Frequency–wavenumber array processing

Melvin J. Hinich[a]

Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061
(Received 21 August 1980; accepted for publication 14 November 1980)

Most array signal processing systems use delay-and-sum beamforming to estimate source bearings. This paper demonstrates the close relationship between beamforming and frequency–wavenumber spectrum analysis. The latter approach has computational advantages over beamforming when the noise is spatially correlated. The wavenumber approach is used to derive the array response of a general linear or planar array to plane wave signals. The statistical properties of the maximum-likelihood estimators of source bearing and amplitude are presented for an array with many elements. Optimal array design is also discussed.

PACS numbers: 43.60.Gk, 43.30.Vh

INTRODUCTION

Suppose that an array of sensors is *receiving* coherent radiation from a distant source. Most array signal processing systems use delay-and-sum beamforming to estimate the source bearing.[1] For sonar arrays, accurate bearing estimation of single or multiple targets is the main design goal; array design and signal processing are subproblems.

There is a close relationship between beamforming and frequency–wavenumber Fourier analysis, which has computational advantages over beamforming when the noise is spatially correlated. This relationship is described in the first part of this paper. Then I show how to use the wavenumber approach to easily calculate the array response for any linear or planar array geometry, and to mitigate the jamming effects of a coherent interfering signal. The final section deals with bearing estimation of a broadband wave when the noise is spatially correlated.

I. WAVENUMBER SPECTRUM AND BEAMFORMING

Let us begin with the simplest model: A linear array of M sensors and a single frequency plane wave signal in complex variable form. Let θ_0 denote the wave's direction of arrival *with respect to the array axis*, let c denote the wave's velocity, and $A=|A|\exp(i\phi)$ is its amplitude. The signal at the kth sensor when there is *no noise* is

$$s(t,x_k)=A\exp[i\omega_0(t-x_k\cos\theta_0/c)]\,, \tag{1}$$

where x_k is the location of the kth sensor $(x_1<x_2<\cdots<x_M)$. The signal in a beam pointed at angle θ (and $-\theta$) is

$$B(t,\theta)=\sum_{k=1}^{M} s(t+\tau_k,x_k)\,, \tag{2}$$

where the kth delay is $\tau_k=x_k\cos\theta/c$. Since a linear array cannot identify between θ_0 and $-\theta_0$, let us arbitrarily assume that $\theta_0>0$.

Since the wavenumber component on the array axis is $\kappa_0=(\omega_0/c)\cos\theta_0$, it follows from (1) and (2) that for

$\kappa=(\omega_0/c)\cos\theta$

$$B(t,\theta)=A\exp(i\omega_0 t)\sum_{k=1}^{M}\exp[i(\kappa-\kappa_0)x_k]=\sum_{k=1}^{M} s(t,x_k)\exp(i\kappa x_k)\,. \tag{3}$$

In other words, beamforming is the same as computing the spatial Fourier transform of the M signals from the array. The spatial frequency $\kappa=(\omega_0/c)\cos\theta$ corresponds to the look angle θ.

In actual practice, a beam is computed from a finite record of the M channels, the beam output is filtered in a narrowband about ω_0, and the filtered signal is squared and summed to give the average energy in the beam for the data set. In frequency–wavenumber analysis, each channel is filtered, and then the spatial Fourier transform is computed. If the received signal is a wave, the square of the magnitude of this transform has a peak of height $(M|A|)^2$ when $\kappa=\kappa_0=(\omega_0/c)\cos\theta_0$.

Now consider a planar array of M sensors, and a plane wave

$$s(t,x_k,y_k)=A\exp\left[i\omega_0\left(t-\frac{x_k\cos\theta_0+y_k\sin\theta_0}{c}\right)\right], \tag{4}$$

where θ_0 is wave's direction with respect to the x axis, and the kth sensor is located at (x_k,y_k). The signal in a beam pointed at angle θ is

$$B(t,\theta)=\sum_{k=1}^{M} s\left(t+\frac{x_k\cos\theta+y_k\sin\theta}{c}\right). \tag{5}$$

Defining $\kappa_x=(\omega_0/c)\cos\theta$ and $\kappa_y=(\omega_0/c)\sin\theta$, it follows from (4) and (5) that

$$B(t,\theta)=\sum_{k=1}^{M} s(t,x_k,y_k)\exp[i(\kappa_x x_k+\kappa_y y_k)]\,, \tag{6}$$

and $|B(t,\theta)|^2$ has a peak of height $(M|A|)^2$ when $\kappa_x=(\omega_0/c)\cos\theta_0$ and $\kappa_y=(\omega_0/c)\sin\theta_0$. Thus $B(t,\theta)$ is the two-dimensional spatial Fourier transform of the data. In terms of the wavenumber components, $\theta=\tan^{-1}\kappa_y/\kappa_x$.

