

RESOLUTIONS MOD I, GOLOD PAIRS

by

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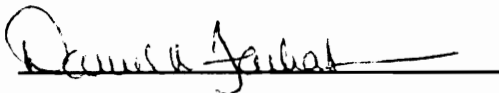
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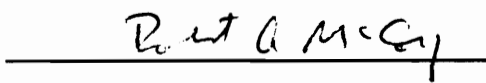
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Mathematics

(ABSTRACT)

Let R be a commutative ring, I be an ideal in R and let M be a R/I -module. In this thesis we construct a R/I -projective resolution of M using given R -projective resolutions of M and I . As immediate consequences of our construction we give descriptions of the canonical maps $\text{Ext}_{R/I}^n(M, N) \rightarrow \text{Ext}_R^n(M, N)$ and $\text{Tor}_n^R(M, N) \rightarrow \text{Tor}_n^{R/I}(M, N)$ for a R/I -module N and we give a new proof of a theorem of Gulliksen [6] which states that if I is generated by a regular sequence of length r then $\coprod_{n=0}^{\infty} \text{Tor}_n^{R/I}(M, N)$ is a graded module over the polynomial ring $R/I[X_1 \cdots X_r]$ with $\deg X_i = -2$, $1 \leq i \leq r$. If I is generated by a regular element and if the R -projective dimension of M is finite, we show that M has a R/I -projective resolution which is eventually periodic of period two. This generalizes a result of Eisenbud [3].

In the case when $R = (R, \mathfrak{m})$ is a Noetherian local ring and M is a finitely generated R/I -module, we discuss the minimality of the constructed resolution. If it is minimal we call (M, I) a Golod pair over R . We give a direct proof of a theorem of Levin [10] which states that if (M, I) is a Golod pair over R then $(\Omega_{R/I}^n(M), I)$ is a Golod pair over R where $\Omega_{R/I}^n(M)$ is the n^{th} syzygy of the constructed R/I -projective resolution of M . We show that the converse of the last theorem is not true and if $(\Omega_{R/I}^1(M), I)$ is a Golod pair over R then we give a necessary and sufficient condition for (M, I) to be a Golod pair over R .

Finally we prove that if (M, I) is a Golod pair over R and if $a \in I - \mathfrak{m}I$ is a regular

element in R then $(M, (a))$ and $(I/(a), (a))$ are Golod pairs over R and $(M, I/(a))$ is a Golod pair over $R/(a)$. As a corollary of this result we show that if the natural map $\pi : R \rightarrow R/I$ is a Golod homomorphism (this means $(R/\mathfrak{m}, I)$ is a Golod pair over R ,Levin [8]), then the natural maps $\pi_1 : R \rightarrow R/(a)$ and $\pi_2 : R/(a) \rightarrow R/I$ are Golod homomorphisms.

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Dedicated

to the loving memory

of my Father

and to my Mother

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Chapter I. Introduction

We begin by defining the concept of a projective resolution of a module M over a commutative ring R . Recall that a R -module F is called a free module if there exists a subset $\{f_i\}$ of F such that every $f \in F$ has a unique expression $f = \sum r_i f_i$ with $r_i \in R$. A projective R -module is a direct summand of a free R -module and a projective resolution of the R -module M is a sequence (possibly infinite) of R -modules and homomorphisms

$$\dots \rightarrow Q_n \xrightarrow{e_n} Q_{n-1} \xrightarrow{e_{n-1}} \dots \rightarrow Q_0 \xrightarrow{e_0} M \rightarrow 0$$

such that

i) Q_0, Q_1, \dots are projective R -modules,

ii) e_0 is surjective and $\ker e_n = \text{im } e_{n+1}, n \geq 0$, (i.e. the sequence is exact). We denote the above resolution by (\mathbf{Q}, e) and when Q_0, Q_1, \dots are free R -modules we say that (\mathbf{Q}, e) is a free resolution.

Using projective resolutions of the R -module M we define the projective dimension of the R -module M as the smallest nonnegative integer n such that there exists a projective resolution of length n as follows.

$$0 \rightarrow Q_n \xrightarrow{e_n} Q_{n-1} \rightarrow \dots \rightarrow Q_0 \xrightarrow{e_0} M \rightarrow 0$$

If M does not have a "finite resolution" as above then we say that projective dimension of M is infinite.

The study of projective resolutions of a R -module M is sometimes useful in getting some information about M or R . The projective dimension of M measures, in some sense, how far M is from being projective.

For example if every R -module M has a resolution of the type

$$0 \rightarrow Q_1 \xrightarrow{e_1} Q_0 \xrightarrow{e_0} M \rightarrow 0$$

then every ideal I in R is a projective R -module. Using projective resolutions of a R -module M we define, R -modules, $\text{Tor}_n^R(M, N), \text{Ext}_R^n(M, N)$ for $n \geq 0$ which depend only on the R -modules M and N , as follows.

Let (\mathbf{Q}, e) be a projective resolution of a R -module M as above and consider the following sequences of R -modules and homomorphisms,

$$\cdots \rightarrow Q_{n+1} \otimes_R N \xrightarrow{e_{n+1} \otimes 1_N} Q_n \otimes_R N \xrightarrow{e_n \otimes 1_N} Q_{n-1} \otimes_R N \rightarrow \cdots \rightarrow Q_0 \otimes_R N$$

and

$$\cdots \leftarrow \text{Hom}_R(Q_{n+1}, N) \xleftarrow{\text{Hom}_R(e_{n+1}, N)} \text{Hom}_R(Q_n, N) \xleftarrow{\text{Hom}_R(e_n, N)} \text{Hom}_R(Q_{n-1}, N) \leftarrow \cdots \leftarrow \text{Hom}_R(Q_0, N)$$

which are complexes, i.e.

$$\text{im } e_{n+1} \otimes 1_N \subset \ker e_n \otimes 1_N \text{ and } \text{im } \text{Hom}(e_n, N) \subset \ker (e_{n+1}, N).$$

We now define $\text{Tor}_0^R(M, N) = Q_0 \otimes_R N / \text{im } e_1 \otimes 1_N \simeq M \otimes_R N$ and for $n \geq 1$ we define $\text{Tor}_n^R(M, N) = \ker e_n \otimes 1_N / \text{im } e_{n+1} \otimes 1_N$

Similarly, we define $\text{Ext}_R^0(M, N) = \ker \text{Hom}_R(e_0, N) \simeq \text{Hom}_R(M, N)$ and for $n \geq 1$ $\text{Ext}_R^n(M, N) = \ker \text{Hom}_R(e_{n+1}, N) / \text{im}(e_n, N)$

The modules $\text{Tor}_n^R(M, N)$ and $\text{Ext}_R^n(M, N)$ measure, how far the above complexes are from being exact. They are sometimes useful in getting some information about M, N and R . For example if $\text{Ext}_R^1(M, N) = (0)$ for every R -module N then M is a projective R -module.

Now assume that R is a commutative ring, I is an ideal in R and M is a module over the ring R/I . The natural map $R \rightarrow R/I$ makes M into a R -module. We will be considering projective resolutions of M as a R -module as well as of M as a R/I -module. We call these resolutions respectively R -projective resolutions and R/I -projective resolutions of M . In this thesis we construct a specific R/I -projective resolution $(\overline{\mathbf{U}}, \overline{h})$ of M using

given R -projective resolutions of M and I . As a first application of this construction we give a description of the canonical maps

$$\mathrm{Tor}_n^R(M, N) \rightarrow \mathrm{Tor}_n^{R/I}(M, N) \text{ and } \mathrm{Ext}_{R/I}^n(M, N) \rightarrow \mathrm{Ext}_R^n(M, N)$$

Another result which follows from our construction and some results of Eisenbud [3] is the following theorem of Gulliksen [6]. Suppose I is generated by a regular sequence x_1, x_2, \dots, x_r . This means that x_1 is a nonzero divisor in R and the image of x_i in $R_i = R/(x_1, \dots, x_{i-1})$ is a non zero divisor in R_i , $2 \leq i \leq r$. Gulliksen's theorem states that for all R/I -modules M and N ,

$\mathrm{Tor}_n^{R/I}(M, N) = \prod_{n=0}^{\infty} \mathrm{Tor}_n^{R/I}(M, N)$ is a module over the polynomial ring $R/I[X_1, \dots, X_r]$ such that $X_i \cdot \mathrm{Tor}_n^{R/I}(M, N) \subset \mathrm{Tor}_{n-2}^{R/I}(M, N)$. Next we prove the following theorem which was proved by Eisenbud [3] in the case when R is a regular local ring, (e.g. the ring of formal power series over complex numbers). We show that if R is a commutative ring, I is an ideal in R which is generated by a nonzero divisor x in R and if M is a $R/(x)$ -module such that projective dimension of the R -module M is $n < \infty$, then M has a $R/(x)$ -projective resolution,

$$\cdots \rightarrow \bar{U}_{n+1} \xrightarrow{\bar{h}_{n+1}} \bar{U}_n \xrightarrow{\bar{h}_n} \bar{U}_{n-1} \rightarrow \cdots \rightarrow \bar{U}_0 \rightarrow M \rightarrow 0$$

such that for $s \geq 0$ $\bar{U}_{n-1+2s} = \bar{U}_{n-1}$, $\bar{U}_{n+2s} = \bar{U}_n$, $\bar{h}_{n+2s} = \bar{h}_n$, $\bar{h}_{n+2s+1} = \bar{h}_{n+1}$.

[The above resolution is said to be eventually periodic of period 2.]

The remainder of the thesis is concerned with the applications of our construction to the case when $R = (R, \mathfrak{m})$ is a Noetherian local ring (that is R is a ring with the unique maximal ideal \mathfrak{m} and every ideal $J \subset R$ is finitely generated), I is an ideal in R and M is a finitely generated R/I -module. Let (\mathbf{Q}, e) and (\mathbf{P}, e) be minimal R -projective resolutions of M and I respectively (this means, $\mathrm{im} e_n \subset \mathfrak{m}Q_{n-1}$, $\mathrm{im} d_n \subset \mathfrak{m}P_{n-1}$, $n \geq 1$). We then investigate the question of minimality of the constructed R/I -projective resolution of M . We say that (M, I) is a Golod pair over R if the constructed R/I -projective resolution of

M is minimal. If $M = R/\mathfrak{m}$ then we show that (M, I) is a Golod pair over R if and only if I is a Golod ideal [8,9]. The notion of a Golod ideal was introduced by Golod [4] to study the question of rationality of the Poincaré series of a local ring. Recall that the Poincaré series of a local ring (R, \mathfrak{m}) is the formal power series $\sum a_n z^n$ where a_n is dimension of the R/\mathfrak{m} -vector space $\text{Tor}_n^{R/I}(R/\mathfrak{m}, R/\mathfrak{m})$ and it is said to be rational if it can be represented as a quotient of two polynomials. Several classes of local rings have rational Poincaré series. However Anick [1] constructed a local ring whose Poincaré series is not rational. We define the Poincaré series of a finitely generated R -module M to be the formal power series $\sum b_n z^n$ where b_n is dimension of the R/\mathfrak{m} -vector space $\text{Tor}_n^R(M, R/\mathfrak{m})$. In section 2 we give a direct proof of a theorem of Levin [10] which states that if (M, I) is a Golod pair over R , (so that the constructed R/I -projective resolution $(\bar{U}_\cdot, \bar{h}_\cdot)$ of M is minimal), then $(\Omega_n^{R/I}(M), I)$ is a Golod pair over R where $\Omega_n^{R/I}(M)$ is the n^{th} R/I -syzygy of M . We investigate the question if $(\Omega_1^{R/I}(M), I)$ is a Golod pair over R is (M, I) a Golod pair over R ? Another result we prove is that if $\bar{h}_n(\bar{U}_n) \subset \bar{\mathfrak{m}}\bar{U}_{n-1}$ then $\bar{h}_{n-2}(\bar{U}_{n-2}) \subset \bar{\mathfrak{m}}\bar{U}_{n-4}$.

In Chapter III we assume we have a Noetherian local ring R and ideals $J \subset I$. We study the relationship of the following properties:

- i) (M, I) is a Golod pair (over R)
- ii) (M, J) is a Golod pair (over R)
- iii) $(M, I/J)$ is a Golod pair (over R/J)

We prove the following result in this case. If (M, I) is a Golod pair over R and if \mathfrak{x} is a nonzero divisor in R then

- (a) $(M, (\mathfrak{x}))$ is a Golod pair (over R)
- (b) $(I/(\mathfrak{x}), (\mathfrak{x}))$ is a Golod pair (over R)

and

- (c) $(M, I/(\mathfrak{x}))$ is a Golod pair (over $R/(\mathfrak{x})$)

Chapter V contains the proof of the main theorem of Chapter II.

Other than the generality of the main theorem and its consequences, our approach leads to straightforward and elementary proofs of results that originally had less direct proofs which usually relied on arguments requiring the use of spectral sequences.

We conclude the introduction with the basic fact that maps from projective modules to epimorphic images lift. For completeness and clarity, we state this result without proof (which can be found in any standard text on homological algebra).

LEMMA 0. Consider the following exact diagram of modules over a ring R .

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \uparrow p_A & & & & \uparrow p_C & & \\
 & & P_A & & & & P_C & & \\
 & & \uparrow & & & & \uparrow & & \\
 & & \ker(p_A) & & & & \ker(p_C) & & \\
 & & \uparrow & & & & \uparrow & & \\
 & & 0 & & & & 0 & &
 \end{array}$$

Assume that P_A and P_C are projective R -modules and let $P_B = P_A \amalg P_C$. Then there exists a surjective homomorphism $p_B : P_B \rightarrow B$ such that the following diagram is exact.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \uparrow p_A & & \uparrow p_B & & \uparrow p_C & & \\
 0 & \longrightarrow & P_A & \xrightarrow{\text{inclusion}} & P_B & \xrightarrow{\text{projection}} & P_C & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & \ker(p_A) & \longrightarrow & \ker(p_B) & \longrightarrow & \ker(p_C) & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
 \end{array}$$

LEMMA 0'. With the same notation as in the above lemma assume further that R is a G -graded ring (where G is an abelian group), the R -modules A, B, C, P_A, P_C are graded

modules and p_A and p_C are degree 0 maps, then P_B is a graded module and p_B can be chosen to be a degree 0 map.

PROOF: Note that in the above diagram p_B can be replaced by the map $\sum a_g \rightarrow \sum (p_B(a_g))_g$, ($a_g \in P_B^g$, the g^{th} graded component of P_B).

