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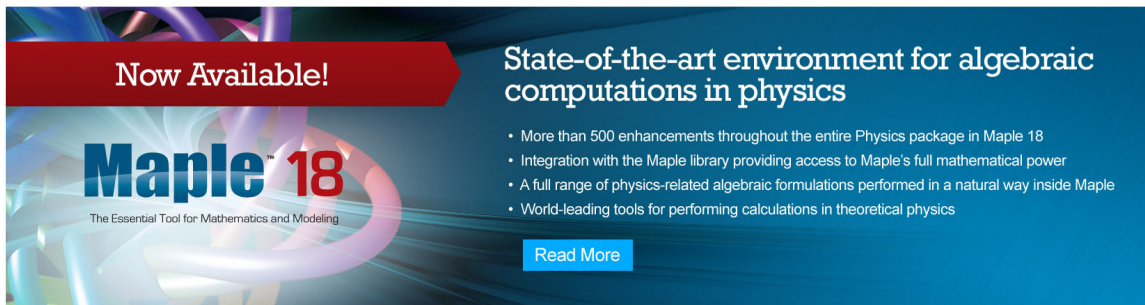
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Continuity of the S matrix for the perturbed Hill's equation

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The behavior of the scattering matrix associated with the perturbed Hill's equation as the spectral parameter approaches an endpoint of a spectral band is studied. In particular, the continuity of the scattering matrix at the band edges is proven and explicit expressions for the transmission and reflection coefficients at those points are derived. All possible cases are discussed and our fall-off assumptions on the perturbation are weaker than those made by other authors.

I. INTRODUCTION

On $L_2(\mathbf{R})$ we consider the Schrödinger operators

$$H_0 = -\frac{d^2}{dx^2} + P(x)$$

and

$$H = -\frac{d^2}{dx^2} + P(x) + V(x),$$

where $P(x)$ and $V(x)$ are real-valued potentials such that $P(x) \in L^1([0, 1])$, $P(x+1) = P(x)$, and

$$\int_{-\infty}^{\infty} (1 + |x|) |V(x)| dx < \infty. \quad (1.1)$$

It is well known¹⁻⁴ that the spectrum $\sigma(H_0)$ of H_0 is absolutely continuous and is the union of closed intervals

$$\sigma(H_0) = \bigcup_{n=0} [E_{2n}, E_{2n+1}],$$

where $-\infty < E_0 < E_1 \leq E_2 < E_3 \cdots$. Each point E_n is an eigenvalue of a boundary value problem of the form $H_0\psi = E\psi$ on $[0, 1]$ with either periodic or antiperiodic boundary conditions. The eigenvalues associated with the periodic boundary conditions coincide with the points E_n for $n = 4k$ and $n = 4k + 3$, the eigenvalues associated with the antiperiodic boundary conditions coincide with the points E_n for $n = 4k + 1$ and $n = 4k + 2$ ($k = 0, 1, 2, \dots$). The intervals $[E_{2n}, E_{2n+1}]$ ($n = 0, 1, \dots$) are often referred to as "bands" and the intervals (E_{2n+1}, E_{2n+2}) as "gaps." If $E_{2n+1} = E_{2n+2}$, we say that the corresponding gap is closed. The continuous part of the spectrum of H is also absolutely continuous and it agrees with the spectrum of H_0 . In addition, H may have at most a finite number of eigenvalues in any gap. We refer the reader to Refs. 5, 6, and 7 for more information about the eigenvalues.

In this article, the points E_n which are endpoints of an open gap, will be classified as "generic" or "exceptional" according to the following definition. The generic case is said to occur

at $E = E_n$ if the equation $H\psi = E_n\psi$ has no bounded nontrivial solution. The exceptional case is said to occur at $E = E_n$ if $H\psi = E_n\psi$ has a bounded nontrivial solution. In the exceptional case, the point E_n is also referred to as a half bound state. The equations $H_0\psi = E\psi$ and $H\psi = E\psi$ are called Hill's equation and perturbed Hill's equation, respectively.

Associated with the operators H_0 and H is the scattering matrix

$$S(E) = \begin{bmatrix} T(E) & R(E) \\ L(E) & T(E) \end{bmatrix},$$

where $T(E)$ is the transmission coefficient and $L(E)$ and $R(E)$ are the reflection coefficients from the left and from the right, respectively. The object of the present article is to prove that $S(E)$ is continuous at the points E_n , both in the generic case and the exceptional case, and to obtain the correct leading asymptotic behavior as $E \rightarrow E_n$.

In the generic case, the continuity of the scattering matrix at E_n is known.^{8,9} It is also known that $T(E_n) = 0$. For the reflection coefficients the results have been incomplete. As Theorem 3.3 below shows, there are two possibilities: $R(E_n) = L(E_n) = -1$ or $R(E_n) = L(E_n) = 1$. To the best of our knowledge, only the first possibility has been noted in the literature.⁹ In the exceptional case, the continuity of $S(E)$ has only been established⁸ under the stronger condition that $\int_{-\infty}^{\infty} (1+x^2)|V(x)|dx < \infty$. However, no explicit expression for $S(E_n)$ is given in Ref. 8. In Ref. 9 the perturbed Hill's equation is studied under the assumptions that $|x||V(x)| \in L^1(\mathbf{R})$ and $V(x) \in L^2(\mathbf{R})$ [which together imply (1.1)]. However, in the exceptional case, the expressions for $T(E_n)$, $R(E_n)$, and $L(E_n)$ given in Ref. 9 do not agree with those of Theorem 3.3. We should also mention that the situation regarding the continuity of $S(E)$ at E_n for the perturbed Hill's equation is similar to that in the case when $P(x) = 0$, where a half bound state can occur at $E = 0$. This latter case was studied in Ref. 10. The method used in the present article is a generalization of the method of Ref. 10 to the perturbed Hill's equation. We also rely on some results from Ref. 11, where the Titchmarsh–Weyl m coefficient and its connection with half bound states is studied for the perturbed Hill's equation on the half line.

The article is organized as follows. In Sec. II we establish the notation and prove two technical lemmas. In Sec. III we prove the continuity of the scattering matrix.

II. PRELIMINARIES

Let $\phi_0(x, E)$ and $\theta_0(x, E)$ denote the solutions of the equation

$$H_0\psi = E\psi$$

that satisfy the conditions

$$\phi_0(0, E) = \theta_0'(0, E) = 0, \quad \text{and} \quad \phi_0'(0, E) = \theta_0(0, E) = 1.$$

