Exploring Middle School Students' Heuristic Thinking About Probability

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ABSTRACT

This descriptive qualitative study examines six eighth-grade students' thinking while solving probability problems. This study aimed to gather direct information on students' problem solving processes informed by the heuristics and biases framework. This study used purposive sampling (Patton, 1990) to identify eighth-grade students who were knowledgeable about probability and had reached the formal operational stage of cognitive development. These criterion were necessary to reduce the likelihood of students' merely guessing answers and important so that the researcher could distinguish between reasoning and intuition.

The theoretical framework for this study was informed by Kahneman and Fredrick's (2002) recent revision to the heuristics and biases framework grounded in the research of Amos Tversky and Daniel Kahneman. Kahneman and Fredrick (2002) drew on dual process theory to explain systematic and predictable heuristic ways of thinking. Dual process theory hypothesizes that human thinking is divided into two different modes of processing. One mode, called System 1, is fast and linked to intuition, and the other, called System 2, is slow and linked to reasoning (Evans, 2008; Stanovich & West, 2000). Within dual process theory, System 1 thinking provides a credible system for explaining why people use heuristic thinking (Kahneman & Frederick, 2002). The recent revision to the heuristics and biases framework is focused on three heuristics, representativeness, conjunction fallacy, and availability. These three heuristics are believed to share the same mental process identified by Kahneman and Fredrick (2002), as the *attribute substitution process*.

The clinical task based interview method was used in this study. This technique allowed the researcher to better observe and interact with the participants while exploring the students' probability thinking. The researcher also used think-aloud protocols to better reveal the organic thinking patterns of the students in real time (Ericsson & Simon, 1980; Fox, Ericsson, & Bets, 2010; Van Someren, Barnard, & Sandberg, 1994). The data from the interviews were analyzed using the constant comparison method (Glaser, 1965). This analysis revealed three categories that were combined with other analyses to create profiles for various thinking patterns observed by the researcher.

The researcher identified patterns of thinking by students that were consistent with System 1 thinking and associated with the attribute substitution process (Kahneman & Fredrick, 2002). There were also situations in which students demonstrated ways of thinking consistent with System 2 thinking. However, unexpected ways of thinking were also identified by the researcher. For example, there were occasions when students substituted their fraction knowledge when solving probability problems and even seemed to equate probability with fractions. This type of thinking was referred to as the *content substitution process* in this study. This process occurred when students were using System 1 thinking as well as other types of thinking. In addition, the researcher observed students with thinking patterns that contained characteristics of both System 1 and System 2, which is referred to as *slow intuition* in this study. S*low intuition* seemed to

affect students' problem solving strategies as they wavered between multiple problem solving strategies that included either of the two substitution processes: attribute substitution and content substitution.

This study contributes to the body of knowledge related to probabilistic thinking. In particular, this study informs our understanding of heuristic thinking used by eighth-grade students when solving probability problems. Further, teaching practices that draw on Fischbein's (1975, 1987) general notion of intuition might be developed and used to improve probability reasoning skills. These teaching practices target students that depend on the attribute substitution process and/or the content substitution process. Each of these heuristic ways of thinking may require different instructional techniques to help students develop more sound ways of thinking about probability. Regardless, teachers need to be informed of the extent that some students rely on their fraction knowledge when solving probability problems.

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Chapter 1

Introduction

This descriptive qualitative study explored the way that heuristic thinking affected six eighth grade students' problem solving strategies when answering probability problems. The study used many of the same problems situations and analytic constructs as previous studies. However, this study was informed by recent revisions to the heuristic and biases theory to incorporate recent developments in the neuroscience community (Kahneman, 2003; Kahneman & Fredrick, 2002). In addition, this study used a smaller number of participants and interview methodology to allow for more direct exploration into student thinking compared to many of the past studies that relied on indirect measures, such as multiple-choice questions to accommodate the larger number of participants (e.g. Cox & Mouw, 1992; Fischbein & Schnarch, 1997; Fischbein & Gazit, 1996; Kahneman & Tversky, 1971; Kahneman & Tversky, 1974; Tversky & Kahneman, 1983). The goal of this study is to examine students' problem solving processes and their thinking processes when solving probability problems. In particular, the students were asked to solve problems associated with a particular group of misconceptions linked to the revised heuristics and biases framework. In turn, the findings contribute to the existing literature on middle school students' conception of probability and inform teaching practices associated with students' probability learning.

Existing Research

In order to address teaching and learning, it is important to understand how people think. Prior research exploring students' probabilistic thinking was typically segregated into these familiar categories: students understanding of probability, their cognitive development, and teaching practices –particularly focused on misconceptions. There are many research studies focused on students conceptual understanding associated with a particular probability concept, such as contingency tables, randomness, or sample space or in general that related to instruction (e.g. Batanero & Serrano, 1999; Chernoff & Zazkis, 2011; Groth, 2006; Hansen, McCann, Myers, 1985; Hawkins & Kapadia, 1984). Other researchers have focused on children's implicit cognitive processes and their notions about probability that they bring to the classroom (e.g. Fischbein, 1975; Fischbein 1987). Many other research studies were longitudinal so as to map out students' developmental capabilities associated with particular probability concepts such as randomness and single probability events (e.g. Fischbein, et al., 1970a; Hoemann & Ross, 1971; Jones, et al., 1999; Piaget & Inhelder, 1975). Still, other researchers examined students' misconceptions about probability that compelled the researchers to offer instructional interventions to address the misconceptions or gaps in understanding (e.g. Cox & Mouw, 1992; Fischbein & Gazit, 1984; Fischbein & Schnarch, 1997; Flores, 2006; Shaughnessy, 1981; Wilkins & Reese, 2003). Many of the interventions recommended experiments, hands-on mathematics activities, technology based activities, or direct instruction to address the misconceptions or to promote conceptual understanding (e.g. Fischbein, et al., 1970b; Flores, 2006; Shaughnessy, 1971; Wilkins & Reese, 2003). The current study takes a different perspective, one that does not focus on mastering skills associated with a particular stage of development, any particular probability concepts, or promoting a particular intervention. Instead, it focused on students' misconceptions associated with the heuristics and biases framework and how students' conceptions manifest themselves in situations of uncertainty.

Heuristics and biases is an area of research that is a closely related to research associated with judgment theory. Judgment heuristics, particularly, judgment under uncertainty, tries to identify and explain correct and incorrect judgments people make that is based on the same mental process. Heuristics and biases research took a slightly different perspective by attempting to identify and explain correct and incorrect responses associated with probability problems based on different mental processes (Kahneman, et. al. 1982). To understand how heuristics and biases frame this study, it is important to understand the underpinnings of judgment theory.

Kahneman et al. (1982) explain that judgment research was based on three areas of research that originated in the 1950's and the 1960's. The first area, initiated by Paul Meehl, was research that compared the accuracy between clinical predictions (experts) with statistical calculations (cited by Kahneman et al., 1882). The evidence showed that statistical predictions were superior to the intuitive judgments of experts (cited by Kahneman et al., 1982).

The second area of research was based on Bayesian statistics that was introduced to the psychology community by Ward Edwards that provided an ideal model to compare human judgments to statistical norms (cited by Kahneman, et al., 1982). This research revealed the biases that are prone to inductive inferences and the ways to offset them (cited by Kahneman, et al., 1982). This was a major part of the judgments under uncertainty research and it was met with criticism (Hertwig & Gigerenzer, 1999; Gigerenzer, 2005). An outspoken critic and experimental psychologist, Gigerenzer (2005), studied human errors and he argued that cognitive errors viewed in terms of logical thinking was a narrow focus because attention is focused on performance or accuracy and not on how the mind works. Gigerenzer (2005) claimed that "good errors" (p.197) allow a researcher to see into the mind and analyze the mind in a natural and organic state. Good errors are defined as "indispensable and functional" and foster growth in knowledge (p. 196). They are prominent among children, and Gigerenzer (2005) gives the following example to demonstrate a good error. A three year old child used the phrase, 'I gived,' instead of 'I gave,' which is incorrect English, but no one expects the small child to know which verbs are regular and which ones are irregular. This type of error shows the child's effort to extend beyond what he already knows about verb tense. The characteristic of a good error is an attempt to reach "a goal more quickly, or attaining it at all" (Gigerenzer, 2005, p. 196). Essentially, Gigerenzer (2005) believes it is better to try and fail than to not try at all; growth comes from errors.

The third area of research was cognitive psychology, which had a major impact on judgment research. It included strategies in thinking that was explored by Jerome Brunner and heuristic reasoning and bounded rationality that was explored by Herbert A. Simon (cited by Kahneman, et al., 1982). Both of these research camps were interested in the way people reduced the complexity of tasks so they could be addressed or answered (cited by Kahneman, et al. (1982).

The heuristics and biases framework helped to situate this current study to shed light on the various probabilistic ways middle school students think about probability without making judgments about the student's cognitive development, probabilistic inabilities, or correlating their answers to correct responses or expected norms. In this regard, the researcher shared Gigerenzer's (2005) position about not judging children's thinking errors, which was the original position of the heuristics and biases camp, which also saw value in exploring errors in human thinking.

Studies based on heuristic judgment and the heuristics and biases framework that focused on the individuals thinking without judging, have been done in the past. However, this researcher suggests, to increase the effectiveness of unpacking individuals' thinking patterns, it requires a shift in methodology; a shift to qualitative methods. Quantitative studies were used more frequently in the 1970's when Amos Tversky and Daniel Kahneman began their work. The heuristics and biases framework was built on the results that came out of these quantitative studies that used instruments that were developed to identify heuristic thinking for judgments under uncertainty. For example, Tversky and Kahneman (1974, 1983) developed problems that addressed three, of several, types of heuristics: availability, representativeness, and anchoring. The instruments used multiple choice answers to facilitate large numbers of participants and were analyzed using quantitative methods. Kahneman and Tversky (1996) claimed that between-subjects were more appropriate ways to study biases associated with heuristics versus Hertwig's preference for using within-subjects design, also called repeated measures when exploring individual's using the conjunctive effect heuristic. For example, in an "adversarial collaboration" (p. 269), Kahneman and Hertwig joined forces to evaluate the conjunctive effect using the frequency format in three experiments (Mellers, et al., 2001). This collaboration pitted the within-subjects design, preferred by Hertwig, against the between-subjects design, preferred by Kahneman. In the end, the two camps agreed on some points such as that filler items in the problem statement increases conjunctive effects, but disagree on the way individuals interpret "and" as a union operation because of the design methods (Mellers, et al., 2001).

Gigerenzer's (1999) preferred research focusing on cognitive processes instead of correlating the person's thinking to statistical norms, which was popular with the heuristic and biases research. Several other researchers also held this view and focused on cognitive processes on a micro level. For example, Konold (1989, 1991) used qualitative methods to explore the way students' think when addressing probability problems. He claimed that students' were attempting to predict the outcome of a probability experiment instead of using probability expectations to determine the most likely outcome. He called this way of thinking the *outcome approach* and suggested that students using the representativeness heuristic may be attempting to predict an outcome (Konold, 1993).

This current study differs from Konold's (1989, 1991) and other researchers' work because it was based on the recent revisions or modernization of the heuristics and biases framework originally developed by Amos Tversky and Daniel Kahneman (Kahneman & Fredrick, 2002). Recently, Kahneman and Fredrick (2002), informed by dual process theory, hypothesized that the dual mode of thinking may explain people's inclination towards heuristic thinking (Kahneman & Fredrick, 2002). In particular, System 1 thinking would be responsible for heuristic thinking because it was linked to intuition, a way of thinking that occurred quickly and associative in nature (Evans, 2008; Kahneman & Fredrick, 2002; Stanovich & West, 2000). System 1 was associated with subjective probability that had roots in the Bayesian statistical model that was used in the judgments research framework. The other mode labeled System 2 was linked to reasoning that occurred more slowly and effortful and it was linked to correct answers based on statistical norms (Broccas & Carrillo, 2012; Evans, 2008; Schneider & Chen, 2003; Schneider & Schifrin, 1977; Stanovich & West, 2000). The revised framework of heuristics and biases was based on three heuristics –availability, representativeness, and the conjunctive effect- in which individuals draw on the same heuristic thinking pattern referred to as the *attribute substitution process* (Kahneman & Fredrick, 2002). The attribute substitution process explains how the errors in thinking occur when an attribute in a problem is inadvertently switched out for another attribute that is more easily assessable.

Significance of This Study

It is important for adults to understand and critique data so they can be involved and active citizens in the 21st century (Gal, 2005; Paulos, 1988; Steen, 2001; Steen, 1997; Wilkins, 2000). Today, more and more information is presented in numeric or graph formats in the media. This requires new knowledge and skills to read and understand information about our communities, our nation, and our world. The mathematics education community responded by putting their research into practice.

In the *Handbook of Research on Mathematics Teaching and Learning*, Shaughnessy (1992) reviewed the research addressing students understanding about probability and statistics and he included the research on heuristics and biases. He noted that much of the research focusing on the ways people think came from psychologists and the research attempting to influence the ways people think came primarily from mathematics educators. Although, mathematics educators are aware that they need to understand students' thinking and prior knowledge before they can attempt to influence their erroneous thinking patterns (Shaughnessy, 1992). He recommended that future research be performed collaboratively between mathematics educators *and* cognitive psychologists to more effectively understand how students think and how best to develop instructional strategies (Shaughnessy, 1992). The spirit of the current research project is aligned with this notion, that mathematics educators need to have a full understanding of the ways student think when solving probability problems before the development of teaching strategies.

Responding to the importance of quantitative literacy and drawing on mathematics educational research, the National Council of Teachers of Mathematics (NCTM) produced three documents in the late 1980's to the mid 1990's (cited in NCTM, 2000). In these three documents, the NCTM attempted to articulate their goals for both policy makers and teachers (NCTM, 2000). The three documents included the *Curriculum and Evaluation Standards* (1989), *Professional Standards for Teaching Mathematics*, (1991), and *Assessment Standards for School*

Mathematics (1995) (cited in NCTM, 2000). The NCTM recognized that improving mathematics education was on ongoing process, which required revising former standards documents (cited in NCTM, 2000). In 1995 they initiated the Goals 2000 project, that culminated in the *Principle and Standards for School Mathematics* (NCTM, 2000) that includes probability and data analysis content strand for all grades, K-12 that is grounded in research (cited in NCTM, 2000).

As Shaughnessy (1992) suspected with the new NCTM document, *Principle and Standards for School Mathematics* (NCTM, 2000), research addressing students probability and statistics knowledge and skills increased. In the *Second Handbook of Research on Mathematics Teaching and Learning* (Lester, 2007), probability and statistics was split into two chapters. Shaughnessy (2007) addressed the recent research on statistical understanding and teaching practices, while Jones, Langrall, and Mooney (2007) focused attention on probability understanding and classroom practices. During this same time period, more research emerged that was focused on probability and statistics. In particular, NCTM published their sixty-eighth yearbook that focused solely on research that addressed probability and statistics in education (Burrill & Elliot, 2006).

The mathematics education community came together to strengthen the educational experiences for students in K-12 that included a focus on probability and statistics. The research community informed current teaching practices and current curriculum development. The graduating high school student will be armed with the skills to understand, interpret, and critique probability and statistical information. These young adults are positioned to understand the world around them and take their place as active and involved citizens. This research study aims to support the next generation of students by informing new teaching practices to better prepare students as they enter and take an active role as adults.

Rationale for This Study

The current study is important because it focuses on the ways students think when solving probability problems that build on the current knowledge about the ways eighth-grade students think when solving probability problems. In turn, this can inform teaching practices. The study builds on the hypotheses and conceptual constructs from past research but aims to gain more direct information on students' problem solving processes and thinking processes in general when answering probability problems. Jones, et al. (2007) recommended that future research should "focus on subjective probability and the way that students conceptualize it" (p. 947). This study attempts to do this by using the revised heuristics and biases framework that includes dual mode processing. As a result, the findings may provide additional insights for probability teaching practices that may be missed when attention is placed on a particular interpretation of probability, a particular misconception, or the use of traditional assessments that are intended for a large number of participants.

Research Questions and Methods for This Study

Like the previous work by Daniel Kahneman and Amos Tversky, this study attempts to explore heuristic thinking. The revised heuristic and biases framework and direct contact with the participants permitted the following research questions to be addressed:

1. In what ways are students' problem solving strategies influenced by System 1 or System 2?
2. How are students' problem solving strategies perturbed from System 1 to System 2?
3. How stable or strong are students' problem solving strategies associated with System 1 or System 2?

In order to answer these research questions, the study explored the thinking patterns of six eighth grade students during late Fall 2012 to early Spring 2013. The study was located in a rural area in one of the mid-Atlantic states of the United States. Initially 67 students from three eighth grade classrooms -Math 8, Algebra I, and Geometry- were asked to participate in the study. The study had three requirements for the participants: (1) they understand probability at their grade level, (2) they have reached the formal cognitive stage of development (Piaget, 1968), and (3) they are prone to use heuristic thinking some of the time. To address these needs, a purposeful selection (Patton, 1990) process was used in this study that had two stages. The first stage used the Probability Content Knowledge Instrument (Appendix A) to assess students' probability knowledge. The first interview, using the Cognitive Development and Heuristic Thinking instrument (Appendix B), assessed students' cognitive stage of development, as informed by Piaget and Inhelder (1975), on a cursory level, and it assessed students' inclination to use heuristic thinking. The second interview explored the students' probabilistic ways of thinking in greater detail using the Perturbation Activities Instrument (Appendix C). Both interview sessions used the clinical task-based interview method (Goldin, 1997; Koichu & Harel, 2007) coupled with the think-aloud verbal response protocol (Ericsson & Simon, 1980; Fox, Ericsson, & Bets, 2010). The think-aloud protocol was used to overcome the potential issue Fischbein and Gazit (1984) identified when they were exploring upper elementary and middle school student's misconceptions. They claimed that students were inclined to revise their initial thought process when reflecting back on what they had done. The think-aloud protocol provided a way to reduce self-reflection in order to produce more accurate data by promoting a stream of consciousness form of self-reporting. This particular kind of interview and verbal response protocol positioned the research team to increase the possibility of witnessing the students' organic or natural probability thinking. The data collected included the transcribed interviews, the students' jottings when they were answering the probability problems, the researchers' notes as well as the answers from the three instruments.

Organization of This Study

Chapter 2 provides a review of the literature and outlines the theoretical framework used in this study. The first section describes alternative ways to interpret probability: classical (theoretical), relative frequency (experimental), subjective, and a priori, in which the first three were of primary importance because of their prominence in most K-12 mathematics curricula in the United States. The cornerstone for this project, heuristics and biases, is discussed next. This is followed by a discussion of dual process theory that helped to differentiate this study from other studies by focusing on a select group of heuristics that evoke the same thinking process. Next, children's cognitive development is explained in light of probability and statistical thinking, which informed the methods used in this study. These components formed the foundation and theoretical framework for the study.

Chapter 3 describes the methods for this study. Securing participants is addressed first with a description and reasons for implementing a purposive selective process. The participant pool and the two stage selection process are described to show how the participants were filtered to the final six participants used in the study. The chapter describes the data collection process based on the task based interview and the think aloud protocol. The last section describes the data analysis process based on the constant comparison method (Glaser, 1965).

Chapter 4 shares the findings that are organized into three sections. The first section shows the results of the participant selection process. It also presents the initial data collected on the students –response times and answers- that were used in the initial analysis. The second section explains the data coding process that was used to create profiles of the various thinking patterns that students used when solving the probability problems. The last section shares the findings of the analysis with discussion that describes and documents students' various ways of thinking and shifts in their thinking patterns.

Chapter 5 is divided into four sections. The first section answers the research questions in addition to drawing conclusions associated with the expected and the unexpected findings. This is followed by a discussion of the limitations identified in this study. The next section shares the implications of the study for mathematics teachers, mathematics education and psychology researchers, and the mathematics education community in general. The last section offers recommendations for future research on the ways eighth grade students make sense of probability.

Chapter Two

Literature Review

This chapter reviews the literature that laid the foundation for this study. The first section describes alternative ways to interpret probability: classical (theoretical), relative frequency (experimental), subjective, and a priori. These interpretations have different philosophical roots as well as different ways to calculate probability. The cornerstone for this study was the heuristics and biases theoretical framework and dual process theory, which is discussed in the next sections. Dual process theory was used to revise the heuristics and biases framework (Kahneman & Fredrick, 2002) that explained the expected errors associated with heuristic thinking. The revised heuristics and biases framework was used to ground the current study to explore the ways 8th-grade students think about probability. This required that the students' cognitive development for probability and statistical thinking had reached a certain level of cognitive development, which informed the purposeful selection process (Patton, 1990) used in this study.

Probability Interpretations

The interpretations of probability draw from an epistemological perspective and a physical perspective, where both were used when defining or interpreting probability (Hacking, 1971; Hájek, 2001; Weatherford, 1982). These two perspectives were used by the early mathematicians forming this new area of mathematics that included notable mathematicians such as LaPlace, Bernouilli, and Cournot (Hacking, 1971). During this time when this new area of mathematics was being developed, the meaning and usage of the word probability also began to change (Hacking, 1971). In the 1500's the word probability was used to assess theological disagreements (Hacking, 1971). By the early part of the seventeenth century, the word *probability* was used to investigate new ideas or concepts in terms of being possible (Hacking, 1971). By the middle of the century probability was quantified (Hacking, 1971). That is, probability was influenced by the changing meaning and usage of the word *possibility* that became the basis for the idea of equal probability that surfaced by mid-century (Hacking, 1971).

The epistemological interpretation of "probability expresses a relation between a hypothesis *a* and some evidence *b* [which is a] function of [the] logical characteristics of the propositions *a* and *b*" (Hacking, 1971, p. 340). This means that probability reflected the amount of support the hypothesis received from the evidence, which could be influenced by someone's belief in the evidence, or more simply put, probability is concerned with the "relations between propositions and beliefs" (Hacking, 1971, p. 342). This was based on a person's confidence, which can change over time and confidence levels differs from person to person (Hacking, 1971).

The physical interpretation of probability was based on trials, which described "the state of things in the world …[drawing]…on limits of sequences" (Hacking, 1971, p. 342). This interpretation had its own issues because probability quantities differed based on the length of the sequences used.

Before the different perspectives are discussed, it is necessary at this point to address the rules that were developed to quantify probabilities. The rules or axioms of probability [see Table 2.1] include: (1) probabilities are non-negative values that lie between and includes 0 and 1; (2) every certain event or tautology has a probability of 1; (3) the disjunction rule, or the addition law for mutually exclusive events includes events that are either independent or dependent; (4) the conjunction rule, or the multiplication law of events includes events that are either independent or dependent (Weatherford, 1982; Greenland, 1998). Rules three and four are used to find the probabilities of compound events. These rules are revisited when discussing the different interpretations or perspectives of probability.

Table 2.1
Probability Rules

1.	$0 \leq$ Probability ≤ 1; 0 the event cannot occur
2.	Probability = 1, tautology (sum of all events) or one certain event
3.	P(A or B) = P(A) + P(B) for mutually exclusive events P(A or B) = P(A) + P(B) – P(A and B) when P(A and B) = 0
4.	P(A and B) = P(A) * P(B) for independent events P(A and B) = P(A) * P(B|A) for dependent events

Probability has been classified in many different ways. Weatherford (1982) categorized probability into four ways and this study focused on three of them. The four interpretations of probability include: (1) the classical; (2) the *A priori* (an epistemological interpretation, also identified by Hacking [1971] that is also called logical probability because it states a logical relationship between a hypothesis and the evidence); (3) the relative frequency; and (4) the subjective (which also has an epistemological interpretation [Weatherford, 1982]). In this study, there were three interpretations that were relevant: the classical, relative frequency and the subjective interpretations because they are tied to the K-12 NCTM mathematics education curriculum (NCTM, 1989; 2000).

Classical Interpretation

Based on the *classical approach* probabilities are defined as the number of favorable cases out of the total number of possible cases. This interpretation is interesting due to its historical roots that came from games of chance and the way probability was used mathematically as a model to address uncertainty. The development of probability as a part of mathematics included some of the famous early European mathematicians, such as Jacob Bernoulli, his brother Johann, Pierre Simon de Laplace, and Gottfried Leibniz (Roncaglia, 2009;

Weatherford, 1982). These early European mathematicians developed the calculations for probability based on their shared notions about uncertainty that was informed by games of chance and the actuary tables (Roncaglia, 2009; Weatherford, 1982). This perspective of probability placed it outside the realm of a metaphysical reality which means it is a human invention that helps people deal with the world and is not associated with supernatural beings (Weatherford, 1982).

This early definition of probability was used in many modern mathematics texts, such as Marks (1964) definition, "probability - The ratio of the number of ways in which an event can occur in a specified form to the total number of ways in which the event can occur" (as cited in Weatherford, 1982, p. 74), which is a part-whole interpretation. For example, the probability of .8 can be rewritten as 8/10, where we have 8 parts out of the whole, which are 10. Uvarov, Chapman and Isaacs (1964) defines mathematical probability as follows, "if an event can happen in *a* ways and fail in *b* ways, and, except for the numerical difference between *a* and *b,* it is likely to happen to fail, the mathematical probability of its happening is $a/(a+b)$ and of its failing, $b/(a+b)$" (as cited in Weatherford, 1982, p. 74). This definition of probability is basically looking at the events as if they were odds and showing how it can be translated to probabilities. For example, the odds that the home team will win the game is 3:5, the probability is 3/(3+5) or 3/8. Likewise, the odds that the home team will lose the game is 5:3, the probability is 5/(5+3) or 5/8. Compare these last two definitions to the definition given by Jacob Bernoulli in his *Ars conjectandi,* published after his death in 1705, which is fundamentally, "the number of favorable cases over the number of possible cases" (as cited in Roncaglia, 2009, p. 490), or to the definition given by Laplace as cited by Hájek (2001):

> The theory of chance consists [of] reducing all the events of the same kind to a certain number of cases equally possible, that is to say, … we may be equally undecided … in regard to their existence, and in determining the number of cases favorable to the event whose probability is sought. The ratio of this number to that of all the cases possible is the measure of this probability, which is thus simply a fraction whose numerator is the number of favorable cases and whose denominator is the number of all the cases possible. (pp. 6–7)

The classical definition of probability follows from a part-whole reasoning, which is useful but there is one characteristic that has been debated for centuries, the *Principle of Indifference*. This principle is also known as *equal probable*, *equal possible*, or *equally likelihood* of an event (Weatherford, 1982; Roncaglia, 2009). This principle takes the sample space for all possible outcomes and divides it into equal partitions. This premise seemed to set up circular reasoning when defining the probability of an event because it uses the term equally likely outcome which reflected the notion of probability (Magidor, 2003). Another issue connected with this interpretation of probability is the probability values for all possible outcomes are given in advance, which means there is no way to adjust the probabilities when experiments are conducted (Weatherford, 1982). For example, if a person did not know the probability of an event, then assuming each outcome has the same chance of occurring seems reasonable, which meant the initial probability value was known before the number of favorable events occurred. This interpretation is limited because it failed to support irrational probabilities, which are found in quantum physics and mathematics problems (Weatherford, 1982).

Relative Frequency Interpretation

The *relative frequency* interpretation of probability is tied to empirical data and it is inductive in nature because there is no formula that produces the same result each time. This means, probability is linked to a physical possibility that changes from experiment to experiment (Magidor, 2003). The basic ideas surrounding this interpretation are as follows: (1) the probability is an empirical and objective measurement about the real world, (2) the probability value is an objective property like weight or height, and (3) all probabilities can only be known after the fact (Weatherford, 1982). It is this last idea that placed probability on the same level with physical science as a way to gain useful and reliable knowledge about the world (Weatherford, 1982). The relative frequency definition for probability means the probability of an event is the ratio of the frequency for the successful event to the limiting number of identical and independent trials (Roncaglia, 2009). The term "limiting number" refers to an infinite number of trials, in which the relative frequency of the given event became more stable and collected around a value as the number of trials increased, which is the law of large numbers (Roncaglia, 2009). For example, when a balanced coin is flipped 500 times, the number of heads that appears divided by the 500 flips of the coin represents the probability of obtaining a head. As the number of trials increases, then the relative frequency for the event of flipping heads will eventually stabilize, reach equilibrium, or reach the limit of the "true" probability value of one-half.

There is one major advantage of the relative frequency approach and three criticisms identified by Weatherford (1982). The advantage is its scientific approach because it relies on empirical data to establish initial probabilities (Weatherford, 1982). In fact, the probabilities tend to be connected with the objects themselves so the probabilities can be discovered by using experiments like the sciences (Weatherford, 1982). The three criticisms include: the notion of the real or true relative frequencies probabilities can never be known, known to exist, nor the value confirmed or disconfirmed (Weatherford, 1982).

Eberhardt and Glymour (2011) chronicled Hans Reichenbach's philosophical work with probability. He used the birth of new stars to demonstrate the idea that probabilities cannot be confirmed. His example goes as follows.

Consider the probability that a large cloud of gas and dust in space will give birth to a new star. The size of the cloud is measured to find out when it begins to shrink since shrinkage indicates the cloud is beginning to form its own gravity. As the shrinkage continues the cloud became warmer to supports the birth of a new star. Now, suppose measurements were taken from 10,000 cloud clusters in the universe and the data showed that out of the 10,000 clusters, 1,000 of these clusters created a new star. This is a probability of .10. However, if observations were taken for the next 10,000 clusters and the new data showed 1,500 clusters birthed new stars, then the probability is .15. This means the first data collected compared to the second data collected showed a 50% change in the probabilities. This would be interpreted as an error of 50%. This is unacceptable, since there is no measurement error associated with the method or the instrument (Eberhardt & Glymour, 2011).

Many opponents dismiss the relative frequency interpretation of probability because the probability values are theoretically dependent on the infinite sequences of events (Roncaglia, 2009). In reality, the sequences are finite and they can have different results for the same situation as noted with the example above (Roncaglia, 2009). The relative frequentist proponents claim the series will converged, but this may not happen, as mathematically not all series converge; some diverge, which suggests the relative frequency interpretation of probability cannot be confirmed (Weatherford, 1982).

Likewise, Hájek (2001) explained the issue of the single case which was problematic to any variation of the relative frequency interpretation, when considering the appropriate class from which to collect the sequences of events. The following example, informed by Hájek (2001) clarifies this shortcoming. Suppose I want to know the probability that I will develop heart disease. Should I be in a group with people who are or were smokers? Since I am not a smoker or ever have smoked, I will eliminate the smokers from my group. However, should I be in a group that includes men? I remove the men from my group. I have reduced my group membership to include female who are non-smokers. However, should I be in a group that includes females with high cholesterol, since I do not? I can reduce my group size further to include only non-smoking females with normal levels of cholesterol. Should I be in a group that includes non-smoking normal cholesterol females that have a family history of heart disease, since I do not? Eventually, this line of reasoning reduces my target group to only one member - me.

The relative frequency interpretation is problematic for single event probabilities. The classical interpretation handles this by assigning a probability value to every possible outcome in the sample space. For example, defining the experiment as tossing one die once, then the probability of rolling the number two can be calculated from the sample space. There is only one outcome out of a total of six that can show two on the die. Thus, the probability is 1/6. However, with the relative frequency interpretation of probability a single event makes no sense because as von Mises said, [The relative frequency]…can only be applied to collectives (Roncaglia, 2009, p. 492). If a single event is defined as a finite series, then the probabilities could not be implied.

Roncaglia (2009) gave the following example, one coin is tossed 10 times and each time a head appeared. If we repeated the experiment but changed the number of tosses for that one coin to 1000 times, we cannot be certain that the next toss would result in one head appearing. This notion of choosing an appropriate group or class from which to collect the data opens the door to many interpretations in order to identify the appropriate group that can quickly diminish to only a single member group.

Subjective Interpretation

The *subjective* interpretation of probability involves "personal degrees of belief, based on personal judgment and information about experiences related to a given outcome" (Batanero, Henry, & Parzysz, 2005, p. 24). This interpretation is linked to a psychological notion about possibility. Tversky and Kahneman (1974) drew heavily on this interpretation in their heuristics and biases research. Likewise, De Finetti (1974), another proponent of this interpretation, claims probability does not exist, it is an abstract concept that lacks objectivity because he believes the numerical value of a probability measure depends on many factors such as, the observer's knowledge, the context of the observation, and the availability of data to be collected (as cited in Batanero, Henry, & Parzysz, 2005). Sylla (1998) explained, that at first blush, it looks like Bernoulli had a subjective view on probability because he also believed there was "no 'objective probability' because for him 'probable' was a moral term, applied to human statements and behavior, not to the physical world" (p. 55). It is important to point out that for Bernoulli, subjective probability was rational, which differs from how the term is typically used today (Sylla, 1998). Today, subjective probability reflects a personal belief that may or may not be rational (Sylla, 1998).

The subjective interpretation was simple and uncomplicated as it provided an easy way to address initial and single event probabilities (Weatherford, 1982). Although the approach appears primitive, it requires the person to be coherent with their beliefs, feelings, and it requires the person be willing to act on their beliefs (Weatherford, 1982). Subjective probability is found several disciplines that included Bayesian statistics, experimental psychology, and decision theory that includes heuristics and biases research. Subjective probability is tied to heuristic and biases research because Kahneman and Tversky (1979) found that many times participants would make decisions under uncertainty based on their emotions. In particular, they studied people's decision making processes when confronted with economic decisions (Kahneman & Tversky, 1979). Yet, determining how to measure degrees of belief or a person's intensity of feelings for the subjective interpretation of probability is challenging (Weatherford, 1982).

There were three benefits associated with the subjective interpretation of probability: (1) it accommodates a broader range of events to be considered, such as individual or unique events; (2) it opens the door for more uses of the Bayes' Theorem; and (3) it contributes to decision theory by showing people's behaviors are more heuristic in nature (Weatherford, 1982). The major criticism is the theory is not appropriate to probability, but it is appropriate to study human behavior (Weatherford, 1982).

In brief, the classical interpretation considers each of the outcomes in the sample space as equally likely. The relative frequency uses empirical data to calculate the probability of an event. The subjective probability is based on personal judgment, emotions, or beliefs. These three interpretations: classical, relative frequency, and subjective are compared in Table 2.2 below.

Table 2.2

Comparing Three Interpretations of Probability

Probability Interpretation	Definition	Strengths	Weakness
Classical	The number of favorable cases over the number of possible cases	Easy to calculate a probability estimate	Equally likeliness for each outcome
Relative Frequency	The ratio of the relative frequency of an event to the limiting number of identical and independent trials	It is based on experiments	Infinite number of trials required for it to converge on the true probability
Subjective	The person's judgment about a given outcome	It allows researchers to study people's behavior that is informed by their beliefs	It is not appropriate to study probability

Note: Table created by author to summarize discussion points

Mathematics Curriculum

The classical interpretation and the relative frequency interpretation are the dominate interpretations of probability endorsed by the National Council of Teachers of Mathematics (NCTM) K-12 curriculum guide, *Principles and Standards for School Mathematics* (2000; also see 1989). The classical interpretation is referred to as theoretical probability and the relative frequency interpretation is referred to as experimental probability (NCTM, 2000; also see 1989). The subjective probability interpretation is used minimally in the earlier grades as a way to draw students into probability discussions. Teachers ask the students what they believe is the likelihood of a particular event occurring. For example, the NCTM (2000; also see 1989) suggests that teachers ask their students if it will snow. Based on the location of the classroom and the time of year, the students discuss seasons, weather, and the conditions required for snow.

The middle school content strand for data analysis and probability proposed by the NCTM (2000; also see 1989) focused on students developing ideas about chance events. To meet this objective, the curriculum uses the relative frequency interpretation by having the students engage in simulations and experiments. It also includes simple circumstances and advanced to inferences about the likelihood of an outcome that were evaluated with frequency histograms or tree diagrams that were created from a simulation or experiment (NCTM, 2000; also see 1989).

This summary of the mathematics middle school curriculum highlights the dominate probability interpretations deemed appropriate for the middle school students to explore and learn.

Heuristics and Biases

There are times when using heuristics to reduce complex tasks, such as calculating probabilities and predicting outcomes, is beneficial but these tactics can lead to systemic and serious errors (Kahneman, 2003). The person that uses a heuristic and answers incorrectly produces a gap between the answer the person *should* have stated with the answer the person *actually* stated; this gap is called the bias. The exploration of systematic errors was the work of Amos Tversky and Daniel Kahneman before Amos Tversky's untimely death (Kahneman & Fredrick, 2002). Their heuristics and biases research occurred expectantly when they noted the erroneous ways of thinking held by statistical knowledgeable people (Kahneman, 2003; Tversky & Kahneman, 1971). The word heuristic tales on two meanings: as a noun it refers to a cognitive process and when it is used as an adjective, such as a heuristic attribute, it refers to a type of an attribute that was being replaced or substituted (Kahneman, 2003). This section includes a brief historical review of the heuristic and biases research. This is followed by a discussion that focuses on a specific group of heuristics that were linked to a particular mental process when the heuristics and biases framework was recently revised (Kahneman, 2003; Kahneman & Fredrick, 2002).

Early Research

One of Tversky and Kahneman's earliest studies presented two sampling size questions to 84 statistically savvy researchers who attended one of their two professional conferences (Tversky & Kahneman, 1971). They asked the participants the following question: "The mean IQ [for a] population of eighth graders in a city is known to be 100. You have selected a random sample of 50 children for a study of educational achievements. The first child tested has an IQ of 150. What do you expect the mean IQ to be for the whole sample" (Tversky & Kahneman, 1971, p. 106)? The correct response is 101, but Tversky and Kahneman (1971) found that most of the respondents, (no number of participants was given), believed the answer was 100 (Tversky & Kahneman, 1971). Tversky and Kahneman (1971) explained that the participants expected the sampling "to be a self-correcting process" (Tversky & Kahneman, 1971, p. 106). This means, that the remaining sample points would be lower to produce the average of 100. Tversky and Kahneman (1971) hypothesis that individuals believed the data elements in a sample would resemble the target population and that all samples were similar to each other. The dismal results from this experience inspired Tversky and Kahneman (1971) to explore this phenomena, so they focused their attention on intuitive judgments (Kahneman, 2003).

Recently, when Kahneman (2003) explains heuristic thinking he adopted the term accessibility that is used among memory researchers and social cognition psychologists. He uses this term because he believes the study of intuition needs a common concept to reach across different disciplines. The accessibility of a thought requires two components: one is the characteristics of the cognitive process that produces the thought and the second is the characteristics of the stimuli and events that induce it (Kahneman, 2003). Accessibility is connected to a context but Kahneman (2003) uses the word more broadly than the memory researchers and social cognition psychologists use the word (Kahneman, 2003). For Kahneman (2003) the term is linked to the "different aspects and elements of a situation, the different objects in a scene, and the different attributes of an object" (p. 699). The determining factors of accessibility include a "stimulus salience, selective attention, specific training, associative activation, and priming" (Kahneman, 2003, 699). Basically, it is a general term used to describe or measure the degree that an intuitive thought is easily brought to mind.

Types of Heuristics

There are several heuristics that the research team, Amos Tversky and Daniel Kahneman identified and explored since 1971. In this discussion three heuristics are identified and explained. The first is the representativeness heuristic in which biases are associated with statistical reasoning or stereotypes. The second heuristic is called the conjunction fallacy, in which people view compound events more probable than single events. The third heuristic is the availability heuristic in which people select the answer based on how easily it comes to mind. These three heuristics are described in the following paragraphs that include the characteristic error that these heuristics tend to produce (Kahneman, 2003).

Representativeness Heuristic. The representativeness heuristic is similar to the concept of perceptual similarity and tends to be used when events are described according to their general characteristics (Kahneman & Tversky, 1972). The representativeness heuristic is defined by Kahneman and Tversky (1972) as:

> Subjective probability of an event or a sample is determined by the degree to which it: (i) is similar to essential characteristics to its parent population and (ii) reflects the salient features of the process by which it is generated. (p. 430)

This heuristic came into play when people were confronted with the following types of questions: “What is the probability that object A belongs to class B? What is the probability that event A originates from process B? What is the probability that process B will generate event A” (Tversky & Kahneman, 1974, p. 1124; Tversky & Kahneman, 1982a, p. 6)? These questions address different situations such as associations, subsets, and random processes that generated an outcome, or vice versa. Fundamentally, people make judgments based on how similar event A is to event B. Tversky and Kahneman (1982a) explain that to be “representative, it was not sufficient that an uncertain event be similar to its parent population. The event should also reflect the properties of the certain process by which it [was] generated, that is, it should appear random” (p. 35). Randomness is viewed as unpredictable but fair, which means, people expect samples to maintain the same or close to the same proportions found in the population. As mentioned earlier, people usually think of randomness as a self-correcting process over time instead of the long sequence of repeated trials diluting a single outcome or pattern (Tversky & Kahneman, 1982a).