Chapter II. General Results

Throughout this section we assume that R is a commutative (not necessarily Noetherian) ring with 1, I is an ideal in R and M is an R/I -module. Fix R -projective resolutions of M and I ,

$$\begin{array}{l} (\mathbf{Q}, e.) \quad \cdots \longrightarrow Q_n \xrightarrow{e_n} Q_{n-1} \longrightarrow \cdots \longrightarrow Q_0 \xrightarrow{e_0} M \longrightarrow 0 \\ (\mathbf{P}, d.) \quad \cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{d_0} I \longrightarrow 0 \end{array}$$

Our main result is the construction of an R/I -projective resolution of M

$$\cdots \longrightarrow \bar{U}_n \xrightarrow{\bar{h}_n} \bar{U}_{n-1} \longrightarrow \cdots \longrightarrow \bar{U}_0 \xrightarrow{\bar{h}_0} M \longrightarrow 0$$

from $(\mathbf{Q}, e.)$ and $(\mathbf{P}, d.)$. Not only do we completely describe the R/I -projective modules \bar{U}_n but we also provide information about the nature of the maps $\bar{h}_n : \bar{U}_n \rightarrow \bar{U}_{n-1}$ and $\bar{h}_0 : \bar{U}_0 \rightarrow M$.

Let $U_0 = Q_0$ and $U_1 = Q_1$. Inductively define U_n for $n \geq 2$, by

$$U_n : Q_n \amalg \left(\prod_{i=0}^{n-2} (P_{n-i-2} \otimes_R U_i) \right).$$

Note that, for $i \geq 0$, U_i is a projective R -module. All tensor products will be over R unless otherwise stated.

We now state the main theorem and present a proof in Section 4. In the case when $R = (R, \mathfrak{m})$ is a local ring, $M = R/\mathfrak{m}$, a similar construction due to Eagon and Northcott is given in [7].

THEOREM 2.1. *Let R be a commutative ring with 1, $I \subset R$ be an ideal and M be a R/I -module. Suppose $(\mathbf{Q}, e.)$ and $(\mathbf{P}, d.)$ are R -projective resolutions of M and I respectively. Let $U_0 = Q_0$, $U_1 = Q_1$ and $U_n = Q_n \amalg (\prod_{i=0}^{n-2} (P_{n-i-2} \otimes_R U_i))$ for $n \geq 2$. Then there exist R -homomorphisms $h_0 : U_0 \rightarrow M \rightarrow 0$ and $h_i : U_i \rightarrow U_{i-1}$ for $i \geq 1$ such that*

$$A). \quad \cdots \longrightarrow \bar{U}_n \xrightarrow{\bar{h}_n} \bar{U}_{n-1} \longrightarrow \cdots \longrightarrow \bar{U}_0 \xrightarrow{\bar{h}_0} M \longrightarrow 0'$$

is a R/I -projective resolution of M denoted by $(\bar{\mathbf{U}}, \bar{h}.)$, (" $\bar{}$ " means $R/I \otimes_R *$),*

B). The maps h_n satisfy,

(a) $h_0 = e_0, h_1 = e_1,$

(b) for $n \geq 2$, if ϕ_n is the following composition

$$P_0 \otimes U_{n-2} \xrightarrow{\text{inclusion}} U_n \xrightarrow{h_n} U_{n-1} \xrightarrow{h_{n-1}} U_{n-2}$$

and if ψ_n is the following composition

$$P_0 \otimes U_{n-2} \xrightarrow{d_0 \otimes 1_{U_{n-2}}} I \otimes U_{n-2} \xrightarrow{\text{canonical}} IU_{n-2} \xrightarrow{\text{inclusion}} U_{n-2}$$

then $\phi_n = (-1)^s \psi_n$ where $n = 2s$ or $n = 2s + 1,$

(c) for $n \geq 2, h_n(U_n) = h_{n-1}^{-1}(IU_{n-2})$

(d) for $n \geq 2$ the composition

$$Q_n \xrightarrow{\text{inclusion}} U_n \xrightarrow{h_n} U_{n-1}$$

is equal to the composition

$$Q_n \xrightarrow{e_n} Q_{n-1} \xrightarrow{\text{inclusion}} U_{n-1}$$

(e) for $2 \leq r \leq n - 1, n \geq 3$

$$h_n \left(Q_n \coprod \left(\coprod_{i=0}^{r-2} (P_{n-i-2} \otimes U_i) \right) \right) = \ker h_{n-1} |_{Q_{n-1}} \coprod \left(\coprod_{i=0}^{r-2} (P_{n-i-2} \otimes U_i) \right)$$

(f) for $2 \leq r \leq n - 1, n \geq 3$ the composition

$$P_{n-r} \otimes U_{r-2} \xrightarrow{\text{inclusion}} U_n \xrightarrow{h_n} U_{n-1} \xrightarrow{\text{projection}} P_{n-r-1} \otimes U_{r-2}$$

$$\text{is } d_{n-r} \otimes 1_{U_{r-2}} : P_{n-r} \otimes U_{r-2} \rightarrow P_{n-r-1} \otimes U_{r-2}$$

(g) for $n \geq 2, Y_n = \ker h_n$ has the filtration $0 \subset Y_{n,1} \subset Y_{n,2} \subset \dots \subset Y_{n,n} = Y_n$ such that

(i) $Y_{n,1} = \ker h_n |_{Q_n}$

(ii) for $2 \leq r \leq n$ $Y_{n,r} = \ker h_n |_{Q_n \amalg (\amalg_{i=0}^{r-2} (P_{n-i-2} \otimes U_i))}$

(iii) for $2 \leq r \leq n$ the sequence

$$0 \longrightarrow Y_{n,r-1} \xrightarrow{\text{inclusion}} Y_{n,r} \xrightarrow{\text{projection}} \ker d_{n-r} \otimes U_{r-2} \longrightarrow 0$$

induced by the exact sequence (for $r = 2$)

$$0 \rightarrow Q_n \xrightarrow{\text{incl.}} Q_n \amalg P_{n-2} \otimes U_0 \xrightarrow{\text{proj.}} P_{n-2} \otimes U_0 \rightarrow 0$$

and (for $r > 2$),

$$0 \rightarrow Q_n \amalg \left(\amalg_{i=0}^{r-3} (P_{n-i-2} \otimes U_i) \right) \xrightarrow{\text{incl.}} Q_n \amalg \left(\amalg_{i=0}^{r-2} (P_{n-i-2} \otimes U_i) \right) \xrightarrow{\text{proj.}} P_{n-r} \otimes U_{r-2} \rightarrow 0$$

is exact,

(h) for $2 \leq r \leq n-1$, $n \geq 4$ the composition

$$P_{n-r} \otimes U_{r-2} \xrightarrow{\text{inclusion}} U_n \xrightarrow{h_n} U_{n-1} \xrightarrow{\text{projection}} P_{n-r} \otimes U_{r-3}$$

is equal $(-1)^{n+1} 1_{P_{n-r}} \otimes h_{r-2} : P_{n-r} \otimes U_{r-2} \rightarrow P_{n-r} \otimes U_{r-3}$,

(i) for $n \geq 3$ the composition

$$P_0 \otimes U_{n-2} \xrightarrow{\text{inclusion}} U_n \xrightarrow{h_n} U_{n-1} \xrightarrow{\text{projection}} P_0 \otimes U_{n-3}$$

is $(-1)^{n+1} 1_{P_0} \otimes h_{n-2} : P_0 \otimes U_{n-2} \rightarrow P_0 \otimes U_{n-3}$.

There is a graded version of the theorem.

THEOREM 2.1'. Let G be an abelian group, R a G -graded commutative ring with 1, I a homogeneous ideal in R and M a G -graded R/I -module. Suppose that $(\mathbf{Q}, e.)$ and $(\mathbf{P}, d.)$ are graded R -projective resolutions of M and I respectively (with degree 0 maps). Then the conclusions of Theorem 2.1 hold with the additional requirement that the maps $h_n : U_n \rightarrow U_{n-1}$ and $h_0 : U_0 \rightarrow M$ are degree 0 maps. In particular

$$\cdots \longrightarrow R/I \otimes_R U_n \xrightarrow{1 \otimes h_n} R/I \otimes_R U_{n-1} \longrightarrow \cdots \longrightarrow R/I \otimes_R U_0 \xrightarrow{1 \otimes h_0} M \longrightarrow 0$$

is a G -graded projective R/I -resolution of M .

For the remainder of this paper, if $(\mathbf{Q}\cdot, e\cdot)$ and $(\mathbf{P}\cdot, d\cdot)$ are R -projective resolutions of M and I respectively, $(\overline{\mathbf{U}}\cdot, \overline{h}\cdot)$ will denote the R/I -projective resolution of M

$$\cdots \longrightarrow R/I \otimes_R U_n \xrightarrow{1 \otimes h_n} R/I \otimes_R U_{n-1} \longrightarrow \cdots \longrightarrow R/I \otimes_R U_0 \xrightarrow{1 \otimes h_0} M \longrightarrow 0$$

given in Theorem 2.1.

We end this section with some applications of the theorem. First assume that we are given two R/I -modules M and N . It is well-known that for each $n \geq 0$, there are natural maps:

$$T_n : \text{Tor}_n^R(M, N) \longrightarrow \text{Tor}_n^{R/I}(M, N) \text{ and}$$

$$E_n : \text{Ext}_{R/I}^n(M, N) \longrightarrow \text{Ext}_R^n(M, N).$$

Given R -projective resolutions of $(\mathbf{Q}\cdot, e\cdot)$ and $(\mathbf{P}\cdot, d\cdot)$ of M and I respectively we present explicit descriptions of T_n and E_n . Consider the following diagram:

$$\begin{array}{ccccc} Q_{n+1} \otimes_R N & \xrightarrow{e_{n+1} \otimes 1} & Q_n \otimes_R N & \xrightarrow{e_n \otimes 1} & Q_{n-1} \otimes_R N \\ \alpha_{n+1} \otimes 1 \downarrow & & \alpha_n \otimes 1 \downarrow & & \alpha_{n-1} \otimes 1 \downarrow \\ \overline{U}_{n+1} \otimes_{R/I} N & \xrightarrow{\overline{h}_{n+1} \otimes 1} & \overline{U}_n \otimes_{R/I} N & \xrightarrow{\overline{h}_n \otimes 1} & \overline{U}_{n-1} \otimes_{R/I} N \end{array}$$

where α_n is the composition $Q_n \xrightarrow{\text{inclusion}} U_n \xrightarrow{\text{canonical}} \overline{U}_n$. Theorem 2.1 B(a) and (d) imply that the diagram is commutative. The commutativity of the diagram implies $\alpha_n \otimes 1$ induces a map $\tilde{T}_n : \ker(e_n \otimes 1) \rightarrow \ker(\overline{h}_n \otimes 1)$ so that $\tilde{T}_n(\text{im}(e_{n+1} \otimes 1)) \subseteq \text{im}(\overline{h}_{n+1} \otimes 1)$.

Hence we get a description of the maps

$$T_n : \text{Tor}_n^R(M, N) \rightarrow \text{Tor}_n^{R/I}(M, N).$$

Similarly, Theorem 2.1 B(a) and (d) also imply the commutativity of the diagram

$$\begin{array}{ccccc} \text{Hom}_R(Q_{n+1}, N) & \xleftarrow{\text{Hom}_R(e_{n+1}, N)} & \text{Hom}_R(Q_n, N) & \xleftarrow{\text{Hom}_R(e_n, N)} & \text{Hom}_R(Q_{n-1}, N) \\ \text{Hom}_R(\alpha_{n+1}, N) \uparrow & & \text{Hom}_R(\alpha_n, N) \uparrow & & \text{Hom}_R(\alpha_{n-1}, N) \uparrow \\ \text{Hom}_{R/I}(\overline{U}_{n+1}, N) & \xleftarrow{\text{Hom}_{R/I}(\overline{h}_{n+1}, N)} & \text{Hom}_{R/I}(\overline{U}_n, N) & \xleftarrow{\text{Hom}_{R/I}(\overline{h}_n, N)} & \text{Hom}_{R/I}(\overline{U}_{n-1}, N) \end{array}$$

and we get induced maps

$$E_n : \text{Ext}_{R/I}^n(M, N) \rightarrow \text{Ext}_R^n(M, N).$$

Next we turn to the special case where I is a principal ideal generated by a nonzero divisor.

THEOREM 2.2. *Let R be a commutative ring, a be a nonzero divisor in R , and M be an $R/(a)$ -module. Let (\mathbf{Q}, e) be an R -projective resolution of M and let (\mathbf{P}, d) be the following R -projective resolution of the ideal (a) , $0 \rightarrow R \xrightarrow{d_0} (a) \rightarrow 0$, where $d_0(1) = a$. Then there exists an $R/(a)$ -projective resolution, $(\bar{\mathbf{U}}, \bar{h})$ of M , such that*

A). $\bar{U}_0 = Q_0/aQ_0$, $\bar{U}_1 = Q_1/aQ_1$ and for $n \geq 2$, $\bar{U}_n = (Q_n/aQ_n) \amalg \bar{U}_{n-2}$,

B). the maps \bar{h}_0 and \bar{h}_1 are induced by e_0 and e_1 respectively, and, for $n \geq 2$ the composition

$$0 \longrightarrow Q_n/aQ_n \xrightarrow{\text{inclusion}} \bar{U}_n \xrightarrow{\bar{h}_n} \bar{U}_{n-1} \xrightarrow{\text{projection}} Q_{n-1}/aQ_{n-1}$$

is equal to the map induced by $e_n : Q_n \rightarrow Q_{n-1}$,

C). there exists an exact sequence

$$0 \rightarrow (\ker(e_1))/a(\ker(e_1)) \rightarrow \ker \bar{h}_1 \rightarrow M \rightarrow 0$$

D). there exist exact commutative diagrams

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & (\ker(e_1))/a(\ker(e_1)) & \longrightarrow & \ker(\bar{h}_1) & \longrightarrow & M \longrightarrow 0 \\
 & & \uparrow \bar{e}_2 & & \uparrow \bar{h}_2 & & \uparrow \bar{h}_0 \\
 0 & \longrightarrow & Q_2/aQ_2 & \xrightarrow{i} & (Q_2/aQ_2) \amalg (\bar{U}_0) & \xrightarrow{p} & \bar{U}_0 \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & (\ker(e_2))/a(\ker(e_2)) & \xrightarrow{i} & \ker(\bar{h}_2) & \longrightarrow & \ker(\bar{h}_0) \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

and for $n \geq 3$,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Q_{n-1}/aQ_{n-1} & \xrightarrow{i} & \bar{U}_{n-1} & \xrightarrow{p} & \bar{U}_{n-3} & \longrightarrow & 0 \\
& & \uparrow \bar{e}_n & & \uparrow \bar{h}_n & & \uparrow \bar{h}_{n-2} & & \\
0 & \longrightarrow & Q_n/aQ_n & \xrightarrow{i} & \bar{U}_n & \xrightarrow{p} & \bar{U}_{n-2} & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & (\ker(e_n))/a(\ker(e_n)) & \xrightarrow{i} & \ker(\bar{h}_n) & \longrightarrow & \ker(\bar{h}_{n-2}) & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}$$

PROOF: Consider the following R -resolution of M

$$(\mathbf{Q}_., e.): \quad \cdots \rightarrow Q_n \xrightarrow{e_n} Q_{n-1} \rightarrow \cdots \rightarrow Q_1 \rightarrow Q_0 \xrightarrow{e_0} M \rightarrow 0.$$

Tensoring with $R/(a)$ and noting that the R -projective dimension of $R/(a)$ is 1 we obtain exact sequences

$$\begin{aligned}
0 &\rightarrow e_1^{-1}(aQ_0)/(aQ_1) \xrightarrow{i} Q_1/(aQ_1) \xrightarrow{\bar{e}_1} M \rightarrow 0 \\
\cdots &\rightarrow Q_n/(aQ_n) \xrightarrow{\bar{e}_n} Q_{n-1}/(aQ_{n-1}) \rightarrow \cdots \rightarrow Q_3/(aQ_3) \xrightarrow{\bar{e}_3} Q_2/(aQ_2)
\end{aligned}$$

Here, for $n \geq 1$, \bar{e}_n is induced by e_n .

