar motions much. For values of μ around the critical value 0.06, the unstable region is broken into two sections. But checking the nonplanar response curves, we find that there is only one nonplanar response curve. Increasing the damping coefficient above a critical value stabilizes the planar motion.

If the amplitude of base motion increases to $\beta_0 f = 0.012$, we find that Hopf bifurcations begin to appear in the nonplanar response curves. Between these Hopf bifurcation points the unstable fixed-point solutions correspond to amplitude- and phase-modulated motions, which will be investigated later.

To evaluate the influence of nonlinear geometric terms, we let $\alpha_i = \beta_i = 0, i = 1, 2, 3, 4$, thereby deleting the nonlinear geometric terms from Eqs.(14)-(17). The results show that, without the nonlinear geometric terms, the planar response curves of the first mode are bent to the left and are qualitatively as well as quantitatively wrong, as compared with those in Fig. 5.4. Whereas in the absence of the nonlinear geometric terms, nonplanar motions can be predicted in the case of external excitations but cannot be predicted in the case of parametric excitations (see Chapter III), the predicted amplitudes and frequency range are erroneous. These results confirm the fact that the nonlinear geometric terms are important for predicting the response of the first mode.

Figures 5.7 and 5.8 show the response curves of the second mode with δ_2 being equal to -0.02 and 0.002, respectively. It follows from these figures that the nonplanar bifurcation point x' is on the unstable or stable branch of the planar response curve depending on whether $\delta_2 < 0$ or $\delta_2 > 0$, respectively. In Figure 5.7, the branch Ay' is unstable with respect to out-of-plane disturbances. When δ_2 is close to zero, the unstable region Ay' in Fig. 5.7 or $x'y'$ in Fig. 5.8 is large. Furthermore, point B in Fig. 5.7 is independent of the value of δ_2 , which is different from the results of Haight and King [38]. The effect of the natural frequency bias δ_2 on the unstable frequency range xy of Fig. 5.7, of planar motions is shown in Fig. 5.9. We note that the bifurcation point y' is always on the stable branch of the planar response curve (solid line) and it monotonically shifts to the left as δ_2 decreases. But the bifurcation point x' is on the stable branch when $\delta_2 > 0$ and on the unstable branch (dashed line) when $\delta_2 < 0$. Also we note that x' cannot lie on the stable segment of the left planar response curve (dotted line), which means that the branch AB is always stable irrespective of the value of δ_2 . This character is different from that of the first mode shown in Fig. 5.2.

Furthermore, both Figures 5.5 and 5.9 show that nonplanar motions can only exist in a small range of δ_2 if $\delta_2 > 0$. But for $\delta_2 < 0$, bending in the y-direction is more flexible than that in the z-direction. Hence, $v(s,t)$ can be easily autoparametrically excited, and nonplanar motions exist in a larger range of δ_2 .

Figure 5.8 shows that when the control parameter σ (i.e., the excitation frequency) changes, the fixed-point solutions may experience three types of bifurcations: (a) saddle-node, (b) pitchfork, and (c) Hopf bifurcations. A saddle-node bifurcation occurs when an eigenvalue of the Jacobian matrix of the modulation equations crosses the imaginary axis along the real axis into the right-half of the complex plane and a stable and an unstable fixed point collide and annihilate each other, e.g., points z, E, G, and M in Fig. 5.8. After a saddle-node bifurcation, the solution jumps onto another attractor. Pitchfork bifurcations are associated with symmetric systems. The symmetry of the system studied here is an inversion symmetry. Similar to a saddle-node bifurcation, a pitchfork bifurcation is associated with an eigenvalue crossing the imaginary axis along the real axis into the right-half of the complex plane. There are two kinds of pitchfork bifurcations: supercritical pitchfork bifurcation, such as point y in Fig. 5.8, and subcritical pitchfork bifurcation, such as point x in Fig. 5.8. A Hopf bifurcation [61] occurs when two complex conjugate eigenvalues transversely cross the imaginary axis into the right-half of the complex plane with a nonzero speed, e.g., points K and H in Fig. 5.8. Based on the Hopf bifurcation theorem [39], one expects amplitude- and phase-modulated motions for values of σ near H and K. To investigate the behavior of the system for values of the parameters corresponding to these unstable fixed-point solutions, we use a fifth- and sixth-order Runge-Kutta routine to integrate (5.22) for a long period of time to determine the attractors of the modulation equations. Figures 5.10-5.13 show the

obtained attractors for $\sigma = -0.01117$, -0.005231 , -0.005131 , and -0.002 , corresponding to points 1-4 in Fig. 5.8. Figures 5.10(b)-5.13(b) show that the energy is continuously exchanged between the two modes. Moreover, the phase difference is not equal to 180° as in the case of internal resonance of free vibrations [22]. The plusses in Figs. 5.10(b), 5.11(b), and 5.12(b) denote the unstable fixed-point solutions predicted by the perturbation solution. Figures 5.10-5.12 show that changes in the amplitudes are periodic and hence the response of the beam consists of two periods, which are not commensurable in general, and hence the response is a motion on a torus. The spectra of the amplitude a_{2n} are shown in Figs. 5.10(a), 5.11(a), and 5.12(a), which confirm that a_{2n} is periodic. We note that when $\sigma = -0.002$, the amplitudes seem to be chaotic as shown in Fig. 5.13. The broad-band character of the spectrum of a_{2n} in Fig. 5.13(a) suggests that the attractor is chaotic. It follows from the harmonic components with the lowest frequency in the spectra in Figs. 5.10-5.13 that the periods of the wave form of a_{2n} are approximately 250, 300, 167, and ∞ . Since the linear natural period of the second mode $T = 0.285$, the amplitude-modulated motions shown in Figs. 5.10-5.12 vary periodically but very slowly. Hence, these variations may be easily overlooked in experiments. Here we want to indicate that, in performing the Fourier harmonic analysis, we did not use any window (or taper) functions, and the zero-frequency components were eliminated in all the figures.

Since the amplitudes and phases are not constant but periodic with a period that is larger than that corresponding to free oscillations, the resulting motion is nonperiodic having two periods (i.e., motion on a torus). This motion can be better visualized by plotting the motion of the tip-end of the beam, as shown in Fig. 5.14. Figures 5.14(a), (b), and (c) show that the elliptical route keeps changing the lengths of axes and direction, and they also show the twisting motion. But the twisting frequency is very

low because the period of the wave amplitude is very large as indicated above. Because of the nonlinear terms, the inertia force in the y-direction is not proportional to $v(s,t)$ and the inertia force in the z-direction is not proportional to $w(s,t)$, and hence the resultant inertia force is not parallel to the total displacement in the y-z plane and it induces a twisting moment on the beam. This is a whirling motion of the beating type. Figure 5.14(d) shows the steady whirling motion for comparison. We note that the tip-end of the beam steadily follows an elliptical route, which is the steady whirling motion. Although the base motion is in the z-direction, a_{1n} is larger than a_{2n} in this case. We know that the linear perfect resonant points are $\sigma = 0$ and $\delta_2/2(1 + \delta_0)$ for w and v , respectively. But the nonlinear softening effects decrease the natural frequency of the second mode, and hence the perfect resonant point of v is moved closer to $\sigma = 0$ than that of w , which results in $a_{1n} > a_{2n}$. It follows from Fig. 5.14(d) that, for the forced vibration case, the phase difference between w and v is not 90° . In order to "make" the tip-end twist faster for a clearer picture, we put $\varepsilon = 3$ in obtaining Fig. 5.14. If ε is smaller, the tip-end will twist slowly, the path lines will get closer, and the amplitude will become smaller as can be seen from (5.20). But the shape will not change. Another point is that the motion is asymmetric even though the geometry of the structure is symmetric. With different initiations of the twisting motion, the steady twisting motion can exist within the other part of the y-z plane and the route will be the one obtained by rotating the figures shown in Fig. 5.14 by 180° with respect to the z axis, which is due to the inversion symmetry of v .

Neglecting the nonlinear geometric terms, we find that the response curves of the second mode are similar to those in Fig. 5.8 but the nonlinear geometric terms increase the amplitude of planar responses and narrow down the frequency range of nonplanar motions. As expected, the importance of the nonlinear geometric terms

decreases as the response frequency increases. Moreover, we find that nonplanar motions can be unstable even in the absence of nonlinear geometric terms, in disagreement with Hyer [49].

5.3.2 Rectangular Cross Section

In this case, $\delta_0 = -0.9745$ (i.e., $b/h \simeq 6.2673$) and $\beta_y = 0.0352$, and we let $\mu = 0.04$. For this beam, $\omega_{21} = \omega_{12}$. We investigate the possibility of nonplanar vibrations consisting of the first mode in the z-direction and the second mode in the y-direction.

When $\delta_z = 0.02$, Figure 5.15 shows that the nonplanar response curve consists of two branches, which end at a value of σ far below zero. Except for a small range in which amplitude- and phase-modulated motions exist, the upper branch is unstable. But the lower branch is stable within a very large frequency range below $\sigma = 0$; the amplitudes in both bending directions are of the same order although the flexural rigidity of bending in the z-direction is much larger than that in the y-direction. This phenomenon results from the participation of the nonlinear geometric terms due to coupling between torsional and lateral vibrations and the low torsional rigidity of a rectangular compact beam. When $v = 0$, only flexural bending can create a displacement in the z-direction because the torsional motion does not create any displacement in the z-direction. However, the large flexural rigidity in the z-direction resists such bending. When $v \neq 0$, small disturbances in the z-direction can induce large-amplitude vibrations in the z-direction via torsional motion because the torsional rigidity is low. This also explains why the route of the tip-end of the beam is a long ellipse, as shown in Fig. 5.16. The gap inside the curve is due to the base

motion in z-direction. It follows from the positions of the cross section indicated in this figure that the frequency of the torsional motion is equal to 2Ω . Since we neglected the torsional inertia, the rotation angle here is due to the twisting moment induced by inertia forces. Hence, the torsional inertia may not be negligible for a rectangular beam.

When $\delta_2 = -0.001$, the response curves are shown in Fig. 5.17. We note that there is only one unstable region in the planar response curves due to out-of-plane disturbances. Checking four fixed-point solutions, points 1-4 in Fig. 5.15, we find that all of these unstable fixed-point solutions having a pair of complex conjugate eigenvalues with positive real parts correspond to chaotic motions.

Neglecting the nonlinear geometric terms, we find that the frequency range of nonplanar motions is different and the bending directions of nonplanar response curves are opposite. It confirms again that the nonlinear geometric terms are not negligible for low-frequency responses.

In this chapter, we investigated the dynamic behavior of out-of-plane, one-to-one modal interactions. It should be noted that a three-to-one in-plane modal coupling between the second and third modes may occur in this system because $\omega_3/\omega_2 \simeq 3$ and the governing equations contain cubic nonlinearities [72]. For rectangular beams, the torsional inertia may have an influence on the responses because the nonplanar motion is induced by torsional motions.

Figure 5.1. Coordinate systems : $x-y-z$ = inertial reference frame; $\xi - \eta - \zeta$ = principal axes of the beam's cross section at position s .

Figure 5.2. Planar response curves of the first mode for a beam with an aspect ratio $b/h \simeq 1.0$: mode (1,1) , $\omega_{11} = \omega_{21}$, $\delta_0 = 0.0$, $\delta_2 = 0.0$, $\mu = 0.04$, $\beta_y = 0.6489$.

Figure 5.3. Planar response curves of the second mode for a beam with an aspect ratio $b/h \approx 1.0$: mode (2,2) , $\omega_{12} = \omega_{22}$, $\delta_0 = 0.0$, $\delta_2 = 0.0$, $\mu = 0.04$, $\beta_y = 0.6489$.

Figure 5.4. Response curves of the first mode for a beam with an aspect ratio $b/h \approx 1.0$: mode (1,1) , $\omega_{11} = \omega_{21}$, $\delta_0 = 0.0$, $\mu = 0.04$, $\beta_3 f = 0.002$, $\beta_1 = 0.6489$; a_2 = planar response amplitude; a_{1n}, a_{2n} = nonplanar response amplitudes; (a) $\delta_2 = -0.01$, (b) $\delta_2 = 0.004$.

Figure 5.5. Variation of the unstable frequency range of planar responses with the natural frequency bias δ_2 : $b/h \simeq 1.0$, mode (1,1) , $\omega_{11} = \omega_{21}$, $\delta_0 = 0.0$, $\mu = 0.04$, $\beta_1 f = 0.002$, $\beta_1 = 0.6489$; a_2 = planar response amplitude (for reference).

Figure 5.6. Variation of the unstable frequency range of planar responses with the damping coefficient μ : $b/h \simeq 1.0$, mode (1,1) , $\omega_{11} = \omega_{21}$, $\delta_0 = 0.0$, $\delta_2 = -0.01$, $\beta_1 f = 0.002$, $\beta_1 = 0.6489$; a_2 = planar response amplitude (for reference).

Figure 5.7. Response curves of the second mode for a beam with an aspect ratio $b/h \approx 1.0$: mode (2,2) , $\omega_{12} = \omega_{22}$, $\delta_0 = 0.0$, $\delta_2 = -0.02$, $\mu = 0.04$, $\beta_s f = 0.00006$, $\beta_y = 0.6489$; a_2 = planar response amplitudes; a_{1n} , a_{2n} = nonplanar response amplitudes.

Figure 5.8. Response curves of the second mode for a beam with an aspect ratio $b/h \approx 1.0$: mode (2,2) , $\omega_{12} = \omega_{22}$, $\delta_0 = 0.0$, $\delta_2 = 0.002$, $\mu = 0.04$, $\beta_s f = 0.00006$, $\beta_r = 0.6489$; a_2 = planar response amplitudes; a_{1n} , a_{2n} = nonplanar response amplitudes.

Figure 5.9. Variation of the unstable frequency range of planar responses with the natural frequency bias δ_2 : $b/h \simeq 1.0$, mode (2,2) , $\omega_{12} = \omega_{22}$, $\delta_0 = 0.0$, $\mu = 0.04$, $\beta_1 f = 0.00006$, $\beta_1 = 0.6489$; a_2 = planar response amplitude (for reference).

Figure 5.10. (a) Fourier harmonic analysis of a_{2n} (b) a projection of the trajectory onto the $a_1 - a_2$ plane : $b/h \simeq 1.0$, mode (2,2) , $\omega_{12} = \omega_{22}$, $\delta_0 = 0.0$, $\delta_2 = 0.002$, $\mu = 0.04$, $\beta_v f = 0.00006$, $\beta_v = 0.6489$, $\sigma = -0.01117$.

Figure 5.11. (a) Fourier harmonic analysis of a_{2n} (b) a projection of the trajectory onto the $a_1 - a_2$ plane : $b/h \simeq 1.0$, mode (2,2) , $\omega_{12} = \omega_{22}$, $\delta_0 = 0.0$, $\delta_2 = 0.002$, $\mu = 0.04$, $\beta_1 f = 0.00006$, $\beta_1 = 0.6489$, $\sigma = -0.005231$.

Figure 5.12. (a) Fourier harmonic analysis of a_{2n} (b) a projection of the trajectory onto the $a_1 - a_2$ plane : $b/h \simeq 1.0$, mode (2,2) , $\omega_{12} = \omega_{22}$, $\delta_0 = 0.0$, $\delta_2 = 0.002$, $\mu = 0.04$, $\beta_f = 0.00006$, $\beta_y = 0.6489$, $\sigma = -0.005131$.

Figure 5.13. (a) Fourier harmonic analysis of a_{2n} , (b) a projection of the trajectory onto the $a_1 - a_2$ plane : $b/h \simeq 1.0$, mode (2,2) , $\omega_{12} = \omega_{22}$, $\delta_0 = 0.0$, $\delta_2 = 0.002$, $\mu = 0.04$, $\beta_1 f = 0.00006$, $\beta_1 = 0.6489$, $\sigma = -0.002$.

Figure 5.14. Path of the tip-end of the beam for the case of an amplitude- and phase-modulated motion : $b/h \simeq 1.0$, mode (2,2) , $\omega_{12} = \omega_{22}$, $\delta_0 = 0.0$, $\delta_2 = 0.002$, $\mu = 0.04$, $\beta_0 f = 0.00006$, $\beta_1 = 0.6489$, $\varepsilon = 3$, (a) $\sigma = -0.01117$, (b) $\sigma = -0.005231$, (c) $\sigma = -0.005131$, (d) $\sigma = 0$ (steady whirling motion).

Figure 5.15. Response curves of a beam with an aspect ratio $b/h \simeq 6.2673$: mode (2,1) , $\omega_{12} = \omega_{21}$, $\delta_0 = -0.9745$, $\delta_2 = 0.02$, $\mu = 0.04$, $\beta_0 f = 0.005$, $\beta_1 = 0.0352$; a_2 = planar response amplitude; a_{1n}, a_{2n} = nonplanar response amplitudes.

Figure 5.16. The path of the tip-end of the beam for the case of a steady whirling motion : $b/h \simeq 6.2673$, mode (2,1) , $\omega_{12} = \omega_{21}$, $\delta_0 = -0.9745$, $\delta_2 = 0.02$, $\mu = 0.04$, $\beta_f = 0.005$, $\beta_v = 0.0352$, $\varepsilon = 1$, $\sigma = -0.05$.

Figure 5.17. Response curves of a beam with an aspect ratio $b/h \simeq 6.2673$: mode (2,1) , $\omega_{12} = \omega_{21}$, $\delta_0 = -0.9745$, $\delta_2 = -0.001$, $\mu = 0.04$, $\beta_y f = 0.005$, $\beta_y = 0.0352$; a_2 = planar response amplitude; a_{1n} , a_{2n} = nonplanar response amplitudes.

CHAPTER VI

DYNAMIC CHARACTERISTICS OF SLEWING METALLIC AND COMPOSITE BEAMS

6.1 *Introduction*

To control a very flexible manipulator where sensors are not collocated with the actuators, one needs an accurate dynamic model of the system [13]. Cannon and Schmitz [13] also indicated that the speed of the response to commands is ultimately limited by the inherent wave-propagation delay for the beam. Meirovitch et al. [63] compared pole allocation, linear optimal control, and nonlinear on-off control methods with coupled control and independent modal-space control. They showed that the independent modal-space control method is better than the coupled control method. Baz and Poh [9] used a single piezo-electric actuator to control the vibrations of a flexible cantilevered beam. They presented an active control

algorithm, which is a combination of the independent modal-space control method and the positive position feedback strategy, with time sharing capabilities provided.

Because the control action added to control a few low-frequency modes may result in spill-over effects and excite some uncontrolled high-frequency modes, a satisfactory mathematical model may require more than fifty vibration modes [63]. Moreover, Hughes and Skelton [51] proved that a necessary and sufficient condition for controllability of this system with a single actuator at the hub is that the eigenvalues are distinct and the modal rotation angle at the hub of every single mode must be nonzero. Hence, obtaining accurate mode shapes becomes an important step in the analysis. In view of this, we take rotary inertia into consideration because it has a definite influence on the vibration of high-frequency modes. Usually numerical methods (e.g., finite-element method) and approximate procedures (e.g., assumed modes method [62]) provide accurate natural frequencies but rough mode shapes. Hence, we solve the eigenvalue problem of a slewing isotropic beam and obtain the exact frequencies and mode shapes by including the effects of the distributed rotary inertia and the hub's inertia. Some dynamic characteristics are studied.

We use a combination of the state-space concept and the fundamental-matrix method to obtain the exact frequencies and mode shapes of a composite beam exhibiting bending-twisting coupling and consisting of three prismatic segments. The problem is also solved by using a finite-element method and the results are compared with those obtained by using the state-space concept.