II. ARRAY RESPONSE FOR RECEIVED WAVES

Let us use the wavenumber approach to compute the response of a given array, starting with a linear geom-

[a] This work was supported by the Office of Naval Research (Statistics and Probability Program) under contract.

 0001-4966/81/030732-06$00.80

etry. To simplify the discrete Fourier transforms, select the origin of the array axis so that $x_1 = 0$. Let us make the modest assumption that there exists a distance d such that $x_k = n_k d$, where n_k is a positive integer for each $k = 2, \ldots, M$. This will hold if the x_k are rational numbers.

The question of array aliasing (grating lobes) now arises.[2] Let D denote the greatest common divisor of the integers $\{n_2, \ldots, n_M\}$. From Hinich and Weber,[3] the discrete Fourier transform DFT (3) is periodic with period $2\pi/Dd$. Its principle domain is either $-\pi/Dd \leq \kappa \leq \pi/Dd$ or $0 \leq \kappa \leq 2\pi/Dd$, depending on the convention used. This means that a real wave is not aliased if its wavelength $\lambda_0 = 2\pi c/\omega_0 \geq 2Dd$. For example, if $x_1 = 0$, $x_2 = 11d$, $x_3 = 19d$, and $x = 30d$, the wave is not aliased if $\lambda_0 > 2d$. Assume that the n_k are such that $D = 1$.

Let $s(x)$ denote the filtered plane wave $A \exp[-i(2\pi/\lambda_0)x \times \cos\theta_0]$ at a point x on the axis. Define the counting sequence

$$r(j) = 1, \quad \text{if there is a sensor at } jd, \qquad = 0, \quad \text{otherwise}. \tag{7}$$

From (3), the filtered transform can be written

$$B(\theta) = \sum_{j=0}^{N-1} r(j) s(jd) \exp(i\kappa jd), \tag{8}$$

where $N \geq n_M$. This transform can be computed for the grid $\{\kappa_l = 2\pi l/Nd\colon\ l = 0, 1, \ldots, N-1\}$ using the FFT algorithm. Defining L to be the integer closest to Nd/λ_0, the associated *look angle* grid is $\{\theta_l = \cos^{-1}(\lambda_0 l/Nd)$ if $0 \leq l \leq L$, and $\theta_l = \cos^{-1}[\lambda_0(l-N)/Nd]$ if $N - L < l < N-1\}$. For example $\theta_0 = \pi/2$ and $\theta_L = \cos^{-1}(1) = 0$ if Nd is divisible by λ_0.

These arccosine equations give a mapping between the κ_l and a set of N look angles in the interval $[0, \pi]$ that can accomodate *any* $\lambda_0 \geq 2d$. For example if $\lambda_0 = 2d$, then $L = N/2$ (N even) or $L = (N-1)/2$ (N odd).

Since the right-hand side of (8) is

$$N^{-1} \sum_{m=0}^{N-1} R(l-m) S(m)$$

where

$$R(l) = \sum_{j=0}^{N-1} r(j) \exp\left(\frac{i2\pi jl}{N}\right) \tag{9a}$$

and

$$S(m) = \sum_{j=0}^{N-1} s(jd) \exp\left(\frac{i2\pi jm}{N}\right), \tag{9b}$$

then

$$B(\theta_l) = \frac{1}{N} \sum_{m=0}^{N-1} R(l-m) S(m). \tag{10}$$

The response at look angle θ_l is defined to be $B(\theta_l)$ for the wave signal. The beam power pattern is the sequence $\{|B(\theta_l)|^2\}$.

For example, suppose there is an integer l_0 such that $\theta_0 = \cos^{-1}(\lambda_0 l_0/Nd)$ or $\theta_0 = \cos^{-1}[\lambda_0(l - N/Nd)]$. Then

$$S(m) = \sum_{j=0}^{N-1} A \exp\left(-i2\pi \frac{jl_0}{N}\right) \exp i2\pi \frac{jm}{N}, \quad = NA, \quad \text{if } m = l_0, \quad = 0, \quad \text{if } m \neq l_0. \tag{11}$$

Normalizing by setting $A = 1$, it follows from (9), (10), and (11) that the response is

$$B(\theta_l) = \sum_{j=0}^{N-1} r(j) \exp\left(i2\pi \frac{j(l-l_0)}{N}\right). \tag{12}$$

The peak response is

$$M = \sum_{j=0}^{N-1} r(j) \text{ at } l = l_0,$$

which is clearly independent of the spacing of the M sensors.

