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# Beam propagation in focusing media with random-axis misalignments: Second- and higher-order moments<sup>a)</sup>

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A novel statistical technique which allows the asymptotic evaluation of second- and higher-order averaged observables related to the stochastic complex parabolic equation is applied to the problem of beam propagation in a focusing medium characterized by random-axis misalignments. Analytical and numerical results concerning on- and off-axis statistics (e.g., the variance of intensity fluctuations, modal power transfer, the probability distribution density of the log-irradiance, etc.) are presented, and comparisons are made with previously reported findings.

## 1. INTRODUCTION

In the quasioptical regime, the propagation of beamed signals along the  $z$  direction in focusing media with random-axis misalignments is described exceedingly well by the stochastic complex parabolic equation<sup>1,2</sup>

$$\frac{i}{k} \frac{\partial}{\partial z} \psi(\mathbf{x}, z; \alpha) = -\frac{1}{2k^2} \nabla_{\mathbf{x}}^2 \psi(\mathbf{x}, z; \alpha) + V(\mathbf{x}, z; \alpha) \psi(\mathbf{x}, z; \alpha), \quad z > 0, \quad (1.1a)$$

$$V(\mathbf{x}, z; \alpha) = \frac{1}{2} g^2 [\mathbf{x} - \mathbf{a} \delta H(z; \alpha)]^2, \quad \mathbf{x} \in R^2, \quad (1.1b)$$

$$\psi(\mathbf{x}, 0; \alpha) = \psi_0(\mathbf{x}). \quad (1.1c)$$

Here,  $k$  is a reference wavenumber,  $g$  is a spatial frequency (units: radians/meter), and  $\mathbf{a}$  is a fixed vector quantity. The potential field given in (1.1b) corresponds to a parabolically focusing medium whose equilibrium axis is perturbed via the zero-mean,  $z$ -dependent, real random function  $\delta H(z; \alpha)$ . The latter, as well as the slowly varying, complex, random, amplitude function  $\psi(\mathbf{x}, z; \alpha)$ , depends on a parameter  $\alpha \in A$ ,  $(A, F, P)$  being an underlying probability measure space.

It is our goal in this exposition to examine the boundary-value problem (1.1) in an unbounded (with respect to  $\mathbf{x}$ ) domain. It should be pointed out, however, that this idealized problem provides a good approximation to the forward propagation of low-order modes in a fiber lightguide having a parabolically graded refraction index, with random-axis misalignment of microbending.<sup>3,4</sup> It can also give some insight into the problem of forward propagation of low-order acoustic modes near an idealized, randomly perturbed, underwater sound channel axis, provided that transverse (with respect to  $z$ ) statistical fluctuations due to internal waves can be ignored.

There exist physical situations which require that the initial condition (1.1c) be random (e.g., aberrations in a lens through which a laser beam passes before it enters into the random medium). However, a generalization of the discussion in this paper to account for such an initially partially coherent beam presents no fundamental difficulties. It is, also, relatively straightforward

to account for random deformations along the channel axis of the more general form  $\delta H(z; \alpha)$ , where  $\delta H(z; \alpha)$  is a zero-mean, vector-valued, real random function.

The problem (1.1) has already been investigated by Besieris *et al.*<sup>5</sup> from the point of view of a quantum mechanical harmonic oscillator whose equilibrium position is randomly perturbed.<sup>6</sup> This was done using a kinetic approach at the level of second-order statistical moments. Marcuse (cf. Ref. 3) has also studied an initial-boundary-value problem closely resembling (1.1). His problem (related to fiber-optical propagation) is more realistic than (1.1). As a result, his approach (a modal analysis) is more difficult to justify with estimates of accuracy. The only carefully derived results to date dealing with higher-order statistics of the problem (1.1) are those reported by McLaughlin.<sup>7</sup> Using the diffusion approximation (cf. Refs. 8–10), he has studied the average intensity and the intensity fluctuations on the beam axis, as well as the decay of mean power from the fundamental mode of the unperturbed focusing medium, and mean power transitions to higher modes.

It is our intent to study (1.1) by means of a new statistical technique which allows the asymptotic evaluation of second- and higher-order statistical observables without having to derive first equations for second- or higher-order coherence functions. It will be shown that in the special case where  $\delta H(z; \alpha)$  in (1.1) is a wide-sense stationary,  $\delta$ -correlated, Gaussian random process, a certain class of even moments of the wavefunction  $\psi(\mathbf{x}, z; \alpha)$  can be computed exactly. More importantly, it will be shown that these quantities can be computed asymptotically (e.g., in the long-range regime), even under realistic assumptions about the statistical fluctuations of the medium. Our main findings will be compared to those of previous workers, especially McLaughlin's (cf. Ref. 7). We shall obtain, in addition, several new results, such as off-axis statistics, the variance connected with beam wandering, the probability distribution density function of the log irradiance, etc.

The structure of the paper can be outlined as follows: A basic conservation law pertaining to the stochastic parabolic equation (1.1) is developed in Sec. 2. A fundamental ansatz on which the proposed technique is based is then introduced in Sec. 3 for the general case of two-

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dimensional beam propagation in a parabolically focusing medium with random-axis misalignments. The one-dimensional version of this problem is discussed in Sec. 4. Finally, following an analysis of the basic statistical problem (cf. Sec. 5), several new results linked to second- and higher-order observables are computed in Secs. 6 and 7.

## 2. BASIC CONSERVATION LAW

Corresponding to the stochastic parabolic equation (1.1), let

$$i(\mathbf{x}, z; \alpha) = \psi^*(\mathbf{x}, z; \alpha) \psi(\mathbf{x}, z; \alpha) \quad (2.1)$$

and

$$\mathbf{j}(\mathbf{x}, z; \alpha) = \frac{i}{2k} [\psi(\mathbf{x}, z; \alpha) \nabla_{\mathbf{x}} \psi^*(\mathbf{x}, z; \alpha) - \psi^*(\mathbf{x}, z; \alpha) \nabla_{\mathbf{x}} \psi(\mathbf{x}, z; \alpha)] \quad (2.2)$$

denote the *intensity* (or irradiance) and *intensity flux densities*, respectively. By virtue of the self-adjointness of the operator  $-(1/2k^2) \nabla_{\mathbf{x}}^2 + (1/2)g^2[\mathbf{x} - \mathbf{a}\delta H(z; \alpha)]^2$  in (1.1), we have the following conservation law:

$$\frac{\partial}{\partial z} i(\mathbf{x}, z; \alpha) + \nabla_{\mathbf{x}} \cdot \mathbf{j}(\mathbf{x}, z; \alpha) = 0 \quad \forall \alpha \in A. \quad (2.3)$$

As a consequence, the total intensity  $I(z; \alpha)$ , defined by

$$I(z; \alpha) = \int_{R^2} d\mathbf{x} i(\mathbf{x}, z; \alpha), \quad (2.4)$$

is conserved for every realization  $\alpha \in A$ , viz.,

$$\frac{d}{dz} I(z; \alpha) = 0, \quad (2.5)$$

or

$$I(z; \alpha) = I(0; \alpha) = \int_{R^2} d\mathbf{x} \psi^*(\mathbf{x}, 0; \alpha) \psi(\mathbf{x}, 0; \alpha). \quad (2.6)$$

In the sequel, we shall assume that  $I(0; \alpha)$  is normalized to unity for every realization  $\alpha \in A$ .

We define, next, a vector  $\mathbf{s}(\mathbf{x}, z; \alpha)$  by the relationship

$$\mathbf{s}(\mathbf{x}, z; \alpha) = i(\mathbf{x}, z; \alpha) \hat{\mathbf{z}} + \mathbf{j}(\mathbf{x}, z; \alpha), \quad \mathbf{z} \equiv \mathbf{z}/|\mathbf{z}|. \quad (2.7)$$

The conservation law (2.3) can be rewritten in terms of  $\mathbf{s}(\mathbf{x}, z; \alpha)$  as follows:

$$\nabla \cdot \mathbf{s}(\mathbf{x}, z; \alpha) = 0, \quad \nabla = \nabla_{\mathbf{x}} + (\partial/\partial z) \hat{\mathbf{z}}. \quad (2.8)$$

From physical considerations,  $\mathbf{s}(\mathbf{x}, z; \alpha)$  may be interpreted as a *power flux density*. On the strength of the divergence theorem, one has the identity

$$\int_V \int \nabla \cdot \mathbf{s}(\mathbf{x}, z; \alpha) dV = \int_{S_c} \int \mathbf{s}(\mathbf{x}, z; \alpha) \cdot \hat{\mathbf{n}} dA = 0, \quad (2.9)$$

where  $V$  is the volume bounded by a regular closed surface  $S_c$  and  $\hat{\mathbf{n}}$  is a unit outwardly directed normal vector. From a more practical point of view, the power intercepted by a detector (indicated by an open  $S_0$ ) can be written as follows:

$$\int_{S_0} \int \mathbf{s}(\mathbf{x}, z; \alpha) \cdot \hat{\mathbf{n}} dA. \quad (2.10)$$

## 3. THE FUNDAMENTAL ANSATZ

In the stochastic parabolic equation (1.1) we make a change of the transverse (with respect to  $z$ ) spatial variable corresponding to a "moving" coordinate system,

$$\mathbf{y}(z; \alpha) = \mathbf{x} - \mathbf{u}(z; \alpha), \quad (3.1)$$

and represent the wavefunction  $\psi(\mathbf{x}, z; \alpha)$  in the form

$$\psi(\mathbf{x}, z; \alpha) = \phi[\mathbf{y}(z; \alpha), z] \exp\{ik[\dot{\mathbf{u}}(z; \alpha) \cdot \mathbf{y}(z; \alpha) + \gamma(z; \alpha)]\}, \quad (3.2)$$

where  $\mathbf{u}(z; \alpha)$  and  $\gamma(z; \alpha)$  are as yet unspecified random functions. The dot over  $\mathbf{u}(z; \alpha)$  in (3.2) designates a derivative with respect to  $z$ . To avoid unnecessary complexity in notation, we shall not write the arguments of  $\mathbf{y}$ ,  $\mathbf{u}$ , and  $\gamma$  out explicitly, unless there is ambiguity.

