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The Riemann–Hilbert problem for nonsymmetric systems

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A comparison of the Riemann–Hilbert problem and the Wiener–Hopf factorization problem arising in the solution of half-space singular integral equations is presented. Emphasis is on the factorization of functions lacking the reflection symmetry usual in transport theory.

I. INTRODUCTION

Wiener–Hopf integral equations, which arise in many physical applications, in particular, in transport theory, can be solved either by the classical Wiener–Hopf factorization approach, or alternatively, by conversion to a Riemann–Hilbert problem. Both methods are described in detail by Roos.[1] In the scalar case, a demonstration of the equivalence of the two methods, for a class of transport problems, has been given by Aamodt and Case.[2,3]

As a practical matter, analytical solutions to either of these problems are difficult to evaluate numerically, and one generally uses an iterative evaluation of nonlinear nonsingular integral equations. These integral equations arise from the Wiener–Hopf factorization of the symbol of the integral equation (the *dispersion function* in the language of transport theory).[4] This represents an advantage of the Wiener–Hopf approach; in fact, the proof that the solution to the Riemann–Hilbert problem provides a Wiener–Hopf factorization is a crucial part of the analysis of Ref. 2 (also see Ref. 3, p. 126).

In the present paper, we study the Wiener–Hopf factorization problem for symbols that do not possess the usual time-reversal invariance $\Lambda(z) = \Lambda(-z)$. A physical model leading to such a symbol is given, for example, in Ref. 5. Similarly, the Vlasov equation with nonsymmetric distribution function lacks this time-reversal invariance.[6] In the next section we give the solution of the factorization problem for scalar functions. Although this problem is well understood from the classical analysis of Muskhelishvili,[7] it is not often encountered in the transport theory literature.

In Sec. III we describe the analysis for the vector case, considering the matrix Riemann–Hilbert problem and the matrix Wiener–Hopf factorization for a dispersion matrix $\Lambda(z)$, which is not assumed to be an even function of z. Here there is a distinct advantage to the Wiener–Hopf point of view, rather than the Riemann–Hilbert approach, as we shall indicate.

II. THE SCALAR EQUATION

We shall assume throughout that $\Lambda(z)$ is an analytic function, except for a branch cut on which it has continuous boundary values, is bounded at infinity, and is real analytic,[8] $\Lambda(\bar{z}) = \overline{\Lambda(z)}$, as well as an integrability condition on arg Λ specified below. For simplicity we shall assume that Λ is nonvanishing on the branch cut and that the branch cut is the interval $[-L_-, L_+]$ of the extended real line. Then the Riemann–Hilbert problem is the determination of two bounded complex functions X and Y, satisfying

$$X^+(\nu)/X^-(\nu) = \Lambda^+(\nu)/\Lambda^-(\nu), \quad \nu\in[0,L_+] \tag{1}$$

and

$$Y^-(\nu)/Y^+(\nu) = \Lambda^+(-\nu)/\Lambda^-(-\nu), \quad \nu\in[0,L_-], \tag{2}$$

with X and Y of finite order at infinity, analytic outside the indicated intervals, and with continuous boundary values. The superscripts $\pm$ represent boundary values from above and below the branch cut.

Lemma: A solution to (1) is given by

$$X(z) = [P(z)/Q(z)]X_1(z), \tag{3a}$$

with

$$X_1(z) = \exp\left(\frac{1}{\pi}\int_0^{L_+} \Theta(\mu)\,\frac{d\mu}{\mu - z}\right), \tag{3b}$$

$$\Theta(\mu) = \arg\Lambda^+(\mu) - \arg\Lambda^+(L_+), \tag{3c}$$

for arbitrary polynomial P and appropriate monomial Q determined by the behavior of Λ at the end point $z=0$. A solution to (2) is given by

$$Y(z) = [\hat{P}(z)/\hat{Q}(z)]Y_1(z), \tag{4a}$$

with

$$Y_1(z) = \exp\left(\frac{1}{\pi}\int_0^{L_-} \hat{\Theta}(-\mu)\,\frac{d\mu}{\mu - z}\right), \tag{4b}$$

$$\hat{\Theta}(\mu) = -\arg\Lambda^+(-\mu) + \arg\Lambda^+(-L_-), \tag{4c}$$

for arbitrary polynomial $\hat{P}$ and appropriate monomial $\hat{Q}$ determined by the behavior of Λ at the end point $z=0$.