If there does not exist such a l_0 for a given N, then the peak response is smeared over several adjacent wavenumbers. This smearing can be eliminated or reduced by extending the $r(j)$ sequence with zeros and thus increasing N. If N is sufficiently large, there exists l_0 such that $|2\pi l_0/N - \kappa_0|$ is either zero or very small.

Let us now turn to the response of a planar array. Analogous to the linear geometry, assume that $x_k = n_k d$ and $y_k = m_k d$ where n_k and m_k are positive integers for $k = 2, \ldots, M$, and $x_1 = y_1 = 0$.

Let $s(x, y)$ denote the filtered plane wave signal at location (x, y). From (6), the filtered two-dimensional transform is given by

$$B(\theta) = \sum_{k=1}^{M} s(n_k d, m_k d) \exp[i(\kappa_x n_k d + \kappa_y m_k d)], \quad = \sum_{j=0}^{N-1} \sum_{j'=0}^{N-1} r(j, j') s(jd, j'd) \exp[i(\kappa_x jd + \kappa_y j'd)], \tag{13}$$

where $N \geq n_M, m_M$ and

$$r(j, j') = 1, \quad \text{if } j = n_k \text{ and } j' = m_k \text{ for some } k, \qquad = 0, \quad \text{otherwise}.$$

Using the wavenumber grid $\{\kappa_{xl} = 2\pi l/Nd, \kappa_{yk} = 2\pi k/Nd;\ k, l = 0, 1, \ldots, N-1\}$, it follows from (13) that

$$B(\theta_{l,k}) = \frac{1}{N^2} \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} R(l-m, k-n) S(m, n), \tag{14}$$

where

$$R(l, k) = \sum_{j=0}^{N-1} \sum_{j'=0}^{N-1} r(j, j') \exp\left(\frac{i2\pi(jl + j'k)}{N}\right) \tag{15a}$$

and

$$S(m, n) = \sum_{j=0}^{N-1} \sum_{j'=0}^{N-1} s(jd, j'd) \exp\left(\frac{i2\pi(jm + j'n)}{N}\right). \tag{15b}$$

As long as $k^2 + l^2 = L^2$, the look angle is as follows:

$$\begin{aligned} \theta_{l,k} &= \tan^{-1} k/l, \quad \text{if } 0 \leq l \leq L, 0 \leq k \leq L, \\ &= \tan^{-1} k/(l-N), \quad \text{if } N-L < l \leq N-1, 0 \leq k \leq L, \\ &= \tan^{-1}(k-N)/l, \quad \text{if } 0 \leq l \leq L, N-L < k < N-1, \\ &= \tan^{-1}(k-N)/(l-N), \quad \text{if } N-L \quad l < N-1, N-L < k \leq N-1. \end{aligned} \tag{16}$$

These equations provide a mapping between the κ_{xl} and κ_{yk} and a set of N^2 look angles in the interval $(0, 2\pi)$ that can accomodate any wave with $\lambda_0 \geq 2d$.

For example, suppose that the signal is $\exp[-i(2\pi/\lambda_0)(x\cos\theta_0 + y\sin\theta_0)]$, where $\lambda_0^{-1}\cos\theta_0 = l_0/Nd$ and $\lambda_0^{-1}\sin\theta_0 = k_0/Nd$ for some (l_0, k_0), i.e., $l_0 = L\cos\theta_0$ and $k_0 = L\sin\theta_0$. Then from (15b), $S(m,n) = N^2$ for $m = l_0, n = k_0$ and $S(m,n) = 0$, otherwise. From (14), the response to this wave for look angle $\theta_{l,k}$ is

$$B(\theta_{l,k}) = R(l - l_0, k - k_0) . \qquad (17)$$

III. ARRAY DESIGN

If there is no *a priori* knowledge of the bearings of sources of interest, it will now be shown that the optimal geometry for a fixed aperture linear array with a fixed number of sensors is two subarrays, one on each end of the axis.[4] Each subarray has its sensors equally spaced. By optimal, I mean the design conforming to the constraints on aperture and sensors that gives the maximum accuracy for the estimate of a source bearing when noise is present. As is shown in Hinich and Shaman,[5] the accuracy in terms of mean squared error depends on the sharpness (the concavity) of $|R(\kappa)|^2$ at $\kappa = 0$, where

$$R(\kappa) = \sum_{j=0}^{N-1} r(j)\exp(i\kappa j d) .$$

Thus we want to select the $r(j)$'s to maximize

$$C(0) = (d^2/d\kappa^2)\,|R(0)|^2 . \qquad (18)$$

It is easy to show that

$$d^{-2}C(0) = \sum_{j=0}^{N-1} r(j) \sum_{j=0}^{N-1} j^2 r(j) - \left(\sum_{j=0}^{N-1} j r(j)\right)^2 ,$$

$$= M \sum_{j=0}^{N-1}\left(j - M^{-1}\sum_{j=0}^{N-1} j r(j)\right)^2 r(j) . \qquad (19)$$