6.2 Exact Solutions

It follows from (2.113) and (2.114) that the linear equations of motion describing the free, undamped, elastically coupled bending-twisting vibrations of a slewing non-prismatic composite beam (see Fig. 6.1) are

$$m\ddot{w} + (D_{22}w'')' = (D_{12}\gamma')' + (j_\eta \ddot{w}')' + m\ddot{\Theta}s - (j_\eta \ddot{\Theta})' , \quad (6.1a)$$

$$j_\xi \ddot{\gamma} - (D_{11}\gamma')' = - (D_{12}w'')' . \quad (6.1b)$$

The boundary conditions are

$$(D_{22}w'')' - (D_{12}\gamma')' - j_\eta \ddot{w}' + j_\eta \ddot{\Theta} = w'' = \gamma' = 0 \text{ at } s = L , \quad (6.2a)$$

$$w = w' = \gamma = D_{22}w'' - D_{12}\gamma' + I_H \ddot{\Theta} = 0 \text{ at } s = 0 . \quad (6.2b)$$

To solve equations (6.1) and (6.2), we define the absolute flexural displacement $d(s,t)$ as

$$d(s,t) \equiv w(s,t) - s\Theta(t) \quad (6.3)$$

and rewrite (6.1) and (6.2) as

$$m\ddot{d} + (D_{22}d'')' - (D_{12}\gamma')' - (j_\eta \ddot{d}')' = 0 , \quad (6.4a)$$

$$j_\xi \ddot{\gamma} - (D_{11}\gamma')' + (D_{12}d'')' = 0 , \quad (6.4b)$$

$$d = D_{22}d'' - D_{12}\gamma' - I_H \ddot{d}' = \gamma = 0 \text{ at } s = 0 , \quad (6.5a)$$

$$(D_{22}d'')' - (D_{12}\gamma')' - j_\eta \ddot{d}' = d'' = \gamma' = 0 \text{ at } s = L , \quad (6.5b)$$

where $\Theta(t) = -d'(0,t)$ is used.

6.2.1 Isotropic Beams

For an isotropic beam, $D_{12} = 0$ because there is no bending-twisting coupling. Considering a prismatic, uniform beam and letting

$$w(s,t) = W(s)e^{i\omega t}, \quad d(s,t) = D(s)e^{i\omega t}, \quad (6.6)$$

in (6.4) and (6.5), we obtain

$$D_{22}D^{iv} = \omega^2(mD - j_\eta D''), \quad (6.7)$$

$$D = D_{22}D'' + \omega^2 I_H D' = 0 \text{ at } s = 0, \quad (6.8a)$$

$$D'' = D_{22}D''' + \omega^2 j_\eta D' = 0 \text{ at } s = L. \quad (6.8b)$$

Solving this eigenvalue problem, we obtain the mode shapes

$$W(s) = \cosh \beta_2 s - \cos \beta_1 s + c_1 \sin \beta_1 s + c_2 \sinh \beta_2 s - s(c_1 \beta_1 + c_2 \beta_2), \quad (6.9a)$$

where

$$\beta_1 = \sqrt{\frac{j_\eta \omega^2 + \sqrt{j_\eta^2 \omega^4 + 4D_{22}m\omega^2}}{2D_{22}}}, \quad \beta_2 = \sqrt{\frac{-j_\eta \omega^2 + \sqrt{j_\eta^2 \omega^4 + 4D_{22}m\omega^2}}{2D_{22}}},$$

$$c_1 = \frac{(\beta_1^2 + \beta_2^2)\beta_2^2 \sinh \beta_2 L - \frac{\omega^2 I_H}{D_{22}} \beta_2 (\beta_1^2 \cos \beta_1 L + \beta_2^2 \cosh \beta_2 L)}{-\frac{\omega^2 I_H}{D_{22}} (\beta_1 \beta_2^2 \sinh \beta_2 L + \beta_2 \beta_1^2 \sin \beta_1 L)},$$

$$c_2 = \frac{(\beta_1^2 + \beta_2^2)\beta_1^2 \sin \beta_1 L + \frac{\omega^2 I_H}{D_{22}} \beta_1 (\beta_1^2 \cos \beta_1 L + \beta_2^2 \cosh \beta_2 L)}{-\frac{\omega^2 I_H}{D_{22}} (\beta_1 \beta_2^2 \sinh \beta_2 L + \beta_2 \beta_1^2 \sin \beta_1 L)},$$

and the linear circular frequency ω is a root of the characteristic equation

$$2I_H m \omega^4 \left[1 + \left(1 + \frac{j_\eta^2 \omega^2}{2mD_{22}} \right) \cos \beta_1 L \cosh \beta_2 L \right] - I_H j_\eta \omega^5 \sqrt{\frac{m}{D_{22}}} \sin \beta_1 L \sinh \beta_2 L \\ + D_{22} \sqrt{j_\eta^2 \omega^4 + 4D_{22} m \omega^2} (\beta_1^3 \sin \beta_1 L \cosh \beta_2 L - \beta_2^3 \cos \beta_1 L \sinh \beta_2 L) = 0 . \quad (6.9b)$$

Taking limits, we obtain the mode shapes and characteristic equations for four special cases as follows:

(a) $I_H \rightarrow \infty$: (a fixed-free beam with rotary inertias considered)

$$W(s) = \cosh \beta_2 s - \cos \beta_1 s + c_1 \sin \beta_1 s + c_2 \sinh \beta_2 s , \quad (6.10a)$$

$$1 + \left(1 + \frac{j_\eta^2 \omega^2}{2mD_{22}} \right) \cos \beta_1 L \cosh \beta_2 L - \frac{j_\eta \omega}{2\sqrt{mD_{22}}} \sin \beta_1 L \sinh \beta_2 L = 0 , \quad (6.10b)$$

$$\text{where } \frac{c_1}{\beta_2} = -\frac{c_2}{\beta_1} = \frac{\beta_1^3 \cos \beta_1 L + \beta_2^3 \cosh \beta_2 L}{\beta_1 \beta_2^3 \sinh \beta_2 L + \beta_2 \beta_1^3 \sin \beta_1 L} .$$

(b) $j_\eta = 0$:

$$W(s) = \cosh \beta s - \cos \beta s + c_1 \sin \beta s + c_2 \sinh \beta s + \frac{2\sqrt{mD_{22}}}{I_H \omega} s , \quad (6.11a)$$

$$1 + \cos \beta L \cosh \beta L + \frac{m^{1/4} D_{22}^{3/4}}{I_H \omega^{3/2}} (\sin \beta L \cosh \beta L - \cos \beta L \sinh \beta L) = 0 , \quad (6.11b)$$

$$\text{where } \beta_1 = \beta_2 = \left(\frac{m \omega^2}{D_{22}} \right)^{1/4} \equiv \beta .$$

(c) $I_H \rightarrow \infty$ and $j_\eta = 0$: (a fixed-free beam with no rotary inertias considered)

$$W(s) = \cosh \beta s - \cos \beta s + c_1(\sin \beta s - \sinh \beta s) , \quad (6.12a)$$

$$1 + \cos \beta L \cosh \beta L = 0 , \quad (6.12b)$$

$$\text{where } \beta_1 = \beta_2 = \left(\frac{m\omega^2}{D_{zz}} \right)^{1/4} \equiv \beta \text{ and } c_1 = -c_2 = \frac{\cos \beta L + \cosh \beta L}{\sin \beta L + \sinh \beta L} .$$

(d) $I_H = 0$: (a hinged-free beam)

$$W(s) = \frac{\beta_2^2 \sinh \beta_2 L}{\beta_1^2 \sin \beta_1 L} \sin \beta_1 s + \sinh \beta_2 s - s \left(\frac{\beta_2^2 \sinh \beta_2 L}{\beta_1 \sin \beta_1 L} + \beta_2 \right) , \quad (6.13a)$$

$$\beta_1^3 \sin \beta_1 L \cosh \beta_2 L - \beta_2^3 \cos \beta_1 L \sinh \beta_2 L = 0 . \quad (6.13b)$$

We solve the characteristic equation for ω using the false position method [34], and then obtain the mode shapes and corresponding values of Θ . From (6.9b), (6.11b), and (6.13b), we can see that $\omega = 0$ is a solution, which means that there is a rigid-body mode.

6.2.2 Composite Beams

Here we present a method based on the state-space concept and the fundamental-matrix solution to solve for the frequencies and eigenfunctions of a nonuniform, non-prismatic composite beam, which exhibits bending-twisting coupling. For a harmonic motion, we assume that

$$w(s,t) = W(s)e^{i\omega t}, \quad d(s,t) = D(s)e^{i\omega t}, \quad \gamma(s,t) = \Gamma(s)e^{i\omega t}, \quad (6.14)$$

expand (6.4) and (6.5), let

$$x_1 = V, \quad x_2 = V', \quad x_3 = V'', \quad x_4 = V''', \quad x_5 = \Gamma, \quad x_6 = \Gamma', \quad (6.15)$$

and rewrite (6.4) in the matrix form

$$\{X'\} = [A]\{X\}, \quad (6.16)$$

where the prime denotes the derivative with respect to s ,

$$\{X\} = \{x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6\}^T, \quad [A] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & A_{63} & A_{64} & A_{65} & A_{66} \end{bmatrix},$$

$$A_{41} = \frac{m\omega^2}{D_{22} - D_{12}^2/D_{11}},$$

$$A_{42} = \frac{-\omega^2 j_\eta'}{D_{22} - D_{12}^2/D_{11}},$$

$$A_{43} = \frac{-D_{22}'' + 2D_{12}'D_{12}'/D_{11} - 2D_{11}'D_{12}'D_{12}/D_{11}^2 + D_{12}D_{12}''/D_{11} - \omega^2 j_\eta}{D_{22} - D_{12}^2/D_{11}},$$

$$A_{44} = \frac{-2D_{22}' + 2D_{12}'D_{12}/D_{11} - 2D_{11}'D_{12}^2/D_{11}^2 + 2D_{12}D_{12}'/D_{11}}{D_{22} - D_{12}^2/D_{11}},$$

$$A_{45} = \frac{-2D_{12}'j_\xi\omega^2/D_{11} + 2D_{11}'j_\xi D_{12}\omega^2/D_{11}^2 - D_{12}j_\xi'\omega^2/D_{11}}{D_{22} - D_{12}^2/D_{11}},$$

$$A_{46} = \frac{D_{12}'' - 2D_{12}'D_{11}'/D_{11} - D_{12}j_\xi\omega^2/D_{11} + 2D_{11}'D_{11}'D_{12}/D_{11}^2 - D_{12}D_{11}''/D_{11}}{D_{22} - D_{12}^2/D_{11}},$$

$$A_{63} = D_{12}'/D_{11}, \quad A_{64} = D_{12}/D_{11}, \quad A_{65} = -j_\xi\omega^2/D_{11}, \quad A_{66} = -D_{11}'/D_{11}.$$

If the beam is non-prismatic or nonuniform, $[A]$ is not a constant matrix. In order to take advantage of the fundamental-matrix method [5], we divide the beam into n

segments and approximate $[A]$ by a constant matrix $[A_{s_i}^{(i)}]$ over each segment. The i -th segment spans the interval $[s_i - r_i/2, s_i + r_i/2]$, where s_i denotes the coordinate of its center and r_i is its length. Thus,

$$\{X^{(i)}\} = [A_{s_i}^{(i)}]\{X^{(i)}\} , \quad (6.17)$$

over the i th element. If the beam is made of prismatic segments, there is no approximation in $[A_{s_i}^{(i)}]$. According to the fundamental-matrix method [5], the solution of (6.17) can be expressed in the general form

$$\{X^{(i)}\} = [R^{(i)}]e^{[J^{(i)}]s}\{\hat{K}^{(i)}\} , \quad (6.18)$$

where $[R^{(i)}]$ is a 6x6 matrix whose columns are the eigenvectors of $[A_{s_i}^{(i)}]$, $[J^{(i)}]$ is the Jordan canonical form of $[A_{s_i}^{(i)}]$, and $\{\hat{K}^{(i)}\}$ is a 6-element constant vector. In order to overcome the problem of stiffness, we introduce local coordinates over each subinterval by means of the coordinate transformation

$$\xi = s - s_i , \quad (6.19)$$

and obtain

$$\{X^{(i)}\} = [R^{(i)}]e^{[J^{(i)}]\xi}\{K^{(i)}\} , \quad (6.20)$$

where $\{K^{(i)}\} = e^{[J^{(i)}]s_i}\{\hat{K}^{(i)}\}$. By putting the boundary conditions in matrix form, we obtain

$$[B_{-r_1/2}^{(1)}]\{X_{-r_1/2}^{(1)}\} = [E_{-r_1/2}^{(1)}]\{K^{(1)}\} = 0 \quad (6.21)$$

at $s = 0$, where

$$[E_{-r_1/2}^{(1)}] \equiv [B_{-r_1/2}^{(1)}][R^{(1)}]e^{-[J^{(1)}]r_1/2}, [B_{-r_1/2}^{(1)}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & I_H \omega^2 & D_{22} & 0 & 0 & -D_{12} \end{bmatrix}^{(1)}_{-r_1/2},$$

and

$$[B_{r_n/2}^{(n)}]\{X_{r_n/2}^{(n)}\} = [E_{r_n/2}^{(n)}]\{K^{(n)}\} = 0 \quad (6.22)$$

at $s = L$, where $[E_{r_n/2}^{(n)}] \equiv [B_{r_n/2}^{(n)}][R^{(n)}]e^{[J^{(n)}]r_n/2}$,

$$[B_{r_n/2}^{(n)}] = \begin{bmatrix} 0 - j_\eta \omega^2 (D_{12}D_{12}'/D_{11} - D_{23}') (D_{12}^2/D_{11} - D_{22}) (-D_{12}j_\eta \omega^2/D_{11}) (D_{12}' - D_{12}D_{11}'/D_{11}) \\ 0 & 0 & -D_{22} & 0 & 0 & D_{12} \\ 0 & 0 & -D_{12} & 0 & 0 & D_{11} \end{bmatrix}^{(n)}_{r_n/2}.$$

Because D, D', Γ , the bending moment, shear force, and torque are continuous at the juncture of two adjacent elements, we obtain the continuity conditions

$$[B_{r_i/2}^{(i)}]\{X_{r_i/2}^{(i)}\} - [B_{-r_{i+1}/2}^{(i+1)}]\{X_{-r_{i+1}/2}^{(i+1)}\} = [E_{r_i/2}^{(i)}]\{K^{(i)}\} - [E_{-r_{i+1}/2}^{(i+1)}]\{K^{(i+1)}\} = 0 \quad (6.23)$$

at $s = s_i \pm r_i/2$, where $i = 1, 2, \dots, n-1$,

$$[E_{r_i/2}^{(i)}] \equiv [B_{r_i/2}^{(i)}][R^{(i)}]e^{[J^{(i)}]r_i/2}, [E_{-r_{i+1}/2}^{(i+1)}] \equiv [B_{-r_{i+1}/2}^{(i+1)}][R^{(i+1)}]e^{-[J^{(i+1)}]r_{i+1}/2},$$

$$[B^{(n)}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -D_{22} & 0 & 0 & D_{12} \\ 0 - j_\eta \omega^2 (D_{12}D_{12}'/D_{11} - D_{22}') (D_{12}^2/D_{11} - D_{22}) (-D_{12}j_\xi \omega^2/D_{11}) (D_{12}' - D_{12}D_{11}'/D_{11}) \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -D_{12} & 0 & 0 & D_{11} \end{bmatrix}.$$

Combining the boundary and continuity conditions, we obtain

$$[\hat{E}]\{K\} = \{0\}, \quad (6.24)$$

where $[\hat{E}]$ is a $6n \times 6n$ matrix, $\{K\}$ is a $6n \times 1$ vector,

$$\{K\} \equiv \{\{K^{(1)}\}^T \dots \{K^{(n)}\}^T\}^T,$$

$$[\hat{E}] \equiv \begin{bmatrix} [E_{-r_1/2}^{(1)}] & & & & & \\ [E_{r_1/2}^{(1)}] - [E_{-r_2/2}^{(2)}] & & & & & \\ & [E_{r_2/2}^{(2)}] - [E_{-r_3/2}^{(3)}] & & & & \\ & & & 0 & & \\ & & & & & \\ & 0 & & & & \\ & & & & [E_{r_{n-2}/2}^{(n-2)}] - [E_{-r_{n-1}/2}^{(n-1)}] & \\ & & & & [E_{r_{n-1}/2}^{(n-1)}] - [E_{-r_n/2}^{(n)}] & \\ & & & & & [E_{r_n/2}^{(n)}] \end{bmatrix}.$$

For nontrivial solutions the determinant of $[\hat{E}]$ must vanish, which yields the values of the natural frequencies ω .

The solution of this linearly-coupled eigenvalue problem proceeds as follows:

(i) We guess the value of ω .

(ii) We use the DGVCRC subroutine from the IMSL library to calculate $[R^{(n)}]$ and $[J^{(n)}]$ for $[A_{ij}^{(n)}]$ of each element.

(iii) We use the DLFTCG and DLFDCG subroutines from the IMSL library to calculate the determinant of $[\hat{E}]$. If $|\det[\hat{E}]| < \hat{\epsilon}$, where $\hat{\epsilon}$ is a small convergence parameter, the value of ω is the natural angular frequency and then begin step (iv). If $|\det[\hat{E}]| > \hat{\epsilon}$, we use the false position method to estimate the value of ω and then repeat (i) and (ii).

(iv) We assume a value to one of the elements of $\{K\}$ and use the DLSACG subroutine from the IMSL library to solve for the other elements of $\{K\}$.

(v) We use (6.20) to calculate the eigenvectors $\{X^{(n)}\}$.

Because $[A_{ij}^{(n)}]$ is asymmetric, the elements of $[R^{(n)}]$, $[J^{(n)}]$, and $\{K^{(n)}\}$ are in general, complex numbers. In the present case, all frequencies are distinct and hence $e^{i\omega t}$ is a diagonal matrix having the elements $e^{i\omega t}$. Moreover, $\{X\}$ is a real vector because $[R^{(n)}]$, $[J^{(n)}]$, and $\{K^{(n)}\}$ have corresponding complex conjugate pairs.

6.3 Finite-Element Modeling

Using the boundary conditions, we obtain the energy form of the system as

$$\int_0^L (D_{22}D''^2 - 2D_{12}D''\Gamma' + D_{11}\Gamma'^2)ds = \omega^2 \int_0^L (mD^2 + j_\eta D'^2 + j_\xi \Gamma^2)ds + \omega^2 I_H D'^2|_{s=0} \quad (6.25)$$

From (6.25), we note that the rotary inertia j_η and the hub's inertia I_H always decrease the natural frequencies ω , but the torsional inertia j_ξ may decrease or increase the natural frequencies, depending on the eigenfunctions. We Construct a variational formulation [81] for a two-node element of (6.25) and use Hermite cubics as the interpolation functions for $D(s)$ and linear interpolation functions for Γ ; that is

$$D = \{(3\xi^2 - 2\xi^3) (\xi^2 - \xi^3)r (1 - 3\xi^2 + 2\xi^3) (-\xi + 2\xi^2 - \xi^3)r\} \{D_{j-1} D_{j-1}' D_j D_j'\}^T \quad (6.26a)$$

$$\Gamma = \{\xi \ 1 - \xi\} \{\Gamma_{j-1} \ \Gamma_j\}^T, \quad (6.26b)$$

$$\xi = (s_j + r_j/2 - s)/r_j.$$

Thus, we obtain the equation of motion of the j th element as

$$[K^e]\{q^e\} = \omega^2 [M^e]\{q^e\}, \quad (6.27)$$

where $\{q^e\}^T \equiv \{\Gamma_{j-1} D_{j-1} D_{j-1}' \Gamma_j D_j D_j'\}^T$ denotes the displacement vector of the j th element and r is its length. The element stiffness matrix $[K^e]$ and element mass matrix $[M^e]$ are

$$[K^e] = r \int_0^1 [T]^T [H]^T \begin{bmatrix} D_{11} & -D_{12} \\ -D_{12} & D_{22} \end{bmatrix} [H][T] d\xi \quad (6.28a)$$

$$[M^e] = r \int_0^1 [T]^T [H_m]^T \begin{bmatrix} j_\xi & 0 \\ 0 & m \end{bmatrix} [H_m][T] d\xi + r \int_0^1 [T]^T \{\hat{H}_m\} j_\eta \{\hat{H}_m\}^T [T] d\xi \quad (6.28b)$$

where

$$[T] = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -r \\ 0 & 3 & r & 0 & -3 & 2r \\ 0 & -2 & -r & 0 & 2 & -r \end{bmatrix}, [H] = \frac{1}{r^2} \begin{bmatrix} 0 & -h & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 6\xi \end{bmatrix},$$

$$[H_m] = \begin{bmatrix} 1 & \xi & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \xi & \xi^2 & \xi^3 \end{bmatrix}, \{\hat{H}_m\} = \{0 & 0 & 0 & 1 & 2\xi & 3\xi^2\}^T.$$

Assembling and adding I_η to the entry M_{33} , we obtain

$$[K]\{q\} = \omega^2 [M]\{q\}, \quad (6.29)$$

where $[K]$ and $[M]$ are the global stiffness and mass matrices.