Let $\phi_0(E) = \phi_0(1, E)$, $\theta_0(E) = \theta_0(1, E)$ and define the discriminant by

$$\Delta(E) = \frac{1}{2}[\phi_0'(E) + \theta_0(E)]. \quad (2.1)$$

It is well known¹⁻⁴ that $\sigma(H_0) = \{E: |\Delta(E)| \leq 1\}$. The so-called quasimomentum k is defined by

$$k = k(E) = \cos^{-1}[\Delta(E)], \quad (2.2)$$

where the branch of \cos^{-1} is such that $\text{Im } k(E) > 0$ when $E < E_0$ and $k(E_0) = 0$. The global mapping and analyticity properties of the function $k(E)$ have been studied in Ref. 12. For the purpose of this article, we only need certain local properties that pertain to a small neighborhood of a given point E_n . We summarize here those facts that are relevant to the present article. Similar results

were obtained in Ref. 11. The band $[E_{2n}, E_{2n+1}]$ is mapped onto the interval $[n\pi, (n+1)\pi]$ and the interval $(-\infty, E_0)$ goes over into the positive imaginary k axis. In each gap (E_{2n+1}, E_{2n+2}) there is a unique point \tilde{E}_{n+1} such that $\Delta'(\tilde{E}_{n+1})=0$. Then $k(\tilde{E}_{n+1})=(n+1)\pi+i\delta_{n+1}$ ($\delta_{n+1}>0$) and the two intervals $(E_{2n+1}, \tilde{E}_{n+1})$ and $[\tilde{E}_{n+1}, E_{2n+2})$ are both mapped onto the segment $\{k:k=(n+1)\pi+it, 0<t\leq\delta_{n+1}\}$. We will also need the analytic continuation of $k(E)$ into $\text{Im } E < 0$, where, for the present article, it suffices to assume that E is near one of the points E_n . To this end, we introduce the slit disks $D_{2n}=\{E:|E-E_{2n}|<\rho_{2n}\}\setminus[E_{2n}, E_{2n}+\rho_{2n})$ and $D_{2n+1}=\{E:|E-E_{2n+1}|<\rho_{2n+1}\}\setminus(E_{2n+1}-\rho_{2n+1}, E_{2n+1}]$. The positive numbers ρ_{2n} and ρ_{2n+1} are sufficiently small so as to guarantee the following results. Under the map $E \rightarrow k(E)$ the slit disk D_{2n} is mapped one to one onto an open set in the k plane whose boundary consists of the segment $(n\pi-\epsilon_{2n}, n\pi+\epsilon_{2n})$ ($\epsilon_{2n}=\tan^{-1}[\sqrt{1-\Delta^2(E_{2n}+\rho_{2n})}/\Delta(E_{2n}+\rho_{2n})]$) and a curve, the image of the circle $|E-E_{2n}|=\rho_{2n}$, in the upper half of the k plane joining the endpoints of the segment. Similarly, each slit disk D_{2n+1} is mapped onto a domain bounded by a segment $((n+1)\pi-\epsilon_{2n+1}, (n+1)\pi+\epsilon_{2n+1})$ and a curve in the upper half plane joining the endpoints of the segment. The upper edge of the interval $(E_{2n}, E_{2n}+\rho_{2n})$ is mapped onto $(n\pi, n\pi+\epsilon_{2n})$, the lower edge is mapped onto $(n\pi-\epsilon_{2n}, n\pi)$. Similarly, the upper edge of $(E_{2n+1}-\rho_{2n+1}, E_{2n+1})$ is mapped onto $((n+1)\pi-\epsilon_{2n+1}, (n+1)\pi)$ and the lower edge onto $((n+1)\pi, (n+1)\pi+\epsilon_{2n+1})$.

In the sequel the important spectral parameter will be k . So we will henceforth write k in place of E , i.e., $\phi_0(x, k)$ in place of $\phi_0(x, E)$, etc. Let $m_0^{(\pm)}(k)$ denote the Titchmarsh–Weyl m functions associated with the operator H_0^{11} . Then, for $\text{Im } k > 0$, we have that

$$\psi_0^{(+)}(x, k) = \theta_0(x, k) + m_0^{(+)}(k)\phi_0(x, k) \in L_2(0, \infty), \tag{2.3}$$

$$\psi_0^{(-)}(x, k) = \theta_0(x, k) + m_0^{(-)}(k)\phi_0(x, k) \in L_2(-\infty, 0). \tag{2.4}$$

There exist functions $\xi_0^{(\pm)}(x, k)$ with $\xi_0^{(\pm)}(x+1, k) = \xi_0^{(\pm)}(x, k)$, $\xi_0^{(\pm)}(0, k) = 1$, such that

$$\psi_0^{(\pm)}(x, k) = \xi_0^{(\pm)}(x, k)e^{\pm ikx}. \tag{2.5}$$

Hence $\psi_0^{(\pm)}(1, k) = e^{\pm ik}$ and from Eqs. (2.3) and (2.4), we have that

$$m_0^{(\pm)}(k) = \frac{e^{\pm ik} - \theta_0(k)}{\phi_0(k)}. \tag{2.6}$$

Further, by using Eqs. (2.1) and (2.2) in Eq. (2.6), we obtain

$$m_0^{(\pm)}(k) = \frac{\phi_0'(k) - \theta_0(k)}{2\phi_0(k)} \pm \frac{i \sin k}{\phi_0(k)}. \tag{2.7}$$

Evaluating the Wronskian of $\psi_0^{(+)}(x, k)$ and $\psi_0^{(-)}(x, k)$ yields

$$[\psi_0^{(+)}(\cdot, k); \psi_0^{(-)}(\cdot, k)] = m_0^{(-)}(k) - m_0^{(+)}(k),$$

where $[F(\cdot); G(\cdot)] = F(x)G'(x) - F'(x)G(x)$. Thus, by Eq. (2.6)

$$[\psi_0^{(+)}(\cdot, k); \psi_0^{(-)}(\cdot, k)] = -\frac{2i \sin k}{\phi_0(k)}. \tag{2.8}$$

Now suppose that E lies strictly inside a band $[E_{2n}, E_{2n+1}]$, i.e., $k \in (n\pi, (n+1)\pi)$. Then there exist solutions $\Psi^{(\pm)}(x, k)$ of

$$H\psi = E\psi \tag{2.9}$$

such that

$$\Psi^{(+)}(x, k) = \begin{cases} T(k)\psi_0^{(+)}(x, k) + o(1), & x \rightarrow \infty, \\ \psi_0^{(+)}(x, k) + L(k)\psi_0^{(-)}(x, k) + o(1), & x \rightarrow -\infty \end{cases} \quad (2.10)$$

and

$$\Psi^{(-)}(x, k) = \begin{cases} \psi_0^{(-)}(x, k) + R(k)\psi_0^{(+)}(x, k) + o(1), & x \rightarrow \infty, \\ T(k)\psi_0^{(-)}(x, k) + o(1), & x \rightarrow -\infty. \end{cases} \quad (2.11)$$

These asymptotic relations define $T(k)$, $L(k)$, and $R(k)$. In addition to $\Psi^{(\pm)}(x, k)$, we introduce solutions $F^{(\pm)}(x, k)$ of Eq. (2.9) that satisfy the integral equations

$$F^{(+)}(x, k) = \psi_0^{(+)}(x, k) - \int_x^\infty A(x, t; k) V(t) F^{(+)}(t, k) dt, \quad (2.12)$$

$$F^{(-)}(x, k) = \psi_0^{(-)}(x, k) + \int_{-\infty}^x A(x, t; k) V(t) F^{(-)}(t, k) dt, \quad (2.13)$$

where

$$A(x, t; k) = -\frac{1}{[\psi_0^{(+)}(\cdot, k); \psi_0^{(-)}(\cdot, k)]} [\psi_0^{(+)}(x, k)\psi_0^{(-)}(t, k) - \psi_0^{(-)}(x, k)\psi_0^{(+)}(t, k)]. \quad (2.14)$$