Simplifying matters by making M even, it follows from (19) that $C(0)$ is maximized by setting $r(j) = 1$ for $j = 0, \ldots, M/2 - 1, \ldots, N-1$ and $r(j) = 0$, otherwise. This is the two subarray geometry mentioned above. From (12), its beam pattern for a wave with direction θ_0 is

$$|B(\theta_l)|^2 = |R(l - l_0)|^2$$
$$= 4\cos^2\left[\pi\left(1 - \frac{M}{2N}\right)(l - l_0)\right]\frac{\sin^2[\pi M(l - l_0)/2N]}{\sin^2[\pi(l - l_0)/N]} . \qquad (20)$$

If $N \gg M$, the peak to sidelobe ratio at $\kappa \simeq \kappa_0 \pm 2\pi/Md$ is approximately $(2/\pi)^2 = -3.9$ dB. At $\kappa \simeq \kappa_0 \pm 6\pi/Md$, the ratio is approximately $(2/3\pi)^2 = -13.5$ dB.

By appropriately tapering the array (weighting the channels), the peak to first sidelobe ratio can be reduced, but at the expense of the height and width of the main lobe. This means that a tapered array gives a less accurate estimate of θ_0 when noise is present than does the untapered array discussed above. This array gives the best resolution and accuracy for weak sources when M and N are large.

These results are easily extended to planar arrays. If we want the array's response to be the same for all look angles, then the geometry must be circular. The analogy to the two subarray design is a disk where the sensors are placed around its outer edge. In general, we want to select the $r(j,j')$ to maximize the concavity of the main lobe. Using similar calculations as were used to derive (19), we want to maximize

$$\sum_{j=0}^{N-1}\sum_{j'=0}^{N-1}(j - \bar{j})^2 r(j,j') \sum_{j=0}^{N-1}\sum_{j'=0}^{N-1}(j' - \bar{j}')^2 r(j,j') - (\overline{jj'})^2 , \qquad (21)$$

where

$$\bar{j} = M^{-1}\sum_{j=0}^{N-1}\sum_{j'=0}^{N-1} j r(j,j') , \qquad (22a)$$

$$\bar{j}' = M^{-1}\sum_{j=0}^{N-1}\sum_{j'=0}^{N-1} j' r(j,j') , \qquad (22b)$$

and

$$\overline{jj'} = M^{-1}\sum_{j=0}^{N-1}\sum_{j'=0}^{N-1} jj' r(j,j') . \qquad (22c)$$

In the next section, we deal with estimating source bearings when the received signal is a sum of coherent waves plus noise.

IV. SIGNAL PLUS NOISE

The major advantage of the frequency–wavenumber approach is the ease with which it handles spatially correlated noise. Spatially correlated noise makes the signal-to-noise ratio (SNR) direction dependent. Complicated calculations are needed to achieve optimal bearing estimation using time delay methods in this case.[6] The calculations are relatively easy for the ω–κ approach.

Once again it is easier to explain the method for a linear array and a single frequency wave. Suppose that the signal at x_k is a plane wave plus stationary, zero-mean Gaussian noise denoted $\epsilon(t, x_k)$. Filtering in a narrowband about ω_0, the signal is

$$s(x_k) = A\exp[-i(2\pi/\lambda_0)x_k\cos\theta_0] + \epsilon(x_k) ,$$
$$= A\exp(-i\kappa_0 x_k) + \epsilon(x_k) , \qquad (23)$$

where $\epsilon(x_k)$ is the filtered noise. If the noise is spatially correlated, then the wavenumber spectrum of the noise is not flat. Let us go into this in some depth.