This lemma is a slight modification of a well-known formula.[1,4,7] Note that $\arg\Lambda(L_+) = n\pi$ and $\arg\Lambda(-L_-) = m\pi$ for $n, m\in\mathbb{Z}$ if $L_i < \infty$, by the real analytic assumption. It is necessary to impose this condition on Λ if the branch cut is infinite, in which case the renormalization achieved by $\Theta(L_+) = \hat{\Theta}(-L_-) = 0$ guarantees that the integrals defining X_1 and Y_1 will exist. One may observe that X_1 and Y_1 both tend to 1 as $|z|\to\infty$.

Let us suppose that $\Lambda(z)$ has N zeros z_i, each of multiplicity n_i, and, recalling that Λ is bounded at infinity, suppose that $\Lambda(z) \sim K_z^{-M}$, $M\geqslant 0$, as $z\to\infty$. Then, by the argument principle

$$\arg \Lambda^{+}(L_{+}) - \arg \Lambda^{+}(-L_{-}) = \pi(I+M), \quad (5a)$$

where

$$I = \sum_{i=1}^{N} n_i. \quad (5b)$$

We can now state the following

Theorem: Let $X_1(z)$ and $Y_1(z)$ be defined by Eqs. (3b) and (4b), respectively. Then

$$\Lambda(z) = K_z^{-M-I} X_1(z) Y_1(-z) \prod_{i=1}^{N} (z-z_i)^{n_i}. \quad (6)$$

Proof: Consider the function

$$R(z) = K_z^{-M-I} X_1(z) Y_1(-z) \prod_{i=1}^{N} \frac{(z-z_i)^{n_i}}{\Lambda(z)}. \quad (7)$$

Evidently, as $z \to \infty$, $R(z) \to 1$. Further, from Eqs. (2a) and (2b) we easily compute

$$R^{+}(\nu)/R^{-}(\nu) = 1, \quad -L_{-} \leqslant \nu \leqslant L_{+}. \quad (8)$$

Thus, $R(z)$ is an entire function except for a possible pole of order $M+I$ at $z=0$. Now, a standard estimate[4,5] utilizing (3b) and (4b) gives, for z near zero,

$$X_1(z) \sim z^{-\Theta(0)/\pi} \quad (9a)$$

$$Y_1(z) \sim z^{-\hat{\Theta}(0)/\pi}. \quad (9b)$$

Then, near $z=0$,

$$R(z) \sim z^{-(1/\pi)\Theta(0) - (1/\pi)\hat{\Theta}(0) - M - I} \quad (10)$$

Since

$$\Theta(0) = \arg \Lambda^{+}(0) - \arg \Lambda^{+}(L_{-}), \quad (11)$$

$$\hat{\Theta}(0) = -\arg \Lambda^{+}(0) + \arg \Lambda^{+}(-L_{+}), \quad (12)$$

it follows that

$$\hat{\Theta}(0) + \Theta(0) = \arg \Lambda^{+}(L_{+}) - \arg \Lambda^{+}(-L_{-}), \quad (13)$$

and using (13) and (5a) in (7), we see that $R(0) = \text{const}$. Thus, by Liouville's theorem, $R(z) \equiv 1$.

Equation (6) can be written in a more symmetric form by defining new functions $X(z)$ and $Y(z)$, which satisfy Case's end point condition.[4,5,9] Let

$$X(x) = X_1(z)/z^{-\Theta(0)/\pi}, \quad (14)$$

$$Y(z) = Y_1(z)/z^{-\hat{\Theta}(0)/\pi}. \quad (15)$$

Then

$$\Lambda(z) = KX(z)Y(-z) \prod_{i=1}^{N} (z-z_i)^{n_i}. \quad (16)$$

Since $\Theta(0) \leqslant 0$ and $\hat{\Theta}(0) \leqslant 0$, it follows that X and Y are both bounded at infinity. Thus, by Cauchy's theorem,

$$X(z) = X(\infty) + \frac{1}{2\pi i} \int_0^{L_{+}} \frac{X^{+}(\nu) - X^{-}(\nu)}{\nu - z} d\nu$$

$$= X(\infty) + \int_0^{L_{+}} \frac{\eta(\nu) X^{+}(\nu)}{\Lambda^{+}(\nu)(\nu - z)} d\nu, \quad (17)$$

where

$$\eta(\nu) \equiv (1/2\pi i)(\Lambda^{+}(\nu) - \Lambda^{-}(\nu)), \quad (18)$$

and we have used (1). Similarly,

$$Y(z) = Y(\infty) + \frac{1}{2\pi i} \int_0^{L_{-}} \frac{Y^{+}(\nu) - Y^{-}(\nu)}{\nu - z} d\nu$$

$$= Y(\infty) + \int_0^{L_{+}} \frac{\eta(-\nu) Y^{+}(\nu)}{\Lambda^{+}(\nu)(\nu - z)} d\nu. \quad (19)$$