The split exact sequence $0 \rightarrow \ker(e_1) \xrightarrow{i} e_1^{-1}(aQ_0) \xrightarrow{e_1} aQ_0 \rightarrow 0$ induces a short exact sequence of $R/(a)$ -modules

$$0 \rightarrow \ker(e_1)/(a\ker(e_1)) \rightarrow e_1^{-1}(aQ_0)/(aQ_1) \rightarrow (aQ_0)/(a\ker(e_0)) \rightarrow 0$$

Put $U_0 = Q_0$, $U_1 = Q_1$, $h_0 = e_0$, and $h_1 = e_1$ and noting that $aQ_0/a\ker(e_0) \cong M$ we obtain an exact sequence

$$0 \rightarrow \ker(e_1)/(a\ker(e_1)) \rightarrow \ker \bar{h}_1 \rightarrow M \rightarrow 0$$

This proves (C).

Now $\text{coker}(\bar{e}_3) \cong \ker(e_1)/(a\ker(e_1))$, hence we have the following exact diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & \ker(e_1)/(a\ker(e_1)) & \longrightarrow & \ker \bar{h}_1 & \longrightarrow & M \longrightarrow 0 \\
 & & \uparrow \bar{e}_2 & & \uparrow \bar{h}_0 & & \\
 & & Q_2/(aQ_2) & & \bar{U}_0 & & \\
 & & \uparrow & & \uparrow & & \\
 & & \ker \bar{e}_2 & & \ker \bar{h}_0 & & \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Applying Lemma 0 we get the following exact diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & (\ker e_1)/(a\ker e_1) & \longrightarrow & \ker \bar{h}_1 & \longrightarrow & M \longrightarrow 0 \\
 & & \uparrow \bar{e}_2 & & \uparrow \bar{h}_2 & & \uparrow \bar{h}_0 \\
 0 & \longrightarrow & Q_2/aQ_2 & \xrightarrow{i} & (Q_2/aQ_2) \amalg (\bar{U}_0) & \xrightarrow{p} & \bar{U}_0 \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & (\ker e_2)/(a\ker e_2) & \xrightarrow{i} & \ker \bar{h}_2 & \longrightarrow & \ker \bar{h}_0 \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

This proves the first part of (D). Now assume (D) is true for n with $n \geq 2$. Thus we have the following exact diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & Q_n/(aQ_n) & \xrightarrow{i} & \bar{U}_n & \xrightarrow{p} & \bar{U}_{n-2} \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \ker(e_n)/(a\ker(e_n)) & \longrightarrow & \ker(\bar{h}_n) & \longrightarrow & \ker \bar{h}_{n-2} \longrightarrow 0 \\
 & & \uparrow \bar{e}_{n+1} & & \uparrow \bar{h}_{n-1} & & \\
 & & Q_{n+1}/(aQ_{n+1}) & & \bar{U}_{n-1} & &
 \end{array}$$

Again applying Lemma 0 we get an exact diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Q_n/aQ_n & \xrightarrow{i} & \bar{U}_n & \xrightarrow{p} & \bar{U}_{n-2} & \longrightarrow & 0 \\
& & \uparrow \bar{e}_{n+1} & & \uparrow \bar{h}_{n+1} & & \uparrow \bar{h}_{n-1} & & \\
0 & \longrightarrow & Q_{n+1}/aQ_{n+1} & \xrightarrow{i} & \bar{U}_{n+1} & \xrightarrow{p} & \bar{U}_{n-1} & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & (\ker e_{n+1})/(a\ker e_{n+1}) & \xrightarrow{i} & \ker \bar{h}_{n+1} & \xrightarrow{p} & \ker \bar{h}_{n-1} & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & 0 & & 0 & & 0 & &
\end{array}$$

This proves part (D) and the proof is now complete.

With the same notation as in the last theorem put $X_0 = \ker e_0$ and consider the following exact sequence.

$$0 \rightarrow aQ_0/aX_0 \rightarrow X_0/aX_0 \rightarrow X_0/aQ_0 \rightarrow 0$$

Note that $aQ_0/aX_0 \simeq M$, X_0/aQ_0 is the first $R/(a)$ -syzygy of M (which we denote by K_0) and that the following is a $R/(a)$ -resolution of X_0/aX_0 .

$$\dots \rightarrow Q_n/aQ_n \xrightarrow{\bar{e}_n} Q_{n-1}/aQ_{n-1} \rightarrow \dots \rightarrow Q_1/aQ_1 \xrightarrow{\bar{e}_1} X_0/aX_0 \rightarrow 0$$

It follows easily from this, that for all $R/(a)$ -modules N and for all $n \geq 1$, $\text{Tor}_n^{R/(a)}(X_0/aX_0, N) = \text{Tor}_{n+1}^R(M, N)$. Now rewrite the above exact sequence as $0 \rightarrow M \rightarrow X_0/aX_0 \rightarrow K_0 \rightarrow 0$.

Then applying $\text{Tor}^{R/(a)}(\cdot, N)$ we get the following (well known) long exact sequence.

$$\dots \rightarrow \text{Tor}_{n+1}^{R/(a)}(M, N) \rightarrow \text{Tor}_{n-1}^{R/(a)}(M, N) \rightarrow \text{Tor}_n^R(M, N) \rightarrow \text{Tor}_n^{R/(a)}(M, N) \rightarrow \dots$$

Note that in the above sequence the map $\text{Tor}_{n+1}^{R/(a)}(M, N) \rightarrow \text{Tor}_{n-1}^{R/(a)}(M, N)$ and $\text{Tor}_n^R(M, N) \rightarrow \text{Tor}_n^{R/(a)}(M, N)$ are the maps described in the next theorem and on the page 9, respectively.

We now prove the following theorem of Gulliksen [6].

THEOREM 2.3. *Let R be a commutative ring, I be an ideal in R generated by a regular sequence x_1, \dots, x_r , and let M and N be R/I -modules. Then $\text{Tor}^{R/I}(M, N) = \coprod_{n=0}^{\infty} \text{Tor}_n^{R/I}(M, N)$ is a graded module over the polynomial ring $R/I[X_1, \dots, X_r]$ such that $\deg X_i = -2, 1 \leq i \leq r$.*

PROOF: Let (\mathbf{Q}, e) be a R -projective resolution of M and let (\mathbf{P}, d) be the R -projective resolution of I given by the Koszul complex. Thus P_0 is a free R -module with a basis f_1, \dots, f_r such that $d_0 : P_0 \rightarrow I$ maps f_i to $x_i, 1 \leq i \leq r$. Now let $(\bar{\mathbf{U}}, \bar{h})$ be the R/I -projective resolution of M given by Theorem 2.1. Note that

$$\dots \longrightarrow U_n \xrightarrow{h_n} U_{n-1} \longrightarrow \dots \longrightarrow U_0 \xrightarrow{h_0} M \longrightarrow 0$$

is a lifting of the resolution $(\bar{\mathbf{U}}, \bar{h})$ to a sequence of R -modules and homomorphisms in the sense of Eisenbud [3]. Let $t_i^n : U_n \rightarrow U_{n-2}$ be the projection defined by $t_i^n(*, \dots, *, \sum f_j \otimes u_{n-2}^j) = (-1)^s u_{n-2}^i$, if $n = 2s$ or $n = 2s+1$ and $\bar{t}_i^n : \bar{U}_n \rightarrow \bar{U}_{n-2}$ be the map induced by $t_i^n, 1 \leq i \leq r$. Since the map $\bar{d}_1 : \bar{P}_1 \rightarrow \bar{P}_0$ is the zero map (" $\bar{\cdot}$ " denotes mod I) it follows from Theorem 1.1 B(e),(i) that $\bar{t}^i = (\bar{t}_i^n)$ is a chain map of degree -2 on $(\bar{\mathbf{U}}, \bar{h}), 1 \leq i \leq r$. This means, $\bar{h}_{n-2} \circ \bar{t}_i^n = \bar{t}_i^{n-1} \circ \bar{h}_{n-1}$. The chain maps \bar{t}^i induce maps of degree -2 on $\text{Tor}^{R/I}(M, N)$ which commute and are well defined by the results of section 1 in the cited paper of Eisenbud.

Since every module V over a commutative ring R can be treated as a module over the polynomial ring $R[X_1, \dots, X_r]$ with $X_i V = (0), 1 \leq i \leq r$, we have the following corollary.

COROLLARY 2.4. *If V is a module over a commutative ring R then for every $r \geq 1$, and for all R -modules N , $\text{Tor}^R(V, N) = \coprod_{n=0}^{\infty} \text{Tor}_n(V, N)$ is a graded module over the polynomial ring $R[X_1, \dots, X_r]$ with $\deg X_i = -2, 1 \leq i \leq r$.*

The next application generalizes a result of D. Eisenbud [3] where he assumed R is a regular local ring.

PROPOSITION 2.5. *Let R be an arbitrary commutative ring with 1. Let $a \in R$ be a nonzero divisor and suppose M is an R -module such that $aM = (0)$. Suppose the R -projective*

dimension of M is $\leq d$, with $d \geq 1$, and

$$0 \longrightarrow Q_d \xrightarrow{e_d} Q_{d-1} \xrightarrow{e_{d-1}} \cdots \longrightarrow Q_0 \xrightarrow{e_0} M \longrightarrow 0$$

is an R -projective resolution of M . Consider the R -projective resolution $0 \rightarrow R \xrightarrow{a} (a) \rightarrow 0$ of the ideal (a) . Then the $R/(a)$ -projective resolution, (\bar{U}, \bar{h}) of Theorem 1.1, is eventually periodic of period 2. More precisely, there are natural isomorphisms

$$\bar{U}_{d+k} \xrightarrow{\sim} \begin{cases} \bar{U}_{d-1}, & \text{if } k \text{ is an odd nonnegative integer} \\ \bar{U}_d, & \text{if } k \text{ is an even nonnegative integer} \end{cases}$$

Viewing these isomorphisms as identifications then

$$\bar{h}_{d+k} = \begin{cases} \bar{h}_{d-1}, & \text{if } k \text{ is an odd nonnegative integer} \\ \bar{h}_d, & \text{if } k \text{ is an even nonnegative integer} \end{cases}$$

PROOF: Since a is a nonzero divisor and $aM = 0$ it follows that the R -projective dimension of M is at least 1. Thus $d \geq 1$. For $k = 0$ the result is immediate. Assume $k > 0$. Since $Q_s = 0$ if $s \geq d + 1$ and $P_t = 0$ for $t \geq 1$ we see from the definition of U_n that

$$U_{d+k} = P_0 \otimes_R U_{d+k-2}$$

Since $P_0 = R$, the first part of the result follows. Finally, recall Theorem 2.1 B(i) which states

$$P_0 \otimes_R U_{n-2} \xrightarrow{\text{inclusion}} U_n \xrightarrow{h_n} U_{n-1} \xrightarrow{\text{projection}} P_0 \otimes_R U_{n-3}$$

is $1 \otimes h_{n-2} : P_0 \otimes_R U_{n-2} \rightarrow P_0 \otimes_R U_{n-3}$ for $n \geq 3$. But in this case, the inclusion and projection are the identity maps. Using the identification of U_{n-2} with $P_0 \otimes_R U_{n-2}$ the result follows.

Next we prove a result which, in the local case, follows from [11], (although our approach is different). Recall that if M is an R -module and (\mathbf{Q}, e) is an R -projective resolution for M we say the Poincaré series of M with respect to (\mathbf{Q}, e) has *polynomial growth* if the minimal number of generators of Q_n is bounded by a polynomial in n .

PROPOSITION 2.6. *Let R be a commutative ring with 1 and let I be an ideal generated by a finite R -sequence. Suppose M is an R/I -module which has an R -projective resolution whose Poincaré series has polynomial growth. Then there is an R/I -projective resolution of M whose Poincaré series also has polynomial growth.*

PROOF: By induction, it suffices to show that the result holds if I is generated by a nonzero divisor $a \in I$. Then I has an R -projective resolution $0 \rightarrow R \xrightarrow{d_0} I \rightarrow 0$. Suppose $\dots \rightarrow Q_n \xrightarrow{e_n} Q_{n-1} \rightarrow \dots \rightarrow Q_0 \rightarrow M \rightarrow 0$ is an R -projective resolution of M . Let q_n be the minimal number of generators of Q_n and suppose that $\{q_n\}_{n=0}^\infty$ has polynomial growth. By Theorem 2.1 there is an R/I -projective resolution $\dots \rightarrow \bar{U}_n \xrightarrow{\bar{h}_n} \bar{U}_{n-1} \rightarrow \dots \rightarrow \bar{U}_0 \rightarrow M \rightarrow 0$ such that $\bar{U}_n \cong R/I \otimes_R U_n$ where $U_0 = Q_0$, $U_1 = Q_1$ and $U_n = Q_n \oplus U_{n-2}$. Thus if u_n is the number of R/I -generators of \bar{U}_n then $u_0 = q_0$, $u_1 = q_1$ and $u_n = q_n + u_{n-2}$ for $n \geq 2$. It follows that if $\{q_n\}_{n=0}^\infty$ has polynomial growth then so does $\{u_n\}_{n=0}^\infty$.

We end this section with some remarks on the injectivity of the maps T_n in case R is a local ring. Let (R, \mathfrak{m}) be a local Noetherian ring, I an ideal in R and M an R/I -module. Suppose (\mathbf{Q}, e) and (\mathbf{P}, d) are minimal R -projective resolutions of M and I respectively. Let $(\bar{\mathbf{U}}, \bar{h})$ be the R/I -projective resolution of M constructed in Theorem 1.1 with $\bar{U}_n = R/I \otimes_R U_n$ where $U_0 = Q_0$, $U_1 = Q_1$, and $U_n = Q_n \amalg \left(\coprod_{i=0}^{n-2} (P_{n-i-2} \otimes_R U_i) \right)$ for $n \geq 2$. We have the following interpretation of the injectivity of the maps

$$T_n : \text{Tor}_n^R(M, R/\mathfrak{m}) \rightarrow \text{Tor}_n^{R/I}(M, R/\mathfrak{m}).$$

We have $\Omega_{R/I}^1(M) = \ker(e_0)/IU_0$ and $\Omega_{R/I}^n(M) = h_{n-1}^{-1}(IU_{n-2})/(IU_{n-1})$, for $n \geq 1$ where $\Omega_{R/I}^n(M)$ is the n^{th} syzygy of $(\bar{\mathbf{U}}, \bar{h})$ (see Theorem 2.1).