Imposing the geometric boundary condition $D(0) = \Gamma(0) = 0$, shifting the circular frequency (in order to calculate the rigid-body mode), and using the subspace

iteration method, we can solve (6.29) for ω and $\{q\}$ and then use (6.26a) and (6.26b) to obtain the shape functions.

If the element is uniform and prismatic, the element stiffness and mass matrices are obtained as

$$[K^e] = \frac{1}{r^3} \begin{bmatrix} D_{11}r^2 & 0 & -D_{12}r^2 - D_{11}r^2 & 0 & D_{12}r^2 \\ 0 & 12D_{22} & 6D_{22}r & 0 & -12D_{22} & 6D_{22}r \\ -D_{12}r^2 & 6D_{22}r & 4D_{22}r^2 & D_{12}r^2 & -6D_{22}r & 2D_{22}r^2 \\ -D_{11}r^2 & 0 & D_{12}r^2 & D_{11}r^2 & 0 & -D_{12}r^2 \\ 0 & -12D_{22} & -6D_{22}r & 0 & 12D_{22} & -6D_{22}r \\ D_{12}r^2 & 6D_{22}r & 2D_{22}^2 & -D_{12}r^2 - 6D_{22}r & 4D_{22}r^2 \end{bmatrix} \quad (6.30a)$$

$$[M^e] = \frac{mr}{420} \begin{bmatrix} 140j_{\xi}/m & 0 & 0 & 70j_{\xi}/m & 0 & 0 \\ 0 & 156 & 22r & 0 & 54 & -13r \\ 0 & 22r & 4r^2 & 0 & 13r & -3r^2 \\ 70j_{\xi}/m & 0 & 0 & 140j_{\xi}/m & 0 & 0 \\ 0 & 54 & 13r & 0 & 156 & -22r \\ 0 & -13r & -3r^2 & 0 & -22r & 4r^2 \end{bmatrix}$$

$$+ \frac{j_{\eta}}{30r} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 36 & 3r & 0 & -36 & 3r \\ 0 & 3r & 4r^2 & 0 & -3r & -r^2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -36 & -3r & 0 & 36 & -3r \\ 0 & 3r & -r^2 & 0 & -3r & 4r^2 \end{bmatrix} \quad (6.30b)$$

To solve for the eigensolutions of an isotropic beam, one only needs to put $D_{12} = 0$ and set the torsional displacement Γ equal to zero.

6.4 Numerical Results

6.4.1 Isotropic Beams

To investigate the effects of the distributed rotary inertia and the inertia of the hub, we solve the eigenvalue problem of a slewing prismatic aluminum beam having a rectangular cross section with width $b = b_1 = b_2 = b_3$ (see Fig.6.1), thickness h , and length $L = 20$ ft. The other properties of the beam are

Young's modulus $E = 1.4832 \times 10^9 \text{ lbf/ft}^2$,

density $= 5.232 \text{ slug/ft}^3$,

the beam's moment of inertia about the fixed end $I_b \equiv \frac{1}{3} mL^3 = 290.67 \text{ slug} - \text{ft}^2$.

Figures 6.2(a) and (b) show the influence of the rotary inertia j_r on the values of the first 45 natural frequencies of a beam with $b = 0.5$ ft and $h = 0.5$ in and a beam with $b = 0.5$ in and $h = 0.5$ ft, respectively, when $I_H = I_b$. We do not show the rigid-body mode, whose frequency is zero. We note that the rotary inertia is negligible when the beam is vibrating along the direction of weak flexural rigidity, but the rotary inertia obviously decreases the natural frequencies of a beam which is vibrating along the direction of strong flexural rigidity, especially the high frequencies. Although the rotary inertia decreases the natural frequencies, it increases the values of Θ , especially those of high-frequency modes. Moreover, we note that the rotary inertia has more

effect on Θ than on the frequency. For example, it decreases the natural frequency of the tenth flexural mode by 2.4 % but increases the value of Θ by 4.7 % for a beam with $b = 0.5$ in and $h = 0.5$ ft and $I_H = 290.67 \text{ slug} - \text{ft}^2$. Hence, we include the rotary inertia in the following analysis.

Figures 6.3(a) and (b) show the influence of the hub's inertia I_H on the first five natural frequencies of a beam with $b = 0.5$ ft and $h = 0.5$ in and a beam with $b = 0.5$ in and $h = 0.5$ ft, respectively. We note that increasing I_H decreases all natural frequencies, especially the frequencies of higher modes. But when I_H increases beyond a very small value ($\approx 5 \text{ slug} - \text{ft}^2$), its effect on the natural frequencies becomes negligible. It confirms the fact that point boundary conditions provide just some local effects for higher modes, as indicated in Refs.[27] and [31]. Moreover, for vibrations in both the weak and strong directions, the variations of the frequencies with the hub's inertia have the same trend.

Figure 6.4 shows the variation of the values of Θ of the first five modes with I_H when $b = 0.5$ ft and $h = 0.5$ in and the mode shapes are normalized so that $D(L) = 1$. We note that the values of Θ are small and decrease with increasing mode number when $I_H > 13 \text{ slug} - \text{ft}^2$. But, except for the first mode, the values of Θ are large and change rapidly when $I_H < 15 \text{ slug} - \text{ft}^2$. Owing to the characteristics of small values of Θ and high frequencies, a high frequency and resolution actuator is needed to control the tip motion if the actuator is located at the hub.

If we use the characteristic length L and the characteristic time $L^2 \sqrt{m/D_{22}}$ to nondimensionalize the variables as

$$s^* = s/L, D^* = D/L, t^* = t \sqrt{D_{22}/mL^4}, \omega^* = \omega \sqrt{mL^4/D_{22}},$$

$$\dot{j}_\eta = j_\eta / mL^2 = (h/L)^2, \dot{I}_H = \frac{I_H}{mL^3} = \frac{I_H}{3I_B},$$

then (6.7) becomes

$$D^{*iv} = \omega^{*2}(D^* - \dot{j}_\eta D^{*''}) \quad (6.31)$$

and the boundary conditions (6.8a) and (6.8b) become

$$D^* = D^{*''} + \omega^{*2} \dot{I}_H D^{*'} = 0 \quad \text{at } s^* = 0 \quad (6.32a)$$

$$D^{*''} = D^{*'''} + \omega^{*2} \dot{j}_\eta D^{*'} = 0 \quad \text{at } s^* = 1 \quad (6.32b)$$

One can see from (6.31) and (6.32) that the eigensolutions depend only on the values of $\dot{I}_H$ and $\dot{j}_\eta$. In other words, if two systems have the same ratios $\frac{\dot{I}_H}{\dot{j}_\eta}$ and $\frac{h}{L}$, then they have the same mode shapes and nondimensionalized natural frequencies, and the dimensional natural frequencies are different only by a constant factor $\sqrt{mL^2/D_{22}}$. But, the rotary inertia is negligible for the low-frequency modes, as shown in Fig.6.2. This explains why Figs. 6.3(a) and (b) are almost the same. Similarly, the variation of $\dot{I}_H$ with Θ for the first five modes of a beam with $b = 0.5$ in and $h = 0.5$ ft are almost the same as those in Fig.6.4.

Figures 6.5(a)-(d) show the first four flexural mode shapes for different values of $\dot{I}_H$. Here the rotation angle at $s = 0$ is much larger than the real angle because D and s were scaled differently. We note that $\dot{I}_H$ dominates the flexural shapes around the point $s = 0$, but it does not affect very much the shapes around the tip-end of the beam, especially for the higher modes. It confirms again that $\dot{I}_H$ has a small, local influence on the mode shapes.

There are several reasons that make us conclude that a small hub's inertia is in favor of control. First, the values of Θ and the natural frequencies are large when $\dot{I}_H$ is small. Second, because of the problem of wave propagation [62], only systems with high natural frequencies can be manipulated at high speeds. Third, a system is more

controllable if the value of Θ for each mode is large. Lastly, a small I_H implies that a small power is needed.

6.4.2 A Symmetrically Laminated Composite Beam

We consider a graphite-epoxy composite beam with the layups $[30^\circ_3/60^\circ_3/0^\circ_2]_s$ and having the properties

$$E_1 = 1.92 \times 10^7 \text{ psi}, E_2 = E_3 = 1.56 \times 10^6 \text{ psi},$$

$$G_{23} = 5.23 \times 10^5 \text{ psi}, G_{12} = G_{13} = 8.20 \times 10^5 \text{ psi}, \nu_{12} = \nu_{13} = 0.24, \nu_{23} = 0.49,$$

$$b_1 = 0.6 \text{ in}, b_2 = 0.5 \text{ in}, b_3 = 0.4 \text{ in}, L = 3.0 \text{ ft},$$

$$\text{lamina thickness } t_k = 0.005 \text{ in}, \text{ density } \rho = 3.0 \text{ slug/ft}^3.$$

The beam's moment of inertia about the fixed end is calculated to be $0.20822 \text{ slug} - \text{ft}^2$. Using the method shown in Section 2.4.6, we obtain

for $b = b_1 = 0.6 \text{ in}$:

$$D_{11} = 208.24 \text{ lbf} - \text{in}^2, D_{22} = 159.73 \text{ lbf} - \text{in}^2, D_{12} = -62.862 \text{ lbf} - \text{in}^2,$$

for $b = b_2 = 0.5 \text{ in}$:

$$D_{11} = 170.35 \text{ lbf} - \text{in}^2, D_{22} = 133.10 \text{ lbf} - \text{in}^2, D_{12} = -52.385 \text{ lbf} - \text{in}^2,$$

and for $b = b_3 = 0.4 \text{ in}$:

$$D_{11} = 132.46 \text{ lbf} - \text{in}^2, D_{22} = 106.48 \text{ lbf} - \text{in}^2, D_{12} = -41.908 \text{ lbf} - \text{in}^2,$$

where D_{11} is modified [100] to take into account the warping effect. Because there is no approximation in the $[A_{ij}^0]$ matrix, the results obtained by using the state-space concept and the fundamental-matrix method are exact. However, the finite-element solution is not exact because Hermite cubics were used to approximate the displacement D and linear functions were used to approximate the displacement Γ .

Figures 6.6(a)-(d) show the bending mode shapes and the bending-induced torsional mode shapes of the first four modes. Here $\Gamma(s)$ is called the bending-induced torsional mode shape because its frequency is much smaller than the frequency obtained by neglecting the coupling term $(D_{12}d'')'$ in (6.4b). We note that $\Gamma = 0$ wherever $D' = 0$, which indicates that $j_z\ddot{y}$ is negligible in (6.4b). When $D = 0$, $\Gamma \neq 0$ except at the point $s = 0$. Thus, the nodal lines are not parallel to the y -axis. Moreover, the twisting angle increases with mode number, and the tip-end of the beam usually has the largest absolute twisting angle. Figure 6.7 shows the variations of D'' and D''' of the first two modes along the span. It reveals the discontinuities of D'' and D''' .

To compare the results obtained by using the fundamental-matrix method (FMM) and the finite-element method (FEM), we construct a 12-element finite-element model and obtain the modal frequencies and the corresponding values of Θ for the first eleven flexural-torsional modes, as shown in Figs. 6.8(a) and (b). In obtaining Fig. 6.8(b) we normalized the mode shape so that $D(L) = 1$. The FEM results are accurate for the lower modes, but the error increases with mode number and the frequencies are higher than the exact values. Figure 6.8(b) shows that the values of Θ are accurately predicted by the FEM method except for those corresponding to the ninth mode. Of course, the values of Θ predicted by the FEM analysis are smaller than those predicted by the FMM method because the FEM model is stiffer than the real structure owing to the fact that displacement distributions are assumed in the finite-element modeling. Figures 6.9(a) and (b) compare the third and the fourth mode shapes obtained by using the FMM method (solid lines) with those obtained by using the FEM method (dashed lines). The reason for the error in the predicted torsional mode shapes is the fact that a first-order function (linear interpolation) was

used to approximate a six-order distribution function (because D and Γ are linearly coupled). The errors in predicting the mode shapes increases as the mode number increases. To improve the accuracy, one needs to increase the number of elements or to use elements with more nodes [81]. In either case, the dimensions of the matrices $[M]$ and $[K]$ will increase. However, when D''' , D'''' , and Γ''' are needed in analysis (e.g., nonlinear analyses by using a combination of Galerkin procedure and the method of multiple scales [68]), the FMM method will provide a more accurate solution because it does not use the weak (energy) formulation in the solution procedure.

We show in Figure 6.10 the ninth bending-twisting mode shape. It looks like a pure torsional mode. But, because the bending and torsion are linearly coupled, whenever there is a torsional displacement, there is a flexural displacement, and vice versa. Hence, the ninth mode is not a pure torsional mode, but the torsion-induced flexural displacement is very small. Because the torsional inertia J_t is a very small quantity, the energy of the torsional motion is not large enough to 'drive' the structure through the coupling term $(D_{12}y')''$ to produce a significant flexural displacement. On the other hand, the flexure-induced torsional motion is usually large because the bending motion possesses enough energy to 'drive' the structure through the coupling term $(D_{12}d'')'$ to produce a significant torsional motion. Because the FEM model used here does not satisfactorily predict the torsional motion and a torsional motion dominates the ninth mode, the value of Θ of the ninth mode cannot be accurately calculated by the FEM method, as shown in Fig. 6.8(b).

Figure 6.11 shows the variation of the first five modal frequencies with the hub's inertia. The natural frequencies are not sensitive to the values of I_H when I_H is larger

than a small value ($\simeq 0.001s/ug - ft^2$), but I_H decreases all natural frequencies. This trend is the same as that for isotropic beams. Figure 6.12 (a) shows the variation of the values of Θ with I_H for the first five modes. It shows that increasing I_H results in a decrease in the value of Θ except when $I_H < 0.001s/ug - ft^2$, which is the same trend as that for isotropic beams. Figure 6.12(b) shows the variation of the tip-end twisting angle of the first five modes with I_H . We note that I_H can increase or decrease the twisting angle of different modes, but this effect occurs only in a very small range. When I_H is larger than $0.001 s/ug - ft^2$, $\Gamma(L)$ is not sensitive to the value of I_H . Furthermore, this figure also shows that $\Gamma(L)$ increases with mode number, which is a serious problem in control because the values of Θ are small for high-frequency modes.

6.5 Chapter Summary

The exact eigenfunctions of a slewing isotropic beam are derived, and the effects of distributed rotary inertias and hub's inertia are studied. The results show that the rotary inertia is not negligible for a beam vibrating in the principal direction of strong flexural rigidity and it decreases the natural frequencies and increases the modal rotation angles at the hub. On the other hand, the rotary inertia is negligible for a beam vibrating in the principal direction of weak flexural rigidity. Due to the problem of wave propagation, controllability, the power needed in the action of control, and the fact that the modal rotation angles at the hub and the natural frequencies are large when the hub's inertia is small, a small hub inertia is desirable for controlling a slewing beam. For a composite beam made of three prismatic segments and

exhibiting bending-twisting coupling, a powerful technique based on the state-space concept and the fundamental-matrix solution is used to obtain the exact eigensolutions. The effects of the hub's inertia and the bending-twisting coupling are investigated. The influence of the hub's inertia on the modal frequencies and the mode shapes are similar to those for isotropic beams. But, the bending-twisting coupling results in a serious control problem because it creates a large twisting angle at the tip-end of the beam, especially for higher modes. The results obtained by using the fundamental-matrix method and the finite-element method are compared.

Figure 6.1. Configuration of a slewing beam and the reference coordinate system x - y - z , which is fixed on the hub.

Figure 6.2. Modal frequencies of (a) a beam with $b = 0.5$ ft and $h = 0.5$ in and (b) a beam with $b = 0.5$ in and $h = 0.5$ ft : $I_H = I_B = 290.67$ slug - ft², o in the absence of j_s , + in the presence of j_s .

Figure 6.3. Variation of the first five modal frequencies with the hub's inertia I_H : (a) a beam with $b = 0.5$ ft and $h = 0.5$ in, and (b) a beam with $b = 0.5$ in and $h = 0.5$ ft .

Figure 6.4. Variation of the values of Θ of the first five modes with the hub's inertia I_H : $b = 0.5$ ft, $h = 0.5$ in, (a) $I_H = 0 - 30$ slug-ft², (b) $I_H = 30 - 430$ slug-ft².

Figure 6.5. Mode shapes for different values of I_H : (a) first mode, (b) second mode.

Figure 6.5. continued, (c) third mode, (d) fourth mode.

Figure 6.6. The flexural mode shape $D(s)$, the flexure-induced torsional mode shape $\Gamma(s)$, and the slope $D'(s)$ of the beam when $I_H = 0.02 s/ug - ft^2$: (a) first mode, (b) second mode.

Figure 6.6. continued, (c) third mode, (d) fourth mode.

Figure 6.7. The variation of D'' and D''' along the span when $I_N = 0.02 \text{ slug} - ft^2$: (a) first mode and (b) second mode.

Figure 6.8. (a) The modal frequencies and (b) values of Θ of the first eleven modes: $I_H = 0.02 \text{ slug-ft}^2$, + = fundamental-matrix method (FMM) and o = finite-element method (FEM).

Figure 6.9. The mode shapes $D(s)$ and $\Gamma(s)$ obtained using the fundamental-matrix method (FMM) (solid lines) and finite-element method (FEM) (dashed lines) when $I_n = 0.02 \text{ slug} - \text{ft}^2$: (a) third mode and (b) fourth mode.

Figure 6.10. The ninth flexural-torsional mode shape: $I_N = 0.02 \text{ slug} - \text{ft}^2$.

Figure 6.11. Variation of the first five modal frequencies with the hub's inertia I_H .

Figure 6.12. Variation of the values of (a) Θ and (b) tip-end twisting angle $\Gamma(L)$ of the first five modes with the hub's inertia I_H .

CHAPTER VII

FORCED FLAPWISE VIBRATIONS OF A COMPOSITE BEAM

The forced nonlinear vibration of a symmetrically laminated graphite-epoxy composite beam (see Fig.7.1(a)) is investigated by using the equations developed in Chapter II. We use a technique based on the state-space concept and the fundamental-matrix method to solve for the structurally coupled eigensolutions, and then use the concept of adjoint and the method of multiple scales to derive four equations for the modulation of the amplitudes and phases. The analysis focuses on the case of primary resonance of the first flexural-torsional mode when its frequency is approximately one-half the frequency of the first out-of-plane flexural mode. Period-doubling bifurcations and chaotic motions are addressed.