From Eqs. (2.12), (2.13), and standard estimates [see also Sec. III, Eqs. (3.30) and (3.31)], it follows that $F^{(+)}(x, k) = \psi_0^{(+)}(x, k) + o(1)$ as $x \rightarrow +\infty$ and $F^{(-)}(x, k) = \psi_0^{(-)}(x, k) + o(1)$ as $x \rightarrow -\infty$. Moreover, these relations may be differentiated. In analogy to the case $P(x) = 0$, we call $F^{(\pm)}(x, k)$ the Jost solutions of Eq. (2.9). They are related to $\Psi^{(\pm)}(x, k)$ by

$$\Psi^{(\pm)}(x, k) = T(k)F^{(\pm)}(x, k).$$

By combining Eqs. (2.10), (2.11) with Eqs. (2.12), (2.13) and using Eq. (2.8), we obtain

$$T(k) = -\frac{2i \sin k}{[F^{(+)}(\cdot, k); F^{(-)}(\cdot, k)]\phi_0(k)} = \frac{1}{1 + I^{(\pm)}(k)}, \quad (2.15)$$

$$L(k) = -\frac{[F^{(+)}(\cdot, k); \overline{F^{(-)}(\cdot, k)}]}{[F^{(+)}(\cdot, k); F^{(-)}(\cdot, k)]} = -T(k)J^{(+)}(k), \quad (2.16)$$

$$R(k) = -\frac{[\overline{F^{(+)}(\cdot, k)}; F^{(-)}(\cdot, k)]}{[F^{(+)}(\cdot, k); F^{(-)}(\cdot, k)]} = -T(k)J^{(-)}(k), \quad (2.17)$$

where

$$I^{(\pm)}(k) = -\frac{\phi_0(k)}{2i \sin k} \int_{-\infty}^{\infty} \psi_0^{(\mp)}(t, k) V(t) F^{(\pm)}(t, k) dt, \quad (2.18)$$

$$J^{(\pm)}(k) = -\frac{\phi_0(k)}{2i \sin k} \int_{-\infty}^{\infty} \psi_0^{(\pm)}(t, k) V(t) F^{(\pm)}(t, k) dt. \quad (2.19)$$

By evaluating the Wronskians $[\Psi^{(+)}(\cdot, k); \overline{\Psi^{(+)}(\cdot, k)}]$, $[\Psi^{(-)}(\cdot, k); \overline{\Psi^{(-)}(\cdot, k)}]$, and $[\Psi^{(+)}(\cdot, k); \overline{\Psi^{(-)}(\cdot, k)}]$ as $x \rightarrow \pm\infty$, we obtain, for real k , the unitarity relations

$$\begin{aligned} |T(k)|^2 + |R(k)|^2 &= |T(k)|^2 + |L(k)|^2 = 1, \\ T(k)\overline{R(k)} + L(k)\overline{T(k)} &= 0. \end{aligned} \tag{2.20}$$

Our next goal is to prove two lemmas that will be used in Sec. III. First we need some more notation. Let

$$k_n = n\pi, \quad n = 0, 1, 2, \dots$$

Throughout this section we assume that

$$\phi_0(k_n) \neq 0. \tag{2.21}$$

The case when $\phi_0(k_n) = 0$ will be considered separately in Sec. III. Furthermore, we define

$$g_0(x, k) = \psi_0^{(+)}(x, k) + \psi_0^{(-)}(x, k). \tag{2.22}$$

Lemma 2.1: Let $z(x, k_n)$ be a solution of $H\psi = E_n\psi$ ($n = 0, 1, 2, \dots$) and let $a = z(0, k_n)$, $b = z'(0, k_n)$. Then

(a) $z(x, k_n)$ is bounded for $x \geq 0$ if and only if

$$b - am_0^{(+)}(k_n) + \frac{1}{2} \int_0^\infty g_0(t, k_n)V(t)z(t, k_n)dt = 0; \tag{2.23}$$

(b) $z(x, k_n)$ is bounded for $x \leq 0$ if and only if

$$b - am_0^{(+)}(k_n) - \frac{1}{2} \int_{-\infty}^0 g_0(t, k_n)V(t)z(t, k_n)dt = 0.$$

Proof: We only prove (a); the proof of (b) is similar. First, we note that $g_0(x, k_n)$ and $\phi_0(x, k_n)$ are linearly independent solutions of $H_0\psi = E_n\psi$ and that $g_0(0, k_n) = 2$, $g_0'(0, k_n) = m_0^{(+)}(k_n) + m_0^{(-)}(k_n) = 2m_0^{(+)}(k_n)$, and $[\phi_0(\cdot, k_n); g_0(\cdot, k_n)] = -2$. By the variation of constants formula we have

$$z(x, k_n) = [b - am_0^{(+)}(k_n)]\phi_0(x, k_n) + \frac{a}{2}g_0(x, k_n) + \frac{1}{2} \int_0^x K(x, t; k_n)V(t)z(t, k_n)dt, \tag{2.24}$$

where $K(x, t; k_n) = \phi_0(x, k_n)g_0(t, k_n) - g_0(x, k_n)\phi_0(t, k_n)$. It follows from standard asymptotic results that

$$|\phi_0(x, k_n)| \leq C(1+x), \tag{2.25}$$

$$|z(x, k_n)| \leq C(1+x) \tag{2.26}$$

and hence

$$|K(x, t; k_n)| \leq C(1+x), \quad 0 \leq t \leq x \tag{2.27}$$

for some suitable constant C . By using Eqs. (2.25)–(2.27) and taking $x \rightarrow +\infty$ in Eq. (2.24), it follows that

$$z(x, k_n) = \left[b - am_0^{(+)}(k_n) + \frac{1}{2} \int_0^\infty g_0(t, k_n) V(t) z(t, k_n) dt \right] \phi_0(x, k_n) + o(x). \quad (2.28)$$

Thus, in order for $z(x, k_n)$ to be bounded, it is necessary that the bracketed term in Eq. (2.28) vanishes, i.e., Eq. (2.23) holds. Conversely, if this term vanishes, then $z(x, k_n) = o(x)$ and so $z(x, k_n)$ must be a multiple of $g_0(x, k_n)$; that is, $z(x, k_n)$ must be bounded. Lemma 2.1 is proved. ■

When k is near k_n (with n fixed) it will often be convenient to use the variable

$$\alpha = k - k_n.$$

The properties of the mapping $E \rightarrow k(E)$ [see Eq. (2.2)] imply that $E(k) = E(k_n + \alpha)$ and hence $\phi_0(x, k_n + \alpha)$ and $\theta_0(x, k_n + \alpha)$ are even functions of α . Moreover, when α is real, the following relations hold:

$$\begin{aligned} m_0^{(+)}(k_n + \alpha) &= m_0^{(-)}(k_n - \alpha) = \overline{m_0^{(+)}(k_n - \alpha)} = \overline{m_0^{(-)}(k_n + \alpha)}, \\ \psi_0^{(+)}(x, k_n + \alpha) &= \psi_0^{(-)}(x, k_n - \alpha) = \overline{\psi_0^{(+)}(x, k_n - \alpha)} = \overline{\psi_0^{(-)}(x, k_n + \alpha)}. \end{aligned} \quad (2.29)$$

Lemma 2.2: Let $z(x, k)$ be a solution of $H\psi = E\psi$ with $z(0, k) = a$ and $z'(0, k) = b$, where a and b are independent of k , and suppose that $z(x, k_n)$ is bounded for $x \geq 0$. Then, for k near k_n , k real, we have the estimate

$$|z(x, k) - z(x, k_n)| \leq C(1+x) \left[\alpha^2 + \frac{\alpha^2 x}{1 + |\alpha|x} \right]. \quad (2.30)$$