Our next step is to substitute (3.2) into (1.1) and carry out the indicated operations. This procedure leads to the following expression:

$$\begin{aligned} \frac{i}{k} \frac{\partial}{\partial z} \phi = & -\frac{1}{2k^2} \nabla_{\mathbf{y}}^2 \phi + \frac{1}{2} g^2 \mathbf{y}^2 \phi \\ & + (\ddot{\mathbf{u}} + g^2 \mathbf{u} - g^2 \mathbf{a} \delta H) \cdot \mathbf{y} \phi \\ & + (\dot{\gamma} - \frac{1}{2} \dot{\mathbf{u}}^2 + \frac{1}{2} g^2 \mathbf{u}^2 - g^2 \mathbf{u} \cdot \mathbf{a} \delta H + \frac{1}{2} g^2 \mathbf{a}^2 \delta H^2) \phi. \end{aligned} \quad (3.3)$$

We require that the terms within the parentheses on the right-hand side of (3.3) vanish. This condition gives rise to the following relationships:

$$(i) \frac{i}{k} \frac{\partial}{\partial z} \phi(\mathbf{y}, z) = \frac{1}{2k^2} \nabla_{\mathbf{y}}^2 \phi(\mathbf{y}, z) + \frac{1}{2} g^2 \mathbf{y}^2 \phi(\mathbf{y}, z), \quad (3.4)$$

$$(ii) \ddot{\mathbf{u}}(z; \alpha) + g^2 \mathbf{u}(z; \alpha) = g^2 \mathbf{a} \delta H(z; \alpha), \quad (3.5)$$

$$(iii) \dot{\gamma}(z; \alpha) = \frac{1}{2} \dot{\mathbf{u}}^2 - \frac{1}{2} g^2 \mathbf{u}^2 + g^2 \mathbf{u} \cdot \mathbf{a} \delta H - \frac{1}{2} g^2 \mathbf{a}^2 \delta H^2. \quad (3.6)$$

It is seen that within the framework of this formulation, the new wavefunction  $\phi(\mathbf{y}, z)$  satisfies the parabolic equation characterizing the unperturbed focusing medium. It should be noted, however, that  $\phi$  is a random function by virtue of its implicit dependence on  $\mathbf{u}(z; \alpha)$ , viz.,  $\phi = \phi[\mathbf{x} - \mathbf{u}(z; \alpha), z]$ , which, in turn, satisfies the Langevin-type equation (3.5).

To proceed with our analysis, we shall need appropriate initial conditions for  $\mathbf{u}$  and  $\dot{\mathbf{u}}$ . Toward this end, we set  $z = 0$  on both sides of (3.2):

$$\psi_0(\mathbf{x}) = \phi(\mathbf{x} - \mathbf{u}_0, 0) \exp\{ik[\dot{\mathbf{u}}_0 \cdot (\mathbf{x} - \mathbf{u}_0) + \gamma_0]\}, \quad (3.7)$$

where  $\mathbf{u}_0$ ,  $\dot{\mathbf{u}}_0$ , and  $\gamma_0$  are respectively the values of  $\mathbf{u}(z; \alpha)$ ,  $\dot{\mathbf{u}}(z; \alpha)$ , and  $\gamma(z; \alpha)$  at  $z = 0$ . From (3.6), one has

$$\gamma(z; \alpha) = \int_0^z d\xi \left( \frac{1}{2} \dot{\mathbf{u}}^2 - \frac{1}{2} g^2 \mathbf{u}^2 + g^2 \mathbf{u} \cdot \mathbf{a} \delta H - \frac{1}{2} g^2 \mathbf{a}^2 \delta H^2 \right) + c, \quad (3.8)$$

where  $c$  is a constant of integration. Choosing the initial conditions  $\mathbf{u}_0 = \dot{\mathbf{u}}_0 = 0$ , it follows, then, from (3.7) and (3.8) that

$$\psi_0(\mathbf{x}) = \phi_0(\mathbf{x}) \exp(ikc), \quad (3.9)$$

where  $\phi_0(\mathbf{x}) = \phi(\mathbf{x}, 0)$ . The phase term  $\exp(ikc)$  plays an unimportant role in the evaluation of a large class of "even" moments of the wavefunction  $\psi$ ; it will, therefore, be omitted by taking  $c = 0$ . We have, then, finally,

$$\psi_0(\mathbf{x}) = \phi_0(\mathbf{x}). \quad (3.10)$$

Thus, given the initial value  $\psi_0(\mathbf{x})$  for the stochastic parabolic equation (1.1), the correct initial condition for the "deterministic" parabolic equation (3.4) is

$$\phi(\mathbf{y}, 0) \equiv \phi_0(\mathbf{y}) = \psi_0(\mathbf{y}). \quad (3.11)$$

Our procedure in the sequel can be outlined as

follows: The parabolic equation (3.4) for the unperturbed medium, with the boundary condition (3.11), will be solved first for the wavefunction  $\phi$ . The latter is a functional of the random function  $u(z;\alpha)$  via the relationship  $\phi = \phi[x - u(z;\alpha), z]$ . This solution for  $\phi$  will, be used in the expression (3.2) for the original wavefunction  $\psi(x, z; \alpha)$ . This wavefunction is, in turn, a functional of  $u(z;\alpha)$ ,  $\dot{u}(z;\alpha)$ , and  $\delta H(z;\alpha)$  because of its dependence on  $\phi$  and the presence of the exponential factor in (3.2). Our ultimate goal will be to obtain statistical moments of the random wavefield  $\psi(x, z; \alpha)$  which are linked with physical observables.

Before we proceed any further, we wish to point out that the statistical technique outlined in this section has been motivated by the work of Papanicolaou *et al.*<sup>11</sup> and McLaughlin (cf. Ref. 7). Using “key representations” which are similar to—but distinct from—our basic ansatz (3.2), they have studied the propagation of a Gaussian beam in a randomly perturbed strongly focusing medium, and have derived detailed information, especially in connection with beam-axis statistics, which would have been difficult to obtain by other methods. In the special case of a deterministic perturbation, viz.,  $\delta H(z;\alpha) \rightarrow \delta H(z)$ , our “key representation” (3.2) is an extension of a well-known method in quantum mechanics. Ter Haar,<sup>12</sup> for example, has used it to determine the motion of an one-dimensional harmonic oscillator under the action of an externally applied force. Along the same vein, Svin'in<sup>13</sup> has recently applied this technique to the study of the Brownian motion of an one-dimensional, damped, quantum mechanical harmonic oscillator in an external field. Conceptually, we feel that our technique is also close to recently formulated methods based on the operator Langevin equation (cf. Ref. 14; see, also, remarks in Ref. 13) and Feynman path integration (cf. Refs. 15, 16). This is an important conjecture which we hope to substantiate in the future.

#### 4. SPECIALIZATION TO THE ONE-DIMENSIONAL CASE

To avoid unnecessary complexity which may obscure our main contributions, we shall limit our subsequent work to the one-dimensional version of the stochastic parabolic equation (1.1), viz.,

$$\frac{i}{k} \frac{\partial}{\partial z} \psi(x, z; \alpha) = -\frac{1}{2k^2} \frac{\partial^2}{\partial x^2} \psi(x, z; \alpha) \quad (4.1a)$$

$$+ \frac{1}{2} g^2 [x - a\delta H(z; \alpha)]^2 \psi(x, z; \alpha), \quad z > 0, \quad (4.1b)$$

$$\psi(x, 0; \alpha) = \psi_0(x). \quad (4.1b)$$

Corresponding to (3.1), (3.2), (3.4)–(3.6), (3.8), and (3.11), we have, then, the relations

$$y = x - u(z; \alpha), \quad (4.2)$$

$$\psi(x, z; \alpha) = \phi[y(z; \alpha), z] \exp\{ik[\dot{u}(z; \alpha)y(z; \alpha) + \gamma(z; \alpha)]\}, \quad (4.3)$$

$$\frac{i}{k} \frac{\partial}{\partial z} \phi(y, z) = -\frac{1}{2k^2} \frac{\partial^2}{\partial y^2} \phi(y, z) + \frac{1}{2} g^2 y^2 \phi(y, z), \quad z > 0, \quad (4.4a)$$

$$\phi(y, 0) = \phi_0(y) = \psi_0(y), \quad (4.4b)$$

$$\ddot{u}(z; \alpha) + g^2 u(z; \alpha) = g^2 a \delta H(z; \alpha), \quad z > 0, \quad (4.5a)$$

$$u(0; \alpha) = \dot{u}(0; \alpha) = 0, \quad (4.5b)$$

$$\gamma(z; \alpha) = \int_0^z d\xi \left( \frac{1}{2} \dot{u}^2 - \frac{1}{2} g^2 u^2 + g^2 u a \delta H - \frac{1}{2} g^2 a^2 \delta H^2 \right). \quad (4.6)$$

Consider, next, the parabolic equation (4.4) for the unperturbed medium. This problem is isomorphic to the Schrödinger equation for an one-dimensional quantum mechanical harmonic oscillator whose solution is well known.<sup>17</sup> Let  $G(y, y', z)$  be the Green's function associated with (4.4). In this case, it is given explicitly as follows:

$$G(y, y', z) = (gk/2\pi i \sin gz)^{1/2} \times \exp[-(gk/2i \sin gz)(y^2 \cos gz - 2yy' + y'^2 \cos gz)]. \quad (4.7)$$

This expression provides a link between the wavefunction  $\phi(y, z)$ ,  $z > 0$ , and the boundary condition  $\phi_0(y)$ :

$$\phi(y, z) = \int_{-\infty}^{\infty} dy' G(y, y', z) \phi_0(y'). \quad (4.8)$$

In order to evaluate the wavefunction  $\phi(y, z)$  explicitly, we shall have to decide on a specific boundary condition  $\phi_0(x)$  and, hence,  $\phi_0(y)$ . For simplicity, let us choose the fundamental mode corresponding to the parabolic equation for the background focusing medium, viz.,

$$\psi_0(x) = (gk/\pi)^{1/4} \exp[-(1/2)gkx^2]. \quad (4.9)$$

This initial configuration is normalized to unity [cf. Eq. (2.6)]. In light of the identity  $\phi_0(y) = \psi_0(y)$ , expressions (4.7)–(4.9) lead to the wavefunction

$$\phi(y, z) = (gk/\pi)^{1/4} \exp[-(1/2)gky^2] \exp[-\frac{1}{2}igz]. \quad (4.10)$$

This, of course, is the “ground state” wavefunction (“stationary state”) of the parabolic equation (4.4).

We introduce, next, (4.10) into our fundamental relation (4.3):

$$\psi(x, z; \alpha) = (gk/\pi)^{1/4} \exp[-\frac{1}{2}gky^2] \exp[-\frac{1}{2}igz] \exp[ik(\dot{u}y + \gamma)]. \quad (4.11)$$

This constitutes a solution to the original stochastic complex parabolic equation (4.1) for every realization  $\alpha \in A$ . In general, the computation of the ensemble average of an arbitrary functional of  $\psi(x, z; \alpha)$  requires the joint probability density function of the random functions  $u(z; \alpha)$ ,  $\dot{u}(z; \alpha)$ , and  $\gamma(z; \alpha)$ . The latter can be found from the analysis of the following set of coupled first-order, nonlinear, stochastic, ordinary differential equations:

$$\dot{u}(z; \alpha) = v(z; \alpha), \quad (4.12a)$$

$$\dot{v}(z; \alpha) + g^2 u(z; \alpha) = g^2 a \delta H(z; \alpha), \quad z > 0, \quad (4.12b)$$

$$u(0; \alpha) = v(0; \alpha) = 0, \quad (4.12c)$$

$$\dot{\gamma}(z; \alpha) = \frac{1}{2} v^2(z; \alpha) - \frac{1}{2} g^2 u^2(z; \alpha) + g^2 a u(z; \alpha) \delta H(z; \alpha) - \frac{1}{2} g^2 a^2 \delta H^2(z; \alpha), \quad z > 0, \quad (4.13a)$$

$$\gamma(0; \alpha) = 0. \quad (4.13b)$$

It is important, however, to note that expressions of the form  $\{\psi^*(x_1, z; \alpha) \psi^*(x_2, z; \alpha) \cdots \psi^*(x_m, z; \alpha) (\psi(x'_1, z; \alpha) \times \psi(x'_2, z; \alpha) \cdots \psi(x'_m, z; \alpha))\}$ , linked to a large class of important averaged physical observables (cf. Sec. 6 *et seq.*), are functionals only of  $u(z; \alpha)$  and  $\dot{u}(z; \alpha)$ . Since the latter are governed solely by the Langevin-type, linear, stochastic, ordinary differential equation (4.5)

[cf., also, Eq. (4.12)], our task of computing a large number of physically important moments of the wavefunction  $\psi(x, z; \alpha)$  can be accomplished provided the joint probability density function of the random functions  $u(z; \alpha)$  and  $\dot{u}(z; \alpha)$  is available. This problem is considered in detail in the next section.