Kahneman and Tversky (1982b) contend that the representativeness heuristic is directional. This means, a person “evaluates the input [or]…predicts the outcome” (p. 57). An example of Tversky and Kahneman’s (1982b) a situation that may trigger a person to use representativeness heuristic is as follows: If a person is described as shy, meek, and timid, what would be that person’s occupation? a librarian, a doctor, a sales person or a farmer? Kahneman and Tversky (1972) found that a majority of the respondents chose the librarian as the most probable occupation based on the stereotype of a librarian in our society. Another example, of the a question that may trigger a person to use the representativeness heuristic is cited by Kahneman and Tversky (1972) that involves a maternity ward with five newborn baby boys and two newborn baby girls. The nurses believe the next baby to enter their ward will be a girl so that the number of baby boys and the number of baby girls begin to even out. The belief stemmed from the notion that the proportions in the sample must be equal or at least remain close to the proportions found in the population so that the sample represents the population. That is why the nursed expected the next birth to be girl so that their sample in their hospital ward will reflect the proportions for the gender population. Tversky and Kahneman (1982a) warned that this type of thinking ignored the variability associated with sample size. Smaller samples produced proportions or means that could vary a great deal from the population proportion or mean and larger samples were less susceptible to wide variations and tend to hover closer to the population proportion or mean.

Conjunction Fallacy. The conjunction fallacy, also called the conjunction effect, occurs when a person violates the conjunction law of probability. This means, the person believes the compound events is more probable than the single events included in the compound event. Recall the multiplication law for independent events, which states that two independent events when combined have a smaller probability than either of the single events. For example, the probability that event A occurs is 0.43 and the probability that event B occurs is 0.57. The probability for both events A and B, given they are independent events is 0.43*.57 or 0.245, which was less than either one of the single event probabilities. People tend to think that more information increases the probability or make it more likely. The additional information makes it more specific but it did not make it more likely.

Drawing on an example from Tversky and Kahneman (1982): “Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and she also participated in anti-nuclear demonstrations” (p. 297). Originally, the description was designed to suggest Linda was a feminist but not a bank teller (Tversky & Kahneman, 1983). This description was followed by eight statements, where they focused on the statement that aligned with the description given, *Linda is a feminist*, with the statement that was unlike her, *that she is a bank tell,* and the last statement that combined these two statements (Tversky & Kahneman, 1983). The remaining statements were considered distractors (Tversky & Kahneman, 1983). The participants in the study were asked to rank the statements from most likely to least likely: “Linda is a teacher in an elementary school, Linda works in a bookstore and takes Yoga classes, Linda is active in the feminist movement (F), Linda is a psychiatric social worker, Linda is a member of the League of Women Voters, Linda is a bank teller (T), Linda is an insurance sales person, Linda is a bank teller and is active in the feminist movement (T&F) (Tversky & Kahneman, 1983, p. 297). They observed and evaluated the rankings for three statements that were labeled F, T and T&F (Tversky & Kahneman, 1983). They found that most participants ranked the two single statements marked by (T) and (F) and the joined statement marked as (T&F) as follows: (F) was more likely than (T and F), which was more likely than statement (T) (Tversky & Kahneman, 1983). Examining this through the lens of logic, if the feminist statement is false (F) and the bank teller statement is true (T), the combination of the two statements is false based on the conjunction rule (Tversky & Kahneman, 1983). In terms of probability, given the probabilities for each of these independent events, the conjunction is the product. Thus, the joining of the two events (T&F) would be less likely, than either statement (T) or statement (F).

Hertwig and Gigerenzer (1999) challenged the explanation offered by Tversky and Kahneman (1983) that people fail to use the mathematics laws of probability to address this question. They believed that Tversky and Kahneman (1983) underestimated and inappropriately labeled rational responses as biased (Hertwig & Gigerenzer, 1999). They performed a series of experiments that showed college students held non-mathematical notions associated with the word, *probability*. They followed this up in another experiment that showed problems could be designed in such a way so people will use probability reasoning when they see the word probability (Hertwig & Gigerenzer, 1999). Finally, they also performed an experiment to show that using the word *frequency* reduced the conjunction rule violation (Hertwig & Gigerenzer, 1999). They contended that the wording of the problem could provoke heuristic thinking (Hertwig & Gigerenzer, 1999).

Availability Heuristic. The situation that triggers a person to use the availability heuristic is observed when the person attempts to find the probability for a specific event (Kahneman & Tversky, 1972). The person will estimate the probability in one of two ways: by assessing the availability of the event from their experiences or by associative distance (Tversky & Kahneman, 1971, 1973). In both cases, the researchers found that larger categories are recalled faster than smaller categories (Tversky & Kahneman, 1973). Likewise, incidents that are easy to imagine tend to be viewed as more likely to happen than incidents that are harder to imagine. For example, a person may assess the cancer rate in their community by recalling, or retrieving, the number of people they know who were diagnosed with cancer. Unlike the representativeness heuristic, people when they become aware they are using the availability heuristic, will correct their decision or at times over correct (Kahneman, 2003).

They also found that when two events frequently occurred together, these two events are more easily recalled (Tversky & Kahneman, 1973). This means people are estimating "the numerosity of a class, the likelihood of an event, or the frequency of co-occurrences by assessing the ease with which the relevant mental operation of retrieval, construction, or association can be carried out" (Tversky & Kahneman, 1973, p. 208).

Attribute Substitution Process

In 2002, Kahneman and Frederick revisited the original heuristics and biases research of Kahneman and Tversky from the 1970s. Originally, Tversky and Kahneman's (1974) research lacked an explicit definition for judgmental heuristics, but the definition was alluded to by such terms as "principles, as process, or as a source of cues for judgment" (Kahneman, 2003, p. 707) in many of their studies. Their concept was "invented to accommodate the observation that intuitive judgments of probability are mediated by attributes such as similarity and associative fluency, which are not intrinsically related to uncertainty," such as when one is in a life or death situation, in which only one solution quickly comes to mind (Kahneman, 2003, p. 701). Drawing on more recent research, Kahneman and Frederick (2002) identified three major changes to the earlier work: (1) they identified a process that was common to the representativeness, conjunction fallacy, and the availability heuristic, that they labeled *attribute substitution*, (2) they extended the concept of heuristic beyond the domain of judgment about uncertain events, and (3) they included an explicit treatment of the conditions under which intuitive judgments were modified or overridden by the monitoring operations associated with reasoning (Kahneman, 2003; Kahneman & Fredrick, 2002). In this current study the researcher focused on the first change, the attribute substitution process. The attribute substitution process occurs when the target attribute is switched out by another attribute, and the switch is successful when the target attribute is out of reach, or when it is similar to another attribute either semantically, or it is closely associated with it (Kahneman & Frederick, 2002).

Kahneman and Frederick (2002) identified three conditions for the attribution substitution process that is linked to intuition. The first condition has the target attribute out of reach of the person, but this does not apply to facts retrieved from memory because a substitution would not occur. The second condition has the attribute within easy reach of the person due to perception, the automatic evaluation process or it could be due priming. Kahneman and Frederick (2002) gave this example of priming. A person is thinking about his love life when he is asked to evaluate his level of happiness. He may answer that question based only on how happy he is with his love life and ignore other areas of his life. The last condition for the attribute substitution process is the person fails to notice that the substitution took place so that the reflective system does not correct it. Kahneman and Frederick (2002) give this example: a bat and ball together was $1.10, the bat alone was $1.00 more than the ball. The person is asked to find the price of the ball. Many people answer incorrectly and claim the ball costs $.10 because people fail to work out the solution. Instead they parse out the sum into two parts: $1.00 for the bat and $.10 for the ball. In this case the attribute substituted appeared to be the way the numbers were partitioned.

Summary

The *Heuristics and Biases* section opened with its historical synopsis. Three types of heuristics were identified and described: representativeness, conjunction fallacy, and availability. These three types of heuristics share the same generalized cognitive process. This generalized or generic heuristic process is called the *attribute substitution* process (Kahneman & Frederick, 2002). The substitution occurs when people, unknowingly, switched out a characteristic in the problem for another characteristic – the heuristic attribute – because it readily enters their mind. Basically, people understood the original question and thought they answered it correctly because the switch was plausible and consistent with the situation (Kahneman & Frederick, 2002).

Dual Process Theory

The dual process model caught the attention of researchers within several different disciplines such as, evolution psychologists (Comides & Toobey, 1997), cognitive psychologists (Evans, 2008), and behavioral economists (Kahneman & Frederick, 2002) as they explored the tensions that exist between intuition and rational logical thought. Originating within the information processing framework to explain or model cognition, dual process appeared to address an issue lacking from information processing that Fischbein (1997) believed needed to be addressed: *intuition*. Information processing was defined by Silver (1987) as:

> The cognitive activity …[that] consists of controlling the flow of information into and out of working memory by processes such as receiving information from the sensory buffer and retrieving information from LTM [long term memory]; recognizing, comparing, and manipulating symbols into working memory; and storing information in LTM (p. 39).

Fischbein (1997) explained that in a broad sense, this definition held because "every mental elaboration is based on processes of assimilation and storing of information for immediate and delayed use, and processes of retrieval, selection, coordination, and adaptation" (p. 31). However, in a narrow sense, the computer model paradigm with its focus on algorithms indicated that gained information could be retrieved, combined, and modified for a specific problem based on a specified set of rules (Fischbein, 1997). The issue with this more narrow view was the focus on algorithms because it represented all of the steps required to address a certain problem, and according to Fischbein (1997) this perspective not only ignored the place intuition played but it also failed to acknowledge that each time a cognitive activity was retrieved and used there was a potential for changes although very slight.

Dual process theory seeks to understand both types of cognitive processes: explicit and implicit (Schneider & Shiffrin, 1977; Schneider & Chen, 2003). The implicit cognitive process is associated with research focused on attention, (Schneider & Shiffrin, 1977; Schneider & Chen, 2003), decision making (Tversky & Kahneman, 1974; Kahneman, 2003), and reasoning (Evans, Barston, & Curtis-Holmes, 2005; Leron & Hazzan 2006) to name a few. Dual process theory addresses two cognitive processes or functions: one that operates automatically and one that operates under the deliberate control of the individual.

System 1 and System 2

The naming convention for each of these processes is not universal. The Dual Process community researchers continue to debate whether the dual cognitive processes are separate entities or one entity with two functions (Evans, 2011; Smith and DeCoster, 2000). The terms System 1 (S1) and System 2 (S2) was first coined by Stanovich and West (2000) and was accepted by other leaders in the dual process theory community such as Evans (2008). It was also adopted by the *Heuristics and Biases* community leaders, Kahneman and Frederick (2002) and used among mathematics educators (Leron, 2010). This study used the names, System 1 and System 2 for the two cognitive functions.

Recently, Evans (2011) decided to use new terms for these two functions, Type 1 and Type 2 because it places emphasis on the functional aspect of these two processes. The jury is still out to determine clearly if these two are separate systems, which the names System 1 and System 2 suggest, or one system with two different functions, which the labels Type I and Type

II highlight. Evans (2011) defines “Type 1 [as] fast, high capacity, independent of working memory and cognitive ability [and] Type 2 [as] slow, low capacity, heavily dependent on working memory and related to individual differences in cognitive ability” (p. 87). Type 2’s connection to central working memory includes *controlled attention,* in which Type I’s underlying process works below people’s awareness but people sense a feeling or an intuition and/or an emotion (Evans, 2011). Smith and DeCoster (2000) supported the two separate memory system based on recent psychological and neuropsychological research on memory systems. As two separate systems, each memory systems have its own acquisition, retention, and retrieval processes. In either case, attention plays a role in the ways these two systems or functions work and this is discussed later in this section.

Characteristics of System 1 and System 2

System 1 is the automatic processing system and System 2 is the controlled processing system (Stanovich & West, 2000). Schneider and Shiffrin (1997) found that the two systems are located in different parts of the brain, and suggest they have different evolutionary origins. Some researchers believe that System 2 is a relatively recent evolved system that is influenced by culture but System 1 is the original processing system (Leron & Hazzan, 2006, Comides & Toobey, 1997).

System 1 has the following characteristics: automatic, associative, effortless, rapid (parallel), process is implicit (Kahneman & Frederick, 2002) and it acts on content that is affective, has causal tendencies, either concrete or specific, and prototypes. System 1 holds implied information and associative information that is situated within a context, (Evans, Barston, & Curtis-Holmes, 2005; Kahneman, 2003) that is linked to intuition or beliefs (Kahneman, 2003; Leron, 2010; Evans, 2003).

The *Heuristics and Biases* camp identified three basic characteristics that distinguish one system from the other: "speed, controllability, and the contents on which they operate" (Kahneman & Frederick, 2002, p. 51). Later Kahneman (2003) included the terms "automatic, effortless, associative, implicit, emotionally charged, and governed by habit" (p.698). Informed by Stanovich and West (2002) System 1 is "contextualized, socialized, and personalized" (p. 436). The intuitive nature of System 1 is addressed in greater detail later in this section.

On the other hand, System 2's characteristics are: "controlled, effortful, deductive, slow (serial), self-aware, rule application and it acts on content that is neutral, statistics, abstract, and sets" (Kahneman & Frederick, 2002, p. 51).

Imaging studies found that System 1, the automatic process, is activated when a certain stimuli triggers a parallel sequence of neurons in the brain (Schneider & Shiffrin, 1977). The neuron sequence is a relatively permanent set of connections that requires extensive and consistent training for it to fully develop (Schneider & Chen, 2003). System 2 is home to abstraction and generalization and it processes at a slower rate compared to System 1 (Evans, Barston, & Curtis-Holmes, 2005; Leron & Hazzan 2006; Leron & Hazzan, 2009; Kahneman, 2003). Imaging studies found that System 2, the controlled process, requires the individual's awareness and it triggers a serial sequence of neurons in the brain (Schneider & Chen, 2003). The neuron sequence is a temporary set of connections that has a limited capacity, yet once it is established the sequence can be modified and applied to new situations (Schneider & Chen, 2003).

One of the key differences between System 1 and System 2 is speed (Kahneman, 2003; Evans, et al., 2005). Some researchers believe the two systems compete for control as they each responde to reasoning tasks (Evans, Barston, & Curtis-Holmes, 2005; Leron & Hazzan, 2006;

Leron & Hazzan, 2009) and believe System 2 should monitor the quick, and at times, incorrect responses coming from System 1 (Gilbert, 2002; Stanovich & West, 2002). To demonstrate the fast response coming from System 1, Evans, Barston, & Curtis-Holmes (2005) used syllogisms. In their experiment, they had the participants identify the syllogisms as valid or invalid after each was shown on a computer screen for 10 seconds. Evan, et al. (2005) found that at first the participants used reasoning to evaluate the syllogisms. When the time that the syllogism appeared on the computer screen was reduced, the participants began to experience increasing demand on their working memory. At a certain point, the participants one by one began to evaluate the syllogisms as valid or invalid based on its believability rather than its logical structure (Evans, Barston, & Curtis-Holmes, 2005). The participants relied on the quick response coming from System 1 when the time was reduced. This example aligns with Kahneman and Fredrick's (2002) attribute substitution process because believability is linked to former experiences that are easily accessible to the individual that were used to evaluate the syllogisms.

Intuition

System 1 is a way of thinking that holds implied information and associative information that is tied to a context, (Evans, Barston, & Curtis-Holmes, 2005; Kahneman, 2003). System 1 thinking is linked to intuitions, which is an idea, or a belief (Evan, 2003; Fischbein, 1987; Kahneman, 2003; Leron, 2010). Yet, the definition of intuition among psychologists, philosophers, researchers, and mathematics educators is broad and at times contradictory (See Fischbein, 1987). For example, evolutionary psychologists, Cosmides and Tooby (1997) believe intuitions form a type of decision making process that is effortless and quick. The quick response is believed to be essential to the preservation of the human species, but it is also prone to bias. A second example comes from the mathematics educators, Garfield and Ahlgren (1988) who insist that "students appear to have difficulties developing correct intuitions about fundamental ideas of probability" (p. 47). They define statistical intuition as a familiarity with the content based on a deep understanding and extensive experience. Likewise, the educational psychologists, Fischbein, Tirosh, and Melamed (1981) also focus on this same idea of a *feeling,* or "intuitive acceptance," which has two characteristics, "confidence and degree of obviousness" (p. 491).

The following paragraphs briefly discuss these three interpretations of intuition. The first is the interpretation used by the heuristics and biases psychologists Daniel Kahneman (Kahneman, 2003; Kahneman & Fredrick, 2002). This is followed by the interpretation used by the educational psychologist, Efrim Fischbein (Fischbein, 1975, 1987). The third interpretation of intuition is defined by a cognitive developmental psychologist, Jean Piaget (Piaget, 1968; Beth & Piaget, 1986).

Kahneman

Kahneman (2003) placed System 1 and System 2 on a continuum. Intuition is placed between the automatic operations of perception and System 2. He characterized the process of intuition as "fast, parallel, automatic, effortless, associative, slow learning, and emotionally charged" (Kahneman, 2003, p. 698). Kahneman and Frederick (2002) explain where they placed intuition on the continuum in the following statement:

> Intuitive thinking extends perception [similar to] processing from current sensations to judgment objects that are not currently present including mental representations that are evoked by language. However, the representations on which intuitive judgments operate retain some features of percepts: They are concrete and specific, and they carry causal propensities and an affective charge. (p. 50)

Based on others research, Kahneman and Frederick (2002) identified other aspects that influenced intuitive thinking, such as the amount of time the respondent was given to contemplate (Finucane et al., 2000), the respondent's mood (Isen, Nygren, & Ashby, 1988; Bless, et al., 1996), the respondent's intellectual abilities (Stanovich & West, 2002), and the respondent's content knowledge, specifically statistical knowledge (Nisbett et al, 1983; Agnoli & Krantx, 1989; Agnoli, 1991). These studies were used to identify the characteristics Kahneman and Fredrick (2002) used to describe intuition.

Fischbein

Fischbein (1987) held a notion similar to those in the dual process camp - individuals have one reality that can be addressed from different perspectives, intuition and reasoning - which positions intuition as a type of cognition. Fischbein (1987) believed intuition developed in parallel with reasoning; intuition is not a source of cognition nor is it a method of cognition. Since it is intelligence (or cognition), it develops with the individual like reasoning. He claimed that intuitive knowledge is a "kind of knowledge which is not based on sufficient empirical evidence or on rigorous logical arguments and, despite all this, one tends to accept it as certain and evident" (Fischbein, 1987, p. 26).

Fischbein (1975, 1987) believed Piaget unjustifiably ignored the role that intuition plays in supporting reasoning and believed that intuition fills, at the intellectual level, what perception fills at the sensory motor level. For example, Fischbein (1987) says that he "perceives the table in front of him, he does not need to prove its existence, … [likewise, intuition]… exceeds the given facts and extrapolates beyond the directly assessable information" (p. 13). Fischbein (1975) believed intuition pushes knowledge into action as he explains below:

> The moment of insight is merely the moment at which the cognitive steps are converted into incipient action. In the first (cognitive) stage, I know what I am looking for: this is the searching of intelligent behavior. In the second stage, *I know what to do*. Intuition (in our case, anticipatory intuition) is the moment of transition from the first to the second stage. (p. 15)

Fischbein (1975) was interested in testing his idea that intuition is tied to action. He believed probability is the best vehicle for his investigations for three reasons: (1) probability intuitions involves images, as found with random devices like dice or spinners; (2) probability uses stochastic experiments, the relative frequency interpretation that is action oriented; and (3) human behavior is in essence probabilistic, as are events found in the environment. For instance, Fischbein and Grossman (1997) explored intuition using combinatorics. They collected data using a survey from their 253 participants; students from grades 7, 9, 11, college students, and adults. Twenty five participants were selected for follow up interviews to explain their solutions. Fischbein and Grossman (1997) found the participants used underlying schemata mechanisms. They defined schemata, in the Piagetian sense, as “a program aimed to interpret a certain amount of information and to prepare and control the corresponding reaction” (Fischbein & Grossman, 1997, p. 29). They found a consistent schemata process that was logically based, where the participants used some form of binary multiplication. This suggested that intuition that appeared as guesses, are actually manipulated by underlying schemata - intuitions are based on schemata and tied to action (Fischbein & Schnarch, 1997). As mentioned above, intuitions do not require an explicit justification because the participants believed their evaluation was self-evident and hardly questionable (Fischbein & Gazit, 1984), which seems to align with the characteristics identified by System 1.

Characteristics of Intuition

Fischbein (1987) claimed intuition is complex and has many different interpretations. Therefore, Fischbein (1987) defined intuition by describing its multifaceted characteristics: "self-evidence, intrinsic certainty, perseverance, coerciveness, theory status, extrapolativeness, globality and implicitness" (p. 43). These characteristics represented Fischbein's (1987) attempt to synthesize the thoughts and research of others with his own notions and interpretations about intuition. The following paragraphs explain each of these characteristics.

The first characteristic is *self-evidence*, or it is obvious to the person, which means an individual feel no desire to formally, or empirically to prove a statement or to prove a solution to a problem (Fischbein, 1987). For example, the shortest distance between two points is a straight line requires no formal proof nor does a person feel compelled to prove this statement empirically to his or her own satisfaction (Fischbein, 1987). On the other hand, the notion that the sum of the angles of a triangle equals 180 degrees is not self-evident or obvious to us. Therefore, it requires a formal proof or we engage in experiments that test this notion, which means, this statement was not intuitive (Fischbein, 1987).

Intrinsic certainty is highly correlated to self-evidence but they differ and one does not imply the other (Fischbein, 1987). Many theorems in mathematics are accepted as true but they are not self-evident or obvious, they require proof. When a statement or solution is accepted as intrinsically certain, it is believed to be true, or in other words, we are confident. The identification of an intuitive thought requires a "presence of an intuitive cognition, one has to determine the extent to which it appears to the individual as an intrinsic belief" (p. 46).

The next characteristic is *perseverance*. Once the intuition is established, it is hearty and hard to eradicate (Fischbein, 1987). Formal instruction that includes conceptual understanding has little impact to remove or override a student's erroneous intuitions (Fischbein, 1987).

The characteristic of *coerciveness* means that intuition appears absolute. As a result, considering other alternatives is either not done or it appears unnecessary (Fischbein, 1987). Fischbein (1987) used the example that the set of all even counting numbers is equivalent to the set of all counting numbers. Without a formal proof, this appears impossible because the set of all counting numbers includes both the even counting numbers and the odd counting numbers. The intuition, a subset cannot have the same cardinality as the whole set, is coercive.

The next characteristic is really a description of its *status*. Intuition gained mini theory status, not the status of a major theory, because intuition "is not a pure theory….[it] is a theory expressed in a particular representation using a model: a paradigm, an analogy, a diagram, a behavioral construct …," much like a tree diagram models combinatoric problems (Fischbein, 1987, p. 50). This study uses intuition mini theory because System 1 is tied to it.

The characteristic of *extrapolation* is noted when a person draws a conclusion or makes an inference that reaches beyond the available information (Fischbein, 1987). However, Fischbein (1987) explained that the perceived obviousness or self-evidence of intuitions make it difficult to recognize that extrapolation occurred. Fundamentally, intuition always reaches beyond the available data, but for something to be an intuition, it needs the feeling of certainty joined with the extrapolation. If the certainty component is missing a person is merely guessing (Fischbein, 1987).

The *globality* of intuition means it is a "synthetic view" (Fischbein, 1987, p. 53) which differs from analytical thinking. The globality of intuition is a type of a selection process that sifts through the clues and eliminates the information that are not organized into a single, compact and coherent meaning. This type of coherence makes intuitions robust and resistant to change (Fischbein, 1987). The following examples demonstrate the meaning of globality and these examples also highlight the differences between intuitive and analytical thinking that aligns with System 1 and System 2: “Two liters of juice costs $3, what is the price for 4 liters” (Fischbein, 1987, p. 14)? The intuitive response for this problem is $6. On the other hand, the question that asks, what is the cost for ¾ liter of juice? This solution does not appear intuitive because the multiplication needed to solve the problem lacks a direct and global characteristic, so an individual must use analytical thinking (Fischbein, 1987).

The last characteristic Fischbein (1987) described was *implicitness*. Intuitions are based on tacit processes. If a researcher tries to point out these subsurface strategies or processes to the individual, it immediately destroys the intrinsic certainty, the compactness, and the robustness associated with intuitions (Fischbein, 1987). As a result, the person becomes confused during the reasoning process (Fischbein, 1987).

Classifications. Fischbein (1987) classified intuition in three ways: the origins of the intuition, the roles the intuitions play, and the relationships intuition has to other cognitions. His classifications were expanded from his earlier classifications that had two groupings; affirmatory and anticipatory intuitions (Fischbein, 1975). The following paragraphs described each of these ways to classify intuition based on his earlier and his later work. In this discussion, the focus is on the origins of the intuition and the roles the intuition play in the problem solving process since these two are most pertinent to this study.

Origin. There are two ways to classify the origins of intuitions: primary and secondary (Fischbein, 1975, 1987). Primary intuitions are formed before and independently of systematic instruction while secondary intuitions were formed after formal instruction took place (Fischbein, 1975, 1987). Secondary intuitions are the intuitions that are built from, for example, mathematics instruction, in which an individual gains an intuition about mathematics. Fischbein (1975) hypothesized that there exit a close relationship between action and intuition. As a result, secondary intuition may be reconstructed from the individual experiences that first defined them. This means that educational experiences could redefine secondary intuitions or develop them to the point where intuition becomes an *educated guess.* In this case, an educated guess is based on these past educational experiences.

Fischbein (1975) acknowledged that repetition is required to gain intuition that appears automatic but this kind of intuition differed from automaticity commonly associated with memorized facts or procedures. The fully developed secondary intuition combined several elements of the intellect - motor, imagination, and conceptual components - which means, it requires a deep familiarity and all-encompassing engagement with the situation, problem, or issue (Fischbein, 1975). On the other hand, when a person lacks knowledge, the person relies solely on primary intuition which typically leads to a wrong conclusion. Fischbein believed in order to develop secondary intuitions, the educational process has to use motor, representational and conceptual components, and other representational and motor mechanisms at a higher structural level to encourage its development. Fischbein (1975) defines automaticity of an internalized action as the final stage of intuitive development.

Roles. The second way Fischbein (1975, 1987) categorizes intuitions is by roles, where the roles addressed the relationship between intuitions and the solutions. There are four ways to group intuitions based on roles: (1) affirmatory, (2) conjectural, (3) anticipatory, and (4) conclusive. Anticipatory and conclusive intuitions are the problem solving intuitions (Fischbein, 1987).

The *affirmatory type* of intuition means a person affirms his or her claim about a situation (Fischbein, 1987). There is a feeling of conviction or certainty, where certain facts are viewed as truth and are known to the individual (Fischbein, 1975, 1987). This requires some prior knowledge and personal experience with the information (Fischbein, 1975, 1987). Fischbein drew on the example given earlier: the shortest distance between two points is a straight line. If the truth is self-evident to the person, then it is an intuition (Fischbein, 1987). A*ffirmatory* intuitions are divided into two sub-groups: (1) semantic, relational, inferential and (2) ground and personal (Fischbein, 1987). In the first sub-group, the *semantic* type of affirmatory intuition refers to the meaning of the concepts, where the *relational affirmatory* intuition is expressed in seemingly self-evident and self-consistent statements that are not necessarily correct (Fischbein, 1987). The *inferential affirmatory intuition* is based on the structure of the intuitive thought which is either induction or deduction (Fischbein, 1987). This shows the close relationship Fischbein (1975, 1987) believed existed between intuition and reasoning. The second sub-group of *affirmatory* intuitions is classified as *ground* or *personal* (Fischbein, 1975, 1987). The *ground* refers to the intuitions that developed on their own during childhood and are culturally specific. The *personal* type of affirmatory intuition develops from the individual's personal experiences (Fischbein, 1987). The ground and personal types of affirmatory intuitions are very similar but they have a slightly different focus; the ground affirmatory intuition emerges from the person situated within a context while the personal affirmatory intuition includes experiences that are under the control of the individual, such as formal educational experiences.

On the other hand, *anticipatory intuitions* are preliminary. They occur after the person takes a global view of a problem or situation before that person engages in a full analysis of the problem (Fischbein, 1975; 1987). It is like a conjecture that a person has before solving a problem, where the individual *feel* like he is on the right path (Fischbein, 1975, 1987). *Conjectural* intuitions are an assumption about a future event, incident, or an occurrence (Fischbein, 1975, 1987). Confidence is required for this to be an intuition (Fischbein, 1987). The person *feel* like it is self-evident (Fischbein, 1975). This type of intuition is common among individuals that have many years of experience in a specific field or a person who follows a topic closely so that he could confidently predict a future result (Fischbein, 1987). For example, a person, based on his knowledge and experiences may say "From what I understand about the lack of funding for the Border Patrol, there will be an increase in drugs smuggled into the United States this year." This differed from Piaget's idea of conjecture, which is hypothetical thinking (Fischbein, 1987) and it also marks the formal operational cognitive stage of development.

Anticipatory intuitions represent the initial overall view of the problem before the person engaged in the analysis (Fischbein, 1987). This may seem similar to *affirmatory intuition,* but they differ in the role they play during the problem solving process (Fischbein, 1987). Recall, with *affirmatory intuition* a statement or idea is self-evident to the person (Fischbein, 1987). On the other hand, *anticipatory intuition* "does not simply establish an (apparently) given fact. It appears as a discovery as the solution to the problem and the (apparently) sudden result of a previous solving endeavor" (Fischbein, 1987, p.61). This is equivalent to the idea that a light bulb of understanding is switched on, or "it clicked" for the person.

These categories are placed on a continuum from: affirmatory, conjectural, to anticipatory instead of discrete categories because there is overlap with some components from the neighboring categories (Fischbein, 1987). The distinguishing factor is the situational context because it is implicit or tacit for the *affirmatory intuitions* but explicit for the *conjectural* and *anticipatory intuitions* (Fischbein, 1987). *Anticipatory intuitions* need analytical control. *Anticipatory intuition* appears during the problem solving process in which the subjective ideas are “moments of illumination, as certain, evident, definitive, globally grasped truths” (Fischbein, 1987, pp.63-64). Conjectures on the other hand, are methodical, explicit, deliberate, and fully controlled and as such they are not intuitions (Fischbein, 1987). Piaget called this type of thinking, hypothetical thinking (Fischbein, 1987).

The last category, *conclusive intuitions* globally summarize the basic ideas of a solution to a problem that escaped the person on numerous prior occasions (Fischbein, 1987). The person perceives the solution due to his deep level of familiarity with the problem (Fischbein, 1987). In this situation the person finally understands and can “see” the solution.

Finally, Fischbein (1975) last classification is informed by Piaget’s (1968) cognitive development stages: pre-operational, operational, and post-operational (Fischbein, 1975). In the *pre-operational* stage the individual “synthesizing experience in a given domain, confer speed, adaptability, and efficiency on the appropriate action” (p. 8). Fischbein (1975) gave an example based on spatial intuition drawing on the example given by Piaget (1968). However, Fischbein (1987) believes this stage can last a lifetime. The *operational* stage involves reasoning and there are two sub-categories: the first sub-category is *basic operational intuitions.* The individual says the conclusion is true based on the premises that are typically guided by the rules of formal logic (Fischbein, 1975). The second sub-category is the *derived specific operational intuitions* that occur when an individual anticipates a solution during the problem solving process (Fischbein, 1975). The third stage is the *post-operational* that synthesizes previous experiences so the operational structures are automatized (Fischbein, 1975). For example, a doctor provides an initial diagnosis when presented with a set of symptoms. The initial diagnosis is based on his intuition before the test results are known (Fischbein, 1975). This way to classify intuitions demonstrates that intuition development looks different because of the source of the intuition.

Piaget

As mentioned above, Piaget (1968) dismissed intuition as semi-logic or half logic in the pre-operational stage of cognitive development. He described intuition as representational thinking that depends on perceptual experiences, in which children create their own reasoning based on immature logic that is error prone. Piaget (1968) believed that reasoning occurs at the formal operational stage, when the person exhibits reversibility. Reversibility is defined as the ability to go back to the original position. For example, the reversibility of moving laterally to the right is moving laterally to the left (Piaget, 1968). Intuition for Fischbein (1975, 1987) is considered a developmental process based on mathematical logic, a structure that is a representation of some internalized action that produces an operation from the intuition as soon as that intuition becomes reversible. The following paragraphs explain Piaget's (1966) classification of intuition.

Piaget notions of intuition were seen through the lens of theoretical mathematicians as they developed mathematics relying on their intuition; an informal way of thinking (Beth & Piaget, 1966). His classification is based on structured dichotomies, in which he identifies two categories of intuitions. The first category is the dichotomous pair of intuitions: empirical and operational. Empirical intuitions focus on the physical aspects of an object, such as size, or the psychological aspects of a mental object, such as the intuition of height (Beth & Piaget, 1966). The operational intuitions are actions connected to the object or psychological objects. The empirical intuitions develop from experimentation, while the operational intuition depends on intelligence that goes through “three stages of development: intuitions associated with actions on an object, with actions interiorized in the form of operations (but still applicable to objects) and …with operations independent of all possible action” (Beth & Piaget, 1966, p. 225).

The second category shows the dichotomy between operational intuitions; geometric intuitions that were associated with an image or symbolizing intuition and the operational intuition that focused on the item itself (Beth & Piaget, 1966). Piaget’s classification lacks clarity because of his loose use of the term intuition according to Fischbein (1987). Agreeing with Fischbein’s (1987) assessment, this study drew on Fischbein’s (1975, 1987) ways of understanding and describing intuition.

Comparing Intuition Perspectives

A comparison of the different intuitions will focus on Fischbein (1975, 1987) and Kahneman (2003) with little attention given to Piaget (1968). Kahneman (2003) and Fischbein (1975, 1987) both accepted intuition as a valid way of thinking, separate from reasoning. However, they too had their differences.

Kahneman (2003) claims the attribute substitution process occurs at the beginning of the problem solving process. At that time, an attribute is quickly switched out when the person is using intuition. This switch occurs because System 2 allows it (Kahneman, 2003). Fischbein (1975, 1987) did not limit when intuition is used during the problem solving process; it could be used at any time. Kahneman (2003) focused his attention on biases that were the result of using intuition while Fischbein (1975, 1987) was less concerned with errors during the intuitive developmental process. However, when intuition is fully developed, he expected intuitions to lead to accurate solutions.

There are aspects of intuition that Fischbein (1975; 1987) and Kahneman (2003) agree. Most notably they both agree with its speed - intuitions occur quickly. They both agree on other characteristics such as intuition is a global process that is implicit, self-evident, and extrapolated by the person. They agree that the individual using intuition does not need proof and the individual typically does not consider alternatives, but if they are considered, they are quickly dismissed. Fischbein (1975, 1987) went to great lengths describing the various ways to categorize and describe intuitive thinking. This current study draws on Fischbein's (1987) extensive descriptive accounts of intuition in concert with Kahneman's (2003) notions about intuition. The characteristics of intuition used by Fischbein (1987) and Kahneman (2003) is displayed in Table 2.3.

Table 2.3
Characteristics of Intuition

Fischbein's (1987) Intuition	Kahneman (2003) Tversky and Kahneman (1974, 1983)
Global	Global
Self-evident	Self-evident
Certitude	NA
Immediate	Immediate
Exceed Given Evidence	NA
Extrapolations – starting with known facts	Extrapolations
Cohesive	Affective charge
Perseverance	Associated within a context
Implicitness	Implicitness

Function of Attention

It is necessary at this point to take a closer look at attention because attention can trigger intuitive thinking or reasoning. Fischbein (1975, 1987) and Kahneman and Fredrick (2002) acknowledge the role attention has with initiating intuition or reasoning. In this current study, it is hypothesized that the students' attention that leads to a particular way of thinking may inform the problem solving strategies used by the participants.

Attention, from the constructivist perspective, may influence or trigger the intuitive or reasoning process. Von Glasersfeld (1981) draws on Ceccato's model of attention because it "underpin[ed] Piaget's crucial notion of abstraction" (p.79), in which the "structure of certain abstract concepts could be interpreted as patterns of attention" (p. 85). Fundamentally, this means that the pattern of attention is "pulse-like," so that attention alternates between focused and unfocused (von Glasersfeld, 1981). The pulses of attention and the segments of sensory signals associated with a concept determines whether content is registered or not registered based on the coordination of these pulses of attention with the sensory signals (von Glasersfeld, 1981, p. 85). The alternating pulses between focused or unfocused attention may be facilitated by motivation due to the goal-oriented nature of the activities that "influence behavior by capturing attention" (Reeve, 2005, p. 12). This notion that links attention with motivation aligns with von Glasersfeld's (1995) interpretation of Piaget's idea of knowledge construction because for Piaget (1968), the operation is goal-oriented. As von Glasersfeld explains (1995), the results from an operation are "evaluated primarily by the criterion of success, and success must ultimately be understood in terms of the organism's efforts to gain, maintain, and extend its internal equilibrium in the face of perturbations" (p. 74). This notion of success also ties back to attention because if the complexity of the activity is perceived as being too hard or too easy, the individual lacks motivation to continue with the task and as a result the person becomes inattentive (Fischbein, 1975; Simon, 1986). Gallagher and Reid (2002) explain that Inhelder focused her research on the cues she integrated into her experiments when she worked with children. She focused on the children interacting with objects and as the cues were noticed by the children, they would modify their actions. The attention cues coming from external sources and from internal sources can be used to identify the relevant features of a problem (Posner & Snyder, 2004). The external source refers to the factors in the environment that direct attention and the internal source related to the person's memory (Posner & Snyder, 2004). If the internal attention cues are weak, then more working memory capacity is needed to perform the retrieval (Posner & Snyder, 2004). This seems to suggest that external and internal attention cues may originate from System 1 or System 2. For example, when Leron (2010) gave his opening address to the North American Psychology of Mathematics Education Conference in Columbus, Ohio, he stated that many of the repeated mathematical errors people experience are not due to weak, lacking, or faulty mathematical knowledge but a failure of System 2 to override the error made by System 1. Leron (2010) compared the analytical approach some students use with a non-analytical approach used by other students when solving problems. He examined cueing and its impact on the problem solving process.

As mentioned above, attention cues from external or internal sources trigger a way of thinking. In System 1, the process is fast and requires minimal effort on the part of the individual. System 2 is slow and effortful, which indicates that working memory becomes taxed during the problem solving process. Jarrod, Tam, Harvey & Baddeley (2011) defined working memory as "the ability to maintain information in the face of potential distraction" (p.688). Taxing working memory is based on the notion that "processing and storage compete for a shared and limited, central pool of resources" (Jarrod, et al., 2011, p.688). Jarrod, et al. (2011) described a recent model called time-based resource-sharing (TBRS) (Barrouillet, Bernardin, & Camos, 2004; Barrouillet, Bernardin, Portrat, Vergauwe, & Camos, 2007 cited by Jarrod, et al., 2011) in which attention is split between processing requirements and memory retrieval requirements. The amount of time attention gives to processing and not to retrieving information is associated with an index called cognitive load that measures perceived effort (Jarrod, et al., 2011). It is beyond the scope of this study to tackle cognitive load on a detailed level, but an awareness of perceived effort may help to describe different ways of thinking.

In summary, the perspectives of intuition provided by Kahneman (2003) and Fischbein (1975, 1987) and Piaget (1975, 1986) showed areas of agreement and differences. Each of the perspective provides insight and depth for this current study. The role attention plays with the two modes of thinking is influenced by internal and external cues. Each of these modes of thinking is influenced by the amount time attention gives to processing and to retrieval.