A similar estimate, with x replaced by $|x|$, holds if $z(x, k_n)$ is assumed to be bounded for $x \leq 0$. *Proof:* It suffices to consider $x \geq 0$. The proof when $x \leq 0$ is similar. From Ref. 11 we have the estimates ($x \geq 0$)

$$|\phi_0(x, k) - \phi_0(x, k_n)| \leq C(1+x) \left[\alpha^2 + \left(\frac{\alpha x}{1 + |\alpha|x} \right)^2 \right], \quad (2.31)$$

$$|g_0(x, k) - g_0(x, k_n)| \leq C \left[(1+x)\alpha^2 + \left(\frac{\alpha x}{1 + |\alpha|x} \right)^2 \right]. \quad (2.32)$$

These estimates hold in some interval $k_n - \epsilon \leq k \leq k_n + \epsilon$ ($\epsilon > 0$). Since $g_0(0, k) = 2$ and $g_0'(0, k) = m_0^{(+)}(k) + m_0^{(-)}(k)$, the variation of constants formula yields

$$z(x, k) = \left[b - \frac{a}{2} (m_0^{(+)}(k) + m_0^{(-)}(k)) \right] \phi_0(x, k) + \frac{a}{2} g_0(x, k) + \frac{1}{2} \int_0^x K(x, t; k) V(t) z(t, k) dt,$$

where $K(x, t; k) = \phi_0(x, k) g_0(t, k) - g_0(x, k) \phi_0(t, k)$. By Eqs. (2.7) and (2.29), we have that $m_0^{(+)}(k_n + \alpha) + m_0^{(-)}(k_n + \alpha) = 2m_0^{(+)}(k_n) + O(\alpha^2)$, and hence

$$\begin{aligned} z(k, x) - z(x, k_n) &= [b - am_0^{(+)}(k_n)] [\phi_0(x, k) - \phi_0(x, k_n)] + O(\alpha^2) \phi_0(x, k) + \frac{a}{2} [g_0(x, k) \\ &\quad - g_0(x, k_n)] + \frac{1}{2} \int_0^x [K(x, t; k) - K(x, t; k_n)] V(t) z(t, k_n) dt \end{aligned}$$

$$+\frac{1}{2}\int_0^x K(x,t;k)V(t)[z(t,k)-z(t,k_n)]dt=I_1+I_2+I_3+I_4+I_5. \tag{2.33}$$

We estimate each of the terms I_1, \dots, I_5 . Since there are terms in I_1 and I_4 that need to be combined, we begin with I_4 . We write

$$\begin{aligned} I_4 &= \frac{1}{2} [\phi_0(x,k) - \phi_0(x,k_n)] \int_0^x g_0(t,k_n)V(t)z(t,k_n)dt + \frac{1}{2} \phi_0(x,k) \int_0^x [g_0(t,k) \\ &\quad - g_0(t,k_n)]V(t)z(t,k_n)dt - \frac{1}{2} [g_0(x,k) - g_0(x,k_n)] \int_0^x \phi_0(t,k)V(t)z(t,k_n)dt \\ &\quad - \frac{1}{2} g_0(x,k_n) \int_0^x [\phi_0(t,k) - \phi_0(t,k_n)]V(t)z(t,k_n)dt \\ &= J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Combining I_1 and J_1 with the help of Eq. (2.23), we obtain

$$I_1 + J_1 = -\frac{1}{2} [\phi_0(x,k) - \phi_0(x,k_n)] \int_x^\infty g_0(t,k_n)V(t)z(t,k_n)dt.$$

Therefore

$$|I_1 + J_1| \leq C \left[\alpha^2 + \left(\frac{\alpha x}{1 + \alpha x} \right)^2 \right] \int_x^\infty |V(t)|(1+t)dt, \tag{2.34}$$

where we have used the boundedness of $z(t,k_n)$ and $g_0(t,k_n)$. Turning to the term J_2 , we use Eqs. (2.25), (2.32), and (1.1), to obtain

$$|J_2| \leq C(1+x) \left[\alpha^2 + \frac{\alpha^2 x}{1 + |\alpha|x} \right]. \tag{2.35}$$

Similarly, by using Eqs. (2.31) and (2.32) we get

$$|J_3| \leq C \left[(1+x)\alpha^2 + \left(\frac{\alpha x}{1 + |\alpha|x} \right)^2 \right], \tag{2.36}$$

$$|J_4| \leq C \left[\alpha^2 + \left(\frac{\alpha x}{1 + |\alpha|x} \right)^2 \right]. \tag{2.37}$$

Returning to Eq. (2.33) and estimating the remaining terms, we obtain

$$|I_2| \leq C\alpha^2(1+x), \tag{2.38}$$

$$|I_3| \leq C \left[(1+x)\alpha^2 + \left(\frac{\alpha x}{1 + |\alpha|x} \right)^2 \right], \tag{2.39}$$

$$|I_5| \leq C(1+x) \int_0^x |V(t)||z(t,k) - z(t,k_n)|dt. \tag{2.40}$$

Since

$$\left(\frac{\alpha x}{1+|\alpha|x}\right)^2 \leq (1+x) \frac{\alpha^2 x}{1+|\alpha|x}$$

the estimates (2.34)–(2.40) can be combined with the result that

$$|z(x,k) - z(x,k_n)| \leq C(1+x) \left(\alpha^2 + \frac{\alpha^2 x}{1+|\alpha|x} \right) + C(1+x) \int_0^x |V(t)| |z(t,k) - z(t,k_n)| dt. \quad (2.41)$$

Let

$$h(x,\alpha) = C(1+x) \left(\alpha^2 + \frac{\alpha^2 x}{1+|\alpha|x} \right)$$

and

$$u(x,k) = \frac{|z(x,k) - z(x,k_n)|}{h(x,k)}.$$

Since $z \rightarrow z/(z+1)$ is monotonically increasing, we have that $h(t,\alpha) \leq h(x,\alpha)$ when $t \leq x$. Thus Eq. (2.41) becomes

$$u(x,k) \leq 1 + C \int_0^x (1+t) |V(t)| u(t,k) dt.$$

Applying Gronwall's inequality gives $u(x,k) \leq C$ and Eq. (2.30) follows. Lemma 2.2 is proved. ■

III. ASYMPTOTICS AND CONTINUITY OF THE SCATTERING MATRIX

We can assume that, in addition to Eq. (2.21)

$$F^{(+)}(0,k_n) \neq 0. \quad (3.1)$$

This can always be accomplished by a shift of the origin if necessary. Let $\phi(x,k)$ and $\theta(x,k)$ be solutions of Eq. (2.9) satisfying $\phi(0,k) = \theta'(0,k) = 0$ and $\phi'(0,k) = \theta(0,k) = 1$. Let

$$z(x,k) = F^{(+)'}(0,k_n) \phi(x,k) + F^{(+)}(0,k_n) \theta(x,k) \quad (3.2)$$

so that $z(x,k)$ is the solution of Eq. (2.9) which satisfies $z(0,k) = F^{(+)}(0,k_n)$ and $z'(0,k) = F^{(+)'}(0,k_n)$.