In most applications the signals are transient, and it is then easy to observe the noise alone at each sensor prior to the onset of the signals. From (8), (9), and (10), the spatial Fourier transform of the noise along is

$$B(\theta_l) = N^{-1}\sum_{m=0}^{N-1} R(l - m)U(m) ,$$

where

$$U(m) = \sum_{j=0}^{N-1}\epsilon(jd)\exp\left(\frac{i2\pi jm}{N}\right) . \qquad (24)$$

When N is large, $\{U(0), \ldots, U(N-1)\}$ are (approximately) uncorrelated complex Gaussian variates with zero means.[7] Moreover,

$$EN^{-1}|U(m)|^2 \simeq S_\epsilon(\kappa_m), \tag{25}$$

the wavenumber spectrum of the noise at $\kappa_m = 2\pi m/Nd$.[8] Thus

$$E|B(\theta_l)|^2 \simeq N^{-1}\sum_{m=0}^{N-1} |R(l-m)|^2 S_\epsilon(\kappa_m). \tag{26}$$

Assume that the array has a sidelobe pattern similar to the one given by (20). Since $R(0)=M$, it follows from (26) that

$$E|B(\theta_l)|^2 \simeq \frac{M}{p}\sum_{k=-(p-1)/2}^{(p-1)/2} S_\epsilon(\kappa_{l+k}), \tag{27}$$

where $p=(N/M)$ is assumed to be odd for simplicity. Assuming $S_\epsilon(\kappa)$ is *slowing varying* in the $2\pi p/Nd \simeq 2\pi/Md$ band about κ_l,

$$EM^{-1}|B(\theta_l)|^2 \simeq S_\epsilon(\kappa_l). \tag{28}$$

Using the complex Gaussian nature of the $U(m)$, it can be shown that the distribution of $2|B(\theta_l)|^2/MS_\epsilon(\kappa_l)$ is approximately chi-squared with two degrees of freedom (χ_2^2) and thus the variance of $M^{-1}|B(\theta_l)|^2$ is approximately $S_\epsilon^2(\kappa_l)$. This means that we must average the $M^{-1}|B(\theta_l)|^2$ in some way to obtain an accurate estimate of $S_\epsilon(\kappa_l)$.

If the noise is truly stationary, an accurate estimate can be obtained by averaging $M^{-1}|B(\theta_l)|^2$ computed at different times. For example, if $M^{-1}|B(\theta_l)|^2$ is computed for each l from 100 nonoverlapping records of the array output, the variance of the average $\langle M^{-1}|B(\theta_l)|^2\rangle$ is $S_\epsilon^2(\kappa_l)/100$. Let us assume that M, N, and the sampling time are sufficiently large so that we can treat $S_\epsilon(\kappa_l)$ as *known* for $l=0,\ldots,N-1$.

Returning to the signal, it follows from (12), (23), and (28) that

$$EM^{-1}|B(\theta_l)|^2 = |A|^2 M^{-1}|R(l-l_0)|^2 + S_\epsilon(\kappa_l), \tag{29}$$

if there is an l_0 such that $\theta_0 = \cos^{-1}(\lambda_0 l_0/Nd)$. Since $EM^{-1}|B(\theta_{l_0})|^2 = M|A|^2 + S_\epsilon(\kappa_{l_0})$, the normalized beam pattern has a peak of order $M|A|^2$ against a background of order $S_\epsilon(\kappa)$ for the look angle $\theta_{l_0}=\theta_0$, provided that $M|A|^2 \gg S_\epsilon(\kappa)$ for κ in a band about κ_{l_0}.

It is often useful to have a test statistic to determine the statistical significance of the maximum energy peak in the beam output. Let the null hypothesis be $A=0$, i.e., the peak is due to noise alone. Consider the test statistic

$$X = \max_{l=0,\ldots,N-1} \frac{2|B(\theta_l)|^2}{MS_\epsilon(\kappa_l)}, \tag{30}$$

which is analogous to the statistic of the Fisher test for the presence of a sinusoid in additive noise.[9] The distribution of X is needed to compute the threshold for an α-level test of the null hypothesis. This distribution is hard to obtain since the beam outputs $B(\theta_l)$ are *correlated*. But if the sidelobes rapidly diminish as M increases, as is the case for the optimal array or an equally spaced array, $\{B(\theta_{jp}):\ j=1,\ldots,M\}$ are uncorrelated for large M. Then the distribution of X is approximately the same as the distribution of the maximum of M uncorrelated χ_2^2 variates. The cumulative distribution function (cdf) of this maximum is $[F(x)]^M$, where $F(x)$ is the cdf of a χ_2^2 variate. The cumulative rejected at the α level if $X > x_0$, where x_0 satisfies

$$\alpha = 1 - [F(x_0)]^M. \tag{31}$$

Note that in (30), $|B(\theta_l)|^2$ is divided by the noise wavenumber spectrum. This division is the spatial analogy of prewhitening. If the beam outputs are to be visually inspected for the presence of a wave, then $P(\theta_l) = |B(\theta_l)|^2/S_\epsilon(\kappa_l)$ should be plotted for $l=0,\ldots,N-1$.