7.1 Equations of Motion and Linear Eigensolutions

The problem being studied is the forced vibration of the uniform, inextensional cantilever beam shown in Fig.7.1(a). The equations of motion are obtained by substituting

$$\Theta(t) = D_{12} = D_{23} = 0 \text{ and } \bar{v}(s,t) = v(s,t) - F \cos \Omega t$$

into (2.112)-(2.114). To nondimensionalize the equations, we use the characteristic length L and the characteristic time $L^2\sqrt{m/D_{33}}$ and define

$$\begin{aligned} s^* &= s/L, v^* = \bar{v}/L, w^* = w/L, t^* = t\sqrt{D_{33}/mL^4}, \hat{f} = F/L, \Omega^* = \Omega\sqrt{mL^4/D_{33}}, \\ \mu_1^* &= \mu_1 L^2/\sqrt{mD_{33}}, \mu_2^* = \mu_2 L^2/\sqrt{mD_{33}}, \mu_3^* = \mu_3/\sqrt{mD_{33}}, \\ j_1^* &= j_z/mL^2, j_2^* = j_y/mL^2, j_3^* = j_x/mL^2. \end{aligned}$$

For convenience, we drop the superscript $*$ in the following analysis and use primes and dots to denote $\partial/\partial s^*$ and $\partial/\partial t^*$, respectively. Furthermore, we define

$$\beta_{11} = D_{11}/D_{33}, \beta_{22} = D_{22}/D_{33}, \beta_{13} = D_{13}/D_{33}.$$

The nondimensionalized equations of motion are

$$\begin{aligned} \ddot{v} + v^{iv} + \beta_{13}\gamma''' - j_3\ddot{v}'' &= \varepsilon\{-\mu_1\dot{v} - [v'(v'v'' + w'w'')]'\} - \beta_{11}(\gamma'w'')' \\ &+ (\beta_{22} - 1)[(\gamma w'' - \gamma^2 v'')' - w'''\int_0^s v'v'w' ds] - \frac{1}{2}\left\{v'\int_1^s [\int_0^s (v'^2 + w'^2) ds]\right\}' \\ &+ \beta_{13}(-v''w'' - \frac{1}{2}v'^2\gamma'' - w''^2\gamma + \frac{1}{2}\gamma^2\gamma'' + \gamma\gamma'^2)' + \hat{f}\Omega^2 \cos \Omega t^*, \end{aligned} \quad (7.1)$$

$$\ddot{w} + \beta_{22}w^{iv} - j_2\ddot{w}'' = \varepsilon\{-\mu_2\dot{w} - \beta_{22}[w'(v'v'' + w'w'')]'\} + \beta_{11}(\gamma'v'')'$$

$$\begin{aligned}
& + (\beta_{22} - 1)[(\gamma v'' + \gamma^2 w'')' + v''' \int_0^s v' w'' ds]' - \frac{1}{2} \left\{ w' \int_1^s \left[\int_0^s (v'^2 + w'^2) ds \right]'' ds \right\}' \\
& + \beta_{13}(-\gamma \gamma'' + \gamma v'' w'' + v''^2 - \gamma'^2 - \gamma'' \int_0^s v' w'' ds)' , \tag{7.2}
\end{aligned}$$

$$\begin{aligned}
j_1 \ddot{\gamma} - \beta_{11} \gamma'' - \beta_{13} v''' &= \varepsilon \{ -\mu_3 \dot{\gamma} + (1 - \beta_{22})[(v''^2 - w''^2)\gamma - v'' w''] \\
& + j_1 \left(\int_0^s v'' w' ds \right)'' - j_1 (\dot{v} w')' + (j_2 - j_3)[(\dot{v}^2 - \dot{w}^2)\gamma - \dot{w} \dot{v}] \\
& + \beta_{13}(w''' \gamma + w''' \int_0^s v' w'' ds - \frac{1}{2} v''' \gamma^2 + \frac{1}{2} v''' v'^2 + v''^2 v' + v' w''^2) \} , \tag{7.3}
\end{aligned}$$

where ε is a nondimensional parameter introduced for bookkeeping purposes. The boundary conditions are

$$v = v' = w = w' = \gamma = 0 \text{ at } s = 0 , \tag{7.4}$$

$$v'' = w'' = \gamma' = 0 \text{ at } s = 1 , \tag{7.5a}$$

$$\begin{aligned}
v''' + \beta_{13} \gamma'' - j_3 \ddot{v} &= \varepsilon [(\beta_{22} - 1)(w''' \gamma - v''' \gamma^2 - w''' \int_0^1 v'' w' ds) \\
& - v'(v' v''' + w' w''') + \frac{1}{2} \beta_{13}(\gamma^2 \gamma'' - v'^2 \gamma'')] \quad \text{at } s = 1 , \tag{7.5b}
\end{aligned}$$

$$\begin{aligned}
\beta_{22} w''' - j_2 \ddot{w} &= \varepsilon [(\beta_{22} - 1)(v''' \gamma + w''' \gamma^2 + v''' \int_0^1 w'' v' ds) \\
& - \beta_{22} w'(v' v''' + w' w''') - \beta_{13}(\gamma \gamma'' + \gamma'' \int_0^1 w'' v' ds)] \quad \text{at } s = 1 . \tag{7.5c}
\end{aligned}$$

We consider a graphite-epoxy composite beam, as shown in Fig.7.1(b), with the layups $[10^\circ_s/45^\circ_s/90^\circ_s]_s$, and the properties

$$E_1 = 1.92 \times 10^7 \text{ psi} , E_2 = E_3 = 1.56 \times 10^6 \text{ psi} ,$$

$$G_{23} = 5.23 \times 10^5 \text{ psi} , G_{12} = G_{13} = 8.20 \times 10^5 \text{ psi} , \nu_{12} = \nu_{13} = 0.24 , \nu_{23} = 0.49 ,$$

$$b = 0.37526 \text{ in} , \text{ length } L = 1.5 \text{ ft} , \text{ lamina thickness } t_k = 0.005 \text{ in} , \text{ density } \rho = 96.1 \text{ lb/ft}^3 .$$

The width b of the cross section is designed so that the first natural frequency of $w(s,t)$ is approximately twice the natural frequency of the first coupled mode of vibration of $v(s,t)$ and $\gamma(s,t)$. Using (2.197), we obtain

$$D_{11} = 436.23 \text{ lbf} - in^2, D_{22} = 5547.2 \text{ lbf} - in^2,$$

$$D_{33} = 1532.4 \text{ lbf} - in^2, D_{13} = 252.02 \text{ lbf} - in^2.$$

To solve the linear parts of (7.1) and (7.3) for the coupled vibration modes and frequencies, we make use of the state-space concept and the fundamental-matrix method. To this end, we assume that

$$v(s,t) = V(s)e^{i\omega t}, \gamma(s,t) = \Gamma(s)e^{i\omega t}. \quad (7.6)$$

Substituting (7.6) into the linear parts of (7.1) and (7.3) yields

$$-\omega^2 V + V^{iv} + \beta_{13}\Gamma''' + j_3\omega^2 V'' = 0, \quad (7.7)$$

$$j_1\omega^2 \Gamma + \beta_{11}\Gamma'' + \beta_{13}V''' = 0. \quad (7.8)$$

Substituting (7.6) into (7.4), (7.5a), and the linear parts of (7.5b) and (7.5c) yields

$$V(0) = V'(0) = \Gamma(0) = 0, \quad (7.9)$$

$$V'''(1) + \beta_{13}\Gamma''(1) + j_3\omega^2 V'(1) = 0, \quad (7.10)$$

$$V''(1) = \Gamma'(1) = 0 \quad (7.11)$$

To solve the eigenvalue problem consisting of (7.7) and (7.8) and the boundary conditions (7.9)-(7.11), we cast (7.7) and (7.8) in a system of first-order equations by using the transformation

$$x_1 = V, x_2 = V', x_3 = V'', x_4 = V''', x_5 = \Gamma, x_6 = \Gamma'. \quad (7.12)$$

Then, (7.7) and (7.8) can be expressed as

$$\{X'\} = [A]\{X\}, \quad (7.13)$$

where

$$\{X\} = \{x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6\}^T, \quad [A] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ A_{41} & 0 & A_{43} & 0 & 0 & A_{46} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & A_{64} & A_{65} & 0 \end{bmatrix}, \quad (7.14)$$

$$A_{41} = \frac{\omega^2}{1 - \beta_{13}^2/\beta_{11}}, \quad A_{43} = \frac{-j_3\omega^2}{1 - \beta_{13}^2/\beta_{11}},$$

$$A_{46} = \frac{j_1\omega^2\beta_{13}}{\beta_{11} - \beta_{13}^2}, \quad A_{64} = -\frac{\beta_{13}}{\beta_{11}}, \quad A_{65} = -\frac{j_1\omega^2}{\beta_{11}}. \quad (7.15)$$

According to the fundamental-matrix method [5], the solution of (7.13) can be expressed in the general form

$$\{X\} = [R]e^{[J]s}\{K\}, \quad (7.16)$$

where $[R]$ is a 6x6 matrix whose columns are the eigenvectors of $[A]$, $[J]$ is the Jordan canonical form of $[A]$, and $\{K\}$ is a 6-element constant vector. Implementing the boundary conditions (7.9)-(7.11) results in

$$[T_{01}]\{K\} = \{0\}, \quad (7.17)$$

$$\text{where } [T_{01}] = \begin{bmatrix} [T_0] \\ [T_1] \end{bmatrix}, [T_0] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} [R], \quad (7.18)$$

$$[T_1] = \begin{bmatrix} 0 & j_3 \omega^2 & 0 & 1 - \beta_{13}^2/\beta_{11} & -j_1 \omega^2 \beta_{13}/\beta_{11} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} [R] e^{[J]}. \quad (7.19)$$

For nontrivial solutions the determinant of $[T_{01}]$ must vanish, which yields the natural frequencies ω_m . Because $[A]$ is asymmetric, the elements of $[R]$, $[J]$, and $\{K\}$ are, in general, complex numbers. In the present case, all eigensolutions are distinct and hence $e^{[J]s}$ is a diagonal matrix having the diagonal elements $e^{j_i s}$. The first three natural frequencies are found to be 3.3445, 20.955, and 58.647, and their corresponding mode shapes are

First mode

$$\begin{aligned} V(s) = x_1 &= \sum_{i=1}^6 R_{1i} K_i e^{\lambda_i s} \\ &= -1.0000 \cosh(1.8750s) + 0.73411 \sinh(1.8750s) \\ &\quad + 1.0000 \cos(1.8751s) - 0.73406 \sin(1.8751s) \\ &\quad - 4.6296 \times 10^{-8} \cos(0.04063s) - 7.3628 \times 10^{-10} \sin(0.04063s), \end{aligned}$$

$$\begin{aligned}
\Gamma(s) = x_5 &= \sum_{i=1}^6 R_{5i} K_i e^{\lambda_i s} \\
&= -0.79480 \cosh(1.8750s) + 1.0827 \sinh(1.8750s) \\
&\quad + 0.79557 \cos(1.8751s) + 1.0838 \sin(1.8751s) \\
&\quad - 7.4673 \times 10^{-4} \cos(0.040630s) + 4.6954 \times 10^{-2} \sin(0.040630s) ,
\end{aligned}$$

Second mode

$$\begin{aligned}
V(s) &= -1.0002 \cosh(4.6928s) + 1.0187 \sinh(4.6928s) \\
&\quad + 1.0002 \cos(4.6938s) - 1.0184 \sin(4.6938s) \\
&\quad + 1.9034 \times 10^{-6} \cos(0.25455s) - 1.0037 \times 10^{-7} \sin(0.25455s) , \\
\Gamma(s) &= -2.7535 \cosh(4.6928s) + 2.7035 \sinh(4.6928s) \\
&\quad + 2.7698 \cos(4.6938s) + 2.7201 \sin(4.6938s) \\
&\quad - 1.6248 \times 10^{-2} \cos(0.25455s) - 3.0812 \times 10^{-1} \sin(0.25455s) ,
\end{aligned}$$

Third mode

$$\begin{aligned}
V(s) &= -1.0004 \cosh(7.8493s) + 0.99960 \sinh(7.8493s) \\
&\quad + 1.0004 \cos(7.8541s) - 0.99899 \sin(7.8541s) \\
&\quad - 1.7688 \times 10^{-5} \cos(0.71243s) - 1.2905 \times 10^{-6} \sin(0.71243s) , \\
\Gamma(s) &= -4.4957 \cosh(7.8493s) + 4.4992 \sinh(7.8493s) \\
&\quad + 4.5704 \cos(7.8541s) + 4.5768 \sin(7.8541s) \\
&\quad - 7.4640 \times 10^{-2} \cos(0.71243s) + 1.0230 \sin(0.71243s) .
\end{aligned}$$

The linear mode shapes and their corresponding frequencies for motions in the z direction can be obtained from (7.2), (7.4), and (7.5). The first natural frequency is 6.689 and its corresponding mode shape is

$$\begin{aligned}
W(s) &= \cosh(1.8750s) - 0.73405 \sinh(1.8750s) \\
&\quad - \cos(1.8751s) + 0.73401 \sin(1.8751s) .
\end{aligned}$$

We note that there is a small difference between the first mode shapes of v and w . However, the magnitude of the bending-induced torsion is large and is the same order as that of bending, as shown in Fig.7.2. Figures 7.3(a) and (b) show the mode shapes of the coupled second and third modes of v and γ , respectively. It is apparent that torsion is zero at the locations where the slope of the beam is zero. We note that the flexural induced torsional mode shapes are different from the linear uncoupled torsional mode shapes $\sin \frac{1}{2}(2n-1)\pi s$, where n denotes the mode number. Moreover, the twisting angle induced by bending is large for high-frequency modes. Furthermore, the torsional displacement does not vanish at the locations where the flexural displacement vanishes, which indicates that the nodal lines are not parallel to the z -axis.

7.2 Multiple Scale Analysis

To analyze the solutions of the coupled nonlinear equations (7.1)-(7.3) and boundary conditions (7.4) and (7.5), we employ the method of multiple scales [68,69]. We consider weakly nonlinear vibrations and seek an asymptotically valid expansion for the displacement fields. Thus, we assume that

$$v(s, T_0, T_1; \varepsilon) = v_0(s, T_0, T_1) + \varepsilon v_1(s, T_0, T_1) + \dots, \quad (7.20)$$

$$w(s, T_0, T_1; \varepsilon) = w_0(s, T_0, T_1) + \varepsilon w_1(s, T_0, T_1) + \dots, \quad (7.21)$$

$$\gamma(s, T_0, T_1; \varepsilon) = \gamma_0(s, T_0, T_1) + \varepsilon \gamma_1(s, T_0, T_1) + \dots, \quad (7.22)$$

where $T_0 = t$, a fast scale characterizing motions with the natural frequencies ω_i and Ω ; and $T_1 = \varepsilon t$, a slow scale characterizing the modulations of the amplitudes and phases. Substituting (7.20)-(7.22) into (7.1)-(7.5) and equating coefficients of like powers of ε , we obtain

Order ε^0 :

$$D_0^2 v_0 + v_0^{iv} + \beta_{13} \gamma_0''' - j_3 D_0^2 v_0'' = 0 , \quad (7.23)$$

$$D_0^2 w_0 + \beta_{22} w_0^{iv} - j_2 D_0^2 w_0'' = 0 , \quad (7.24)$$

$$j_1 D_0^2 \gamma_0 - \beta_{11} \gamma_0'' - \beta_{13} v_0''' = 0 , \quad (7.25)$$

$$v_0 = v_0' = w_0 = w_0' = \gamma_0 = 0 \quad \text{at } s = 0 , \quad (7.26)$$

$$\begin{aligned} -v_0''' - \beta_{13} \gamma_0'' + j_3 D_0^2 v_0' &= v_0'' = \gamma_0' \\ &= -\beta_{22} w_0''' + j_2 D_0^2 w_0' = w_0'' = 0 \quad \text{at } s = 1, \end{aligned} \quad (7.27)$$

Order ε :

$$\begin{aligned} D_0^2 v_1 + v_1^{iv} + \beta_{13} \gamma_1''' - j_3 D_0^2 v_1'' \\ = -2D_0 D_1 v_0 + 2j_3 D_0 D_1 v_0'' - \mu_1 D_0 v_0 - \beta_{11} (w_0'' \gamma_0')' \\ + (1 - \beta_{22}) [(v_0'' \gamma_0^2 - w_0'' \gamma_0)' + (w_0''' \int_0^s v_0'' w_0'' ds)'] - [v_0' (v_0' v_0'' + w_0' w_0'')] ' \\ + \beta_{13} (-v_0'' w_0'' - \frac{1}{2} v_0'^2 \gamma_0'' - w_0''^2 \gamma_0 + \frac{1}{2} \gamma_0^2 \gamma_0'' + \gamma_0 \gamma_0'^2)' \\ - \frac{1}{2} [v_0' \int_1^s D_0^2 \int_0^s (v_0'^2 + w_0'^2) ds ds]' + \hat{f} \Omega^2 \cos(\Omega t) , \end{aligned} \quad (7.28)$$

$$\begin{aligned} D_0^2 w_1 + \beta_{22} w_1^{iv} - j_2 D_0^2 w_1'' \\ = -2D_0 D_1 w_0 + 2j_2 D_0 D_1 w_0'' - \mu_2 D_0 w_0 + \beta_{11} (v_0'' \gamma_0')' \\ + (\beta_{22} - 1) [(v_0'' \gamma_0 + w_0'' \gamma_0^2)' + (v_0''' \int_0^s v_0' w_0'' ds)'] - \beta_{22} [w_0' (w_0' w_0'' + v_0' v_0'')] ' \\ + \beta_{13} (-\gamma_0 \gamma_0'' + \gamma_0 v_0'' w_0'' + v_0''^2 - \gamma_0'^2 - \gamma_0'' \int_0^s v_0' w_0'' ds)' \end{aligned}$$

$$-\frac{1}{2} [w_0' \int_1^s D_0^2 \int_0^s (v_0'^2 + w_0'^2) ds ds]' , \quad (7.29)$$

$$\begin{aligned} j_1 D_0^2 \gamma_1 - \beta_{11} \gamma_1'' - \beta_{13} v_1''' \\ = -2j_1 D_0 D_1 \gamma_0 - \mu_3 D_0 \gamma_0 + (1 - \beta_{22}) [v_0''^2 \gamma_0 - w_0''^2 \gamma_0 - v_0'' w_0''] + j_1 D_0^2 \int_0^s v_0'' w_0' ds \\ + \beta_{13} (w_0''' \gamma_0 + w_0''' \int_0^s v_0' w_0'' ds - \frac{1}{2} v_0''' \gamma_0^2 + \frac{1}{2} v_0''' v_0'^2 + v_0''^2 v_0' + v_0' w_0''^2) \\ - j_1 D_0 (w_0' D_0 v_0') + (j_2 - j_3) [(D_0 v_0')^2 \gamma_0 - (D_0 w_0')^2 \gamma_0 - D_0 v_0' D_0 w_0'] , \end{aligned} \quad (7.30)$$

$$v_1 = v_1' = w_1 = w_1' = \gamma_1 = 0 \text{ at } s = 0 , \quad (7.31)$$

$$v_1'' = w_1'' = \gamma_1' = 0 \text{ at } s = 1 , \quad (7.32a)$$

$$\begin{aligned} v_1''' + \beta_{13} \gamma_1'' - j_3 D_0^2 v_1' = (\beta_{22} - 1) (w_0''' \gamma_0 - v_0''' \gamma_0^2 - w_0''' \int_0^1 v_0'' w_0' ds) \\ - v_0' (v_0' v_0''' + w_0' w_0''') + \frac{1}{2} \beta_{13} (\gamma_0^2 \gamma_0'' - v_0'^2 \gamma_0'') \text{ at } s = 1 , \end{aligned} \quad (7.32b)$$

$$\begin{aligned} \beta_{22} w_1''' - j_2 D_0^2 w_1' = (\beta_{22} - 1) (v_0''' \gamma_0 + w_0''' \gamma_0^2 + v_0''' \int_0^1 w_0'' v_0' ds) \\ - \beta_{22} w_0' (v_0' v_0''' + w_0' w_0''') - \beta_{13} (\gamma_0 \gamma_0'' + \gamma_0'' \int_0^1 w_0'' v_0' ds) \text{ at } s = 1 . \end{aligned} \quad (7.32c)$$

The general solutions of (7.23)-(7.27) consist of two infinite sequences of modes corresponding to two infinite sequences of frequencies. However, due to damping all modes that are not directly excited by the loads or indirectly excited by autoparametric resonances decay with time [72]. In this chapter, we consider the case in which the first mode of v and γ is directly excited by a primary resonance and the first mode of w is indirectly excited by a two-to-one autoparametric resonance. Consequently all modes except these modes decay with time and hence we express the solution of the first-order problem as

$$v_0(s, T_0, T_1) = V(s) A_1(T_1) e^{i\omega_1 T_0} + cc , \quad (7.33)$$

$$w_0(s, T_0, T_1) = W(s)A_2(T_1)e^{i\omega_2 T_0} + cc, \quad (7.34)$$

$$\gamma_0(s, T_0, T_1) = \Gamma(s)A_1(T_1)e^{i\omega_1 T_0} + cc, \quad (7.35)$$

where the amplitudes of γ and v are assumed to be the same because they are linearly coupled.