Lemma 3.1: When k is real, the following relations hold:

- (a) $F^{(+)}(0,k_n)[F^{(+)}(\cdot,k); F^{(-)}(\cdot,k)] = F^{(-)}(0,k)[F^{(+)}(\cdot,k); z(\cdot,k)]$
 $\quad - F^{(+)}(0,k)[F^{(-)}(\cdot,k); z(\cdot,k)],$
- (b) $F^{(+)}(0,k_n)[F^{(+)}(x,k); \overline{F^{(-)}(x,k)}] = \overline{F^{(-)}(0,k)}[F^{(+)}(x,k); z(x,k)]$
 $\quad - F^{(+)}(0,k)[\overline{F^{(-)}(x,k)}; z(x,k)].$

Proof: By straightforward calculation. ■

Note that $[F^{(+)}(\cdot,k_n); F^{(-)}(\cdot,k_n)] = 0$ implies that we are in the exceptional case. Then $F^{(+)}(x,k_n)$ and $F^{(-)}(x,k_n)$ are linearly dependent, i.e.,

$$F^{(+)}(x, k_n) = a_n F^{(-)}(x, k_n), \quad (3.3)$$

with $a_n \neq 0$. Hence Eq. (2.9) has a bounded solution for $E = E_n$.

Lemma 3.2: Suppose that a half bound state occurs at $k = k_n$. Then, as $k \rightarrow k_n$, k real, we have

$$(a) \quad [F^{(+)}(\cdot, k); z(\cdot, k)] = (-1)^{n+1} i [\phi_0(k_n)]^{-1} \alpha + o(\alpha),$$

$$(b) \quad [F^{(-)}(\cdot, k); z(\cdot, k)] = (-1)^n a_n i [\phi_0(k_n)]^{-1} \alpha + o(\alpha).$$

Proof: The variation of constants formula gives

$$\phi(x, k) = \phi_0(x, k) + \int_0^x A(x, t; k) V(t) \phi(t, k) dt,$$

where $A(x, t; k)$ is given by Eq. (2.14). Then, using

$$\phi_0(x, k) = - \frac{(\psi_0^{(+)}(x, k) - \psi_0^{(-)}(x, k))}{[\psi_0^{(+)}(\cdot, k); \psi_0^{(-)}(\cdot, k)]}, \quad (3.4)$$

we obtain

$$[F^{(+)}(\cdot, k); \phi(\cdot, k)] = F^{(+)}(0, k) = 1 + \int_0^\infty \psi_0^{(+)}(t, k) V(t) \phi(t, k) dt, \quad (3.5)$$

$$[F^{(-)}(\cdot, k); \phi(\cdot, k)] = F^{(-)}(0, k) = 1 - \int_{-\infty}^0 \psi_0^{(-)}(t, k) V(t) \phi(t, k) dt. \quad (3.6)$$

Similarly, we get

$$[F^{(+)}(\cdot, k); \theta(\cdot, k)] = -F^{(+)\prime}(0, k) = -m_0^{(+)}(k) + \int_0^\infty \psi_0^{(+)}(t, k) V(t) \theta(t, k) dt, \quad (3.7)$$

$$[F^{(-)}(\cdot, k); \theta(\cdot, k)] = -F^{(-)\prime}(0, k) = -m_0^{(-)}(k) - \int_{-\infty}^0 \psi_0^{(-)}(t, k) V(t) \theta(t, k) dt. \quad (3.8)$$

Combining Eq. (3.2) with Eqs. (3.5)–(3.8), we obtain

$$\begin{aligned} [F^{(+)}(\cdot, k); z(\cdot, k)] &= F^{(+)}(0, k) F^{(+)\prime}(0, k_n) - F^{(+)\prime}(0, k) F^{(+)}(0, k_n) \\ &= -m_0^{(+)}(k) F^{(+)}(0, k_n) + F^{(+)\prime}(0, k_n) + \int_0^\infty \psi_0^{(+)}(t, k) V(t) z(t, k) dt \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} [F^{(-)}(\cdot, k); z(\cdot, k)] &= F^{(-)}(0, k) F^{(+)\prime}(0, k_n) - F^{(-)\prime}(0, k) F^{(+)}(0, k_n) \\ &= -m_0^{(-)}(k) F^{(+)}(0, k_n) + F^{(+)\prime}(0, k_n) - \int_{-\infty}^0 \psi_0^{(-)}(t, k) V(t) z(t, k) dt. \end{aligned} \quad (3.10)$$

Consider first the integral on the right-hand side of Eq. (3.9). It can be written as

$$\begin{aligned}
Z &\equiv \int_0^\infty \psi_0^{(+)}(t, k) V(t) z(t, k) dt = \int_0^\infty \psi_0^{(+)}(t, k_n) V(t) z(t, k_n) dt \\
&\quad + \int_0^\infty [\psi_0^{(+)}(t, k) - \psi_0^{(+)}(t, k_n)] V(t) z(t, k_n) dt \\
&\quad + \int_0^\infty \psi_0^{(+)}(t, k) V(t) [z(t, k) - z(t, k_n)] dt \\
&= Z_1 + Z_2 + Z_3.
\end{aligned} \tag{3.11}$$

Since $z(x, k_n) = F^{(+)}(x, k_n)$, Eq. (3.9) gives

$$Z_1 = F^{(+)}(0, k_n) m_0^{(+)}(k_n) - F^{(+)}(0, k_n). \tag{3.12}$$

In order to deal with the term Z_2 , we note that the boundedness of $z(x, k_n)$ together with Eqs. (2.23) and (2.24) imply that, as $x \rightarrow +\infty$

$$z(x, k_n) = \frac{g_0(x, k_n)}{2} \left[F^{(+)}(0, k_n) - \int_0^\infty \phi_0(t, k_n) V(t) z(t, k_n) dt \right] + o(1).$$

On the other hand, by Eqs. (2.12) and (2.22), $z(x, k_n) = F^{(+)}(x, k_n) = \psi_0^{(+)}(x, k_n) + o(1) = g_0(x, k_n)/2 + o(1)$ as $x \rightarrow +\infty$. Thus

$$F^{(+)}(0, k_n) - \int_0^\infty \phi_0(t, k_n) V(t) z(t, k_n) dt = 1. \tag{3.13}$$

Expanding $\xi_0^{(+)}(x, k)$ near $k = k_n$ gives $\xi_0^{(+)}(x, k) = \xi_0^{(+)}(x, k_n) + \dot{\xi}_0^{(+)}(x, k_n) \alpha + O(\alpha^2)$, where the remainder term is uniform in x since $\xi_0^{(+)}(x, k_n)$ is periodic. Then, for small α

$$\begin{aligned}
\psi_0^{(+)}(x, k) - \psi_0^{(+)}(x, k_n) &= \alpha e^{ik_n x} [ix \xi_0^{(+)}(x, k_n) + \dot{\xi}_0^{(+)}(x, k_n)] + (e^{i\alpha x} - 1 - i\alpha x) e^{ik_n x} \xi_0^{(+)}(x, k_n) \\
&\quad + \alpha (e^{i\alpha x} - 1) e^{ik_n x} \dot{\xi}_0^{(+)}(x, k_n) + O(\alpha^2).
\end{aligned} \tag{3.14}$$