Thus $\text{Tor}_0^{R/I}(M, R/\mathfrak{m}) = U_0/(\mathfrak{m}U_0)$ and for $n \geq 1$, $\text{Tor}_n^{R/I}(M, R/\mathfrak{m}) = h_n^{-1}(\mathfrak{m}U_{n-1})/(h_{n+1}(U_{n+1}) + \mathfrak{m}U_n)$. Furthermore, for $n \geq 0$, $\text{Tor}_n^R(M, R/\mathfrak{m}) = Q_n/(\mathfrak{m}Q_n)$

and T_n is defined by

$$T_n(q_n + \mathfrak{m}Q_n) = (q_n, 0, \dots, 0) + h_{n+1}(U_{n+1}) + \mathfrak{m}U_n$$

where $q_n \in Q_n$, for $n \geq 1$. Now T_0 is an isomorphism and T_1 is injective if and only if

$h_2(U_2) \subseteq \mathfrak{m}U_1$. For $n \geq 2$, T_n is injective if and only if for every $u_{n+1} \in U_{n+1}$ such that

$h_{n+1}(u_{n+1}) \in Q_n \amalg (\mathfrak{m}(\amalg_{i=0}^{n-2} P_{n-i-2} \otimes_R U_i))$ we must have $h_{n+1}(u_{n+1}) \in \mathfrak{m}U_n$.

Chapter III. Applications to Local Rings: Golod Pairs

In this chapter we assume R is a Noetherian local ring with maximal ideal \mathfrak{m} . All modules will be assumed to be finitely generated. If I is an ideal in R and M is an R/I -module, we let $\overline{\mathfrak{m}} = \mathfrak{m}/I$ and assume we are given R -projective resolutions $(\mathbf{Q}, e.)$ and $(\mathbf{P}, d.)$ of M and I respectively. The basic question we consider is:

When is the R/I -projective resolution of M , $(\overline{\mathbf{U}}, \overline{h.})$ of Theorem 2.1 minimal?

We say (M, I) is a *Golod pair over R* if $(\overline{\mathbf{U}}, \overline{h.})$ is a minimal R/I -projective resolution of M . We will shortly see that if $M = R/\mathfrak{m}$ then (M, I) is a Golod pair if (and only if) I is a Golod ideal.

If X is an R -module we let $v(X)$ denote the minimal number of generators of X . Recall that if $\dots \longrightarrow S_n \xrightarrow{f_n} S_{n-1} \longrightarrow \dots \longrightarrow S_0 \xrightarrow{f_0} X \longrightarrow 0$ is an R -projective resolution of X then the *Poincaré series of X* with respect of $(\mathbf{S}, f.)$, $\mathcal{P}_{\mathbf{S}}^R(X)$, is

$$\sum_{i=0}^{\infty} v(S_i)z^i$$

where z is an indeterminant. If $(\mathbf{S}, f.)$ is a minimal R -projective resolution of X , we will sometimes denote $\mathcal{P}_{\mathbf{S}}^R(X)$ as simply $\mathcal{P}^R(X)$ and call it *the Poincaré series of X* .

From the description of U_0, U_1 and U_n in Theorem 1.1, it is easy to prove

$$\mathcal{P}_{\overline{\mathbf{U}}}^{R/I}(M) = \frac{\mathcal{P}_{\mathbf{Q}}^R(M)}{1 - z^2 \mathcal{P}_{\mathbf{P}}^R(I)}$$

Since $0 \rightarrow I \rightarrow R \rightarrow R/I \rightarrow 0$ is exact, if we let \mathbf{P}' be

$$\dots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \dots \longrightarrow P_0 \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

then we obtain,

$$\mathcal{P}_{\overline{\mathbf{U}}}^{R/I}(M) = \frac{\mathcal{P}_{\mathbf{Q}}^R(M)}{1 + z - z \mathcal{P}_{\mathbf{P}'}^R(R/I)}$$

If we have formal power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$, and if for $n \geq 0$, $a_n \leq b_n$, we write this as follows.

$$\sum_{n=0}^{\infty} a_n z^n \leq \sum_{n=0}^{\infty} b_n z^n$$

. With the above convention, by the previous calculation we have shown

THEOREM 3.1. (Gover and Salmon [5])

$$\mathcal{P}^{R/I}(M) \leq \frac{\mathcal{P}^R(M)}{1 + z - z\mathcal{P}^R(R/I)}.$$

We see that our definition of Golod pair in the case $M = R/\mathfrak{m}$ is the same as in the literature [8,9]. Namely, if $M = R/\mathfrak{m}$, the following are equivalent:

- (i) $\mathcal{P}^{R/I}(M) = \mathcal{P}_{\bar{U}}^{R/I}(M)$.
- (ii) (\bar{U}, \bar{h}) is a minimal R/I -projective resolution of M .
- (iii) $\mathcal{P}^{R/I}(M) = \frac{\mathcal{P}^R(M)}{1+z-z\mathcal{P}^R(R/I)}$.
- (iv) I is a Golod ideal.

Much of the literature in the area is concerned with the case when $M = R/\mathfrak{m}$. As remarked above, in this case $(R/\mathfrak{m}, I)$ is a Golod pair if and only if I is a Golod ideal. Levin [8] showed that for n sufficiently large, \mathfrak{m}^n is a Golod ideal. Shamash [11] proved that $(M, (a))$ is a Golod pair if $a \in \mathfrak{m}ANN(M)$ where $ANN(X)$ denotes the R -annihilator of X . When $M = R/\mathfrak{m}$, this result was proved by Tate [12]. Avramov [2] has shown that if \mathfrak{z} is a nonzero divisor in R and I is a proper ideal then $(R/\mathfrak{m}, \mathfrak{z}I)$ is a Golod pair. Before constructing new Golod pairs we make some preliminary observations.

Our first result is immediate from the definitions.

LEMMA 3.2. Let (R, \mathfrak{m}) be a local Noetherian ring and I an ideal in R . Suppose M_1 and M_2 are R/I -modules. Then

- (i) (M_1, I) and (M_2, I) are Golod pairs if and only if $(M_1 \amalg M_2, I)$ is a Golod pair.

(ii) $(R/I, I)$ is not a Golod pair and hence if (M_1, I) is a Golod pair then M_1 has no nonzero free summands.

PROPOSITION 3.3. *Let (R, \mathfrak{m}) be a local Noetherian ring, I an ideal in R and M an R/I -module. Suppose (\mathbf{Q}, e) and (\mathbf{P}, d) are minimal R -projective resolutions of M and I respectively. Let $(\bar{\mathbf{U}}, \bar{h})$ be the R/I -projective resolution of M constructed in Theorem 1.1 with $\bar{U}_n = R/I \otimes_R U_n$ where $U_0 = Q_0, U_1 = Q_1$, and $U_n = Q_n \amalg \left(\prod_{i=0}^{n-2} (P_{n-i-2} \otimes_R U_i) \right)$ for $n \geq 2$. Then the following statements are equivalent:*

- (i) $(\bar{\mathbf{U}}, \bar{h})$ is a minimal R/I -projective resolution of M .
- (ii) $IU_0 \subseteq \mathfrak{m} \ker(h_0)$, $h_{n+1} : U_{n+1} \rightarrow h_n^{-1}(IU_{n-1})$ is an R -projective cover and $IU_n \subseteq \mathfrak{m} h_n^{-1}(IU_{n-1})$ for $n \geq 1$.
- (iii) $\bar{h}_n(\bar{U}_n) \subseteq \overline{\mathfrak{m}} \bar{U}_{n-1}$ for $n \geq 1$.
- (iv) $h_n(U_n) \subseteq \mathfrak{m} U_{n-1}$ for $n \geq 1$.
- (v) a. $h_2(U_2) \subseteq \mathfrak{m} U_1$.

b. for $2 \leq r \leq n-1$ and $n \geq 3$.

$$h_n \left(Q_n \amalg \left(\prod_{i=0}^r (P_{n-i-2} \otimes_R U_i) \right) \right) \subseteq \mathfrak{m} \left(Q_{n-1} \amalg \left(\prod_{i=0}^r (P_{n-i-3} \otimes_R U_i) \right) \right).$$

c. for $n \geq 3$, $h_n(P_0 \otimes U_{n-2}) \subseteq \mathfrak{m} U_{n-1}$.

PROOF: The equivalences of (i), (ii), (iii) and (iv) are immediate from Theorem 2.1 and the fact that $\bar{h}_n = 1_{R/I} \otimes h_n$. The equivalence of (v) with (i) follows from Theorem 2.1B(e) and (i).

The next result is not surprising.

PROPOSITION 3.4. *If $(\bar{\mathbf{U}}, \bar{h})$ is a minimal R/I -projective resolution of M then (\mathbf{Q}, e) is a minimal R -projective resolution of M and (\mathbf{P}, d) is a minimal R -projective resolution of I .*

PROOF: Suppose that $(\bar{\mathbf{U}}, \bar{h})$ is a minimal R/I -projective resolution of M . By Proposition 3.3 (iv), $h_n(U_n) \subseteq \mathfrak{m} U_{n-1}$. In particular, by Theorem 2.1B (a) and (d) we must have that

$e_n(Q_n) \subseteq \mathfrak{m}Q_{n-1}$ for $n \geq 1$. Hence (\mathbf{Q}, e) is a minimal R -projective resolution of I . The minimality of (\mathbf{P}, d) follows from this and the construction of the resolution $(\overline{\mathbf{U}}, \overline{h})$, (cf. Section 4).

We now present a new proof of a theorem of Levin [10].

THEOREM 3.5. *Let (R, \mathfrak{m}) be a local Noetherian ring. Suppose (M, I) is a Golod pair and $(\overline{\mathbf{U}}, \overline{h})$ is a minimal R/I -projective resolution of M . Then for $n \geq 0$ $(\Omega_{R/I}^{n+1}(M), I)$ is also a Golod pair, where $\Omega_{R/I}^{n+1}(M)$ is the $(n+1)$ st R/I -syzygy of M .*

PROOF: Let $(\overline{\mathbf{U}}, \overline{h})$ be the R/I -minimal resolution of M constructed in Theorems 2.1 and 2.2. Then

$$\cdots \longrightarrow \overline{U}_{n+m+1} \xrightarrow{\overline{h}_{n+m+1}} \overline{U}_{n+m} \longrightarrow \cdots \xrightarrow{\overline{h}_{n+1}} \overline{U}_{n+1} \xrightarrow{\pi_{n+1}} \Omega_{R/I}^{n+1}(M) \longrightarrow 0$$

is a minimal R/I -projective resolution of $\Omega_{R/I}^{n+1}M$, where π_{n+1} is the map induced by \overline{h}_{n+1} .

We now construct a minimal R -resolution of $\Omega_{R/I}^{n+1}(M)$. Let $V_0 = U_{n+1}$, $V_1 = U_{n+2}$ and $V_m = Q_{n+m+1} \amalg (\amalg_{i=0}^n P_{m+n-i-1} \otimes_R U_i)$ for $m \geq 2$.

Consider

$$\cdots \longrightarrow V_m \xrightarrow{f_m} V_{m-1} \longrightarrow \cdots \longrightarrow V_0 \xrightarrow{f_0} \Omega_{R/I}^{n+1}(M) \longrightarrow 0 \quad (*)$$

where $f_0 : V_0 \rightarrow \Omega_{R/I}^{n+1}(M)$ is the canonical composition

$$U_{n+1} \longrightarrow \overline{U}_{n+1} \xrightarrow{\pi_{n+1}} \Omega_{R/I}^{n+1}(M)$$

$f_1 : V_1 \rightarrow V_0$ is $h_{n+2} : U_{n+2} \rightarrow U_{n+1}$ and, for $m \geq 2$, $f_m : V_m \rightarrow V_{m-1}$ is the map induced from $h_{m+n+1} |_{Q_{n+m+1}} \amalg (\amalg_{i=0}^n P_{m+n-i-1} \otimes_R U_i) : V_m \rightarrow U_{m+n}$ since, by Theorem 2.1 B(e), $\text{im}(h_{m+n+1} |_{V_m}) \subseteq V_{m-1}$.

Now for $n \geq 2$, the exactness of $(*)$ follows from Theorem 2.1 B(e). For $n = 1$ $\Omega_{R/I}^1(M) = \ker e_0/IQ_0$ and again the exactness of $(*)$ follows from Theorem 2.1 B(e) and from the fact that $f_0 = Q_1 \rightarrow \ker e_1/IQ_0$ is induced from $e_1 : Q_1 \rightarrow Q_0$. Thus $(*)$ is an

R -projective resolution of $\Omega_{R/I}^{n+1}(M)$. Furthermore, Proposition 3.3 implies that (*) is a minimal R -projective resolution of $\Omega_{R/I}^{n+1}(M)$.

To prove that $(\Omega_{R/I}^{n+1}(M), I)$ is a Golod pair, we must show that the R/I -module resolution $(\hat{\mathbf{W}}., \hat{h}.)$ of $\Omega_{R/I}^{n+1}(M)$ induced from $(\mathbf{V}., f.)$ and $(\mathbf{P}., d.)$ yields a minimal R/I -projective resolution of M . So, by Theorem 2.1 and the remarks at the beginning of the section, we must prove $\overline{W}_m = R/I \otimes_R V_m \amalg (\amalg_{i=0}^{m-2} (P_{m-i-2} \otimes_R W_i)) = \overline{U}_{m+n+1}$, $m \geq 2$ where $W_0 = V_0$, $W_1 = V_1$ and

$$W_m = V_m \otimes_R \left(\prod_{i=0}^{m-2} (P_{m-i-2} \otimes W_i) \right).$$

We proceed by induction. Since $V_0 = U_{n+1}$ and $V_1 = U_{n+2}$ we see that $\overline{W}_0 = \overline{U}_{n+1}$ and $\overline{W}_1 = \overline{U}_{n+2}$. Assuming $W_i \cong U_{n+i+1}$ we have

$$\begin{aligned} W_m &= V_m \prod_{i=0}^{m-2} \left(\prod_{i=0}^{m-2} (P_{m-i-2} \otimes W_i) \right) \\ &\cong Q_{n+m+1} \prod_{i=0}^n \left(\prod_{i=0}^n P_{m+n-i-1} \otimes U_i \right) \prod_{i=0}^{m-2} \left(\prod_{i=0}^{m-2} (P_{m-i-2} \otimes U_{n+i+1}) \right) \\ &= U_{m+n+1} \end{aligned}$$

This completes the proof.

As a corollary to the last theorem, we obtain another characterization of Golod pairs.

COROLLARY 3.6. *(M, I) is a Golod pair if and only if the n^{th} -syzygy of the resolution $(\overline{\mathbf{U}}., \overline{h}.)$ has no nonzero free summands for $n \geq 0$.*

PROOF: Assume that (M, I) is a Golod pair. It follows from the last theorem that $(\Omega_{R/I}^n(M), I)$ is a Golod pair. Thus, by Lemma 3.2(ii), $\Omega_{R/I}^n(M)$ has no free summands.