Learning and Mathematics Education

The discussion turns to learning and mathematics education that focuses on constructivism and learning models for probability. Constructivism is a learning model that claims knowledge is constructed by a person. Constructivism is the foundation for the cognitive development models that were developed to show how children gain probability knowledge and reasoning skills. In particular, there are two cognitive development models that describe the stages of probability and statistical reasoning development for elementary and middle school students. The two models are Piaget's cognitive development model and Jones, et al. (1999) cognitive development model. In this study, the researcher drew heavily on Piaget and Inhelder's (1975) model and used Jones et al. (1999) model as a secondary source.

Constructivism

Piaget considered learning as a process that is based on a person's experiences that guides the person to construct structures or knowledge (von Glasersfeld, 1995). Piaget borrowed the term *structure* from mathematics because he considered knowledge as a system that had its own self-regulating (equilibration) laws or rules (Gallagher & Reid, 2002). This system is controlled by the individual and the individual is responsible for construction the structure or knowledge with a goal of reaching equilibrium (Gallagher & Reid, 2002). *Equilibrium* is defined as a "balance or level of understanding reached in cognitive development [in which] the balance [is] between assimilation and accommodation or between affirmations and negations, [which was] a fleeting state [that] open[ed] up new possibilities for growth in understanding" (Gallagher & Reid, 2002, p. 233). On the other hand, *disequilibrium* pushed a person beyond their current level of understanding, which prompted the person to reestablish equilibrium (Gallagher & Reid, 2002).

Equilibration is defined as "a process whose mechanisms [are] assimilation and accommodation [in which] external experience [is] incorporated into internal structures in such a way that existing structures [are] conserved and enriched" (Gallagher & Reid, 2002, p. 233). Charles (1974) stated the process more concisely when he said, "Maturation, experience, and socialization come together in order to build new mental structures or rebuild current mental structures" (p.4). The goal to reestablish equilibrium is the force or motivation behind learning because the individual always seeks balance (Gallagher & Reid, 2002).

The knowledge construction process, that seeks to maintain equilibrium, ends with one these results, either expanding the existing scheme or creating a new scheme (von Glasersfeld, 1995). The word scheme as used in this study was influenced by von Glasersfeld (1995) that he described as a process that creates a mental or cognitive object. A scheme has three parts: the recognition template, the action on the object, and the outcome. The outcome is compared to the initial goal to determine if the scheme produced equilibrium.

The expansion of the current scheme or creating a new scheme processes both begin with an experience and the characteristics of that *experienced object* is compared with the recognition template of the current scheme. The recognition template triggers an activity on the new object; if the object is recognized a known action or operation is initiated with an expected result based on prior experiences (von Glasersfeld, 1995). If the individual includes the new object to his or her existing set of objects, the recognition template remains fundamentally unchanged but it is broadened (von Glasersfeld, 1995). This generalizing or broadening process is called *assimilation* (von Glasersfeld, 1995). On the other hand, if the result is unexpected, a perturbation occurs and there are two possible outcomes, both called an *accommodation.* If the unexpected result is desirable a new recognition template is created to include the new characteristics. If the outcome is disappointing, then one or more of the new characteristics modifies the current recognition template, which in turn, changes the circumstances that initiates the activity part of the scheme in the future (von Glasersfeld, 1995). As von Glasersfeld (1995) explained,

> The learning theory that emerges from Piaget's work can be summarized by saying that cognitive change and learning in a specific direction take place when a scheme, instead of producing the expected result, leads to a perturbation and [the] perturbation in turn, to an accommodation that maintains or establishes equilibrium. (p. 68)

However, another component is needed to support cognitive growth. Von Glasersfeld

(1995) believed that knowledge, cognitive structures, are “evaluated primarily by the criterion of success, and success must ultimately be understood in terms of the organism’s efforts to gain, maintain, and extend its internal equilibrium in the face of perturbations” (p. 74). This means, in order to reconstruct knowledge the individual must become engaged with the activity. It is this notion of success that plays a role when discussing dual modes of thinking within constructivism. However, one further aspect of scheme theory needs to be addressed, how does the individual affect the schemes through assimilation or accommodation?

There are two ways to affect a scheme: (1) physically acting or experiencing an object through the senses, or (2) experiencing an object through mental operations (von Glasersfeld, 1995). There are three kinds of mental operations: (1) empirical abstraction that comes through sensory motor experiences that is observable, (2) reflective abstraction which is the persons own coordinating activity, and (3) logical transformation, which means the relations associated with a group structure are changed (von Glasersfeld, 1995). Piaget broke reflective abstraction down further into three types: the first type is similar to the word used in optics, *reflection*. This type of reflective abstraction occurs when an “activity or mental operation develops on one level [and is] abstracted … and applied to a higher level (von Glasersfeld, 1995, p. 105). The second is called *reflected abstraction*, because the person is aware of what is being abstracted (von Glasersfeld, 1995). It is the “projection of something from a preceding level onto a higher one and a cognitive reconstruction or reorganization of what has been transferred” (von Glasersfeld, 1009, p. 105). The third is called *pseudo-empirical* because it requires suitable sensorimotor material that children “are able to re-present … things to themselves but they are not fully on the level of concrete operations” (von Glasersfeld, 1995, p. 106).

Scheme theory describes learning as a construction process initiated by the individual as a means to increase knowledge. The construction process uses objects that are either physical objects or mental objects. Knowledge construction uses two processes accommodation and assimilation as the person seeks to re-establish internal equilibrium.

Cognitive Development Models

Piaget based his cognitive development theory on six fundamental ideas: (1) children want to learn, (2) children organize their learning that comes from their experiences, (3) children's interactions with physical environment and other people are essential for their cognitive development, (4) children adapt to their environment, (5) children seek equilibrium which pushes them towards more complex thinking, and (6) children's qualitative thinking differs according to their age (Ormrod, 2008). Piaget used the term *stage* to define his levels of cognitive development, but it is not intended to be interpreted as a category that describes a set of skills or abilities (Gallagher & Reid, 2002). The term *stage* is intended as a potential starting point for new knowledge construction, not for prediction (Gallagher & Reid, 2002). The stages provide a way to organize the observations of a child's cognitive development, which was Piaget's intent (von Glasersfeld, 1995). Piaget's has four general stages: (1) the motor-sensory stage, (2) the pre-operational stage, (3) the concrete operational stage, and (4) the formal operational stage (Piaget & Inhelder, 1975). In the motor-sensory stage the child coordinates his or her actions. The pre-operational stage child lacks awareness of the reversibility of actions so his or her thinking is considered semi-logical. The child can internalize thought, which is the representation of action in thought. In the concrete operational stage the child begins to sense the reversibility of operations that is observed by conservation because the child can understand the relationships among objects. The formal operational stage is evidenced by the strategies or reasoning methods used by the adolescence while engaged in problem solving (Gallagher & Reid, 2002). At this stage of development, the adolescent is capable of hypothetical-deductive thinking, if-then statements, and the adolescence is capable of returning to the starting point in order to propose another conjecture (Gallagher & Reid, 2002). It is this reversibility that separates the formal operational stage from the concrete operational stage (Piaget & Inhelder, 1975). The adolescent can move between reality and possibility and back again (Piaget, 1970; Piaget & Inhelder, 1975; Gallaher & Reid, 2002).

Piaget and Inhelder's Model. Piaget's interpretation of probability was influenced by the French philosopher and mathematician, Antoine Augustine Cournot (1801 – 1877), whose ideas were shaped by determinism and the classical interpretation of probability that was dominate in the 19th century (San Martin, 2006). Piaget drew from Cournot and used the word fortuitous to refer to random events. The four stages of cognitive development common for other areas of cognition, is reduced for probability. It lacks the first stage of development, the sensory motor stage because Piaget and Inhelder believed children lack the ability to understand the concept about probability (Piaget & Inhelder, 1975). Piaget and Inhelder (1975) performed many experiments exploring the developmental process of children's probability knowledge that includes several different components of probability, from randomness, types of distributions, quantification of probability with one single or two events, to combinatorics to name a few. There are four types of experiments: random mixture, random drawing, quantifying probability, and combinatorics that were used by the researcher to identify suitable participants.

Random Mixture Experiments. The random mixture experiments focus on random drawings of objects. One experiment uses counters or chips that are placed in varying and unequal amounts, from 10 to 20, in two bags, one bag marked A and the other bag marked B.

The counters have a cross marked on one side and a circle marked on the opposite side. The experiment begins by asking the child to predict the outcome of the counters before the counters are tossed on the table. Depending on the specific experiment, some events show all of the counters with the cross side up, or all of the counters show the circle side up, or there is a mixture of crosses and circles showing.

In the first experiment, the experimenter took all of the counters in the second bag marked B. This bag contains between10 to 20 counters but each counter has a cross marked on both sides. The child is asked to predict the outcome of the counters, not knowing the counters are "fixed." The second experiment is similar but the objects are changed from counters to marbles. The bag marked A has 40 marbles in it, with 20 red and 20 blue. The bag marked B contains 40 blue marbles. The participants know the number of red and blue marbles in each bag. Various numbers of marbles are drawn from each bag. It is interesting to note that the children share similar perspectives that do not reflect the true proportions in the bag even when the possible outcomes are obvious, such as drawing one marble from the bag that has all blue marbles. Knowing the population of the marbles for each bag makes it easier to compare the sample of marbles drawn from each bag with their related populations. At the end of their experiment, Piaget and Inhelder (1975) ask the participants to define or describe chance. The children's reply "does not show the actual operations of thought" but it does show some level of understanding about the concept of chance (Piaget & Inhelder, 1975, p. 114-115).

For the game of Heads and Tails, the pre-operational child expects the results to even out, and they also consider the magical powers of the person tossing the coin (Piaget & Inhelder, 1975). In the concrete operational stage, the child begins to see the game in terms of probability or mixture but his ability to quantify the probability is lacking (Piaget & Inhelder, 1975). When the child reaches the formal operations stage the child recognizes that the longer the sequence for an outcome the less likely it is. The researchers also noted from their experiments that the child has a sense of combinations under different situations (Piaget & Inhelder, 1975).

In particular, the pre-operational child is surprised with the outcome for the marbles in the bags experiment even when the child is been told the bag contains only blue marbles. The child lacks a sense about a random mixture and is unable to quantify the probability of selecting a blue marble from the bag containing all blue marbles. The concrete operational child, as demonstrated with the coin tossing experiment, begins to grasp the notion about quantifying probability and the child has an idea about combinations with a mixture. The formal operational child is not described by Piaget and Inhelder (1975). See Table 2.4 for the summary of Piaget & Inhelder's (1975) cognitive development stages for the random mixture experiments.

Table 2.4
Cognitive Development for Random Mixtures

	Pre-Operational Stage I: Intuition of Rarity, but Not of Random Mixture	Concrete Operational Stage II: Global Sense of Probabilities	Formal Operational Stage III: Quantification of Probability
Games of Heads and Tails	Accepts appearance of outcome as reality – No connections to numbers Even out results – Not based on combinatorics knowledge Identical counters not considered strange or a trick – Power of the person throwing the chips	Sees the problem in terms of probability or mixture Random mixture has significance - Chips represent more than material mixing False chips are seen contrary to probability Different mental orientation from Stage I Does not understand different degrees of probability	Intuition of probability includes quantification The longer the sequence of a single outcome the lesser the probability Sense of combinations of different situations vs simple attribute judgments

(continued)

Table 2.4
Cognitive Development for Random Mixtures (continued)

Pre-Operational	Concrete Operational	Formal Operational	Pre-Operational
Stage I: Intuition of Rarity, but Not of Random Mixture	Stage II: Global Sense of Probabilities	Stage III: Quantification of Probability	Stage I: Intuition of Rarity, but Not of Random Mixture
Drawing Marbles from a Jar – Focus on sub-stages and transition	Stage IB No understanding of random mixture Fooled by sack with all same colored marbles Indifference to quantification – yet child counted marbles Bag with only 1 color marbles - able to select another color	Transition to Stage II Idea of combinatorics mixture Beginning of quantification of probability	* Piaget did not describe this stage of the experiment
Child's Definition of Chance	No understanding of chance – "it happens fast" (p.115)	Chance is rarity – "by chance, I did something wrong …because it is something that does not happen often" (p. 115).	By interaction or interference of causal series – "if he has been invited for the first time, and he falls ill on just that day..[or]..I didn't know whether I wanted to go see my grandmother or my aunt, and so I went to see my grandmother and found there my aunt" (p. 115)

Note. Table was created by author to summarize Piaget and Inhelder's (1975) experiments.

Random Drawing of Pairs Experiments. The next set of experiments, Piaget and Inhelder (1975) use random drawing of pairs without replacement. The experiment uses four sets of counters that are placed on the table so the participant can see the counters. Set A has 15 yellow counters, set B has 10 red counters, set C has 7 green counters and set D has 3 blue counters. The number of counters in each of the sets varies for each participant and the sets maintain a decreasing number of counters in each subsequent set. The experimenters replicate the initial proportion by placing all of the counters into a bag that contain a total of 35 counters in which 15 are yellow, 10 are red, 7 are green, and 3 are blue. They ask the child to predict the most probable number of pairs as the child looks at all of the counters placed on the table. For example, in the mixture described above, a child may predict the most probable number of pairs containing a yellow counter and a red counter. Then, the child chooses a pair of counters at random. After each drawing, the pairs are placed on the table so the child can see which counters are drawn and compare that to the remaining counters. The child may note that the total number of counters in the bag at the beginning of the experiment remains the same but they are divided into two groups as the experiment progresses, counters chosen and counters remaining. The experiment is modified for the youngest children. They draw one counter at a time instead of a pair of counters.

The random drawing of pairs experiment has the pre-operational child using 1 counter. The children at this level lack a systematic way to solve the problem. Predicting the color for the next draw is subjective and it is influenced by the colors of the counters, by the number of chips in either of the sets, or the closeness between two shades of color, such as red and rose. Prediction is dominated by personal preference. The concrete operational child, using pairs, begins to consider the pairing of the other colors but fails to compare the number chosen to the number that remains. Prediction is based on fairness, the notion that all colors should take their turn, regardless of which counters remain to be chosen. The formal operational child grasps that adjustment to the remaining counters for this experiment without replacement. Piaget &

Inhelder's (1975) cognitive development stages for their random drawing of pairs experiments are summarized in Table 2.5.

Table 2.5

Cognitive Development for Random Drawing of Pairs

Developmental Stage	Age (years)	Title of Stage	Characteristics
Stage I: Preoperational	4 - 7	Absence of Systematic probability (1 counter)	Does not understand random mixture as a system of chance interactions; frequencies Makes choices based on order of colors of counters on the table Predicts based on the number of counters of a certain color Predicts based on closeness of two chips colors (rose and red) Quantity is seen as a characteristic of a counter Prediction subjective – preference for a color
Stage IB			Overall – lacks a system of operations that are both numeric and combinatorics Predicts correctly some of the time based on proper numerical reasoning
Stage II: Concrete Operational	7-12	Beginning of Quantified probability	Reason on fixed number of counters in each color group – forget to consider the number of counters already used Predicts based on what has not been drawn yet Prediction based on rotation – taking turns of colors drawn The child conceives the whole of the probabilities and the combinations as static sequences, not change over time
Stage III: Formal Operational	12 - up	Quantifying the possibilities	Sees that modifications are needed after each draw

Note. Table was created by author to summarize Piaget and Inhelder's (1975) experiments.

Quantification of Probability Experiments. The next set of experiments directly explores children's quantification of probability. Piaget and Inhelder (1975) use these experiments to understand children's operative quantification. They want to know how children quantify probabilities before exploring how children address combinatorics operations. The children demonstrate their ways of quantifying probabilities when they are engaged in the activities.

The experiments include one set activities and two set activities. The one set activities combines red chips with white chips in varying proportions that require the child to predict which color chip has the greater chance to be selected at random. The two set activity also uses white chips and red chips but some of the chips have crosses on the opposite side. The children choose the set of chips that has the greater chance of selecting a chip with a cross on the other side. Before the children predict the outcome, the children are shown the chips that have the cross on the other side for each of the groups. These activities highlight two aspects that are essential for a child to interpret chance: (1) the child must develop the operations of chance and (2) the child must understand the relationship between the individual cases and the whole distribution that is both physical and mental (Piaget & Inhelder, 1975). The younger children fail this operative quantification problem because they do not recognize the need to count the number of elements in each of these sets, (1) the number of favorable elements, (2) the number of non-favorable elements, and (3) the total number of favorable and unfavorable elements (Piaget & Inhelder, 1975). As the children develop cognitively, they begin to search for relationships between these three sets but they fail to adjust the numbers when they change. Eventually, the children respond appropriately to the change in the number of elements in the sets, but this experiment shows that children are late to recognize numerical relationships between favorable and unfavorable sets and the whole set. The relationships in these sets highlight the logical and arithmetic procedures required for probability judgments (Piaget and Inhelder, 1975).

In the quantification experiments the early pre-operational child is unable to understand the part-whole concept. As the child progresses, he begins to make intuitive comparisons but he continues to struggle with the part-whole concept. The concrete operational child grasps part-whole for a single set but struggles when comparing two sets. The child lacks proportional reasoning, and as a result he tends to vacillate between focusing on the favorable or on the unfavorable outcomes in each set. The formal operational child understands part-whole and proportional reasoning. He uses disjunctive reasoning with reversibility. Piaget and Inhelder's (1975) cognitive developmental stages for these experiments are displayed in Table 2.6.

Table 2.6
Cognitive Development for Quantification of Probability

Developmental Stage	Title of Stage	Characteristics
Stage I A Preoperational	Absence of logical and arithmetic comparisons	Child unable to make comparison based on the quantified relationships for single set
Stage I B Preoperational	Relations intermediary between stages I and II	Child initiates intuitive comparison Child concludes some single sets correctly Child struggles with part-whole concept
Stage II A Concrete Operational	Systematic failure with proportionality questions	Child understands single set Child unable to address proportionality
Stage II B Concrete Operational	Progressive empirical solution of questions of proportionality	Child struggles with proportionality Child perceives proportionality in simple case Child oscillates between focusing on the favorable and unfavorable outcomes within each set
Stage III: Formal Operational	Solution of questions with two variables	Child works with multiplicative relationships, Child links the parts with the whole Child uses disjunctive reasoning with reversibility to quantify probabilities

Note. Table was created by author to summarize Piaget and Inhelder's (1975) experiments.

Combinatory Experiments. Piaget and Inhelder (1975) hypothesize that a thorough understanding of chance requires understanding mixture that includes combinations and permutations in order to advance to the formal operational stage of development. In this study, the researcher also drew on the two-dimensional type of combination problem (X x Y) that was used by English (1993), in which two variables may or may not have the same value. For example, the pants and tops problem, which is a two-dimensional combination problem that is solved by multiplication when there are three tops from which to choose and four pants from which to choose (3 tops x 4 pants = 12 different outfits).

Piaget and Inhelder (1975) explore combinations when they ask their participants to find all the ways that two colored objects could be combined when taken from three colored object or four colored objects. It appears that Piaget and Inhelder (1975) have more than one object associated with each color to help the child avoid repeating combinations but this is unclear from the reading. Piaget and Inhelder (1975) want to know if the children can systematically discover, through the activities, that the number of pairs from three colored objects, C_3^2, which is three and the number of pairs from four colored objects, C_4^2, which is six. For the youngest children the experiment begins with three different colored chips (red, white, blue), in which the chips represent little boys. They asked the children to show all the possible ways that the little boys can leave two at a time. Piaget and Inhelder (1975) extend the experiment to include a fourth color, yellow. The youngest children are unsuccessful finding all combinations. A systematic approach to find the answer is beyond them, they merely pair the chips by chance. They fail to notice that if they repeat a pair, as observed by one boy, by laying down one color and repeatedly place a different color next to it, they can find all of the pairs. The concrete operational children initially use four colors of chips and they are asked to make pairs. The colors of chips are increased to five and then to six different colors. Once the children complete the task for each set of colored chips, they are asked if they can find a faster and more accurate way to identify all of the pairs.

At this stage of cognitive development, Piaget and Inhelder (1975) believed it is apparent why the “pre-scientific mind struggled with the notion of combinations” (p. 169) because these children approach the problem from a juxtaposition position, much like addition. They fail to approach the problem from an intersection or interference perspective or a multiplicative perspective. Some children discover the multiplicative perspective but it is not an idea they have before the experiment.

In stage I, called the Empirical stage, the pre-operational child finds the solution by randomly using trial and error. In stage II, called the Discovering a System stage, the concrete operational children rely on the juxtaposition of the pairs. For example, if a child took four colors, identified as ABCD, and initially made the pairs AB + CD. The next, more sophisticated way of making combinations, the children gradually introduces the intersection of the pairs, but they are still dominantly influenced by juxtaposition. For example, one child uses five colors identified as ABCDE, to generate this initial sequence of pairs: AB, BC, CD, DE. Then the child finds the other pairs empirically. Piaget and Inhelder (1975) call this a cross juxtaposition. In the third level of sophistication, the child pairs the end colors first and then continues with a crossed juxtaposition. For example with five colors one of the children begins with AE, then continues with AB, BC, CD, DE, then skipps one item each time as shown with AC, BD, CE and then uses the empirical approach. The fourth method, continues to rely on juxtaposition, but also draws on the symmetry of pairs. This means, pairing the first color with each of the subsequent colors and doing the same thing for the second color. However, the implementation of the symmetrical method is incomplete. For example, one child shows the following succession of creating pairs with six colors: AB then FE, BC then ED, and ending with CD with the remaining pairs found empirically. The last method is an incomplete implementation of intersection that signals the ends to this stage. This occurs when the child begins to use symmetry: AB, AC, AD, AE, BC, BD, BE, but later rejects the method and falls back on a symmetrical strategy of four. Other children fall back on the empirical strategy. Piaget and Inhelder (1975) marvel at these two prominent tendencies found in this age group, simple linear succession and symmetry, undermine their ability to comprehend probabilities associated with combinations.

There is a way of finding the combinations that delineates Stage II from Stage III, which begins when the child fully comprehended the problem. However, the method to solve the problem is underdeveloped. For example, using six colors, one child begins with AB, AC, AD, AE, AF, then moves to other end of the sequence FE, FD, FC, FB, then went back to the front with BC, BD, BE, then he went back to the end with ED, EC, EB, before finishing empirically. Another child uses symmetry for A and its pairs, B and its pairs and C with its pairs. He notices that with each color there is one less number of pairing so as not to pair two of the same color. The last stage called the Discovery of a System finds the formal operational children successfully using a system where none of the pairs are omitted.

Piaget and Inhelder (1975) explain the underlying roadblock children face finding all possible ways to combine two objects from *n* terms is that the child must "coordinate several different series or correspondences and the child needs to anticipate the scheme of their relationships before making a proper construction," which requires formal operations (p. 172). The formal operational stage finds the child using second order operations, "operations bearing on other operations that can be concrete" (Piaget & Inhelder, 1975, p. 172) to successfully solve combination problems. Piaget and Inhelder's (1975) experiments for the developmental stages for combinations is displayed in Table 2.7.

Table 2.7

Cognitive Development for Combinations

Developmental Stage	Title of Stage	Characteristics
Stage I Pre-Operational	Empirical Operations	Child randomly creates pairs
Stage II Concrete Operational	Search for a System	Child progresses by using juxtaposition, Cross juxtaposition, and symmetry but Completes the solution using empirical means
Between Stage II and Stage III		Child understands the problem under investigation Child attempts to use symmetry to a larger extent but reverts to juxtapositions or empirical means
Stage III: Formal Operational	The Discovery of a System	Child engages in second order operations

Note. The Table was created by the author to summarize Piaget and Inhelder's (1975) experiments.

Jones, Thorton, Langrall, and Mooney's Model. Another cognitive development model that gained prominence in recent years among the mathematics educators and statistics educators is a four level model designed by Jones, Thorton, Langrall, and Mooney (1999). Their four developmental levels address many of the elementary concepts found in probability and statistics curriculum (Jones, et al., 1999; NCTM, 2000). They include sample space, probability of a single event, probability comparisons, and conditional probability. The four levels follow a child's probability and statistics cognitive development from kindergarten or first grade through middle school, which basically aligns with the developmental stages identified by Piaget and Inhelder (1975). The levels are described in general terms since they cover a broader range of concepts associated with probability and statistics (Jones, et al., 1999).

In addition, they lack age association for each level that Piaget and Inhelder (1975) use with their stage model. Piaget and Inhelder (1975) believe cognitive development is tied to the chronological maturation process. Like Piaget and Inhelder (1975), each subsequent level represents a progression in cognitive development. Level 1 is described as the *subjective* level in which the child has no sense of what is possible (Jones, et al., 1999). The term subjective appears to align with the subjective interpretation of probability, in which children's thinking is completely deterministic. (Jones, et al., 1999) The children believe there is a reason for everything that is based on personal opinion (Jones, et al., 1999). Level 2, is called the *transitional* stage because the child holds a non-systematic sense of the possible (Jones, et al., 1999). This stage is highly intuitive because there is a lack of understanding that probabilities are ratios (Jones, et al., 1999). In Level 3, the *informal quantitative* stage, the child possesses a semi-systematic sense of what is possible. He understands two-stage problems, such as flipping 2 coins and he can generate his own informal terminology describing probability events, outcomes, and processes (Jones, et al., 1999). Level 4 is called the *numerical* stage because in this level the child has numerical or formal knowledge about probability and he can analyze situations theoretically (Jones et al., 1999). The following paragraphs describe each level of development for each of the four stochastic concepts: sample space, probability of a single event, comparing probabilities, and conditional probabilities (see Jones, et al., 1999, p. 489).

***Sample Space*.** In Level 1, the child is unable to list all of the possible outcomes but the Level 2 child can list all possible outcomes for a single event (Jones, et al., 1999). The Level 3 child is able to use a systematic strategy to list all of the possible outcomes for a compound event, but the Level 4 child is also able to link the sample space with its probabilities (Jones, et al., 1999).

Probability of One Event. The Level 1 child can only discern probabilities in terms of certain events and impossible events (Jones, et al., 1999). The Level 2 child is able to identify events in more refined terms, least likely or most likely and he can rank the probability outcomes (Jones, et al., 1999). The Level 3 child uses quantitative reasoning and measures to describe the likelihood of events and the Level 4 child can assign numerical probabilities to the events (Jones, et al., 1999).

Probability Comparisons**.** The Level 1 child lacks the ability to compare fair with unfair situations but the Level 2 children can distinguish between them (Jones, et al., 1999). However, the child tends to revert back to subjective judgments (Jones, et al., 1999). The Level 3 children are able to compare the likelihoods of events "using inverted numerical representations" and the Level 4 children use ratios or numerical measure to compare probabilities (Jones, et al., 1999, p. 489).

Conditional Probabilities. The Level 1 child uses subjective judgments but the Level 2 child notices that some event probabilities change when there is no replacement (Jones, et al., 1999). The Level 3 child recognizes that all of the event probabilities change when there is no replacement (Jones, et al., 1999). The Level 4 child can distinguish between independent and dependent events (Jones, et al., 1999).

Summary

This chapter outlined the different interpretations of probability, the recent developments with heuristics and biases research and dual process research that are linked with mathematics education as it pertains to probability knowledge and cognitive development. Recently, Kahneman and Frederick (2002) use dual process theory to explain heuristic thinking for three types of heuristic thinking: representativeness, conjunction fallacy, and availability. Kahneman and Frederick (2002) believe that these three heuristic ways of thinking use the same generalized mechanism that they call the attribute substitution process that it is controlled by System 1. System 1 is fast, implicit, automatic, effortless, emotional, and associative, in a cognitive sense and linked events or ideas (Kahneman & Frederick, 2002). System 2 is slow, explicit, effortful, and it is linked with reasoning (Kahneman & Fredrick, 2002). Fischbein's (1975, 1987) ideas of intuition share some of the same characteristics identified by Kahneman and Fredrick (2002) but Fischbein (1975, 1987) explored intuitive thinking more broadly. Fischbein (1975, 1987) identifies characteristics of intuition in greater detail and he classified intuitions according to their source and their roles.

All of these components are used to meet the goal of this current study which is to explore the way eighth-grade students' think when solving problems. In particular, the students are asked to solve probability problems associated with a particular group of misconceptions linked to the revised heuristics and biases framework. The revised heuristics and biases framework includes dual process theory in which intuition is linked to System 1 that tends to invoke heuristic thinking and reasoning that is linked to System 2 and tends to invoke appropriate problem solving strategies (Kahneman & Fredrick, 2002). This current study aims to describe the problem solving processes and the thinking processes in general of eighth-grade students when they solve probability problems.

Chapter 3

Methods

This is a descriptive qualitative study that explores the thinking patterns of middle school students' when they are solving probability problems. In particular, it focuses on confirming Kahneman and Fredrick's (2002) notion that fast thinking produces systematic errors that are initiated by the attribute substitution process and the slow thinking produces accurate solutions. The researcher hopes to observe and identify particular attributes being substituted within each problem. It is hypothesized that the probability interpretation used in the problem will influence the attribute substituted. It is anticipated that intuitive thought will occur at the beginning of the problem solving process, as theorized by Kahneman and Fredrick (2002). Like Kahneman (2003), the researcher agrees that the dual process theory (Evans, 2008) offers a compelling explanation for heuristic thinking. The researcher will explore the ways that perturbations, initiated by her may move students away from System 1 thinking and towards System 2 thinking. Finally, the stories the students tell when they explain their thinking will be accepted as their mathematics when the researcher interprets and unpacks the students' thinking (Steffe & Kieren, 1994).

The clinical interview methodology will allow the researcher to observe and interact with the participants in order to explore the students' probability thinking. The clinical interview has its roots in Piaget's (1975) research when he explored the knowledge structures and the knowledge development of children (Clement, 2000; Ginsburg, 1981). The clinical interview was challenged as an authentic research method in the early part of the 20th century because using participant's verbal responses was believed to lack legitimacy (Shoenfeld, 2007). Yet, many mathematics education researchers embraced this method, such as Gelman (1980) when she explored prekindergarten children's mathematical thinking about number. Concurrently, the

1980's Nobel Prize winner Herbert A. Simon and his colleague K. A. Ericsson, challenged the low status associated with verbal responses as data, claiming that verbal responses, although not completely accurate, are suggestive of the participant's thinking processes (Ericsson & Simon, 1980). Today, there are many forms of the clinical interview that include: the unstructured open-ended interview or interviews where participants solve predetermined mathematics problems (Clement, 2000) also referred to as clinical task-based interviews (Goldin, 1997; Koichu & Harel, 2007), which are deemed as appropriate methods to explore mathematical thinking (Shoenfeld, 2007).

This study will strategically use the clinical task-based interview with a particular type of verbal response. The goal is to observe the students thinking in real time when they are engaged in probability activities designed to reveal one of these two ways of thinking: intuitive (S1) or reasoning (S2). For this study, the goal is to reduce introspection and initiate thinking in real time that emerges verbally for observation by the researcher. The researcher will use verbal report as data that will be analyzed. The researcher is particularly interested in the *concurrent verbalization* type of verbal report that differs from introspection (Ericsson & Simon, 1980; Ericsson & Fox, 2011). *Concurrent verbalization* occurs when the participant verbalizes his or her thoughts while engaged with a task (Ericsson & Simon, 1980). Ericsson and Simon (1980) identify two types of concurrent verbal responses: *think-aloud* and *talk-aloud*, in which both rely on the participant's short term memory. The *talk-aloud* type of verbal response encourages the participants to respond to the problem or activity verbally just like it enters their short term memory (Ericsson & Simon, 1980; Fox, Ericsson, & Bets, 2010). The *think aloud* differs because the activity may enter the participant's short term memory in non-verbal form, which tends to be the case when solving mathematics problems. This requires the participant to translate their thinking into a verbal format for utterance (Ericsson & Simon, 1980; Fox, Ericsson, & Bets, 2010), in which utterances include sounds other than words. Both types of verbalization occurring during the activity may reflect the participant's natural thinking processes in real time (Ericsson & Simon, 1980; Fox, Ericsson, & Bets, 2010; Van Someren, Barnard, & Sandberg, 1994). The researcher will use the *think aloud* protocol for this current study. She will explain and model the protocol with the participants before the participants engage in the task-based activities. The students are encouraged to verbalize any and all thoughts that enter his or her mind when they are thinking about and answering the questions. Using this protocol may require gentle reminders to the participants to maintain their verbalizations.

Hertwig and Gigerenzer (1999) used the think aloud protocol when they conducted a series of experiments revisiting the conjunctive effect. The college age participants were randomly assigned to one of four groups in which two of the groups used the think aloud protocol and the other two groups did not. They found that the groups using the think aloud protocol showed no difference in the proportion of responses that violated the conjunction rule when compared to the groups that did not use the protocol. This suggests that the protocol does not impact or interfere with the natural mental processing of the students but it may reveal in greater detail their mental processes.

This study will use the *think aloud* protocol because it allows the participants to share their mathematical thinking in real time. It is anticipated that fast thinking may be more easily identified as it is described in Chapter 2. At the beginning of the first interview session, the participants will be given instructions and the researcher will model the *think aloud* protocol. There are two goals for the interview process. The first goal is to evaluate the participants' cognitive development during the screening process and the second goal is to unpack their thinking. The cognitive development component addressed proportional reasoning, randomness, and combinations based on Piaget and Inhelder's (1975) studies during the screening process in order to identify the students who had reached the formal operational stage of cognitive development. The research team anticipates that the students' thinking could be unpacked to reveal heuristic (intuitive thinking) or probabilistic reasoning associated with System 1 and System 2 respectively.

In addition to collecting the utterances and jottings of the participants, a research journal will be kept by the researcher to promote credibility, dependability, and confirmability through triangulation (Anfara, Brown, Mangione, 2002). The journal will provide a space for reflexive practice in which overt, emerging biases or perspectives that may influence the research study can be brought into the open for reflection and analysis. This current study will include several types of memoing: embedded, extended, analytical, and procedural, which will also create a space for continual analysis, bracketing, and documentation.

Participants

There are 67 eighth grade students attending a middle-school located in a rural area within a mid-Atlantic state in the United States. The school population is predominately white and middle class. The study will begin with 20 students who are enrolled in a Mathematics eighth-grade course, 20 students who are enrolled in an Algebra I course, and 27 students who are enrolled in a Geometry course. These students have had experiences with probability concepts, laws, and probability measurement from their 6^{th} and 7^{th} grade curriculum as well as the foundation that was laid during their elementary school years as recommended by the NCTM (2000) that was found in the curriculum used in the elementary schools in the county. The number of participants will be reduced through a screening process that is discussed in the procedures section.

Data Collection

The data was collected during the Fall of 2012 and early in the Spring of 2013. First, the Probability Content Knowledge Assessment, discussed later in the chapter, contains 14 multiple choice questions was administered to the three eighth-grade classes taught by one middle school teacher. Next, two interview sessions, using the Cognitive Development and Heuristic Thinking and the Perturbation Activities, were videotaped. Each interview session lasted between 20 minutes and 35 minutes. The videotaped episodes were transcribed by the researcher. In addition, the participant's jottings were collected and the researcher's jottings and reflections were recorded in a research journal during and following the interviews. Additional memos were generated during the data analysis stage that was an ongoing process while the data was being collected.

Instruments

There are three major instruments used in the study, the Probability Content Knowledge Assessment used before the first interview session, the Cognitive Development and Heuristic Thinking used in the first interview sessions, and the Perturbation Activities used in the second interview sessions. The following paragraphs explain the purpose for the instruments and describe each in detail.

Probability Content Knowledge Assessment

The Probability Content Knowledge Assessment instrument, (See appendix A), assesses students' probability content knowledge. It is necessary in this study for the students to hold a strong knowledge of probability in order to explore reasoning and heuristic thinking. The instrument is used to identify the students, among the 67 students enrolled in the three classes that had strong probability knowledge.

The instrument contains 14 multiple choice questions and the students were encouraged to show their work. The questions came from the 6th grade, 7th grade, and 8th grade released 2009 and 2010 State-wide assessments for middle school students (Virginia, DOE). Seven questions focus on the probability of a single event. There are a variety of single event problems in the assessment that include, selecting a particular color of a marble from a bag when the contents are known, calculating the probability of a particular outcome from tossing one fair die, calculating the outcome from one spin on a multi-colored spinner, and determining the outcome when one card is drawn at random from a known set of cards. The solutions for the single events are represented in fractions, percentages, and identifying the more likely outcome. Four questions address combinations that include making different outfits, selecting elective courses, choosing pizza toppings, and choosing different types of soccer knee pads. The solutions to these problems are displayed in a list format (see Appendix A, problem 2), a tree-like diagram (see Appendix A problem 3), identifying the total number of combinations (see Appendix A problem 7), or identifying the correct operation to calculate the total number of combinations (see Appendix A problem 9). One question focuses on the Law of large numbers, in which experimental results are expected to approach theoretical values when comparing a coin flipped 30 times to the same coin flipped an additional 300 times. There is one question in the instrument that describes an independent compound event using a penny, nickel, and quarter. The question asks the students to identify the probability if all three coins were to land on tails after one toss. The last question asks the students to find the number of goldfish in a pond when a sample of 40 fish include 2/5 that are goldfish.

Cognitive Development and Heuristic Thinking Assessment

The Cognitive Development and Heuristic Thinking Assessment instrument, (see Appendix B) has 11 probability activities that are divided into two parts. The first part assesses the student's cognitive development level based on four activities drawn from Piaget and Inhelder (1975) experiments, in which the ideal participant will perform at the formal operational stage. The second part determines if the students relies on heuristic thinking based on seven multiple-choice questions. The multiple-choice responses are coded as: heuristic that correlates to heuristic thinking, correct that correlates to correct reasoning, or incorrect that correlates to the student lacking probability knowledge for that question. It is necessary to remind the reader that these labels are not used to judge the students cognitive abilities. It is a way to link the instrument to the literature. Six of these questions are drawn from previous research (Fischbein and Schnarch, 1997; Shaughnessy, 1975; Tversky and Kahneman, 1974, 1982) and one is created by the researcher.

The first four activities (see Appendix B) focus on cognitive development based on Piaget and Inhelder's (1975) probability experiments and they are open ended questions. One activity was vital to identify the students who had reached the formal operational stage of cognitive development (Piaget & Inhelder, 1975). This activity has the students compare a group of two black chips in which one chip has a sticker on it to a group of four red chips in which two chips have a sticker on each of them. This is followed by several statements and the student will identify the true statement. The second probability activity is a multi-stage event in which the student will identify the most likely event; flipping one fair coin three times and it landing on heads each time or flipping one fair coin four times and it landing on heads each time. The third activity is a combinations problem. The students are asked to find the number of pairs they could make from four different colored chips: red, blue, yellow, and green, and each chip in the pair must be a different color. The last activity addresses independent and dependent events. The students are asked to compare one draw from a bag versus two draws from the same bag without replacement. The students are shown a bag containing 35 counters. There are 15 yellow counters, 10 red counters, 7 green counters, and 3 blue counters. Originally, Piaget and Inhelder (1975) designed this problem so the children drew two counters at the same time. It is modified for this study because the types of compound event questions used on the standardized tests did not reflect this level of complexity.

The remaining seven multiple-choice questions are used to assess the participant's reliance on three types of heuristic thinking: representativeness, the conjunctive effect, or availability that uses the same thinking process, the attribute substitution process. The first question shows one form of the representativeness heuristic associated with randomness. The students compare the likelihood between two sequences of coin tosses, T H H T H T and T T T T T T, that result from flipping one fair coin six times. If the person used the representativeness heuristic thinking, he would say the first sequence is more likely because the outcomes appear random (Tversky & Kahneman, 1974). Many times individuals believe short runs from coin tosses should reflect the probability found in the population. This means, the number of heads and tails in the sequence should be approximately the same, since the coin is fair (Tversky & Kahneman, 1974). In the second activity, the participants are asked to determine the most likely outcome when comparing –pulling one red ball from an urn that contains 10 red balls and 90 white balls or pulling four red balls in a row, with replacement, from an urn that contains 50 red balls and 50 white balls (Shaughnessy, 1977). The person using heuristic thinking will claim that getting the four red balls with replacement is more likely to occur. Shaughnessy (1977) found that individuals ignore the different number of balls in the urns and focused their attention on the equal proportions between the red balls and the white balls.