Using the estimates $|e^{i\alpha x} - 1| \leq C|\alpha|x/(1+|\alpha|x)$ and $|e^{i\alpha x} - 1 - i\alpha x| \leq C\alpha^2 x^2/(1+|\alpha|x)$, we can write Eq. (3.14) as

$$\psi_0^{(+)}(x, k) - \psi_0^{(+)}(x, k_n) = \frac{i(-1)^n \phi_0(x, k_n)}{\phi_0(k_n)} \alpha + O\left(\frac{\alpha^2 x^2}{1+|\alpha|x}\right) + O\left(\frac{\alpha^2 x}{1+|\alpha|x}\right) + O(\alpha^2), \tag{3.15}$$

where we have also used the relations

$$\dot{\xi}_0^{(-)}(x, k_n) = e^{2ik_n x} \dot{\xi}_0^{(+)}(x, k_n)$$

and

$$\phi_0(x, k_n) = (-1)^{n+1} i \phi_0(k_n) e^{ik_n x} [ix \xi_0^{(+)}(x, k_n) + \dot{\xi}_0^{(+)}(x, k_n)],$$

which follow from Eqs. (2.3)–(2.5) and (3.4) on letting $k \rightarrow k_n$. On inserting Eq. (3.15) in Z_2 and appealing to the Lebesgue dominated convergence theorem, it follows that

$$Z_2 = \frac{i(-1)^n \alpha}{\phi_0(k_n)} \int_0^\infty \phi_0(t, k_n) V(t) z(t, k_n) dt + o(\alpha)$$

and hence, by Eq. (3.13)

$$Z_2 = \frac{i(-1)^n \alpha}{\phi_0(k_n)} [F^{(+)}(0, k_n) - 1] + o(\alpha). \quad (3.16)$$

In order to estimate Z_3 we use Eq. (2.30) and the Lebesgue dominated convergence theorem. Then

$$|Z_3| \leq C \int_0^\infty |V(t)| (1+t) \left(\alpha^2 + \frac{\alpha^2 t}{1+|\alpha|t} \right) dt = o(\alpha). \quad (3.17)$$

Therefore, putting together Eqs. (3.12), (3.16), and (3.17) yields

$$Z = F^{(+)}(0, k_n) m_0^{(+)}(k_n) - F^{(+)}(0, k_n) + \frac{i(-1)^n \alpha}{\phi_0(k_n)} [F^{(+)}(0, k_n) - 1] + o(\alpha). \quad (3.18)$$

Moreover, by Eq. (2.7)

$$m_0^{(+)}(k) = m_0^{(+)}(k_n) + \frac{i(-1)^n}{\phi_0(k_n)} \alpha + O(\alpha^2). \quad (3.19)$$

By using Eqs. (3.18), (3.19), and (3.9), we obtain the relation (a) of the lemma. Part (b) follows similarly by analyzing the integral on the right-hand side of Eq. (3.10), where one also has to make use of Eq. (3.3). Alternatively, part (b) can be reduced to part (a) by means of the substitution $x \rightarrow -x$. We omit the details. ■

In preparation of the proof of the next theorem we collect some more details about the case when $\phi_0(k_n) = 0$ and the corresponding gap is open. By means of the variation of constants formula (see Ref. 3, p. 28), we find that

$$\phi_0(k) = -\frac{1}{2\Delta'(E_n)} \left(\int_0^1 \phi_0^2(x, k_n) dx \right) \alpha^2 + O(\alpha^4). \quad (3.20)$$

In deriving Eq. (3.20) we have also used the facts that $\theta_0(k_n) = \pm 1$ and $E - E_n = \mp (2\Delta'(E_n))^{-1} \alpha^2 + O(\alpha^4)$, where the upper (lower) sign is to be chosen when $\Delta(E_n) = +1$ ($\Delta(E_n) = -1$). Thus, by Eq. (2.7), the imaginary part of $m_0^{(\pm)}(k)$ blows up as $k \rightarrow k_n$. On the other hand, the real part of $m_0^{(\pm)}(k)$ tends to a finite limit because $\phi_0'(k_n) = \theta_0(k_n) (= \pm 1)$. As a result, the solutions $y_0^{(\pm)}(x, k)$ blow up and so do the solutions $F^{(\pm)}(x, k)$. Therefore, if we define

$$\chi^{(\pm)}(x, k) = \frac{F^{(\pm)}(x, k)}{m_0^{(\pm)}(k)} \quad (3.21)$$

then the solutions $\chi^{(\pm)}(x, k)$ have finite limits as $k \rightarrow k_n$. The limiting functions $\chi^{(\pm)}(x, k_n)$ obey the integral equations

$$\chi^{(\pm)}(x, k_n) = \phi_0(x, k_n) - \int_x^{\pm\infty} [\phi_0(x, k_n) \theta_0(t, k_n) - \theta_0(x, k_n) \phi_0(t, k_n)] V(t) \chi^{(\pm)}(t, k_n) dt. \quad (3.22)$$

Since the function $\phi_0(x, k_n)$ is periodic [if $\Delta(k_n)=1$] or antiperiodic [if $\Delta(k_n)=-1$] and hence bounded, Eq. (3.22) implies that $\chi^{(\pm)}(x, k_n) = \phi_0(x, k_n) + o(1)$ as $x \rightarrow \pm\infty$. The solutions $\chi^{(\pm)}(x, k_n)$ are linearly independent if and only if k_n is not a half bound state. If k_n is a half bound state, we define b_n by

$$\chi^{(+)}(x, k_n) = b_n \chi^{(-)}(x, k_n). \tag{3.23}$$

Since $m_0^{(+)}(k)/m_0^{(-)}(k) \rightarrow -1$ as $k \rightarrow k_n$, we conclude from Eqs. (3.21) and (3.23) that

$$b_n = - \lim_{k \rightarrow k_n} \frac{F^{(+)}(x, k)}{F^{(-)}(x, k)}. \tag{3.24}$$

The following theorem is our main result. We obtain the leading asymptotic behaviors of the entries of $S(k)$ as $k \rightarrow k_n$; in particular, we prove that $S(k)$ is always continuous at k_n . Note the difference between the statements regarding $T(k)$ and $R(k)$ or $L(k)$. In the case of the transmission coefficient k is allowed to be complex with $\text{Im } k \geq 0$, while in the case of the reflection coefficients k is required to be real. This is a consequence of the fact that under our assumption (1.1) the transmission coefficient has an analytic continuation to a small neighborhood of k_n in the upper half plane, whereas generally the reflection coefficients do not have such a continuation.