These are generalizations of Case's identity A.[4]

We can compute $X^{+}(\nu)/\Lambda^{+}(\nu)$ and $Y^{+}(\nu)/\Lambda^{+}(-\nu)$ from (16), which when substituted into (17) and (19) leads to coupled equations for X and Y, the analog of Case's identity C[5] and the Chandrasekhar H equations:[3,4]

$$X(z) = X(\infty) + \frac{1}{K} \int_0^{L_{+}} \frac{\eta(\nu)}{Y(-\nu)(\nu - z)\Pi_{i=1}^{N}(\nu - z_i)^{n_i}} d\nu, \quad (20a)$$

$$Y(z) = Y(\infty) + \frac{1}{K} \int_0^{L_{-}} \frac{\eta(-\nu)}{X(-\nu)(\nu - z)\Pi_{i=1}^{N}(-\nu - z_i)^{n_i}} d\nu. \quad (20b)$$

Note that $X(\infty)$ and $Y(\infty)$ are either 1 or 0.

The existence and uniqueness of solutions of Eqs. (20) are intimately related to the existence and uniqueness of solutions of the convolution equation, and can be proved for various classes of symbols Λ, for example, those satisfying a contractive condition on $\Lambda - I$. This will be discussed further in the next section.

Corollary: If Λ is an even function, then X and Y may be chosen such that $Y(z) = X(-z)$.

Under the hypothesis of the corollary, the results of the theorem reduce to those of Ref. 2.

III. THE VECTOR EQUATION

Transport equations as coupled linear integrodifferential equations arise in the description of multiphase systems, or moment approximations of single phase systems, where components of the dependent variable correspond to phase densities, or to various moments. Typical of such equations is the multigroup neutron transport equation, which in stationary one-dimensional geometry may be written as[10]

$$\mu \frac{\partial \Psi}{\partial x} + \Sigma \Psi = B(\mu) \int_{-1}^{1} C(s) \Psi(x,s) ds, \quad x \in \mathbb{R}_{+}. \quad (21)$$

Here, the components $\psi_i(x,\mu)$ of $\Psi = \{\psi_i\}_{i=1}^{n}$ are the neutron densities in the ith energy group. The matrices $B(\mu)$ and $C(s)$ may be, respectively, $n \times nm$ and $nm \times n$ matrices,

$$B(\mu) = (B_1(\mu) \cdots B_m(\mu)), \quad C(s) = \begin{pmatrix} C_1(s) \\ \vdots \\ C_m(s) \end{pmatrix},$$

thus representing m-term separable collision kernels of the form $\Sigma_{j=1}^{m} B_j(\mu) \int_{-1}^{1} C_j(s) \Psi(x,s) ds$. The cross section matrix Σ is generally taken to be a diagonal positive constant matrix.

It is instructive to review briefly the derivation of the solution of (21), with $\Sigma = I$ for simplicity. For more general Σ the analysis is similar, however, with the degeneracy of the spectrum varying along the interval $[-1,1]$.[11] Proceeding with the standard separation of variables technique, the assumption

$$\Psi(x,\mu) = e^{-x/\nu}\Phi_\nu(\mu) \tag{22}$$

leads to the eigenvalue problem

$$(\nu-\mu)\Phi_\nu(\mu) = \nu B(\mu)\int_{-1}^{1} C(s)\Phi_\nu(s)ds, \tag{23}$$

whose solution may be written as

$$\Phi_\nu(\mu) = P[\nu/(\nu-\mu)]B(\mu)Q(\nu) + \lambda(\nu)\delta(\nu-\mu), \tag{24}$$

for $\nu\in[-1,1]$, with

$$Q(\nu) = \int_{-1}^{1} C(s)\Phi_\nu(s)ds. \tag{25}$$

The dispersion function Λ is defined as the $mn\times mn$ matrix,[12]

$$\Lambda(z) = I - z\int_{-1}^{1} C(s)B(s)\frac{1}{z-s}ds, \tag{26}$$

whose boundary values above and below the cut $[-1,1]$ are given by

$$\Lambda^{\pm}(\mu) = I - \mu\int_{-1}^{1} P\frac{1}{\mu-s}C(s)B(s)ds \pm \pi i\mu C(\mu)B(\mu). \tag{27}$$