The third activity tests the conjunction effect. The student using heuristic thinking believes that two combined independent events has a higher probability of occurring than the probability values for each of the single events contained in the compound statement. This violates the *conjunction* law of probability, which states, P(A and B) = P(A)*P(B) for independent events. Tversky and Kahneman (1982) tested this by using a description that stereotyped a person. The stereotyping used in this assessment was modified to align with middle school students experiences. Originally, Tversky and Kahneman (1982) gave a short description of a person that aligned with the stereotype of a librarian. This description was followed by eight statement and the participants were asked to rank the probability of these eight statements from most probable to least probable. Tversky and Kahneman (1982) focused their attention on three statements. One statement aligned with the description of the person given in the vignette, the second statement did not align with the description, and the third statement joined these two opposing statements together. The remaining five statements were considered distractors. One was also a compound statement but it did not contain any of the other distractor statements in it. A person influenced by heuristic thinking will rank the compound statement higher than one or both of the single statements that is used in the compound statement (Tversky & Kahneman, 1982). The modified version used for this study is similar. It differs in three ways: there are six total statements instead of eight, the three distractor statements are all single statements, and the compound statement joins two statements that are both plausible based on the description in the problem. The students are asked to rank the six statements from most probable to least probable.

The fourth activity tests the representativeness heuristic thinking also called the negative regency effect (Fischbein & Schnarch, 1997). This activity shows a family with four male children. Given that the birth of a boy and a girl to be about equal, the participants are asked to determine the gender of the fifth child (Fischbein & Schnarch, 1997). Typically, the individual relying on heuristic thinking believes the next child will be a girl since none of the prior children were girls; this means a girl is due to be born (Fischbein & Schnarch, 1997).

The fifth activity also focuses on the representativeness heuristic thinking. It addresses sample sizes with probabilities. The participants compare two scenarios of outcomes, getting two heads from three tosses of a fair coin with getting 200 heads from 300 tosses of a fair coin (Fischbein & Schnarch, 1997; Tversky & Kahneman, 1982). Individuals influenced by heuristic thinking say the chances are the same for both situations (Fischbein & Schnarch, 1997; Tversky & Kahneman, 1982). The people who demonstrated heuristic thinking focus their attention on the proportions associated with the population and dismiss the sizes of the two samples.

The sixth and seventh activities focus on the availability heuristic thinking. The first of these two has the participant forming two different committees from a given number of people. Specifically, the question asks the participant to find the number of ways that you can make a committee with two members who are chosen from 10 people, and this is smaller than, equal to, or greater than the number of committees you can make with eight people who are chosen from 10 people (Fischbein & Schnarch, 1997; Tversky & Kahneman, 1974). The person influenced by heuristic thinking will claim that the number of ways to create two member committees is greater than eight member committees because it is easier to envision more ways to create two member committees (Fischbein & Schnarch, 1997; Tversky & Kahneman, 1974). The second availability scenario is a combinatorics problem designed by two members on the research team. This activity uses marbles and boxes. The goal is to determine the number of different ways the marbles can be placed into the boxes. Specifically, the participant compares person A, who has two marbles, one red and one green, and four boxes with a second person B who had four marbles, one red, one green, one blue and one yellow, and two boxes. The participant are asked to determine who has more ways to place his marbles into his boxes; person A, person B, or both people had the same number of ways to place their marbles into their boxes. The person drawing on heuristic thinking would state that the person with two marbles has more ways to place his two marbles into his four boxes. This seems reasonable because it is easier to envision more ways to place two marbles into four boxes.

Perturbation Activities

The Perturbation Activities instrument (see Appendix C) includes four activities. Two activities have multiple-choice responses, and two activities have open-ended responses. The instrument has two aims: to perturb the participant's reliance on heuristic reasoning or sound probabilistic reasoning and to reveal the participant's stability of thinking associated with each. That is, if a participant demonstrates heuristic thinking in the first session, can that participant be perturbed away from System 1 thinking towards System 2 thinking in this session. Likewise, vice versa, if a participant draws on sound probabilistic reasoning in the first interview session, is the reasoning stable to resist attempts to disturb it during this session.

The probability problem solving activities include: the Hospital problem (Fischbein & Schnarch, 1997; Shaughnessy, 1977, 1981; Tversky & Kahneman, 1974), the Chips and Randomness problem (Tversky & Kahneman, 1974), the Description of an Individual problem that is modified but informed by Tversky & Kahneman, (1974), and the Marbles and Boxes problem created by the researcher but informed by Fischbein and Schnarch (1997), Shaughnessy

(1977) and Tversky and Kahneman, (1973). In the following paragraphs, each activity is explained and they include initial ways to perturb the students thinking. They are reported as cues removed or as cues added. These were included into the activities as a guide for the researcher when interviewing the students. They will not be included in the analysis per se.

The Hospital problem (see Appendix C) is a multiple-choice activity that is designed to invoke the representativeness heuristic thinking when comparing two hospitals that are different sizes. Specifically, the question asks the participants to determine which hospital has more days throughout the year, in which the proportion of baby boys born each day would be more than a given percent; a small hospital, a large hospital, or they were the same. The problem has two different formats. A more recent version includes a statement on the size of each hospital and a specific number of days (Fischbein & Schnarch, 1997), but the earlier version used by the mathematics education researcher, Shaughnessy (1977) is more simply stated. It lacks the numeric sizes of the hospitals and the particular time period is changed to "more days." In this current study, Shaughnessy's problem is used and it is stated as follows: "The chance that a baby is born a boy is about half. Over the course of an entire year, would there be more days when at least 60% of the babies born were boys in: (a) a large hospital, (b) a small hospital, (c) makes no difference" (Shaughnessy, 1977, p. 311). An attention cue is added to the problem by changing the 60% to 100%. Going to the extreme with the 100% may perturb a student's thinking. Reducing the percentages into smaller increments, such as 90%, 80%, 70% towards the initial 60% may locate the point where their understanding and reasoning ends and heuristic thinking begins.

The Chips and Randomness problem (see Appendix C) is an activity that was a multiple-choice response used by Cox and Mouw (1992) to reveal the representativeness heuristic thinking. Originally, the problem was stated so that the participants were to “imagine two urns, each containing thirty identical poker chips. One side of each poker chip was marked with a number while on the other side was marked with a symbol” (p. 167). There are five chips chosen at random from each urn. To perturb the participants thinking, the cues added approach has the student observe two sets of randomly drawn chips with the symbol side up (Cox & Mouw, 1992). The participants are asked to select the more likely outcome between the two sequences or that they were both equally likely in which both sequences of chips appeared random. After the participants responded, the other sides of the chips were shown on the instrument that showed two sequences of numbers (Cox & Mouw, 1992). One set had a consecutive numerical sequence, one through five, and the other had a random numerical sequence. The participants were asked to select the multiple-choice response that was more likely. In the cues removed group, Cox and Mouw (1992) administered the task showing the numbers first and then showed the participants the random symbols on the opposite side of the chips. In this current study, the activity is acted out in front of the participant and the students are asked to determine the more likely sequence or if both sequences are equally likely.

The third probability activity is called the Description of an Individual (see Appendix C) that addresses the conjunction effect. It is a modified version of the “Linda is a bank teller” problem from Tversky and Kahneman (1983) that was used in the first interview session but the description differs from the version used in the first interview session. As was done before, the participants are to rank the probabilities associated with each statement. The stereotyping is consistent with middle-school student’s experiences with social grouping among their classmates. For example, the typical social groups included the high academic achievers, the athletes, and the musicians. In this activity, a fictitious student named Stacey is described as a high academic achiever; a 13 year old middle-school girl who is outspoken, very bright, she loves her science class and particularly enjoys the topics related to biology and the eco-system. Following this description there are six statements that the participants are asked to rank from the most probable to the least probable. One statement has two of the attributes describing Stacey combined to form a compound statement. The remaining statements include several filler statements in which some are consistent with the description of Stacy and other that are not consistent with the description of Stacy. The attention cueing draws on the Cox and Mouw (1992) design in which the cues removed approach has the compound statement at the end of the list, while the attention cues added approach has the compound statement listed first. An alternative cues added approach was designed in which the words "statement 1," "statement 2," and so forth, were replaced with "event 1," "event 2" and the compound statement is identified specifically as "event 3 and event 5." This alternative approach is used in response to the dismal results observed from the pilot study performed by the researcher during the Summer 2013 in which none of the participants were perturbed as suggested by Cox and Mouw (1992). It is believed that the attention cue that focused on the location of the compound statement was too subtle for the middle-school students that participated in the pilot study.

The fourth activity, called the Making a Committee problem (see Appendix C), was designed to prompt the availability heuristic thinking as first described by Tversky and Kahneman (1974) and used later by Fischbein and Schnarch (1997). This activity is a multiple-choice response and it is modified from the same type of problem used in the first interview session. The modifications are subtle. The first modification replaces the word *teams* with *committees* and the second modification reduced the number of total members from 10 to 6. In this activity, the participants are asked to determine if the number of ways that you can make a committee with two members who were chosen from six people were: smaller than, equal to, or greater than the number of ways that you can make a team with four members who were chosen from the six people. The attention cues removed approach has the problem stated without a recommendation to create a tree diagram or table or other pictorial diagram, while the attention cues added approach includes hints to create a diagram or provided blocks for the student to use. The Venn diagram was the cueing used to perturb the students thinking. The other recommendations shown in Appendix C are available if needed. A summary of the instruments for this study is shown in Table 3.1.

Table 3.1
Instruments and Their Purpose

Instrument Name	Number of Questions	Types of Questions
Probability Content Knowledge	14	Single probability events Compound probability events Combinations
Cognitive Development (4) and Heuristic Thinking (7)	11	Comparing single probability events Compound probabilities events Combinations Representativeness Conjunction effect Availability
Perturbation Activities	4	Representativeness Conjunction effect Availability

Procedures

The data collection procedure is divided into two sections. This first section describes a screening process. The first part of the screening process uses the Probability Content Knowledge Assessment Instrument (see Appendix A) to 67 eighth-grade students. The students with the highest scores advance to the second part of the screening process. The second part of the screening process uses the Cognitive Development and Heuristic Thinking Instrument (see Appendix B). In addition, this section describes how each student's response time is documented. A Decision Tool is created to identify the six students to participate in this study. The second section describes the procedures that were used when administering the Perturbation Activities to these six students.

Screening Procedures

The data collection associated with the screening process had three phases. The first phase administered the Probability Content Knowledge Assessment instrument (See Appendix A) to 67 eighth grade students. The results from this instrument were used to identify the students with the strongest probability knowledge to advance to the next screening phase. The second screening phase is conducted during the first interview session using the Cognitive Development and Heuristic Thinking instrument (See Appendix A). This instrument evaluates the students' level of cognitive development, their tendency to use heuristic thinking and their response times. The response time measures the length of time that elapse from when the student completes reading the problem until the student gives an answer. This is used to create a *fast* and *slow* response time profile for each student.

The third phase ranks students using a Decision Tool that was designed by the researcher for this study. This tool is described in the Design and Data Analysis section. The Decision Tool identifies the most suitable students to advance to the second interview to engage in the Perturbation Activities (see Appendix C).

Perturbation Procedures

The students' initial answers to these activities are used as a starting place to determine how best to perturb their thinking depending on their answer. If the student responds with an answer that is labeled as a heuristic response then intuitive thinking is suspected, with an answer that is labeled as a correct responses then reasoning is suspected. If the student answers with a response labeled as incorrect, then either heuristic thinking or faulty reasoning is suspected. The researcher attempts to perturb the student's thinking and may draw on the guidance from Cox and Mouw's research (1992). The goal is to explore the perturbations experienced by the students that moves the students from intuition toward reasoning or moves the student away from reasoning. For example, if the student responds heuristically, as determined by their answer, questions are asked by the researcher in order to perturb the student into using reasoning that leads to the correct answer.

The instruments used in this study are used to collect the data. They each have a specific purpose and they are administered at a particular time during the data collection process. A summary of the instruments is shown in Table 3.2.

Table 3.2
Administration of the Instruments and Their Purpose

Administered	Instrument Name	Purpose
Pre-Interviews	Probability Content Knowledge	Screen Students for Interview Session 1
Interview Session 1	Cognitive Development and Heuristic Thinking	Screen Students for Interview Session 2
	Five Number Summary	Calculate Response Times
	Decision Tool	Identify students for Interview Session 2
Interview Session 2	Perturbation Activities	Perturbing Student thinking Explore Thinking

Design and Data Analysis

The design and data analysis section is divided into three major sections. The first section explains the screening process and the second section explains the qualitative analysis and the third section contains the research questions. The first section describes the two stage screening process and identifies the six students that are used in this study. The second section explains the qualitative analysis and it describes the thinking profiles based on the data coding process. The third division is the research questions and the section explains how the research questions may be answered based on the analysis.

Screening Process

The screening process for the most suitable candidates has two stages. The first stage uses the Probability Content Knowledge Assessment instrument (see Appendix A) administered to three eighth grades classes that consist of 67 students. The students with high scores will advance to the first interview in which the Cognitive Development and Heuristic Thinking instrument (see Appendix B) is administered. Prior to the second interview, six students will be identified and asked to participate in the second interview sessions. The following paragraphs describe this process in detail.

Stage One: Pre-Interview Assessment. The Probability Content Knowledge Assessment had 14 multiple-choice questions that are worth one point each. The goal is to choose16 students with the highest scores to advance to the first interview session. In the case that more than 16 students shared the same high scores the written work would be examined to further distinguish between the candidates. For example, when examining the work it may be the case that a student circled the correct answer for the wrong reason or due to an error in calculation, which would eliminate the student from the candidate list. The original goal is to have all potential candidates score above 82% in order to provide access to the study for the students enrolled in the Math-8 class. The researcher believed this was the lowest level of competency for probability knowledge that reduces guess work.

Stage Two: First Interview. The Cognitive Development and Heuristic Thinking instrument is used during the first interview. The students are assessed in three areas: (1) response time, (2) level of cognitive development, based on the correctness of their solution, and (3) use of heuristic thinking based on the response selected. Each of these areas is addressed below.

***Response Time*.** The time to answer a question is measured from the time the participant completes reading the question aloud to the moment when the first answer was given. It is anticipated that most of the responses will be verbal but there may be times when the participants answer the question by writing the solution or circling one of the multiple-choice answers. This data will also be used to identify participants' heuristic thinking (Kahneman, 2003).

This study focuses on *how* and *what* the participants are thinking and that requires the think aloud protocol to differentiate between heuristic thinking and reasoning. Time may be a promising indicator of heuristic thinking but the think aloud protocol slows down the observable response time since the participants are engaged in two mental activities at the same time: (1) responding to the probability activity and (2) attempting to communicate their thoughts in real time that includes translating non-verbal thinking to verbal utterances. This requires a different way to calibrate and distinguish *fast thinking* from *slow thinking* that is unique to each participant. Descriptive data analysis based on the *Five-Number Summary* provides a way to distinguish fast response times from slow response times that are unique for each participant. The *Five-Number Summary* (Johnson & Kuby, 2005) includes: median, first quartile, third quartile, highest value in the data set, and the lowest value in the data set. The first quartile is selected as the demarcation for fast response times. This means, response times that are less than the first quartile measure, the lower 25% of the response time data, are identified as fast responses. The third quartile is selected as the demarcation for the slow response times. This means, response times that are greater than the third quartile measure, more than 75% of the response time data, are identified as slow responses. The responses between these two values are identified as a typical or normal response time for the person and offer no insight to distinguish fast thinking from slow thinking based on time. The *Five-Number-Summary* is appropriate because this technique is less sensitive to large variations and outliers in the data as opposed to calculating averages and standard deviations. In addition, the number of data points is too few to explore an underlying normal distribution for response times.

If the response time fails to indicate slow thinking or fast thinking, additional characteristics may be observed that aligned with heuristic thinking (intuition). For example, heuristic thinking appears to be marked by a failure of a person to think deeply about a problem due to a quick response. The individual who engages in heuristic thinking may inadvertently substitute an attribute given in the question with something else or the individual may answer a different question. If a substitution appears absent, the participant language may suggest that intuition is dominating his thinking. For example, if he says, “I don’t know why, I just know the answer is one-half.”

Cognitive Development. The participant’s cognitive level of development is assessed using four questions drawn from Piaget and Inhelder’s (1975) experiments as described above. The solutions to the cognitive development activities are scored 1 point each, either right or wrong. In addition, the transcripts and videos are used to identify the characteristics as outlined in each stage of development for the four areas of probability as described by Piaget and Inhelder (1975). Based on the participant's solutions and the characteristics shown, the participant is evaluated as belonging to the pre-operational, the concrete operational, or the formally operational stage (Piaget & Inhelder, 1975). It is intended for all participants who are identified to advance to the interviews based on their Probability Content Knowledge Pre-assessment, would be performing at the formal operational stage in all areas.

Heuristic Thinking. The participants' heuristic thinking is evaluated using the seven activities. The solutions to the activities, the response times, and the transcripts are used to identify heuristic thinking. First, the solutions provide the major clue to a heuristic response as most of these activities have been used by other researchers in their heuristic research (Fischbein & Schnarch, 1997; Shaughnessy, 1977, 1981; Tversky & Kahneman, 1974, 1982). This means one of the three selections for the multiple-choice questions was identified as a heuristic response or also called the main misconception. The other two selections were identified as correct - suggesting correct reasoning and wrong – suggesting incorrect thinking (Fischbein & Schnarch, 1997; Shaughnessy, 1977, 1981; Tversky & Kahneman, 1974, 1982). If a person scores four or more incorrect out of the seven questions then the person may be more inclined to use heuristic thinking since this is more than half. During the interview, the researchers will ask for clarification of a solution to validate heuristic thinking and attempt to identify the attribute being substituted by the participant. This may be identified by asking the participant, which part of the question is attracting her attention. The transcripts and the video of non-verbal responses are necessary to help identify the attribute being substituted.

Identifying Advancing Students

The Decision Tool uses five variables or criterion to evaluate the student: the mathematics class the student is enrolled in at the time of the study, probability content knowledge score, cognitive development score, heuristic thinking score. Next to the heuristic thinking score the number of *incorrect solutions selected* and the number of times heuristic thinking is used is noted. Weights are assigned to each of the five criterion to rank their importance. The weights for each criterion are greater than zero and less than 1 and the five weights summed to 1. Created in an excel spreadsheet file, the decision tool allows the researcher to perform sensitivity analysis by altering the weights that clarify the rankings in an effort to identify the most suitable students for this study. It is the case that another researcher may have different notions on the criteria and the weighting. The following paragraphs discuss these five variables and their final weighted values.

Course enrollment is an important criterion to promote access to the study for all of the students enrolled in the host teacher's three eighth-grade mathematics courses: Math 8, Algebra I, and Geometry. To promote an equitable means for inclusiveness, Math 8 is given the highest value of 3, Algebra I a value of 2, and Geometry a value of 1. These values are inverted because it is anticipated that most of the students in the Algebra I and Geometry classes will score higher on the Probability Knowledge Assessment. The weight assigned to the course enrollment criterion is 0.05. The value of 0.05 is chosen because of the tension that may arise between inclusiveness and choosing the most suitable students for the study.

The participant's probability knowledge and cognitive development level are judged to be the two most important criteria and are given a weight of 0.35 for each. The value of 0.35 is chosen for each because each of these two components is equally important for the study. It is necessary that the participants hold a high level of probability knowledge and the ability to use that knowledge in novel situations.

Identifying students who demonstrate heuristic thinking is important and it is weighted 0.15. This value is used because if the students fail to show heuristic thinking, then they will not be suitable for the goals of this project. This value may appear low compared to the other criteria due to the focus of the study, however, it is anticipated that most of the participants will show heuristic thinking on three or more occasions.

The last criterion identifies the number of questions the students answer wrong and it is given a weight of 0.1. This weight provides another way to evaluate the students' probability knowledge. Specifically, the "wrong" answer may have been a guess due to a lack of knowledge. Guessing an answer based on intuition differs from guessing an answer based on a lack of knowledge, although intuition may potentially direct a person to a wrong solution. Therefore, it is necessary to consider these incorrect responses.

Qualitative Analysis

The qualitative analysis section describes how the data collected from the first and second interview will be analyzed. The analysis is expected to generate useful categories that can be used to answer the research questions. The following paragraphs describes the data-coding process.

Data-coding

The Constant Comparison Method is used on all transcribed video recording for the six students in order to analyze their ways of thinking (Charmaz, 2006; Glaser, 1965). The coding process begins with line coding to help the researcher remain "open to the data and see nuances in it…and refocus later interviews" (Charmaz, 2006, p. 50). Typically, when using the constant comparison method, the next iteration of coding uses the terms that resulted from the line coding, also called focused coding, that will eventually lead to conceptual categories that are also called themes (Charmaz, 2006). The constant comparative method is used "to aid analysts … in generating a theory which is integrated, consistent, plausible, close to the data, and in a form which is clear enough to be readily, if only partially, operationalized for testing in quantitative research" (p. 437-438.). Furthermore, Glaser (1965) states that the method is not designed so that two different people working independently using the same data would produce the same results. Glaser (1965) explains that the constant comparison method attempts to generate many properties and hypotheses about a general phenomenon. The properties include such things as conditions, consequences, dimensions, types, process, to name a few, but they must be integrated into an overarching theory, which is not generalized beyond the scope of the study nor is the overarching theory to be proved in the study (Glaser, 1965).

This current study begins with line coding, in which each line is reduced to a descriptive word or a short phrase. The goal of these descriptive words or phrases is to generate as many categories as possible as suggested by Glaser (1965). New data are compared to the previous coded data within a category to enrich a current category or create a new category (Glaser, 1965). This constant comparison of the incidents to previous categories begins developing the theoretical properties for that category. The properties of a category opens it up so the researcher can see the width, breadth, and conditions that impact that category along with its connection to other categories (Glaser, 1965). During the analysis the researcher will write analytic memos recording ideas that spring from the analysis. As the process continues the unit of analysis changes from incidents to incident comparison then to the comparison of the incidents to the properties of that category. At this final point the property of a category begins to be relatable in many different ways so that it is consistent within the category and connections are made between categories (Glaser, 1965). This means as the researcher breaks the data apart into categories the researcher also begins to bring the categories back together to show the relationship between them.

The process, if unchecked, can keep growing so that the analysis becomes overwhelming. Fortunately, “delimiting features” contains the process to prevent this from occurring (Glaser, 1965, p. 441). There are two ways that can be used to contain the process, which will be used in this study. The first way relates to the theory and the second way is based on the original list of categories used for coding (Glaser, 1965). For the first way, when the theory begins to solidify, the modifications become fewer and fewer until there are no more modifications made to the categories except for clarification (Glaser, 1965). This solidification process helps the researcher stay within the boundaries of the emerging theory. The second way pertains to the original list of categories that helps to focus the researcher because the more discriminating the researcher is with choosing coding and analyzing incidents and comparing important incidents into smaller sets of categories, the more saturated the category (Glaser, 1965). The advantage of this method is its close connection to relevant data throughout the process; relevant to the categories and the theory.

Specific to this study, all of the eleven interviews in the first interview session are transcribed after the interviews. The six students who advance to the second interview has their second interview session transcribed after the interviews and both interview sessions are line coded shortly after they are transcribed (See Appendix D - Procedural Memo 3). The line coding is performed electronically with the *Comment* feature under the *Review* module in Microsoft Word. The words or terms that can be used in the line coding are written on sticky notes in order to generate categories. The sticky notes and a large sheet of paper or a white board can be used to generate categories. The manual process is preferred in order to stay close to the data, visually see the relationships between categories as they developed, and regroup the categories to refine category codes is expected to support the analysis.

In this study, each problem or activity is identified as an incident in the researcher's field notes. This seems appropriate because it will produce a simple way to locate data about a particular student based on the interview session and the activity. For example, the participant named Ben performs the fifth activity in the first interview. When the data is transcribed the episode will be labeled as B1.5, in which the letter B represents the students name, Ben, the number 1 represents the interview session, and the number 5 represents the episode. This will facilitate the data analysis.

Research Questions

In this study there are three categories that were noted in the research questions that came from the theory. The three categories were: (1) thinking type, which refers to intuitive thinking (S1) or reasoning (S2) that will be used when the students solve the probability problems; (2) perturbations, which refers to the perturbations that will be initiated by the researcher in order to move the students' from intuitive thinking (S1) to reasoning (S2); and the last category is (3) stability that refers to the commitment level the students have with their initial answers. The data analysis will identify if new categories or themes emerge to support a deeper understanding of the student's thinking in order to address the following research questions:

1. In what ways are participants' problems solving strategies influenced by System 1 or System 2?
2. How are participants' problems solving strategies perturbed from System 1 to System 2?

3. How stable or strong are participants' problem solving strategies associated with System 1 or System 2?

Summary

This qualitative study uses the task-based clinical interview with the think-aloud protocol that provides a way to investigate eighth grades students thinking when solving probability problems that are designed to invoke heuristic thinking. In order to witness intuitive thinking or reasoning when solving novel probability tasks, it is necessary that the participants have the skills and the knowledge to perform at the formal operational level (Piaget & Inhelder, 1975). As a result, a screening process is developed that uses two instruments. The two interviews from the six students selected for the study are videotaped and transcribed by the researcher. The transcriptions are line coded and analyzed using the constant comparison method. The analysis is supported with the student's written work, jottings, and the researcher's memos in order to answer the three research questions.

Chapter 4

Results

Chapter four is divided into three major sections: the results of the participant selection process, explanation and results of the data coding process, and findings with respect to the research questions. The participant selection process describes how the six participants were selected for the final interview. The second section discusses the results from the data coding process based on the constant comparison method (Glaser, 1965). The coding process highlighted the importance of confidence, internal perturbations, and misapplication of fraction knowledge in characterizing students' ways of thinking and their solution strategies. These and other results from the data coding were then used to create profiles of behavior consistent with System 1 thinking, System 2 thinking, and other salient ways of thinking that were identified by the researcher. The third section shares the findings related to each of these profiles and discusses them in respect to the research questions.

Participant Selection

Stage One: Probability Content Knowledge

The researcher administered the Content Knowledge Assessment instrument (see Appendix A) to 67 students from three 8th grade classes: 20 Algebra students, 20 Math 8 students, and 27 Geometry students. The students' scores were ranked from highest to lowest; the highest performing students were candidates to advance to the interview portion of the study. Ideally, 16 students with scores at the higher end of the 90% range would have participated in the study. However, there were only 12 students who scored in the 90% range and were interested in participating in the study. As a result, the threshold was lowered to a score of 80% in order to expand the candidate pool of students to enlist the 16 students for the study. However, the researcher was still unable to recruit 16 participants. In addition, to promote inclusiveness, three Math 8 students were asked to advance to the first interview even though their performance scores fell short of the 80% score. In the end, only one Math 8 student agreed to be interviewed. This student achieved the highest score in the Math 8 class, 71%. Several students who were initially identified to participate in the study, changed their mind, or failed to return the parental consent form. In the end, 11 students participated in the first interview.

Stage Two: Identifying Students for the Second Interviews

The goal of stage two was to identify six students that would be the focus of the study. There were aspects of the evaluation of the students that were identified as necessary for this study that extended beyond their probability and statistics knowledge that was assessed in the first stage of the identification process. It was necessary for the students to have reached the formal operational stage of cognitive development because this would potentially enable the students to use their probability knowledge in novel ways. In addition, the students needed to show a greater tendency to use heuristic thinking in comparison to wrong thinking, which should have been minimized by this stage of the identification process due to the use of high scores on the Cognitive Knowledge Assessment (Appendix A) as a selection criterion. In order to identify the most suitable six students, the Cognitive Development and Heuristic Thinking instrument (Appendix B) was used to assess their stage of cognitive development and their tendency to use heuristic thinking. This instrument was divided into two sections. The first section had four questions that assessed the students' level of cognitive development for probabilistic reasoning (Jones, et al., 1999; Piaget & Inhelder, 1975). The four questions addressed different aspects of probability that included calculating probabilities, finding combinations without a formula, understanding the nature of randomness, and understanding probabilities for compound events. A student's response was evaluated as either satisfactory (worth one point) or unsatisfactory (worth zero points) based on the correctness of their response and the explanation accompanying their solution, in which both had to be satisfactory to earn the one point.

The second section had seven questions that evaluated the students' inclination towards heuristic thinking. The seven questions focused on the *representativeness*, *conjunction fallacy*, and *availability* heuristics. Each question had three multiple choice answers: one was the correct solution, one was the incorrect solution, and one indicated heuristic thinking (Fischbein & Schnarch, 1997; Shaughnessy, 1975; Tversky & Kahneman, 1974, 1982). The ideal student would have answered several of the items in the second section of the instrument with a heuristic response or a correct response. Therefore, the incorrect choice was deemed inferior to heuristic thinking because it signified a lack of probability knowledge. The heuristic response indicated probability knowledge that was misapplied as expected based on prior research. The results from the 11 participants, to whom Cognitive Development and Heuristic Assessment (Appendix B) was administered are presented in Table 4.1. Each student chose a pseudonym, which is used throughout Chapter 4 and 5.

Table 4.1
Cognitive Development and Heuristic Assessment Results

Name	Gender	Class	Probability Knowledge Assessment (n = 14)	Cognitive Development Assessment (n = 4)	Heuristic Assessment - Number of Heuristic Responses (n = 7)
Ben	M	Geometry	100%	100%	4 (0 incorrect choices)
Billie Jones	F	Geometry	89.2%	50%	5 (1 incorrect choice)
Brittany	F	Algebra I	92.8%	75%	4 (1 incorrect choice)

(continued)

Table 4.1

Cognitive Development and Heuristic Assessment Results (continued)

Name	Gender	Class	Probability Knowledge Assessment (n = 14)	Cognitive Development Assessment (n = 4)	Heuristic Assessment - Number of Heuristic Responses (n = 7)
Chelsea	F	Geometry	92.8%	75%	4 (1 incorrect choice)
Diane	F	Geometry	92.8%	50%	4 (0 incorrect choices)
Elise	F	Algebra I	100%	75%	4 (1 incorrect choice)
Guinea Pig	F	Geometry	100%	100%	4 (0 incorrect choices)
Josh	M	Geometry	92.8%	100%	4 (0 incorrect choices)
Maggie	F	Math 8	71%	50%	3 (2 incorrect choices)
Maree	F	Algebra I	85.7%	50%	3 (2 incorrect choices)
Ryan	M	Geometry	92.8%	100%	4 (0 incorrect choices)

Immediately, four students stood out Ben, Guinea Pig, Josh, and Ryan -because of their high scores on the Cognitive Development Instrument section and the Heuristic Thinking section of the Cognitive Development and Heuristic Thinking instrument (Appendix B). The selection of the last two participants was less clear cut so a decision tool was created. The decision tool ranked the participants by weighting various criteria as indicated in Table 4.2. The scores from the students' probability knowledge assessments and the cognitive development assessment were given the greatest weight of 0.35 because it was necessary that the students have strong content knowledge and have reached the formal operational stage of cognitive development in order to allow for the possibility of appropriate System 2 reasoning. In addition, it was important that the students demonstrated a tendency to use heuristic thinking since this study was focused on the impact of heuristic thinking on problem solving strategies. Therefore, this was given a weight of 0.15. However, answers that were identified as incorrect were considered red flags because they suggested the student lacked probability understanding or the student's cognitive development stage was lower than the formal operational stage of development or both. Therefore, the number of incorrect responses was given a weight of 0.1 (In retrospect, the weight should have been 0 or -0.1 but changing the weighting of these values would have no effect on the students' final rank order in this study). The lowest weighted criterion was course enrollment and it was given a weight of 0.05, in which Math 8 was given a preference score of 3, Algebra a preference score of 2 and Geometry a preference score of 1. The preference score provided an opportunity for the Math 8 students to be competitive with the students from the other two courses. When the students' scores were tied, the student with the higher score on the Probability Knowledge Assessment was ranked higher. The results of the selection process are shown in Table 4.2.

Table 4.2
Decision Tool to Select Participants for the Second Set of Interviews

	Weights	0.05	0.35	0.35	0.15	0.1		
Name	Gender	Course	Probability Knowledge	Cognitive Development	Heuristic	Wrong	Ranking Value	Ranking Order
Ben	M	1	1.000	1.000	4	0	1.35	1
Billie Jones	F	1	0.892	0.500	5	1	1.19	5
Brittany	F	2	0.928	0.750	4	1	1.19	5
Chelsea	F	1	0.928	0.750	4	1	1.14	
Diane	F	1	0.928	0.500	4	1	1.05	
Elise	F	2	1.000	0.750	4	0	1.31	3
Guinea Pig	F	1	1.000	1.000	4	1	1.25	4
Josh	M	1	0.928	1.000	4	0	1.32	2
Maggie	F	3	0.710	0.500	3	0	1.02	
Maree	F	2	0.857	0.500	3	2	0.82	
Ryan	M	1	0.928	1.000	4	0	1.32	2

Course Key: Math 8 = 3; Algebra I = 2; Geometry = 1

Based on the ranking, Ben (B), Josh (JS), Ryan (R), Elise (E), and Guinea Pig (GP) were the first five participants selected for the study. Brittney (Br) was the sixth participant chosen for the study because her Probability Knowledge Assessment score was higher than Billie Jones. As such, these six students were invited to participate in the second interview session and were the participants whose ways of thinking was explored in order to answer the research questions.

Development of the Data Codes

The development of the data codes began with the response times. This section describes the three types of response times: fast, slow, and inconclusive. An initial analysis was done using the response time and the students' initial answer to get an overview of the results. The initial analysis highlighted instances that required further analysis that was achieved using the data codes that were developed.

Response Times

During the first interview session the research team recorded the time it took for each student to answer each of the questions. These response times were used to create response time profiles for each of the six participants. The response times (in seconds) were measured starting from the time the students completed reading the question, which sometimes included multiple readings, or from the time the students received a response to their last clarifying question. The ending time was measured in one of two ways: audibly when the student verbalizes the answer or visually when the student selected an answer on the instrument or wrote an answer. The response times were summarized using a five-number summary: the student's quickest response time, slowest time, median time, and the first and third quartile response times.

The five-number summaries were used as a guide to distinguish between fast and slow thinking. Response times at or below the first quartile times suggested fast thinking and response times at or above the third quartile times suggested slow thinking. The response times that fell within the interquartile range weakly endorsed slow or fast thinking. For example, Ben had a first quartile value of 13 seconds, which meant if he responded to a question in 13 seconds or less, his thinking was labeled fast. He had a third quartile value of 24 seconds, which meant his thinking was labeled as slow if it took him 24 or more second to respond. A response time that measured between 13 seconds and 24 seconds was labeled as inconclusive, meaning it was neither particularly fast thinking nor particularly slow. The results are shown in Table 4.3.

Table 4.3
Five Number Summary Results

Name	Low	Q1	Median	Q3	High
Ben	2	13	20	24	61
Brittany	1	8	14	32	65
Elise	1	2	4	20	28
Guinea Pig	2	4	9	22	58
Josh	4	11	15	53	78
Ryan	2	5	10	38	148

The information shown in Table 4.3 was used segregate response times into three different categories. The three different categories are: slow, inconclusive, and fast. These response time are used shortly in the initial analysis.

Table 4.4
First Instrument – Response Time Profiles (in seconds)

Student	Fast (F)	Inconclusive (I)	Slow (S)
Ben	≤ 13 sec	14 - 23	≥ 24
Brittany	≤ 8	9 – 31	≥ 32
Elise	≤ 2	3 – 19	≥ 20
Guinea Pig	≤ 4	5 – 21	≥ 22
Josh	≤ 11	12 – 52	≥ 53
Ryan	≤ 5	6 – 37	≥ 38

These divisions were a general guideline, and responses in the inconclusive range would occasionally be considered when identifying an example of System 1 or System 2 thinking if the researcher observed behaviors that aligned with the other indicators (see Profile *S-I* and profile *S*-R). The category response times summaries for the six students are shown in Table 4.4. The inconclusive response times were an indication that the students' responses would need to be examined more closely, which was done through the qualitative analysis that is discussed later in the chapter.

Initial Analysis

There were seven questions from the first set of interviews and four questions from the second set of interviews that were meant to elicit System 1 thinking. In this first analysis of the data, the students' way of thinking – System 1 or System 2 – was first assessed by the answer they provided – heuristic for System 1 or correct for System 2 – and their response time – fast for System 1 and slow for System 2. The students' answers (before attempts to externally perturb their thinking) with their relative speed are shown in Table 4.5 for the first interview session and Table 4.6 for the second interview session. The letter F suggested a fast response, the letter S suggested a slow response and the letter I suggested an inconclusive response time.

Table 4.5
Interview Session One – Initial Analysis

Name	Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Question 7
Ben	Correct F	Correct* I	Heuristic I	Correct I	Heuristic I	Heuristic S	Heuristic I
Brittany	Correct I	Heuristic S	Heuristic I	Correct F	Heuristic F	Wrong I	Heuristic I
Elise	Heuristic F	Heuristic F	Correct I	Correct F	Heuristic I	Wrong I	Correct* I
GP	Heuristic F	Correct* S	Correct F	Correct I	Heuristic I	Heuristic I	Correct S
Josh	Heuristic F	Heuristic I	Heuristic I	Correct I	Heuristic F	Correct S	Correct S
Ryan	Heuristic F	Heuristic I	Correct S	Correct I	Correct I	Heuristic I	Correct* S

* Math wrong

As shown in Table 4.5, in the first interview session there were seven instances when the students quickly gave a heuristic answer indicating System 1 thinking, and there were four instances when they slowly gave a correct answer, indicating System 2 thinking. Note that the response times were calculated using all 11 of the items in the first instrument, but only the heuristic section with its seven items are shown in the table.

In table 4.6 the students' answers and their response times are shown, which were based on the profiles developed from the first interview. Josh was the only participant who responded quickly to two activities. One of his responses was correct, which may suggest automaticity. The second fast response time was answered with the heuristic answer. It would be expected if the response times were randomly distributed that each student would have one response time categorized as fast, one as slow and two as inconclusive. Yet this pattern was not observed, with far fewer fast responses and more inconclusive responses than would be expected.

Table 4.6
Interview Session Two - Initial Analysis

Name	Activity 1	Activity 2	Activity 3	Activity 4
Ben	Correct S	Correct I	Correct S	Heuristic I
Brittany	Heuristic I	Correct I	Correct I	Heuristic S
Elise	Heuristic I	Correct S	Correct S	Incorrect I
Guinea Pig	Correct I	Correct I	Correct I	Heuristic S
Josh	Heuristic F	Correct F	Heuristic I	Correct I
Ryan	Incorrect I	Correct I	Heuristic I	Incorrect S

This initial analysis was insufficient both in terms of interpreting the response times that fell within the inconclusive range and in interpreting response times that did not match the answer choices as expected.

Constant Comparison Coding

The three coding categories – ways of thinking, perturbations, and stability of response – were implied by the original research questions. Hence related codes were attended to throughout the initial analysis and coding processes. Anticipated codes for the ways of thinking category were System 1 thinking and System 2 thinking. The perturbations category originally referred to the alternative ways the research team would disrupt the students' thinking process. The stability of response category would be anticipated to include codes that showed the level of commitment the student had toward their initial answer.

Although the researcher attended to these categories during coding, she also engaged in a formal coding process, the constant comparison method (Glaser, 1965), using both sets of interviews. The details of the coding process are included in Appendix G, and the final coding results are shown in Table 4.7. Although, the final categories that emerged from the coding process – emotion, perturbations, and mathematics used – did not themselves contribute to the current analysis, some of their sub-categories provided insights into the data that were important for the current analysis. In particular, the emotions category provided behavioral indicators that distinguished between confident and timid responses, which were used to characterize the students' ways of thinking. The perturbation category was split into external and internal perturbations, which again, were used to characterize students' ways of thinking. Finally, the mathematics used category resulted in the identification of many students' responses that relied on their fraction knowledge, which was a useful characterization of problem solving strategies addressed later in the analysis. The following subsections describe in greater detail the components of the categories that impacted the current study.