Suppose that $P(\theta_l)$ has a peak at θ_{l_0} that is statistically significant. Then θ_{l_0} is the natural estimate of θ_0. If $(2\pi/\lambda_0)\cos\theta_0$ is not equal to a κ_{l_0}, but falls between two grid points, there is a *quantization error* in θ_{l_0} of order $1/Nd$.

When N is large, the root mean-square error of θ_{l_0} due to noise is approximated by

$$\mathrm{rmse}\,\theta_{l_0} \simeq \lambda_0/2\pi(2\gamma\rho M)^{1/2} x_M \sin\theta_0, \tag{32}$$

where

$$\gamma = M^{-1}\sum_{k=1}^{M}(x_k - \bar{x})^2$$

is a geometry factor, x_M is the array length, and ρ is the power SNR in a narrowband about ω_0.[10] Since $\lim_{M\to\infty} M^{1/2}\,\mathrm{rmse}\,\theta_{l_0}$ is equal to the Cramer–Rao bound for the asymptotic variance of a consistent estimator of θ_0, θ_{l_0} is approximately maximum likelihood for large M and N. For a large aperture array with many sensors, this bearing error component is often less than the quantization error.

Now consider the problem of estimating the complex amplitude A. If the noise is *spatially uncorrelated*, it is shown in the Appendix that the maximum-likelihood estimator of A is

$$\hat{A} = (1/M)B(\theta_{l_0}). \tag{33}$$

Since $EB(\theta_{l_0}) = AR(0) = AM$, it follows that $\hat{A}$ is unbiased, i.e., $E\hat{A}=A$. From (29) the variance of $\hat{A}$ is

$$E|\hat{A}-A|^2 = \sigma_\epsilon^2/M, \tag{34}$$

where $\sigma_\epsilon^2 = E\epsilon^2(x_k)$.

If $S_\epsilon(\kappa)$ is not flat, $\hat{A}$ is still unbiased.[11] Its variance is approximately given by

$$E|\hat{A}-A|^2 = [S_\epsilon(\kappa_{l_0})/M]. \tag{35}$$

Thus $\hat{A}$ is a precise estimator of A when M is large, and is useful in removing the effect of a coherent jamming signal.

The planar array processing is a simple straightforward extension of the linear case, using two indices and two sums. For example, applying (14), (15), and (17) to (33), the estimator of A using a planar array is

$$\hat{A} = (1/M)B(\theta_{l_0,k_0}), \tag{36}$$

where (l_0, k_0) jointly maximize $|B(\theta_{l,k})|^2/S_\epsilon(\kappa_{xl}, \kappa_{yk})$, and $S_\epsilon(\kappa_{xl},\kappa_{yk})$ is the noise's wavenumber spectrum at $(\kappa_{xl},\kappa_{yk})$. The variance of $\hat{A}$ is approximately

$M^{-1}S_\epsilon(\kappa_{xl_0}, \kappa_{yk_0})$.

The estimator of θ_0 (θ_{l_0,k_0}) is given by the appropriate equation in (16) with l and k replaced by l_0 and k_0. When N is large,[12] θ_{l_0,k_0} is approximately maximum likelihood and

$$\mathrm{rmse}\theta_{l_0,k_0} \simeq \lambda_0/2\pi(2\gamma\rho M)^{1/2}x_M. \qquad (37)$$

V. BLOCKING A JAMMING SIGNAL

Suppose the filtered signal for the kth sensor is

$$s(x_k) = A\exp[-i(2\pi/\lambda_0)x_k\cos\theta_0] + A_J\exp[-i(2\pi/\lambda_J)x_k\cos\theta_J] + \epsilon(x_k),$$

where A_J, λ_J, and θ_J are the amplitude, wavelength, and direction, respectively, of a wave that is interfering with the wave of interest. If $\lambda_J \neq \lambda_0$, then all or most of the jamming energy will be filtered out. Thus let $\lambda_J = \lambda_0$, and assume that A_J and θ_J are unknown.

If $|A_J| \gg |A|$, then the beam pattern will be dominated by the jammer. If this is the case, then θ_J is estimated by the angle θ_{l_J} such that $|B(\theta_{l_J})|^2/S_\epsilon(\kappa_{l_J})$ is a maximum for $l = 0, \ldots, N-1$. The amplitude A_J is estimated using (33) with l_J in place of l_0.