The discrete eigenvalues ν_i obey $\det\Lambda(\nu_i)=0$. We shall assume here that $\det\Lambda$ is nonvanishing on $[-1,1]\cup\{\infty\}$. If the solution Ψ obeys the reflection invariance $\Psi(x,\mu) = \Psi(-x,-\mu)$, then $\det\Lambda(z) = \det\Lambda(-z)$ and with the real analytic assumption this implies that the eigenvalues occur in quartets $\pm\nu_i$, $\pm\bar{\nu}_i$. Reflection invariance does not imply, however, that Λ itself is an even function, but only the symmetry condition $B(\mu)C(s) = B(-\mu)\times C(-s)$.

Multiplication of (24) by $C(\mu)$ and integration leads to

$$R(\nu)Q(\nu) = C(\nu)\lambda(\nu), \tag{28}$$

where

$$R(\nu) = \tfrac{1}{2}(\Lambda^{+}(\nu) + \Lambda^{-}(\mu)). \tag{29}$$

The n-fold degeneracy of the eigenvalue problem is thus evident. Choosing the standard basis e_i in $\mathbb{R}^n$ for λ and assuming R is invertible, one finds for the n degenerate *generalized eigenvectors* (Case's singular eigenfunctions):

$$\Phi_\nu^{(i)}(\mu) = P[\nu/(\nu-\mu)]B(\mu)R^{-1}(\nu)C(\nu)e_i + \delta(\nu-\mu)e_i, \quad i=1,\ldots,n. \tag{30}$$

The solution of the half-space boundary value problem (21) with boundary value

$$\Psi(x=0,\mu) = \Phi_0(\mu), \quad \mu>0, \tag{31}$$

requires expansion of the datum Ψ_0 in terms of the eigenfunctions $\Phi_\nu^{(i)}$ with $\nu\geqslant 0$,

$$\Phi_0(\mu) = \sum_{i=1}^{n}\int_0^1 \Phi_\nu^{(i)}(\mu)A(\nu)d\nu = \int_0^1 \Phi_\nu(\mu)A(\nu)d\nu, \tag{32}$$

where Φ_ν is the $n\times n$ matrix whose ith column is $\Phi_\nu^{(i)}$ and A is an n vector of expansion coefficients. After some manipulation, this can be written as

$$\mu C(\mu)\Phi_0(\mu) = \Lambda^{+}(\mu)N^{+}(\mu) - \Lambda^{-}(\mu)N^{-}(\mu), \quad \mu>0, \tag{33}$$

where the Hilbert transform $N(z)$ is defined as the mn vector

$$N(z) = \frac{1}{2\pi i}\int_0^1 \frac{\nu}{\nu-z}R(\nu)^{-1}C(\nu)A(\nu)d\nu. \tag{34}$$

Then expansion coefficients A can be written in terms of $N(\nu)$ as the solution of any n independent equations of the overdetermined set of equations,

$$C(\nu)A(\nu) = (1/\nu)R(\nu)(N^{+}(\nu) - N^{-}(\nu)). \tag{35}$$

Given a matrix function $\Lambda(z)$ that is analytic on the finite complex plane except for a branch cut on $[-L_-, L_+]\subset\mathbb{R}$, bounded at infinity, and real analytic, the (homogeneous) matrix Riemann–Hilbert problem on the half line $\mathbb{R}_+$, associated with the function $\Lambda(z)$ is written as

$$X^{+}(\mu) = X^{-}(\mu)\Lambda^{-}(\mu)^{-1}\Lambda^{+}(\mu), \quad \mu>0, \tag{36}$$

for a matrix function $X(z)$, analytic on $\mathbb{C}/\mathbb{R}_+$, of finite order at infinity, with continuous boundary values and satisfying (36). The adjoint Riemann–Hilbert problem on the halfline $\mathbb{R}_-$ is written as

$$Y^{+}(-\mu) = \Lambda^{-}(\mu)\Lambda^{+}(\mu)^{-1}Y^{-}(-\mu), \quad \mu<0, \tag{37}$$