## 5. ANALYSIS OF THE BASIC STATISTICAL PROBLEM

The second-order, Langevin-type, stochastic ordinary differential equation (4.5) may be recast into the form

$$\frac{d}{dz} v(z; \alpha) + g^2 u(z; \alpha) = g^2 a \delta H(z; \alpha), \quad (5.1a)$$

$$\frac{d}{dz} u(z; \alpha) = v(z; \alpha), \quad (5.1b)$$

$$u(0; \alpha) = v(0; \alpha) = 0. \quad (5.1c)$$

This problem is closely related to the Brownian motion of a randomly forced, classical, harmonic oscillator.

The "fine-grained" density, or classical "phase-space" distribution function, associated with (5.1) is introduced next as follows:

$$f_c(u, v, z; \alpha) = \delta[u - u(z; \alpha)] \delta[v - v(z; \alpha)], \quad (5.2a)$$

$$f_c(u, v, 0; \alpha) = \delta(u) \delta(v). \quad (5.2b)$$

It obeys the continuity, or Liouville equation, which reads<sup>18</sup>

$$\frac{\partial}{\partial z} f_c(u, v, z; \alpha) = L f_c(u, v, z; \alpha); \quad (5.3a)$$

$$L f_c(u, v, z; \alpha) = -v \frac{\partial}{\partial u} f_c(u, v, z; \alpha) + \theta f_c(u, v, z; \alpha); \quad (5.3b)$$

$$\theta f_c(u, v, z; \alpha) = \left[ g^2 u \frac{\partial}{\partial v} - g^2 a \delta H(z; \alpha) \frac{\partial}{\partial v} \right] f_c(u, v, z; \alpha). \quad (5.3c)$$

We shall embark next on a statistical analysis of (5.3). Using only the first-order smoothing approximation (cf. Refs. 19, 20; see, also, Refs. 21 and 5), we obtain the following kinetic equation for the ensemble average of the density function:

$$\begin{aligned} & \left( \frac{\partial}{\partial z} + v \frac{\partial}{\partial u} - g^2 u \frac{\partial}{\partial v} \right) E\{f_c(u, v, z; \alpha)\} \\ &= g^4 a^2 \frac{\partial}{\partial v} \left[ \int_0^z d\xi \Gamma(\xi) \left( \frac{\sin g\xi}{g} \frac{\partial}{\partial u} + \cos g\xi \frac{\partial}{\partial v} \right) \right. \\ & \quad \times E\left\{ f_c \left( u \cos g\xi - \frac{1}{g} v \sin g\xi, v \cos g\xi \right. \right. \\ & \quad \left. \left. + g u \sin g\xi, z - \xi; \alpha \right) \right\} \Big]. \end{aligned} \quad (5.4)$$

In deriving this expression it has been assumed that  $\delta H(z; \alpha)$  is a zero-mean, wide-sense stationary random process, with correlation function  $\Gamma(\xi) = E\{\delta H(z; \alpha) \delta H(z - \xi; \alpha)\}$ . The kinetic equation (5.4) is uniformly valid in range. The right-hand side of (5.4) contains generalized operators (nonlocal, with space "memory") in phase space.

For a random process  $\delta H(z; \alpha)$  which is  $\delta$ -correlated in range, viz.,  $\Gamma(\xi) = S \delta(\xi)$ , where  $S$  is a constant, the integration over  $\xi$  in (5.4) can be carried out explicitly.

The resulting transport equation is

$$\begin{aligned} & \left( \frac{\partial}{\partial z} + v \frac{\partial}{\partial u} - g^2 u \frac{\partial}{\partial v} \right) E\{f_c(u, v, z; \alpha)\} \\ &= D \frac{\partial^2}{\partial v^2} E\{f_c(u, v, z; \alpha)\}, \end{aligned} \quad (5.5)$$

where  $D = g^4 a^2 S$ . If, in addition to the above assumptions,  $\delta H(z; \alpha)$  is a Gaussian random process, (5.5) is the *exact* statistical equation for  $E\{f(u, v, z; \alpha)\}$ . This can be established by means of the Donsker–Furutsu–Novikov functional formalism.<sup>22–24</sup> In the long-range Markovian approximation (cf. Ref. 25; see, also, Refs. 21b and 5), (5.4) reduces to the simpler transport equation

$$\begin{aligned} & \left( \frac{\partial}{\partial z} + v \frac{\partial}{\partial u} - g^2 u \frac{\partial}{\partial v} \right) E\{f_c(u, v, z; \alpha)\} \\ &= D_1 \frac{\partial^2}{\partial v^2} E\{f_c(u, v, z; \alpha)\} + D_2 \frac{\partial^2}{\partial u \partial v} E\{f_c(u, v, z; \alpha)\}. \end{aligned} \quad (5.6)$$

The diffusion coefficients are given by the expressions

$$D_1 = g^4 a^2 \int_0^\infty d\xi \Gamma(\xi) \cos g\xi, \quad (5.7a)$$

$$D_2 = g^3 a^2 \int_0^\infty d\xi \Gamma(\xi) \sin g\xi. \quad (5.7b)$$

The quantity  $E\{f_c(u, v, z; \alpha)\}$  is nonnegative; as such, provided that it is normalized to unity, it can be considered as the joint probability distribution density of  $u$  and  $v$ . A requirement of our statistical formulation (cf. Sec. 4) is that  $E\{f_c(u, v, z; \alpha)\}$  be known explicitly. In general, no exact solution seems to be possible for the kinetic equation (5.4) [augmented by the boundary condition  $E\{f_c(u, v, 0; \alpha)\} = \delta(u) \delta(v)$ ]; this, however, is not the case for the Fokker–Planck equations (5.5) and (5.6), as it will be shown below.

The Fokker–Planck equation (5.5) has been studied extensively (cf., e.g., Refs. 26 and 27). Its exact solution is a two-dimensional Gaussian distribution in  $u$  and  $v$ .

Let

$$\Sigma = E\{[\mathbf{w}(z; \alpha) - E\{\mathbf{w}(z; \alpha)\}][\mathbf{w}(z; \alpha) - E\{\mathbf{w}(z; \alpha)\}]^T\} \quad (5.8)$$

be the autocovariance matrix of the two-dimensional vector process  $\mathbf{w}(z; \alpha) = [u(z; \alpha), v(z; \alpha)]$ . It is given explicitly as follows:

$$\Sigma = \{\sigma_{ij}^2\}, \quad i, j = 1, 2, \quad (5.9a)$$

$$\sigma_{11}^2 = D[(z/g^2) - (1/2g^3) \sin 2gz], \quad (5.9b)$$

$$\sigma_{12}^2 = \sigma_{21}^2 = (D/g^2) \sin^2 gz, \quad (5.9c)$$

$$\sigma_{22}^2 = D[z + (1/2g) \sin 2gz]. \quad (5.9d)$$

In deriving (5.9), use has been made of the fact that  $E\{u\} = E\{v\} = 0$ , and hence,  $E\{\mathbf{w}\} = 0$ . Accounting for this property, the general form of the desired normal distribution density function is

$$E\{f_c(\mathbf{w}, z; \alpha)\} \equiv F(\mathbf{w}, z) = (2\pi)^{-1} (\det \Sigma)^{-1/2} \exp(-\frac{1}{2} \mathbf{w}^T \Sigma^{-1} \mathbf{w}), \quad (5.10)$$

where  $\Sigma^{-1}$  is the inverse of the covariance matrix. More explicitly,

$$F(\mathbf{w}, z) = (2\pi)^{-1} (\det \Sigma)^{-1/2} \times \exp \left[ -\frac{1}{2} \frac{1}{\det \Sigma} (\sigma_{22}^2 u^2 + \sigma_{11}^2 v^2 - 2\sigma_{12}^2 uv) \right], \quad (5.11)$$

where the  $\sigma_{ij}^2$ ,  $i, j=1, 2$ , are the entries of the covariance matrix [cf. Eqs. (5.9b)–(5.9d)] and

$$\det \Sigma = (D^2/g^2)z^2 - (D^2/g^4)\sin^2 gz. \quad (5.12)$$

Finally, the “marginal” distribution density functions of the random functions  $u(z; \alpha)$  and  $v(z; \alpha)$  can be readily found from (5.10):

$$F(u, z) = \int_{-\infty}^{\infty} dv F(v, u, z) = (2\pi)^{-1/2} \sigma_{11}^{-1} \times \exp \left( -\frac{1}{2} \frac{1}{\sigma_{11}^2} u^2 \right), \quad (5.13)$$

$$F(v, z) = \int_{-\infty}^{\infty} du F(u, v, z) = (2\pi)^{-1/2} \sigma_{22}^{-1} \times \exp \left( -\frac{1}{2} \frac{1}{\sigma_{22}^2} v^2 \right). \quad (5.14)$$

The more complicated Fokker–Planck equation (5.6) based on the long-range Markovian approximation is a variant of Kramers’ equation (cf. Ref. 28). As such, it can be integrated by the method of characteristics. Its solution constitutes a two-dimensional normal distribution of the form (5.11), with covariance matrix

$$\Sigma = \{\sigma_{ij}^2\}, \quad i, j=1, 2, \quad (5.15a)$$

$$\sigma_{11}^2 = (D_2/g^2)\sin^2 gz + D_1[(z/g^2) - (1/2g^3)\sin 2gz], \quad (5.15b)$$

$$\sigma_{12}^2 = \sigma_{21}^2 = (D_2/2g)\sin 2gz + (D_1/g^2)\sin^2 gz, \quad (5.15c)$$

$$\sigma_{22}^2 = D_1[z + (1/2g)\sin 2gz] - D_2\sin^2 gz. \quad (5.15d)$$

Several general results presented in the following two sections hold for both Fokker–Planck equations (5.5) and (5.6). This is due to the fact that the solutions to these equations, i.e., the respective joint probability distribution density functions, have the same functional form [cf. Eq. (5.11)]. For the sake of simplicity, however, many specific analytic and numerical results are based on the assumption that  $\delta H(z; \alpha)$  is a wide-sense stationary,  $\delta$ -correlated random process.