On the other hand, if $\Omega_{R/I}^n(M)$ has no free summands then the resolution $(\overline{\mathbf{U}}., \overline{h}.)$ is minimal. Hence (M, I) is a Golod pair.

Before investigating the converse of Theorem 3.5, we apply the remarks following Theorem 2.1. It is clear from the construction of the maps that if (M, I) is a Golod pair then

$T_n : \text{Tor}_n^R(M, R/\mathfrak{m}) \rightarrow \text{Tor}_n^{R/I}(M, R/\mathfrak{m})$ is injective for $n \geq 0$. It follows from this and Theorem 3.5 that if (M, I) is a Golod pair then for $m \geq 1, i \geq 0$ the natural maps

$$T_n^m : \text{Tor}_n^R(\Omega_{R/I}^m(M), R/\mathfrak{m}) \longrightarrow \text{Tor}_n^{R/I}(\Omega_{R/I}^m(M), R/\mathfrak{m})$$

are injective. Conversely, Levin [10] proved that if

$$T_n : \text{Tor}_n^R(M, R/\mathfrak{m}) \longrightarrow \text{Tor}_n^{R/I}(M, R/\mathfrak{m})$$

and

$$T_n^1 : \text{Tor}_n^R(\Omega_{R/I}^1(M), R/\mathfrak{m}) \longrightarrow \text{Tor}_n^{R/I}(\Omega_{R/I}^1(M), R/\mathfrak{m})$$

are injective for $n \geq 0$ then (M, I) is a Golod pair.

If $T_n : \text{Tor}_n^R(M, R/\mathfrak{m}) \longrightarrow \text{Tor}_n^{R/I}(M, R/\mathfrak{m})$ is injective then for $n \geq 2$ the image of the following composition

$$Q_n \coprod (P_{n-2} \otimes_R U_0) \xrightarrow{\text{inclusion}} U_n \xrightarrow{h_n} U_{n-1}$$

is contained in $\mathfrak{m}U_{n-1}$ (or, more precisely, in $\mathfrak{m}Q_{n-1} \coprod (\mathfrak{m}P_{n-3} \otimes_R U_0) \coprod \underbrace{0 \coprod \cdots \coprod 0}_{n-3} \subseteq U_{n-1}$). In particular, we see that the R -projective resolution (*) of $\Omega_{R/I}^1(M)$ constructed in Theorem 3.5 is minimal. If $T_n : \text{Tor}_n^R(R/\mathfrak{m}, R/\mathfrak{m}) \rightarrow \text{Tor}_n^{R/I}(R/\mathfrak{m}, R/\mathfrak{m})$ is injective for all $n \geq 0$ then Avramov [2] calls I a small ideal. Thus, if I is a small ideal then we have an explicit R -projective resolution of $\Omega_{R/I}^1(R/\mathfrak{m})$. Although the converse of Theorem 3.5 is false in general, (see remarks after Corollary 4.4 in the next section), the next result provides a special case where the converse holds.

THEOREM 3.7. *Let (R, \mathfrak{m}) be a Noetherian local ring, I an ideal in R and M an R/I -module. Suppose (\mathbf{Q}, e) is a minimal R -projective resolution of M and (\mathbf{P}, d) is a minimal R -projective resolution of I . Let K denote $\Omega_{R/I}^1(M)$. Then*

- (1) *there is an R -projective resolution (\mathbf{Q}', e') of $K = \ker(1 \otimes e_0) : (R/I) \otimes_R Q_0 \rightarrow (R/I) \otimes_R M$ such that $Q'_0 = Q_1$ and for $i \geq 1, Q'_i = Q_{i+1} \coprod (P_{i-1} \otimes_R Q_0)$, and*

(2) (M, I) is a Golod pair if and only if (K, I) is a Golod pair and (\mathbf{Q}', e') is a minimal R -projective resolution of K .

PROOF: The proof of part (1) follows immediately from (*) in the proof of Theorem 3.5. If (M, I) is a Golod pair, it also follows from the proof of Theorem 3.5 that the R -projective resolution (\mathbf{Q}', e') of K in (1) is minimal and that (K, I) is a Golod pair.

Finally, suppose that (K, I) is a Golod pair and (\mathbf{Q}', e') is a minimal R -projective resolution of K . We wish to prove (M, I) is Golod pair. We prove this using Poincaré series and equivalence of (M, I) being a Golod pair with $\mathcal{P}^{R/I}(M) = \mathcal{P}^R(M)/1 + z - z\mathcal{P}^R(R/I)$. We have $\mathcal{P}^{R/I}(M) = v(Q_0) + z\mathcal{P}^{R/I}(K)$. By assumption that (K, I) is a Golod pair, we get

$$\begin{aligned} & \mathcal{P}^{R/I}(M) \\ &= v(Q_0) + \frac{z\mathcal{P}^R(K)}{1 + z - z\mathcal{P}^R(R/I)} \\ &= \frac{v(Q_0) + v(Q_0)z + z(\mathcal{P}^R(K) - v(Q_0)\mathcal{P}^R(R/I))}{1 + z - z\mathcal{P}^R(R/I)} \\ &= \frac{v(Q_0) + v(Q_0)z + z[(v(Q_1) - v(Q_0) + v(Q_2)z + v(Q_3)z^2 + \dots)]}{1 + z - z\mathcal{P}^R(R/I)} \\ &= \frac{\mathcal{P}^R(M)}{1 + z - z\mathcal{P}^R(R/I)}. \end{aligned}$$

where the third equality follows from (1). Hence (M, I) is a Golod pair.

COROLLARY 3.8. Let (R, \mathfrak{m}) be a Noetherian local ring and M an R -module. Suppose $Q \xrightarrow{\pi} M$ is a projective cover of M and $K = \ker \pi/xQ$ where $x \in \text{ANN}(M)$ is a nonzero divisor in R such that $xQ \in \mathfrak{m}\ker \pi$. Then if $(K, (a))$ is a Golod pair then $(M, (a))$ is a Golod pair.

PROOF: Let (\mathbf{Q}, e) be a minimal resolution of M . Then, since xQ_0 is a projective R -module, the following short exact sequence splits

$$0 \longrightarrow \ker e_1 \longrightarrow e_1^{-1}(xQ_0) \xrightarrow{e_1} xQ_0 \longrightarrow 0.$$

It follows that we get a minimal R -projective resolution of K of the form

$$\cdots \longrightarrow Q_4 \xrightarrow{e_4} Q_3 \xrightarrow{(e_3, 0)} Q_2 \amalg Q_0 \xrightarrow{e'_1} Q_1 \longrightarrow K \longrightarrow 0$$

where $e'_1 : Q_2 \amalg Q_0 \rightarrow Q_1$ is given by $e' = \begin{pmatrix} e_2 \\ g \end{pmatrix}$ where g is the composition $Q_0 \longrightarrow e_1^{-1}(xQ_0) \xrightarrow{\text{inclusion}} Q_1$ with the first map being an isomorphism $Q_0 \rightarrow xQ_0$ and then splitting of e_1 . The result now follows from Theorem 3.7.

The next result provides an interesting property of the R/I -resolution of M , (\bar{U}, \bar{h}) .

PROPOSITION 3.9. *Let (\bar{U}, \bar{h}) be the resolution of M constructed in Theorem 2.1 and let $\bar{\mathfrak{m}} = \mathfrak{m}/I$. If for some $n \geq 2$, $\bar{h}_n(\bar{U}_n) \not\subseteq \bar{\mathfrak{m}}\bar{U}_{n-1}$ then $\bar{h}_{n+2}(\bar{U}_{n+2}) \not\subseteq \bar{\mathfrak{m}}\bar{U}_{n+1}$. Hence if $\bar{h}_n(\bar{U}_n) \subseteq \bar{\mathfrak{m}}\bar{U}_{n-1}$ then $\bar{h}_{n-2}(\bar{U}_{n-2}) \subseteq \bar{\mathfrak{m}}\bar{U}_{n-3}$ (where $\bar{U}_{-1} = M$). In particular if $\bar{h}_n(\bar{U}_n) \subseteq \bar{\mathfrak{m}}\bar{U}_{n-1}$ and $\bar{h}_{n-1}(\bar{U}_{n-1}) \subseteq \bar{\mathfrak{m}}\bar{U}_{n-2}$ then*

$$\bar{U}_{n-1} \xrightarrow{\bar{h}_{n-1}} \bar{U}_{n-2} \longrightarrow \cdots \longrightarrow \bar{U}_0 \longrightarrow M \longrightarrow 0$$

is the beginning of a minimal R/I -projective resolution of M .

PROOF: Suppose $\bar{h}_n(\bar{U}_n) \not\subseteq \bar{\mathfrak{m}}\bar{U}_{n-1}$. It follows from Theorem 2.1(i) that the following composition

$$\frac{P_0 \otimes_R U_n}{I(P_0 \otimes_R U_n)} \xrightarrow{\text{inclusion}} \bar{U}_{n+2} \xrightarrow{\bar{h}_{n+2}} \bar{U}_{n+1} \xrightarrow{\text{projection}} \frac{P_0 \otimes_R U_{n-1}}{I(P_0 \otimes_R U_{n-1})}$$

is $\overline{1p_0 \otimes h_n}$, (where $\overline{1p_0 \otimes h_n}$ is the map induced by $1p_0 \otimes h_n : P_0 \otimes_R U_n \rightarrow P_0 \otimes_R U_{n-1}$).

Hence

$$\overline{1p_0 \otimes h_n} \left(\frac{P_0 \otimes_R U_n}{I(P_0 \otimes_R U_n)} \right) \not\subseteq \bar{\mathfrak{m}} \left(\frac{P_0 \otimes_R U_{n-1}}{I(P_0 \otimes_R U_{n-1})} \right).$$

In particular $\bar{h}_{n+2}(\bar{U}_{n+2}) \not\subseteq \bar{\mathfrak{m}}\bar{U}_{n+1}$. The other implications follow immediately from this.

We now note the following result.

PROPOSITION 3.10. *Let R be a Noetherian commutative ring (not necessarily local), $I \subset R$ be an ideal and let M be a finitely generated R/I -module. Let (\mathbf{Q}, e) and (\mathbf{P}, d) be R -free*

resolutions of M and I respectively. Let (\bar{U}, \bar{h}) be the R/I -free resolution of M given by Theorem 2.1. Choose bases B_n of the free modules \bar{U}_n and suppose that $\bar{h}_n : \bar{U}_n \rightarrow \bar{U}_{n-1}$ is represented by a matrix $[x_{ij}^n]$ over R/I relative to these bases. Then the ideals $J_n = (x_{ij}^n)$ in R/I satisfy, $J_1 \subset J_3 \subset \dots$, and $J_2 \subset J_4 \subset \dots$. In particular if $J^o = \cup_{n, \text{odd}} J_n$, $J^e = \cup_{n, \text{even}} J_n$ and if $J = J^o + J^e$ is a proper ideal then for each prime ideal $\mathfrak{p} \supset J$, the $(R/I_{\mathfrak{p}})$ -projective resolution (\bar{V}, \bar{k}) of $M_{\mathfrak{p}}$ obtained by localizing the resolution (\bar{U}, \bar{h}) at \mathfrak{p} is minimal. In particular $(M_{\mathfrak{p}}, I_{\mathfrak{p}})$ is a Golod pair over $R_{\mathfrak{p}}$.

PROOF: The first part of the proposition follows from Theorem 2.1 B(i). The second part is obvious.

Unfortunately, the next example shows that we may have

$$\bar{U}_{n-1} \xrightarrow{\bar{h}_{n-1}} \bar{U}_{n-2} \longrightarrow \Omega_{R/I}^{n-2}(M) \xrightarrow{\bar{h}_{n-2}} 0$$

a minimal R/I -projective presentation of $\Omega_{R/I}^{n-2}(M)$ and yet the resolution (\bar{U}, \bar{h}) of Theorem 2.1 need not be minimal.

Example 3.11 Let $R = K[[x, y]]$ be a power series ring in two variables over a field K . Let $M = k$ and $I = (x^2, y^2)$. A minimal R -projective resolution of M is $0 \rightarrow \wedge^2 R \rightarrow R^2 \rightarrow (x^2, y^2) \rightarrow 0$. Thus $U_0 \cong R$, $U_1 \cong R^2$, $U_2 = \wedge^2 R^2 \coprod (R^2 \otimes_R R)$ and $U_3 = (R^2 \otimes_R R^2) \coprod (\wedge^2 R \otimes_R R)$. In particular U_0 is rank 1, U_1 is rank 2, U_2 is rank 3 while U_3 is rank 5.

Since the minimal R/I -projective resolution $\dots \rightarrow \bar{V}_3 \rightarrow \bar{V}_2 \rightarrow \bar{V}_1 \rightarrow \bar{V}_0 \rightarrow K \rightarrow 0$ of K has the property that the minimal number of R/I -generators of \bar{V}_i is $i + 1$ we see that $\bar{U}_2 \xrightarrow{\bar{h}_2} \bar{U}_1 \xrightarrow{\bar{h}_1} \bar{U}_0 \longrightarrow K \longrightarrow 0$ is part of a minimal R/I -projective resolution of K while $\bar{U}_3 \xrightarrow{\bar{h}_3} \bar{U}_2 \xrightarrow{\bar{h}_2} \bar{U}_1 \longrightarrow K \longrightarrow 0$ is not.