To quantitatively describe the nearness of the autoparametric resonance, we introduce a detuning parameter δ defined by

$$\omega_2 = 2\omega_1(1 + \varepsilon\delta), \quad (7.36)$$

where δ depends on b , the ply angles, and/or the stacking sequence. The nearness of the external resonance is represented by another detuning parameter σ defined by

$$\Omega = \omega_1(1 + \varepsilon\sigma). \quad (7.37)$$

Substituting (7.33)-(7.37) into (7.28)-(7.30), we obtain

$$D_0^2 v_1 + v_1^{iv} + \beta_{13}\gamma_1''' - j_3 D_0^2 v_1'' = H_1(s, T_1)e^{i\omega_1 T_0} + cc + NST, \quad (7.38)$$

$$D_0^2 w_1 + \beta_{22}w_1^{iv} - j_2 D_0^2 w_1'' = H_2(s, T_1)e^{i\omega_2 T_0} + cc + NST, \quad (7.39)$$

$$j_1 D_0^2 \gamma_1 - \beta_{11}\gamma_1'' - \beta_{13}v_1''' = H_3(s, T_1)e^{i\omega_1 T_0} + cc + NST, \quad (7.40)$$

where the H_m are functions of the A_m , V , Γ , and W ; cc stands for the complex conjugate of the preceding terms; and NST stands for terms that do not produce secular terms. Because the corresponding homogeneous problem of (7.38)-(7.40), (7.31), and (7.32) has non-trivial solutions similar to (7.33)-(7.35), the inhomogeneous problem has a

solution only if solvability conditions are satisfied [68]. To determine these solvability conditions, we seek a particular solution in the form

$$v_1 = V_1(s, T_1)e^{i\omega_1 T_0} + cc, \quad \gamma_1 = \Gamma_1(s, T_1)e^{i\omega_1 T_0} + cc, \quad w_1 = W_1(s, T_1)e^{i\omega_2 T_0} + cc. \quad (7.41)$$

Substituting (7.41) and (7.33)-(7.35) into (7.38)-(7.40), (7.31), and (7.32) and equating the coefficients of $\exp(i\omega_1 T_0)$ and $\exp(i\omega_2 T_0)$ on both sides, we obtain

$$\beta_{22}W_1^{iv} + j_2\omega_2^2W_1'' - \omega_2^2W_1 = H_2, \quad (7.42)$$

$$W_1 = W_1' = 0 \text{ at } s = 0, \quad (7.43)$$

$$W_1'' = 0 \text{ at } s = 1, \quad (7.44a)$$

$$\begin{aligned} \beta_{22}W_1''' + j_2\omega_2^2W_1' = \\ 2[(\beta_{22} - 1)(W_1''''\Gamma^2 + V_1'''' \int_0^1 W_1''V_1' ds) - \beta_{22}W_1'V_1'''' - \beta_{13}\Gamma_1'' \int_0^1 W_1''V_1' ds]^1 A_1 \bar{A}_1 A_2 \\ + [(\beta_{22} - 1)V_1''''\Gamma - \beta_{13}\Gamma_1\Gamma_1'']^1 A_1^2 e^{-2i\omega_1 \delta T_1} - 3[\beta_{22}W_1'^2 W_1''']^1 A_2^2 \bar{A}_2 \quad \text{at } s = 1, \end{aligned} \quad (7.44b)$$

$$V_1^{iv} + \beta_{13}\Gamma_1''' + j_3\omega_1^2V_1'' - \omega_1^2V_1 = H_1, \quad (7.45)$$

$$\beta_{11}\Gamma_1'' + j_1\omega_1^2\Gamma_1 + \beta_{13}V_1''' = -H_3, \quad (7.46)$$

$$V_1 = V_1' = \Gamma_1 = 0 \text{ at } s = 0, \quad (7.47)$$

$$V_1'' = \Gamma_1' = 0 \text{ at } s = 1, \quad (7.48a)$$

$$\begin{aligned} V_1''' + \beta_{13}\Gamma_1'' + j_3\omega_1^2V_1' = \\ [(\beta_{22} - 1)W_1''''\Gamma]^1 A_2 \bar{A}_1 e^{2i\omega_1 \delta T_1} - 2[(\beta_{22} - 1)W_1'''' \int_0^1 V_1''W_1' ds + V_1''W_1'W_1''']^1 A_2 \bar{A}_2 A_1 \\ + [3(1 - \beta_{22})V_1''''\Gamma^2 - 3V_1'^2 V_1'''' - \frac{3}{2}\beta_{13}V_1'^2 \Gamma'' + \frac{3}{2}\beta_{13}\Gamma^2 \Gamma'']^1 A_1^2 \bar{A}_1 \quad \text{at } s = 1. \end{aligned} \quad (7.48b)$$

We note that the problem governing W_1 is decoupled from that governing V_1 and Γ_1 .

To determine the solvability condition of (7.42)-(7.44), we multiply (7.42) with the so-called adjoint function W^* , integrate the result by parts from $s = 0$ to $s = 1$ to transfer the derivatives from W_1 to W^* , and obtain

$$\begin{aligned} & \left[\beta_{22}(W_1'''W^* - W_1''W'' + W_1'W''' - W_1W''') + j_2\omega_2^2(W_1'W^* - W_1W'^*) \right]_0^1 \\ & + \int_0^1 W_1 [\beta_{22}W^{*iv} + j_2\omega_2^2W''' - \omega_2^2W^*] ds = \int_0^1 H_2W^* ds. \end{aligned} \quad (7.49)$$

The equation governing the adjoint W^* is obtained by setting the coefficient of W_1 in the integrand in (7.49) equal to zero. The result is

$$\beta_{22}W^{*iv} + j_2\omega_2^2W''' - \omega_2^2W^* = 0. \quad (7.50)$$

Using (7.50) and the boundary conditions (7.43) and (7.44a), we have

$$\begin{aligned} & \left[\beta_{22}W_1'W''' - W_1(\beta_{22}W'''' + j_2\omega_2^2W'') + W^*(\beta_{22}W_1''' + j_2\omega_2^2W_1') \right]^1 \\ & - \beta_{22}[W_1'''W^* - W_1''W'']^0 = \int_0^1 H_2W^* ds. \end{aligned} \quad (7.51)$$

Because $W_1'(1)$, $W_1(1)$, $W_1'''(0)$, and $W_1''(0)$ are arbitrary, we define the adjoint boundary conditions by setting each of their coefficients equal to zero. The result is

$$W^* = W'' = 0 \text{ at } s = 0, \quad (7.52a)$$

$$W''' = \beta_{22}W'''' + j_2\omega_2^2W'' = 0 \text{ at } s = 1. \quad (7.52b)$$

Because the adjoint problem (7.50) and (7.52) is the same as that of w_0 , $W^* = W(s)$.

Having defined the adjoint, we return to (7.51). Using (7.52) and (7.44b) in (7.51), we find that the solvability condition of (7.42)-(7.44) is

$$\begin{aligned}
\int_0^1 H_2(s, T_1)W(s)ds = \\
2[(\beta_{22} - 1)W(W''''\Gamma^2 + V'''' \int_0^1 W''V' ds) - \beta_{22}WW'V'V'''' - \beta_{13}W\Gamma'' \int_0^1 W''V' ds]^1 A_1 \bar{A}_1 A_2 \\
+ [(\beta_{22} - 1)WV''''\Gamma - \beta_{13}W\Gamma\Gamma'']^1 A_1^2 \theta^{-2i\omega_1 \delta T_1} - 3[\beta_{22}WW'^2 W'''']^1 A_2^2 \bar{A}_2 . \quad (7.53)
\end{aligned}$$

Substituting (7.33)-(7.37) into the right-hand side of (7.29), keeping the terms that are proportional to $\exp(i\omega_2 T_0)$ and hence determining H_2 , substituting the result into (7.53), and rearranging, we obtain

$$2i(A_2' + \hat{\mu}_2 A_2) + 4\Lambda_2 A_1^2 \theta^{-2i\delta\omega_1 T_1} + 8S_{21} A_1 \bar{A}_1 A_2 + 8S_{22} A_2^2 \bar{A}_2 = 0 , \quad (7.54)$$

where $\hat{\mu}_2$, Λ_2 , S_{21} , and S_{22} are defined in Appendix D.

To determine the solvability condition of the coupled problem (7.45)-(7.48), we add (7.45) times V^* to (7.46) times $-\Gamma^*$, integrate the result by parts from $s = 0$ to $s = 1$ to transfer the derivatives from V_1 and Γ_1 to V^* and Γ^* , and obtain

$$\begin{aligned}
& \int_0^1 V_1 [V^{*iv} + \beta_{13}\Gamma^{*''''} + j_3\omega_1^2 V^{*'''} - \omega_1^2 V^*] ds + \int_0^1 \Gamma_1 [-\beta_{11}\Gamma^{*''''} - j_1\omega_1^2 \Gamma^* - \beta_{13}V^{*''''}] ds \\
& + [V_1'''V^* - V_1''V^{*'} + V_1'V^{*''} - V_1V^{*'''}]_0^1 - \beta_{13}[\Gamma_1''V^* - \Gamma_1'V^{*'} + \Gamma_1V^{*''}]_0^1 \\
& + j_3\omega_1^2[V_1'V^* - V_1V^{*'}]_0^1 - \beta_{11}[\Gamma_1'\Gamma^* - \Gamma_1\Gamma^{*'}]_0^1 - \beta_{13}[V_1''\Gamma^* - V_1'\Gamma^{*'} + V_1\Gamma^{*''}]_0^1 \\
& = \int_0^1 (V^*H_1 + \Gamma^*H_3) ds . \quad (7.55)
\end{aligned}$$

The adjoint problem can be defined by considering the homogeneous problem; that is, by putting $H_1 = H_3 = 0$. The adjoint equations are obtained by setting each of the coefficients of Γ_1 and V_1 in the integrands in (7.55) equal to zero. The result is

$$V^{*iv} + \beta_{13}\Gamma^{*''''} + j_3\omega_1^2 V^{*'''} - \omega_1^2 V^* = 0 , \quad (7.56)$$

$$\beta_{11}\Gamma''' + j_1\omega_1^2\Gamma^* + \beta_{13}V''' = 0. \quad (7.57)$$

Using the boundary conditions (7.47) and (7.48a) and a procedure similar to that used above, we obtain the adjoint boundary conditions

$$V^* = V'' = \Gamma^* = 0 \text{ at } s = 0, \quad (7.58a)$$

$$V''' = \Gamma'' = V''' + \beta_{13}\Gamma''' + j_3\omega_1^2V'' = 0 \text{ at } s = 1. \quad (7.58b)$$

Because the adjoint problem (7.45)-(7.48) is the same as that of γ_0 and v_0 ,

$$V^* = V \text{ and } \Gamma^* = \Gamma.$$

Having defined the adjoint, we return to (7.55). Using the definition of the adjoint and the boundary conditions (7.47) and (7.48) in (7.55), we find the solvability condition

$$\begin{aligned} \int_0^1 (H_1V + H_3\Gamma)ds = \\ [(\beta_{22} - 1)VW''''\Gamma]^1 A_2 \bar{A}_1 e^{2i\omega_1 \delta T_1} - 2[(\beta_{22} - 1)VW'''' \int_0^1 V''W' ds + VV'W'W'''']^1 A_2 \bar{A}_2 A_1 \\ + [3(1 - \beta_{22})VV''''\Gamma^2 - 3VV'^2V''' - \frac{3}{2}\beta_{13}VV'^2\Gamma'' + \frac{3}{2}\beta_{13}V\Gamma^2\Gamma'']^1 A_1^2 \bar{A}_1. \end{aligned} \quad (7.60)$$

Substituting (7.33)-(7.37) into (7.28) and (7.30), collecting the coefficients of $\exp(i\omega_1 T_0)$ and hence determining H_1 and H_3 , substituting the results into (7.60), and rearranging, we obtain

$$2i(A_1' + \hat{\mu}_1 A_1) + 4\Lambda_1 A_2 \bar{A}_1 e^{2i\delta\omega_1 T_1} + 8S_{11}A_1^2 \bar{A}_1 + 8S_{12}A_2 \bar{A}_2 A_1 + Re^{i\sigma\omega_1 T_1} = 0, \quad (7.61)$$

where $\hat{\mu}_1$, Λ_1 , S_{11} , S_{12} , and R are defined in Appendix D.

Substituting the polar form

$$A_1 = \frac{1}{2} a_1(T_1) e^{i\theta_1(T_1)} \quad \text{and} \quad A_2 = \frac{1}{2} a_2(T_1) e^{i\theta_2(T_1)} \quad (7.62)$$

into (7.54) and (7.61) and separating real and imaginary parts, we obtain

$$a_1' + \hat{\mu}_1 a_1 + \Lambda_1 a_1 a_2 \sin v_1 + R \sin v_2 = 0 , \quad (7.63)$$

$$a_1 v_2' - \sigma \omega_1 a_1 + \Lambda_1 a_1 a_2 \cos v_1 + S_{11} a_1^3 + S_{12} a_1 a_2^2 + R \cos v_2 = 0 , \quad (7.64)$$

$$a_2' + \hat{\mu}_2 a_2 - \Lambda_2 a_1^2 \sin v_1 = 0 , \quad (7.65)$$

$$a_2(v_1' - 2v_2') + 2\omega_1(\sigma - \delta)a_2 - \Lambda_2 a_1^2 \cos v_1 + S_{21} a_1^2 a_2 + S_{22} a_2^3 = 0 , \quad (7.66)$$

where

$$v_1 = \theta_2 - 2\theta_1 + 2\omega_1 \delta T_1 , \quad v_2 = \omega_1 \sigma T_1 - \theta_1 . \quad (7.67)$$

Then, to the first approximation,

$$v(s,t) = V(s) a_1(\varepsilon t) \cos(\Omega t - v_2) + \dots , \quad (7.68)$$

$$w(s,t) = W(s) a_2(\varepsilon t) \cos(2\Omega t + v_1 - 2v_2) + \dots , \quad (7.69)$$

$$\gamma(s,t) = \Gamma(s) a_1(\varepsilon t) \cos(\Omega t - v_2) + \dots , \quad (7.70)$$

where the a_m and v_m are given by (7.63)-(7.67). Alternatively, one can express the A_n in the Cartesian form

$$A_1 = \frac{1}{2} (p_1 - iq_1) e^{i\sigma\omega_1 T_1} \quad \text{and} \quad A_2 = \frac{1}{2} (p_2 - iq_2) e^{2i\omega_1(\sigma-\delta)T_1} . \quad (7.71)$$

Substituting (7.71) into (7.54) and (7.61) and separating real and imaginary parts, we obtain

$$p_1' = -\hat{\mu}_1 p_1 - \sigma \omega_1 q_1 - \Lambda_1(q_1 p_2 - q_2 p_1) + S_{12}(p_2^2 + q_2^2)q_1 + S_{11}(p_1^2 + q_1^2)q_1, \quad (7.72)$$

$$q_1' = -\hat{\mu}_1 q_1 + \sigma \omega_1 p_1 - \Lambda_1(q_1 q_2 + p_2 p_1) - S_{12}(p_2^2 + q_2^2)p_1 - S_{11}(p_1^2 + q_1^2)p_1 - R, \quad (7.73)$$

$$p_2' = -\hat{\mu}_2 p_2 - 2\omega_1(\sigma - \delta)q_2 + 2\Lambda_2 p_1 q_1 + S_{21}(p_1^2 + q_1^2)q_2 + S_{22}(p_2^2 + q_2^2)q_2, \quad (7.74)$$

$$q_2' = -\hat{\mu}_2 q_2 + 2\omega_1(\sigma - \delta)p_2 + \Lambda_2(q_1^2 - p_1^2) - S_{21}(p_1^2 + q_1^2)p_2 - S_{22}(p_2^2 + q_2^2)p_2. \quad (7.75)$$

Periodic solutions of the beam correspond to the fixed points (i.e., constant solutions) of (7.63)-(7.67), which in turn correspond to $a_m' = v_m' = 0$ or $p_m' = q_m' = 0$. Because the modulation equations (7.72)-(7.75) are first-order autonomous ordinary-differential equations, the stability of a particular fixed point with respect to an infinitesimal disturbance proportional to $\exp(\lambda T_1)$ is determined by the eigenvalues of the Jacobian matrix of the right-hand sides of (7.72)-(7.75). A given fixed point is stable if and only if all the eigenvalues are in the left-half of the complex plane. When a pair of complex conjugate eigenvalues has a positive real part, the fixed-point solutions of the modulation equations are unstable and the response of the beam is an amplitude- and phase-modulated motion.

7.3 Numerical Results

The case under study is primary resonance of the first bending-twisting mode of v and y in the presence of a two-to-one internal resonance between the first modes of

w and v and γ . We assume that $\mu_1 = 0.07$, $\mu_2 = 0.13$, and $\mu_3 = 0.0003$, which are obtained by assuming that the modal damping ratios of v, w, and γ are 0.01, 0.01, and 0.03, respectively. Moreover, we let the force amplitude $f \equiv \hat{f} \int_0^1 V ds = 0.01$ in the numerical analysis.

Because the trace of the Jacobian matrix of (7.72)-(7.75) is equal to $-2(\hat{\mu}_1 + \hat{\mu}_2)$,

$$\sum_{i=1}^4 \lambda_i = -2(\hat{\mu}_1 + \hat{\mu}_2) = -0.20047,$$

where the λ_i are the eigenvalues of the Jacobian matrix and $\hat{\mu}_1$ and $\hat{\mu}_2$ are defined in Appendix D. Hence, the volume of the trajectories in the four-dimensional state space contracts. Thus, any state solution must ultimately lie in a subspace of dimension smaller than four.