Table 4.7
Data Coding Results

Categories	Sub-Categories	Key Terms	Codes from the Data
Emotions	Emotions: Non-verbal	Body Movements	Learning forward, finger point, re-reading, hands on face, leans on hands, rushes through problems, nodding head
		Facial expressions	Eyes enlarge, squints, grimaces, smiles

(continued)

Table 4.7
Data Coding Results (continued)

Categories	Sub-Categories	Key Terms	Codes from the Data
	Emotions: Verbal	Tone in language	Timid (Lacks confidence) – answers tentatively, limited or no explanations, no examples, speculates Confident – not change answer, affirming, convinces self, emphatic, sticks to original answer, unwavering, justifies answer to self, reassures self, tests situations, no need to assess answer
		Utterances – Non words	Sigh, chuckle, laughing, "Ah," "hmm," "Um"
Perturbations	Perturbations: Internal (May occur any time during the problem solving process)	Beginning of the problem solving process	Asks clarifying questions, confusion, hesitating, grappling, uncertain, unsure, stuck
		During the problem solving process	Hesitating, stopping, redoing, erasing, rejecting, revising, grappling, switching, self-correcting, adjusting thinking, reconsidering, realizing, uncertain, unsure, stuck, questioning, confused
		End of the problem solving process	Straddling between two answers, vacillates, different answers-initial and answers occurring later in the process
	Perturbations: External	During the problem solving process	Designed into the study – Second interview predominately
Use of Mathematics	Type of Mathematics Used	Proportional knowledge	Percentages, fractions, ratio, proportion, multiplicative relationships, easier/harder (for likelihood), chance

(continued)

Table 4.7
Data Coding Results (continued)

Categories	Sub-Categories	Key Terms	Codes from the Data
		Assembling information	Ranking, comparing, sizing, ordering, grouping, arranging
		Operations (calculating)	Multiplying, dividing, subtracting, averaging, simplifying, estimating, converting, combining, evaluating, devising, combining
		Relationships/definition	Equivalence, set/subset, compliment of a set, making connections, sequence defined, independent events
		Representations Concrete Pictorial	Used blocks, chips, coins Diagrams, graphs, other

Emotions Category. The most useful component of the emotions category was the differentiation between students' behaviors that indicated they were confident or timid. Confidence and lack of confidence were integral parts of the characterization of the various ways of thinking, which are discussed in the Profiles section in this chapter. Elise showed confidence when she solved the Two Heads versus Two Hundred Heads problem (Appendix B), while Brittney showed a lack of confidence or timidity when she solved the Making a Team problem (Appendix C).

An immediate answer was often an indication of confidence, but often times the students indicated confidence in their response through their general demeanor. This could occur both in the case of the students who clearly understood a problem and confidently answered it correctly and in the case of the students who *thought* they clearly understood the problem and confidently answered it heuristically. Ben offers an example of the first case when solving the Description of an Individual problem (Appendix C) his confidence seemed to stem from his mathematical understanding of the situation. He stated, “I guess I would put that as 3 (statement 3) because of having both (ranks statement 6 as 4), two is less likely than having one out of 2.” In this case, Ben said, “I guess” as an indication that he was justifying the answer to himself, not actually guessing. His confidence in this situation was evident by his demeanor and his ease when explaining his thinking. Examples of the second case include students who quickly and heuristically answered the Six Fair Coins problem, which is discussed later (see Protocol 4).

In contrast, Brittney exhibited behaviors suggesting that she lacked confidence when answering the Making a Team problem. She was indecisive and claimed she was wrong before offering an explanation about her thinking. Her indecisiveness and hesitation were strong indicators that she lacked confidence with her answer.

Perturbation Category. Perturbations were identified by moments in time when the students’ problem solving appeared to be interrupted. There were two sources of interruptions identified in the data: internal and external. The external perturbations were expected because they were designed into the second interview to provoke students thinking. Internal perturbations that were not explicitly provoked by the research team were also identified and could appear during any stage of the problem solving process. They were marked by hesitation, self-correction, or asking clarifying questions. Many times the research team hypothesized that internal perturbations were present because there seemed to be a change in direction in a student’s thinking. This was evidenced by behaviors such as the student switching an answer or vacillating between answers. The internal perturbation category was also observed when the students abruptly stopped their speech or actions during the problem solving process and then appeared to immediately know the answer to the problem. This again suggested that an internal perturbation interrupted the students' previous thought patterns.

Brittney and GP indicated these two different and distinct results of an internal perturbation, one that seemed to open the door to confusion and the other that opened the door to clarity. In the first example, Brittney's internal perturbation opened the door to confusion when she was asked to solve the Hospital problem (Appendix B). She changed her answer at the beginning of the problem solving process several times, vacillating between two answers: the two different sizes of the hospitals were equally likely to have at least 60% boy babies born or the larger hospital was more likely. This vacillation was interpreted as an internal perturbation or struggle. In contrast, GP experienced an internal perturbation that looked more like an insight when solving the Marbles and Boxes problem (Appendix B). GP drew a diagram of the boxes and as she explained her thinking, she suddenly stopped and stared at her drawing. After a few moments of silence she claimed they must be the same. From the researcher's perspective, this sudden cessation of action and speech was an indicator of an internal perturbation.

In contrast to the internal perturbations, the external perturbations were initiated by the research team. Some of the external perturbations were planned ahead of time, while others occurred spontaneously during the interview. The following example shows Elise, who was initially confident but became frustrated when she was provoked by the research team. She was answering the Making a Team problem (Appendix C), in which she had to decide if more two member or four member teams could be formed out of a group of six people. Her problem solving strategy was based on misapplied mathematics in that she multiplied the size of each team with the total number of people to make up the teams and then, compared the products. She was externally perturbed when she was asked to demonstrate her solution using six cubes. She correctly found the number of ways with two member teams, which was different from her first response. She then lost track of counting the combinations for the four member teams and became frustrated and confused, which seemed to indicate that Elise was no longer confident. This was further observed when she stopped her systematic process as she slid all of the cubes in front of her and cupped her hands around the cubes. Essentially, she stopped answering the question. In this case, the external perturbation changed her way of thinking but she was unable to create a new viable problem solving strategy. In other cases, the external perturbations did not change the student's way of thinking.

The external perturbations were expected to have a positive effect on the students that would lead them to a viable and correct problem solving strategy. In contrast, there were unexpected effects on the students that led the students to various problem solving strategies with unfavorable results. This will be discussed later in the chapter. Internal perturbations were not expected but they corresponded with ways of thinking that were not anticipated in the original research questions. The perturbation category proved helpful in identifying students' ways of thinking by observing points when their thinking patterns were interrupted and changed direction.

Use of Mathematics Category. The coding process identified expected and unexpected ways the students used mathematics when solving the probability problems. The researchers observed students misapplying their mathematics or failing to use their probability knowledge. This was expected based on prior research identifying systematic errors due to heuristic thinking (e.g. Tversky & Kahneman, 1974). There were a few occasions when the students created an unexpected heuristic strategy not linked to the three heuristics identified in the literature: representativeness, conjunction effect, or availability. In particular, the coding process identified a type of problem solving strategy in which students used their fraction knowledge, often equivalent fraction schemes, in place of probability knowledge. They appeared to ignore the probabilistic nature of the problem and instead restructured the probability problem into a fraction format before it was solved. In this study, this will be referred to as the *content substitution* process.

Ben demonstrated the *content substitution* process when solving the Two Heads versus Two Hundred Heads problem (Appendix B). Ben explained, “Ok so, 2 in 3. Let’s see. That’s at least twice in three and that’s 200 in 300, which is simplified to 2 over 3 so (nods head in the affirmative).” Ben simplified the fraction correctly but it was not an appropriate solution in that he is comparing the ratios of the outcomes and not attending to how probabilities in each event affect the answer. The goal was to have the students realize that smaller samples have greater variability than larger samples. However, Ben saw two fractions that were equivalent.

The *content substitution* process parallels the attribute substitution process but it is broader. The individual using the attribute substitution process would inadvertently use an attribute associated with the population or a class and apply it to the sample or specific situation. The students using the *content substitution* process inadvertently switch out probability content for fraction content. This was surmised from their fraction language and fraction operations. In particular, Elise was explicit when she claimed (Chapter 4, p. 25) that when they are equal in fractions then then they are probably equal in probability because probability uses fractions.

Profiles

It was necessary to identify and characterize different ways of thinking in order to address the research questions. As a result, profiles were developed using the students’ answers, response times, and the three coded categories - emotions, perturbations, and mathematics used – that show the general ways the students were thinking as they applied their problem solving strategies. The following paragraphs describe the profiles that correspond to System 1 thinking, System 2 thinking, Fischbein's (1975, 1987) anticipatory intuition and a fourth profile, which describes a way of thinking that differs from the other three and was prevalent in this study. It will be referred to as *slow intuition*. Examples for each of these profiles and how they were affected by external perturbations will be shared in the "Findings" section.

S-I. System 1 thinking was described as fast, implicit, and associative (Evans, 2008, 2011; Kahneman & Fredrick, 2002; Stanovich & West, 2000; Tversky & Kahneman, 1972) and it was linked to heuristic responses. Therefore, student responses that were relatively fast and resulted in a heuristic solution were identified as System 1 for the sake of this study. This is in line with Kahneman and Fredrick (2002) but the use of the term System 1 was originally coined by Stanovich and West (2000) and widely adopted by the dual process community based to describe more general differences in brain patterns in neuroscience studies. Other characteristics that were observed in this study, together with these indicators, will collectively, be referred to as the *S-I* profile in this study.

The emotional behaviors associated with the *S-I* profile included a lack of frustration, ease when answering the question, and confidence. Furthermore, the *S-I* profile lacked internal perturbations. Most times, the mathematics used was inappropriate for the given problem. This was expected due to heuristic thinking associated with this profile. In summary, the essential characteristics of the *S-I* profile include the heuristic response, confidence, a lack of internal perturbation, and a misuse of the mathematics. Many times the response times were fast, but there were a few occasions when the response times were inconclusive.

F-I. In contrast to the attribute substitution process associated with System 1 thinking, Fischbein (1975, 1987) indicated that anticipatory intuition, hereafter referred to as *F-I,* does not need to occur at the beginning of the problem solving process. Furthermore, anticipatory intuition did not correspond with an unconscious heuristic solution but with an insight that leads the person to the correct response (Fischbein, 1987). The internal perturbation indicator for *F-I* seemed to indicate a flash of insight. The emotional characteristics of the profile for anticipatory intuition were consistent with the profile for *S-I* after the insight occurred. The students were confident with their responses and they exhibited behaviors that indicated that they knew what to do or they had the correct answer clearly in their mind. The essential characteristics for the *F-I* profile included an internal perturbation that indicated a flash of insight followed by confidence and the correct answer.

S-R. System 2 was characterized as slow and effortful and linked to reasoning (Evans, 2008, 2011; Kahneman & Fredrick, 2002; Stanovich & West, 2000). In this study it is hereafter called the *S-R* profile. Like the *S-I* profile, the *S-R* profile includes confidence. However, unlike the *S-I* profile, confidence was based on mathematical understanding. The mathematics used to arrive at the correct answer often included a sound and correct explanation. The students *S-R* response sometimes included internal perturbation and sometimes did not. In addition, the response times were generally within the slow range but not always.

Slow Intuition. The research team noted another way of thinking that they called *slow intuition*. *Slow intuition* was a combination of System 1 and System 2 thinking. Typically, the response time was slower, in the inconclusive range, and the answers were heuristic. Unlike the *S-I* or *S-R* profile, the responses lacked confidence. Slow intuition includes internal perturbations observed by hesitations, wavering, and confusion. Many times the internal perturbations marked when the students made a change in their thinking process. Slow intuition could result from internal perturbations at any time during the problem solving process. Other times slow intuition could result from an external perturbation. When using slow intuition, the students seemed to understand that something was amiss, but they were unable to identify it. The essential characteristics associated with *slow intuition* include internal perturbations, a heuristic response at some point during the problem solving process, timidity and confusion, and a slower response time.

Findings with Discussion

The four profiles, *S-I, F-I, S-R* and *slow intuition*, are used to characterize salient ways that students were thinking when answering the probability questions. Within several of these ways of thinking, there were a variety of problem solving strategies and reactions to perturbations. These details will be described in this section.

S-I

All six students indicated *S-I* thinking. Representative examples will be shown from the responses by Elise, Brittney, Ryan, Josh, and Ben. These responses illustrate heuristic responses that used the attribute substitution (Kahneman & Fredrick, 2002) and those that used content substitution. The effect of external perturbations on these responses is discussed as well.

Elise gave a typical *S-I* response, including the use of attribute substitution when she answered the Six Fair Coins problem (Appendix B) -comparing the likelihood of two given sequences: “It’s A because it’s a 50% chance that you’re more likely to get closer to an even amount of [or] equal count of heads and tails because they both have equal chance of getting it each time.” Mathematically, Elise used the equally likely outcomes for one toss of an unbiased coin based on the population parameter, $\rho = 0.5$. This was observed when she said, “Because it’s a 50% chance,” referring to the chance of heads or tails appearing after one toss. Elise applied the population proportion of 0.5 for one outcome to the total number of heads and the total number of tails she expected to see among the six coins. This was noted when she said "[You are] more likely to get closer to an even amount." Elise's way of thinking was consistent with prior research (Tversky & Kahneman, 1972) because she did not mention randomness associated with her choice when she compare it to the alternative sequence of six tails, nor did she consider the correct response, that both sequences were equally likely. Elise's pattern of thinking seemed to align with the recently identified *attribute substitution* process, linked to heuristic thinking. She substituted one attribute in the problem –the population parameter for one fair coin toss– for another –the expected proportions of heads and tails in one outcome of the six coin tosses. In addition to the heuristic nature of her solution methods, Elise answered quickly and confidently. She experienced no internal perturbations, as evidenced by her lack of hesitation. Neither did she consider any alternative explanations. Since this episode occurred during the first interview, the research team did not perturb her thinking.

Ryan's response to the Description of an Individual problem (Appendix C) provides a good example of *S-I* thinking with an inconclusive response time. His response also illustrates that external perturbations can reinforce a student's heuristic thinking instead of inducing a change to *S-R.* The speed of Ryan's thinking fell at the lower end of his inconclusive range, suggesting the he may still be engaged in *S-I* thinking. (R is Ryan; J is the Researcher)

Protocol 1

R: Um, I believe, um, event 3 is more likely of Stacy, and then event 3 and 5 and then event 3.

J: You already said 3

R: Three and five are like, oh gosh, 5 (He erases 3 to change it to 5.)

J: Yeah

R: So, um, I believe event 5 is the next most likely and then event 4 and then event 1.

Ryan initially demonstrated confidence by his lack of internal perturbations or hesitations. Although he ranked statement 3 and ranked it again, this did not demonstrate a change in thinking, just that he lost track. After being corrected for ranking statement 3 twice, Ryan continued to confidently rank the statements. This was observed when he said, “Oh gosh, 5,” followed by, “I believe event 5 is the next most likely and then event 4 and then event 1.” Although he did not verbalize event 2 he did write it down.

His problem solving strategy was based on comparing each statement to the vignette, in which the closer the match the more likely the statement. This was observed when he said, “more likely of Stacy.” This led Ryan to rank the conjunctive statement - *Stacey writes a column in her school newspaper about recycling and she is active in her community animal shelter*- higher than either of the single statements contained in it. His ranking violated the Multiplication Rule: The probability of a compound event cannot exceed the probability of a single event contained in that compound event. This heuristic response, combined with his confidence, relative speed and lack of internal perturbations strongly indicate *S-I* thinking.

In the next two protocols, the research team attempted to provoke Ryan’s thinking. First, Ryan was asked to draw a Venn diagram that was intended to externally perturb his thinking. He was told the probability for each statement contained in the compound statement was the same, $p = 0.5$. He was asked to calculate the probability for the combined event. Recall that Ryan initially ranked the compound statement as more likely than one of the single statements contained in the compound statement.

Protocol 2

R: Ok, so (he draws two circles and labels one circle event 3 and the other circle event 5) and so event 3 column newspaper about recycling, and event 5 she ah, is active in her community animal shelter, and um, they overlap 'cause she enjoys the ecosystem, related to the ecosystem so recycling kind of you know affects the ecosystem. And then the study of biology is a study of animals so she would enjoy going to an animal shelter 'cause she can study about animals and stuff, so that's when they overlap because those are some of her particularly enjoyable topics.

J: Ok, so if event 3 let's say, 50% probable … and event 5 was 50%, what would be the probability for event 3 and 5, in the middle?

R: So event 3 is 50 and event 5 is 50

J: Uh huh

R: Ah, well the percentages is that both of them will happen is almost 100%

J: Oh, ok.

R: Because you would have to combine the percents, I believe (he says the last 2 words in a low whisper).

Figure 1: Ryan's Venn diagram

Figure 1 shows the Venn diagram Ryan created. The Venn diagram, with the accompanying probabilities for each event in the compound statement, was intended to provoke Ryan to realize his ranking of the compound statement was incorrect because it conflicted with the mathematical calculation for the probability of the compound event. However, Ryan misapplied the conjunction rule and added the probabilities, “Ah, well the percentage is that both of them will happen is almost 100%.”

Initially, Ryan demonstrated confidence and certainty with his response. This was observed when he explained his calculation, “Because you would have to combine the percents.” At this point, Ryan appeared to experience a perturbation that was evident by him lowering his voice on the next two words following this statement, “I believe.” However, this internal perturbation lacked strength to have him reconsider his answer.

In Protocol 3 Ryan’s attention was directed to his diagram (see Figure 1) and he was asked to explain the overlap, the intersection for the two events. This external perturbation was used to direct Ryan’s attention to the inconsistency shown with his calculation and his diagram. This did not induce a change in his thinking. He claimed that the overlap was not drawn to scale, that it should be larger to represent the 100% likelihood for both statements. Ryan was confident with his answer and he re-ranked the compound statement as more likely.

Protocol 3

J: Ok, so by looking at that diagram would that represent that?

R: Yes, I believe so because they overlap, because they overlap and they are two of her most enjoyable topics.

J: Ok, so you just changed your answer

R: Yeah I guess so, ‘cause I went back

J: Ok

R: And I wrote it in and it made sense to me more

J: Ok, because I was thinking that by looking at your picture, the overlap area is smaller.

R: Well it should be larger, but, it's the same, I guess I can draw it bigger, but, um, but the fact is that she enjoys biology and the study of the ecosystem.

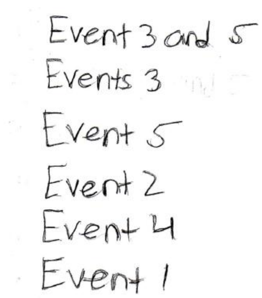

Figure 2: Ryan's final rankings

In Protocol 3, surprisingly Ryan was more confident that the conjunctive statement was the most likely. This was observed when Ryan said "I wrote it in and it made sense to me more." He re-ranked the compound statement as the most likely (see Figure 2) and claimed "The fact is … she enjoys biology and the study of the ecosystem." His use of the words and his tone saying, "The fact is," seemed to emphasize his confidence with his rankings.

On the other hand, Ryan's ranking could be interpreted from a different perspective. He may have thought of the relationships in terms of a set and a subset of the given set. In this case, the ecosystem is a subset of the set, biology. This was considered much later by the researchers when rereading his statements, "the fact is … she enjoys biology and the study of the ecosystem" If this were the case, Ryan would still be operating heuristically since the conjunction would still be less likely.

The research team's perturbation of Ryan's thinking induced surprising results. For example, when Ryan was asked to draw a Venn diagram his response was surprising because he did not reconsider his answer. Ryan remained committed to the notion that the greater amount of detail contained in the statement, the greater the likelihood. A second time, the research team attempted to perturb his thinking it also provoked a surprising result. When Ryan was asked to calculate the probability of the combined statement, his miscalculation reinforced his heuristic way of thinking. He re-ranked the combined statement to the most likely position from the second most likely position.

In the next example Josh used *S-I* thinking with the content substitution process when answering the Two Heads versus Two Hundred Heads problem (see Appendix B). His response time was in his fast range and he was confident with his answer. He compared the likelihoods of the two situations based on his equivalent fraction knowledge, which demonstrated another type of substitution, content substitution. (Recall, JS is Josh)

Protocol 4

> JS: It's going to be equal to because if you have if you take always 100 tosses from each of from the 300, if you take away 100 from 200 and 100, er … if you … divide by 100, yeah, if you divide by 100 on those you would still get 2 out of 3. So, it would be equal to. You could also take your 2 out 3 here and times those by 100 and [you] still get 200 times out of 300 and so they would be equal.

Initially, Josh appeared to be momentarily confused by his mathematics operation but he quickly clarified his thinking and answered heuristically. The researchers interpreted his language, "Yeah" to mean that he had quickly clarified his thinking. Josh explained that the two situations were the same because he could reduce the proportion of heads that appeared out of all of the tosses, 200 out of 300 in the second situation, to 2 out of 3 in the first situation, by dividing both numbers by 100. He demonstrated confidence in his mathematics by volunteering that he could reverse the process, "You could also take your 2 out 3 here and times those by 100 and still get 200 times out of 300 and so they would be equal." Josh used the content substitution process to produce his heuristic answer in which he used equivalent fraction knowledge (2/3 = 200/300) instead of probability knowledge in which the variability is greater for the smaller sample than the larger sample. The notion that he could look at one sample and transform it into the other sample using multiplication seemed to confirm his answer in his mind. His confidence together with the use of a heuristic suggested he was engaged in *S-I* thinking.

When answering the same problem, Elise used fractions like Josh but she made her content substitution even more explicit by saying, "So, if they are equal in fractions then they are probably equal in probability because probability uses fractions so much."

The next example shows *S-I* thinking in which heuristic thinking led to the correct choice, but it did not represent correct reasoning. This is also the first case in which *S-I* thinking was perturbed into *slow intuition.* In this example, Ben answered the Making a Team problem (Appendix C). (Recall B is Ben; J is the Researcher)

Protocol 5

B: Ok, 2 out of 6. So you would have 1, 2, 3, 4, 5, 6 (as he writes these numbers in a vertical line on his paper). Ok, so you can have these two, these two, these two, I mean you can have any number of the two.

J: Right

B: But with 4 you can only have these four (1 – 4), these 4 (2 – 5), these 4 (3 – 6). I mean going down with 2 you can, I mean … this one and this one, this one and this one, this one and this one (he moves his pencil quickly as he links the pairs. It looks like he links 1-2, 2-3, 3-4, 4-5, 5 -6 along with his original linking of pairs drawn on the sheet: 1-2, 2-3, 3-4, 5-6) but with 4 you only have 3 different options on how you can put them together. I mean, (silence)

B: If you have 2 over 3 (he writes 2/3) and one of three (he writes 1/3) I mean, it's saying (1/3) you only have to choose one of the three people, I mean, in there you would have 3 choices and you only have to choose 1 but this one, you have 3 choices you can choose 3 you have 3 different choices, where in this one (2/3) you only have (he sketches 3 short vertical marks on his paper) Let's see you can [have] this one or this one or this one, but with this one (2/3) you can have these two (he circles the first two marks). Well, I don't know actually. I [will] think about it…I am not sure.

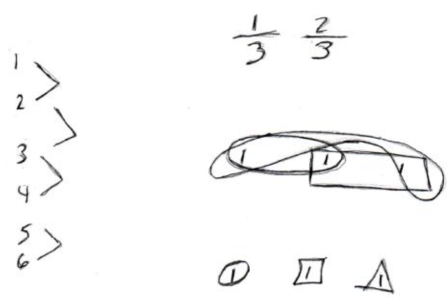

Figure 3: Ben's groupings

Ben initially solved the problem by identifying the number of ways to group two people together and then four people together from six people. This was observed when he said, as he pointed to his diagram (his diagram that showed six vertical lines that represented the six people (see Figure 3), "With 2 you can, I mean … this one and this one, this one and this one, this one and this one but with 4 you only have 3 different options on how you can put them together." Ben claimed there were more ways to create two member teams than four member teams based on his few examples. This was the expected heuristic response when a person used the attribute substitution process. As noted from the literature, substituting a few examples for a more exhaustive enumeration of the number of ways was the heuristic often used (Tversky & Kahneman, 1973).

After Ben gave his answer, he began to explain his thinking using his diagram in which he appeared to experience an internal perturbation when he said, “I mean.” It was subtle but it appeared to change the direction in his problem solving strategy because after saying this, Ben introduced fractions as he explained his new problem solving strategy. This was observed when he said, “If you have 2 over 3 (he writes 2/3) and one of three (he writes 1/3).” Ben stopped grouping the lines in his diagram when he introduced fractions into his problem solving process, which suggested Ben changed from attribute substitution to content substitution. It appeared Ben used fractions as a way to reduce the number of ways to group the teams. Ben reduced two people chosen from six people to one person chosen from three people and he reduced four people chosen out of six people to two people chosen from three people. In essence, he reduced the fraction 2/6 to 1/3 and the fraction 4/6 to 2/3. Using the reduced sizes, Ben returned to the grouping scheme. This was noted when he said, “I mean, its saying (for 1/3) … you would have 3 choices….where in this one (2/3) you only have (he sketches 3 short vertical marks on his paper) …. You can have these two (he circles the first two marks).” Ben was unable to arrive at an answer when he began to use this new strategy. This was noted when he said, “Well, I don’t know actually. I [will] think about it…I am not sure.” His lack of confidence with his new process and his perceived confusion suggested Ben was in *slow intuition*.

In the next protocol, Ben was drawn back to the problem in its original form and given six blocks. After several moments of silence, Ben reaffirmed that the number of ways was the same for both teams.

Protocol 6

J: Are there more choices with the two people or with 4 people?

B: (He begins to group the blocks into pairs) Two, two, two and you would have (he makes 3 disjoint sets of 2 immediately). Then these two (he points to 2 other ways to group them) I mean, (as he points out another 2 combinations) I think there is, I think they're the same number of choices.

J: Ok

(A few moments later)

B: Right. Yes, I guess they would be the same number of choices. (He marks the sheet) I mean, I can't think of a way that they wouldn't be.

Protocol 6 showed Ben grouping the blocks. It appeared he experienced an internal perturbation. This was noted when he said, "I mean (as he points out another 2 combinations)." This internal perturbation appeared to change the way Ben was thinking. Ben realized that both teams had the same number of ways. This was noted when he said, "I think they're the same number of choices." He affirmed his choice when he said, "I can't think of a way that they wouldn't be." The internal perturbation changed Ben's behaviors that appeared consistent with the *S-R* profile because his response time was slower and his answer was correct. In addition, Ben appeared confident with his answer and lacked internal perturbations. However, it may be the case that Ben's final answer was influenced when he used fractions to reduce the sizes of the team. It is uncertain, if Ben could have found the correct number of ways when the two team sizes were not reduced. If this were the case, he would be in *slow intuition*.

When answering the Making a Team problem (Appendix C) heuristically, Ben appeared to demonstrate behaviors consistent with the *S-I* profile based on his confidence, his lack of internal perturbations, and his heuristic answer. The interesting aspect of this interview was the way Ben created his own heuristic after he experienced an internal perturbation. In particular, Ben modified his problem solving strategy by introducing fractions that suggested he switched from the attribute substitution process to content substitution. This change appeared to be influenced by *slow intuition*, which was observed by his behaviors. Ben lacked confidence in his new answer and he appeared to be in a state of disequilibrium. It was surprising to note that Ben's heuristic led him to the correct answer, although if he were asked to quantify the number of teams, his heuristic would fail.

Both Brittney and Elise provide additional examples of *S-I* thinking in which an external perturbation moved the student from *S-I* thinking into *slow intuition* when solving the Hospital problem (Appendix C). Brittney's response is included as representative of both. She gives a response that began consistent with the *S-I* profile using the *attribute substitution* process. Her response time was slightly above her median time due to her momentary hesitation at the beginning of the problem solution process. However, after this hesitation, she quickly eliminated one of the answers -the correct answer- and almost concurrently chose the heuristic response. Her confidence was reflected both by her quickness and in her tone when she explained her heuristic answer. (Recall Br is Brittney and J is the Researcher)

Protocol 7

Br: Well, ok so I think it would be huh, I know B is not the answer. I would say it doesn't really matter.

J: Ok

Br: Because, it doesn't make a difference, because um, I mean, 60% of a small hospital could be boys, 60% of the large hospital can be boys, but I mean like, it could also be like 60% of girls, too, so.

In Protocol 8 the question was restated to Brittney by the researcher. This was a subtle external perturbation that did not change her thinking. Again, Brittney responded that the likelihood for each hospital was the same. Then, suddenly, she seemed to experience an internal perturbation. The internal perturbation changed her thinking. It appeared she changed her thinking in a way that aligned with *slow intuition*, which becomes more apparent in the next protocol.

Protocol 8

J: Right. Ok… over a course of a year … [are] there more days that the large hospital would get 60% boys or more … or the small hospital …? Or would it be the same?

Br: It would be it would probably be the same. Well…like do they have the amount?

J: What are you thinking?

Br: I'm thinking, well, if they have different amounts of people

J: Yes, they would have different amounts of people. The large hospital would have more, a larger number of babies born because it is serving a larger population. And a small hospital would be like a rural area, where not as many babies are born each day. So, would,

Br: Yeah, I would probably go with large then (she erases her first response and circles the larger hospital response).

J: Ok,

Br: Because like, I mean, a small hospital [has] not that much children, well babies born, I mean.

The internal perturbation was observed when she asked the researcher about the number of people in each hospital. Her question was answered in the affirmative. This caused Brittney to change her response to the incorrect answer, the large hospital. Brittney shifted her focus from proportions to the numbers of babies born each day in each of the hospitals.

In the next protocol, the researcher restated the question for Brittney. Brittney said she was torn between two answers – the heuristic answer and the incorrect answer. This indecisiveness is a characteristic of *slow intuition.* In addition, Brittney experienced an internal perturbation that drew her attention back to the problem in which she rereads it. This internal perturbation is also a characteristic of *slow intuition.* This internal perturbation preceded a change in her way of thinking consistent with a move back to *S-I* thinking.

Protocol 9

Br: Um, I am still going between 'the large' and 'it makes no difference'. Um, ok, well, um, (she rereads the first part of the question aloud softly and reads the rest silently) So, I would say … yeah, I am going to go with 'it makes no difference', now.

J: Ok.

Br: Well, you never know, it's a 50-50 chance if you have a baby boy or not.

J: Yeah.

Br: You could have, like you could have like 90% boys. I mean,

J: Right

Br: It’s a possibility. I mean, no one knows what will happen.

In Protocol 9 Brittney was conflicted between two responses the heuristic response, the size of the hospitals is not important, and the large hospital. After saying, “I am still going between ‘the large’ and ‘it makes no difference,” Brittney hesitated and seemed to experience an internal perturbation that focused her attention back to the question. She reread the question silently as observed when she said, “Um, ok, well, um.” This change in thinking caused her to reread the question. She began to read softly aloud and then drifted into silence as she read the rest of the question. Brittney immediately and confidently reverted back to her initial response, marking a return to *S-I* thinking. This was observed when she said, “I am going to go with it makes no difference.”

In summary, the above protocols showed students whose thinking patterns aligned with the *S-I* profile. They answered the questions quickly, confidently, and with a heuristic answer. These students problem solving strategies either reflected the attribute substitution process (Kahneman & Fredrick, 2002), which was expected, or the content substitution process which was not expected. When a *S-I* way of thinking was perturbed by the research team, it was expected that the students would show a pattern of thinking that aligned with the S-R profile described in the following section. However, several students showed unexpected patterns of thinking when they were perturbed. They included Ryan who remained committed to his heuristic way of thinking and Ben and Brittney, who both experienced internal perturbations that moved them both into *slow intuition* with different results. An internal perturbation led Ben to his own heuristic way of thinking, which was the content substitution process, while Brittney experienced changes in thinking that reverted back to the attribute substitution process.

S-R

S-R was characterized by a slower response time and methodical and effortful reasoning that resulted in a correct response. Responses by Ben, GP, and Josh were selected to demonstrate S-R thinking. The effect of external perturbations on their correct thinking is also described.

GP and Ben both demonstrated correct reasoning when they answered the Description of an Individual problem (Appendix C). Ben's response is used as an example, because it illustrates the *S-R* profile well. He was confident with his answer, he lacked internal perturbations, and his answer was correct. His mathematical reasoning was correct because he understood the conjunction rule in probability. Ben understood that it was less likely to have the compound event occur when compared to the probability of each of the two single statements that comprised the conjunction. Furthermore, Ben's reaction to the external perturbation is the kind of observed behavior that was anticipated: when externally perturbed, he remained committed to his initial answer.

Initially, Ben answered, "Well, volunteering and active, those are both close for number 1 to me (statements 2 and 5). Well, I guess I would put that as 1 (statement 5) and that one is 2 (statement 2). And then, recycling *and*, or just recycling (debating statement 6 the conjunctive statement) um I guess I would put that as 3 (statement 3) because of having both, two is less likely than having one out of 2. And then I would put 5 (statement 1) and there 6 (statement 4)."

Ben showed that he understood the significance of the compound statement when he said, "I would put (statement 3) as 3 because having both, two is less likely than having one out of two." This suggested that Ben realized that the probability for a compound event would be less likely than for either of the single events that comprised it. He demonstrated a slow and methodical way of thinking that appeared more effortful as he ranked each statement. This was observed by this tone when he said, “Well, volunteering and active, those are both close for number 1 to me.” It was also observed that he compared the statements with the description of the individual given in the vignette.

Ben’s thinking was perturbed by the research team to determine his level of confidence with his answer. It was expected that Ben would not change his thinking and this was the case. Ben reacted to the external perturbation, in which he was asked to draw a Venn diagram. He drew the two intersecting sets. When Ben was asked to consider the probability of the intersecting set when each set was given a probability of 50%, he explained the combined statement would be less likely than either of the single statements. He linked the combined probability to a realistic expectation of the individual in the problem having time to do both activities. (Recall B is Ben and J is the researcher).

Protocol 10

J: Ok, Could you draw a Venn diagram on this? Do you remember those?

B: (He draws two circles as disjoint sets and labels one 3 and then be erases the second circle that he has not yet labeled and redraws to show an overlap) Sorry.

B: So, both of them have to with science (he writes science in the space where the circles overlap). But, with [statement] 5 you’re really not outspoken in middle school, you’re, well, I’ve noticed with people is they don’t quite get along with people their [own] age but in an animal shelter, you’re dealing with animals and people would like that. People who are smart with science would tend to like to interact with animals. That’s where I get that so it’s like (he writes inside the set labeled 5 ‘interaction without people’)

B: I mean, in this one (points to set 5) she isn't, she's not really trying to tell people something. She is just trying to help but in that one (points to set 3) she is trying to tell people something, but people might not listen.

J: Right. Ok, so if we're like putting numbers to this, would this make sense based on what you are talking about here if like that was 50% and the other was [50%]?

B: And then for both, I mean, yeah they both have science but if you're outspoken you still might do it but, the chances of having time to do both is even, could be less (as he points to the center where the two sets overlap).

The research team attempted to perturb Ben's thinking when he was asked to draw a Venn diagram. This attempt failed and was noted when Ben drew the two intersecting sets. Ben explained the elements in each set and how the intersection set represented science. His explanation suggested his thinking was unchanged which was noted when he said, "So, both of them have to with science….In this one (he points to set 5) she isn't, she's not really trying to tell people something. She is just trying to help but in that one (he points to set 3) she is trying to tell people something, but people might not listen." When he was asked to consider the probability of the intersection when set three and set five were given probability measures of 50%, Ben claimed that it would be less. This was observed when he said, "And then for both, I mean, yeah they both have science but if you're outspoken you still might do it but, the chances of having time to do both is even, could be less (as he points to the center where the two sets overlap)." Ben's reasoning based on his earlier statements about the probabilities of an intersecting set, showed he understood the probability would be less as he translated that to the context of this problem. In this case, Ben related it to having the time to do both activities.

GP provided another good example of *S-R* thinking for the Hospital problem (Appendix C), but her reaction to the external perturbation was not expected. Her new way of thinking did not align with any of the profiles developed because she answered rather quickly with the wrong answer. The incorrect answer would typically indicate that the person lack probability understanding. This was surprising since her initial response suggested she understood probability. Subsequent external perturbations moved GP into other unexpected directions. However, in the end, she went back to the correct response. Here is her initial response:

> Um, I don't know about this one. I think a small hospital because it takes less of the baby boys. It takes less of them to make more than there are girls. Because if there were only like 10 babies, it would only take six boys to make like 60% but if there were 100 babies it would take 60.

When GP first answered the question, her response time was not within her slow range, it was closer to her median response time, but her momentary hesitation and her initial doubt seemed to suggest she was not engaged in *S-I* thinking because it appeared she was collecting her thought. In her next statement, she confidently claimed the small hospital was more likely to have more baby boys born, which was the correct answer, and she was able to explain her answer based on her correct mathematical knowledge.

She methodically relied on her mathematical knowledge when she applied daily birth rates to each hospital. This was observed when she said, "If there were only like 10 babies, it would only take six boys to make like 60% but if there were 100 babies it would take 60." She appeared to understand that the number of successes, baby boys, was less likely in a large hospital than a small hospital. She appeared to understand this by focusing on the minimum number of baby boys required to reach the 60% or more mark.

In the next several protocols, GP's thinking was perturbed when she was asked to consider different percentages of baby boys born in each hospital. These external perturbations moved GP into many different directions. This was unexpected since her initial answer suggested she understood the impact of variability on sample sizes. (Recall, GP is Guinea Pig, J is the researcher, and O is the observer).

Protocol 11

J: Ok, so how about if for the small hospital and the large hospital only 10% of the babies were boys? Which do you think would be most likely?

GP: Probably a large hospital because there would still be a lot of baby boys.

J: Ok, even with only 10%?

GP: Yeah. There would be more than a small hospital. I don't know if that made any sense. I think it would be more likely, I don't know.

O: So, what you mean is, in a large hospital there are more boys than in the small hospital the number?

GP: Yeah

O: Ok

In Protocol 11 GP was asked to consider 10% or less of the babies born were boys. GP answered incorrectly, with a hint of uncertainty and confusion in her voice, the large hospital. This was observed when she said, "There would be more (in a large hospital) than a small hospital. I don't know if that made any sense. I think it would be more likely, I don't know." A member of the research team asked for clarification, because it appeared GP answered the question by comparing the possible number of baby boys born in each hospital. GP clarified by stating "Yeah," that she was comparing the number of baby boys born in each hospital. In this case, GP failed to consider the probability aspect of the problem.

In Protocol 12 the problem was restated. She was also reminded to consider the chances instead of the magnitude of the hospital sizes. This time, GP changed her answer to the heuristic response. This meant that the sizes of the hospitals did not matter based on proportions given in the problem. This response was consistent with the *attribute substitution* process in which she substituted the population of the proportion of baby boys born for each sample. She focused her attention on the population proportion of 0.5. Her lack of confidence at this point suggests a pattern of thinking that was consistent with the *slow intuition* profile, which is described later in this chapter as opposed to the *S-I* profile.

Protocol 12

J: Ok, but how about if there's at least 10% of those babies that [are] boys

O: Not number, chance.

GP: Um, I think the, I don't know, they would probably be about equal because

… the proportions would be about the same.

GP lacked confidence with her response. This was noted when she hesitated and said, "Um, I think the, I don't know." GP turned to fraction thinking when she drew on the population proportions for gender births. This was evident when she said, "Because of the proportions would be about the same." GP answered the question based on fractions, when previously she focused on the sizes of the two hospitals. In this case GP showed behaviors consistent with the *slow intuition* profile because she answered heuristically using the *content substitution* process and she lacked confidence.

In Protocol 13, GP was perturbed again. She was asked to consider the question when all of the babies born were boys. She answered correctly, the small hospital, and explained that consistency in one gender was more easily achieved when there were a lesser number of babies involved. At first it appeared she was moved back to a pattern of thinking consistent with the *S-R* profile. Although she may still be in slow intuition because of her remark after she stated her answer, “I don’t know if that makes sense.” This suggested she may have doubts about her answer or her ability to explain her answer.

Protocol 13

GP: Um the small hospital.