We conclude this section with some remarks about construction of a minimal resolution and about the Poincaré series in a special case. It follows from a theorem of Levin ([9], Thm 4.6, page 61) that for $n \gg 0$, $(R/\mathfrak{m}^{n-1}, \mathfrak{m}^n)$ is a Golod pair. Choose minimal projective R -resolutions (\mathbf{Q}, e) and (\mathbf{P}, d) of R/\mathfrak{m}^{n-1} and \mathfrak{m}^n respectively. Since $(R/\mathfrak{m}^{n-1}, \mathfrak{m}^n)$ is

a Golod pair, we have

$$T_j : \text{Tor}_j^R(R/\mathfrak{m}^{n-1}, R/\mathfrak{m}) \longrightarrow \text{Tor}_j^{R/\mathfrak{m}^n}(R/\mathfrak{m}^{n-1}, R/\mathfrak{m})$$

is injective for $j \geq 0$. The following R -projective resolution of $\Omega_{R/\mathfrak{m}^n}^1(R/\mathfrak{m}^{n-1}) \cong \mathfrak{m}^{n-1}/\mathfrak{m}^n$ is minimal

$$\cdots \rightarrow V_m \xrightarrow{f_m} V_{m-1} \rightarrow \cdots \rightarrow V_0 \rightarrow \Omega_{R/\mathfrak{m}^n}^1(R/\mathfrak{m}^{n-1}) \rightarrow 0$$

(see (*) of Theorem 3.5). Recall that $V_0 = Q_1$ and $V_m = Q_{m+1} \amalg (P_{m-1} \otimes_R Q_0) \cong Q_{m+1} \amalg P_{m-1}$. Thus we have a new construction of a minimal R -projective resolution of a vector space of dimension $\dim_{R/\mathfrak{m}}(\mathfrak{m}^{n-1}/\mathfrak{m}^n)$. It follows that if $\mathcal{P}^R(*)$ denotes the Poincaré series of $*$ in the indeterminate z , then

$$\dim_{R/\mathfrak{m}}(\mathfrak{m}^{n-1}/\mathfrak{m}^n)\mathcal{P}^R(R/\mathfrak{m}) = \mathcal{P}^R(\mathfrak{m}^{n-1}) + z\mathcal{P}^R(\mathfrak{m}^n).$$

Chapter IV. Further Results and Intermediate Ideals

In this section we continue to assume that (R, \mathfrak{m}) is a Noetherian local ring. If $J \subseteq I$ are ideals and M is an R/I -module, we consider the relationship of the following three conditions:

- (i) (M, I) is a Golod pair (over R)
- (ii) (M, J) is a Golod pair (over R)
- (iii) $(M, I/J)$ is a Golod pair (over R/J)

Gover and Salmon [5] showed that if $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n$ is a chain of ideals in R and $\mathbf{x}_1, \dots, \mathbf{x}_n$ is a sequence of elements in R such that \mathbf{x}_1 is R -regular and for $j = 2, \dots, n$, \mathbf{x}_j is $R/(\mathbf{x}_1 I_1 + \cdots + \mathbf{x}_{j-1} I_{j-1})$ -regular then, setting $\mathbf{k} = R/\mathfrak{m}$, $(\mathbf{k}, \mathbf{x}_1 I_1 + \cdots + \mathbf{x}_n I_n)$ is a Golod pair over R . In particular, for $2 \leq r \leq n$, $(\mathbf{k}, \mathbf{x}_1 I_1 + \cdots + \mathbf{x}_r I_r)$ and $(\mathbf{k}, \mathbf{x}_1 I_1 + \cdots + \mathbf{x}_{r-1} I_{r-1})$ are Golod pairs over R and $(\mathbf{k}, (\mathbf{x}_1 I_1 + \cdots + \mathbf{x}_r I_r)/(\mathbf{x}_1 I_1 + \cdots + \mathbf{x}_{r-1} I_{r-1}))$ is a Golod pair over $R/(\mathbf{x}_1 I_1 + \cdots + \mathbf{x}_{r-1} I_{r-1})$.

If X is a finitely generated R -module, we let $v(X)$ denote the minimal number of generators of X .

PROPOSITION 4.1. *Suppose $J \subseteq I \subseteq \mathfrak{m} \subseteq R$ and M is a finitely generated R/I -module such that*

- (i) (M, I) is a Golod pair over R
- (ii) (M, J) is a Golod pair over R and
- (iii) $(M, I/J)$ is a Golod pair over R/J .

Then $v(I) = v(J) + v(I/J)$.

PROOF: Suppose $(\mathbf{Q}\cdot, e\cdot)$, $(\mathbf{P}\cdot, d\cdot)$ and $(\mathbf{P}'\cdot, d'\cdot)$ are minimal R -projective resolutions of M , I and J respectively. Suppose that $(\hat{\mathbf{P}}\cdot, \hat{d}\cdot)$ is a minimal projective R/J -projective resolution of I/J . Using $(\mathbf{Q}\cdot, e\cdot)$ and $(\mathbf{P}'\cdot, d'\cdot)$, we apply Theorem 2.1 to get an R/J -resolution $(\mathbf{U}'\cdot, h'\cdot)$ of M . By hypothesis, $(\mathbf{U}'\cdot, h'\cdot)$ is minimal. Note that $U'_2 = R/J \otimes_R (Q_2 \amalg (P'_0 \otimes_R Q_0))$.

Using (\mathbf{U}', h') and $(\hat{\mathbf{P}}, \hat{d})$ we apply Theorem 2.1 to get an $(R/J)/(I/J) \simeq R/I$ -projective resolution $(\hat{\mathbf{U}}, \hat{h})$ of M . By hypothesis $(\hat{\mathbf{U}}, \hat{h})$ is minimal. Note that

$$\begin{aligned}\hat{U}_2 &= (R/J)/(I/J) \otimes_{R/I} [U'_2 \coprod (\hat{P}_0 \otimes_{R/I} U'_0)] \\ &\approx (R/J)/(I/J) \otimes_{R/I} [R/J \otimes_R [Q_2 \coprod (P'_0 \otimes_R Q_0)] \coprod [\hat{R}_0 \otimes_{R/I} (R/I \otimes_R Q_0)]]\end{aligned}$$

Thus $v(\hat{U}_2) = v(Q_2) + v(P'_0) \cdot v(Q_0) + v(\hat{P}_0) \cdot v(Q_0)$.

Finally, using (\mathbf{Q}, e) and (\mathbf{P}, d) , we apply Theorem 1.1 to obtain an R/I -projective resolution $(\bar{\mathbf{U}}, \bar{h})$ of M . By hypothesis $(\bar{\mathbf{U}}, \bar{h})$ is minimal. Note that

$$\bar{U}_2 = R/I \otimes_R (Q_2 \coprod (P_0 \otimes_R Q_0)).$$

Thus $v(\bar{U}_2) = v(Q_2) + v(P_0) \cdot v(Q_0)$. By minimality of the resolutions, we conclude $v(\hat{U}_2) = v(\bar{U}_2)$ and hence $v(P_0) = v(P'_0) = v(\hat{P}_0)$. Again, by minimality of the resolutions (\mathbf{P}, d) , (\mathbf{P}', d') and $(\hat{\mathbf{P}}, \hat{d})$ we have $v(P_0) = v(I)$, $v(P'_0) = v(J)$ and $v(\hat{P}_0) = v(I/J)$ and we are done.

In general the converse is not true. The next result is a partial converse for the case where J is a principal ideal generated by a nonzero divisor of R .

THEOREM 4.2. *Let (R, \mathfrak{m}) be a local Noetherian ring and (M, I) a Golod pair over R . Let x_1, \dots, x_s , $s > 1$ be a minimal set of generators of I and assume x_1 is a nonzero divisor in R . Let $J = (x_1)$. Then*

- (i) (M, J) is a Golod pair (over R).
- (ii) $(I/J, J)$ is a Golod pair (over R).
- (iii) $(M, I/J)$ is a Golod pair (over R/J).

PROOF: Let (\mathbf{Q}, e) and (\mathbf{P}, d) be minimal R -projective resolutions of M and I respectively. Since (M, I) is a Golod pair, applying Theorem 1.1, we obtain a minimal R/I -projective resolution $(\bar{\mathbf{U}}, \bar{h})$ of M . We have $v(\bar{U}_0) = v(Q_0)$, $v(\bar{U}_1) = v(Q_1)$ and, for

$n \geq 2$

$$v(\bar{U}_n) = v(Q_n) + \sum_{i=0}^{n-2} v(P_{n-i-2})v(\bar{U}_i).$$

Next we construct an $\frac{R/(\mathbf{x}_1)}{I/(\mathbf{x}_1)}$ -projective resolution of M , $(\bar{V}., \bar{k}.)$. For this we need an $R/(\mathbf{x}_1)$ -projective resolution $(\mathbf{W}., l.)$ of M and an $R/(\mathbf{x}_1)$ -projective resolution $(\mathbf{X}., m.)$ of $I/(\mathbf{x}_1)$.

Consider the short exact sequence $0 \rightarrow (\mathbf{x}_1) \rightarrow I \rightarrow I/(\mathbf{x}_1) \rightarrow 0$. Choose a generating set y_1, \dots, y_s of P_0 such that $d_0(y_1) = \mathbf{x}_1$. Then we have the following commutative diagram with exact rows and columns.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & (\mathbf{x}_1) & \longrightarrow & I & \longrightarrow & I/(\mathbf{x}_1) \longrightarrow 0 \\
 & & \uparrow d_{01} & & \uparrow d_0 & & \uparrow d'_0 \\
 0 & \longrightarrow & (y_1) & \longrightarrow & P_0 & \longrightarrow & P_0/(y_1) \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & 0 & \longrightarrow & \ker d_0 & \longrightarrow & \ker d'_0 \longrightarrow 0 \\
 & & & & \uparrow & & \uparrow \\
 & & & & 0 & & 0
 \end{array}$$

Since $P_0 = (y_1) \amalg P_0/(y_1)$, we see $\ker d_0 \simeq \ker d'_0$. Thus we have a minimal R -projective resolution $(\mathbf{P}'., d'.)$ of $I/(\mathbf{x}_1)$ with $v(P'_0) = v(P_0) - 1$ and $v(P'_i) = v(P_i)$ for $i \geq 1$. Using $0 \longrightarrow R \xrightarrow{\mathbf{x}_1} (\mathbf{x}_1) \longrightarrow 0$ as the minimal R -projective resolution of (\mathbf{x}_1) , we apply Theorem 1.1 to $(\mathbf{Q}., e.)$ to obtain a $R/(\mathbf{x}_1)$ -projective resolution $(\mathbf{W}., l.)$ of M and apply Theorem 1.1 to $(\mathbf{P}'., d'.)$ to obtain an $R/(\mathbf{x}_1)$ -projective resolution $(\mathbf{X}., m.)$ of $I/(\mathbf{x}_1)$. We have that

- (a) $v(W_0) = v(Q_0)$, $v(W_1) = v(Q_1)$ and for $n \geq 2$ $v(W_n) = v(Q_n) + v(Q_{n-2}) + \dots$ and
- (b) $v(X_0) = v(P'_0) = v(P_0) - 1$, $v(X_1) = v(P'_1)$ and for $n \geq 2$ $v(X_n) = v(P'_n) + v(P'_{n-2}) + \dots$.

Apply Theorem 2.1 to $(\mathbf{W}., l.)$ and $(\mathbf{X}., m.)$ to obtain an $R/(\mathbf{x}_1)/I/(\mathbf{x}_1) \simeq R/I$ -projective resolution $(\bar{V}., \bar{k}.)$ of M . Since if $\bar{V}_n = R/(\mathbf{x}_1)/I/(\mathbf{x}_1) \otimes_{R/(\mathbf{x}_1)} V_n$ then $V_0 = W_0$, $V_1 = W_1$

and for $n \geq 2$ $V_n = W_n \amalg (\amalg_{i=0}^{n-2} X_{n-i-2} \otimes_{R/(\mathfrak{x}_1)} V_i)$ we get the formulas

$$v(\overline{V}_0) = v(W_0), \quad v(\overline{V}_1) = v(W_1)$$

and for $n \geq 2$

$$v(\overline{V}_n) = v(W_n) + \sum_{i=0}^{n-2} v(X_{n-i-2})v(V_i).$$

Since $(\overline{U}, \overline{h})$ is a minimal R/I -projective resolution of M , to prove $(M, I/(\mathfrak{x}_1))$ is a Golod pair over $R/(\mathfrak{x}_1)$, it suffices to show $v(\overline{V}_n) = v(\overline{U}_n)$. Once this is done, by Proposition 3.4, it will follow that both (W, l) and (X, m) are minimal $R/(\mathfrak{x}_1)$ -projective resolutions and hence both $(M, (\mathfrak{x}_1))$ and $(I/(\mathfrak{x}_1), (\mathfrak{x}_1))$ are Golod pairs.

It is easy to verify that $v(\overline{V}_i) = v(\overline{U}_i)$ for $i \leq 4$ directly. We finish the proof by induction. Assume $v(\overline{V}_i) = v(\overline{U}_i)$ for $i < n$, and $n \geq 4$. We prove $v(\overline{V}_n) = v(\overline{U}_n)$.

Consider $v(\overline{V}_n) - v(\overline{U}_n)$.

$$\begin{aligned} & v(\overline{V}_n) - v(\overline{U}_n) \\ &= v(W_n) - v(Q_n) + \sum_{i=0}^{n-2} v(X_{n-i-2})v(\overline{V}_i) - \sum_{i=0}^{n-2} v(P_{n-i-2})v(\overline{U}_i). \end{aligned}$$

By induction we conclude

$$\begin{aligned} & v(\overline{V}_n) - v(\overline{U}_n) \\ &= v(W_n) - v(Q_n) + \sum_{i=0}^{n-2} (v(X_{n-i-2}) - v(P_{n-i-2}))v(\overline{U}_i). \end{aligned}$$

By (a)

$$\begin{aligned} & v(\overline{V}_n) - v(\overline{U}_n) \\ &= (v(Q_{n-2}) + v(Q_{n-4}) + \cdots) + \sum_{i=0}^{n-4} [v(X_{n-i-2}) - v(P_{n-i-2})]v(\overline{U}_i) \\ & \quad + [v(X_1) - v(P_1)]v(\overline{U}_{n-3}) + [v(X_0) - v(P_1)]v(\overline{U}_{n-2}). \end{aligned}$$

Now $v(X_1) = v(P_1)$ and $v(X_0) - v(P_0) = -1$. Furthermore, for $n \geq 4$, $v(X_{n-i-2}) -$

$v(P_{n-i-2}) = v(X_{n-i-4})$. Thus we have

$$\begin{aligned} & v(\overline{V}_n) - v(\overline{U}_n) \\ &= (v(Q_{n-2}) + v(Q_{n-4}) + \cdots) + \sum_{i=0}^{n-4} [v(X_{n-i-2}) - v(P_{n-i-2})]v(\overline{U}_i) \\ & \quad - v(\overline{U}_{n-2}). \end{aligned}$$

Since $v(\overline{U}_{n-2}) = v(W_{n-2}) + \sum_{i=0}^{n-4} v(P_{n-i-4})v(\overline{U}_i)$ we conclude

$$\begin{aligned} & v(\overline{V}_n) - v(\overline{U}_n) \\ &= (v(Q_{n-4}) + v(Q_{n-6}) + \cdots) + \sum_{i=0}^{n-4} [v(X_{n-i-4}) - v(P_{n-i-4})]v(\overline{U}_i). \end{aligned}$$

But this is the formula for $v(\overline{V}_{n-2}) - v(\overline{U}_{n-2})$ and hence by induction, $v(\overline{V}_n) - v(\overline{U}_n) = 0$.

This completes the proof.

Recall that the canonical surjection $\pi : R \rightarrow R/I$ is a *Golod homomorphism* if $(R/\mathfrak{m}, I)$ is a Golod pair [8,9].

COROLLARY 4.3. *Let (R, \mathfrak{m}) be a local Noetherian ring and suppose an ideal I has a nonzero divisor \mathfrak{x} among a minimal set of generators. If the canonical surjection $\pi : R \rightarrow R/I$ is a Golod homomorphism then the canonical surjections $R \rightarrow R/(\mathfrak{x})$ and $R/(\mathfrak{x}) \rightarrow R/I$ are also Golod homomorphisms.*

PROOF: The result follows from Theorem 4.2 by setting $M = R/\mathfrak{m}$.