We use the false position method [34] to solve (7.63)-(7.66) for the fixed points. Then, we determine the stability of the fixed points by examining the eigenvalues of the Jacobian matrix of (7.72)-(7.75). Figures 7.4(a) and (b) show the frequency-response curves for the internal detuning parameter values $\delta = -0.1$ and 0.1 , respectively. It follows from (7.36) and (7.37) that perfectly tuned resonances correspond to $\sigma = 0$ and $\delta = 0$. Hence, when δ is large, the two flexural modes are greatly detuned and the 'gate' through which the energy can be pumped from the in-plane mode into the out-of-plane mode is narrowed down. Thus, out-of-plane responses are small. But when the natural frequencies of the two flexural modes are closely tuned, Fig. 7.5(a) shows that there are two dominant peaks in the frequency-response curve and the fixed point loses its stability through a Hopf bifurcation at $\sigma = \sigma_1 = -0.01322$ and $\sigma = \sigma_2 = 0.01235$. The effective nonlinearity is of the hardening

type, indicating the dominance of the geometric nonlinearity over the inertial nonlinearity.

Because $D_{13} \neq 0$, v and γ are linearly coupled. Hence, whenever a nontrivial flexural deflection v is produced by an external excitation, a torsional displacement γ is produced by this linear coupling. Moreover, the phase angle between the excitation force and v is the same as that between the excitation force and γ . We note from (7.2) that the quadratic terms

$$\beta_{11}(\gamma'v'')', (\beta_{22} - 1)(\gamma v'')'', \beta_{13}(-\gamma\gamma'' + v''^2 - \gamma'^2)',$$

act as forcing terms in the equation of w . Therefore, the beam vibration is nonplanar due to these nonlinear bending-twisting couplings. Because $\gamma < 0$ when $v > 0$ (see Fig.7.2(a)), if the ξ -axis is constrained on the x - y plane, an extra internal tension stress is created at the side of the beam with $\xi < 0$ and an extra compression stress is created at the side with $\xi > 0$ in order to keep the cross section a plane. Thus, if there is no constraint on the neutral axis, there must be some deflection in the negative z -direction to relieve the tension stress on one side and the compression on the other side. Therefore, if v is nontrivial, γ and w must be nontrivial. The perturbation solution also exhibits this characteristic for the special case of resonant vibration. It follows from (7.65) and (7.66) that there are no solutions for which $a_1 \neq 0$ while $a_2 = 0$; that is, the planar motion is not a solution. Hence the out-of-plane flexural vibration w is a forced rather than a parametric response and the vibration is nonplanar. This behavior is quite different from that of isotropic beams, where the response may be planar or nonplanar depending on the disturbance parameters and initial conditions.

Figure 7.5(b) reveals that for values of σ far below $\sigma = 0$, the phase angle v_2 of v is close to 0° , while the phase angle $2v_2 - v_1$ of w is close to -180° , as predicted by linear theory. Similarly, the lower branch of the response curve for large values of σ has phase angles close to 180° and 0° for v and w , respectively. But for responses with values of σ close to zero or responses corresponding to the upper branch, the phase angles are different from those predicted by linear theory.

Inspecting the values of S_{ij} shown in Appendix D, we note that S_{21} is larger than Λ_2 and S_{12} and S_{11} are larger than Λ_1 . Consequently, the cubic terms may strongly influence the response. Neglecting the cubic nonlinear terms, we obtain the frequency-response curve shown in Fig.7.6. Comparing Fig. 7.5(a) with Fig.7.6 shows that the influence of the cubic terms is not negligible if the response amplitudes are not very small. In Fig. 7.7, we present typical force-response curves. Again, neglecting the cubic terms may result in an overestimation of the response amplitudes and overlooking amplitude- and phase-modulated motions. Hence we include the cubic terms in the analysis.

Next, we investigate the long-time behavior of the solutions of the modulation equations (7.72)-(7.75) when the fixed points are unstable with a pair of complex eigenvalues having a positive real part. To save CPU time and locate stable and unstable limit cycles, we use the algorithm proposed by Chua and Lin [14]. First, we transform the initial-value problem into a boundary-value problem with periodic boundary conditions. Then we use a shooting technique coupled with a Newton-Raphson iteration method to determine the period and a set of initial conditions on the orbit. Because the period is calculated accurately, the leakage problem in the FFT (fast Fourier transform) analyses of the long-time response can

be avoided. But, because the Newton-Raphson method is a first-order approximation, a good guess of the initial conditions and period is needed for convergence. To locate the stable orbits, we integrate (7.72)-(7.75) for a given set of parameters for a long time and use a point on the orbit as well as its period as a guess for locating nearly stable orbits.

Figure 7.8 shows the long-time behavior of the solutions of the modulation equations for $\sigma = 0.00028571$. To investigate the behavior of the system for other values of σ in $[\sigma_1, \sigma_2]$, we calculated the long-time solutions of the modulation equations for fifty values of σ . In Figs. 7.9(a) and (b) we show, respectively, the variation of the period and amplitude of the limit cycle with σ . We note that the period of the limit cycle changes continuously with σ , indicating the absence of period-multiplying bifurcations. This is confirmed by the magnitudes and phase angles of the Floquet multipliers shown in Fig.7.10. Because (7.72)-(7.75) are autonomous, one of the four Floquet multipliers is always unitary, which is not shown in Fig.7.10. We note that all other Floquet multipliers are within the unit circle in the complex plane. At σ_1 and σ_2 , one of the Floquet multipliers leaves the unit circle through $+1$, indicating a jump from quasi-periodic to periodic solutions. Because there are no large-amplitude limit cycles for values of σ near σ_1 and σ_2 , these bifurcations are supercritical Hopf bifurcations [61].

As σ increases beyond σ_1 , a symmetric limit cycle with small amplitude is born, which indicates that it is a supercritical Hopf bifurcation point [61]. Figure 7.11 shows that the small limit cycles born around σ_1 are symmetric with respect to the unstable fixed point, which is indicated by a plus sign. A similar bifurcation occurs at $\sigma = \sigma_2$. As σ increases far beyond σ_1 , the limit cycle increases in size and becomes asymmetric,

as shown in Fig.7.12(a). The appearance of even harmonics in the power spectral density (PSD) of a_1 confirms the asymmetry of the attractor. Figures 7.12(b)-(e) show the limit cycles obtained at different values of σ and the corresponding power spectral densities of the a_1 signal. We note that, although the period of the limit cycle and the number of peaks vary with σ , there are no chaotic motions or period-doubling bifurcations.

For values of σ in the interval (σ_1, σ_2) , the motion of the beam tip is an amplitude- and phase-modulated motion, as shown in Figs.7.13(b)-(e) for the conditions in Figs. 7.12(a), (b), (c), and (e). Here we did not include the component of the base motion $f \cos \Omega t$, which is very small. Figures 7.13(a) and (f) show a periodic tip motion at $\sigma = -0.015664 < \sigma_1$ and $\sigma = 0.014411 > \sigma_2$, respectively. We note that the motion type changes continuously from a downward to an upward motion. This trend also explains the presence of peaks in the response curve. The time period included in Figs. 7.13(b)-(e) is three times the modulation period. The tip motion may be periodic or two-period quasi-periodic, depending on whether the excitation frequency Ω is or is not commensurate with the modulation frequency of the amplitudes and phases.

Checking the tip motion corresponding to the lower branch of the response curve for $\sigma > \sigma_2$, we find that the path of the beam tip is like a horizontal line curved a little bit upward.

For an externally excited nonlinear system, damping significantly influences its response, especially around the resonance conditions. Changing the damping value by a ratio R_{ratio} , we found that the location of the Hopf bifurcation points changes, as shown in Fig. 7.14. Thus, amplitude- and phase-modulated motions can be avoided by increasing the damping coefficient above critical values.

Next, we investigate the influence of decreasing the damping coefficients to about one-half those used in Fig. 7.5(a). Figure 7.15 shows the frequency-response curves when $\mu_1 = 0.03$, $\mu_2 = 0.06$, and $\mu_3 = 0.0002$. Using the shooting technique, we studied the amplitude- and phase-modulated motions at fifty different input frequencies. Figure 7.16(a) shows that the period of the amplitudes and phases does not vary continuously with σ . The jumps correspond to period-doubling bifurcations. Chaotic motions were also found. The dotted lines in Fig. 7.16(b) correspond to chaotic solutions. Figure 7.16(b) shows that $\sigma = \sigma_2 = 0.0176$ is a supercritical Hopf bifurcation point but $\sigma = \sigma_1 = -0.016768$ is a subcritical Hopf bifurcation point.

As σ decreases below σ_2 , a symmetric limit cycle with small amplitude is born, but the symmetry is broken at a value of σ not far from σ_2 , as shown in Fig. 7.17(a). Figures 7.17(b)-(f) show other attractors for different values of σ . From the power spectral density of a_1 shown on the right-hand side, we can see clearly the change in period, the asymmetric property, and the chaotic behavior. To confirm the chaotic nature of the motion corresponding to Fig. 7.17(d), we generated its corresponding Poincare section at $p_2 = 0$ and obtained 2000 points whose projection on the $p_1 - q_1$ plane is shown in Fig. 7.18.

As σ is decreased below σ_1 , we found a period-doubling sequence as shown in Fig. 7.19. The values of σ corresponding to these bifurcations are $\sigma = -0.0172878$, -0.0173411 , -0.0173495 , and -0.0173513 . Using these values, we obtain

$$\frac{\sigma_3 - \sigma_2}{\sigma_4 - \sigma_3} = 4.667 \quad ,$$

which is close to the Feigenbaum universality constant 4.66920. As σ decreases below -0.017365, the solution becomes chaotic and then jumps onto the fixed-point attractor at $\sigma = -0.01738$.

For values of σ greater than but very close to σ_1 , a large-amplitude attractor, shown in Fig. 7.20(a), coexists with a small-amplitude attractor, shown in Fig. 7.20(b). However, the small-amplitude attractor exists only within a small frequency range.

Figure 7.1. Coordinate systems : (a) $x-y-z$ = inertial reference frame, $\xi-\eta-\zeta$ = principal axes of the beam's cross section at position s , (b) $1-2-3$ = material coordinates.

Figure 7.2. (a) The first flexural mode shape $V(s)$ (solid line) and the first flexure-induced torsional mode shape $\Gamma(s)$ (broken line) and (b) the first flexural mode shape $W(s)$.

Figure 7.3. (a) The second flexural mode shape $V(s)$ (solid line) and the second flexure-induced torsional mode shape $\Gamma(s)$ (broken line) and (b) the third flexural mode shape $V(s)$ (solid line) and the third flexure-induced torsional mode shape $\Gamma(s)$ (broken line).

Figure 7.4. Frequency-response curves: (a) $\delta = -0.1$ and (b) $\delta = 0.1$.

Figure 7.5. (a) Frequency-response curves and (b) phase angles : $\delta = -0.01$.

Figure 7.6. Frequency-response curves in the absence of the cubic terms:
 $\delta = -0.01$.

Figure 7.7. Force-response curves when $\delta = -0.01$, $\sigma = 0.02$: (a) in the presence of the cubic terms and (b) in the absence of the cubic terms.

Figure 7.8. Long-time histories of the amplitudes of the two modes when $\sigma = 0.00028571$ and $\delta = -0.01$.

Figure 7.9. (a) Period of the modulation of the amplitudes and (b) the maxima and minima of the amplitudes: $\delta = -0.01$.

Figure 7.10. (a) magnitudes and (b) phases of three Floquet multipliers: $\delta = -0.01$.

Figure 7.11. The two-dimensional projection of the attractor at $\sigma = -0.01322$ onto (a) the $a_1 - a_2$ plane, (b) the $p_1 - p_2$ plane, and (c) the $p_1 - q_1$ plane; and (d) the power spectral density of p_1 ; (+ denotes the unstable fixed-point solution).

Figure 7.12. The two-dimensional projection of the attractor onto the $a_1 - a_2$ plane and the PSD (power spectral density) of amplitude a_1 , when $\delta = -0.01$: (a) $\sigma = -0.012286$, (b) $\sigma = -0.006$, (c) $\sigma = 0.00028571$.

Figure 7.12. continued, (d) $\sigma = 0.0065714$, (e) $\sigma = 0.010571$.

Figure 7.13. The path of the tip end of the beam when $\delta = -0.01$: (a) $\sigma = -0.015664$, (b) $\sigma = -0.012286$, (c) $\sigma = -0.006$, (d) $\sigma = 0.00028571$, (e) $\sigma = 0.010571$, and (f) $\sigma = 0.014411$.

Figure 7.14. The influence of damping on the Hopf bifurcation points when $\delta = -0.01$ and $\mu_1, \mu_2, \mu_3 = R_{\text{ratio}}(0.07, 0.13, 0.0003)$.

Figure 7.15. Frequency-response curves when $\delta = -0.01$, $\mu_1 = 0.03$, $\mu_2 = 0.06$, and $\mu_3 = 0.0002$.

Figure 7.16. (a) Period of the modulation of the amplitudes and (b) the maxima and minima of the amplitudes: $\delta = -0.01$, $\mu_1 = 0.03$, $\mu_2 = 0.06$, and $\mu_3 = 0.0002$.

Figure 7.17. The two-dimensional projection of the attractor onto the $p_1 - p_2$ plane and the PSD (power spectral density) of amplitude a_1 when $\delta = -0.01$, $\mu_1 = 0.03$, $\mu_2 = 0.06$, $\mu_3 = 0.0002$: (a) $\sigma = 0.015091$, (b) $\sigma = 0.01$, (c) $\sigma = 0.0074545$.

Figure 7.17. continued, (d) $\sigma = -0.0056364$, (e) $\sigma = -0.010364$, (f) $\sigma = -0.015455$.

Figure 7.18. The two-dimensional projection of the Poincaré' section taken at $p_2 = 0$ when $\delta = -0.01$, $\mu_1 = 0.03$, $\mu_2 = 0.06$, $\mu_3 = 0.0002$, and $\sigma = -0.0056364$.

Figure 7.19. The two-dimensional projection of the large-amplitude attractors at (a) $\sigma = -0.017$, period T , (b) $\sigma = -0.0173$, period $2T$, (c) $\sigma = -0.017343$, period $4T$, (d) $\sigma = -0.0173508$, period $8T$, (e) $\sigma = -0.01736$, period $16T$, (f) $\sigma = -0.017365$, period $32T$: $\delta = -0.01$, $\mu_1 = 0.03$, $\mu_2 = 0.06$, $\mu_3 = 0.0002$, (+ denotes the stable fixed-point solution).

Figure 7.20. The two-dimensional projection of two different, stable limit-cycle attractors at $\sigma = -0.01659$ and the corresponding path of the beam's tip end when $\delta = -0.01$, $\mu_1 = 0.03$, $\mu_2 = 0.06$, and $\mu_3 = 0.0002$: (a) the large-amplitude attractor and (b) the small-amplitude attractor.

CHAPTER VIII

FORCED CHORDWISE VIBRATIONS OF A COMPOSITE BEAM

In this chapter we investigate the response of the cantilevered composite beam studied in Chapter VII (see Fig. 7.1) subject to a harmonic base-excitation along the z direction. The analysis focuses on the case of primary resonance of the first flexural mode when its frequency is approximately twice the frequency of the first out-of-plane flexural-torsional mode. Hopf bifurcations, symmetry-breaking bifurcations, amplitude- and phase-modulated motions, period-multiplying sequences, and chaotic motions are studied.

8.1 Multiple Scale Analysis

Expressing the relative displacement $\bar{w}(s,t)$ as

$$\bar{w}(s,t) = w(s,t) - F \cos \Omega t$$

and using the characteristic length L and the characteristic time $L^2 \sqrt{m/D_{33}}$ to normalize the variables, one can obtain the equations of motion, which are the same as (7.1)-(7.3) except that the forcing term $\hat{f} \Omega^2 \cos \Omega t$ appears in the equation governing $w(s,t)$ instead of the equation governing $v(s,t)$.

Because the input frequency is different from the one used in Chapter VII, the detuning parameters σ and δ are defined by

$$\Omega = \omega_2(1 + \varepsilon \sigma) , \quad (8.1a)$$

$$\omega_2 = 2\omega_1(1 + \varepsilon \delta) . \quad (8.1b)$$

Following the same procedure used in Chapter VII, we obtain the first-order asymptotic solution

$$v(s,t) = V(s)a_1(\varepsilon t) \cos\left(\frac{\Omega}{2}t - \frac{v_1 + v_2}{2}\right) + \dots , \quad (8.2a)$$

$$w(s,t) = W(s)a_2(\varepsilon t) \cos(\Omega t - v_2) + \dots , \quad (8.2b)$$

$$\gamma(s,t) = \Gamma(s)a_1(\varepsilon t) \cos\left(\frac{\Omega}{2}t - \frac{v_1 + v_2}{2}\right) + \dots . \quad (8.2c)$$

The equations governing the amplitudes a_i and phases v_i are

$$a_1' + \hat{\mu}_1 a_1 + \Lambda_1 a_1 a_2 \sin v_1 = 0 , \quad (8.3a)$$

$$\frac{1}{2} a_1 (v_1' + v_2') - (\sigma + \delta) \omega_1 a_1 + \Lambda_1 a_1 a_2 \cos v_1 + S_{11} a_1^3 + S_{12} a_1 a_2^2 = 0 , \quad (8.3b)$$

$$a_2' + \hat{\mu}_2 a_2 - \Lambda_2 a_1^2 \sin v_1 + \hat{R} \sin v_2 = 0 , \quad (8.3c)$$

$$a_2 v_2' - 2\omega_1 \sigma a_2 + \Lambda_2 a_1^2 \cos v_1 + S_{21} a_1^2 a_2 + S_{22} a_2^3 + \hat{R} \cos v_2 = 0 , \quad (8.3d)$$

where $\hat{\mu}_1, \hat{\mu}_2, \Lambda_1, \Lambda_2, S_{11}, S_{12}, S_{21}, S_{22}$, and $\hat{R}$ are defined in Appendix D, and

$$v_1 = \theta_2 - 2\theta_1 + 2\omega_1 \delta T_1 , \quad v_2 = \omega_2 \sigma T_1 - \theta_2 . \quad (8.4)$$

Periodic solutions of the beam correspond to the fixed points (i.e., constant solutions) of (8.3a)-(8.3d), which in turn correspond to $a_1' = a_2' = v_1' = v_2' = 0$.

Introducing the Cartesian coordinates p_i and q_i defined by

$$p_1 = a_1 \cos\left(\frac{v_1 + v_2}{2}\right) , \quad q_1 = a_1 \sin\left(\frac{v_1 + v_2}{2}\right) , \quad (8.5a)$$

$$p_2 = a_2 \cos v_2 , \quad q_2 = a_2 \sin v_2 , \quad (8.5b)$$

into the averaged equations (8.3), we obtain

$$p_1' = -\hat{\mu}_1 p_1 - \omega_1 (\sigma + \delta) q_1 - \Lambda_1 (q_1 p_2 - q_2 p_1) + S_{12} (p_2^2 + q_2^2) q_1 + S_{11} (p_1^2 + q_1^2) q_1 , \quad (8.6a)$$

$$q_1' = -\hat{\mu}_1 q_1 + \omega_1 (\sigma + \delta) p_1 - \Lambda_1 (q_1 q_2 + p_2 p_1) - S_{12} (p_2^2 + q_2^2) p_1 - S_{11} (p_1^2 + q_1^2) p_1 , \quad (8.6b)$$

$$p_2' = -\hat{\mu}_2 p_2 - 2\omega_1 \sigma q_2 + 2\Lambda_2 p_1 q_1 + S_{21} (p_1^2 + q_1^2) q_2 + S_{22} (p_2^2 + q_2^2) q_2 , \quad (8.6c)$$

$$q_2' = -\hat{\mu}_2 q_2 + 2\omega_1 \sigma p_2 + \Lambda_2(q_1^2 - p_1^2) - S_{21}(p_1^2 + q_1^2)p_2 - S_{22}(p_2^2 + q_2^2)p_2 - \hat{R} . \quad (8.6d)$$

8.2 Numerical Results

We study the case of primary resonance of the first mode of w in the presence of a two-to-one internal resonance between this mode and the first bending-twisting mode of v and γ . We assume that the damping coefficients are $\mu_1 = 0.07$, $\mu_2 = 0.13$, and $\mu_3 = 0.0003$.