for a matrix function $Y(z)$, analytic on $\mathbb{C}/\mathbb{R}_+$, of finite order at infinity, with continuous boundary values and satisfying (37). We shall assume that $\Lambda^{+}(0) = \Lambda^{-}(0)$, which is obviously valid for Λ, an even function of z, and appears to be the case for other transport models studied in the literature. Then the index I_+ (resp. I_-) of $\Lambda(z)$ defined on the subinterval $[0,L_+]$ (resp. $[-L_-,0]$) may be computed by a contour around the appropriate half-plane. Minimal solutions[13] (or fundamental solutions of normal form at infinity) of the Riemann–Hilbert problems (36) and (37) are solutions $X_0(z)$, $Y_0(z)$, which are invertible on their regions of analyticity and, as $z\to\infty$, have the asymptotic behavior

$$X_0(z)\sim D_+(z)M, \quad Y_0(z)\sim ND_-(z), \tag{38}$$

where M, N are constant matrices and $D_\pm(z)$ are diagonal matrices with diagonal entries $d_{\pm ii}(z) = z^{-\kappa_i^{\pm}}$ for *partial indices* $\kappa_i^{\pm}$ satisfying $\Sigma_i\kappa_i^{\pm} = I_\pm$. If a minimal solution of (36) exists, and $X(z)$ is another solution of (36), then by a Liouville argument one may see that $X(z) = X_0(z)P(z)$, where $P(z)$ is a matrix function with polynomial entries. Similarly, $Y(z) = P(z)Y_0(z)$.

In the approach of Gohberg *et al.*[14] to homogeneous Riemann problems, given a barrier function $\Gamma(\mu)$ on $\mathbb{R}$, a solution to the Riemann problem is a pair of functions $W(z)$, $U(z)$, analytic in the upper/lower half-planes, of finite order at infinity, with continuous boundary values and satisfying

$$\Gamma(\mu)W^{+}(\mu) = U^{-}(\mu), \quad \mu\in\mathbb{R}. \tag{39}$$

The connection with (36) is to define $\Gamma(\mu) = \Lambda^{T-}(\mu)\Lambda^{T+}(\mu)^{-1}$ on $[0,L_+]$ and $\Gamma(\mu) = I$ on the remainder of the closed contour $\mathbb{R}\cup\infty$. Then, since the functions W, U are continuous across the negative real axis, they define a single function $X^T(x)$, analytic in the left half-plane and satisfying (36). [The Wiener–Hopf factorization (42) below may be obtained by defining $Y(-z) = \Lambda(z)X(z)^{-1}$ for $z \in \mathbb{C}/[-L_-,0]$.]

Because $\Lambda(z)$ defined by (26) has its cut along $[-1,1]$ and $N(z)$ only along $[0,1]$, an explicit representation of $N(z)$ can be written in terms of the solution of the matrix Riemann–Hilbert problem (36). Indeed, from (33) one has

$$\Gamma(\mu) = X^+(\mu)N^+(\mu) - X^-(\mu)N^-(\mu), \quad \mu > 0, \tag{40}$$

with $\Gamma(\mu) = \mu X^+(\mu)\Lambda^+(\mu)^{-1}C(\mu)\Phi_0(\mu)$, which leads to the solution for N:

$$N(z) = \frac{1}{2\pi i} X(z)^{-1} \int_0^1 \Gamma(\mu)\frac{1}{\mu - z}\,d\mu. \tag{41}$$

Note that, since the Hilbert transform $N(z)$ must be analytic in $\mathbb{C}/[0,1]$, the poles introduced by the zeros of det X must be removed; alternatively, if the minimal solution X_0 is chosen, then the divergent behavior of X_0^{-1} at infinity must be controlled. In either case, this may be done in more or less standard form by introducing the discrete eigenfunctions into (41).

There are two difficulties associated with this approach to matrix singular integral equations, which do not arise in the scalar case. In the first place, the Riemann–Hilbert function $X(z)$ cannot usually be obtained in closed form (but see Refs. 15–17 for solutions of the model described above in the case $n = 2$, $m = 1$, and symmetric Λ). Further, analytic behavior at the endpoints of the interval $[0,L_+]$ and at infinity require that conditions related to the partial indices of the barrier function be fulfilled. Although these obstacles cannot entirely be circumvented, a Wiener–Hopf factorization approach presents several computational advantages.