The response of the jammer can be then subtracted from $B(\theta_l)$, and the adjusted beam pattern defined by

$$|\tilde{B}(\theta_l)|^2 = |B(\theta_l) - \hat{A}_J R(l - l_J)|^2 \qquad (38)$$

can be used to estimate the bearing θ_0.[13]

If $|A_J|$ is of the order of $|A|$, we need to know θ_J (if $\theta_J \neq \theta_0$) to block the jammer. If so, compute (38) with l_J as the integer that makes θ_{l_J} closest to θ_J.

VI. BROADBAND SIGNAL PROCESSING

Until now the signal has been assumed to be a single frequency plane wave. To exposite the processing of a broadband wave, let the array be linear and let

$$s(t, x_k) = s(t - c^{-1}x_k\cos\theta_0) + \epsilon(t, x_k), \qquad (39)$$

where $s(t)$ is a bandlimited signal whose upper frequency is ω_u. Once again the noise is assumed to be Gaussian and to be stationary in time and space.

All signals are transient. Select the time origin so that $s(t) = 0$ for $t < 0$ and $t > T$, where T is the signal duration. Let H be the largest integer less than or equal to $T\omega_u/2\pi$. The signal has the simple Fourier representation

$$s(t) = \sum_{j=-H}^{H} A(\omega_j)\exp(i\omega_j t), \qquad (40)$$

where $\omega_j = 2\pi j/T$ and

$$A(\omega_j) = \frac{1}{T}\int_0^T s(t)\exp(-i\omega_j t)\,dt. \qquad (41)$$

Assume $A(0) = 0$.

Suppose that $s(t, x_k)$ is sampled at times $t_n = n\Delta (n = 0, \ldots, N_T - 1)$ where $\Delta = \pi/\omega_u$ and $N_T = (T/\Delta)$.[14] If $\cos\theta_0 > 0$, part of the leading edge of the signal is lost for $k \geq 1$. The trailing edge is lost when $\cos\theta_0 < 0$. These end effects are negligible if $x_M/c \ll T$. Then from (40),

$$s_j(x_k) = N_T^{-1}\sum_{n=0}^{N_T-1} s(n\Delta, x_k)\exp(-i\omega_j n\Delta) = A(\omega_j)\exp(-i\kappa_j x_k) + \epsilon_j(x_k), \qquad (42)$$

where $\kappa_j = (\omega_j/c)\cos\theta_0$ and $\epsilon_j(x_k)$ is a zero-mean complex Gaussian variate. Its variance for large N_T is

$$E|\epsilon_j(x_k)|^2 \simeq N_T^{-1}S_\epsilon(\omega_j), \qquad (43)$$

where $S_\epsilon(\omega)$, the power spectrum of the noise, is independent of x_k. Thus the DFT of the received signals yields H single frequency waves plus filtered noise. For each $j = 1, \ldots, H$, compute

$$B(\theta_l, \omega_j) = \sum_{k=1}^{M} s_j(x_k)\exp i\kappa_l x_k, \qquad (44)$$

for the κ_l grid discussed in Sec. II. Concentrating on bearing estimation let $\hat{\theta}_j = \theta_{l_0}(\omega_j)$ denote the look angle associated with the maximum $|B(\theta_{l,\omega_j})|^2/S_\epsilon(\kappa_l)$. For large M and N the maximum-likelihood estimator of θ_0, denoted $\hat{\theta}_0$, is approximated by

$$\sum_{j=1}^{H}\sigma_j^{-2}\hat{\theta}_j \Big/ \sum_{j=1}^{H}\sigma_j^{-2}$$

where σ_j^2 is the variance of $\hat{\theta}_j$.[15] Using the large sample approximation given by (32) with ρ replaced by $\rho_j = T_N|A_j|^2/S_\epsilon(\omega_j)$ and λ_0 by $\lambda_j = 2\pi c/\omega_j$,

$$\hat{\theta}_0 \simeq \sum_{j=1}^{H}\omega_j^2\rho_j\hat{\theta}_j \Big/ \sum_{j=1}^{H}\omega_j^2\rho_j \qquad (45)$$

and its root-mean-square error is

$$\mathrm{rmse}\hat{\theta}_0 \simeq \left[c\left(\sum_{j=1}^{H}\omega_j^2\rho_j\right)^{-1/2} \Big/ (2\gamma M)^{1/2}x_M\sin\theta_0\right]. \qquad (46)$$

The SNR ρ_j is estimable since $A(\omega_j)$ is precisely estimated by $M^{-1}B(\theta_{l_0}, \omega_j)$ when M is large.