Theorem 3.3: The asymptotic behaviors of $T(k)$, $R(k)$, and $L(k)$ are as follows:

(a) If $k = k_n$ is not a half bound state, then, as $k \rightarrow k_n$ with $0 \leq \arg(k - k_n) \leq \pi$

$$T(k) = ic_n(k - k_n) + o(k - k_n), \tag{3.25}$$

where c_n is real and nonzero, and, as $k \rightarrow k_n$ through real values

$$R(k) = L(k) = \begin{cases} -1 + o(1), & \text{if } \phi_0(k_n) \neq 0, \\ 1 + o(1), & \text{if } \phi_0(k_n) = 0. \end{cases} \tag{3.26}$$

(b) If $k = k_n$ is a half bound state, then, as $k \rightarrow k_n$ with $0 \leq \arg(k - k_n) \leq \pi$

$$T(k) = \begin{cases} \frac{2a_n}{1+a_n^2} + o(1), & \text{if } \phi_0(k_n) \neq 0, \\ \frac{2b_n}{1+b_n^2} + o(1), & \text{if } \phi_0(k_n) = 0 \end{cases} \tag{3.27}$$

and, as $k \rightarrow k_n$ through real values

$$L(k) = \begin{cases} \frac{a_n^2 - 1}{a_n^2 + 1} + o(1), & \text{if } \phi_0(k_n) \neq 0, \\ \frac{1 - b_n^2}{1 + b_n^2} + o(1), & \text{if } \phi_0(k_n) = 0, \end{cases} \tag{3.28}$$

$$R(k) = \begin{cases} \frac{1 - a_n^2}{1 + a_n^2} + o(1), & \text{if } \phi_0(k_n) \neq 0, \\ \frac{b_n^2 - 1}{1 + b_n^2} + o(1), & \text{if } \phi_0(k_n) = 0. \end{cases} \tag{3.29}$$

(c) If the n th gap is closed (i.e., $E_{2n-1} = E_{2n}$ for some $n \geq 1$), then $T(k)$, $R(k)$, and $L(k)$ are continuous at k_n and $T(k_n) \neq 0$.

Proof:

(a) First assume that $\phi_0(k_n) \neq 0$. Let $\alpha = k - k_n$ be complex with $\text{Im } \alpha \geq 0$. The following estimates are consequences of Eqs. (2.5), (2.8), and (2.14), and hold in a sufficiently small neighborhood of $\alpha = 0$:

$$|A(x, t)| \leq C e^{(\text{Im } \alpha)|x-t|} (1 + |x-t|) \tag{3.30}$$

and therefore, by iteration using Eqs. (2.12) and (2.13)

$$|F^{(\pm)}(x, k)| \leq C e^{\mp(\text{Im } \alpha)x} (1 + \max\{\mp x, 0\}), \tag{3.31}$$

where C is a suitable constant. It follows from Eqs. (2.12), (2.18), and (3.31) that $[F^{(+)}(\cdot, k); F^{(-)}(\cdot, k)]$ and $T(k)$ have analytic continuations into the upper half plane near the point k_n . Therefore, by using Eq. (2.15) we conclude that, in the generic case

$$T(k) = ic_n \alpha + o(\alpha), \tag{3.32}$$

where

$$c_n = \frac{2(-1)^{n+1}}{[F^{(+)}(\cdot, k_n); F^{(-)}(\cdot, k_n)] \phi_0(k_n)}.$$

This proves (3.25), provided $\phi_0(k_n) \neq 0$.

Now suppose that $\phi_0(k_n) = 0$. We make a shift $x \rightarrow x + a$ and consider the perturbed Hill's equation with potentials $P(x; a) = P(x + a)$ and $V(x; a) = V(x + a)$. Associated with the potentials $P(x; a)$ and $V(x; a)$ are solutions analogous to those in Eqs. (2.3), (2.4) and (2.12), (2.13), namely,

$$\psi_0^{(\pm)}(x, k; a) = \frac{\psi_0^{(\pm)}(x + a, k)}{\psi_0^{(\pm)}(a, k)}$$

and

$$F^{(\pm)}(x, k; a) = \frac{F^{(\pm)}(x + a, k)}{\psi_0^{(\pm)}(a, k)}. \tag{3.33}$$

Furthermore, we have the relations

$$\phi_0(x, k; a) = -\phi_0(a, k) \theta_0(x + a, k) + \theta_0(a, k) \phi_0(x + a, k), \tag{3.34}$$

$$\phi_0(1 + a, k) = \theta_0(a, k) \phi_0(k) + \phi_0(a, k) \phi_0'(k), \tag{3.35}$$

$$\theta_0(1 + a, k) = \theta_0(a, k) \theta_0(k) + \phi_0(a, k) \theta_0'(k). \tag{3.36}$$

By using Eqs. (3.34)–(3.36) and the fact that $\phi_0'(k_n) = \theta_0(k_n)$, we obtain

$$\phi_0(1, k_n; a) = -\phi_0^2(a, k_n) \theta_0'(k_n). \tag{3.37}$$

Since the gap is open, $\theta_0'(k_n) \neq 0$. We now choose a such that $\phi_0(a, k_n) \neq 0$ and hence $\phi_0(1, k_n; a) \neq 0$. We remark that it is well known⁵ that one can reduce the case $\phi_0(k_n) = 0$ to the case $\phi_0(k_n) \neq 0$ by a shift of the origin, but the argument given here is more explicit than that in Ref. 5. Also, we need Eq. (3.37) below. It is also easy to see that the discriminant (2.1) and hence the function $k(E)$ are invariant under the shift. Then, by using Eqs. (3.33) and (2.15), we get

$$T(k) = -\frac{2i \sin k}{[F^{(+)}(\cdot, k; a); F^{-}(\cdot, k; a)] \phi_0(1, k; a)} \left[\frac{\phi_0(1, k; a)}{\psi_0^{(+)}(a, k) \psi_0^{(-)}(a, k) \phi_0(k)} \right].$$

After some manipulation, using Eqs. (3.34)–(3.36) and (2.1)–(2.7), we see that the term within brackets is equal to 1. Thus

$$T(k) = -\frac{2i \sin k}{[F^{(+)}(\cdot, k; a); F^{-}(\cdot, k; a)] \phi_0(1, k; a)} = T(k; a). \tag{3.38}$$

In view of Eqs. (3.21) and (3.33), we have that

$$[F^{(+)}(\cdot, k; a); F^{-}(\cdot, k; a)] = \frac{m_0^{(+)}(k) m_0^{(-)}(k)}{\psi_0^{(+)}(a, k) \psi_0^{(-)}(a, k)} [\chi^{(+)}(\cdot; k); \chi^{(-)}(\cdot; k)]$$

and therefore, by Eqs. (2.3) and (2.4)

$$\lim_{k \rightarrow k_n} [F^{(+)}(\cdot, k; a); F^{-}(\cdot, k; a)] = \frac{1}{\phi_0^2(a, k_n)} [\chi^{(+)}(\cdot; k_n); \chi^{(-)}(\cdot; k_n)]. \tag{3.39}$$

Thus, using Eqs. (3.37), (3.38), and (3.39), we infer that Eq. (3.32) holds with

$$c_n = \frac{2(-1)^n}{[\chi^{(+)}(\cdot; k_n); \chi^{(-)}(\cdot; k_n)] \theta_0'(k_n)}.$$

This proves Eq. (3.25) when $\phi_0(k_n) = 0$.

Now we turn to Eq. (3.26) and, again, first assume that $\phi_0(k_n) \neq 0$. By Lemma 2.1 [with $z(x, k) = F^{(+)}(x, k)$], we have that

$$\int_{-\infty}^{\infty} \psi_0^{(+)}(t, k_n) V(t) F^{(+)}(t, k_n) dt = 0$$

if and only if k_n is a half bound state. Hence this integral is nonzero in the case under consideration. Since $\psi_0^{(+)}(x, k_n) = \psi_0^{(-)}(x, k_n)$, we deduce from Eqs. (2.18) and (2.19) that $I^{(+)}(k)/J^{(+)}(k)$ and 1 and $1/J^{(+)}(k) \rightarrow 0$ as $k \rightarrow k_n$. Together with Eqs. (2.16) and (2.17), this implies that $L(k) \rightarrow -1$ and $R(k) \rightarrow -1$ as $k \rightarrow k_n$, proving Eq. (3.26) when $\phi_0(k_n) \neq 0$. If $\phi_0(k_n) = 0$, we use a shift as in the proof of Eq. (3.25). Then Eqs. (2.16) and (3.33) imply that

$$L(k) = L(k; a) \frac{\overline{\psi_0^{(-)}(a, k)}}{\psi_0^{(-)}(a, k)}. \tag{3.40}$$

Since $\overline{\psi_0^{(-)}(a, k)}/\psi_0^{(-)}(a, k) \rightarrow -1$ as $k \rightarrow k_n$ and $L(k_n; a) = -1$, we infer that $L(k_n) = 1$. A similar argument shows that $R(k_n) = 1$.