A canonical Wiener–Hopf factorization is a factorization of the form

$$\Lambda(z) = Y(-z)X(z), \quad z \in \mathbb{R}, \tag{42}$$

where X and Y are analytic on the open left half-plane, invertible on its closure and bounded at infinity. In general, if $\Lambda(z) = Y(-z)X(z)$ is a canonical Wiener–Hopf factorization, then it is clear that $X(z)$ satisfies the Riemann–Hilbert problem (36) for $z > 0$, and $Y(-z)$ satisfies the adjoint Riemann–Hilbert problem (37) for $z < 0$. On the other hand, if $X_0(z)$ is a minimal solution of the Riemann–Hilbert problem (36) for $z > 0$, then it is easy to see that $\Lambda(z)X_0(z)^{-1}$ solves the adjoint Riemann–Hilbert problem, when

$$\Lambda(z) = Y_0(-z)P(z)X_0(z), \quad z \in \mathbb{R}. \tag{43}$$

For (43) to provide a canonical Wiener–Hopf factorization, it is necessary that the zeros of det $\Lambda(z)$ be distributed with total order $\Sigma_{i=1}^{mn}\kappa_i^{\pm}$ in each of the left/right half-planes, and that the partial indices $\kappa_i^{\pm}$ be non-negative.[18] However, this is not sufficient. For although it is always possible to factor $P(z) = R(-z)Q(z)$ with polynomial matrices R and Q having nonvanishing determinants in the left half-plane (See Ball *et al.*[19]), the directions determined by the partial indices must be matched in the factorization of $P(z)$ so that the functions $X(z) = Q(z)X_0(z)$ and $Y(-z) = Y_0(-z)R(-z)$ have proper behavior at infinity.

Under various conditions on matrix functions, it is possible to prove, from the general theory, that a Wiener–Hopf factorization exists. For example, if $\Lambda(z)$ is a rational function on the complex plane, then the existence of a canonical Wiener–Hopf factorization is equivalent to the invertibility of an operator defined in terms of projections and determined by the barrier function $\Gamma(\mu)$ on $\mathbb{R}_+$. Although a similar theorem seems likely to be true for the class of functions $\Lambda(z)$ analytic on $\mathbb{C}/[-L_-,L_+]$ treated here, such a result is not yet known. However, we shall see below that for a variety of dispersion functions arising in transport theory, invertibility of (the inverse of) the albedo operator is equivalent to the existence of a Wiener–Hopf factorization. In these cases, as an application of Cauchy's theorem, one can derive coupled integral equations (analogs of the H equations of Chandrasekhar[3]) for X and Y, assuming nonsingular behavior of $\Lambda(z)$ at $z = L_\pm$:

$$X(z) = X(\infty) + \frac{1}{2\pi i}\int_0^{L_+} Y^{-1}(-\nu)\{\Lambda^+(\nu) - \Lambda^-(\nu)\} \times [1/(\nu - z)]d\nu \tag{44a}$$

$$Y(z) = Y(\infty) + \frac{1}{2\pi i}\int_0^{L_+} \{\Lambda^+(-\nu) - \Lambda^-(-\nu)\} \times X^{-1}(-\nu)[1/(\nu - z)]d\nu. \tag{44b}$$

One may show also that, given $X(\infty)$ and $Y(\infty)$, the equations are uniquely solvable. The advantage of this approach, then, is that one may use functional analytic tools to prove that a Wiener–Hopf factorization must exist, and then obtain the solutions from the integral equations (44), without requiring a detailed analysis of the partial indices.

It follows easily from (36) and (37) that under the symmetry condition $\Lambda(-z) = \Lambda(z)^T$ satisfied by many of the kinetic models studied in the transport theory literature (but not the example detailed above), one may take $Y(z) = X(z)^T$. It is also interesting to note that a left half-space problem, such as (21) for $x \in \mathbb{R}_-$ with (31) for $\mu < 0$, leads to a different Wiener–Hopf factorization problem,

$$\Lambda(z) = \hat{Y}(z)\hat{X}(-z), \quad z \in \mathbb{R}, \tag{45}$$

and, likewise, to the Riemann–Hilbert problem

$$\hat{X}^+(-\mu) = \hat{X}^-(-\mu)\Lambda^+(\mu)^{-1}\Lambda^-(\mu), \quad \mu > 0, \tag{46}$$

and the adjoint problem

$$\hat{Y}^+(\mu) = \Lambda^+(\mu)\Lambda^-(\mu)^{-1}\hat{Y}^-(\mu), \quad \mu < 0. \tag{47}$$

In these equations, $\hat{X}$ and $\hat{Y}$ are analytic in the *right* half-plane. One obtains in like fashion variants of the X equations (44).

The Wiener–Hopf factorization problem associated with Eq. (21) with $(\Sigma)_{ij} = \sigma_i\delta_{ij}$ has been analyzed in Refs. 10 and 20. The existence of a canonical factorization is implied by a contractive condition on the operator $S(\mu) \equiv \Sigma_i C(\sigma_i\mu)e_i \otimes e_i^T B(\sigma_i\mu)$.