## 6. SECOND-ORDER OBSERVABLES

### A. Basic second-order moments

We set as our first task the computation of the *mean intensity*  $E\{i(x, z; \alpha)\}$ , where  $i(x, z; \alpha) = \psi^*(x, z; \alpha)\psi(x, z; \alpha)$  is the irradiance function [cf. Eq. (2.1)]. Toward this end, we note from (4.11) that

$$i(x, z; \alpha) = (gk/\pi)^{1/2} \exp[-gk[x - u(z; \alpha)]^2]. \quad (6.1)$$

Since the last quantity is a functional only of  $u(z; \alpha)$ ,

$$E\{i(x, z; \alpha)\} = \int_{-\infty}^{\infty} du i(x, u) F(u, z), \quad (6.2)$$

where  $F(u, z)$  is the marginal probability density function given in (5.13). The integration in (6.2) can be performed easily, yielding

$$E\{i(x, z; \alpha)\} = \pi^{-1/2} [gk/(1 + 2gk\sigma_{11}^2)]^{1/2} \times \exp[-gk/(1 + 2gk\sigma_{11}^2)]x^2. \quad (6.3)$$

On the beam axis (i.e.,  $x=0$ ), (6.3) specializes to

$$E\{i(0, z; \alpha)\} = \pi^{-1/2} [gk/(1 + 2gk\sigma_{11}^2)]^{1/2}. \quad (6.4)$$

Since  $\sigma_{11}^2(z) > 0$ ,  $\forall z > 0$  [cf. (5.9b)], it is seen that the on-axis mean intensity decreases monotonically as a function of the range  $z$ . This is due to the transfer of mean power from the fundamental mode to higher modes, a subject that will be considered in detail later on.

The *mean centroid* of the beam, defined by

$$X(z) = \int_{-\infty}^{\infty} dx x E\{i(x, z; \alpha)\}, \quad (6.5)$$

is zero for all  $z$  since the integrand is an odd function of  $x$ .

A measure of the *spreading of the beam* can be found by using the definition

$$\frac{1}{2}\sigma_x^2(z) = \int_{-\infty}^{\infty} dx [x - X(z)]^2 E\{i(x, z; \alpha)\}. \quad (6.6)$$

It turns out that

$$\sigma_x^2(z) = (1 + 2gk\sigma_{11}^2)/gk. \quad (6.7)$$

Introducing (6.7) into (6.3), the mean intensity can be written more succinctly as follows:

$$E\{i(x, z; \alpha)\} = [\pi\sigma_x^2(z)]^{-1/2} \exp[-x^2/\sigma_x^2(z)]. \quad (6.8)$$

Comparing (6.8) with the initial intensity, viz.,

$$i(x, 0; \alpha) = (gk/\pi)^{1/2} \exp(-gkx^2), \quad (6.9)$$

it is seen that the mean intensity remains Gaussian for  $z > 0$ . Since  $\psi_0(x)$  is taken to be the fundamental mode of the unperturbed problem,  $i(x, z) = (gk/\pi)^{1/2} \exp(-gkx^2)$  in the absence of random perturbations. In this case, the original beam would not spread at all. In the presence of random fluctuations, however, the variance of the transverse coordinate  $x$  changes from  $(gk)^{-1}$  to the quantity  $\sigma_x^2(z)$  given in (6.7).

If the expression for  $\sigma_{11}^2$  given in (5.9b) is substituted in (6.7), one has

$$\sigma_x^2(z) = [1 + 2gkL(z/g^2 - \sin 2gz/2g^3)]/gk. \quad (6.10)$$

In terms of the dimensionless quantities

$$\xi = gz, \quad c = kD/g^2. \quad (6.11)$$

Eq. (6.10) can be brought into a form convenient for numerical investigation. Specifically,

$$\sigma_x^2(\xi; c)gk = 1 + c(2\xi - \sin 2\xi). \quad (6.12)$$

A plot of this expression versus  $\xi$ , with  $c$  (a dimensionless measure of the strength of the fluctuations) as a parameter, is shown in Fig. 1. The monotonic increase of the beam spreading is clearly evident.

The intensity flux density defined in (2.2) can be written more compactly as

$$j(x, z; \alpha) = \text{Re}[-(i/k)\psi^*(x, z; \alpha)(\partial/\partial x)\psi(x, z; \alpha)]. \quad (6.13)$$

Using (4.11) and performing the operations indicated in (6.13), we obtain

$$j(x, z; \alpha) = (gk/\pi)^{1/2} v(z; \alpha) \exp[-gk[x - u(z; \alpha)]^2], \quad (6.14)$$

a functional of both  $u(z; \alpha)$  and  $v(z; \alpha)$ . The *mean intensity flux density* is, therefore, given by

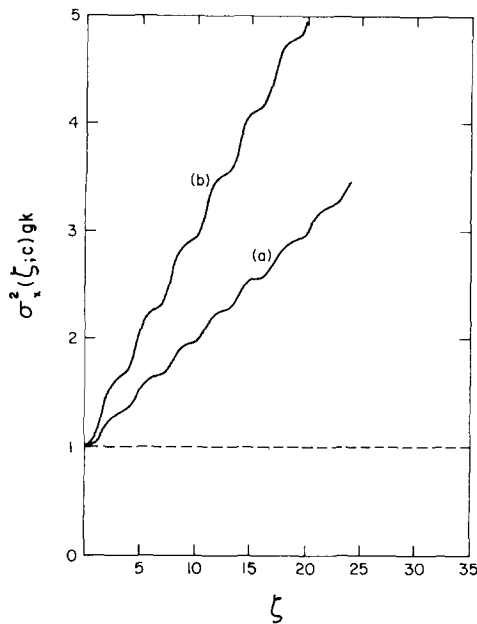


FIG. 1. Variance of transverse beam displacements. (a)  $c = 0.05$ ; (b)  $c = 0.1$ .

$$E\{j(x, z; \alpha)\} = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv j(x, u, v) F(u, v, z), \quad (6.15)$$

where  $F(u, v, z)$  is the joint probability density function in (5.11). In this case, the integration in (6.15) can be performed without difficulty. The final result is

$$E\{j(x, z; \alpha)\} = 2\pi^{-1/2} \sigma_{12}^2 [gk / (1 + 2gk\sigma_{11}^2)]^{3/2} \times \exp\{-[gk / (1 + 2gk\sigma_{11}^2)]x^2\}. \quad (6.16)$$

As a consequence of the conservation law (2.3) which holds for every realization  $\alpha \in A$ , one has by inspection the relation

$$\frac{\partial}{\partial z} E\{i(x, z; \alpha)\} + \frac{\partial}{\partial x} E\{j(x, z; \alpha)\} = 0. \quad (6.17)$$

The latter means that the total mean intensity is conserved, i.e.,

$$\frac{d}{dz} \left[ \int_{-\infty}^{\infty} dx E\{i(x, z; \alpha)\} \right] = 0. \quad (6.18)$$

The conservation law (6.17) can be found by taking the ensemble average of the one-dimensional version of (2.7). Alternatively, it can be obtained by combining our results for  $E\{i(x, z; \alpha)\}$  and  $E\{j(x, z; \alpha)\}$  given in (6.3) and (6.16), respectively.

Having established both  $E\{i(x, z; \alpha)\}$  and  $E\{j(x, z; \alpha)\}$ , we are in a position to compute one more observable: the *mean power flux density* [cf. Eq. (2.7)]. Specifically

$$E\{s(x, z; \alpha)\} = E\{i(x, z; \alpha)\} \hat{z} + E\{j(x, z; \alpha)\} \hat{x}. \quad (6.19)$$

On the basis of a statement made earlier (cf. Sec. 2), the quantity

$$\int_{S_0} E\{s(x, z; \alpha)\} \cdot \hat{n} dA \quad (6.20)$$

can be interpreted as the *total mean power* intercepted by a detector characterized by the open surface  $S_0$ .

We shall undertake next the computation of the *spatial mutual coherence function*  $E\{\psi^*(x_2, z; \alpha)$

$\psi(x_1, z; \alpha)\}$ . We form first the quantity  $\psi^*(x_2, z; \alpha)$   $\psi(x_1, z; \alpha)$  using the expression for  $\psi(x, z; \alpha)$  in (4.11):

$$\begin{aligned} \psi^*(x_2, z; \alpha) \psi(x_1, z; \alpha) &= (gk/\pi)^{1/2} \exp\{-\frac{1}{2}gk[x_2 - u(z; \alpha)]^2 \\ &+ [x_1 - u(z; \alpha)]^2\} \times \exp[-ikv(z; \alpha)(x_2 - x_1)] \\ &\equiv m[x_2, x_1, u(z; \alpha), v(z; \alpha)]. \end{aligned} \quad (6.21)$$

Its ensemble average is given by

$$\begin{aligned} E\{\psi^*(x_2, z; \alpha) \psi(x_1, z; \alpha)\} \\ = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv m(x_2, x_1, u, v) F(u, v, z). \end{aligned} \quad (6.22)$$

The integration can be performed exactly, yielding the following expression for the spatial mutual coherence function:

$$\begin{aligned} E\{\psi^*(x_2, z; \alpha) \psi(x_1, z; \alpha)\} \\ = \pi^{-1/2} [gk / (1 + 2gk\sigma_{11}^2)]^{1/2} \exp[-\frac{1}{2}gk(x_2^2 + x_1^2)] \\ \times \exp[\frac{1}{2}[(gk\sigma_{11}^2) / (1 + 2gk\sigma_{11}^2)](x_2 + x_1)^2] \\ \times \exp[-\frac{1}{2}k^2(\sigma_{12}^4 / \sigma_{11}^2)(1 + 2gk\sigma_{11}^2)^{-1}(x_2 - x_1)^2] \\ \times \exp[-\frac{1}{2}k^2(\det \Sigma / \sigma_{11}^2)(x_2 - x_1)^2] \\ \times \exp[-ikgk\sigma_{21}^2(1 + 2gk\sigma_{11}^2)^{-1}(x_2^2 - x_1^2)]. \end{aligned} \quad (6.23)$$

In the special case that  $x_2$  and  $x_1$  coalesce, i.e.,  $x_2 = x_1 = x$ , the spatial mutual coherence must coincide with the mean intensity found earlier [cf. Eq. (6.3)]. A simple calculation shows that this is, indeed, the case.

## B. Modal power transfer

The unperturbed  $[\delta H(z; \alpha) \rightarrow 0]$  parabolic equation (4.1) is characterized by a set of modes  $h_p(x)$ ,  $p = 0, 1, 2, \dots$ , satisfying the eigenvalue problem

$$-\frac{1}{2k^2} \frac{d^2}{dx^2} h_p(x) + \frac{1}{2} g^2 x^2 h_p(x) = E_p h_p(x). \quad (6.24)$$

It is well known (cf. e.g., Ref. 29) that

$$E_p = g(p + \frac{1}{2})/k \quad (6.25)$$

and

$$h_p(x) = 2^{-p/2} (p!)^{-1/2} (gk/\pi)^{1/4} \exp(-\frac{1}{2}gkx^2) H_p[(gk)^{1/2}x]. \quad (6.26)$$

Here,  $H_p(\xi)$  denotes the  $p$ th Hermite polynomial, viz.,

$$H_p(\xi) = (-1)^p \exp(\xi^2) (d^p/d\xi^p) \exp(-\xi^2). \quad (6.27)$$

The eigenfunctions given in (6.26) are orthonormal, i.e.,

$$\int_{-\infty}^{\infty} dx h_{p'}(x) h_p(x) = \delta_{p', p}. \quad (6.28)$$

The boundary condition  $\psi_0(x)$  specified in (4.9) is the fundamental mode  $h_0(x)$ . This follows easily from (6.26).