COROLLARY 4.4. *Let (R, \mathfrak{m}) be a local Noetherian ring, I an ideal in R and M an R/I -module. Suppose (M, I) is a Golod pair. If $\mathfrak{x} \in I$ is a nonzero divisor of R then $(M, (\mathfrak{x}))$ is a Golod pair.*

PROOF: If $\mathfrak{x} \in I - \mathfrak{m}I$ then the result follows from Theorem 4.2. If $\mathfrak{x} \in \mathfrak{m}I$ then $\mathfrak{x} \in \mathfrak{m}ANN(M)$ and the result follows from Shamash [11].

Next we show that converse to Theorem 3.5 is false. Suppose that (M, I) is a Golod pair and $\mathfrak{x} \in I - \mathfrak{m}I$ is a nonzero divisor in R . Theorem 4.2 implies that $(I/(\mathfrak{x}), (\mathfrak{x}))$ is a

Golod pair. Consider

$$0 \rightarrow I/(\mathbf{x}) \rightarrow R/(\mathbf{x}) \rightarrow R/I \rightarrow 0.$$

It follows that $I/(\mathbf{x}) = \Omega_{R/(\mathbf{x})}^1(R/I)$. But $(R/I, (\mathbf{x}))$ is not a Golod pair. For example, if $R = K[[x, y]]$ and $I = (x, y)^2$. Then it is not hard to show that I is a Golod ideal (i.e. (K, I) is a Golod pair). Hence $(I/(\mathbf{x}^2), (\mathbf{x}^2))$ is also a Golod pair. Now $I/(\mathbf{x}^2)$ is the first $R/(\mathbf{x}^2)$ -syzygy of R/I . Since $I/(\mathbf{x}^2)$ can be generated by two elements a minimal $R/(\mathbf{x}^2)$ -projective resolution of R/I begins $\cdots \rightarrow R/(\mathbf{x}^2) \amalg R/(\mathbf{x}^2) \rightarrow R/(\mathbf{x}^2) \rightarrow R/I \rightarrow 0$. But the minimal R -resolution of R/I -begins $\cdots R \amalg R \amalg R \rightarrow R \rightarrow R/I \rightarrow 0$. Hence $(R/I, (\mathbf{x}^2))$ is not a Golod pair.

Example 4.5 We construct a Golod ideal J in a zero dimensional Gorenstein local ring T such that $\text{soc}(T)$, the socle of T , is properly contained in J .

Let (R, \mathfrak{m}) be a regular local ring of dimension $r > 2$ and let $\mathbf{x}_1, \dots, \mathbf{x}_r$ be a system of generators of \mathfrak{m} . As remarked in Section 2, for sufficiently large n , $(R/\mathfrak{m}, \mathfrak{m}^n)$ is a Golod pair. For $1 \leq i \leq r$ let $T_i = R/(\mathbf{x}_1^n, \dots, \mathbf{x}_i^n)$. Let $\mathfrak{m}_i = \mathfrak{m}/(\mathbf{x}_1^n, \dots, \mathbf{x}_i^n)$. By Theorem 3.2, since $\mathbf{x}_i^n \in \mathfrak{m}^n - \mathfrak{m}^{n+1}$, $(R/\mathfrak{m}, \mathfrak{m}_i^n)$ is a Golod pair over T_1 . Assuming $(R/\mathfrak{m}, \mathfrak{m}_i^n)$ is a Golod pair for $i < r$ over T_i , since $\bar{\mathbf{x}}_{i+1}^n \in \mathfrak{m}_i^n - \mathfrak{m}_i^{n+1}$ where $\bar{\mathbf{x}}_{i+1} = \mathbf{x}_{i+1} + (\mathbf{x}_1^n, \dots, \mathbf{x}_i^n)$, Theorem 3.2 implies $(R/\mathfrak{m}, \mathfrak{m}_{i+1}^n)$ is a Golod pair over T_{i+1} .

Now let $T = T_r$ and $J = \mathfrak{m}_r^n$ is the required example.

We end this section with a result about Poincaré series.

THEOREM 4.6. *Let (R, \mathfrak{m}) be a local ring, I an ideal in R and let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a short exact sequence of R/I -modules such that*

i). (B, I) is a Golod pair over R and

ii). $\mathcal{P}^R(B) = \mathcal{P}^R(A) + \mathcal{P}^R(C)$,

then (A, I) and (C, I) are Golod pairs over R and $\mathcal{P}^{R/I}(B) = \mathcal{P}^{R/I}(A) + \mathcal{P}^{R/I}(C)$.

PROOF: We have

$$\begin{aligned}
\mathcal{P}^{R/I}(B) &\leq \mathcal{P}^{R/I}(A) + \mathcal{P}^{R/I}(C) \\
&\leq \mathcal{P}^R(A)/(1+z-z\mathcal{P}^R(R/I)) + \mathcal{P}^R(C)/(1+z-z\mathcal{P}^R(R/I)), \quad (\text{by Theorem 2.1}) \\
&= [\mathcal{P}^R(A) + \mathcal{P}^R(C)]/(1+z-z\mathcal{P}^R(R/I)) \\
&= \mathcal{P}^R(B)/(1+z-z\mathcal{P}^R(R/I)). \quad \text{by ii) }
\end{aligned}$$

The two end terms of the above inequality are equal by i) and we have

$$[\mathcal{P}^R(A)/(1+z-z\mathcal{P}^R(R/I)) - \mathcal{P}^{R/I}(A)] + [\mathcal{P}^R(C)/(1+z-z\mathcal{P}^R(R/I)) - \mathcal{P}^{R/I}(C)] = 0.$$

It follows by Theorem 3.1 that each bracket is zero, or equivalently, that (A, I) and (C, I) are Golod pairs. The last statement of the theorem now follows.

Chapter V. Proof of the main theorem

In this section we prove Theorem 2.1. Throughout this section we assume that R is a commutative ring, I is an ideal in R and M is an R/I -module. We introduce some new notation that will be useful in the proof of Theorem 1.1. Fix R -projective resolutions of M and I ,

$$\begin{aligned} (\mathbf{Q}, e.) \quad & \cdots \rightarrow Q_n \xrightarrow{e_n} Q_{n-1} \rightarrow \cdots \rightarrow Q_0 \xrightarrow{e_0} M \rightarrow 0 \\ (\mathbf{P}, d.) \quad & \cdots \rightarrow P_n \xrightarrow{d_n} P_{n-1} \rightarrow \cdots \rightarrow P_0 \xrightarrow{d_0} I \rightarrow 0 \end{aligned}$$

Let $U_0 = Q_0$, $U_1 = Q_1$ and for $n \geq 2$

$$U_n = Q_n \amalg \left(\prod_{i=0}^{n-2} (P_{n-i-2} \otimes_R U_i) \right)$$

be projective R -modules. Let $U_{n,r}$ be the sum of the "first r components" of U_n for $1 \leq r \leq n$. Thus $U_{n,1} = Q_n$ and for $2 \leq r \leq n$, $U_{n,r} = Q_n \amalg (\prod_{i=0}^{r-2} (P_{n-i-2} \otimes_R U_i))$. In particular, $U_{n,n} = U_n$. We restate and prove Theorem 2.1 using this notation.

THEOREM 2.1. *Let R be a commutative ring with 1, $I \subset R$ be an ideal in R and M be an R/I -module. Suppose that $(\mathbf{Q}, e.)$ and $(\mathbf{P}, d.)$ are R -projective resolutions of M and I respectively. Let $U_0 = Q_0$, $U_1 = Q_1$ and $U_n = Q_n \amalg (\prod_{i=0}^{n-2} (P_{n-i-2} \otimes_R U_i))$ for $n \geq 2$ be projective R -modules. Set $U_{n,1} = Q_n$ and for $2 \leq r \leq n$ let $U_{n,r} = Q_n \amalg (\prod_{i=0}^{r-2} (P_{n-i-2} \otimes_R U_i))$. Then there exists R -homomorphisms $h_0 : U_0 \rightarrow M$ and $h_n : U_n \rightarrow U_{n-1}$ for $n \geq 1$ such that*

A).

$$\cdots \rightarrow \bar{U}_n \xrightarrow{\bar{h}_n} \bar{U}_{n-1} \rightarrow \cdots \rightarrow \bar{U}_0 \xrightarrow{\bar{h}_0} M \rightarrow 0$$

is an R/I -projective resolution of M where " $\bar{*}$ " denotes $R/I \otimes_R *$ for modules and $1_R \otimes_r *$ for homomorphisms. We denote this R/I -resolution of M by (\bar{U}, \bar{h}) .

B). The homomorphisms h_n satisfy

a) $h_0 = e_0$ and $h_1 = e_1$,

b) for $n \geq 2$ if ϕ_n is the composition

$$P_0 \otimes_R U_{n-2} \xrightarrow{\text{inclusion}} U_n \xrightarrow{h_n} U_{n-1} \xrightarrow{h_{n-1}} U_{n-2}$$

and if ψ_n is the composition

$$P_0 \otimes_R U_{n-2} \xrightarrow{d_0 \otimes 1_{U_{n-2}}} I \otimes_R U_{n-2} \xrightarrow{\text{canonical}} IU_{n-2} \xrightarrow{\text{inclusion}} U_{n-2}'$$

then $\phi_2 = -\psi_2$ and if $\phi_{n-1} = -\psi_{n-1}$ then $\phi_n = (-1)^n \psi_n$, whereas if $\phi_{n-1} = \psi_{n-1}$ then $\phi_n = (-1)^{n+1} \psi_n$ (and hence if $n = 2s$ or $n = 2s + 1$ then $\phi_n = (-1)^s \psi_n$).

c) for $n \geq 2$, $h_n(U_n) = h_{n-1}^{-1}(IU_{n-2})$,

d) if $h_{n,r}$ is the restriction of h_n to $U_{n,r}$ then $h_{n,r}(U_{n,r}) \subseteq U_{n-1,r}$ for $2 \leq n$ and $2 \leq r \leq n$ and $h_{n,r}(U_{n,r}) = \ker(h_{n-1,r})$, for $2 \leq n$ and $1 \leq r \leq n-1$,

e) $h_{n,1} = e_n$,

f) for $n \geq 3$ and $2 \leq r \leq n-1$, the composition

$$P_{n-r} \otimes_R U_{r-2} \xrightarrow{\text{inclusion}} U_{n,r} \xrightarrow{h_{n,r}} U_{n-1,r} \xrightarrow{\text{projection}} P_{n-r-1} \otimes_R U_{r-2}$$

is $d_{n-r} \otimes 1_{U_{r-2}}$,

g) for $2 \leq n$ and $2 \leq r \leq n$ the sequence

$$0 \longrightarrow \ker(h_{n,r-1}) \xrightarrow{\text{inclusion}} \ker(h_{n,r}) \xrightarrow{\text{projection}} \ker(d_{n-r}) \otimes_R U_{r-2} \longrightarrow 0$$

induced by the exact sequence

$$0 \longrightarrow U_{n,r-1} \xrightarrow{\text{inclusion}} U_{n,r} \xrightarrow{\text{projection}} P_{n-r} \otimes_R U_{r-2} \longrightarrow 0$$

is exact,

h) for $n \geq 4$ and $3 \leq r \leq n-1$ the composition

$$P_{n-r} \otimes_R U_{r-2} \xrightarrow{\text{inclusion}} U_{n,r} \xrightarrow{h_{n,r}} U_{n-1,r} \xrightarrow{\text{projection}} P_{n-r} \otimes_R U_{r-3}$$

is $(-1)^{n+1}1_{P_{n-r}} \otimes_R h_{r-2}$,

i) for $n \geq 3$ the composition

$$P_0 \otimes U_{n-2} \xrightarrow{\text{inclusion}} U_n \xrightarrow{h_n} U_{n-1} \xrightarrow{\text{projection}} P_0 \otimes_R U_{n-3}$$

is $(-1)^{n+1}1_{P_0} \otimes h_{n-2}$.

PROOF: Set $h_0 = e_0$ and $h_1 = e_1$. To construct h_2 consider the following diagram of exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & \ker(h_1) & \xrightarrow{\text{inclusion}} & h_1^{-1}(IU_0) & \xrightarrow{h_1} & IU_0 \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow -d_0 \otimes 1_{U_0} \\
 & & Q_2 & & P_0 \otimes_R U_0 & & \\
 & & \uparrow & & \uparrow & & \\
 & & \ker(e_2) & & \ker(d_0) \otimes_R U_0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

Note that $IU_0 \subseteq \ker(h_0) = h_1(U_1)$ and $I \otimes_R U_0 \simeq IU_0$. Apply Lemma 0 to the above diagram. We get the following commutative diagram of exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \ker(h_1) & \xrightarrow{\text{inclusion}} & h_1^{-1}(IU_0) & \xrightarrow{h_1} & IU_0 \longrightarrow 0 \\
 & & \uparrow e_2 & & \uparrow h_2 & & \uparrow -d_0 \otimes 1_{U_0} \\
 0 & \longrightarrow & Q_2 & \xrightarrow{\text{inclusion}} & Q_2 \amalg (P_0 \otimes_R U_0) & \xrightarrow{\text{projection}} & P_0 \otimes_R U_0 \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \ker(e_2) & \longrightarrow & \ker(h_2) & \longrightarrow & \ker(d_0) \otimes_R U_0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Note that the bottom row of the above diagram is

$$0 \longrightarrow \ker h_{2,1} \longrightarrow \ker h_{2,2} \longrightarrow \ker(d_0) \otimes_R U_0 \longrightarrow 0.$$

It is now clear that h_2 satisfies b), c), d), e), and g). To construct h_3 , first consider the following diagram of exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & & & 0 \\
 & & \uparrow & & & & \uparrow \\
 0 & \longrightarrow & \ker(h_{2,1}) & \xrightarrow{\text{inclusion}} & \ker(h_2) & \xrightarrow{\text{projection}} & \ker(d_0) \otimes_R U_0 \longrightarrow 0 \\
 & & \uparrow e_3 & & & & \uparrow d_1 \otimes 1_{U_0} \\
 & & Q_3 & & & & P_1 \otimes_R U_0 \\
 & & \uparrow & & & & \uparrow \\
 & & \ker(e_3) & & & & \ker(d_1) \otimes_R U_0 \\
 & & \uparrow & & & & \uparrow \\
 & & 0 & & & & 0
 \end{array}$$

Again Lemma 0 yields the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \ker(h_{2,1}) & \xrightarrow{\text{inclusion}} & \ker(h_2) & \xrightarrow{\text{projection}} & \ker(d_0) \otimes_R U_0 \longrightarrow 0 \\
 & & \uparrow e_3 & & \uparrow h_{3,2} & & \uparrow d_1 \otimes 1_{U_0} \\
 0 & \longrightarrow & Q_3 & \xrightarrow{\text{inclusion}} & Q_3 \amalg (P_1 \otimes_R U_0) & \xrightarrow{\text{projection}} & P_1 \otimes_R U_0 \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \ker(e_3) & \longrightarrow & \ker(h_{3,2}) & \longrightarrow & \ker(d_1) \otimes_R U_0 \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0
 \end{array} \tag{D}$$

Finally we consider the following exact diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \ker(h_2) & \xrightarrow{\text{inclusion}} & h_2^{-1}(IU_1) & \xrightarrow{h_2} & IU_1 \longrightarrow 0 \\
& & \uparrow h_{3,2} & & \uparrow -d_0 \otimes 1_{U_1} & & \\
& & Q_3 \coprod (P_1 \otimes_R U_0) & & P_0 \otimes_R U_1 & & \\
& & \uparrow & & \uparrow & & \\
& & \ker(h_{3,2}) & & \ker(d_0) \otimes_R U_1 & & \\
& & \uparrow & & \uparrow & & \\
& & 0 & & 0 & &
\end{array}$$

As before we note that $h_2(U_2) = h_1^{-1}(IU_0) \supseteq IU_1$ and $IU_1 \simeq I \otimes_R U_1$.

