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Layne T. Watson and ChangYi Wang

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# Deceleration of a rotating disk in a viscous fluid

Layne T. Watson

Department of Computer Sciences, Virginia Polytechnic Institute, Blacksburg, Virginia 24061

Chang-Yi Wang

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824

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A disk rotating in a viscous fluid decelerates with an angular velocity inversely proportional to time. It is found that the unsteady Navier-Stokes equations admit similarity solutions which depend on a nondimensional parameter  $S = \alpha/\Omega_0$  measuring unsteadiness. The resulting set of nonlinear ordinary differential equations is then integrated numerically. The special case of  $S = -1.606699$  corresponds to the decay of rotation of a free, massless disk in a viscous fluid.

## I. INTRODUCTION

The full, unsteady Navier-Stokes equations with all the unsteady, nonlinear, and viscous terms are very difficult to solve and exact solutions are rare. The existing exact solutions can probably be grouped into three types: (a) Well-known parallel or concentric flows where the nonlinear terms vanish identically<sup>1</sup>; (b) flows without solid boundaries where the nonlinear terms partially cancel<sup>2</sup>; and (c) similarity solutions where the velocities vary as the inverse of time. The solutions of type (c) include the unsteady stagnation point flow found by Yang,<sup>3</sup> the squeezing of a fluid between circular or two-dimensional plates by Wang,<sup>4</sup> and the squeezing of a fluid-filled tube by Uchida and Aoki.<sup>5</sup>

In this paper we shall study an important problem of type (c): the unsteady deceleration of a rotating disk in a viscous fluid. We shall investigate the region near the center of the disk, such that both the disk and the fluid may be considered infinite in dimension. In order to obtain exact unsteady similarity solutions, we require the angular velocity of the disk be inversely proportional to time

$$\Omega(t) = \Omega_0(1 - \alpha t)^{-1}, \quad (1)$$

where  $\alpha$  is a constant and  $\Omega_0$  is a positive constant. When  $\alpha = 0$ , the problem reduces to the case of the steady rotation of a disk in a fluid introduced by von Kármán.<sup>6</sup> We shall study the case when  $\alpha$  is negative (slowing down).

## II. FORMULATION

Let the velocity components be  $u, v, w$  in the cylindrical polar coordinates  $r, \theta, z$ , respectively. The  $z$  axis is the axis of rotation of the disk, with  $z = 0$  on the surface of the disk. Our experience with exact solutions of type (c) suggests the following transformation:

$$u = \frac{\Omega_0 r}{(1 - \alpha t)} f'(\eta), \quad (2)$$

$$v = \frac{\Omega_0 r}{(1 - \alpha t)} g(\eta), \quad (3)$$

$$w = \frac{-2(\nu\Omega_0)^{1/2}}{(1 - \alpha t)^{1/2}} f(\eta), \quad (4)$$

$$p = \frac{-\rho\nu\Omega_0}{(1 - \alpha t)} P(\eta). \quad (5)$$

Here,  $p$  is the pressure,  $\rho$  is the density,  $\nu$  is the kinematic viscosity, and

$$\eta = \left(\frac{\Omega_0}{\nu}\right)^{1/2} \frac{z}{(1 - \alpha t)^{1/2}}. \quad (6)$$

The Navier-Stokes equations then reduce to a set of nonlinear ordinary differential equations

$$f'''(\eta) + 2ff'' + g^2 - f'f' = S\left(\frac{1}{2}\eta f'' + f'\right), \quad (7)$$

$$g''(\eta) - 2f'g + 2fg' = S\left(\frac{1}{2}\eta g' + g\right), \quad (8)$$

$$P' = 2f'' + 4ff' - S(\eta f' + f), \quad (9)$$

where  $S = \alpha/\Omega_0$  is a nondimensional number, measuring unsteadiness.

The boundary conditions are

$$f(0) = 0, \quad f'(0) = 0, \quad g(0) = 1, \quad (10)$$

$$f''(\infty) = 0, \quad g(\infty) = 0, \quad P(0) = P_0. \quad (11)$$

One can solve Eqs. (7) and (8) first, then Eq. (9) yields the function  $P(\eta)$

$$P = P_0 + 2f'(\eta) + 2f^2 - S\eta f. \quad (12)$$

## III. ASYMPTOTIC BEHAVIOR

For large  $\eta$ , we set

$$f(\eta) = c + \bar{f}(\eta), \quad (13)$$

$$g(\eta) = \bar{g}(\eta), \quad (14)$$

TABLE I. Boundary values of  $f$  and  $g$  for various values of  $S$ .