We shall determine next the portion of the beam which remains in the fundamental mode. The complex amplitude of the fundamental mode is defined by

$$q_0(z; \alpha) = \int_{-\infty}^{\infty} dx \psi(x, z; \alpha) h_0(x). \quad (6.29)$$

The quantity of interest is the ensemble average of the

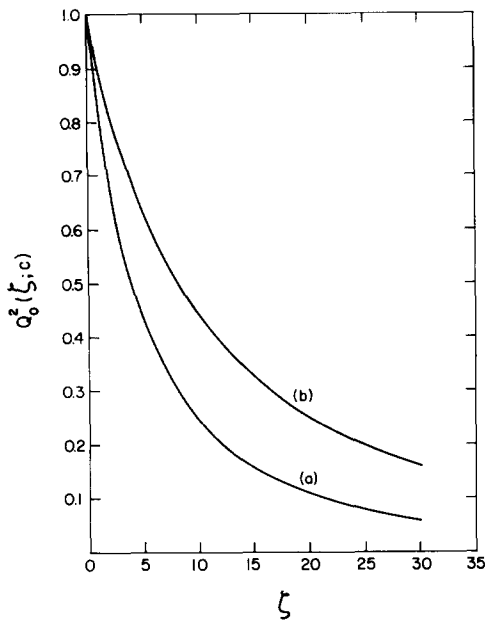


FIG. 2. Decay of mean power in the fundamental mode. (a)  $c=0.1$ ; (b)  $c=0.05$ .

absolute square of  $q_0(z; \alpha)$ , viz.,

$$Q_0^2(z) \equiv E\{|q_0(z; \alpha)|^2\}. \quad (6.30)$$

We note, first, that

$$|q_0(z; \alpha)|^2 = \exp(-\frac{1}{2}gku^2) \exp[-\frac{1}{2}(k/g)v^2]. \quad (6.31)$$

Integrating this quantity over  $u$  and  $v$ , with the joint probability density  $F(u, v, z)$  as a weight, we find

$$Q_0^2(z) = [1 + gk\sigma_{11}^2 + (k/g)(\sigma_{22}^2 + gk\det\Sigma)]^{-1}. \quad (6.32)$$

In the absence of random fluctuations,  $\sigma_{11}^2 \rightarrow 0$ ,  $\sigma_{22}^2 \rightarrow 0$ , and  $\det\Sigma \rightarrow 0$ ; hence,  $Q_0^2(z) \rightarrow 1$ ,  $\forall z$ . In other words, the power remains entirely in the fundamental mode. On the other hand, the general expression (6.32) shows how power (in a mean sense) leaks out of the excited fundamental mode into higher modes.

Using the expressions for  $\sigma_{11}^2$ ,  $\sigma_{22}^2$ , and  $\det\Sigma$  [cf. Eqs. (5.9b), (5.9d), and (5.12), respectively] into (6.32),  $Q_0^2(z)$  can be written as

$$Q_0^2(\xi; c) = [1 + 2c\xi + c^2(\xi^2 - \sin^2\xi)]^{-1} \quad (6.33)$$

in terms of the dimensionless quantities  $\xi$  and  $c$  [cf. Eq. (6.11)]. A plot of  $Q_0^2(\xi; c)$  versus  $\xi$ , with  $c$  as a parameter, is shown in Fig. 2.

Analogously to (6.30), we define the expected modal power transferred from the fundamental mode at  $z=0$  to the  $p$ th mode at distance  $z$  as follows:

$$Q_p^2(z) = E\{|q_p(z; \alpha)|^2\}, \quad (6.34)$$

where  $q_p(z; \alpha)$  is the  $p$ th-order complex amplitude function, viz.,

$$q_p(z; \alpha) = \int_{-\infty}^{\infty} dx \psi(x, z; \alpha) h_p(x). \quad (6.35)$$

The square modulus  $|q_p(z; \alpha)|^2$  required in (6.34) is found to be

$$|q_p(z; \alpha)|^2 = 2^{-p}(p!)^{-1}[(k/g)v^2 + gku^2]^p \times \exp[-\frac{1}{2}(k/g)v^2] \exp(-\frac{1}{2}gku^2). \quad (6.36)$$

Finally,

$$Q_p^2(z) = 2^{-p}(p!)^{-1}(gk)^p(2\pi)^{-1}(\det\Sigma)^{-1/2}(\pi/a)^{1/2}(\pi/a')^{1/2} \times \sum_{s=0}^p \sum_{\rho=0}^s \binom{p}{s} g^{-2s} (2s)! (b'/a)^{2s} [l!(2s-2l)!]^{-1} \times (a/4b'^2)^l [2(p-l-1)!] / (2a')^{p-1}, \quad (6.37a)$$

where

$$a = \frac{1}{2}[(k/g) + (\sigma_{11}^2/\det\Sigma)], \quad (6.37b)$$

$$a' = \frac{1}{2}[gk + (gk\sigma_{11}^2)^2 + (k\sigma_{22}^2)^2 + gk^3\det\Sigma](gk\sigma_{11}^2 + k^2\det\Sigma)^{-1} \quad (6.37c)$$

$$b' = \frac{1}{2}(\sigma_{12}^2/\det\Sigma). \quad (6.37d)$$

Particular values of modal power transfer functions can be obtained from the general expression (6.37) by restricting the values of the index  $p$ . Setting  $p=0$ , for example, specializes (6.37) to the quantity  $Q_0^2(z)$  found earlier. For  $p=1$ , on the other hand, one has

$$Q_1^2(z) = \frac{1}{2}[gk\sigma_{11}^2 + (k/g)\sigma_{22}^2 + 2k^2\det\Sigma] \times [1 + gk\sigma_{11}^2 + (k/g)(\sigma_{22}^2 + gk\det\Sigma)]^{-3/2}, \quad (6.38)$$

a relationship exhibiting how mean power is transferred from the fundamental mode to the first one. Since,  $\sigma_{11}^2$ ,  $\sigma_{22}^2$ , and  $\det\Sigma$  become zero at the boundary  $z=0$ , it follows that  $Q_1^2(0)=0$ . This, of course, is expected since at  $z=0$  all the mean power is stored in the fundamental mode.

In the special case that  $\delta H(z; \alpha)$  is  $\delta$ -correlated,  $Q_1^2(z)$  can be written in the dimensionless form

$$Q_1^2(\xi; c) = [c\xi + c^2(\xi^2 - \sin^2\xi)][1 + 2c\xi + c^2(\xi^2 - \sin^2\xi)]^{-3/2}. \quad (6.39)$$

A plot of  $Q_1^2(\xi; c)$  versus  $\xi$  for two values of the parameter  $c$  is shown in Fig. 3. The average power contained in the first mode rises from zero to a maximum, and, then, decays monotonically to zero as  $\xi \rightarrow \infty$ . It is, also, seen that the position of the maximum, as well as the rate of decay, depends on the strength of the random fluctuations.

Qualitatively, one would expect the same general behavior for the average modal power transfer from the fundamental mode of  $z=0$  to the  $p$ th mode at distance  $z$ . The rate of transfer, however, would depend on the order of the mode.

### C. Degree of coherence

Given the random wavefunction  $\psi(x, z; \alpha)$ , the *degree of coherence*,  $D(z)$ , is defined as follows:

$$D^2(z) = \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_1 |E\{\psi^*(x_2, z; \alpha)\psi(x_1, z; \alpha)\}|^2. \quad (6.40)$$

It can be calculated by using the expression for the spatial mutual coherence [cf. Eq. (6.23)]. The final result is

$$D^2(z) = [1 + 2gk\sigma_{11}^2 + 2(k/g)(\sigma_{22}^2 + 2gk\det\Sigma)]^{-1/2}. \quad (6.41)$$

For a random process  $\delta H(z; \alpha)$  which is  $\delta$ -correlated, (6.41) assumes the following dimensionless form:



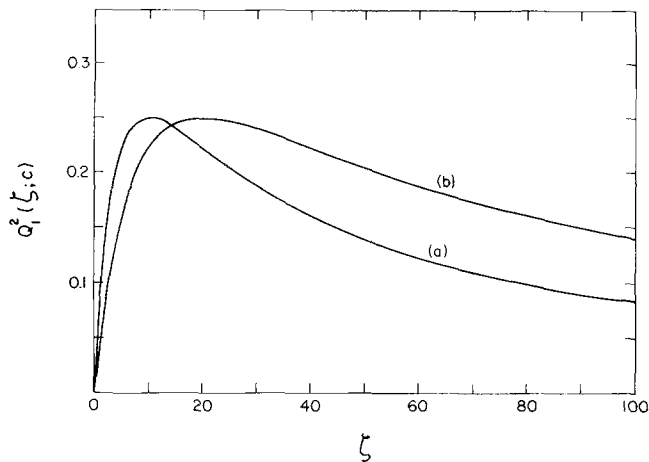


FIG. 3. Mean power transfer to the first mode. (a)  $c = 0.1$ ; (b)  $c = 0.05$ .

$$D^2(z; c) = [1 + 4cz + 4c^2(z^2 - \sin^2 z)]^{-1/2}. \quad (6.42)$$

The latter is shown graphically in Fig. 4.

The degree of coherence is a measure of the irreversible loss of coherence due to the presence of random fluctuations in the medium. At  $z = 0$ ,  $D(z) = 1$ , i.e., the beam is coherent. As the range  $z$  is increased, there is a loss of coherence, i.e.,  $D(z) < 1$ , and finally,  $D(z) \rightarrow 0$  as  $z \rightarrow \infty$ , i.e., the beam is rendered completely incoherent.

In the absence of random fluctuations in the medium,  $D^2(z)$  and, hence,  $D(z)$  would be conserved, i.e.,  $D(z) = D(0) = 1$ ,  $\forall z$ . If the beam were partially coherent at  $z = 0$  due, for example, to aberrations in a focusing optical system, then  $D(0) < 1$ . In this case,  $D(z) \leq D(0)$ ,  $\forall z \geq 0$ , the equality holding for a nonrandom medium.

The degree of coherence defined in (6.40) is by no means the only quantity exhibiting the irreversibility due to the random fluctuations in the medium. Another quantity which describes this irreversible loss of coherence (or information) is the  $H$ -function used in statistical mechanics.<sup>30</sup> This function is intimately related to the thermodynamic (or information-theoretic) concept of entropy. To construct an appropriate definition for the  $H$ -function, we shall require a few preliminary results.