J: Ok, why?

GP: Because to, it doesn’t take as many babies in a small hospital as it does a large hospital to be consistent about one gender.

J: Ok

GP: I don’t know if that makes sense.

All of the students showed behaviors consistent with the *S-R* profile when answering the first part of the Chips and Randomness problem (Appendix C). This problem was designed so the students would answer the first part correctly. The problem showed two sequences of chips. One side showed various sports balls: football, baseball, soccer ball, tennis balls, and basketball, one kind on each chip. The other side showed one number, one through five. Each sequence was a hypothetical result from a random drawing from two bags containing 30 chips each. Each bag generated its own sequence. The students were shown the sports ball side first. Then, to intentionally perturb the students thinking, the chips were turned over to show the numbers. As expected, Ben, GP, and Ryan did not change their reasoning when externally perturbed, whereas Brittney, Elise, and Josh moved into heuristic thinking.

Josh was chosen to demonstrate his change into *slow intuition* when he was externally perturbed. By the end of the interview, Josh moved back into correct *S-R* thinking.

Protocol 14

J: Ok..…We have these 2 sequences (I moved two sequences that were previously laid out towards Josh)

JS: They look about the same to me

J: Ok

JS: It looks like the same stuff

J: Yeah because they had the same kind of sports balls in each bag.

JS: Oh, and they got the same number of each one?

J: Yes they did.

JS: They look like they're the same. You draw one of each. It's kind of hard to get them like exactly the same, like in the same spot.

J: Oh, in the same spot.

JS: Yeah.

JS: So like, because if you wanted to make them the same sequence you would have to like change them around to make it like that. It would really be hard to get the same, like if you drew out a baseball that time, it would be hard to get a baseball. Well it wouldn't be that hard to get a baseball at first, but it would be hard to get it the same throughout the whole thing (as he points to the sequence on the left).

J: Ok, let’s turn them over and see what you think (the chips were turned over to reveal a random set of numbers without repetition in one set and a descending order in the second set).

JS: Turn these over?

J: So, what do you think? Were there any surprises?

JS: That these were in order (left sequence), like 5 to 1 and these (right sequence) are like all mixed up.

J: So, which one seems more likely?

JS: This one (right sequence) because it would be like hard to get it all in a row.

J: But, we saw the other side of them they looked the same

JS: Yeah, so it could go, I mean, if it’s like, I don’t know

Josh was asked to evaluate the likelihood between two sequences. Josh initially answered correctly. He said the two sequences that showed the various sports balls were equally likely because they both appeared random. Josh was intentionally perturbed when the chips were turned over. Josh was asked which sequence was more likely to occur. Josh changed his answer and said the sequence that showed a random pattern. This was observed when he said, “This one (referring to the non-ordered sequence) because it would be like hard to get it all in a row.” He was reminded that when the sports balls sides was showing he claimed they looked the same. At this moment it seemed to be apparent that Josh was confused by what he had seen. This was observed when he said, “Yeah, so it could, I mean, if it’s like, I don’t know.” It appeared Josh was experiencing an internal perturbation because he was unable to reconcile the equally likelihood for the two sequences that showed the sports balls with the non-equally likelihood for the two sequences that showed the numbers. This resulted in a change of his way of thinking from *S-R* to *slow intuition.*

Later in Protocol 15, Josh was asked to consider the non-sequential number sequence. He was asked if that were the desired outcome, would that be less or more likely than the sequential numbered sequence. At this point, Josh realized that it would also be hard to get and understood that both sequences were equally likely. (Recall JS is Josh, J is the researcher, and O is the observer).

Protocol 15

JS: I think not (that now they are equally likely) because that would be really hard to get 1, 2, 3, 4, 5 in a row and they were like (as he places his hand on the second chip in the right sequence)

J: Even though it seemed ok to get (I turn the left sequence over) soccer ball, baseball, basketball, football, volley ball?

JS: Yeah I know it looks like a random selection. But when you turn them over (he turns them over to the number side) it's like perfectly in order. But that (looking at the other sequence) looked like a random selection too. But, when you turn them over they are (referring to the numbers looking random) but these are not (looking at the ordered sequence).

J: My bag had these 30 things in them and I had 6 ones, 6 twos, 6 threes, 6 fours, and 6 fives (I raise my hands and hold my left hand over the left sequence – ordered – and my right hand over the right sequence – not ordered – and then drop my hands in silence) Would these sequences be equally likely to choose at random?

JS: as the other side?

J: Uh huh

JS: Yeah, probably because they would still be hard to get them in the same spot.

J: Ok what do you mean they would still be hard to get them in the same spot?

JS: Because like on the other side, it would be hard to draw baseball same as you drew the other baseball, and so if you drew like so say these were like (as he changes the right numbered side of the chips into order, 1, 2, 3, 4, 5) 1, 2, 3, 4, 5 and then so the first one I pulled here and got this so my second draw would be a 2 and a 3 then a 4 and if I did this side and did the same thing, that would be kind of hard to do

J: OK

O: How about now. I want this (1, 3, 2, 5, 4 sequence) this is what I want. And you randomly drew from a jar. Do you think it very difficult to get this than that one (the 1, 2, 3, 4, 5 sequence) or very easy. I want this one, I don't want, 1, 2, 3, 4, 5.

JS: That would still be just as hard as this (the 1 – 5 ordered sequence) to get this (the 1, 3, 2, 5, 4 sequence) because you have them, like the right number. And there are the same amount of chips in the bag as this so you like have to get it perfect to get that right.

Josh said it was harder to get the ordered sequence of numbers, which was noted when he said, "I think … [it] would be really hard to get 1, 2, 3, 4, 5 in a row." Josh was asked to consider the two sequences from a different perspective. He was asked to consider the situation when the preferred outcome was the sequence 1, 3, 2, 5, 4. He was asked if this sequence would be hard or easy to get. Josh said it would be hard because he realized that getting the chips in a specific order would be hard to achieve. This was observed when he said, "That would still be just as hard as this (the 1 – 5 ordered sequence) to get this (the 1, 3, 2, 5, 4 sequence) because you have them, like the right number. And there are the same amount of chips in [each] bag." When Josh stated, "You have to get it perfect to get that right," suggested he had clarified the problem so he understood the solution. Josh realized that any sequence was equally as hard to get because each sequence was unique. Josh was confident with his final response and he demonstrated behavior that suggested he was certain about his answer and his answer was self-evident to him. This suggested that Josh returned to a pattern of thinking consistent with the *S-R* profile.

In sum, the students whose thinking patterns that aligned with the *S-R* profile answered the problems slowly, with effort, and correctly. When they were perturbed by the research team it was expected that their thinking would remain unchanged, as happened to Ben. However, there were unexpected results such as GP who changed from a correct answer to an incorrect answer before moving back to correct thinking when perturbed a third time. Josh was moved to a heuristic answer because he believed the ordered sequence was less likely when compared to the random sequence. He was externally perturbed several times before he was eventually perturbed back to a correct response. These wavering's between answers were not expected from the students who initially demonstrated thinking patterns consistent with the *S-R* profile.

F-I

Fischbein's (1975, 1987) described intuition as a way of thinking that could be developed and used to support reasoning. As a result, intuition according to Fischbein (1987) would produce correct solutions when fully developed. There were two students that seemed to demonstrate behaviors consistent with one of the roles Fischbein (1987) identified: anticipatory intuition.

In the first example, Josh was addressing the Marbles and Boxes problem (Appendix C). In Protocol 16, the excerpt opens with Josh explaining his initial solution. He explained that the friend with four marbles and two boxes had more ways of placing his marbles into the boxes than he would have with his two marbles and four boxes. Josh used colored blocks to represent the marbles and the squares he drew represented the boxes.

Protocol 16

JS: So, I think your friend with the 4 marbles has more ways

J: Why?

JS: Because he's got, if you have this, he's got more marbles to move around than this guy. You can only have, this one, this one, these, don't really matter, on what marbles they are because this marble, like you have a marble there and a marble there a marble there or a marble there. And then, you can put them both in this one, and not have any in these or you can put both of them in here, here…Or you can have different ones, you could have 1 here and 1 here, 1 in here or 1 in here. Ah, ok, so I figured it out you can have the same number because if you look at it, it will be 4 and 2 and 4 and 2. That is just the way I look at it. You can change these [in]to boxes and these [in]to marbles. I look at it like that.

J: Oh, ok

JS: And then you'll have the same amount. The same chances as this because they would be equal on both sides, so you have the same number.

Josh initially thought the friend had more ways. He was in the process of explaining his answer when the correct approach occurred to him. This was observed when Josh interrupted himself and said, "Ah, ok, so I figured it out you can have the same number." He explained it by noting the complimentary relationship between the two situations. This was noted when he said, “If you look at it, it will be 4 and 2 and 4 and 2….You can change these [in] to boxes and these [in] to marbles.” Josh did not quantify the number of ways mathematically but he did understand the complimentary structure of the problem for these two situations that meant they were the same. He *felt* like he knew the answer, it was self-evident to him and he was confident with his final answer. This was shown when he said, “And then you’ll have the same amount. The same chances as this because they would be equal on both sides, so you have the same number.” He recognized the complementary relationship of the two scenarios – probability knowledge. Josh discovered the solution to the problem after he worked on the problem for a few minutes. At that one specific moment in time, he immediately understood the problem and his behavior demonstrated confidence with his revised solution when he realized he had the wrong answer. These behaviors seemed to suggest Josh used *anticipatory intuition.*

GP demonstrated *anticipatory intuition* when she answered the same problem, Marbles and Boxes (Appendix C). GP was systematically evaluating the number of different ways of placing two marbles into four boxes compared with the number of different ways of placing four marbles into two boxes. GP seemed to experience an internal perturbation when she was explaining her thinking in which she suddenly stopped talking and gazed at the blocks and her sketch. When she was asked to verbalize her thinking she immediately responded with the correct answer without hesitation and with confidence.

Protocol 17

J: So what are you thinking?

GP: You can put the red and the green one in the same boxes or separate boxes. So you could do 8 with the same box and I guess you could do 14. But for this one he only has…(she writes)…(she stops and looks at the problem)

J: So, what are you thinking?

GP: I think they would be about equal because you have more boxes but he has more marbles, so the combinations that you both could make would even up, I think. So, I would say they have about the same number.

GP began the activity by showing the number of different ways of placing the blocks into the boxes. She demonstrated a systematic way of moving the blocks that was demonstrating a mathematical pattern of movement. This was shown when she said, "You can put the red and the green one in the same boxes or separate boxes" as she moved the blocks. GP suddenly experienced insight to the problem. This was observed when she stopped and looked at the problem, "But for this one he only has...(she writes…and then stops)." When prompted to verbalize her thinking, she said, "I think they would be about equal." She realized they were the same which was shown when she said, "You have more boxes but he has more marbles, so the combinations that you both could make would even up, I think. So, I would say they have about the same number." This moment of insight and the language she used suggested she experienced anticipatory intuition. She demonstrated confidence at the end when she said, "So, I would say they have about the same number."

Anticipatory intuition has its own characteristics. Unlike, *S-I* the intuition can occur anytime during the problem solving process as was observed with Josh and GP. Similar to *S-I* and S-R, these students were confident with their solution. Similar to *S-R*, the confidence was grounded on a logical insight so that the students knew what to do next, although the reasoning behind the next logical step may remain implicit similar to *S-I.* In both of these examples, the problem did not have the students quantify the results but merely determine if they were both equally likely or if one scenario was more likely than the other. Josh and GP both anticipated that the situations were equal, which I interpreted that they understood the next steps when using the blocks.

Slow Intuition

In the *S-I* and the *S-R* sections, there were examples of *slow intuition* in which students changed their way of thinking after they were externally perturbed by the research team or they were internally perturbed. In this section *slow intuition* is discussed first when it was observed prior to a perturbation. The later examples show *slow intuition* as a result of a perturbation. As noted earlier, *slow intuition* is characterized by internal perturbations and as a result the students took longer to answer. When using *slow intuition*, students exhibited behaviors that suggested they were confused and lacked confidence in their answer or they wavered between solutions. Implicitly, the students seemed to understand that something was amiss but they were unable to identify it.

Brittney, Elise, and Ryan responses are examples of *slow intuition*. The first example shows Brittney's initial slow intuitive thinking. Ryan also demonstrated *slow intuition* at the onset of the problem solving process. The last two examples, from Brittney and Elise, show *slow intuition* after the research team intentionally perturbed their thinking. In general, external perturbations could result in *S-I* thinking, *S-R* thinking, or move to an incorrect response.

In the following example, Brittney's behaviors appeared consistent with *slow intuition* at the onset. She was timid, she wavered, and she was hesitant when answering the Two Heads versus Two Hundred Heads problem (Appendix C). She initially gave the heuristic response for the question that was followed by internal perturbations that moved her in other directions. First, she either deliberated on her answer, wavering on whether it was correct or not. Then she hesitated before she changed her answer to the incorrect response after she was asked to share her thinking. In the end, she settled on the heuristic answer based on the probability for one coin toss because she attended to one of the scenarios in the problem. She was more confident in the end, which suggested she moved into *S-I* thinking. (Recall Br is Brittney and J is the researcher).

Protocol 18

Br: Um … I think it is equal again.

J: Ok,

Br: Well, (hesitates) no…yes, (hesitates), yes, I am. Because, they could be heads, tails, tails. But it could also be like tails, heads, heads

J: Uh hmm

Br: I mean, it's just an equal … probability that it could be either one.

J: Ok, so that's the same, as getting [heads] 200 times when tossing 300 coins?

Br: Um … it's likely. I guess.

J: So what are you thinking, now?

Br: I think it would most likely be like …(hesitates) … it may be a greater chance because if you flip a coin mostly it always lands on heads for some reason. I don't know why.

J: Ok, in your experience

Br: Yes, whenever I flip a coin it comes up heads and I always pick tails. Hmm…

J: Ok, so we're looking at this, I want to make sure you understand what is being asked. So, we're getting heads at least twice when we toss 3 coins. Is that going to be smaller chance of getting that, or equal to, or is it going to be greater than getting 200 heads when you toss the coin 300 times.

Br: I still think it is equal, actually. Because you know, still, it's equal to have still like, see it's having a coin a heads and a tails and then you have to, you like x those out. And for the middle coin it's a 1 in 2 chance of getting it. So, that is why I think it will be equal to.

Brittney appeared to struggle to understand the question, which was noted when she used her pencil to point as she read the question. After Brittany reread the question a second time, she gave the heuristic answer, that both scenarios were equally likely. It appeared she experienced a series of internal perturbations as she almost immediately said her answer was wrong and just as quickly she changed her mind and said it was correct and appeared to reaffirm her answer to herself. This was noted when she said, "Well, no…yes … yes." Brittney appeared to focus her reasoning on the two possible sequences for the two heads outcome from three coins, which she claimed were equally likely.

At this point, the researcher asked her if the probability was the same for the other scenario, Brittney appeared hesitant and timid when she gave her heuristic answer. This was noted when she said, "Um … it's likely. I guess." Brittney appeared to be contemplating her answer. When she was asked to share her thinking, Brittney experienced an internal perturbation in which she changed her answer based on her personal experiences. This was observed when she said, "I think it would most likely be like…(hesitates) … it may be a greater chance because if you flip a coin mostly it always lands on heads for some reason. I don't know why…. whenever I flip a coin it comes up heads and I always pick tails." The researcher stated the question to Brittney to ensure she understood. Brittney returned to her initial heuristic answer, both scenarios were equally likely. This was observed when she said, “I still think it is equal, actually.” It also appeared that Brittney based her conclusion on two heads from three coins scenario in which she focused on the equal likelihood for the outcome for each coin. This was observed when she said, “Because you know, still, it’s equal to have …. A [one] coin a heads and a tails and then you have to, you like x those out (referring to the HTT or THH possible outcomes she mentioned earlier). And for the middle coin it’s a 1 in 2 chance of getting it. So, that is why I think it will be equal to.”

Brittney appeared to be more confident at this point. This was observed when she said, “I still think it is equal, actually.” This suggested that she favored the heuristic answer throughout the process. In the end, Brittney’s explanation was presented with a confident tone and her final statement, “So, that is why I think it will be equal to,” seemed to suggest she was satisfied with her answer and her problem solving process.

The remaining examples revisit the episodes when the students were perturbed into *slow intuition.* The first example opens with Brittney when she initially gave a heuristic answer to the Hospital problem (Appendix C) that was discussed under the *S-I* section. It was mentioned earlier that she experienced two internal perturbations in which the first one changed her way of thinking so that it aligned with *slow intuition*. The second internal perturbation caused Brittney to reread the question and this time she moved back into the *S-I* pattern of thinking. The following sets of protocols shows Brittney’s response when she experienced the external perturbations initiated by the research team.

In Protocol 19 the research team perturbed Brittney’s thinking when the question was changed from asking about the likelihood of a 60% proportion of boys born to the likelihood of

100% of babies being boys. Brittney was asked which hospital would have more days with 100% of the babies born would be boys.

Protocol 19

J: Right. Ok, let's cross out the 60% …. Would the large hospital have a better chance of having more days with all the babies being born are boys or would the small hospital have a better chance [of having more days where] all the babies born [were] boys.

Br: The small hospital because there's not as much babies being born and the large hospital, I mean like, there is going to be lots of them. I mean like even 1% could have a baby girl in there in a large hospital but in a small hospital you have more of a chance of having all boys.

J: Ok, what if we changed it to 75%? Would the large hospital or the small hospital would be most likely to have 75% of their babies being born that are boys.

Br: The small hospital because there's not as much babies being born and the large hospital, I mean like, there is going to be lots of them. I mean like even 1% could have a baby girl in there in a large hospital but in a small hospital you have more of a chance of having all boys. (Several moments passed) Hmm, that makes me want to change my answer for the first one.

J: What does it make you want to change it to?

Br: The small hospital? Because, there is more of a chance for it, because in a large hospital, I mean, it would be like, yeah, ah, hum. Yeah, I think it would be a small hospital (she says this as a statement, not as a question as before), because like, there is small amounts of people, like small amounts of babies being born, so

I think it would have a better chance of boys being born there. So, (she circles that response) I am actually changing my answer to small.

J: Oh, even for the first one?

Br: Yeah,

When the percentages were changed to 75% again Brittney said the small hospital was more likely. She said “in a small hospital you have more of a chance of having all boys.” Brittney appeared to experience an internal awareness about the problem as demonstrated by her remark, “Hmm, that makes me want to change my answer for the first one.” She hesitated with her answer but then claimed with confidence, “Yeah, I think it would be a small hospital”. This was observed by the inflection in her voice. She stated her answer as a statement, not as a question suggesting more uncertainty.

Brittney demonstrated behaviors that suggested she lacked confidence with her answer. In fact, she was wavering between two answers. In the heuristic one her mathematics was based on the population proportion for gender, and with the incorrect response, she shifted her attention to the number of children in each hospital. After multiple perturbations, these two components – proportion of births and number of births– were coordinated and Brittney settled on the correct response, demonstrating confidence in her answer.

In the next example, Ryan used an idiosyncratic heuristic that included fractions. Unlike Ben in Protocols 4 and 5, Ryan showed behaviors consistent with the *slow intuition* profile at the onset. In particular, he claimed when tossing a fair coin three times it was more likely to get all heads than tossing the fair coin four times and getting all heads. This problem was not one of the problems that was specifically designed to induce heuristic thinking from the Cognitive Development section of the Cognitive Development and Heuristic Thinking instrument

(Appendix B). This problem was intended to test general probabilistic reasoning. (Recall, R is Ryan and J is the researcher).

Protocol 20

R: Um … 3 heads 'cause the chances go up that you won't get it once it moves on. Like it starts out with about 33% you get a 33% chance that it will happen then it goes to a 25% chance that it will happen.

J: Oh, ok. Do you want to explain how you figured that out mathematically?

R: Um well you almost have to know your percents like 1/3 and 2/3,

J: Ok, so how did you come up with 33%?

R: Well, like 1/3, well I don't know. 33% I'm sorry I'm not really sure but it just works so

J: Oh, ok

R: Yeah

J: 'Cause I was just wondering how you came up with the 1/3 with 3 heads and with the 25% with the 4? Is that where you (Ryan interrupted)

R: Yeah, it kind of made sense to me. So, yeah

After Ryan paused for several moments after he read the question aloud, the researcher prompted Ryan to verbalize his thinking. Ryan was unable to explain why his method was appropriate, which indicated the tacit nature of this thinking. This differed from the *S-I* profile, in which the students were confident with their answers and appeared confident that their method was appropriate to produce an answer they perceived as correct.

Ryan's method included fractional thinking mixed with the notion that the longer sequence of coins showing the same outcome are less likely. Ryan claimed that the chance to get three heads was 1/3 or 33% and the chance to get four heads was 1/4 or 25%. Ryan appeared to link the number in the sequence with the denominator of the fraction and the numerator of the fraction was based on one of the two possible outcomes, heads or tails. That is, getting heads is one out of two possibilities. This fraction showed the probability of getting that outcome in which the probability of getting all heads gets smaller as the sequence increases. Ryan's heuristic addressed this but his method would fail if he were asked to quantify the probability of getting three heads when tossing three coins or four heads when tossing four coins. Ryan behaviors suggested that he lacked understanding when explaining his method, but he was insistent that his method worked. Because of Ryan's slower response time and his lack of confidence to understand his heuristic method, although it led him to the correct answer, he was considered as using *slow intuition.* His method included factional thinking that suggested he used the *content substitution* process before he evaluated the probability for each scenario.

In sum, the students who exhibited a profile consistent with *slow intuition* had slower response times that typically fell within the inconclusive range. Their thinking was tacit and they lacked confidence with their answers or wavered between two answers. Their solutions were not self-evident to them. Many times slow intuition was initiated by internal perturbations, as shown in examples from the *S-I* section. Brittney, Elise, and Ryan showcase slow intuition, although this way of thinking was observed by all of students at one time or another during the data collection process.

Summary

Chapter four contained the results from this study that showed the various ways that some eighth grade students think when answering probability problems. Six students were chosen to be the focus of this study. There was a two-stage screening process to identify the most suitable students based on their probability and statistics content knowledge, their cognitive development level, and their tendency to use heuristic thinking.

The results of the study were based on two levels of analysis. The first level included an initial overview that was based on the students' response times and their initial answers to the problems. This approach highlighted behaviors that were unexpected, which required a more in depth analysis of the data. The in-depth analysis was based on the constant comparison method (Glaser, 1965) and produced three major categories: emotions, perturbations, and mathematics used. The analysis drew on the key terms in these categories in conjunction with the literature, the students' response times, and their answers to create four profiles to reflect the students' way of thinking. The four profiles included: *S-I, S-R, F-I,* and *slow intuition*.

The detailed analysis highlighted unexpected thinking patterns. In particular, an unexpected problem solving strategy, the *content substitution* process, was associated with the *S-I* profile. In this type of solution, the students' inadvertently replaced probability content with fraction content.

In addition, the research team observed thinking patterns that differed from the expected patterns associated with System 1 or System 2 as defined in this study. This other pattern of thinking was identified as *slow intuition* in this study. This way of thinking shared some characteristics associated with System 1, as defined in this study, and some characteristics associated with System 2, as defined in this study. Characteristics associated with *slow intuition* included internal perturbations that may occur any time during the problem solving process, a heuristic answer, an inconclusive response time, timidity and confusion.

The students' thinking was intentionally perturbed by the research team. There were expected responses such as the students who were moved from System 1 towards System 2, or the students who were engaged in System 2 who remained committed to their answers. However, the external perturbations also produced many unexpected results. Some students remained committed to their heuristic answer. Other times the perturbation shifted their thinking into slow intuition which found them wavering between competing answers that included wrong answers.

Chapter 5

Conclusions

In this study the researcher explored the different ways that six eighth-grade students thought about probability using a heuristics and biases framework that includes a schema called the attribute substitution process (Kahneman & Fredrick, 2002). In particular, the researcher was interested in student's problem solving strategies and the effects of perturbations on two different ways of thinking, System 1 and System 2. According to previous researchers, System 1 is linked to intuition and System 2 is linked to reasoning (Evans, 2008; Kahneman & Fredrick, 2002). In conceptualizing System 1[1], intuitive thinking also drew upon Fischbein's (1975, 1987) notions about intuition. Fischbein's (1975, 1989) intuitive thinking was characterized in one of two ways: 1) by the attending to the source of the intuition and 2) by attending to the role intuition played during the problem solving process.

There were several expectations held by the researcher when designing this project. There was a level of expectation that intuitive thought would occur at the beginning of the problem solving process for the attribute substitution process (Kahneman & Fredrick, 2002; Kahneman, 2003). Likewise, there was a level of expectation based on dual process theory (Evans, 2008) that heuristic thinking occurs when a person responds too quickly because his attention is focused on a particular attribute in the problem (Kahneman & Fredrick, 2002). Joining these two notions, the researcher anticipated many students would inadvertently switch out an attribute associated with the population (or sample) and contribute it to the sample (or population) depending on the wording of the problem (Kahneman & Fredrick, 2002). It was also anticipated that some of the students would reason correctly about the situation using System 2 thinking as informed by Kahneman and Fredrick (2002).

[1] *System 1 thinking* refers to a subset of System 1 thinking that involve heuristic answer

Finally, there was a level of expectation that the research team could move the students away from System 1 thinking and into System 2 thinking. The students' responses to the researcher team's attempts to influence their way of thinking were used to determine their level of confidence in their responses and the stability of their problem solving strategies. Some of the anticipated results were found. However, there were some unanticipated results as well.

In particular there was a greater diversity in the ways of thinking than expected. The identified ways of thinking included intuitive thinking that was more in line with Fischbein's (1975, 1987) anticipatory intuition than with Kahneman and Frederick's (2002) characterizations involving System 1 thinking. Anticipatory intuition was a way of thinking in which an insight comes suddenly any time during the problem solving process. In addition, there was a way of thinking that involved intuitive thought that differed from Kahneman and Fredrick's (2002) and from Fischbein's (1975, 1987) characterizations of intuition. This way of thinking will be referred to as *slow intuition* in this study.

In this chapter, the researcher uses the findings to answer the original research questions 1 and 3.

1. In what way was a participant's problem solving strategy influenced by System 1 or System 2?

3. How stable or strong was a participant's problem solving strategies associated with System 1 or System 2?

The findings with respect to the student's reactions to the research team's external perturbations were more varied than expected. Therefore, the original second research question, "How is the participant's problem solving strategy perturbed from System 1 to System 2?" will be replaced with a more general research question:

2. What effect did the attempted perturbations have on the students' ways of thinking? The researcher additionally addressed the first research question with respect to the two other ways of thinking identified above--Fischbein's (1975, 1987) intuition and *slow intuition*.

Research Question One

In what way was a participant's problem solving strategy influenced by System 1 or System 2? (or Fischbein's [1975, 1987] intuition and slow intuition)

There were students who were influenced by System 1 thinking as identified by a fast and heuristic response. In line with previous research findings, most of these students' problem solving strategies used the attribute substitution process (Kahneman & Fredrick, 2002). However, there was another problem solving strategy that other students used that was similar to the attribute substitution process. They used a problem solving strategy that was identified in this study as the *content substitution process*. The *content substitution* process occurred when the students inadvertently substituted non-probabilistic content, such as equivalent fractions for probabilistic content. Fractions were the most common content that replaced probability content.

There were students who were influenced by System 2 thinking as evidenced by their slow and methodical thinking process. It was anticipated that patterns of thinking would emerge in these students' problem solving strategies, such as their attending to a particular part of the problem. However, this was not the case. Therefore, the research question cannot be addressed with respect to System 2 thinking.

Recall that *slow intuition* was an unexpected way of thinking that differed from thinking associated with System 1 and System 2. *Slow intuition* was characterized by the students' struggles to find an appropriate problem solving strategy to answer the probability problems. More specifically, instances of slow intuition were identified by internal perturbations and solution speeds located within the interquartile range as described in Chapter 3. At times, the slow intuition emerged organically. Other times it emerged when the research team attempted to perturb the students. In either case, because these students were struggling, it should not be surprising that the students would change and waver between different solutions during the problem solving process and, at some point in their problem solving process, the students would engage in incorrect mathematics that led to incorrect answers. It was not expected that students would be moved towards heuristic strategies, such as the attribute substitution process because based on Kahneman and Frederick's (2002) characterization, attribute substitution always happened subconsciously and at the beginning of the problem solving process.

System 1

System 1 thinking was identified in the *S-I* profile that was developed by the researcher based on the characterizations from the research literature (Evans, 2008, 2011; Kahneman & Fredrick, 2002; Stanovich & West, 2000). The *S-I* profile included certain emotional behaviors and a lack of internal perturbations and the use of a heuristic to reduce the complexity of the problem. In the *S-I* profile the problem solving strategy is conceived immediately, it is effortless, and the answer is self-evident to the student. In particular, confidence was an emotional indicator. *S-I* was marked by a fast response.

In general, when a student's problem solving strategy was influenced by System 1 thinking, confidence eliminated the potential for internal perturbations. In addition, due to their inadvertent attribute or content substitution, the mathematics the student used was generally inappropriate. *S-I* thinking was observed for all of the students and the resulting problem solving strategies negatively impacted their solutions. In the subsections below, the attribute substitution process and the content substitution process are laid out in greater detail.

Attribute Substitution Process. It was observed that all six participants used the attribute substitution process while in System 1 at some point during the two interview sessions. During System 1 thinking, this problem solving strategy was observed for two types of the three standard heuristics, representativeness and the conjunction fallacy. It was observed for one idiosyncratic heuristic observed with the representativeness task that lacked consistency with the documented representativeness type of substitution. The examples for the representativeness and the conjunction fallacy were drawn from Elise, Ryan, and Josh. These students demonstrated the attribute substitution process for the representativeness, conjunction effect, and availability heuristics and for one idiosyncratic heuristic.

Representativeness. The representativeness heuristic was defined as a subjective probability for an event or a sample (Tversky & Kahneman, 1972). If the student were using the representativeness heuristic the student would solve the problem in one of two ways: how closely it resembles the characteristics of the parent population or how closely it resembles the essential features of the process that produced it (Kahneman & Tversky, 1972). Several students used the representativeness heuristic when under the influence of System 1 thinking when they solved the Six Coins problem (Appendix B). Elise exemplified this way of thinking best.

Elise (Chapter 4, p. 25) expected the same number of heads and tails, three of each, because the probability of getting heads or tails on one toss with one balanced coin is 0.5. In this case, the expected probability for the given outcome of a fair coin was substituted for the expected ratio of heads to total coins in each sequence. Then, she compared the ratio of heads to total coins in each sequences. The sequence that showed the three heads and the three tails was consistent with her expectation that three of the coins would show heads and three of the coins would show tails. Her way of thinking matched that in the earlier work of Tversky and Kahneman (1974).

Conjunction Fallacy. All of the students gave a heuristic answer for the Description of Marcia problem (Appendix B) from the first interview or the Description of an Individual problem (Appendix C) from the second interview session. In many cases their times fell within the inconclusive range hovering around the median. There was a case when the response time was fast and slow, but in all cases the students relied on the same problem solving strategy. They each ranked the statements based on how closely they aligned with the description of the individual in the vignette. Ryan is representative of this way of thinking.

Ryan's problem solving strategy illustrated the attribute substitution process for the conjunction fallacy when he solved the Description of an Individual problem (Chapter 4, Protocols 1-3). Ryan ranked the last statement, the conjunction, of the previous statements as more likely to occur than one of the single statements contained within it. He ranked the statement as more likely because it contained greater specificity about the person and it drew on his stereotypes as opposed to probability knowledge. That is, the greater the amount of information contained in a statement that was consistent with the description, the more probable that statement was deemed. This resulted in him violating the conjunction rule for two independent events. This heuristic response was identified in earlier research (Kahneman & Tversky, 1983).

Content Substitution Process. There was another form of substitution that was observed. It was called the *content substitution* process in this study. In contrast to the attribute substitution process, the content substitution process was observed when students used fraction knowledge in place of probability knowledge. This type of thinking was observed by most of the students and most clearly witnessed when the students solved the Two Heads versus Two Hundred Heads problem (Appendix B) based on their use of fractions or their fraction language. Elise exemplified this phenomenon.

Elise (Chapter 4, p. 25) was the most articulate about her use of fractions. She reduced 200 out of 300 to the lowest fraction form. She cited the close relationship between probability and fractions to justify her use of fractions. She seemed to suggest that they were one in the same in her mind. In these examples, the students perceived that the two ratios of coins represented equivalent fractions: one group, 200 hundred heads out of 300 coins, could be reduced to its simplest form that matched the other perceived fraction form, two heads out of three heads. A fraction comparison was hence substituted for a probabilistic comparison. It should be noted that this way of thinking was not interpreted as being new. Fischbein (1999) noted students' heavy reliance on fraction knowledge, "as a non-adequate schema: the schema of proportion" (p.45). What is unique is identifying it as another type of substitution within the heuristics and biases revised framework.

System 2

System 2 thinking in this study was identified by the *S-R* profile that was developed based on the extant research literature (Evans, 2008, 2011; Kahneman & Fredrick, 2002; Stanovich & West, 2000). In order to fit the *S-R* profile, a student's emotional behaviors had to include effortful thinking, an explicit explanation, and methodical way of thinking. Furthermore, *S-R* required a slow response.

In all cases where *S-R* was identified, the students were able to explain their problem solving strategy in a way that led to the correct answer. In all cases the mathematics the students used was appropriate and correctly applied. In this study there was no commonality in problem solving strategies found other than the expected ways of thinking. There were several instances in which S-R was observed: Ben when he answered the Description of an Individual problem (Chapter 4 protocol 9) and GP when she answered the Hospital problem (Chapter 4, Protocols 10-12). However, students had a variety of reactions when perturbed by the research team, which will be summarized in the discussion of the second research question.

Slow Intuition

The following paragraphs discuss slow intuition and its influence on problem solving strategies that led to heuristic answers for some students. There were a few cases when slow intuition occurred organically at the beginning of the problem solving process. However, it was more prevalent after the research team perturbed the students thinking or the student was internally perturbed after giving an initial answer. The phrase *slow intuition* was coined to describe a common pattern in the students' ways of thinking that was distinct from both *S-I* and *S-R* profiles. The most striking difference was that the students using slow intuition experienced internal perturbations throughout the problem solving process, which led to changes in their problem solving strategies during the course of the problem solving process. At times their strategies were based on mathematics that were misapplied and led to the solutions that usually corresponded to heuristic answers. Unlike with System 1 thinking, students were not comfortable with their solution. They wavered and attempted to correct their solutions, although the correct solution remained out of their reach at times. Brittney and Ryan (Protocols 18-20) demonstrated slow intuition that led to two of the three types of heuristics: representativeness and availability.

Heuristic and Incorrect Thinking. Brittney was an example of a student who was under the influence of slow intuition and ended up using both the representativeness heuristic and incorrect thinking. This occurred when she was solving the Hospital problem (Appendix C, Ch. 4, Protocols 6-8). Brittney's problem solving strategy was two-fold. First, she rejected the small hospital as the correct answer without explanation. Second, Brittney focused her attention on the population proportion for gender births and since the 60% was close to 50%, she claimed the size of the hospitals was not important. However, she was not sure of her answer and for the remainder of the time spent on this problem Brittney experienced internal perturbations that changed her problem solving strategies. She wavered between the heuristic way of thinking and a wrong way of thinking.

Two Kinds of Heuristic Thinking. Ben was chosen as the example because he first used the availability heuristic and he then experienced an internal perturbation that shifted him into a content substitution process while under the influence of slow intuition. Therefore, he shifted between two different heuristics. This occurred when he was solving the Making a Team problem (Chapter 4, Protocols 4 and 5).

When Ben was asked to find the number of ways to create two member teams and four member teams from six people, he stopped short of considering all possibilities because he quickly noted that there were more ways to make two member teams than four member teams given the six people. Specifically, he created three teams of two people and one team with four people and he compared the number of teams. He claimed there were more ways to make two member teams than four member teams. This type of a response was an example of the availability heuristic that was noted in earlier research (Tversky & Kahneman, 1973).

He then experienced an internal perturbation that changed his problem solving strategy from forming groups to creating ratios. This led to a different idiosyncratic heuristic: Ben used the number of people chosen for a team as the numerator and the total number of people available for the teams as the denominator. He used these numbers to create two ratios. The first ratio was two out of four and the second ratio was four out of six. He reduced the two people out of six people to one person out of three people. He equated choosing four people out of six people to choosing two people out of three people. He compared the resulting two ratios, one out of three with two out of three. At this point he counted the number of ways to make a one-member team with the number of ways to make a two-member team from three people. He realized in this reduced form that both scenarios had three different ways of grouping the teams.

This final problem solving strategy involved a heuristic in that he used the ratio as a way to reduce the complexity of the problem (Tversky & Kahneman, 1973). However, unlike documented heuristics, his was based on his equivalent fraction knowledge. Although his answer to the problem was correct, his heuristic would not generally work because the number of ways to choose two from six and the number of ways to choose one from three, for example, are not equivalent. This means Ben failed to correctly calculate the actual number of ways to make different team. Ben employed a content substitution process when solving this problem because he first translated the problem into a fraction format in which he reduced the fractions before he found the number of combinations to create different one-member teams and two-member teams from a pool of three people.

Fischbein's Intuition

Concerning Fischbein's (1975, 1987) intuition, Josh and GP's problem solving strategies were influenced by *anticipatory intuition* (Fischbein, 1975, 1987) when they solved the Marbles and Boxes problem (Chapter 4, Protocols 15 and 16). Recall that this problem had the students compare the number of way to place two marbles into four boxes or four marbles into two boxes. Josh worked on the problem for a few minutes and then suddenly he realized the number of ways to place the marbles into the boxes were the same. Prior to this moment of insight, Josh initially struggled to understand the problem. He appeared to be using slow intuition. Then, he showed a few examples of placing four marbles into two boxes and two marbles in four boxes. Josh concluded that there were more ways to place the four marbles into the two boxes, which was the incorrect answer. When he was asked to explain his thinking, during his explanation he experienced this sudden insight. He was confident that he found the answer because he saw the complimentary relationship between the boxes and marbles.

GP (Chapter 4, Protocol 16) had a similar experience on the same problem, although she had not stated a solution. Her moment of insight occurred as she was explaining her thinking during the problem solving process. GP appeared to be in System 2 thinking when the moment of insight occurred. GP was confident because she saw the compliment of the relationships between the boxes and marbles. Josh and GP both experienced a moment when the correct solution occurred to them when they moved to Fischbein's (1975, 1987) intuition. They were confident that they had the answer.

Summary

Summing up research question one, the students' problem solving strategies were by definition going to differ across System 1 and System 2. In many cases no aspect of the problem solving strategies stood out other than those that were expected. The influence of System 1 was expected to produce heuristic responses as identified by Tversky and Kahneman (Kahneman & Tversky, 1972, 1982; Tversky & Kahneman, 1971, 1973, 1974) that were generated by the *attribute substitution process* (Kahneman & Fredrick, 2002). Some of the students' problems solving strategies were linked to the *content substitution process*, an unexpected, substitution process identified in this study.

The original research question assumed that System 1 and System 2 would be the main ways of thinking observed. However, there were numerous instances of a distinct way of thinking referred to as *slow intuition* in this study. It was marked by a response time that was neither particularly slow nor fast and by students' behaviors that indicated that they lacked confidence with their answers. The effect of *slow intuition* on the students' problem solving strategies was that they wavered between strategies that included one of the substitution processes. There were two occasions when another way of thinking, anticipatory intuition as defined by Fischbein (1987), was observed. In both instances, anticipatory intuition led the students to a correct problem solving strategy and a correct result. This differed from System 2 because the solution occurred quickly to the student. The solution was not methodically thought out. Finally, the students' problem solving strategies that were influenced by System 2 led the students to apply mathematical techniques and reasoning appropriately. This would lead to correct solutions.

Research Question Two

What effect did the attempted perturbations have on the students' ways of thinking?