$S$	$f''(0)$	$g'(0)$	$c = f(\infty)$
0	0.510 225	-0.615 917	0.441 7
-0.5	0.614 283	-0.428 406	0.425 5
-1	0.719 787	-0.236 575	0.417 7
-1.606 699	0.848 244	0.000 000	0.412 1
-2	0.931 507	0.154 981	0.409 6
-5	1.562 797	1.360 850	0.400 6
-10	2.600 801	3.413 860	0.395 7
-20	4.646 424	7.579 882	0.392 3

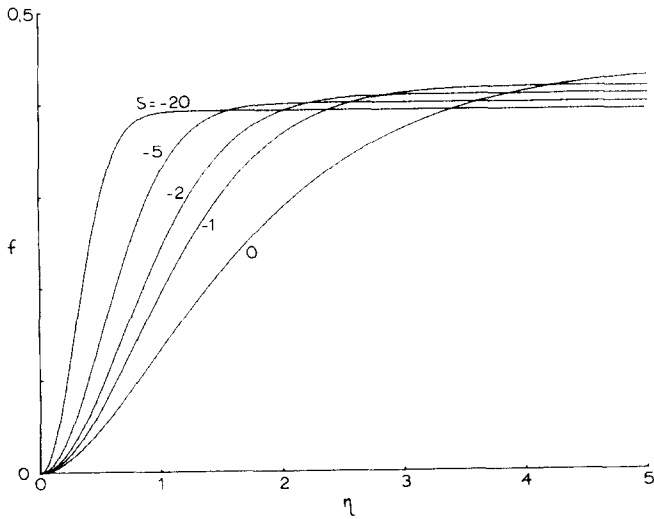


FIG. 1. Axial flow velocity distribution.

where  $c$  is a constant and  $\tilde{f}, \tilde{g}$  are small. Substituting into Eqs. (7) and (8) and linearizing, we find

$$\tilde{f}''' + 2c\tilde{f}'' = S(\frac{1}{2}\eta\tilde{f}'' + \tilde{f}'), \quad (15)$$

$$\tilde{g}'' + 2c\tilde{g}' = S(\frac{1}{2}\eta\tilde{g}' + \tilde{g}). \quad (16)$$

Integrating Eq. (15) twice, we obtain

$$\tilde{f}' + 2c\tilde{f} = S\frac{1}{2}\eta\tilde{f} + c_1 + c_2\eta. \quad (17)$$

The integration constants  $c_1$  and  $c_2$  are set to zero since  $\tilde{f}$  vanishes at infinity. Equation (17) is then integrated once more to give

$$\tilde{f} = c_3 \exp(\frac{1}{4}S\eta^2 - 2c\eta). \quad (18)$$

This shows exponential decay, or the existence of a boundary layer on the disk, is possible only if  $S \leq 0$  or  $\alpha \leq 0$  (slowing down). The function  $\tilde{g}$  shows a similar behavior.

#### IV. NUMERICAL RESULTS

Equations (7), (8), (10), and (11) are solved by a non-linear least-squares algorithm, with the semi-infinite

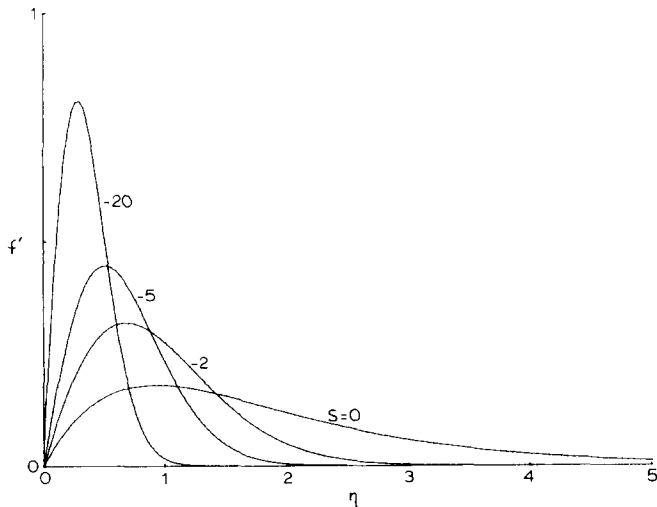


FIG. 2. Radial flow velocity distribution.

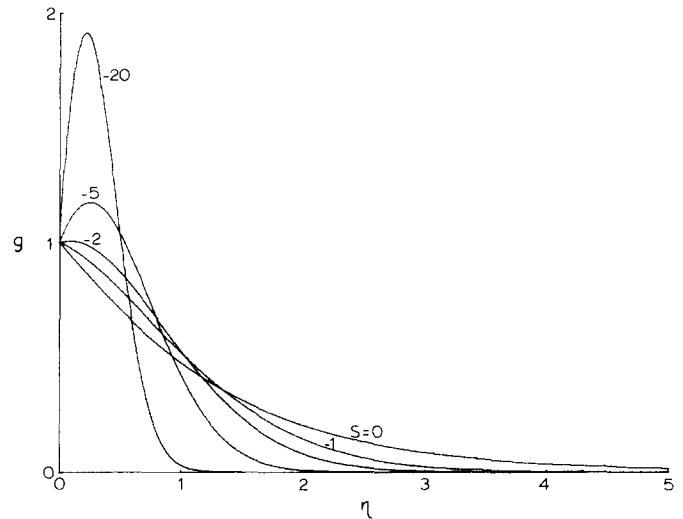


FIG. 3. Circumferential flow velocity distribution.