When the excitation amplitude $f \equiv \hat{f} \int_0^1 W ds = 0.005$, Figs. 8.1(a) and (b) show the frequency-response curves when the internal-detuning parameter $\delta = -0.11$ and -0.093 , respectively. When δ is a large negative number, the two flexural modes are greatly detuned and there is only one branch for the out-of-plane response curve, which is around the resonant point of $v(s,t)$. When δ is increased to -0.093 , another branch of the out-of-plane response curve appears, as shown in Fig. 8.1(b). As δ is increased to 0.05 , the two branches of the nonplanar response curves merge, as shown in Fig. 8.2(a). The fixed-point solution loses its stability through a Hopf bifurcation at $\sigma = \sigma_1 = -0.00725$ and $\sigma = \sigma_2 = 0.033588$, and amplitude- and phase-modulated motions are expected in the interval $\sigma_1 < \sigma < \sigma_2$. We note that the out-of-plane motion is excited through the autoparametric resonance. Figure 8.2(a) shows that the nonlinearity is of the hardening type. The possible planar response a_2 is essentially the linear solution because the amplitude of the base motion is small and the bending stiffness in the z -direction is large. The variation of the phase v_2 with

σ shown in Fig. 8.2(b) also confirms the linearity of the chordwise planar vibration because the phase angle changes from 0° to 180° around the resonant point $\delta = 0$. Furthermore, for nonplanar motions, the amplitude a_2 of the directly excited mode is much smaller than that of the indirectly excited mode a_1 through the autoparametric resonance. It indicates that most of the input energy is spilled over into the lower mode. Figure 8.3 shows the frequency-response curves in the absence of the cubic nonlinear terms. Comparing Fig.8.2(a) with Fig.8.3 shows that the cubic terms reduce the maximum value of a_1 , increase the frequency range between the Hopf bifurcation points σ_1 and σ_2 , and change the left branch of the nonplanar response curve from a softening into a hardening type.

Using the excitation amplitude f as a bifurcation parameter, we obtained the force-response curves shown in Figs. 8.4 and 8.5. Figure 8.4 shows representative curves [72] for $\chi < 0$, where $\chi = 2\sigma\omega_i^2(\sigma + \delta) - \hat{\mu}_1\hat{\mu}_2$, and Fig.8.5 shows representative curves for $\chi > 0$. Figure 8.4(b) shows that, without the cubic terms, the amplitude a_2 of the in-plane mode saturates when the forcing amplitude increases beyond the threshold value f_1 ; the extra energy is spilled over into the lower mode. However, Fig. 8.4(a) shows that the cubic terms reduce the threshold value of f_1 for the onset of out-of-plane motions, destabilize some nonplanar periodic motions by producing amplitude- and phase-modulated motions, and reduce the amplitude of the directly excited mode when the input force increases beyond the threshold value f_1 . Furthermore, for large levels of excitation, a second branch of out-of-plane motions is produced by the cubic terms. In addition to the saturation phenomenon, Fig. 8.5(b) exhibits jumps at f_2 and f_3 . Again, Fig. 8.5(a) shows that the cubic terms reduce the threshold value for the onset of out-of-plane motions, destabilize some nonplanar periodic motions by producing amplitude- and phase-modulated motions, and reduce

the amplitude of the directly excited mode when the input force increases beyond f_3 . We note that $f = f_3$ is a subcritical (reversed) pitchfork bifurcation point at which one of the real eigenvalues of the Jacobian matrix of (8.6) changes sign and there is a change in the number of fixed points. The value $f = f_2$ is a nonlocal bifurcation point at which there is a change in the number of fixed points. For $f_2 < f < f_3$, there are three possible solutions: one stable planar solution, one stable nonplanar solution, and an unstable nonplanar solution. The response of the beam will depend on the initial conditions. For $f > f_3$, the only possible response is nonplanar because the planar one is unstable. For $f < f_2$, a planar flexural vibration is the only solution, and it is linear.

Changing the damping coefficient by a ratio R_{ratio} , we found that the locations of the Hopf bifurcation points change, as shown in Fig. 8.6. We note that increasing the damping coefficient above a critical value results in the elimination of amplitude- and phase-modulated motions.

We studied amplitude- and phase-modulated motions for fifty different input frequencies and obtained Fig. 8.7, which shows the periods and the maxima and minima of the amplitudes of limit cycles. Between $\sigma = -0.005$ and $\sigma = 0.01$, we note that when σ increases, the system responds with increasing amplitudes but with an almost constant frequency. It shows that the response is an amplitude- and phase-modulated motion. Because there are no large-amplitude limit cycles around the Hopf bifurcation points $\sigma_1 = -0.00725$ and $\sigma_2 = 0.033588$, these bifurcations are supercritical Hopf bifurcations. In Fig. 8.8 we show the magnitudes and phase angles of the Floquet multipliers. Because (8.6) is autonomous, one of the four Floquet multipliers is always unity, which is not shown in Fig. 8.8. For $0.016 < \sigma < 0.024$, the

Floquet multipliers change very quickly when σ is varied, and one of them leaves the unit circle in the complex plane through $+1$ or -1 very frequently. And, at σ_1 and σ_2 , one of the Floquet multipliers leaves the unit circle through $+1$, which indicates that the solutions jump from quasi-periodic to periodic ones.

Inspecting the modulation equations (8.6), we note that they are invariant under the transformation $(p_1, q_1, p_2, q_2) \rightarrow (-p_1, -q_1, p_2, q_2)$. Thus, there is an inversion symmetry in the responses of $v(s,t)$ and $\gamma(s,t)$. Because $w(s,t)$ is externally excited, it does not possess an inversion symmetry and the phase angle ν_2 between the input force and $w(s,t)$ is unique. Figures 8.9(a)-(n) show representative solutions existing between the Hopf bifurcation points σ_1 and σ_2 . On the right-hand side is the power spectral density of p_1 . As σ increases beyond σ_1 , a small limit cycle, which is symmetric with respect to the unstable fixed-point solution, is born, indicating that $\sigma = \sigma_1$ is a supercritical Hopf bifurcation point. The symmetry of the limit cycle is broken at a value of σ not far away from σ_1 , as shown in Fig. 8.9(a), where the two plus signs indicate a pair of unstable fixed-point solutions. Figure 8.9(b) shows a pair of asymmetric attractors, which coexist in the interval $0.011457 < \sigma < 0.013226$. Then, these two attractors merge and become a symmetric attractor with respect to $p_1 = 0$, as shown in Fig. 8.9(c). When σ increases to 0.015658, the attractor loses its symmetry, as shown in Fig. 8.9(d). As σ increases further, the attractor becomes chaotic in the interval $0.015658 < \sigma < 0.019638$; Fig. 8.9(e) shows the chaotic attractor at $\sigma = 0.015879$. At $\sigma = 0.019638$, it becomes periodic, as shown in Fig. 8.9(f). When $0.019638 < \sigma < 0.022955$, the attractor undergoes a few period-doubling bifurcations and then a few reverse period-doubling bifurcations, as shown in Figs. 8.9(g)-(i). When $\sigma > 0.022955$, the attractor becomes chaotic as shown Fig. 8.9(j). It is a Lorenz type chaotic attractor since it wanders about the two unstable fixed-point solutions.

Increasing σ slightly, we find that the chaotic attractor splits into two, as shown in Fig. 8.9(k), each of them wanders about one of the fixed-point solutions. They are of the Rossler type. As σ increases further, these two attractors become periodic and undergo a sequence of reverse period-doubling bifurcations, as shown in Figs. 8.9(l)-(n). When σ increases beyond $\sigma_2 (= 0.033588)$, the small limit-cycle symmetric attractors jump onto point attractors.

Figures 8.10(a) and (f) show the periodic tip motion for $\sigma = -0.01 < \sigma_1$ and $\sigma = 0.04 > \sigma_2$, respectively. The periodic responses are stationary and appear as a figure eight since the period of one mode is exactly twice that of the other mode. Figure 8.10(b)-(e) show the trajectories of the tip end of the beam, corresponding to the amplitude- and phase-modulated motions shown in Figs. 8.9(a), (c), (m), and (n). The figure-eight response evolves continuously because the two modes are constantly exchanging energy.

Checking the tip motion corresponding to the left branch of the nonplanar response curve in Fig. 8.2(a), we find that the path of the tip end is curved downward, as shown in Fig. 8.11. It can be seen by considering the bending-twisting coupling that this motion needs less energy for its excitation than the one curved upward.

Figure 8.1. Frequency-response curves: (a) $\delta = -0.11$, (b) $\delta = -0.093$.

Figure 8.2. (a) Frequency-response curves and (b) phase angles : $\delta = 0.05$.

Figure 8.3. Frequency-response curves obtained by neglecting the cubic terms:
 $\delta = 0.05$.

Figure 8.4. Force-response curves obtained by (a) including the cubic terms and (b) neglecting the cubic terms: $\delta = 0.05$, $\sigma = 0.0$.

Figure 8.5. Force-response curves obtained by (a) including the cubic terms and (b) neglecting the cubic terms: $\delta = 0.05$, $\sigma = 0.015$.

Figure 8.6. The influence of damping on the Hopf-bifurcation points: $\delta = 0.05$, $\mu_1, \mu_2, \mu_3 = R_{\text{ratio}} \chi(0.07, 0.13, 0.0003)$.

Figure 8.7. (a) Period of the amplitude and phase modulations and (b) the maxima and minima of amplitudes: $\delta = 0.05$.

Figure 8.8. (a) the magnitudes and (b) the phase angles of three Floquet multipliers: $\delta = 0.05$.

Figure 8.9. The two-dimensional projection of the attractor onto the $p_1 - p_2$ plane and the PSD (power spectral density) of p_1 , when $\delta = 0.05$: (a) $\sigma = -0.0062857$, (b) $\sigma = 0.012572$, (c) $\sigma = 0.015143$, (+ denotes the unstable fixed-point solution).

Figure 8.9. continued, (d) $\sigma = 0.015658$, (e) $\sigma = 0.015879$, (f) $\sigma = 0.019638$.

Figure 8.9. continued, (g) $\sigma = 0.020286$, (h) $\sigma = 0.02205$, (i) $\sigma = 0.022734$.

Figure 8.9. continued, (j) $\sigma = 0.023584$, (k) $\sigma = 0.023617$, (l) $\sigma = 0.02381$.

Figure 8.9. continued, (m) $\sigma = 0.02406$, (n) $\sigma = 0.028857$.

Figure 8.10. The path of the tip end of the beam: $\delta = 0.05$, (a) $\sigma = -0.01$, (b) $\sigma = -0.0062857$, (c) $\sigma = 0.015143$, (d) $\sigma = 0.02406$, (e) $\sigma = 0.028857$, and (f) $\sigma = 0.04$.

Figure 8.11. The path of the tip end of the beam: $\delta = 0.05$, $\sigma = -0.064548$.

CHAPTER IX

CONCLUSIONS AND RECOMMENDATIONS

9.1 *Summary and Conclusions*

A combination of Newton's second law and a transformation using three consecutive Euler angles is used to develop complete equations describing the flexural-flexural-torsional vibration of metallic and composite beams. Taylor series and an ordering scheme are then used to expand the obtained transcendental PDE's into third-order nonlinear PDE's. The definition of twisting curvature and a specified geometric boundary condition are used to define a physical twisting variable, which makes the equations of motion unique and independent of the rotation sequence of the three Euler angles used in the mathematical modeling.

A numerical-perturbation approach is used to obtain first-order asymptotic solutions of these equations. First, the linear equations of motion are solved for the linear

eigenfunctions and natural frequencies. For a metallic beam, the method of undetermined coefficients is used to solve for the linear eigensolutions with or without rotary inertias. For a composite beam exhibiting bending-twisting coupling, a combination of the state-space concept and the fundamental-matrix method is used to solve for eigensolutions. For a nonuniform beam, a finite-element method can be used to calculate the eigensolutions. Then, the method of multiple scales is used to construct a set of nonlinear autonomous first-order ordinary-differential equations describing the slow-time modulation of the amplitudes and phases of the interacting modes.

The inversion symmetry, D_4 symmetry, and $O(2)$ symmetry of the system are identified from the modulation equations. Three types of solutions of the modulation equations are found: (a) fixed-point solutions corresponding to constant-amplitude harmonic vibrations, (b) limit-cycle solutions corresponding to periodically modulated or quasi-periodic vibrations, and (c) chaotically modulated oscillations. To study the behavior of these solutions, we use the frequency and amplitude of excitation as bifurcation (control) parameters. The local stability of a fixed point is determined by the eigenvalues of the Jacobian matrix of the modulation equations. A fixed point can lose stability in one of two ways: (a) a real eigenvalue crosses the imaginary axis into the right-half of the complex plane along the real axis, and (b) a pair of complex conjugate eigenvalues crosses the imaginary axis transversely into the right-half plane. The former is associated with the saddle-node bifurcation (jump) while the latter is associated with the Hopf bifurcation. An unstable fixed point with a pair of complex conjugate eigenvalues having a positive real part corresponds to an amplitude- and phase-modulated motion, and the modulations may be periodic or chaotic. Limit-cycle attractors are located and their periods are calculated by using

a fifth- and sixth-order Runge-Kutta algorithm and a shooting technique coupled with a Newton-Raphson iteration method. The stability of a limit-cycle attractor is determined by using Floquet theory. If all the Floquet multipliers (except one of them being one due to the fact that the modulation equations are autonomous) are within the unit circle in the complex plane, the periodic solution is stable; when at least one of the Floquet multipliers is outside the unit circle, the periodic solution is unstable. As a control parameter is varied, if any Floquet multiplier leaves the unit circle along the real axis through $+1$, a saddle-node (cyclic-fold) bifurcation occurs; If any Floquet multiplier leaves the unit circle along the real axis through -1 , a period-doubling bifurcation occurs; and if a pair of complex conjugate Floquet multipliers leaves the unit circle, a Neimark bifurcation (secondary Hopf bifurcation) occurs. Because the period of a chaotic attractor (strange attractor) is infinite, it cannot be located by the shooting technique. In this thesis, we identify a chaotic attractor by its broadband frequency spectrum obtained from an FFT algorithm and by its Poincare map.

We study the effects of nonlinear inertia, nonlinear curvature, bending-twisting couplings, gravitational force, rotary inertia, modal dampings, one-to-one and two-to-one internal resonances, frequency bias, principal-parametric and primary-resonant excitations, and the excitation amplitude and frequency. We also identify several nonlinear phenomena: jumps, pitchfork bifurcations, saturation, quenching, supercritical and subcritical Hopf bifurcations, limit cycles, symmetry-breaking bifurcations, period-multiplying bifurcations, saddle-node bifurcations, and chaos.

9.1.1 Parametrically Excited Flexural-Flexural Vibrations of an Isotropic Beam

The proposed numerical-perturbation approach is used to analyze the nonlinear response of an inextensional cantilever beam to a principal parametric base excitation. The analysis focuses on the modal interaction of two flexural modes in the two principal planes in the presence of a one-to-one internal (autoparametric) resonance. Because the beam is very long, the first torsional frequency is relatively higher than those of the studied flexural modes and hence there is no torsional mode sustained due to the presence of torsional damping. Four first-order ordinary-differential equations are derived for the modulation of the amplitudes and phases of the two interacting modes with damping, cubic inertial and geometric nonlinearities, and the parametric resonance. Owing to the inversion symmetry of the bendings in both principal planes, there coexist four possible solutions to the modulation equations. For a fixed point, these four solutions reduce to two.

The results show that motions in one plane may be destabilized by disturbances in the second plane, resulting in vibrations in both planes. Moreover, for certain beam and load parameters, nonplanar periodic responses consisting of steady whirling motions are possible. Furthermore, as one of the control parameters varies, the latter steady whirling motions may lose their stability via a Hopf bifurcation, resulting in quasi-periodic whirling motions whose amplitudes and phases vary with time rather than being constant. For certain beam and load parameters, the whirling response may be a chaotically-modulated motion.

The inertia nonlinearity is of the softening type, whereas the geometric nonlinearity is of the hardening type. Consequently, the overall effective nonlinearity may be hardening or softening depending on the order of the mode. For the first mode, the effective nonlinearity is of the hardening type, whereas for the second and higher modes it is of the softening type. Therefore, neglecting the geometric nonlinearity leads to qualitatively erroneous frequency-response curves for the first mode and underpredicts the response amplitudes of the second and higher modes. The error decreases as the mode number increases. These conclusions hold for planar motions in which only one mode is excited. However, neglecting the geometric nonlinearity cannot account for nonplanar motions for all modes, at least for the cases considered, and hence produces erroneous results. Consequently, the nonlinear geometric terms must be included in any analysis of cantilever beams.

9.1.2 Parametrically Excited Flexural-Flexural-Torsional Vibrations of an Isotropic Beam

The numerical-perturbation method is used to study the nonlinear vibration of an inextensional cantilever beam subject to an axial harmonic excitation. The study focuses on the case of principal parametric vibrations of two flexural modes in the presence of a one-to-one internal resonance when the first torsional mode is indirectly excited by the two flexural modes through a two-to-one internal resonance. Six first-order ordinary-differential equations are derived for the modulation of the amplitudes and phases of the three interacting modes with damping, inertial and geometric nonlinearities, and the parametric resonance.

Some symmetry properties of the system are studied using the modulation equations. Although the modulation equations are invariant under three different transformations, there are only four possible solutions because the torsional mode is indirectly excited by the two flexural modes. The analysis shows that torsional motions of beams with near square cross-sections cannot be excited by a longitudinal principal parametric base-excitation.

Only the numerical results for a beam with a near square cross-section are presented. The results are almost the same as those obtained in Chapter III. But, it is shown that the rotary inertias have negligible effects on the fixed-point solutions, but shift slightly the natural frequencies and change the frequency range where limit-cycle solutions exist and the periods, amplitudes, and period-multiplying sequences of the limit cycles.

9.1.3 Forced Flexural-Flexural Vibrations of an Isotropic Beam

The nonlinear planar and nonplanar motions of a long cantilever beam subject to a lateral harmonic base excitation is analyzed. The analysis focuses on the primary resonance of the in-plane flexural mode with a one-to-one autoparametric resonance between the in-plane mode and an out-of-plane flexural mode. Four first-order ordinary-differential equations are derived for the modulation of the amplitudes and phases of the two interacting flexural modes with damping, nonlinearities, and resonances.

Because of the inversion symmetry of the out-of-plane vibration, there are two possible solutions to the modulation equations. The influence of gravity is found to

be negligible. Because the rotary inertias are small, the rotary inertia moment is only important for very high-frequency modes. Furthermore, the natural frequency detuning (bias) δ_2 controls the shift in the unstable frequency range of the planar response curve and hence the development of an interval of nonplanar motions. Damping can suppress and stabilize planar responses. By changing the natural frequency detuning and damping coefficient, one can suppress nonplanar motions.

The results show that in-plane vibrations may be destabilized by out-of-plane disturbances, resulting in vibrations in both planes. For certain beam and load parameters, nonplanar responses can be (a) steady whirling motions (the principal axes of the elliptical route are not parallel to the y- or z-axes), (b) unsteady whirling motions (lengths and directions of the axes of the elliptical route continuously change and the motion also shows a low-frequency twisting motion), or (c) chaotically modulated motions. The unsteady whirling motions may be periodic if the period of the amplitudes is commensurable with that of the base motion.