To treat a larger class of vector equations with matrix coefficients, or indeed with Hilbert space operator coefficients, the *abstract* transport equation,

$$\frac{\partial}{\partial x} T\psi(x) = -(I-B)\psi(x), \quad 0<x<\infty, \qquad (48)$$

has been studied with half-space boundary conditions

$$Q_+ \psi(x=0) = \Psi_0, \quad \|\psi(x)\| = O(1)(x\to\infty). \quad (49)$$

The operator T associated with the streaming term is usually quite simple, e.g., multiplication by μ in Eq. (21), or by $\mu\Sigma^{-1}$, in order to keep B as simple as possible. The boundary value problem has been studied under the assumption that T is (possibly unbounded) self-adjoint and injective. The projection Q_+ onto the positive eigenspaces of T merely specifies that (49) is an incoming flux boundary condition.

Under suitable restrictions on B, it can be shown that the boundary value problem (48) and (49) is equivalent to the convolution equation

$$\psi(x) - \int_0^\infty H(x-y)B\psi(y)dy = e^{-xT^{-1}}\Phi_0, \qquad (50)$$

with the propagator function, defined by

$$H(x) = \pm T^{-1}e^{-xT^{-1}}Q_\pm, \quad \pm x>0. \qquad (51)$$

Thus, it is clear that the symbol W of the convolution equation,

$$W(z) = I - \int_{-\infty}^{\infty} e^{x/z}H(x)B\,dx = I - B - T(z-T)^{-1}B, \qquad (52)$$

should play an essential role in developing a solution to the boundary value problem. Actually, there is a considerable excess of dimensionality as the convolution equation is posed. Indeed, the convolution equation (50), and equivalently the boundary value problem (48) and (49), are solvable if and only if the convolution equation

$$\varphi(x) - \int_0^\infty pH(x-y)Bj\psi(y)dy = pe^{-xT^{-1}}\Phi_0 \quad (53)$$

is solvable, where pj is the orthogonal projection onto any subspace containing range B^* and jp is the identity. It is the function $pW(z)j$, the symbol of the reduced convolution equation, which is the dispersion function of transport theory, and whose factorization is, of course, necessary for the solution of half-space problems.

The most powerful results on the existence of a Wiener–Hopf factorization of $\Lambda(z)$ are applications of the factorization principle for transfer functions of the form $L(z) = \mathscr{A} - \mathscr{C}(z-\mathscr{T})^{-1}\mathscr{B}$ on a Banach space Z, due to Bart *et al.*[21,22] For such functions the existence of a factorization can be related to the direct sum decomposition of Z in terms of certain closed invariant subspaces. A typical class of functions Λ that arise in transport theory and that are of transfer function type are those given by $\mathscr{A} = I-B$, $\mathscr{C} = \mathscr{T} = T$, $\mathscr{B} = B$, and $\Lambda(z) = pL(z)j$ on a Hilbert space H, and with T injective self-adjoint with domain $D(T)$. Then if $Q_\pm$, $P_\pm$ are the positive/negative maximal projections associated with the spectral operators T^{-1}, $T^{-1}\{I-B\}$, a Wiener–Hopf factorization exists if and only if H has the direct sum decomposition $H = \operatorname{Ran} P_+ \oplus \operatorname{Ran} Q_-$, or, equivalently if and only if the operator $V = Q_+P_+ + Q_-P_-$ is invertible. A study of the structure of this operator leads to the existence of a factorization under any of the following conditions: A strictly positive; B compact contractive; A invertible with $A^{-1}B$ contractive.[23]

Integral equations for the Wiener–Hopf factors analogous to (44) can be obtained in the general operator case as well. Under the assumption B compact, for example, one finds [with the normalization $X(0) = Y(0) = I$]

$$X(z) = I + z\int_0^\infty \frac{1}{t-z} Y(-t)^{-1}p\sigma(dt)Bj, \qquad (54a)$$

$$Y(z) = I + z\int_0^\infty \frac{1}{t-z} p\sigma(-dt)BjX(-t)^{-1}, \qquad (54b)$$

where σ is the spectral measure of T.