A two (transverse)-point field density function is introduced first as follows in terms of the wavefunction  $\psi(x, z; \alpha)$ :

$$\rho(x_2, x_1, z; \alpha) = \psi^*(x_2, z; \alpha) \psi(x_1, z; \alpha). \quad (6.43)$$

The "phase-space" analog of the density function is provided by the Wigner distribution density function which is defined as follows<sup>31</sup>:

$$f(x, \theta, z; \alpha) = \frac{k}{2\pi} \int_{-\infty}^{\infty} dy e^{ik\theta y} \rho(x + \frac{1}{2}y, x - \frac{1}{2}y, z; \alpha). \quad (6.44)$$

This quantity is real, but not necessarily positive everywhere. It can be shown (cf. Appendix A of Ref. 5; also, Ref. 32), in general, that  $|f(x, \theta, z; \alpha)| \leq (k/\pi)$  for any realization  $\alpha \in A$ . Provided that  $f(x, \theta, z; \alpha)$  is normalized to unity, this means that the Wigner distribution density function is different from zero in a region of

which the volume in phase space is at least equal to  $(\pi/k)$ . Hence,  $f(x, \theta, z; \alpha)$  can never be sharply localized in  $z$  and  $\theta$ . This situation is reminiscent of the quantum mechanical uncertainty principle. It is also analogous to the ambiguity arising in Fourier optics,<sup>33,34</sup> and the radar ambiguity discussed originally by Woodward.<sup>35</sup>

The mean Wigner distribution density function can be determined by ensemble-averaging both sides of (6.44):

$$E\{f(x, \theta, z; \alpha)\} = \frac{k}{2\pi} \int_{-\infty}^{\infty} dy e^{ik\theta y} E\{\rho(x + \frac{1}{2}y, x - \frac{1}{2}y, z; \alpha)\}. \quad (6.45)$$

The quantity  $E\{\rho(x + \frac{1}{2}y, x - \frac{1}{2}y, z; \alpha)\}$  in the integrand is simply the spatial mutual coherence function [cf. Eq. (6.23)] expressed in terms of "center of mass" and difference coordinates. Taking this into consideration, we find that

$$\begin{aligned} E\{f(x, \theta, z; \alpha)\} &= (k/\pi) [1 + 2gk\sigma_{11}^2 + 2(k/g)(\sigma_{22}^2 + 2gk\det\Sigma)]^{-1/2} \\ &\times \exp\left\{-\frac{k}{g}(1 + 2gk\sigma_{11}^2)[1 + 2gk\sigma_{11}^2 + 2(k/g) \right. \\ &\times (\sigma_{22}^2 + 2gk\det\Sigma)]^{-1} \\ &\times [\theta - 2gk\sigma_{11}^2(1 + 2gk\sigma_{11}^2)^{-1}x]^2\} \exp\left\{-\frac{gk}{(1 + 2gk\sigma_{11}^2)}x^2\right\}. \end{aligned} \quad (6.46)$$

The  $H$ -function is defined next in terms of the mean Wigner distribution density function as follows:

$$H(z) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\theta E\{f'(x, \theta, z; \alpha)\} \ln E\{f'(x, \theta, z; \alpha)\}. \quad (6.47)$$

Here,  $f'(x, \theta, z; \alpha) = f(x, \theta, z; \alpha)/k$ . It has already been pointed out that the Wigner distribution density function (and, hence, its ensemble average), although real, may not necessarily be positive everywhere. Consequently, the definition of the  $H$ -function in (6.47) cannot possibly be valid in general, by virtue of the logarithmic term which is not defined for negative values of  $E\{f(x, \theta, z; \alpha)\}$ . [It is this specific difficulty which is surmounted by the

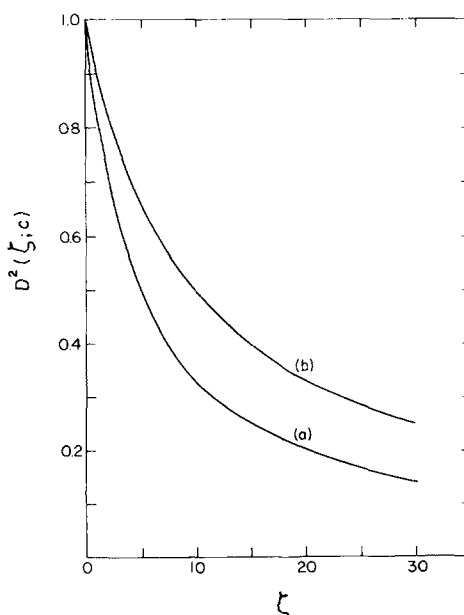


FIG. 4. Variation of the square of the degree of coherence. (a)  $c = 0.1$ ; (b)  $c = 0.05$ .

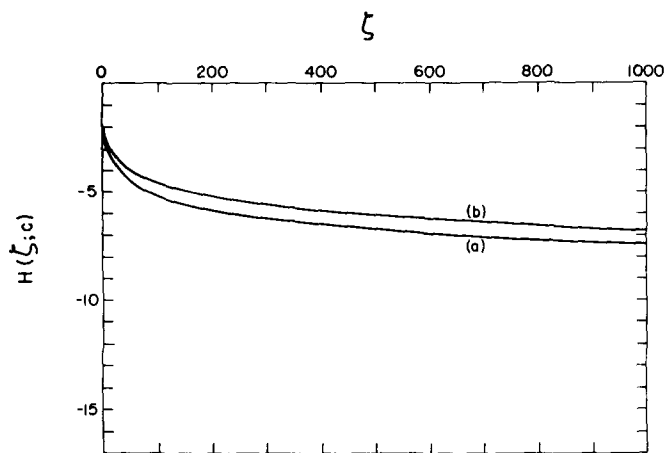


FIG. 5. Axial variation of the  $H$ -function. (a)  $c = 0.1$ ; (b)  $c = 0.05$ .

definition of the degree of coherence given in (6.40)]. However, for the particular problem under consideration here,  $f(x, \theta, z; \alpha)$  (and, hence, its ensemble average) is nonnegative. This can be easily seen by inspecting the explicit solution for  $E\{f(x, \theta, z; \alpha)\}$  given in (6.46). We are, therefore, fully justified in using the definition (6.47). Carrying out the operations indicated on the right-hand side of (6.47) results in the following expression for the  $H$ -function:

$$H(z) = \ln[D^2(z)/\pi] - 1. \quad (6.48)$$

Interestingly, it is a functional of the square of the degree of coherence determined earlier [cf. Eq. (6.41)].

An important property characterizing  $H(z)$  is assumed in the  $H$ -theorem, viz.,

$$(d/dz)H(z) \leq 0, \quad (6.49)$$

which is widely used in statistical mechanics (see, e.g., Ref. 30). The validity of this theorem follows readily from the result (6.48) for  $H(z)$ , in conjunction with the expression for the degree of coherence,  $D(z)$ , given in (6.41) and the specific forms of  $\sigma_{11}^2, \sigma_{22}^2$ , and  $\det \Sigma$  derived in Sec. 5.

The entropy,  $S(z)$ , is defined as the negative  $H$ -function. Corresponding to the  $H$ -theorem (6.49), we have, then,

$$(d/dz)S(z) \geq 0. \quad (6.50)$$

This relation is a manifestation of the second law of thermodynamics, and states that the total entropy of the system (resp. beam) cannot decrease.

The  $H$ -function in (6.48) can be written in terms of the dimensionless quantities  $\xi$  and  $c$  as follows:

$$H(\xi; c) = \ln\{\pi^{-1}[1 + 4c\xi + 4c^2(\xi^2 - \sin^2 \xi)]^{-1/2}\} - 1. \quad (6.51)$$

The latter is depicted in Fig. 5. The non-increasing property of the  $H$ -function incorporated into the  $H$ -theorem is clearly evident in this plot.

#### D. Effective coherence distance

We shall close this section by defining and computing a characteristic scale called the *effective coherence*

*distance*. [In the case of a two-dimensional problem (circular beam), this characteristic scale is referred to as the "effective coherence radius" of the beam (cf. Ref. 36)]. This quantity is intimately related to the angular width of the beam.

Let us begin by integrating  $E\{\psi^*(x + \frac{1}{2}y, z; \alpha)\psi(x - \frac{1}{2}y, z; \alpha)\}$  over  $x$ . The result is

$$\begin{aligned} \gamma(y, z) &\equiv \int_{-\infty}^{\infty} dx E\{\psi^*(x + \frac{1}{2}y, z; \alpha)\psi(x - \frac{1}{2}y, z; \alpha)\} \\ &= \exp\{-\frac{1}{2}[\frac{1}{2}gk + k^2\sigma_{22}^2]y^2\}. \end{aligned} \quad (6.52)$$

The effective coherence distance,  $y_e$ , is defined as that value of  $y$  at which  $\gamma(y, z)$  has become  $e$  times smaller than  $\gamma(0, z)$ , viz.,

$$\gamma(y_e, z) = e^{-1}\gamma(0, z). \quad (6.53)$$

It is clear from (6.52) that  $\gamma(z, z) = 1$ . Hence,

$$\gamma(y_e, z) = e^{-1}. \quad (6.54)$$

The desired quantity  $y_e$  can be easily determined by taking the natural logarithm of (6.54):

$$y_e(z) = [2^{-1}(2^{-1}gk + k^2\sigma_{22}^2)]^{-1/2}. \quad (6.55)$$

The corresponding dimensionless quantity  $y_e(\xi; c)$  is plotted in Fig. 6 for two values of the parameter  $c$ . It has the value of unity at the initial boundary, and decreases monotonically to zero as  $\xi \rightarrow \infty$ .

## 7. HIGHER-ORDER MOMENTS

### A. Fourth-order moments

We consider first the *correlation of the field intensity at two transverse points*, viz.,  $E\{i(x_2, z; \alpha)i(x_1, z; \alpha)\}$ . Using the expression found earlier for  $i(x, z; \alpha)$  [cf. Eq. (6.1)], we have

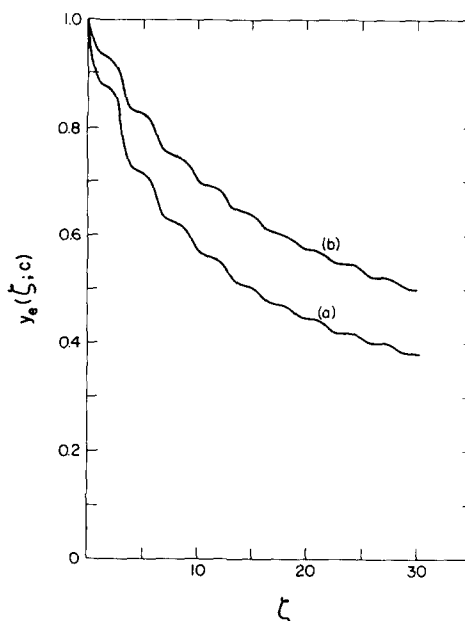


FIG. 6. Effective coherence distance of the beam (a)  $c = 0.1$ ; (b)  $c = 0.05$ .