For a rectangular beam, steady whirling motions exist in a large frequency range below $\sigma = 0$ (σ is the detuning of the primary resonance) because the torsional rigidity of a rectangular compact beam is very small. However, the torsional inertia may not be negligible for rectangular beams. Around the resonant point $\sigma = 0$, most of the nonplanar motions correspond to chaotically modulated motions for the cases we investigated.

9.1.4 Forced Flapwise Vibrations of a Composite Beam

Three nonlinear integro-differential equations of motion are used to investigate the forced nonlinear vibration of a symmetrically laminated graphite-epoxy composite beam. The analysis focuses on the case of primary resonance of the first flexural-torsional mode when its frequency is approximately one-half the frequency of the first out-of-plane flexural mode. A combination of the fundamental matrix method and the method of multiple scales is used to derive four first-order ordinary-differential equations describing the modulation of the amplitudes and phases of the interacting modes with damping, nonlinearity, and resonances. The eigenvalues of the Jacobian matrix of the modulation equations are used to determine the stability of constant solutions, and Floquet theory is used to determine the stability and bifurcations of limit-cycle solutions. Hopf bifurcations, symmetry-breaking bifurcations, amplitude- and phase-modulated motions, period-multiplying sequences, and chaotic motions are studied.

Because of the bending-twisting coupling, the motion is nonplanar although the input force is planar. The quadratic nonlinearity of the bending-twisting coupling breaks the inversion symmetry of the out-of-plane vibration. The two Hopf bifurcation points on the frequency-response curve are found to be supercritical. For an excitation frequency between these two bifurcation points, quasi-periodic vibrations are found and the motion type is changed continuously. Increasing the damping coefficients decreases the frequency range of amplitude- and phase-modulated motions. When the damping coefficients decrease by about a half, one of the Hopf bifurcation points becomes subcritical and large-amplitude limit cycles are found in its neighborhood. Furthermore, around the subcritical Hopf bifurcation point the large-amplitude limit

cycle goes through a cascade of period-doubling bifurcations culminating in chaos, and the bifurcation values of the input frequency confirm the Feigenbaum universality constant. Some chaotic attractors are found between two frequency ranges of limit-cycle attractors having finite periods, which indicates a saddle-node (cyclic-fold) bifurcation.

For certain beam and load parameters, nonplanar responses may be periodic whirling motions with the path of the end of the beam being like a figure eight, amplitude- and phase-modulated motions, or chaotically modulated motions.

9.1.5 Forced Chordwise Vibrations of a Composite Beam

The forced chordwise nonlinear vibration of a symmetrically laminated graphite-epoxy composite beam is studied. The analysis focuses on the case of primary resonance of the first flexural mode along the chord direction when its frequency is approximately two times the frequency of the first out-of-plane flexural-torsional mode. A combination of the fundamental-matrix method and the method of multiple scales is used to derive four first-order ordinary-differential equations describing the modulation of the amplitudes and phases of the interacting modes with damping, nonlinearity, and resonances. The eigenvalues of the Jacobian matrix of the modulation equations are used to determine the stability of constant solutions, and Floquet theory is used to determine the stability and bifurcations of limit-cycle solutions. Hopf bifurcations, symmetry-breaking bifurcations, amplitude- and phase-modulated motions, period-multiplying sequences, and chaotic motions are studied.

Because of the inversion symmetry of the out-of-plane flexural-torsional vibration, two solutions coexist. The results show that the motion can be planar and/or nonplanar although the input force is planar, and the steady response depends on their basins of attraction. Nonplanar responses may be periodic, amplitude- and phase-modulated, or chaotically modulated motions.

In the presence of the cubic coupling terms, the response of the directly excited flexural mode decreases instead of being saturated as the amplitude of the input force increases. Both of the Hopf bifurcation points on the frequency-response curve are supercritical, and limit cycles and chaotic attractors exist between these two bifurcation points. Two routes to chaos are found. First, an attractor loses its inversion symmetry and then jumps onto a chaotic attractor. Second, an attractor undergoes a sequence of period-doubling bifurcations, culminating in a chaotic attractor.

9.2 *Recommendations*

Some issues recommended for further study are:

(a) Investigate the dynamic response of unsymmetrically laminated composite beams, including extension-twisting structural couplings, by accounting for extensionality in the formulation.

(b) Experimentally check the present results of modal couplings and determine a criterion for the closeness of the two natural frequencies for an internal resonance to exist.

(c) Develop a better model to extend the present formulation to include pretwist, first-order shear deformation, and offsets between area centroid, mass centroid, and elastic centroid axes. It is expected that the equations of motion will be many and long. A symbolic manipulator is recommended for this work.

(d) Use experiments to determine the damping coefficients of the various modes of a composite beam because they usually change from mode to mode and control the frequency range of nontrivial parametric responses. Investigate the influence of general damping because it might produce more complex dynamic behaviors than those we found. For a slewing beam with pretwist angle (e.g., helicopter rotor blades), air damping or aerodynamic loads need to be included.

(e) Determine a strategy for controlling the identified undesirable dynamic behaviors, such as, amplitude-modulated motions and chaotically modulated motions.

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Appendix A

DEFINITION OF CURVATURES

According to the definition of curvatures, we write

$$\tilde{\rho}_\xi \equiv \frac{\partial \vec{i}_\eta}{\partial \tilde{s}} \cdot \vec{i}_\zeta, \quad \tilde{\rho}_\eta \equiv -\frac{\partial \vec{i}_\xi}{\partial \tilde{s}} \cdot \vec{i}_\zeta, \quad \tilde{\rho}_\zeta \equiv \frac{\partial \vec{i}_\xi}{\partial \tilde{s}} \cdot \vec{i}_\eta, \quad (A1)$$

where a dot signifies the inner product of two vectors. The sign of the curvatures $\rho_\xi, \rho_\eta, \rho_\zeta$ are defined in accordance with the angular velocities $\omega_\xi, \omega_\eta, \omega_\zeta$, respectively.

Substituting (2.1) and (2.2) into (A1) and using the fact that the base vectors along coordinates x, y , and z are not functions of $\tilde{s}$, we obtain equations (2.7)-(2.9).

Appendix B

COEFFICIENTS FOR EQUATIONS (3.18) and (5.18)

$$\beta_1 = \int_0^1 W_n [V_m'' \int_1^s V_m'' W_n'' ds - V_m''' \int_0^s W_n'' V_m' ds]' ds$$

$$\beta_2 = \int_0^1 W_n [V_m'' \int_0^s \int_1^s V_m'' W_n'' ds ds]' ds$$

$$\beta_3 = \int_0^1 W_n [W_n' (V_m' V_m'')]'] ds$$

$$\beta_4 = \int_0^1 W_n [W_n' (W_n' W_n'')]'] ds$$

$$\beta_5 = \int_0^1 W_n [W_n' \int_1^s \int_0^s W_n'^2 ds ds]' ds$$

$$\beta_6 = \int_0^1 W_n [W_n' \int_1^s \int_0^s V_m'^2 ds ds]' ds$$

$$\beta_7 = \int_0^1 [(s-1)W_n W_n'' + W_n W_n'] ds$$

$$\beta_8 = \int_0^1 W_n ds$$

$$\alpha_i \text{ for mode } (m,n) = \beta_i \text{ for mode } (n,m)$$

$$R_1 = \mu \omega_{1m}$$

$$R_2 = \frac{1}{4} [-2\alpha_6 \omega_{2n}^2 + \delta_0 \alpha_1 + \frac{\delta_0^2}{\beta_\gamma} \alpha_2 + (1 + \delta_0) \alpha_3]$$

$$R_3 = 2\omega_{1m} \omega_{2n} \sigma - \delta_2 z^4$$

$$R_4 = \frac{1}{2} \alpha_5 \omega_{1m}^2 - \frac{3}{4} (1 + \delta_0) \alpha_4$$

$$R_5 = \frac{1}{2} (\delta_0 \alpha_1 + \frac{\delta_0^2}{\beta_\gamma} \alpha_2 + (1 + \delta_0) \alpha_3)$$

$$R_6 = 2\alpha_7 f \omega_{2n}^2$$

$$E_1 = \mu \omega_{2n}$$

$$E_2 = \frac{1}{4} (2\beta_6 \omega_{1m}^2 + \delta_0 \beta_1 - \frac{\delta_0^2}{\beta_\gamma} \beta_2 - \beta_3)$$

$$E_3 = 2\omega_{2n}^2 \sigma$$

$$E_4 = \frac{1}{2} \beta_5 \omega_{2n}^2 - \frac{3}{4} \beta_4$$

$$E_5 = \frac{1}{2} (\delta_0 \beta_1 - \frac{\delta_0^2}{\beta_\gamma} \beta_2 - \beta_3)$$

$$E_6 = 2\beta_7 f \omega_{2n}^2$$

Appendix C

COEFFICIENTS FOR EQUATIONS (4.45)-(4.50)

$$R_2 = 2\alpha_4 f \omega_{1m}^2$$

$$R_3 = \frac{1}{2} [-\beta_y \alpha_{11} + (1 - \beta_y) \alpha_{12}]$$

$$R_4 = \frac{1}{4} [(\beta_y - 1) \alpha_7 - \beta_y \alpha_8 + 4\alpha_9 \omega_{2n}^2]$$

$$R_5 = -2\omega_{1m}^2 \sigma + \delta_2 \alpha_2 + j_3 \omega_{1m}^2 \alpha_3$$

$$R_6 = \frac{1}{4} (-3\beta_y \alpha_5 + 4\alpha_6 \omega_{1m}^2)$$

$$R_7 = \frac{1}{2} [(\beta_y - 1) \alpha_7 - \beta_y \alpha_8]$$

$$R_8 = \frac{1}{2} (1 - \beta_y) \alpha_{10}$$

$$E_2 = 2\alpha_{24} f \omega_{2n}^2$$

$$E_3 = \frac{1}{2} [\beta_y \alpha_{31} + (1 - \beta_y) \alpha_{32}]$$

$$E_4 = \frac{1}{4} [(1 - \beta_y) \alpha_{27} - \alpha_{28} + 4\alpha_{29} \omega_{1m}^2]$$

where

$$\alpha_2 = \int_0^1 V_m' V_r ds$$

$$\alpha_3 = \int_0^1 V_m'' V_r ds$$

$$\alpha_4 = \int_0^1 [(s - 1) V_m'' + V_m'] V_r ds$$

$$E_5 = \omega_{2n}^2 (j_2 \alpha_{23} - 2\sigma)$$

$$E_6 = \frac{1}{4} (-3\alpha_{25} + 4\alpha_{26} \omega_{2n}^2)$$

$$E_7 = \frac{1}{2} [(1 - \beta_y) \alpha_{27} - \alpha_{28}]$$

$$E_8 = \frac{1}{2} (1 - \beta_y) \alpha_{30}$$

$$L_2 = \frac{1}{2} \left[-4\omega_{1m}^2 \alpha_{47} + \frac{1 - \beta_y}{j_1} \alpha_{48} \right. \\ \left. + \alpha_{49} \omega_{1m}^2 \left(2 + \frac{j_2 - j_3}{j_1} \right) \right]$$

$$L_3 = \omega_{3k}^2 \sigma + \delta_1 \alpha_{42}$$

$$L_4 = \frac{1}{2} \left(\frac{\beta_y - 1}{j_1} \alpha_{43} + \frac{j_2 - j_3}{j_1} \alpha_{44} \omega_{1m}^2 \right)$$

$$L_5 = \frac{1}{2} \left[\frac{1 - \beta_y}{j_1} \alpha_{45} - \frac{j_2 - j_3}{j_1} \alpha_{46} \omega_{2n}^2 \right]$$

$$\alpha_5 = \int_0^1 [V_m' (V_p' V_q'')] V_r ds$$

$$\alpha_6 = \frac{1}{2} \int_0^1 (V_m' \int_1^s \int_0^s V_p' V_q' ds ds) V_r ds$$

$$\alpha_7 = \int_0^1 (W_n''' \int_0^s V_m'' W_p' ds) V_r ds$$

$$\alpha_8 = \int_0^1 [V_m'(W_n'W_p'')]'V_r ds$$

$$\alpha_9 = \frac{1}{2} \int_0^1 (V_m' \int_1^s \int_0^s W_n'W_p' ds ds)'V_r ds$$

$$\alpha_{10} = \int_0^1 (V_m''\Gamma_k\Gamma_p)''V_r ds$$

$$\alpha_{11} = \int_0^1 (W_n''\Gamma_k')'V_r ds$$

$$\alpha_{12} = \int_0^1 (W_n''\Gamma_k)''V_r ds$$

$$\alpha_{23} = \int_0^1 W_n''W_r ds$$

$$\alpha_{24} = \int_0^1 [(s-1)W_n'' + W_n']W_r ds$$

$$\alpha_{25} = \int_0^1 [W_n'(W_p'W_q'')]'W_r ds$$

$$\alpha_{26} = \frac{1}{2} \int_0^1 (W_n' \int_1^s \int_0^s W_p'W_q' ds ds)'W_r ds$$

$$\alpha_{27} = \int_0^1 (V_m''' \int_0^s V_p'W_n'' ds)'W_r ds$$

$$\alpha_{28} = \int_0^1 [W_n'(V_m'V_p'')]'W_r ds$$

$$\alpha_{29} = \frac{1}{2} \int_0^1 (W_n' \int_1^s \int_0^s V_m'V_p' ds ds)'W_r ds$$

$$\alpha_{30} = \int_0^1 (W_n''\Gamma_k\Gamma_p)''W_r ds$$

$$\alpha_{31} = \int_0^1 (V_m''\Gamma_k')'W_r ds$$

$$\alpha_{32} = \int_0^1 (V_m''\Gamma_k)''W_r ds$$

$$\alpha_{42} = \int_0^1 \Gamma_k''\Gamma_r ds$$

$$\alpha_{43} = \int_0^1 \Gamma_k V_m''V_p''\Gamma_r ds$$

$$\alpha_{44} = \int_0^1 \Gamma_k V_m'V_p'\Gamma_r ds$$

$$\alpha_{45} = \int_0^1 \Gamma_k W_n''W_p''\Gamma_r ds$$

$$\alpha_{46} = \int_0^1 \Gamma_k W_n'W_p'\Gamma_r ds$$

$$\alpha_{47} = \int_0^1 (\int_0^s V_m''W_n' ds)\Gamma_r ds$$

$$\alpha_{48} = \int_0^1 V_m''W_n''\Gamma_r ds$$

$$\alpha_{49} = \int_0^1 V_m'W_n'\Gamma_r ds$$

Appendix D

COEFFICIENTS FOR EQUATIONS (7.63)-(7.66) and (8.3)

$$\hat{\mu}_1 = R_2/R_1 (= 0.035230)$$

$$\Lambda_1 = R_3/R_1 (= 3.0908)$$

$$S_{12} = -R_4/R_1 (= 7.7105)$$

$$S_{11} = -R_5/R_1 (= 4.3686)$$

$$\hat{\mu}_2 = E_2/E_1 (= 0.065002)$$

$$\Lambda_2 = -E_3/E_1 (= 0.77311)$$

$$S_{21} = -E_4/E_1 (= 1.6597)$$

$$S_{22} = -E_5/E_1 (= 0.51760)$$

$$R = -\frac{1}{2} \hat{f} \omega_1^2 \int_0^1 V ds / R_1$$

$$\hat{R} = -2 \hat{f} \omega_1^2 \int_0^1 W ds / E_1$$

where

$$R_1 = \omega_1 \int_0^1 (j_1 \Gamma^2 + V^2 - j_3 V'' V) ds$$

$$R_2 = \frac{1}{2} \omega_1 \int_0^1 (\mu_3 \Gamma^2 + \mu_1 V^2) ds$$

$$R_3 = \frac{1}{4} \int_0^1 [(1 - \beta_{22}) V'' W'' \Gamma - \beta_{13} W'''' \Gamma^2 + j_1 \omega_1^2 (\Gamma \int_0^s V'' W' ds + V' W' \Gamma)]$$

$$+ 2(j_2 - j_3)\omega_1^2 V' W' \Gamma + \beta_{11}(W'' \Gamma')' V + (1 - \beta_{22})(W'' \Gamma)'' V + \beta_{13}(V'' W''')' V] ds \\ + \frac{1}{4} [(\beta_{22} - 1) V W'''' \Gamma]^{s=1}$$

$$R_4 = \frac{1}{4} \int_0^1 [(\beta_{22} - 1) W'''' \Gamma^2 + \beta_{13} \Gamma W'''' \int_0^s V' W'' ds + \beta_{13} V' W'''' \Gamma \\ - 4(j_2 - j_3)\omega_1^2 W''^2 \Gamma^2 + (1 - \beta_{22}) V(W'''' \int_0^s V'' W' ds)' - (V'(W' W'''))' V - \beta_{13} V(W'''' \Gamma)'] ds \\ + \frac{1}{4} [(\beta_{22} - 1) V W'''' \int_0^1 V'' W' ds + V V' W' W''''']^{s=1}$$

$$R_5 = \frac{1}{16} \int_0^1 [6(1 - \beta_{22}) V''^2 \Gamma^2 + \beta_{13}(-3 V'''' \Gamma^3 + 3 V'''' V'^2 \Gamma + 6 V'''' V' \Gamma) \\ + 2(j_2 - j_3)\omega_1^2 V'^2 \Gamma^2 + 6(1 - \beta_{22}) V(V'' \Gamma^2)'' - 6 V(V'(V' V''))' \\ + \beta_{13}(-3 V(V'^2 \Gamma'')' + 3 V(\Gamma^2 \Gamma'')' + 6 V(\Gamma \Gamma'^2)' + 4 \omega_1^2 V(V' \int_1^s \int_0^s V' V' ds ds)')] ds \\ - \frac{3}{8} [(1 - \beta_{22}) V V'''' \Gamma^2 - V V'^2 V'''' - \frac{1}{2} \beta_{13} V V'^2 \Gamma'' + \frac{1}{2} \beta_{13} V \Gamma^2 \Gamma'']^{s=1}$$

$$E_1 = 2\omega_1 \int_0^1 (W^2 - j_2 W W'') ds$$

$$E_2 = \mu_2 \omega_1 \int_0^1 W^2 ds$$

$$E_3 = \frac{1}{4} \int_0^1 [\beta_{11} W(V'' \Gamma')' + (\beta_{22} - 1) W(V'' \Gamma)'' + \beta_{13}(-W(\Gamma \Gamma'')' + W(V''^2)' - W(\Gamma'^2)')] ds \\ - \frac{1}{4} [(\beta_{22} - 1) W V'''' \Gamma - \beta_{13} W \Gamma \Gamma'']^{s=1}$$

$$E_4 = \frac{1}{4} \int_0^1 [(\beta_{22} - 1)(W(W'' \Gamma^2)'' + W(V'''' \int_0^s V' W'' ds)') - \beta_{22} W(W'(V' V''))' \\ + \beta_{13}(W(\Gamma V'' W'')' - W(\Gamma'' \int_0^s V' W'' ds)')] ds \\ - \frac{1}{4} [(\beta_{22} - 1) W(W'''' \Gamma^2 + V'''' \int_0^1 W'' V' ds) - \beta_{22} W W' V' V'''' - \beta_{13} W \Gamma'' \int_0^1 W'' V' ds]^{s=1}$$

$$E_5 = \int_0^1 [-\frac{3}{8} \beta_{22} W(W'(W' W''))' + \omega_1^2 W(W' \int_1^s \int_0^s W'^2 ds ds)'] ds + \frac{3}{8} \beta_{22} [W W'^2 W''''']^{s=1}$$

Vita

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