The techniques used by the research team to perturb the students' System 1 thinking in order to engender System 2 thinking included the following: modifying parameters of the probability problem, asking the students to quantify the probabilities in the problem, asking the students to demonstrate their problem solving strategies using manipulatives, and presenting counter-intuitive examples. The research team expected the students using System 1 thinking to being surprised that they had misapplied their probability knowledge and quickly changed their answer to the correct response. This was observed with Brittney when she solved the Hospital problem (Chapter 4, Protocols 6-8). The research team expected the students under the influence of System 2 thinking to remain under its influence. This was the case with Ben when he solved the Description of an Individual problem (Chapter 4, Protocol 9).

However, there were occasions when the external perturbations affected the students' thinking in unexpected ways. There were three different unexpected scenarios observed by the research team: (1) one student who started with System 1 thinking remained with System 1 thinking, (2) the students who moved from System 2 thinking into *slow intuition,* and (3) the students who moved from System 2 thinking into System 1 thinking.

The only example of the first scenario was Ryan' response when he solved the Description of an Individual problem (Chapter 4, Protocols 1-3). At first, Ryan answered heuristically when he ranked the conjunctive statement as more probable than one of the two statements that constituted the conjunctive statement. To perturb his thinking, Ryan was asked to draw a Venn diagram for the compound statement. It was expected that his diagram would cause him to reconsider his initial answer. However, this was not the case. Ryan was then asked to find the probability of the combined statement. He added the probabilities of each constituting statement and he concluded that the combined statement was most likely. Ryan was asked to reconcile his mathematics with his diagram and he claimed his diagram was inaccurate. In the end, he ranked the combined statement as the most likely statement and the external perturbations were unable to move Ryan's thinking away from his initial heuristic response influenced by System 1.

GP offers an example of the second scenario when she was moved from System 2 into *slow intuition*. GP initially answered the Hospital problem (Chapter 4, Protocols 10-12) correctly and otherwise demonstrated behaviors that suggested her problem solving strategies were influenced by System 2. It appeared GP understood that smaller samples had greater variability than larger samples because she explained that it would be easier to get more of one gender in the small hospital than in a large hospital. Recall that GP was initially asked to consider which hospital was more likely to have 60% or more of the babies born to be boys. The research team attempted to perturb her thinking by changing the percentage in the problem to 10% for the birth of baby boys. She responded by changing her answer to the incorrect response using a problem solving strategy of comparing the number of babies born in each hospital. Specifically, she compared the 10% number from each hospital and she was choosing the larger hospital because the number was larger. She was reminded that this was a probability problem, which was expected to move her back to her initial problem solving strategy. This was not the case and she changed her answer again to the heuristic response, which was that the sizes of the hospital were not important. Her problem solving strategy had shifted to attribute substitution in which she considered the population proportion for gender births and she dismissed the sizes of the hospitals. Near the end of the interview, the researchers asked her to return to her original example. In her original example she assigned a numeric value to the average number of babies born in each of the hospitals per day. She was then asked to consider which hospital was more likely to have all girl babies. This time GP moved back into System 2. She answered correctly and her problem solving strategy reverted back to the strategy she used when she initially answered the question. GP's intermediate way of thinking was a good example of slow intuition in that she wavered between varying problem solving strategies and experienced uncertainty linked to internal perturbations.

Josh gives an example of the third scenario when he moved from System 2 to System 1 thinking due to an external perturbation as he solved the Chips and Randomness problem (Chapter 4, Protocols 13 and 14). Recall in this problem that the students were shown two sequences of five chips that show various sports balls on one side and asked which sequence was more likely to occur. Josh seemed to be using System 2 thinking when he answered the first part of the activity correctly: both sequences were equally likely. Josh's thinking was perturbed when the chips were turned over. Using the reverse side of the chips, one sequence showed a pattern of numbers one through five and the other sequence did not. Josh changed his answer and claimed the sequence that showed the sequential order to the numbers one through five was less likely, which was the heuristic response and indicated he was influenced by System 1. His problem solving strategy was based on the appearance of randomness. Further provoking pushed him into slow intuition. In the end, Josh moved back to System 2 when the research team externally perturbed him by asking him to consider how likely it would be if the sequence lacking a pattern were preferred.

In conclusion, for research question two, the researchers could sometimes perturb the students' thinking from either System 1 or from System 2. The ability to move the students away from System 1 and towards System 2 was expected. It was also expected that attempting to move the students away from System 2 thinking would fail. There were three scenarios observed that were unexpected. The first was a student that remained using System 1 thinking even after an attempt was made to externally perturb him. The second were when some students were moved from System 2 to *slow intuition*. The third was when the students were moved from System 2 to System 1.

Research Question Three

How stable or strong was a participant's problem solving strategies associated with System 1 or System 2?

The strength or stability of a student's problem solving strategy was observed by looking at the student's level of commitment to the initial answer when they were externally perturbed. Recall that the research team initially had two expectations when they provoked the students thinking. The first expectation was that the students under the influence of System 1 would have a weak commitment to their problem solving strategy. It was anticipated the external perturbations would move the students into System 2 thinking because the external perturbation would induce them to scrutinize their answer, realize their error as a result of the scrutiny, and change their problem solving strategy to produce the correct answer. As mentioned in the previous section, Brittney gave an example of this when she solved the Hospital problem. However, several perturbations were required before she arrived at the correct answer. At that point, she did realize the error in her thinking and was no longer committed to her initial answer. In this example, Brittney was weakly commitment to her initial answer and problem solving strategy as she considered several alternative strategies during the interview.

The second expectation was that the students under the influence of System 2 thinking would have a strong commitment to their problem solving strategy because their problem solving strategy was based on their sound understanding of the mathematics. Ben was an example of this when he solved the Description of an Individual problem (Chapter 4, Protocol 9). The attempted external perturbation was unable to change his thinking. As expected, Ben was strongly committed to his initial problem solving strategy when provoked by the research team.

There were scenarios that were unexpected. For example, Ryan remained committed to his initial heuristic problem solving strategy when he answered the Description of an Individual problem (Chapter 4, Protocols 1-3). Some System 1 thinking was more stable than anticipated including Brittney (Chapter 4, Protocols 6-8) who would revert back to her initial heuristic answer. When externally perturbed by the research team, Ryan remained committed to his heuristic strategy that inadvertently re-enforced his heuristic thinking. Brittney, although she recanted the heuristic problem solving strategy in the end, vacillated between answers that included a return to the heuristic strategy.

It was expected that the students under the influence of System 2 would be more stable and this was true for some students like Ben (Chapter 4, Protocol 9). However, this was not the case for Josh when he was solving the Chips and Randomness problem (Chapter 4, Protocols 13 and 14). It was unexpected that a student under the influence of System 2 would relinquish their initial response. Josh readily relinquished his attachment to the correct strategy, when externally perturbed by the research team before he returned to it at the end of the interview.

Limitations

There are two limitations identified in this study. The first limitation is the heuristics and biases framework used in this study. The second limitation is the nature of intuitive thinking. In the first case, the heuristics and biases framework failed to support the goal of the study due to the other ways of thinking that were observed. In the second case, attempting to draw out explicit thinking in real time is challenging and much more so when the thinking is implicit. The following paragraphs explain these limitations and their impact on the findings.

The first limitation addresses the heuristics and biases framework used in this study. The initial notions held by the researcher, concerning the general heuristics and biases framework, suggested that the framework would provide a way to separate particular ways of thinking: heuristic thinking, correct thinking, and wrong thinking. The revised heuristics and biases framework appeared to provide an explanation for why heuristic thinking is used. The explanation is informed by dual processing theory, in which System 1 (fast thinking) is linked to intuitive thinking, and System 2 (slow thinking) is linked to reasoning (e.g. Evans, 2008; Stanovich & West, 2000). It was hypothesized that the researcher would be able to identify specific attributes associated with the content, context, or structure of the probability problems that initiated either System 1 or System 2 thinking. However, unexpected results confounded the analysis since many of the students did not fall cleanly into these initial patterns of thinking. Thus, the framework required a broader perspective when analyzing the various thinking patterns observed.

It is necessary at this point to remind the reader of two points associated with the heuristics and biases framework. First, in general, heuristic thinking is a way to solve a complex task in which the individual cannot mentally perform the calculations or the magnitude of the problem prevents the mental calculation. As a result, the individual uses a heuristic to reduce the task into a manageable form. Second, the heuristics and biases research evolved from decision theory, in which decisions were characterized dichotomously. These responses were compared to statistical norms that were used to label the responses as either correct or incorrect. The researchers, Amos Tversky and Daniel Kahneman, found that within certain situations of uncertainty there were common incorrect responses, and they focused their attention on this phenomenon later identified as their heuristics and biases research.

Returning to the current research study, when analyzing the eighth-grade students' various ways of thinking, it became evident to this researcher that the heuristics and biases framework did not provide the preliminary separation of the students' thinking to the degree that was expected. As identified in the findings there were many students who did not follow the expected patterns. Likewise, when the researcher perturbed the students' thinking it was expected that intuitive thinking would give way to reasoning. This notion is based on the revised heuristic and biases framework (Kahneman & Fredrick, 2002), in which it was hypothesized that slowing down the thinking process would allow the students to recognize their erroneous thinking. However, this was not always the case and this brought into question the labeling of the students' answers: correct, heuristic, or wrong. The labeling seemed to reflect a judgment on the students' thinking, which was not the intent of this researcher. This researcher agrees with the non-judgmental attitude shared among other researchers who explore students' thinking (e.g. Gigerenzer, 2005; Cobb & Steff, 1983). In the end, the researcher may have witnessed various levels of cognitive development so that labeling the students' answers is inappropriate.

In particular, this researcher observed a few occasions when students' were engaged in System 1 thinking that resulted in correct answers. The quick and correct response suggests automaticity, which is favorable among mathematics educators because the student has reduced cognitive load so that new knowledge can be built. Mathematics educators wish to move students from System 2 thinking that is slow, to System 1 thinking that is fast, a possible indication that learning has occurred. The notion that System 2 is linked to reasoning with correct answers was also problematic because some System 2 thinking can be fruitless, which requires the researcher to perturb thinking into a more productive direction. There were also cases in which the students were engaged in System 2 reasoning but minor calculation errors produced a wrong answer. This example highlights the ineffective labeling of the students' ways of thinking. There were also occasions in which students were thinking methodically and slowly but their reasoning was based on their own heuristic that led to a correct answer. If the students were asked to quantify their answers, their heuristics would fail them. There were several other occasions that were observed by the researcher in which students demonstrated thinking patterns between these two extreme ways of thinking, System 1 and System 2. As described earlier, this was referred to as slow intuition. In addition, content substitution was observed in which students attributed or equated characteristics of other content areas with probability and used this knowledge to solve the probability problems. For example, some students inappropriately substituted their understandings of fractions to solve probability problems.

In the end, the framework did not meet the goals of the study, to parse out eighth-grade students' ways of thinking into one of three groups for further analysis. The six eighth-grade students used different patterns of thinking and their ways of thinking failed to always align with their answers. Labeling the answers as correct, heuristic, or wrong seemed to be an over simplification that carried judgmental overtones on the ways the students were thinking. The revised heuristics and biases framework may prove more effective when the constraints of the categories and labeling of the answers are either relaxed or renamed.

The second limitation addresses the implicit nature of intuitive thinking. As a result, it is difficult to recognize and examine intuitive thinking. The *think aloud* method was specifically chosen to gain increased information about the students' intuitive thinking. Inherent with this technique, and interviews in general, was the issue associated with self-reporting. In this study, the research team had to remind the students to verbalize their thinking. Many times this was met with additional silence and incomplete thoughts. This could be remedied in future studies by modifying the instructions to the students about the protocol. In this study the researcher modeled the think aloud protocol after reading an example problem. The researcher verbalized thinking patterns that demonstrated reasoning. The modeling should have included another example of an implicit response. This example would show incomplete thoughts that appeared unfocused and underdeveloped. Essentially, a response that was less verbal. The researcher could show the participants that they could circle words or terms in the question that caught their attention when words failed them. However, there are no guarantees that the participants' response would allow the research team to clearly observe their various ways of thinking, particularly their intuitive thinking.

Implications

This study contributes to the body of knowledge associated with eighth grade students various ways of thinking and how that impacted their problem solving strategies. In particular, teaching practices can be refined to address different intuitive ways of thinking. For example, Kahneman and Fredrick (2002) intuition suggests an impulsive thought process that substitutes one attribute for another with the attribute substitution process. Fischbein's (1975, 1987) general notion about intuition suggests that it can be developed and used to support reasoning. In addition, teaching practices may need to target some student's inclination to use the content substitution process. The content substitution process was observed when the students were using System 1 thinking and when the students were using slow intuition. Each of these ways of thinking may require a different teaching practice to counter the students' inclination to substitute content. In either case, teachers need to be informed of the extent that some students rely on their fraction knowledge when solving probability problems.

Recalling prior research, Shaughnessy (2003) mentioned the 1996 National Assessment of Educational Progress (NAEP) report. The report showed students' performance trends with probability problems that highlighted students' difficulty with the ratio format for probability problems. Counter to the NAEP (1996) position was the literature review performed by Peter Bryant and Terezinha Nunes (2009) addressing children's understanding of probability in which they suggested ratios may be the better way to address probability instead of a part-whole approach. This study found the students using the term ratio as a synonym for part whole and thus the debate continues.

Recommendations for Future Research

Future research that expands on this study, could investigate the problem solving strategies of students who have reached the preoperational stage and the concrete operational stages of cognitive development (Piaget, 1968). This would begin to create a trajectory of these various ways of thinking: System 1 thinking, System 2 thinking, or the *slow intuition*. In addition, future research could explore problem solving strategies for probability problems by creating different instruments for the study. Data collection instruments that were technology based may open implicit thinking patterns more effectively. For example, probability problems that are designed as interactive computer activities may be more effective to differentiate between System 1 and System 2 thinking.

In particular, the researcher for this study intends to expand on this research by focusing on the content substitution process. It is hypothesized that fraction knowledge may need to undergo some type of reorganization so that problem solving strategies retains probability knowledge. In addition, it is hypothesized that the reorganization may look different for different probability content. For example, a problem focused on randomness may have a reorganization scheme that differs from a problem focused on variability. These notions about fraction reorganization form the bases for the next research study.

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Appendix A

Probability Content Knowledge Assessment

Name:_Key_________

Date:___Sept 1________

Purpose: Assess participant's probability knowledge in order to identify the more suitable candidates to progress to the next level in the research study. It is necessary for participants to hold an adequate level of probability knowledge in order to pit reasoning against heuristic thinking. The following problems were obtained from the released 2009 and 2010 Statewide assessments for middle school students, grades 6, 7 and 8.

Section 1: 6th Grade

1. A bag contains 4 red, 5 green, 3 blue, and 6 yellow tiles of equal size and shape. One tile is randomly selected from the bag. What is the probability that the tile selected is blue?
   a. 5/6
   b. 1/3
   c. 1/5
   d. 1/6

2. Andy is packing clothes for a camping trip. Based on the chart, which of the following shows all of Andy's possible clothing combinations of 1 color of shirt, 1 type of pants, and 1 type of shoes?

Andy's Clothes

Shirt	Pants	Shoes
Blue	Shorts	Boots
Green	Jeans	Sandals

A. blue shirt, shorts, boots blue shirt, shorts, sandals blue shirt, jeans, boots blue shirt, jeans, sandals green shirt, shorts, boots green shirt, shorts, sandals green shirt, jeans, boots green shirt, jeans, sandals	B. blue shirt, shorts, boots blue shirt, jeans, sandals green shirt, shorts, boots green shirt, jeans, sandals
C. blue shirt, shorts, boots blue shirt, shorts, sandals blue shirt, jeans, boots blue shirt, jeans, sandals green shirt, shorts, boots green shirt, jeans, boots red shirt, shorts, sandals red shirt, jeans, sandals	D. blue shirt, jeans, sandals Green shirt, shorts, boots

3. Students at Wilson Middle School must choose one elective from each group in the table.

Middle School Electives

Sport	**Clubs**	**Language**
Soccer	Ecology	French
Basketball	Drama	Spanish
Track		

Which tree diagram shows all possible combinations of choosing one elective from each group?

A

French
Ecology <
Spanish
Soccer <
French
Drama <
Spanish

French
Ecology <
Spanish
Track <
French
Drama <
Spanish

B

French
Ecology <
Spanish
Soccer <
French
Drama <
Spanish

French
Ecology <
Spanish
Basketball <
French
Drama <
Spanish

C.

French
Ecology <
Latin
Soccer <
French
Drama <
Latin

French
Ecology <
Latin
Basketball <
French
Drama <
Latin

French
Ecology <
Spanish
Track <
French
Drama <
Spanish

D.

French
Ecology <
Spanish
Soccer <
French
Drama <
Spanish

French
Ecology <
Spanish
Basketball <
French
Drama <
Spanish

French
Ecology <
Spanish
Track <
French
Drama <
Spanish

4. A deck of 50 cards for a math game has 9 red cards, 13 blue cards, 18 green cards, and 10 yellow cards. What is the probability that a card randomly selected from the deck will be a blue card?

a. 13%
b. 26%
c. 74%
d. 87%

5. Ivan has a fair number cube numbered 1 through 6. He will roll the cube one time. What is the probability that the number shown on the top face is a 2?
a. 1/6
b. 1/3
c. 2/3
d. 5/6

Section 2: 7th Grade

6. A bag has 15 white marbles and 12 blue marbles. Allison will randomly select 1 marble from this bag. What is the probability that she will select a blue marble?
a. 5/4
b. 4/5
c. 5/9
d. 4/9

7. The school soccer team is ordering new knee pads for their uniforms. The knee pads come in 4 different colors, 6 sizes, and 2 styles. How many different outcomes of knee pads are available?
a. 12
b. 24
c. 40
d. 48

8. Kenan recorded the outcomes he got when he flipped a fair coin 30 times.
If he flips the same coin 300 more times, then he should expect that for these 300 flips —

a. less than 60 flips will land tails up
b. close to 150 flips will land heads up
c. between 250 and 300 flips will land tails up
d. more than 250 flips will land heads up

9. Randall wants to buy a pizza. He can select from 5 different sizes, 4 types of crust, and 12 toppings for his pizza. Which of the following shows how to find all the different possible choices of 1 size, 1 crust, and 1 pizza topping Randall can buy?

a. 5* 4*12
b. 5*4 + 12
c. 5 * (4+12)
d. 5 + 4 + 12

Section 3: 8th Grade

10. This table shows the number of marbles in a bag by color. If one marble is randomly selected from the bag, what is the probability that it will be a blue or a red marble?

a. 24%
b. 26%
c. 38%
d. 52%

Marbles in a bag

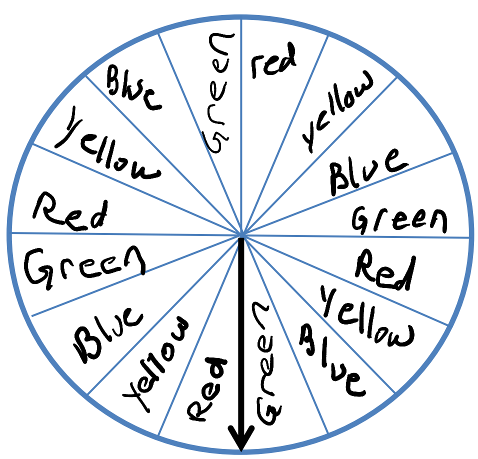
Color	Number
Blue	7
Green	10
Yellow	14
Red	19

11. A fair cube used in a game has 1 yellow side and 5 green sides. Emily will win the game if the cube lands on a green side on her next roll. Which statement best describes Emily's chance of winning the game?

a. Certainly will win
b. Certainly will lose
c. Most likely will win
d. Most likely will lose

12. Jill is playing a game with a spinner like the one pictured below. If the sections are congruent, what is the probability that on Jill's next spin the arrow will land on a section labeled red?

a. 1/3
b. 1/4
c. 1/15
d. 1/16



13. In Happy Hills Pond, 2 out of 5 fish are goldfish. If 40 fish were randomly chosen from the pond, which is most likely the number of goldfish chosen?

a. 8
b. 16
c. 24
d. 37

14. The table shows all of the possible outcomes when Juan tosses a penny, a nickel, and a quarter at the same time. What is the probability that Juan's result will be 3 tails on his first toss?

a. 1/8
b. 1/4
c. 1/3
d. 1/2

Penny	**Nickel**	**Quarter**
Heads	Heads	Heads
Heads	Heads	Tails
Heads	Tails	Heads
Heads	Tails	Tails
Tails	Heads	Heads
Tails	Heads	Tails
Tails	Tails	Heads
Tails	Tails	Tails

Appendix B

Cognitive Development and Heuristic Thinking

Name:___Key_______

Date:_Sept 15_______

Part A: Cognitive Development

1. If you have a group of two black chips where 1 has a sticker on it and a group of four red chips where 2 have a sticker on it, which of the following statements is true? Why?

   a. both groups of chips have the same chance of selecting a chip with a sticker on it
   b. the red group has a greater chance of selecting a chip with a sticker on it
   c. the black group has a greater chance of selecting a chips with a sticker on it
   Piaget & Inhelder (1975) quantification of probability – proportions; ½ vs 2/4, which is equal

2. Three Heads or Four Heads
   If you have one coin. Is it more likely to get 3 heads after three tosses or more likely to get a four heads after 4 tosses? Why?
   Piaget & Inhelder (1975) random – probability for sequences decreases; ½*1/2*1/2 = 1/8 vs ½*1/2*1/2*1/2 = 1/16. Compound events where we want each event to occur P(A and B and C and D) = P(A)*P(B)*P(C)*P(D)

3. You have four chips, where each chip is a different color: red, blue, yellow, and green. How many pairs can you make where the two chips are different colors? Why?
   Piaget & Inhelder (1975) random mixtures; Chart r, b, y, g rows and columns, deselect same colors and since combo, only choose above or below diagonal, 6
   Combination formula n!/r!(n-r)! = 4!/2!(4-2)! = 4!/2!2! = 24/4 = 6.

4. A bag contains 35 counters where 15 are yellow, 10 are red, 7 are green, and 3 are blue. What is the most likely outcome when selecting one counter at random? What is the most likely outcome when selecting a second counter at random without replacement? Why?
   Piaget & Inhelder (1975) combinations; Modified this to a single event from the original with choosing a pair which has the following as the solution. This was modified since it was found to be oversimplified by the participants in the pilot study.
   Yellow: 15/35; If yellow chosen first, then 14/34, if not, 15/34
   Red: 10/35; If red chosen first, then 9/34, if not 10/34
   Green: 7/35; If green chosen first, then 6/34, if not then 7/34
   Blue: 3/35; If blue chosen first, then 2/34, if not then 3/34

Part B: Heuristic Thinking

1. Six Fair Coins
   Given two sequences of a fair coin, toss, which of the following is more probable? Give a reason for you answer.

   a. T H H T H T
   b. T T T T T T
   c. They are equally likely
   Tversky & Kahneman (1974), People expect short runs to show randomness and to be in the appropriate proportions that match the population.

2. Red and White Balls
   What is more likely to occur?

   a. Pulling one red ball from a jar containing 10 red and 90 white balls.
   b. Pulling four red balls in a row from a jar containing 50 red and 50 white balls (with replacement)
   c. The chances are about equal
      Representativeness - Shaughnessy, 1977
      1 Red ball: 10/100 = 1/10
      4 Red balls: ½*1/2*1/2*1/2 = 1/16 (misconception)

3. Description of Marcia
   Marcia has a small core of close friends with whom she goes out on rare occasions, but for the most part during her free time she prefers to spend time at home working Sudoku puzzles. She is a good student and recently was awarded a small scholarship for academic excellence. Please rank the following statements in terms of how likely you believe them to be, using a 1 for most probable and a 5 for least probable.

   Statement 1: Marcia volunteers at the local public library
   Statement 2: Marcia enjoys reading historic novels
   Statement 3: Marcia spends her weekends competing in road running events
   Statement 4: Marcia has a sports car
   Statement 5: Marcia is a library volunteer and enjoys writing short stories
   Conjunction fallacy – similar to Tversky and Kahneman (1982) "Linda" problem but changed to be more assessable to middle school students. Like the Linda problem, there are 3 fillers (b, c, d). Item a and e are the stereotype or distractors where e contains the conjunction.

4. Fifth Child
   A certain family is expecting the birth of their fifth child. The first four children were boys. Assuming that the probability for the birth of a boy and a girl to be about equal, which of the following statements is true?

a. The child will probably be a girl – main misconception, girl is due
b. The child will probably be a boy
c. The probability for the child to be a boy or a girl is equal - correct
Representativeness (gambler's fallacy or negative regency) F&S 1999

5. Two Heads versus Two Hundred Heads
The likelihood of getting heads at least twice when tossing three coins is:

a. Smaller than - incorrect
b. Equal to - incorrect, main misconception
c. Greater than - correct
the likelihood of getting heads at least 200 times when tossing 300 coins
F&S 1997, they refer to T&K, 1982 – effect of sample size

6. Making a Committee
The number of ways that you can make a committee with 2 members who are chosen from 10 people is (fill in the blank) ____

a. Smaller than – main misconception
b. Equal to - correct
c. Greater than

the number of ways that you can make a committee with 8 members who are chosen from 10 people.
Availability - T&K 1974; F&S 1997
2 members: 10!/2!(10-2)! = 10!/2!8!
8 members: 10!/8!(10-8)! = 10!/8!2!

7. Marbles and Boxes
You randomly place your two marbles, 1 red and 1 green, into 4 boxes, where the marbles can be placed into the same box. Your friend randomly places his 4 marbles, 1 red, 1 green, 1 blue, and 1 yellow, into 2 boxes. Which of the following is true?

a. You have more ways to place your 2 marbles into your 4 boxes?
b. Your friend has more ways to place his 4 marbles into 2 boxes?
c. You and your friend have the same number of ways of placing your marbles into your own boxes?
Availability - Joyce and I made up, which is a contextual problem to the grid number of paths problem. $4^2 = 2^4 = 16$. The researcher will ask how the student selected the correct response, since 4*2 = 8 and 2*4 =8, which is what we found in the pilot study.

Appendix C

Perturbation Activities

Name: ____Key__________

Date: ___Sept 15_________

Purpose: To engage with the participant with activities so as to perturb his or her heuristic thinking. Choose the appropriate scenario for the participant based on the findings from the first interview. This page includes additional information for me when I interview the participants.

Activity I: Hospital Problem (Representativeness Heuristic)

The chance that a baby is born a boy is about 1/2. Over the course of an entire year, would there be more days when at least 60% of the babies born were boys in:

(a) a large hospital
(b) a small hospital
(c) makes no difference

Cues added approach
Researcher: The chance that a baby is born a boy is about ½. Over the course of an entire year, would there be more days when at least 100% of the babies born were boys in:

Cues removed approach
Researcher: The chance that a baby is born a boy is about ½. Over the course of an entire year, would there be more days when at least 60% of the babies born were boys in:

(a) a large hospital
(b) a small hospital
(c) it makes no difference between the sizes of the hospital

Alternative Cues Removed Group: Show the following chart with each option, 60% of the babies are boys and indicate that the large hospital has on average 20 (12 babies are boys) babies born and the small hospital on average has 5 babies born (3 babies are boys). I can use a coin to represent a birth since it is close to the 60% and flip it 5 times and 20 times so they can see which is more likely to occur more often.

Days ->	1	2	3
Large			
Small			

……

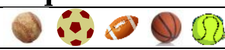
360	361	362	363	364	365

Activity II: Chips and Randomness (Representativeness Heuristic)

Cues added approach

We have two jars that have 30 chips in them. We have taken out 5 chips from each jar. We have this *sequence* of chips that we drew randomly from the first jar and we have this second *sequence* of chips drawn randomly from the second jar. What does the word "sequence mean" to you? Which *sequence* do you think is more likely to occur? Or do you think they are the same?

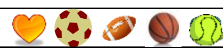
Researcher: (Turns over the chips) How does your answer now compare with your response when you saw the symbols?

Cues Added	**Sequence 1**	**Sequence 2**
Chip Sequence – side A		
Chip Sequence – side B	1 2 3 4 5	3 5 1 2 4

Cues removed approach

We have two jars that have 30 chips in them. We have taken out 5 chips from each jar. The first *sequence* of numbers came from the first jar and the second *sequence* of numbers on the chips came from the second jar. Which *sequence* do you think is more likely to occur? Or do you think they are the same?

Researcher: (Turns over the chips) How does your answer now compare with your estimate when you saw the numbers?

Cues Removed	**Sequence 1**	**Sequence 2**
Chip Sequence – side A	1 2 3 4 5	3 5 1 2 4
Chip Sequence – side B		

Activity III: Description of an Individual (Conjunction Effect)

Cues added approach

Stacey is 13 years old middle school girl who is outspoken and very bright. She loves her science class. She particularly enjoys the topics related to biology and the eco-system. Please rank the order of these statements from least probable to most probable.

	General to Specific
Statement 1	Stacey runs in 5K events.
Statement 2	Stacey is active in the community animal shelter.
Statement 3	Stacey wants to become a fitness trainer when she grows up.
Statement 4	Stacey writes a column in her school newspaper about recycling.
Statement 5	Stacey volunteers on most weekends in the local hospital.
Statement 6	Stacey is active in the community animal shelter and she writes a column in her school newspaper about recycling.

Cues removed approach

Stacey is 13 years old middle school girl who is outspoken and very bright. She loves her science class. She particularly enjoys the topics related to biology and the eco-system. Please rank the order of these statements from least probable to most probable.

	Specific to General
Statement 1	Stacey is active in the community animal shelter and she writes a column in her school newspaper about recycling.
Statement 2	Stacey volunteers on most weekends in the local hospital.
Statement 3	Stacey writes a column in her school newspaper about recycling.
Statement 4	Stacey wants to become a fitness trainer when she grows up.
Statement 5	Stacey is active in her community animal shelter.
Statement 6	Stacey runs in 5K events.

Alternative Cues removed approach

Stacey is 13 years old middle school girl who is outspoken and very bright. She loves her science class. She particularly enjoys the topics related to biology and the eco-system. Please rank the order of these statements from least probable to most probable.

	Specific to General
Event 1	Stacey runs in 5K events.
Event 2	Stacey volunteers on most weekends in the local hospital.
Event 3	Stacey writes a column in her school newspaper about recycling.
Event 4	Stacey wants to become a fitness trainer when she grows up.
Event 5	Stacey is active in her community animal shelter.
Events 3 and 5	Stacey writes a column in her school newspaper about recycling and she is active in her community animal shelter.

Activity IV: Making a Team (Availability)

Is the number of ways that you can make a team with 2 members who are chosen from 6 people is (fill in the blank) ____

d. Smaller than – main misconception
e. Equal to - correct
f. Greater than

the number of ways that you can make a team with 4 members who are chosen from 6 people?
Availability - T&K 1974; F&S 1997 Modification to the original problem
Team with 2 members: 6!/2!(6-2)! = 6!/2!4! =
Team with 4 members: 6!/4!(6-4)! = 6!/4!2!

Cues added approach (to reduce confusion – attention cues to help the person)

Two person team:
The cues added have the students consider a tree diagram. This is consistent with what they have learned in school.

Alternative cues added approach
Perhaps another suggestion can be a table where they consider all of the ways people can be matched.

	Person 1	Person 2	Person 3	Person 4	Person 5	Person 6
Person 1		X	X	X	X	X
Person 2			X	X	X	X
Person 3				X	X	X
Person 4					X	X
Person 5						X
Person 6						

Tree Diagram:
First branch Second branches
Person 1 – Person 2, or 3 or 4 or 5 or 6
Person 2 – Person 3 or 4 or 5 or 6
Person 3 – Person 4 or 5 or 6
Person 4 – Person 5 or 6
Person 5 – Person 6
Person 6 – has already been paired with each of the others

We count these up to find there are 15 ways to create a team of 2 people from 6 people.

Four Person Team:

	Branch 1	Remaining	Branches ->	(alternatives	To get 4)	
Ways	Person 1	Person 2	Person 3	Person 4	Person 5	Person 6
1	X	X	X	X		
2	X	X	X		X	
3	X	X	X			X
4	X	X		X	X	
5	X	X		X		X
6	X	X			X	X
7	X		X	X	X	
8	X		X	X		X
9	X		X		X	X
10	X			X	X	X
		Branch 1	Remaining	Branches ->	(alternatives	To get 4)
11		X	X	X	X	
12		X	X	X		X
13		X	X		X	X
14		X		X	X	X
			Branch 1	Remaining	Branches	
15			X	X	X	X

We can count all of the ways that are shown in the rows. This matches the combinations formula for selecting four out of 6 (6!/4!(6-4)! = 6!/4!2! = 15)

APPENDIX D

Bracketing Memos

Bracketing Memo #1

Opinions and ideas I have concerning 8th grade students as participants**.**

A. Behavior
   1. Somewhat friendly to adults
   2. Honest and open with their opinions when asked
   3. Patient with all the questions
   4. Engage in the activities
   5. Open my eyes to alternative ways that students think
   6. Put forth an honest effort to answer questions
   7. Sit across from me
   8. Can feel comfortable being video-taped
   9. Willingness to restate thinking so I can understand them
B. Dress
   1. Clean casual clothing
   2. Groomed but clothing and hair may be untidy
   3. May smell low levels of perspiration or body odor

Ideas about 8th grade students that is shown in movies that may distort my notions

1. Rowdy
2. Smart mouth
3. Tolerates authority figures to lack of respect for authority
4. Shuts out adults
5. Disrespectful
6. Impulsive
7. Self-centered
8. Bullies
9. Cliques – exclusive oriented
10. Humor behavior oriented vs. intellectually oriented

Bracketing Memo #2

Date: April 17, 2013
Topic: Tensions in the Data: Evaluative and Epistemological

Evaluative:
It occurred to me that I may have used line codes that were "evaluative" in nature. For example, I am reflecting on my thoughts when I read the transcripts and notice that I felt like I did not perturb the students "correctly" because they changed their answer to the "wrong" one when they

began with a heuristic answer or worse, I thought, when I perturbed them and they were correct and they went to either a heuristic or wrong response. I think I use too much evaluative thinking and focus on the “right” answer, which may have skewed my word choice for codes.

I will re-evaluate the coding again, this time looking for evaluative words and changing them.

Methodological Differences:

I am thinking there is an epistemological disconnect using the instruments from H&B with qualitative research methods. This came to while running this afternoon, after I re-read Kahneman & Frederick (2002) and I read the article that they referenced Hertwig & Gigerenzer (1999) where Kahneman said that qualitative methods don’t work as proved by this study. But Hertwig and Gigerenzer did not conclude that. So, I began thinking about the tensions between the two camps and wonder if that is underlying the issue. H&B comes out of Decision Theory which is quantitatively biased. It uses quantitative methods to help make better managerial decisions – manage with data. Therefore, this may be something I may need to consider.

APPENDIX E

After Interview Memos

Memo #1

Dec 6, 2012
First Set of Interviews:
Participants: G.P., Elise, Brittney
Think about random. It seems like they can apply it sometimes but other times they cannot. Is the notion of randomness limited because of what you mechanically do – toss coins, throw dice? Is randomness not recognized when you are comparing samples? Could a "deterministic" thinking be substituted in and "random" pushed out? Think about probability as potentially two components: varying and proportions of outcomes. How may the Classical definition undermine the idea of randomness?

Dec 13, 2012
First Set of Interviews:
Participants: Maggie, Ryan, Billy Jones, Ben
Two noticed the compliment of the set up for the marbles and boxes problem. They believed the number of placements were the same for both. It appeared they were drawing on intuition – not the fast kind that slips past the template and does not need to be checked – but the ability to mathematically check was beyond their reach. Check out other characteristics of intuition to see if this fits. Maybe a lack of knowledge means they have pushed their reasoning abilities to their limit. Hmmm.

Dec 14, 2012
First Set of Interviews:
Participants: Chelsea, Diane
These two participants changed their minds several times when they understood the question. This seems to suggest they were answering a different question either by misreading or misunderstanding the problem situation or what was being asked of them. But, was this making the problem simpler???

Dec 18, 2012
First Set of Interviews:
Participant:

One student said she changed her mind before speaking, which indicates it is very difficult to "catch" intuition in real time or at all. The participant indicated that "more times" means "more likely" to her.

I need to consider revisiting a question at the beginning of the second interview.

Seeing fraction thinking is heavily involved in their thinking, as expected but…
Personal experiences play heavily into this too: Maree acted out one of the problems and tossed a coin 12 times and got heads 10 times. This was normal for her – she said she almost always gets heads.

Considering the word “sequence” because it appears the students do not address it as an entity but as a string of separate entities. This needs to be included as a follow up question.

APPENDIX F

Analytic Memos

Analytic Memo #1
Date: Jan 3, 2013
Topic: Representativeness

I am thinking about representativeness, its components and perturbation. Representativeness has 2 components; process generation, which is the randomness and population proportions, which is where the samples link to the similarity of the population. It seems like each one of these components could trigger the heuristic. This means, the perturbation must address the part that is being triggered.

Perhaps, to differentiate between understanding compound events from heuristic thinking, have the students calculate the probabilities for two people tossing a coin in the following way:

Person A: H, next toss -> HT …. HHTHTT

Person B: T, next toss -> TT …. TTTTTT

Analytic Memo: 2
Date: Jan 3, 2013
Topic: Compound Events

I am thinking about compound events that go unnoticed in a mathematical sense. That is, the proportion of the population is the focus where the number of trials becomes the numerator for the proportions. For example, the 1 pull with 10 red and 90 white versus 4 pulls for red where 50 red and 50 white. The comparison of fractions becomes 10/90 (stated as a ratio) and 4/50 or 4/100.

Perturb with 1 pull from 25 red and 75 white and compare with 2 pulls where there is 50 red and 50 white. In this situation I suspect the fraction comparison will be 25/100 to ½ or 2/50 or 2/ 100 instead of the ½, where they will reduce the fractions to ¼ vs ½, 1/25, 1/50.

Analytic Memo: 3
Date: Jan 3, 2013
Topic: Conjunctive Effect and Compound Events

They seem not to be making a connection between probability and set theory. I don't think they have been introduced to that – need to check on that. They have been introduced into compound events, $P(A \cap B)$ as indicated in 8^{th} grade SOL's. But, not all of these students are taking 8^{th} grade math. The Algebra and Geometry students may not have this knowledge.

Analytic Memo: 4
Date: Jan 4, 2013
Topic: Scheme Theory – Assimilation/Accommodation/Abstraction

I am finding that I need scheme theory to help me unpack their thinking. It offers me a framework to identify and explain what I see. I need to review and check chapter 2 about this discussion.

Assimilation – fits an experience into a conceptual structure it already has. This means, it disregards what doesn't fit into the existing conceptual structure. But to make it fit, to re-establish equilibrium, perception modifies or adapts it so it does fit.

Learning
Motor-sensory or mental thought comes in -> assimilated into the recognition template, if it is fixed or a new one is made, that is accommodation. Otherwise, the assimilation process continues with the action and the checking. This reminds me on some level of single cell reproduction – hmmm.

Assimilation reduces new experiences to current conceptual structures. This seems to mean that using heuristics is an assimilation process – perhaps recognizing this can highlight or identify current conceptual structures that are limited.

Attribute substitution process is a generic scheme, Kahneman's scheme, where all along the 3 stages assimilation occurs. The recognition template results in assimilation: starting part of the scheme (RT), then the activity produces a result that the person tries to assimilate to its expected result. A perturbation has the person review the assimilation that took place during the RT part that is reviewing the characteristics. This review can result in the modification of the RT or make a new one. That includes the different characteristic(s). This is where a new scheme is made – accommodation.

Later in evening: I had the students work out situations today using manipulatives. I am wondering if I am confounding my data by doing this. I was thinking it may be a way to review their thinking but now I wonder. I am thinking this because of the different types of abstraction.

Abstraction – note that mental operations are the basic elements of reflective abstraction. Piaget's mental operations are abstraction (isolated chunks of information that can be compared to other – pg. 91). There are several kinds: empirical when abstraction is made from sensory motor materials (manipulatives). Reflective abstraction is the person's own mental activity, or logical transformations (changing the relations that characterize a group structure).
In addition there are 3 kinds of reflective abstraction: (1) reflective – level of a coordination of persons operations (activity) that projects or reorganizes on another conceptual level, (2) reflected – person is aware of what has been done, (3) pseudo empirical – only suitable sensory motor materials available. This seems to suggest manipulatives.