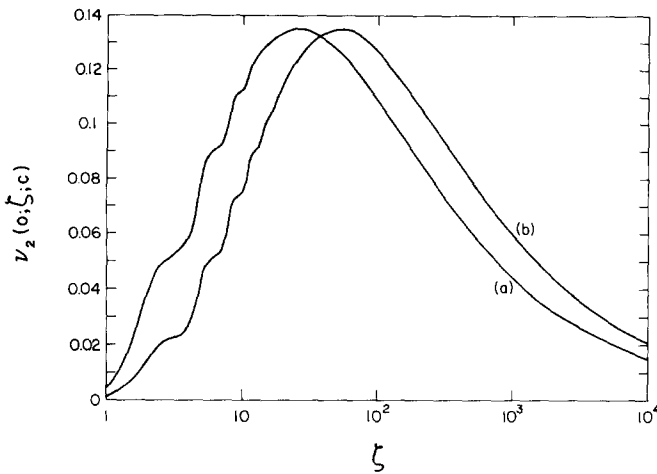


FIG. 7. Variance of on-axis intensity fluctuations. (a)  $c=0.1$ ; (b)  $c=0.05$ .

$$i(x_2, z; \alpha) i(x_1, z; \alpha) = (gk/\pi) \exp[-gk[x_2 - u(z; \alpha)]^2] \times \exp[-gk[x_1 - u(z; \alpha)]^2], \quad (7.1)$$

from which, upon averaging over the probability density function  $F(u, v, z)$ , we obtain

$$E\{i(x_2, z; \alpha) i(x_1, z; \alpha)\} = (gk/\pi)(1 + 4gk\sigma_{11}^2)^{-1/2} \times \exp[-gk[(1 + 2gk\sigma_{11}^2)/(1 + 4gk\sigma_{11}^2)](x_2^2 + x_1^2)] \times \exp[4\sigma_{11}^2[(gk)^2/(1 + 4gk\sigma_{11}^2)]x_1x_2]. \quad (7.2)$$

As an illustration of a physical situation which requires the special fourth-order moment given in (7.2), consider the problem of "beam wandering." Associated with the stochastic parabolic equation (4.1), we define the "centroid" of the beam by

$$x(z; \alpha) = \int_{-\infty}^{\infty} dx x i(x, z; \alpha), \quad \forall \alpha \in A. \quad (7.3)$$

(As explained earlier, we assume that the total intensity is normalized to unity for each realization  $\alpha \in A$ .) The mean centroid,  $X(z)$ , of the beam has already been defined [cf. Eq. (6.5)] and found to be equal to zero. It is interesting, however, to consider also the variance of the random function  $x(z; \alpha)$ , viz.,

$$\text{var}\{x(z; \alpha)\} = E\{[x(z; \alpha) - X(z)]^2\} = \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_1 x_2 x_1 E\{i(x_2, z; \alpha) i(x_1, z; \alpha)\}. \quad (7.4)$$

Introducing (7.2) into the last equation and carrying out the integration we find

$$\text{var}\{x(z; \alpha)\} = \sigma_{11}^2, \quad (7.5)$$

an expression which vanishes at  $z=0$  and becomes unbounded as  $z \rightarrow \infty$ . This behavior is a manifestation of the instability of the beam due to the presence of random-axis misalignments.

Setting  $x_2 = x_1 = x$ , (7.2) reduces to the *intensity autocorrelation function*

$$E\{i^2(x, z; \alpha)\} = (gk/\pi)(1 + 4gk\sigma_{11}^2)^{-1/2} \times \exp[-2gk/(1 + 4gk\sigma_{11}^2)]x^2. \quad (7.6)$$

This result can be used in conjunction with the expression for the mean intensity [cf. Eq. (6.3)] in order to compute the *variance* (or *fluctuation*) of the intensity:

$$E\{[i(x, z; \alpha) - E\{i(x, z; \alpha)\}]^2\} \equiv \nu_2(x, z) = (gk/\pi) \{ (1 + 4gk\sigma_{11}^2)^{-1/2} \exp[-2gk/(1 + 4gk\sigma_{11}^2)]x^2 - (1 + 2gk\sigma_{11}^2)^{-1} \exp[-2gk/(1 + 2gk\sigma_{11}^2)]x^2 \}. \quad (7.7)$$

Figure 7 shows the behavior of the on-axis fluctuations of intensity  $\nu_2(0, \xi; c)$  for two values of the parameter  $c$ . As expected from physical considerations,  $\nu_2(0, \xi, c)$  vanishes at the plane of the source ( $\xi=0$ ). It is also seen, however, that  $\nu_2(0, \xi, c)$  decays monotonically to zero as  $\xi \rightarrow \infty$ , after it rises to a maximum at an intermediate range. On- and off-axis intensity fluctuations are shown graphically in Fig. 8. Here,  $\nu_2(\xi; c; d)$  is plotted versus  $\xi$  for one value of  $c$  and five values of the new dimensionless parameter  $d = (gk)^{1/2}x$ . In general, one might expect that for fixed values of the range  $\xi$  and the strength of the random fluctuations (incorporated into the parameter  $c$ ),  $\nu_2(\xi; c; d)$  would decrease monotonically as  $d$  was increased. This is definitely not the case, as it is clearly shown in Fig. 8. Specifically, for  $\xi=10$  and  $c=0.1$ ,  $\nu_2(\xi; c; d)$  increases for values of  $d$  up to about 0.75, and decreases for higher values of this parameter. A plausible explanation of this "anomalous" behavior could be the "elastic" influence of the background deterministic focusing medium.

## B. Higher-order moments

Statistical moments beyond the fourth order level can be calculated without too much difficulty. For example,

$$E\{i^n(x, z; \alpha)\} = (gk/\pi)^{n/2} (1 + 2ngk\sigma_{11}^2)^{-1/2} \times \exp[-n(gk/(1 + 2ngk\sigma_{11}^2))x^2], \quad (7.8)$$

where  $n$  is a positive integer. Furthermore,

$$E\{i^n(x_2, z; \alpha) i^n(x_1, z; \alpha)\} = (gk/\pi)^n (1 + 4ngk\sigma_{11}^2)^{-1/2} \times \exp[-ngk[(1 + 2ngk\sigma_{11}^2)/(1 + 4ngk\sigma_{11}^2)](x_2^2 + x_1^2)] \times \exp[4\sigma_{11}^2[(ngk)^2/(1 + 4ngk\sigma_{11}^2)]x_1x_2]. \quad (7.9)$$

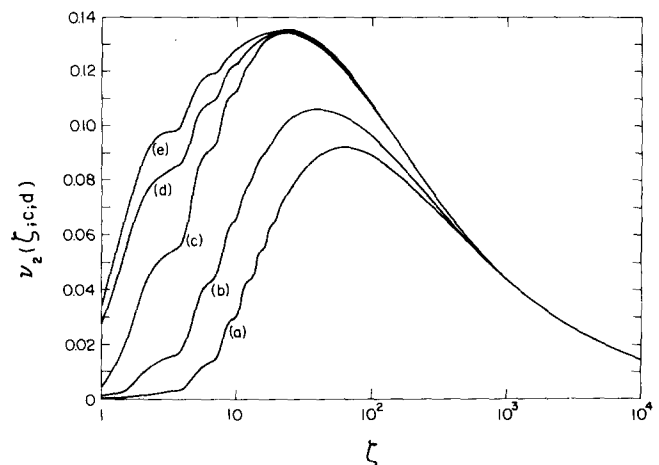


FIG. 8. Variance of on- and off-axis intensity fluctuations. (a)  $d=2.25$ ; (b)  $d=2$ ; (c)  $d=0$  (on-axis); (d)  $d=0.5$ ; (e)  $d=0.75$ . All curves are computed for  $c=0.1$ .

Finally, analogously to (7.7), we find that

$$\begin{aligned} E\{[i^n(x, z; \alpha) - E\{i^n(x, z; \alpha)\}]^2\} &= \nu_{2n}(x, z) \\ &= (gk/\pi)^n [(1 + 4ngk\sigma_{11}^2)^{-1/2} \exp\{-2n[gk/(1 + 4ngk\sigma_{11}^2)]x^2\} \\ &\quad - (1 + 2ngk\sigma_{11}^2)^{-1} \exp\{-2n[gk/(1 + 2ngk\sigma_{11}^2)]x^2\}]. \end{aligned} \quad (7.10)$$

### C. Probability density function of the log irradiance

The probability density function  $p(i)$  of the intensity (or irradiance)  $i = \psi^* \psi$  can be expressed in terms of the moments  $E\{i^\nu(x, z; \alpha)\}$  as follows:

$$p(i) = (2\pi i)^{-1} \int_{-i\infty-\epsilon}^{i\infty-\epsilon} d\nu i^{-\nu-1} (x, z; \alpha) E\{i^\nu(x, z; \alpha)\}, \quad (7.11)$$

where  $\epsilon$  is an infinitesimal positive number. The conditions for the applicability of (7.11), viz., that  $E\{i^\nu(x, z; \alpha)\} \leq \exp(\nu E_0)$ ,  $\nu \sim \infty$ ,  $E_0$  being a real constant, and  $E\{i^\nu(x, z; \alpha)\}$  be analytic on the right half-plane of  $\nu$  [satisfied by our expression given in (7.8)], are fully discussed in Refs. 37 and 38.

We introduce, next, the definitions  $E = \ln[i(x, z; \alpha)/i_0]$ ,  $i_0 = (gk/\pi)^{1/2}$ ,  $E_0 = gkx^2$ , and  $\sigma_E = 2gk\sigma_{11}^2$ . (We have kept here unaltered the notation introduced by Furutsu in Ref. 37.) We have, then,

$$\begin{aligned} p(E) &= p(i)(d/dE)i \\ &= (2\pi i)^{-1} \int_{-i\infty-\epsilon}^{i\infty-\epsilon} d\nu (1 + \nu\sigma_E)^{-1/2} \\ &\quad \times \exp\{-\nu[(1 + \nu\sigma_E)^{-1}E_0 - E]\}. \end{aligned} \quad (7.12)$$

The contour integration can be carried out, with the result

$$p(E) = \begin{cases} 0, & E > 0, \\ (-\pi E\sigma_E)^{-1/2} \exp[(E - E_0)/\sigma_E] \cosh 2[(-EE_0)^{1/2}/\sigma_E], & E < 0, \end{cases} \quad (7.13a)$$

$$(7.13b)$$

(7.13a) being a direct consequence of the analyticity of the integrand in (7.12) on the right half-plane of  $\nu$ . The probability density function  $p(E)$  of the log irradiance is nonnegative; furthermore, it satisfies the normalization property

$$\int_{-\infty}^0 dE p(E) = 1. \quad (7.14)$$

On the beam axis ( $x=0$  or, equivalently,  $E_0=0$ ), (7.13) reduces to the simpler form

$$p(E) = \begin{cases} 0, & E > 0, \\ (-\pi E\sigma_E)^{-1/2} \exp(E/\sigma_E), & E < 0. \end{cases} \quad (7.15a)$$

$$(7.15b)$$

Having established an analytical expression for  $p(E)$ , various moments of the log irradiance can be found by direct integration. For example,

$$M_1 \equiv E\{E\} = \int_{-\infty}^0 dE E p(E) = -(\frac{1}{2}\sigma_E + E_0) \quad (7.16)$$

and

$$\begin{aligned} M_2 &\equiv E\{(E - E\{E\})^2\} \\ &= \int_{-\infty}^0 dE (E - E\{E\})^2 p(E) = \frac{1}{2}\sigma_E^2 + 2E_0\sigma_E \end{aligned} \quad (7.17)$$

for the general case, i.e.,  $x \neq 0$ . Higher-order moments of the log irradiance can be computed in the same manner.