We now lift $-d_0 \otimes 1_{U_1}$ to $h_2^{-1}(IU_1)$ and construct h_3 as in Lemma 0. Let $\beta : P_0 \otimes_R U_1 \rightarrow U_1$ be the map so that for $p_0 \in P_0$ and $u_1 \in U_1$,

$$\beta(p_0 \otimes u_1) = -d_0(p_0)u_1 - h_2(0, p_0 \otimes h_1(u_1))$$

Since h_2 satisfies b) we have

$$h_1(\beta(p_0 \otimes u_1)) = -d_0(p_0)h_1(u_1) + d_0(p_0)h_1(u_1) = 0.$$

Thus the image of β is contained in $\ker(h_1)$. Let $\tilde{\beta}$ be a lifting of β to Q_2 so that the following diagram commutes:

$$\begin{array}{ccccc}
& & P_0 \otimes_R U_1 & & \\
& & \downarrow \beta & & \\
Q_2 & \xrightarrow{\tilde{\beta}} & \ker(h_1) & \rightarrow & 0 \\
& \xrightarrow{e_2} & & &
\end{array}$$

Then $P_0 \otimes_R U_1 \xrightarrow{(\tilde{\beta}, 1_{P_0} \otimes h_1)} Q_2 \coprod (P_0 \otimes_R U_0)$ is the required lift which we denote by α . This makes sense since for $p_0 \in P_0$ and $u_1 \in U_1$

$$\begin{aligned}
h_2 \circ \alpha(p_0 \otimes u_1) &= h_2(\tilde{\beta}(p_0 \otimes u_1), p_0 \otimes h_1(u_1)) \\
&= h_2(\tilde{\beta}(p_0 \otimes u_1), 0) + h_2(0, p_0 \otimes h_1(u_1)) \\
&= \beta(p_0 \otimes u_1) + h_2(0, p_0 \otimes h_1(u_1)), && \text{by d)} \\
&= -d_0(p_0)u_1, && \text{by the definition of } \beta
\end{aligned}$$

We now define

$$h_3 : (U_3 =) Q_3 \coprod (P_1 \otimes_R U_0) \coprod (P_0 \otimes_R U_1) \rightarrow h_2^{-1}(IU_1)$$

by requiring $h_3(x, y, z) = h_{3,2}(x, y, 0) + \alpha(0, 0, z)$ where $x \in Q_3, y \in P_1 \otimes_R U_0$ and $z \in P_0 \otimes_R U_1$. As in Lemma 0 we have the following exact commutative diagram D')

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \ker(h_2) & \xrightarrow{\text{inclusion}} & h_2^{-1}(IU_1) & \xrightarrow{h_2} & IU_2 & \longrightarrow & 0 \\
 & & \uparrow h_{3,2} & & \uparrow h_3 & & \uparrow -d_0 \otimes 1_{U_1} & & \\
 0 & \longrightarrow & Q_3 \coprod (P_1 \otimes_R U_0) & \longrightarrow & U_3 & \longrightarrow & P_0 \otimes_R U_1 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 0 & \longrightarrow & \ker(h_{3,2}) & \longrightarrow & \ker(h_3) & \longrightarrow & \ker(d_0) \otimes_R U_1 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
 \end{array}$$

It is immediate from the diagrams D) and D') that h_3 satisfies the conditions b) through h).

We finish the proof by induction. Assume that for $n \geq 4$ and $2 \leq i < n$, h_i is constructed as desired. In order to construct h_n we first coconstruct $h_{n,r} : U_{n,r} \rightarrow U_{n-1,r}$ by induction on r for $1 \leq r \leq n-1$. Let $h_{n,1} = e_n$. Construction of $h_{n,2}$ is similar to the corresponding construction in the case when $n = 3$ and we omit the proof. Now assume that $h_{n,r-1}$ is constructed for $3 \leq r \leq n-1$ such that f), g), and h) hold. Since the induction hypotheses

imply that h_{n-1} satisfies g), we have the following exact diagram:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \ker(h_{n-1,r-1}) & \xrightarrow{\text{inclusion}} & \ker(h_{n-1,r}) & \xrightarrow{\text{projection}} & \ker(d_{n-r-1}) \otimes_R U_{r-2} \longrightarrow 0 \\
& & \uparrow h_{n,r-1} & & \uparrow d_{n-r} \otimes 1_{U_{r-2}} & & \\
& & U_{n,r-1} & & P_{n-r} \otimes_R U_{r-2} & & \\
& & \uparrow & & \uparrow & & \\
& & \ker(h_{n,r-1}) & & \ker(d_{n-r}) \otimes_R U_{r-2} & & \\
& & \uparrow & & \uparrow & & \\
& & 0 & & 0 & &
\end{array}$$

Again we lift $d_{n-r} \otimes 1_{U_{r-2}}$ to $\ker(h_{n-1,r})$ as follows. Define $\beta : P_{n-r} \otimes U_{r-2} \rightarrow U_{n-1,r}$ so that for $p_{n-r} \in P_{n-r}$ and $u_{r-2} \in U_{r-2}$

$$\beta(p_{n-r} \otimes u_{r-2}) = (\underbrace{0, \dots, 0}_{r-2}, (-1)^{n+1} p_{n-r} \otimes h_{r-2}(u_{r-2}), d_{n-r}(p_{n-r}) \otimes u_{r-2})$$

We claim that the image of $h_{n-1,r} \circ \beta$ is contained in $\ker(h_{n-2,r-2})$. If $n = 4$ or $n > 4$ and $r = n - 1$ this follows from b), d) and i). If $n > 4$ and $3 \leq r < n - 1$, the desired containment follows from the following argument.

$$\begin{aligned}
h_{n-1,r} \circ \beta(p_{n-r} \otimes u_{r-2}) &= h_{n-1}(\underbrace{0, \dots, 0}_{r-2}, (-1)^{n+1} p_{n-r} \otimes h_{r-2}(u_{r-2}), 0) \\
&\quad + h_{n-1}(\underbrace{0, \dots, 0}_{r-1}, d_{n-r}(p_{n-r}) \otimes u_{n-r}) \\
&= (\underbrace{*, \dots, *}_{r-2}, (-1)^{n+1} p_{n-r} \otimes h_{r-2}(u_{r-2}), 0) \\
&\quad + (\underbrace{*', \dots, *'}_{r-2}, (-1)^n d_{n-r}(p_{n-r}) h_{r-2}(u_{n-r}), d_{n-r-1} \circ d_{n-r}(p_{n-r}) \otimes u_{n-r})
\end{aligned}$$

(This follows from the inductive hypothesis that h_{n-1} satisfies f) and h)).

$$= (\underbrace{*', \dots, *'}_{r-2}, 0, 0)$$

The last element belongs to $\ker(h_{n-2,r-2})$ since, by induction, h_{n-1} satisfies d). Let $\tilde{\beta}$ be a lifting of $h_{n-1,r} \circ \beta$ to $U_{n-1,r-2}$ as follows.

$$\begin{array}{ccc} & & P_{n-r} \otimes_R U_{r-2} \\ & & \downarrow h_{n-1,r} \circ \beta \\ & \tilde{\beta} & \\ U_{n-1,r-2} & \xrightarrow{h_{n-1,r-2}} & \ker(h_{n-2,r-2}) \end{array}$$

Now define $\alpha : P_{n-r} \otimes_R U_{r-2} \rightarrow U_{n-1,r}$ such that for $p_{n-r} \in P_{n-r}$ and $u_{r-2} \in U_{r-2}$ we have

$$\alpha(p_{n-r} \otimes u_{r-2}) = (-\tilde{\beta}(p_{n-r} \otimes u_{r-2}), (-1)^{n+1} p_{n-r} \otimes h_{r-2}(u_{r-2}), d_{n-r}(p_{n-r}) \otimes u_{r-2}).$$

It follows that α is the required lifting of $d_{n-r} \otimes u_{r-2}$ to $\ker(h_{n-1,r})$ since

$$\begin{aligned} h_{n-1,r} \circ \alpha(p_{n-r} \otimes u_{r-2}) &= h_{n-1,r}(-\tilde{\beta}(p_{n-r} \otimes u_{r-2}), 0, 0) \\ &\quad + h_{n-1,r}(\underbrace{0, \dots, 0}_{r-2}, (-1)^{n+1} p_{n-r} \otimes h_{r-2}(u_{r-2}), d_{n-r}(p_{n-r}) \otimes u_{r-2}) \\ &= -h_{n-1,r-2} \circ \tilde{\beta}(p_{n-r} \otimes u_{r-2}) + h_{n-1,r} \circ \beta(p_{n-r} \otimes u_{r-2}) \\ &= -h_{n-1,r} \circ \beta(p_{n-r} \otimes u_{r-2}) + h_{n-1,r} \circ \beta(p_{n-r} \otimes u_{r-2}) \\ &= 0 \end{aligned}$$

As in the proof of Lemma 0 we have the following exact commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \ker(h_{n-1,r-1}) & \xrightarrow{\text{inclusion}} & \ker(h_{n-1,r}) & \xrightarrow{\text{projection}} & \ker(d_{n-r-1}) \otimes_R U_{r-2} \longrightarrow 0 \\ & & \uparrow h_{n,r-1} & & \uparrow h_{n,r} & & \uparrow d_{n-r} \otimes 1_{U_{r-2}} \\ & & U_{n,r-1} & \xrightarrow{\text{inclusion}} & U_{n,r} & \xrightarrow{\text{projection}} & P_{n-r} \otimes_R U_{r-2} \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ & & \ker(h_{n,r-1}) & \longrightarrow & \ker(h_{n,r}) & \longrightarrow & \ker(d_{n-r}) \otimes_R U_{r-2} \\ & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & & 0 & & 0 \end{array}$$

From the above diagram and the induction hypothesis for $h_{n,r-1}$ we see that $h_{n,r}$ satisfies conditions f), g), and h). This finishes the construction of $h_{n,r}$ for $1 \leq r \leq n-1$. Note that

$$h_{n,1} = e_n, \quad h_{n,r-1} = h_{n,r}|_{U_{n,r-1}}, \quad \text{for } 2 \leq r \leq n-1.$$

We now construct h_n in the case when $\phi_{n-1} = -\psi_{n-1}$, (cf. b)). The proof in the case when $\phi_{n-1} = \psi_{n-1}$ is similar and will only be indicated. Consider the following exact diagram.

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \ker(h_{n-1}) & \xrightarrow{\text{inclusion}} & h_{n-1}^{-1}(IU_{n-2}) & \xrightarrow{h_{n-1}} & IU_{n-2} \longrightarrow 0 \\
& & \uparrow h_{n,n-1} & & \uparrow (-1)^n d_0 \otimes 1_{U_{n-2}} & & \\
& & U_{n,n-1} & & P_0 \otimes_R U_{n-2} & & \\
& & \uparrow & & \uparrow & & \\
& & \ker(h_{n,n-1}) & & \ker(d_0) \otimes_R U_{n-2} & & \\
& & \uparrow & & \uparrow & & \\
& & 0 & & 0 & &
\end{array}$$

We now lift $(-1)^n d_0 \otimes 1_{U_{n-2}}$ to $h_{n-1}^{-1}(IU_{n-2})$ and construct

$$h_n : (U_n =) U_{n,n-1} \coprod (P_0 \otimes_R U_{n-2}) \longrightarrow h_{n-1}^{-1}(IU_{n-2})$$

as in Lemma 0.

Define $\beta : P_0 \otimes_R U_{n-2} \rightarrow U_{n-2}$ such that for $p_0 \in P_0$ and $u_{n-2} \in U_{n-2}$ we have

$$\beta(p_0 \otimes u_{n-2}) = (-1)^n d_0(p_0)u_{n-2} - h_{n-1}(0, \dots, 0, (-1)^{n+1} p_0 \otimes h_{n-2}(u_{n-2}))$$

Since $\phi_{n-1} = -\psi_{n-1}$ we have $h_{n-2} \circ \beta(p_0 \otimes u_{n-2}) = 0$ so that the image of β is contained in $\ker(h_{n-2})$.

Let $\tilde{\beta}$ be lifting of β to $U_{n-1,n-2}$ as follows.

$$\begin{array}{ccc}
& & P_0 \otimes_R U_{n-2} \\
& & \downarrow \beta \\
U_{n-1,n-2} & \xrightarrow{h_{n-1,n-2}} & \ker(h_{n-2})
\end{array}$$

Now choose the required lifting α of $(-1)^n d_0 \otimes 1_{U_{n-2}}$ as follows.

$\alpha(p_0 \otimes u_{n-2}) = (\tilde{\beta}(p_0 \otimes u_{n-2}), (-1)^{n+1} p_0 \otimes h_{n-2}(u_{n-2}))$ as in the case $n = 3$. [In the case when $\phi_{n-1} = \psi_{n-1}$, in order to lift $(-1)^{n+1} d_0 \otimes 1_{U_{n-2}}$ we define

$$\beta(p_0 \otimes u_{n-2}) = (-1)^{n+1} d_0(p_0)u_{n-2} - h_{n-1}(0, \dots, 0, (-1)^{n+1} p_0 \otimes h_{n-2}(u_{n-2})).$$

The rest of the proof is similar.] We now have the following exact commutative diagram by Lemma 0.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \ker(h_{n-1}) & \xrightarrow{\text{inclusion}} & h_{n-1}^{-1}(IU_{n-2}) & \xrightarrow{h_{n-1}} & IU_{n-2} \longrightarrow 0 \\
 & & \uparrow h_{n,n-1} & & \uparrow h_n & & \uparrow (-1)^n d_0 \otimes 1_{U_{n-2}} \\
 0 & \longrightarrow & U_{n,n-1} & \xrightarrow{\text{inclusion}} & U_n & \xrightarrow{\text{projection}} & P_0 \otimes_R U_{n-2} \longrightarrow 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \ker(h_{n,n-1}) & \longrightarrow & \ker(h_n) & \longrightarrow & \ker(d_0) \otimes_R U_{n-2} \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

Here h_n is defined by $h_n(x_1, \dots, x_n) = h_{n,n-1}(x_1, \dots, x_{n-1}) + \alpha(0, \dots, 0, x_n)$. It is clear from the above diagram and the construction of $h_{n,n-1}$ that h_n satisfies conditions b) through i).

This completes the proof.

We note that Theorem 2.1' can be proved similarly, using Lemma 0' instead of Lemma 0.

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VITA

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