Analytic Memo: 5
Date: Jan 13, 2013
Topic: Sequence

I am again concerned with the definition of sequence. Calculus book (Larson, Hotetler, & Edwards) define it as "mathematically sequence is defined as a function whose domain is the set of positive integers. Although sequence is a function, it is common to represent sequences by subscript notation rather than the standard function notation". This is where I played with the mapping and notation to recall that it is an onto for probability. Onto – a function from x to y is onto if its range consists of all of y (p. 11). This means all the elements in the range are matched up with the x's in the domain, or whatever number or value you find in the range. But, looking at the VA SOL 6.17 and 7.2 (Algebra section) they show geometric and arithmetic sequences that focuses on the changes between each output, not focus on the relationship between the domain and the range. Since probability has a function, which I think it is called random, the mapping of each trial to an item in the range lacks a pattern. (I then play with both types of sequences to think more deeply about the relationship for these and probability).

Non-mathematically speaking, sequence is an order of things, where the order in which things are arranged, actions carried out or events happen. Synonyms: succession, string, progression, order, arrangement. How is sequence used later in probability? May need to check this out by looking at mathematical probability – the higher level mathematics associated with it.

Analytic Memo: 6
Date: Jan 14, 2013
Topic: Combinations

When making new teams I thought the students would realize a new team only requires one person to change. But, I found that some thought they all had to be different. This was a surprise and this would definitely impact their decision for the correct option in a multiple choice situation.

I need to check out the participants with the assessment and interview 1 where they make combinations to see if they used blocks and if so, were they the same color or different. I was not aware that this might have an impact. For example, Brittney showed a possible confusion due to this. In Q3A she found the number of combinations correctly using different colored blocks, although she rejected what she found. Then she used all yellow blocks which I need to recheck.

I am also concerned with the wording of the problem – "You have 6 different people, can you make less than, greater than, or an equal number of 2 member teams where the two people are different than 4 member teams where at least 1 of the members is different," or something like that.

I then played with combination formulas, tree diagrams, tables, slots, binomials.

Analytic Memo: 7
Date: Jan 16, 2013
Topic: Ranking Heuristics by Complexity

I am wondering if these heuristics should be ordered by complexity. For example, if the student's notion about randomness is heuristically driven by appearance or by population proportion, this will carry over to more complex problems such as the hospital problem. Maybe this isn't the correct link but I'm trying to indicate that some heuristics may carry over to other heuristics, that is, more than using representativeness in the conjunction (disjunction versus subset type of thing). I think I need to know how these heuristics screw up probability thinking and how it can carry over to other situations.

Hospital seem linked to getting 3H or 4H easier; 100% adjustment all H. Maybe readdress with this situation and use biased coins.

Analytic Memo: 8
Date: Jan 18, 2013
Topic: Diagramming/Thinking about diagrams

I was playing with diagraming the combination problem for finding the number of pairs with 6 cubes. Network works well for pairs because the links connect two ends. The complement is 4 cubes in a group. You can pair again, but look at what is not used, which are the groups of 4.

Then I diagramed a sequence to show the function from the domain to the range is generated by the random function. The domain is the ordered whole numbers and the range is the sample space. Then I looked at the mathematics definition of random online. The function is non-deterministic.

I revisited the mathematical function for expectation and variance, in terms of the random variable. The random variable has 2 important characteristics – location and dispersion. Location is where it will cluster, that is, central tendency. Dispersion is the typical or average range of values, which is the standard deviation, from the central tendency – expectation.

I thought about the uniform distribution and the sampling distribution, which changes the sample space so the sampling distribution becomes more normal in shape with a narrower variance. If people don't know this, would this have them fall back on what they know – how to work with fractions.

Analytic Memo: 9
Date: Jan 23, 2013
Topic: Hypothesizing on flips in thinking

I really want to know – are the students really substituting something in? What is it and why? Can they be perturbed and if so how stable, why would it be unstable?

They seem to flip thinking in cyclic way: S1 – S2 (Correct) – Wrong. I also need to know how permanent or transitory it is. I need to remember the **students are self-reporting** so they may not know what they are thinking to verbalize it or how to verbalize it if they "kinda" know what they are doing. So, I need to think about it, so see if I can shed some light on this.

What is their thinking under the different triggers?

If heuristic what led them to it? It was attribute substitution. What does it look like? It depends on the heuristic (I think). What is the result? Answering a simpler question, so what is removed to make it easier> Substituting in the population characteristics for the sample, or ignoring what is deemed irrelevant (stated by Josh), or failing to notice information is relevant (the definition of sequence – think of it as just a list). They can also want to quit or jumping to conclusions, perhaps with the availability heuristic. Also, their personal definition for significant words in the problem, like "AND" was also promoted by Hertwig. (This is the disjunction/conjunction debate).

Analytic Memo: 10
Date: Jan 23, 2013
Topic: Exploring Heuristics Mathematically – may impact analysis of data

I was pondering the sequence of 6 flips of a coin according to the mathematical definition of sequence. I was also thinking about the distribution when the numbers of heads that appear go from 0 to 6. This is a binomial distribution. I explore the symmetric nature of the distribution because of the probability for each event being .5.

I thought of this with the hospital problem and the wording. I was looking at the distributions based on the small hospital with 10 kids born each day and the large hospital with 100 kids born each day and the proportion that are boys. The students don't see the "tails" of the distribution with the "more than" or "less than". They don't seem to know that the exact probability is 0 because we are transferring a discrete situation into a continuous situation. The probability of a point is 0 (no area under the curve).

Then exploring 2 out of 3 versus 200 out of 300 problem is also binomial but hard to explore since the numbers for the second sample overflows the calculator. But it seems the students ignore the size and believe equivalent fractions = equivalent probabilities.

I looked at the committee problem. They don't see the compliment of the set. This was Piaget's point for probability but it seems like he was looking for a systematic way of finding the number of combinations. So, the students make a decision on partial information because of the level of effort needed to find the number of ways to make the larger committee. There must be a way to perturb their thinking to focus on "who was kicked out" so to speak.

Analytic Memo: 11
Date: Feb 13, 2013
Topic: Preparing Last Interview – not required

I plan to ask each of the four questions again but in a different way.

1. Hospital – I will use coins. I have that sheet made up.

2. Sequence of outcomes – This time I will say it comes from a bag with only 6 chips.

3. Conjunctive effect – I will use the rewording like with Hertwig and see what happens

4. Combinations – I will do the paths from Fischbein and Schnarch, but a smaller number

Analytic Memo: 12
Date: April 3, 2013
Topic: Categories/Themes

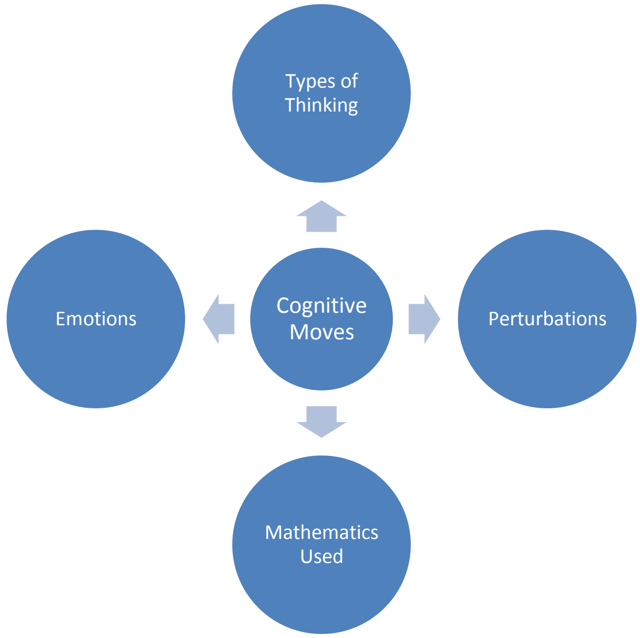
I have not been happy with my data analysis as it seems to lack rigor. I have been struggling with it since I turned in chapter 4 and 5 at the end of February. Recently, Jay indicated that data analysis section in chapter 3 needed more work, not surprised.

My meeting with Steve, Beth, David, and Joyce did not help me as much as I thought it did at first. They focused my attention on details, but I realized my problem was generalizing the results.

Since this meeting I have revisited the data, also anticipating redoing the line coding but realized that I was doing the same thing so I stopped. I then started to consider my categories from the coding: system 1 thinking, system 2 thinking, reasoning answer, heuristic answer, wrong answer, fast response, slow response, internal perturbations, external perturbations, sticking to initial response, changing initial response. I also coded emotional responses that included confident, hesitant, guessing.

<table>
<tr><th colspan="14">Cognitive Moves</th></tr>
<tr><th colspan="6">Response Time</th><th colspan="3">Perturbations</th><th colspan="2">Emotions</th><th colspan="3">Math Used</th></tr>
<tr><td colspan="2">S1 - Fast</td><td colspan="2">S2 - Slow</td><td colspan="2">Inconclusive</td><td colspan="2">Internal</td><td>External</td><td rowspan="3">Confident</td><td rowspan="3">Timid</td><td rowspan="3">Fractions</td><td rowspan="3">Percents</td><td rowspan="3">Ratios</td></tr>
<tr><td rowspan="2">Intuition</td><td rowspan="2">Reason</td><td rowspan="2">Intuition</td><td rowspan="2">Reason</td><td rowspan="2">Intuition</td><td rowspan="2">Reason</td><td colspan="3">Results</td></tr>
<tr><td>Stick to initial</td><td>Change from initial</td><td>Wavers</td></tr>
</table>

I can see the initial coding can create these levels. The theme is cognitive moves, where the initial response is timed. It is fast, slow, or inconclusive. The student is either responding with confidence or timidly when drawing the type of thinking with prior knowledge of mathematics.



This figure shows that the theme is influenced by the students type of thinking, emotional state, and mathematics used in the problem, and the perturbations experienced.

<u>Analytic Memo: 13</u>
Date: April 3, 2013
Topic: Identifying categories/themes – continued

This is an extension from Analytic Memo 12, but with a different focus. It is rewriting notes I jotted down when meeting with Beth and Joyce on April 2.

1. Another chart to show my cognitive moves by heuristic problem. Example, row is problems and columns are cognitive moves, where it is labeled S1 -> S2. In the cell I would place initials of the participants; J, GP, R.

2. I need to show how the categories related to each other. I am thinking I did that on a high level, but it could also be the case on a lower level. Not sure, need to re-evaluate that.

3. After line coding, the result is my cognitive moves, where I describe in general at first. I include transformations of thinking which led to the moves, that is sophistication of thinking. So, what was it that led a person to move from S1 to S2? How more sophisticated did their thinking become from a perturbation (internal or external). I will describe each of these moves by the categories I identify as mentioned in number 1 where I make that chart. I will go to the data to

show how the transformation was made. Glaser and Strauss used to describe to do constant comparison method.

4. I am really doing that so use the articles Beth gave to me.

5. I will not be using scheme theory to describe this stuff – that is, scheme theory will not be used to describe the transformations. I may have to think about this more – am I really looking at transformations outside of perturbations???

<u>Analytic Memo: 14</u>
Date: April 12, 2013
Category: Intuition

Intuition for this study will focus on the characteristics that are tied to attention since triggers a recognition template that initiates a process, or action. I need to identify the characteristics that are associated with this.

It occurred to me today as I was thinking about intuition that I could parse out the initial responses and perturbation using Fischbein' perspectives. *Feeling* includes confidence and degree of obviousness, which may be easier noted in the students than trying to map characteristics to a behavior. That is, I can maybe look at it like this:

Perspectives ->	**Category**		**Origin**		**Knowledge Based**	
	Affirmatory	Anticipatory	Primary	Secondary	Automaticity	Guess
Characteristics | V	Facts seen as true, known; routine way to think	Preliminary - global view of a problem prior to full analysis; conjecture	Out of school	Formal education	Fast with knowledge - motor, abstract concepts, imagination	Fast without knowledge
Self-evident: solution is obvious to student						
Intrinsic certainty: Holds a belief it is true						
Perseverance: The intuition is persistent						
Coerciveness: absolute, alternatives not considered						

Extrapolation: Reaching beyond the information						
Globality: Intuition acts like a filter						
Implicitness: tacit processes, its destruction produces confusion						

I think I will use the terms that align with thinking that I can observe, either through behavior, seen from written work, or heard from verbal utterances. I will go through the list and identify if it will be useful for me and use it in the chart. (I will put this into chapter 4 to show analysis? Or should it be in chapter 2 or 3??) I think I could use this like a check list. Not exactly sure yet, but this does help to structure the discussion on students responses.

My categories may be the top row across and I use the characteristics to distinguish an intuition from reasoning. I will relook at the data and see if any of these show up.

The going back and forth between focused and unfocused I noticed by the students may be the pulse-like attention von Glasersfeld (1981) talked about. Underlying this may be motivation that helps capture attention. Too easy or too hard, a person becomes inattentive, so I may need to get more information on attention viewed through the lens of motivation.

<u>Analytic Memo: 15</u>
Date: April 13, 2013
Category: Struggling - Finding the Relationships Among Categories or Themes

I am struggling with the relationships between the categories that have shown up when I have done the coding:

1. Categories: Based on research questions, determined by timing
<u>Response time</u> (fast, slow, inconclusive – used to identify S1 or S2), <u>Type of Math</u> (not really used), <u>Perturbations</u> (Internal/External), <u>Commitment to Initial Response</u> (Weak/Somewhat weak/ Somewhat strong/ Strong)

2. Theme/sub-theme: (Beth's Office) Theme "cognitive moves" had this before I went to her office.
Perturbations were a sub-theme with the other categories under it.
Categories: <u>Student Response</u> (know/guess; decisive/indecisive), <u>Before Answer Question</u>(Questions/focus), <u>Emotions</u>(Confident/Timid), <u>Prior Knowledge</u> (Math/non-math)

I thought at this time emotions may be the tie into scheme theory.

3. Overarching Theme: Cognitive Moves
Themes: Prior knowledge, perturbations, emotional responses
Categories: (Later that day at home – went too general)
Themes:
Prior Knowledge (in school/out of school), *Perturbations* (internal/external), *Emotional Responses* (Positive/negative)

This seemed too general. I tried it a 4th time and still felt like some of the uniqueness of the study was being left out. I was not getting a different perspective on this. The categories didn't seem to work or work together well.

Now What?
I went back to the literature focusing on Fischbein's discussion on intuition. Jay mentioned that I perhaps should stay focused on intuition and not tied to System 1 and System 2.

I looked at it on the way he described it as *pre-operational*, *operational*, and *post-operational* for the process. But it seems these categories show up all along the 3 stage process. Then I realized (again, like I thought before - that T&K were looking at the recognition template on the scheme, specifically that some information was being assimilated that initiated a process that created a wrong solution.)

I also started to think that my categories may be some aspect of Fischbein's characteristics associated with intuition, which I reiterate below in the chart.

Maybe the problem is the **name** I gave to the category or the themes???

Perspectives ->	**Type of Intuition**		**Origin**		**Knowledge Based**	
	Affirmatory	Anticipatory	Primary	Secondary	Automaticity	Guess
Characteristics | V	Facts seen as true, known; routine way to think	Preliminary - global view of a problem prior to full analysis; conjecture	Out of school	Formal education	Fast with knowledge - motor, abstract concepts, imagination	Fast without knowledge
Self-evident: solution is obvious to student						
Intrinsic certainty: Holds a belief it is true						
Perseverance: The intuition is						

persistent						
Coerciveness: absolute, alternatives not considered						
Extrapolation: Reaching beyond the information						
Globality: Intuition acts like a filter						
Implicitness: tacit processes, its destruction produces confusion						

Fischbein 1997 book:

"Relationship between intuition and intelligence in the ontogenesis of probabilistic thinking"
Translation – Developing probability thinking in a kid where intuition and the ability to learn or reason is related. Yes, he uses the word intelligence to mean reason.
This seems to suggest that intuition comes before reasoning, and intuition can be used to help develop probability reasoning in the kid.
He focuses on the development of the intuitions in relations to the development of operational schemas of the intellect. He sees that intuition is an intermediate step between outward action and inward action (or operation in Piagetian terms) for some intuitions.

Piaget and Inhelder 1951 book:

Focus is on the logical organization of probability and chance concepts. In this work, the interpretive, explanatory and problem solving dispositions that come from intuition is limited and patchy.

Kahneman

His intuition is associative – makes sense since he has these three heuristics aligned with unconscious associations made between the problem and something else – like sample with population characteristics. He also sees it at the level of perception.

Matrix: Codes to Categories

Date: April 16

Analysis: Connecting Codes to Categories

Overarching Theme - Open

Categories:

Emotional Response

Incidents ->									
Non-verbal									
Verbal – tone, words									
Verbal Utterances (not words)									

Mathematics Used/not

Incidents->								
Prior knowledge – math								
Other experiences								

Conflict – internal/external (Later changed to Perturbations – internal and external)

Incidents->								
Hesitation								
Confused								
Not understand question								
Asks Clarification								

Mathematics Knowledge – was demonstrated in pre-assessment in one way

I did not see the value of filling out this chart in since I found out later that I need to mention how all of the participants responded and go into detail for one or two of them. If I were to use only one student, then this would have been more helpful (I think).

Detail in explanation									
Clarity in explanation									
Own diagrams/analogies									
Sophistication in math language									

APPENDIX G

Data Coding

This appendix outlines the data coding process. It includes bracketing before the coding and the several iterations of recoding.

Bracketing

Prior to data collection, time was spent reflecting on ideal middle school participant and the way middle school students were portrayed in the media in order to uncover biased perceptions about middle school students (See Appendix D – Bracketing #1). This was important because it was necessary to be mindful of attitudes and their influences when interpreting data. In this study, some initial notions included the idea that student responses would clearly differentiate between System 1 and System 2 thinking as described by Kahneman and Fredrick (2002). Initially, it appeared that each research questions identified a useful category to interpret the data: type of thinking, perturbations, and stability of their response. The type of thinking category would help to answer the first research question. The perturbations category came from the second research question. The perturbations were ways the research team provoked the students thinking that was done more extensively during the second interview session. It was anticipated that the students would provide the heuristic answer and the research team would provoke their thinking in order to move them from System 1 thinking to System 2 thinking. The third category called stability of their response was pulled from the third research question. This category referred to the students' level of confidence they had with their initial answer. The data analysis produced similar categories and codes that reflected the students' different ways of thinking and how that influenced their problem solving strategies. After the final categories were coded, they were used to create profiles of the students thinking. The profiles were used to communicate the different ways of thinking and how they influenced the students' problem solving strategies.

Category Generation

This section described the results from the repetitive and systematic coding process that were used based on the constant comparison method (Glaser, 1965). The process was repeated five times before the final recoding for three major reasons: (1) to reduce bias, (2) to support technical accuracy and (3) to keep the focus on the research purpose. Repeating the process took several months. During this time numerous analytic memos and procedural memos were written, time was spent revisiting the literature and several discussions took place among the research team, peers, and committee members. The following paragraphs describe the multiple recoding that examined the data from many different perspectives. This included using the second interviews only, to using both interviews. It included alternative ways to initially group the data to generate categories. For example, group the data based on students or based on the four activities from the second interview. The subsequent coding processes replaced the previous coding results.

First Coding. The responses from the second interviews, Perturbation Activities instrument (See Appendix C), were used to generate the first several iterations of coding and recoding. Attention was placed on the second interview because this interview session included the research team perturbing the students thinking. The conversations that developed during the perturbation process allowed the research team to observe the students' thinking in greater depth.

Prior to analyzing the data, the answers the students circled on the multiple choice questions was presumed to be adequate to distinguish heuristic thinking from correct reasoning. This was based on prior research results (Fischbein & Gazit, 1984; Fischbein & Schnarch, 1997;

Kahneman & Tversky, 1972; Shaughnessy, 1988; Tversky & Kahneman, 1971, 1973, 1983). In the revised heuristics and biases framework, heuristic thinking was linked to System 1 and correct reasoning was linked to System 2. The expectation was that intuitive thinking would be demonstrated when the students failed to use or misapplied mathematics when answering the questions. Likewise, the expectation that the students were engaged in reasoning would be exhibited by methodical explanations. These expectations were gleaned from Kahneman and Fredrick's (2002) and Fischbein's (1975, 1989) notions of intuitive thinking and Kahneman and Fredrick's (2002) and Evans (2008) notions about reasoning both within the dual process model. This was not always observed. Many students' response times and their selected responses did not align with Kahneman and Fredrick's (2002) description of these two ways of thinking as indicated above. For example, Brittney responded with a heuristic answer that was accompanied with a slow response time. This was observed with the Making a Team activity (see Appendix C). Brittney was given blocks to demonstrate her thinking when comparing the two combinations. She claimed there were more ways to create different two member teams than four member teams from a group of six people.

Pondering this unexpected result, the mathematics of the activities became a focus for deeper evaluation (Appendix D – EM1.1, EM3.1, EM6.1, EM8A.2, EM12.2; Appendix F – Analytic memo 1 – 3, 5, 6, 8, 9, and 10). These initial findings were discussed with fellow doctoral students after several of the interviews with the students were completed and transcribed. The collaboration helped to clarify the relationship between heuristic thinking and the probability problem structure. The probability problems had different levels of complexity. During this collaboration, the *students'* mathematics was unpacked. Returning to the data, the researcher considered ranking the heuristic activities by complexity (See Appendix F – Analytic

Memo 7). Perhaps, by ranking the probability activities by complexity, it would help to identify deeper aspects of heuristic thinking as it emerged when the problem complexity increased (See Appendix F – Analytic Memo 11). It appeared that the Hospital problem (see Appendix C) was the most complex because of its mathematical structure and its rich context. The mathematical structure of the hospital problem was the same as the structure Two Heads versus Two Hundred Heads problem (see Appendix B). The latter lacked the richer context found with the Hospital problem (See Appendix C). In addition, the Two Heads versus Two Hundred Heads problem (see Appendix B) all of the students responded faster to this problem than to the Hospital problem (see Appendix C). Yet, not all of the multiple choice selections were heuristic with coded fast response times nor were all of the coded slower response times correct. Most of the response times fell within the inconclusive range, which was unexpected. Two students, Brittney and Josh, were coded with fast response times and they selected the heuristic answer. On the other hand, one student, Ryan, answered correctly, but he responded within his inclusive range. The remaining three students, GP, Elise, and Ben, responded with a heuristic answer and a response times that fell within their inconclusive range. The response times failed to support intuitive thinking as suggested by Kahneman and Fredrick (2002), based on dual process theory. Dual process theory was based on neuroscience research that measured brain activity in sub-second units. Nonetheless, it appeared that some of the students were engaged in some type of substitution process because the students selected the heuristic answer.

The literature was revisited to explore intuitive thinking in greater depth. Most of the extant literature pertaining to heuristics and biases examined intuitive thought through the lens of making decisions (Tversky & Kahneman 1971, 1973, 1974, 1982a, 1982b, 1983) or through the lens of dual process theory (Evans, 2011, 2008; Schneider & Chen, 2003; Schneider & Shiffrin, 1977; Smith & DeCoster, 2000; Stanovich & West, 2000). The literature addressing heuristics and biases from an educational psychological perspective was revisited (Fischbein, 1987; Fischbein, 1975; Fischbein & Gazit, 1984; Fischbein & Grossman, 1997; Fischbein & Schnarch, 1997). Revisiting the literature to expand on the nature of intuitive thinking helped to support the purpose of this study, which was also added in chapter 2 of this document. The purpose of this study was to document how System 1 or System 2 influenced the eighth grade students' problem solving strategies while engaged in probability activities. The implicit nature of intuitive thinking (Fischbein, 1975; Kahneman & Fredrick, 2002) and as additional effort required to translate mathematical symbolic thinking into a verbal format (Ericsson & Simon, 1980; Fox, Ericsson, & Bets, 2010) made it more difficult to observe specific attributes being substituted.

The categories that emerged from the first round of coding included: *type of thinking, conflicts,* and *strength of conviction* that referred to the students' confidence with their initial response.

Second Coding. During the second round of recoding, a conversation with a colleague focused on the constant comparison method process. There was concern that the categories may have been influenced by the research questions instead of reflecting the data. This time, both of the interview transcripts for the six students were used to create the categories. The results of the coding process identified four categories. The first two categories identified were the same from the first coding - type of thinking and conflicts. The two new categories were - *emotions* and *type of mathematics used*.

The *emotions* category included verbal and non-verbal responses. For example, when GP was explaining her thinking for the Making a Committee problem (see Appendix B), she explained how she would compare the number of ways for each size of committee. She ended her explanation saying “If the makes sense” and then started to laugh. I interpreted her verbal laugh as an indicator of an internal emotional response of nervousness that seemed connected to her uncertainty with her answer. The emotions category provided behaviors that could be observed and linked to System 1 or System 2.

The *mathematics used* category included the types of mathematics the students used when answering a question. For example, Elise used equivalent fraction knowledge when responding to Two Heads versus Two Hundred Heads activity (see Appendix B). This category included situations when mathematics was not used. These were most evident with the Description of Marcia problem (see Appendix B) and with the Description of an Individual problem (see Appendix C). Both of these problems addressed the Conjunctive Fallacy heuristic and many students failed to draw on probability knowledge for this problem. They ignored the multiplication rule when combining two independent events. This category identified behaviors that could be used to describe problem solving strategies.

In the end, this iteration identified the following categories: *type of thinking used*, *type of mathematics used*, *conflicts* instead of perturbations, and *emotions.* The *conflicts* label identified the internal conflicts the student experienced while engaged in the problem that was observed when they change their minds, made false starts, or made verbal utterances such as “hum”. The *emotions* category identified outwardly displayed behaviors: frustration, confusion, or surprise. This study focused on the impact that System 1 or System 2 had on the students’ problem solving strategies and the next category supported the problem solving strategy component. The category, *type of mathematics used,* (fractions, percentages, ratios, etc.) revealed the students problem solving process. This category appeared promising in order to identify an attribute the students were substituting. During this iteration, the light was placed on the *conflicts* category since so many internal perturbations were witnessed and unexpected.

While contemplating the *conflicts* and *emotion* categories together, initial thoughts were to remove the *emotions* category. However, this was short lived because this category included negative and positive emotions. This range of emotions would further help to identify the System that was influencing the students' problem solving strategy. For example, Ben was internally perturbed when he grimaced and squinted, but he was not perturbed or upset in the common use of the word. He even said, "This is an interesting problem." Internal perturbations prompted the students to ask clarifying questions that signaled they were struggling to make sense of the problem. This was interpreted as an emotional state of confusion. Yet, there was a common element with internal conflicts and emotions; they were both internal to the student. This category was recoded using a previous category name, perturbations. The *perturbation* category was subdivided into *internal* and *external.* Negative emotions and internal conflicts aligned more closely with the broader term, *internal perturbations* because the word *perturbation* signified a lack of equilibrium within the student. Recall that the *external perturbations* were designed into the second interview activities, which were questions or comments made by the researcher. At the end of the second recoding process, the following categories were identified and replaced the names from the prior coding process: *emotions*, *perturbations,* and *type of mathematics used.*

Third Coding and Fourth Coding. The third and fourth iterations were essentially redoing what was done during the second iteration with a different focus to determine if one approach proved superior to the other. The first perspective that was used for the third iteration used the data from the Perturbation Activity Instrument (see Appendix C) used in the second interview. The different colored sticky notes were coordinated to each of the four problems used in the instrument. In the end, the categories remained unchanged from the previous coding. The second approach that was used in the fourth iteration used both interview transcripts but this time the students were differentiated by the color of the sticky notes (See Appendix D – Analytic Memo 12 and 13). The goal of this scheme was to identify one or two students who could serve as a case study. The categories remained unchanged from the coding process. In addition, the richness of the codes was based on all of the students. In the end, using one or two students proved unsuitable for a case study. Therefore, all of the interviews for all six students were used in the analysis for the remaining coding iterations.

Fifth Coding. The fifth time the data was recoded, the category grouping process was extended to combine the categories, the same ones found in the fourth recoding iteration, into more abstract categories. The categories included *perturbations*, *emotions*, and *mathematics used.* These three categories were grouped under the umbrella of an overarching theme called *Cognitive Moves.* It appeared this theme could possibly show the trajectory of the students thinking process over time. Unfortunately, the theme seemed too broad and it appeared it would redirect my thinking away from my research purpose. After sharing these thoughts with a colleague, she suggested that I map the categories to isolate relationships between categories. Accepting the suggesting, the mapping was done on a large white board and repeated on a large sheet of paper. Each time the categories were moved around several times. This proved unsuccessful since all of the categories were linked together. That is, the *emotions* and *perturbations* helped to identify the thinking process that influenced the students' problem solving strategy. The *mathematics used* category helped to identify the problem solving strategy the students used. In the end, this approach was abandon. The linkage between these categories would be discussed in the results section or the conclusion section.

While recoding and pondering the research purpose, thoughts about the many times the students were unaware of their thinking or they were unable to describe their actions came to mind. The literature was revisited to investigate *intuition* again (See Appendix F – Analytic Memo 15) with a focus on Fischbein (1975, 1987), Kahneman (2003), Kahneman and Fredrick (2002), and Beth and Piaget (1996). At this time, attention was placed on the triggers that initiated heuristic thinking that may have differed from Kahneman and Fredrick's (2002) descriptions. Recall that Kahneman and Fredrick (2002) believed the initial and fast associations' people made triggered intuitive (heuristic) thinking at the beginning of the problem solving process. Fischbein (1987) believed intuition was an idea that could be triggered by internal notions any time during the problem solving process. Fischbein's (1975, 1987) perspective offered an alternative way to interpret the ways that students intuitively navigated through the problem solving process. However, like Kahneman and Fredrick (2002), the intuition was triggered by an association. For example, Josh was explaining his heuristic response for the Marbles and Boxes problem (see Appendix B). Suddenly he said, "Ah, I figured it out." At this point he changed his answer to the correct response. This was an example of an intuitive thought coming to Josh instantaneously and the answer was self-evident to him and emotionally he displayed behaviors that indicated he was confident. In this case the trigger was something in the problem that signaled a change in direction for his thinking.

Categories

There were four levels of codes that were created from the data coding process that moved from concrete terms to more abstract terms. They were: line codes, key words, sub-categories, and categories. The line codes described observed behaviors in concrete words, terms, or short phrases. The key terms collected the line codes into broader clusters. The sub-categories gathered the key terms into wider and more abstract groups. The categories assembled the sub-categories into larger and more abstract groups. In the end, there were three categories: 1) emotions, 2) perturbations, and 3) mathematics used. The *emotions* category included verbal and non-verbal behaviors that were observed by the research team. The *perturbation* category included two sub-categories, internal and external. The internal perturbations were behaviors that signaled the student was changing their way of thinking unprovoked by the research team. The external perturbations were the attempts made by the research team to provoke the students thinking. The last category was *mathematics used*, which included the sub-categories *types of mathematics used* and *mathematics not used.* These were observed by the students' verbal response and their written work. The final coding results are shown in Table 4.7.

Appendix H

IRB Approval Letter

VirginiaTech

Office of Research Compliance
Institutational Review Board
2000 Kraft Drive, Suite 2000 (0497)
Blacksburg, VA 24060
540/231-4606 Fax 540/231-0959
email irb@vt.edu
website http://www.irb.vt.edu

MEMORANDUM

DATE: November 14, 2012

TO: Jean May Mistele, Jesse L Wilkins

FROM: Virginia Tech Institutional Review Board (FWA00000572, expires May 31, 2014)

PROTOCOL TITLE: Exploring And Perturbing Heuristic Thinking Governed By The Attribute Substitution Process

IRB NUMBER: **12-894**

Effective November 14, 2012, the Virginia Tech Institution Review Board (IRB) Chair, David M Moore, approved the New Application request for the above-mentioned research protocol.

This approval provides permission to begin the human subject activities outlined in the IRB-approved protocol and supporting documents.

Plans to deviate from the approved protocol and/or supporting documents must be submitted to the IRB as an amendment request and approved by the IRB prior to the implementation of any changes, regardless of how minor, except where necessary to eliminate apparent immediate hazards to the subjects. Report within 5 business days to the IRB any injuries or other unanticipated or adverse events involving risks or harms to human research subjects or others.

All investigators (listed above) are required to comply with the researcher requirements outlined at:

http://www.irb.vt.edu/pages/responsibilities.htm

(Please review responsibilities before the commencement of your research.)

PROTOCOL INFORMATION:

Approved As:	**Expedited, under 45 CFR 46.110 category(ies) 6,7**
Protocol Approval Date:	**November 14, 2012**
Protocol Expiration Date:	**November 13, 2013**
Continuing Review Due Date*:	**October 30, 2013**

*Date a Continuing Review application is due to the IRB office if human subject activities covered under this protocol, including data analysis, are to continue beyond the Protocol Expiration Date.

FEDERALLY FUNDED RESEARCH REQUIREMENTS:

Per federal regulations, 45 CFR 46.103(f), the IRB is required to compare all federally funded grant proposals/work statements to the IRB protocol(s) which cover the human research activities included in the proposal / work statement before funds are released. Note that this requirement does not apply to Exempt and Interim IRB protocols, or grants for which VT is not the primary awardee.

The table on the following page indicates whether grant proposals are related to this IRB protocol, and which of the listed proposals, if any, have been compared to this IRB protocol, if required.

Invent the Future

VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY
An equal opportunity, affirmative action institution

Appendix I

Letter to Parent/Guardian

Dear Parent,

My name is Jean Mistele and I am a student at Virginia Tech in the Mathematics Education PhD program offered through the School of Education under the advisement of Dr. Jay Wilkins, wilkins@vt.edu. I would like to know if you are willing to have your child participate in my dissertation study that explores the thinking processes of 8^{th} grade students when they are engaged in probability activities. I am attaching the parent consent form for you to read. I also have a form for your child but I will read it before we begin the interview and then ask your child to sign the form, if you agree to your child's participation.

I would like to begin the interviews as soon as possible in school during the advising time. The school based interviews will last approximately 20 minutes each. We would plan to meet with your child approximately 6 times. However, I am also willing to come to your home or meet at another location such as a public library if that is better for you and your family after school hours. If this is your choice, the interviews would last approximately 60 minutes each and we would plan to meet with your child approximately 3 times.

Each of the interview sessions will be videotaped so we can learn how your child thinks and understands when engaged in probability activities. Please let me know what will work best for you and your family.

Please do not hesitate to call me or email me with any questions or concerns you may have.

Thank you very much for your consideration of my request.

Jean Mistele

jeanm@vt.edu
(540) 745 – 6771 (home)
(540) 239 – 4615 (cell)
(540) 831- 6273 (office)

Appendix J

Parent Consent Form

VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY

Parental Permission Form

Title of Project: Exploring And Perturbing Heuristic Thinking Governed By The Attribute Substitution Process

Investigators:

Jesse L. M. Wilkins
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I. Purpose of the Project:

We are asking your permission for your child to participate in our study. The purpose of the study is to explore how 8th grade students reason when engaged in solving or addressing probability problems.

II. Procedures

We will video tape your son/daughter with an audio-video camera during each of the interview sessions. We will visit your child in school during the advising time or after school so as not to interrupt instructional time for your child. If we meet with your child after school, we could come to your home or to a public location such as the public library, whichever is the best for you and your son/daughter to administer the interviews. You are not required to be present at the interviews but you are more than welcome to be present and observe. We will visit your child at least 2 times, where each session lasts no longer than 60 minutes. If the interview sessions occur during the advising time during school, where each interview session lasts 20 mintues, we expect to meet with your child approximately 6 times. However, if the interview sessions are held outside of school, the interview sessions will last no more than 60 minutes. We plan to meet with your child approximately 2 times if we use the longer sessions. Your child's total amount of participation is anticipated to be about two hours. Each session will include probability activities that may require your child to use various manipulatives such as colored counters, spinners, colored marbles, or dice. Your child will describe his or her thinking and reasoning during each of the activities.

III. Risks

There are no risks in participating in this research beyond those experienced in everyday life.

IV. Benefits

We hope that by working with your child, we can help other teachers and researchers know how 8th grade students think when solving probability problems in order to help improve teaching practices.

V. Extent of Anonymity and Confidentiality

We will not use your child's name in our transcripts, or any other related documents used in the analysis. In addition, future dissemination opportunities such as journal articles, or conference presentations will not reveal your child's identity. Only the research team listed above will have access to the recordings. However, it is possible that the Institutional Review Board (IRB) may view this study's collected data for auditing purposes. The IRB is responsible for the oversight of the protection of human subjects involved in research. In some situations, it may be necessary for an investigator to break confidentiality if child abuse is known or strongly suspected. As investigators, we are also required to notify the appropriate authorities if a subject is believed to be a threat to him or herself, or others. We will destroy all of the audio-video data at the end of 2023.

VI. Compensation

Neither you nor your child will receive any known direct or indirect benefits from being in the study. No promises or guarantees of personal benefits will be made to you or your child to encourage you or your child to participate.

VII. Freedom to Withdraw

You are free to have your child drop out of the study at any time without penalty.Your child can stop participating at any time or skip any activity or session. Your decision to allow your child to participate will not impact you or your child in any way.

You may ask me any questions you may have concerning this research by contacting me, Jean Mistele, PhD student, either in person, by email (jeanm@vt.edu), or by phone (540 239-4615 – cell; 540 745-6771 - home) at any time or my advisor, Dr. Jay Wilkins who may be reached by email, wilkins@vt.edu, or by his office phone at Virginia Tech, 231-8326.

VIII. Subject's Responsibilities

As a volunteer, agreeing to allow your child to participate in this study, you have the following responsibilities: to have your child meet with the researcher at the agreed upon time and location. If changes in time or location are required, you will notify Jean Mistele as soon as possible via email (jeanm@vt.edu), or by phone (540 239-4615 – cell; 540 745-6771 – home). However, if you agree to allow your child to participate in this study during school hours there are no additional responsibilities for you.

IX. Subject's Permission

I have read the Consent Form and conditions of this project. I have had all my questions answered. I hereby acknowledge the above and give my voluntary consent:

☐ Check this box if you would like to receive a summary of the results

Keep this form for your records and future reference.

Parent’s Permission

I have read the Consent Form and condtions of this study. I have had all my questions answered. I hereby acknowledge the above and give my voluntary permission to allow my child to participate in this study by signing my name and indicating the date below.

____________________________________ ____________________

Parent Signature Date

____

Parent’s Printed Name

____

Child’s Printed Name

____________________________________ ________________

Person Obtaining Consent Date

Appendix K

Child Assent Form

CHILD ASSENT FORM FOR SOCIAL BEHAVIORAL RESEARCH
Virginia Tech

Title of Project: Exploring and Perturbing the Attribute Substitution Heuristic

Principle Investigator: Jesse L. M. Wilkins
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Co-Investigators:

Jean M. Mistele
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We are looking for approximately 12 eighth-grade students to be part of a study about how students think when they solve probability problems. You are invited to be one of those students. If you decide to join, you will work with me and at times with another researcher solving probability problems. You can use colored blocks, colored chips where some are marked on them, dice, coins, or paper to draw diagrams to help you answer the questions. We will videotape your responses so that we can see how you solved the problems. We would meet no longer than 60 minutes either before school, during your study hall, after school, at your home, at another public location like the local library, whichever location is best for you and your family. We will meet with you approximately six times.

If you become frustrated by any of the problems, you can choose not to answer. You can choose to join the study or not. Please discuss the idea with your parents, and feel free to ask us any questions. Then, you can decide. If you decide to join, you can change your mind at any time to stop participating in the study.

You may ask me any questions you may have about this study. You can email me, Jean Mistele, (jeanm@vt.edu), or call me (540 239-4615 – cell; 540 745-6771 - home) at any time. You can also call my advisor, Dr. Jay Wilkins at Virginia Tech if you would like. His email is wilkins@vt.edu, and his office phone number is 231-8326.

I have read the Consent Form and conditions of the project. I have had all my questions answered. I hereby acknowledge the above and give my voluntary consent:

____________________________________ ____________________

Student’s Signature Date

____

Student’s Printed Name

____________________________________ ____________________

Person Obtaining Consent Date

Keep this form for your records and future reference.