(b) Suppose that $\phi_0(k_n) \neq 0$. Lemmas 3.1 and 3.2 yield, for real k

$$[F^{(+)}(\cdot, k); F^{(-)}(\cdot, k)] = \frac{(-1)^{n+1} i a_n^2 + 1}{\phi_0(k_n) a_n} \alpha + o(\alpha) \tag{3.41}$$

and

$$[F^{(+)}(\cdot, k); \overline{F^{(-)}(\cdot, k)}] = \frac{(-1)^n i a_n^2 - 1}{\phi_0(k_n) a_n} \alpha + o(\alpha). \tag{3.42}$$

Now, for real k , Eq. (3.27) follows from Eqs. (2.15) and (3.41). In order to extend the result to complex k , we note that by Eqs. (2.5) and (3.31), for k near k_n ($\text{Im } k \geq 0$), we have the estimate

$$|\psi_0^{(-)}(t, k) V(t) F^{(+)}(t, k)| \leq C |V(t)| (1 + \max\{-t, 0\}). \quad (3.43)$$

By using Eqs. (2.15) and (2.18), we can write

$$\frac{1}{T(k)} = 1 - \frac{\phi_0(k)}{2i \sin k} \int_{-\infty}^{\infty} \psi_0^{(-)}(t, k) V(t) F^{(+)}(t, k) dt. \quad (3.44)$$

Together with Eq. (3.43) this implies that $1/T(k)$ is defined and analytic for k near k_n with $\text{Im } k \geq 0$ ($k \neq k_n$), and from Eqs. (3.43) and (3.44), we have the estimate

$$\left| \frac{1}{T(k)} \right| \leq \frac{C}{|\alpha|} \quad (3.45)$$

for α near 0, where C is a suitable constant. The validity of Eq. (3.27) for real k , along with the estimate (3.45) allow us to appeal to theorems of Phragmén–Lindelöf (see Ref. 13, Theorems 1.4.1 and 1.4.4) and to conclude that $T(k)$ approaches a finite limit as $k \rightarrow k_n$ uniformly in $0 \leq \arg(k - k_n) \leq \pi$. Thus Eq. (3.27) is established. The relations Eqs. (3.28) and (3.29) follow immediately from Eqs. (2.16), (2.17), and (3.42). This proves (b) when $\phi_0(k_n) \neq 0$.

Now suppose that $\phi_0(k_n) = 0$. As in part (a) we make a shift $x \rightarrow x + a$, where a is such that $\phi_0(a, k_n) \neq 0$ and hence $\phi_0(1, k_n; a) \neq 0$. By using Eqs. (3.21), (3.33) and the fact that $\psi_0^{(\pm)}(a, k)/m_0^{(\pm)}(k) \rightarrow \phi_0(a, k)$ as $k \rightarrow k_n$, we obtain

$$\frac{F^{(+)}(x, k_n; a)}{F^{(-)}(x, k_n; a)} = \lim_{k \rightarrow k_n} \frac{[F^{(+)}(x + a, k)/\psi_0^{(+)}(a, k)]}{[F^{(-)}(x + a, k)/\psi_0^{(-)}(a, k)]} = \lim_{k \rightarrow k_n} \frac{\chi^{(+)}(x + a; k)}{\chi^{(-)}(x + a; k)} = b_n$$

and hence

$$F^{(+)}(x, k_n; a) = b_n F^{(-)}(x, k_n; a).$$

Now the assertions follow by using Eqs. (3.38), (3.40) and the results for the case when $\phi_0(k_n) \neq 0$.

(c) If the gap is closed, we have that $\phi_0(k_n) = \theta'_0(k_n) = 0$ and $\phi'_0(k_n) = \theta_0(k_n) = 1$ ($\phi'_0(k_n) = \theta_0(k_n) = -1$) if $\Delta(k_n) = 1$ ($\Delta(k_n) = -1$). Since $\Delta(E)$ has a quadratic maximum or minimum at $E = E_n$, $E - E_n$ now vanishes linearly as a function of α as $\alpha \rightarrow 0$ [$E - E_n = \alpha/\sqrt{|\Delta''(E_n)|} + O(\alpha^2)$]. This implies that in Eq. (2.15) the ratio $\sin k/\phi_0(k)$ approaches a finite nonzero limit as $k \rightarrow k_n$. Since the Wronskian $[F^{(+)}(\cdot, k); F^{(-)}(\cdot, k)]$ is continuous at k_n , it follows that $T(k)$ must approach a nonzero limit as $k \rightarrow k_n$. Similarly, we conclude that $R(k)$ and $L(k)$ are continuous at $k = k_n$. ■

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¹D. B. Hinton and J. K. Shaw, "On the absolutely continuous spectrum of the perturbed Hill's equation," Proc. London Math. Soc. **50**, 175 (1985).

²J. Weidmann, *Spectral Theory of Differential Operators*, Lecture Notes in Mathematics, Vol. 1258 (Springer, Berlin, 1987).

³M. S. P. Eastham, *The Spectral Theory of Periodic Differential Equations* (Scottish Academic, Edinburgh, 1973).

⁴M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV. Analysis of Operators* (Academic, New York, 1978).

- ⁵V. A. Zheludev, in *Topics in Mathematical Physics*, edited by M. Sh. Birman (Consultants Bureau, New York, 1968), Vol. 2, p. 87.
- ⁶V. A. Zheludev, in *Topics in Mathematical Physics*, edited by M. Sh. Birman (Consultants Bureau, New York, 1971), Vol. 4, p. 55.
- ⁷F. Gesztesy and B. Simon, "A short proof of Zheludev's theorem," *Trans. Am. Math. Soc.* **335**, 329 (1993).
- ⁸N. E. Firsova, "Levinson formula for the perturbed Hill operator," *Theor. Math. Phys.* **62**, 130 (1985).
- ⁹R. G. Newton, "Inverse scattering by a local impurity in a periodic potential in one dimension," *J. Math. Phys.* **24**, 2152 (1983).
- ¹⁰M. Klaus, "Low-energy behavior of the scattering matrix for the Schrödinger equation on the line," *Inverse Problems* **4**, 505 (1988).
- ¹¹D. Hinton, K. Shaw, and M. Klaus, "On the Titchmarsh–Weyl function for the half line perturbed periodic Hill's equation," *Q. J. Math. Oxford* **41**, 189 (1990).
- ¹²N. E. Firsova, "Riemann surface of quasimomentum and scattering theory for the perturbed Hill operator," *J. Sov. Math.* **11**, 487 (1979).
- ¹³R. P. Boas, *Entire Functions* (Academic, New York, 1954).