interval replaced by a finite interval  $[0, \tau]$ . The magnitude of  $\tau$  is guided by Eq. (18). For  $\tau$  sufficiently large ( $\tau = 10$  proved to be adequate for all  $S \leq 0$ ), we define

$$F(\mathbf{x}) \equiv F(f''(0), g'(0)) \equiv [f'(\tau)]^2 + [g(\tau)]^2. \quad (19)$$

$F(\mathbf{x})$  is minimized by the the Levenberg–Marguardt algorithm,<sup>7</sup> where values of  $F(\mathbf{x})$  are calculated by a variable order, variable step Adams method. The solution  $\mathbf{x}$  together with Eq. (10) give the complete initial conditions for the integration of Eqs. (7) and (8) by a fourth-order Runge–Kutta algorithm. The results for  $f''(0), g'(0)$  are given in Table I.

The initial values for the steady rotation ( $S=0$ ) agree with the numerical values obtained by Cochran<sup>8</sup> and Benton.<sup>9</sup> Figure 1 shows the axial velocity distribution  $f(\eta)$ . The asymptotic values for  $f$  at large  $\eta$  are also given in Table I. The value  $f(\infty)$  represents the suction due to the centrifugal action of the rotating disk. Figure 2 shows the radial velocity distribution  $f'(\eta)$ . The boundary layer becomes thinner and more prominent for large, negative  $S$ . Figure 3 shows the circumferential velocity distribution  $g(\eta)$ . It is noted, for  $S < S^*$

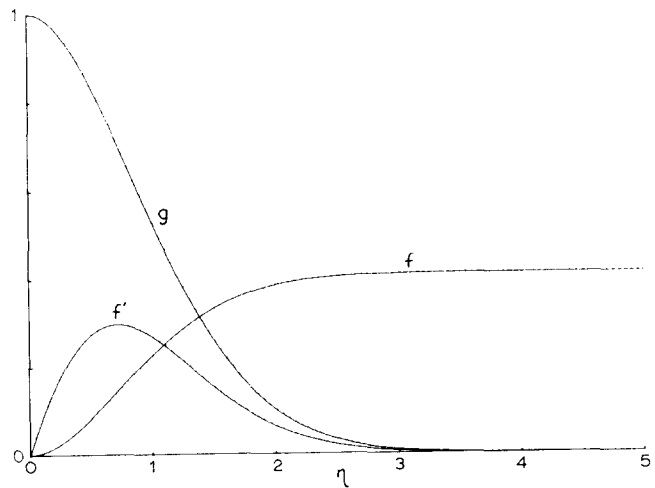


FIG. 4. Velocity profiles for  $S = -1.606\ 699$ .

$\approx -1.606\ 699$ , that the fluid near the disk rotates faster than the disk itself. This is due to the fact that for fast deceleration of the disk (larger negative  $S$ ) the fluid rotation is not able to decay as fast as the disk. We did not find any numerical instability or any inertial oscillations of the fluid. Similarity solutions for  $S > 0$  (acceleration) do not exist.

## V. DISCUSSION

Of interest is the torque experienced by the disk. For a region of radius  $R$ , the torque  $T$  is

$$T = 2\pi\rho\nu \int_0^R r^2 \left. \frac{\partial v}{\partial z} \right|_{z=0} dr$$

$$= \frac{1}{2} \pi\rho\Omega_0 \sqrt{\nu\Omega_0} R^4 (1 - \Omega_0 St)^{-3/2} g'(0). \quad (20)$$

When  $0 \geq S > S^*$ , the rotating disk experiences a resistance since  $g'(0)$  is negative. When  $S < S^*$ ,  $g'(0)$  is positive and the disk experiences a torque in the direction of rotation. The magnitude of this torque decays as  $t^{-3/2}$ .

For the particular case  $S = S^*$ , the torque is zero. This interesting situation corresponds to the decay of rotation of a free, massless disk in an infinite fluid. For example, suppose a light solid disk is rotated by an outside torque on the surface of a viscous fluid. When this outside torque is removed, the disk continues to rotate due to the residual rotation of the fluid. For large times (after initial transients die out), the angular velocity of the disk decays as  $t^{-1}$ , and the fluid motion follows the analysis in this paper with  $S = S^* = -1.606\ 699$ . Figure 4 shows the velocity distributions for this par-

ticular case.

We have studied the unsteady fluid dynamics of a disk rotating with inverse-time angular velocity. Other kinds of unsteady rotation of a disk have been studied, although the solutions are based on perturbation theories. The impulsive rotation from rest was investigated by Thiriot,<sup>10</sup> Benton,<sup>9</sup> and others. The velocities are expanded in a power series in time and solved successively. Another problem is the sudden change from one steady-state rotation (von Kármán solution) to a perturbed, slightly faster steady-state solution.<sup>11</sup> The present paper, being an exact solution of the unsteady Navier–Stokes equations, is not limited by small perturbations.

## ACKNOWLEDGMENT

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