Theorem:[23]

The following are equivalent: (a) The symbol $\Lambda(z)$ has a canonical factorization; (b) the related convolution equation is uniquely solvable on $L_p([0,\infty),H)$, $1\leqslant p\leqslant\infty$; (c) the Hilbert space operator V is invertible on H; and (d) the integrodifferential transport equation in half-space geometry is uniquely solvable on H for each incoming flux boundary condition in $Q_+(D(T))$. In these cases, the coupled nonlinear integral equations (54) have only one solution that are bounded strongly measurable and whose inverses extend to functions analytic on the open half-plane and continuous on its closure. These solutions provide a Wiener–Hopf factorization of the symbol $\Lambda(z)$.

Conservative kinetic models, including critical neutron transport and most gas dynamics equations, correspond generally, in the notation above, to $\rho(B) = 1$, where ρ is the spectral radius. The existence theory for such equations is nontrivial. Greenberg and van der Mee[23,24] have provided a simple algorithm for computing the measures of nonexistence (noncompleteness) and nonuniqueness in terms of the zero root linear manifold N_0 of $T^{-1}(I-B)$, i.e., the zero eigenvectors and generalized eigenvectors. For vanishing behavior at infinity, the general theory developed in the cited references also provides a linear submanifold $\mathscr{M}\subset Q_+(D(T))$ of finite codimension with the property that the boundary value problem is well posed for all $\Phi_0\in\mathscr{M}$. This requirement can be conveniently represented by the restriction

$$(T\Phi_0, Q_+g) = 0, \quad g\in\mathscr{N}, \qquad (55)$$

where $\mathscr{N}\subset N_0$ is a finite-dimensional subspace specified by the Greenberg–van der Mee algorithm. The restriction of the boundary value to a submanifold is required, in physical terms, by the conservation laws. Similarly, for bounded or $O(x)$ behavior at infinity, one can specify a subspace $\mathscr{N}\subset N_0$ and a projection P onto $\mathscr{N}$ with the property that, for each $\Psi_0\in\mathscr{N}$, the boundary value problem with the restriction that the solution Ψ satisfy $P\Psi = \Psi_0$ is well posed. In each of these cases, a representation of the solution then demands, as for the nonconservative case, the solution of the X equations (54).

From a practical point of view, the existence of a unique solution to Eq. (54) whose inverses extend analytically is not very helpful, since the equation itself is not uniquely solvable. However, it is possible to impose a linear constraint on the equation such that the constrained equation has as its unique solution the functions with the analytical behavior indicated in the theorem above. The constraint is imposed by equations

$$I + \nu_r \int_0^\infty Y(-\mu)^{-1} p\sigma(d\mu) Bj \frac{1}{\mu - \nu_r} = 0, \tag{56a}$$

$$I + \nu_l \int_0^\infty p\sigma(-d\mu) Bj X(-\mu)^{-1} \frac{1}{\mu - \nu_l}, \tag{56b}$$

for all simple zeros ν_r, ν_l of det $\Lambda(z)$ in the open right/left half-planes. If A has a nontrivial kernel (conservative case), then the equations become[23,25]

$$I - \int_0^\infty Y(-\mu)^{-1} p\sigma(d\mu) Bj = 0, \tag{57a}$$

$$I - \int_0^\infty p\sigma(-d\mu) Bj X(-\mu)^{-1} = 0. \tag{57b}$$

If $T^{-1}A$ has eigenvalues along the imaginary axis, uniqueness is obtained only if, in addition to these constraints, the principal parts of X^{-1} and Y^{-1} are specified at the imaginary eigenvalues of $T^{-1}A$. These equations can be extended to include the cases of eigenvalues imbedded in the spectrum of T and of nonsimple zeros.[23]

For the example illustrated in the beginning of this section, one has

$$p\sigma(d\mu) Bj = C(\mu) B(\mu) d\mu. \tag{58}$$

An analysis of the factorization and the integral equations for the factors for the matrix function $\Lambda(z) = I - (\Sigma - 2C)^{-1} C \int_{-1}^{1} \mu (z\Sigma - \mu)^{-1} d\mu$ with Σ a positive diagonal matrix and C any constant matrix is given in Ref. 14. The contractive condition in this case is $\|\Sigma^{-1} C\| < 1/2$. Siewert *et al.*[17,18] have given a detailed analysis for a number of important two-dimensional models. Mullikin, in Ref. 26, has carried out the study for the general matrix convolution equation. In both of these works, the equivalence theory is due to a contractive assumption on B. These works, and many others that detail the properties of Wiener–Hopf equations, are based on the pioneering contributions of Muskhelishvili[7] for scalar equations and Gohberg and his collaborators[27,28] for matrix and operator equations.

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