Furutsu and Furuhashi (cf. Ref. 38) have suggested an alternative method for calculating log-irradiance moments. Their technique can be briefly outlined as follows. The  $n$ th moment of  $E$  is given by

$$E\{E^n\} = \int_{-\infty}^0 dE E^n p(E) = -n/(2\pi i) \int_{-i\infty-\epsilon}^{i\infty-\epsilon} d\nu \nu^{-n-1} f(\nu), \quad (7.18)$$

where

$$f(\nu) = (\nu\sigma_E + 1)^{-1/2} \exp[-\nu E_0/(\nu\sigma_E + 1)]. \quad (7.19)$$

The last expression, however, is analytic for  $\text{Re}(\nu) > -\epsilon$  and tends to zero as  $|\nu| \rightarrow \infty$ . Therefore, the second integral on the right-hand side of (7.18) can be carried out by a clockwise contour integration around the pole  $\nu=0$ . Moreover, in the vicinity of this pole,

$$f(\nu) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} E\{E^n\} \nu^n, \quad (7.20)$$

and the log-irradiance moments  $E\{E^n\}$ ,  $n=1, 2, \dots$ , can be obtained by matching (7.20) with the Taylor series expansion of  $f(\nu)$  around  $\nu=0$ .

### D. Saturation phenomena

Consider the expression

$$\sigma_G^{(n)} = [\ln\{E\{i^{2n}(x, z; \alpha)\}/[E\{i^n(x, z; \alpha)\}]^2\}]^{1/2}. \quad (7.21)$$

where  $n$  is a positive integer. Using a result obtained earlier in this section [cf. Eq. (7.8)], as well as the notation introduced in the previous subsection, we may write in the place of (7.21)

$$\begin{aligned} \sigma_G^{(n)} &= [\ln\{(1 + n\sigma_E)(1 + 2n\sigma_E)^{-1/2} \\ &\quad \times \exp[2n^2 E_0 \sigma_E / (1 + 2n\sigma_E)(1 + n\sigma_E)]\}]^{1/2}. \end{aligned} \quad (7.22)$$

For the special cases  $n=1$  and  $n=2$ , we have, respectively,

$$\begin{aligned} \sigma_G^{(1)} &= [\ln\{(1 + \sigma_E)(1 + 2\sigma_E)^{-1/2} \\ &\quad \times \exp[2E_0 \sigma_E / (1 + 2\sigma_E)(1 + \sigma_E)]\}]^{1/2} \end{aligned} \quad (7.23)$$

and

$$\begin{aligned} \sigma_G^{(2)} &= [\ln\{(1 + 2\sigma_E)(1 + 4\sigma_E)^{-1/2} \\ &\quad \times \exp[8E_0 \sigma_E / (1 + 4\sigma_E)(1 + 2\sigma_E)]\}]^{1/2}. \end{aligned} \quad (7.24)$$

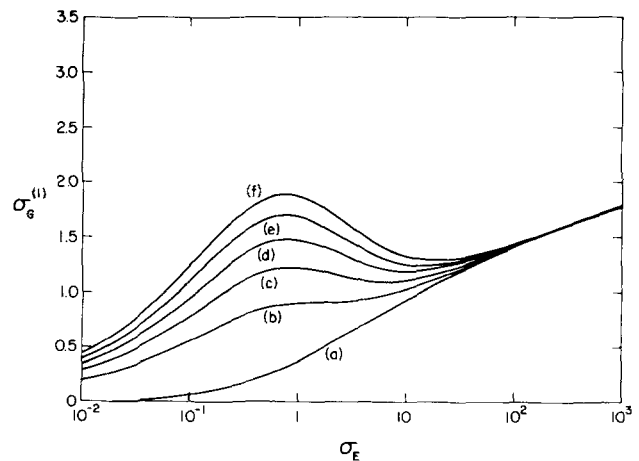


FIG. 9.  $\sigma_G^{(1)}$  as a function of  $\sigma_E$ , with  $E_0$  as a parameter. (a)  $E_0=0$ ; (b)  $E_0=2$ ; (c)  $E_0=4$ ; (d)  $E_0=6$ ; (e)  $E_0=8$ ; (f)  $E_0=10$ .

The quantity  $\sigma_G^{(1)}$  given in (7.23)—the square of which would be the variance of the log irradiance provided that the probability density function  $p(E)$  were assumed to be log-normal—is plotted in Fig. 9 against  $\sigma_E$  for various values of the dimensionless parameter  $E_0$ . It is clear from this graph that  $\sigma_G^{(1)}$  saturates for values of  $\sigma_E$  close to 0.8.

Saturation phenomena associated with the scintillation of waves propagating in turbulent media have been known experimentally. Various theoretical predictions (based for the most part on numerical solutions of the equation for the second moment of irradiance) have, also, been made (cf., e.g., Refs. 39 and 40). Theoretical results very similar to those reported in this paper have been published by Furutsu and Furuhashi (cf. Ref. 38) for the special case of a beam propagating in a deterministically flat (homogeneous) medium, with additively superimposed random fluctuations. The latter are characterized by a simplified (quadratic) Kolmogorov spectrum.

In contradistinction to  $\sigma_G^{(1)}$ , the variance of log irradiance  $M_2$  [cf. Eq. (7.17)] does not exhibit any saturation. This quantity is shown graphically in Fig. 10. An explanation for this behavior of  $M_2$  has been provided by Furutsu and Furuhashi (cf. Ref. 38; also, remarks made in the previous paragraph). The variance of log irradiance  $M_2$  does not saturate since it contains many higher-order moments of the irradiance  $E\{i^\nu(x, z; \alpha)\}$ , and the higher the order  $\nu$  is, the earlier the saturation starts. This situation is partially depicted in Fig. 11, where  $\sigma_G^{(2)}$  [cf. Eq. (7.24)] is plotted versus  $\sigma_E$  for various values of the parameter  $E_0$ . A direct comparison of this graph with the analogous one for  $\sigma_G^{(1)}$  (cf. Fig. 9) shows that the range of values of  $\sigma_E$  for which  $\sigma_G^{(2)}$  saturates is approximately centered around 0.4. Qualitatively, one would expect that this "shifting to the left" of the values of  $\sigma_E$  at saturation would continue as the superscript index in  $\sigma_G$  assumes larger and larger values, i.e.,  $\sigma_G^{(3)}, \sigma_G^{(4)}$ , etc.

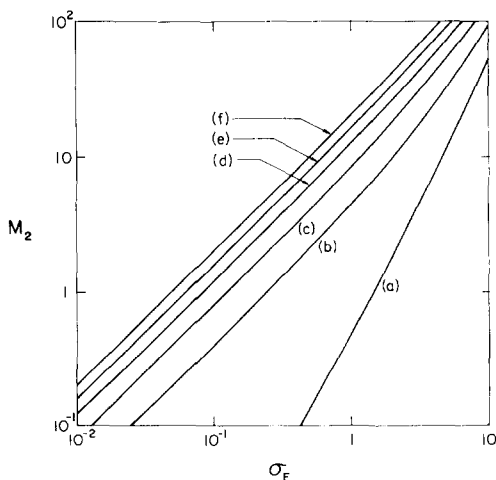


FIG. 10. Variance of the log irradiance. (a)  $E_0 = 0$ ; (b)  $E_0 = 2$ ; (c)  $E_0 = 4$ ; (d)  $E_0 = 6$ ; (e)  $E_0 = 8$ ; (f)  $E_0 = 10$ .

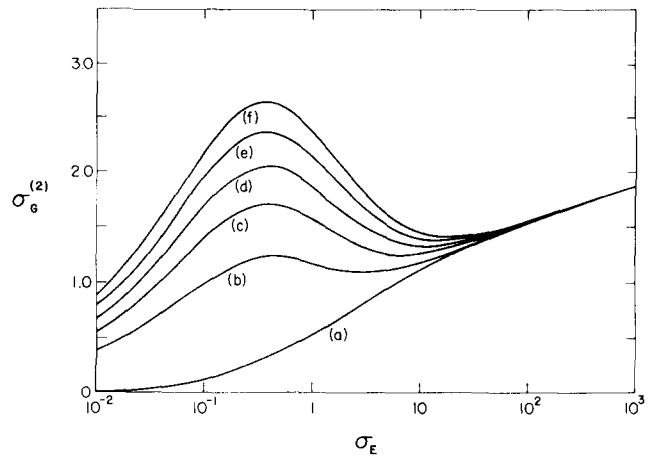


FIG. 11.  $\sigma_G^{(2)}$  as a function of  $\sigma_E$ , with  $E_0$  as a parameter. (a)  $E_0 = 0$ ; (b)  $E_0 = 2$ ; (c)  $E_0 = 4$ ; (d)  $E_0 = 6$ ; (e)  $E_0 = 8$ ; (f)  $E_0 = 10$ .

## 8. CONCLUDING REMARKS

The statistical technique expounded in this paper is based on the fundamental ansatz introduced in Sec. 3. This ansatz constitutes essentially an embedding process: Through the transformations (3.1) and (3.2), the stochastic parabolic equation (1.1) is brought into an one-to-one correspondence with the parabolic equation (3.4) characterizing the unperturbed focusing medium, together with a set of stochastic nonlinear ordinary differential equations [cf. Eqs. (3.5) and (3.6)] which account for the statistical fluctuations in the medium. The basic statistical analysis of the original stochastic partial differential equation is thus simplified significantly since it can now be performed at the level of a set of stochastic ordinary differential equations the mathematical theory of which is already highly developed (cf., e.g., Refs. 8–10).

Conceptually, we feel that the proposed statistical technique, although distinct, is closely related to recently formulated methods based on functional path integration (cf. Refs. 14–16). Similarly to the functional path integration methods, our technique has the distinct advantage that it works on a *global* rather than a *local* level making, thus, easier the algorithmic derivation of asymptotic solutions to higher-order statistical moments.

There seems to exist another important connection between our statistical approach and the path integral technique: The latter can be used to derive systematically *ansätze* (analogous to the one postulated in Sec. 3 of the paper) which can account for more realistic randomly perturbed channels. We are presently investigating along these lines the problem of beam propagation in a focusing medium whose unperturbed frequency is randomly modulated (cf., also, Ref. 11). Proceeding along the same vein, we also hope to make some progress with the more difficult problem of beam propagation in a medium whose parabolically graded deterministic profile is additively perturbed by  $x$ - and  $z$ -dependent random fluctuations.

## ACKNOWLEDGMENT

The author is indebted to his colleague, Professor Werner E. Kohler, for numerous informative discussions related to the subject of this article.

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