Since a maximum-likelihood estimator has minimum mean-square error when the sample size (M) is large, $\hat{\theta}_0$ is *optimal* in a mean-square sense for nonsparse arrays. There is no need for *ad-hoc* bearing estimators for such arrays.

This paper has shown the connection between beamforming and frequency–wavenumber spectral analysis using discrete time and space measurements. A designer of a robust and effective array processing system should have a complete understanding of the relationships between physical models of propagating waves, background noise processes, and the statistical properties of estimators of the key parameters in the signal models.

APPENDIX

If κ_0 is known, the maximum-likelihood estimator of A for the statistical model (23) is given by (33) when $[\epsilon(x_k)]$ are independent Gaussian $N(0, \sigma_\epsilon^2)$ variates.

Proof: Since the errors are Gaussian, the least squares estimator of A is maximum likelihood.[16] The least squares estimator is

$$\frac{\sum_{k=1}^{M} s(x_k)\exp(i\kappa_0 x_k)}{\sum_{k=1}^{M}\exp(-i\kappa_0 x_k)\exp(i\kappa_0 x_k)} = \frac{1}{M}\sum_{j=0}^{N-1} r(j)s(jd)\exp(i\kappa_0 jd)\,,$$

$$= M^{-1}B(\theta_{l_0})\,.$$

[1] B. D. Steinberg, *Principles of Aperture and Array System Design* (Wiley, New York, 1976), Sec. 5.4. His use of symbols differs from mine. For example, he uses θ_0 to denote the direction of arrival from the array normal. See also C. S. Clay and H. Medwin, *Acoustical Oceanography* (Wiley, New York, 1977), Sec. 5.3.2.

[2] Aliasing is defined in M. J. Hinich, "Processing Spatially Aliased Arrays," J. Acoust. Soc. Am. 64, 792–794 (1978). (Also see Steinberg, Sec. 5.2, Ref. 1).

[3] M. J. Hinich and W. E. Weber, "Determination of the Nyquist Frequency for Unequally Spaced Data," ONR Tech. Rep. 17 (revised), Virginia Tech. (1980).

[4] Carter shows that this design minimizes the asymptotic bearing variance for a linear array with a given minimum sensor spacing. See G. C. Carter, "Variance Bounds for Passively Locating an Acoustic Source with a Symmetric Line Array," J. Acoust. Soc. Am. 62, 922–926 (1977). This design is also optimal for range estimation. M. J. Hinich, "Passive Range Estimation Using Subarray Parallax," J. Acoust. Soc. Am. 65, 1229–1230 (1979).

[5] M. J. Hinich and P. Shaman, "Parameter Estimation for An r-dimensional Plane Wave Observed with Additive Independent Gaussian Errors," Ann. Math. Statist. 43, 153–169 (1972).

[6] G. C. Carter and C. H. Knapp, "Time Delay Estimation," in *Proceedings of the* 1976 *IEEE Conference on Acoustics, Speech and Signal Processing* (IEEE, New York, 1976), pp. 357–360. W. J. Bangs and P. M. Schultheiss, "Space-Time Processing for Optimal Parameter Estimation," in *Signal Processing*, edited by J. W. R. Griffiths, P. L. Stocklin, and C. van Schooneveld (Academic, New York, 1973), pp. 577–590.

[7] D. Brillinger, *Time Series, Data Analysis and Theory* (Holt, Rinehart and Winston, New York, 1975), Sec. 4.4.

[8] This spectrum is the one-dimensional spatial analog to the spectrum of a stationary discrete-time stochastic process. See C. M. Bennett, "A Directional Analysis of Sea Waves from Bottom Pressure Measurements," in *Transactions: Ocean Sciences and Engineering of the Atlantic Shelf* (Marine Technology Society, 1968), pp. 71–87.

[9] P. Bloomfield, *Fourier Analysis of Time Series: An Introduction* (Wiley, New York, 1976), Chap. 5.

[10] Levin heuristically derives the maximum-likelihood bearing estimator and its properties for a general three-dimensional array, assuming Gaussian noise. M. J. Levin, "Least-Squares Array Processing for Signals of Unknown Form, Radio Electron. Eng. 29, 213–222 (1965). These results are rigorously derived for a uniformly spaced array by Hinich and Shaman, Ref. 5. The Cramer–Rao bound for the asymptotic bearing variance for a linear array is given by V. H. MacDonald and P. J. Schultheiss, "Optimum Passive Bearing Estimation in a Spatially Incoherent Noise Environment," J. Acoust. Soc. Am. 46, 37–43 (1969). These rigorous derivations match Levin's results.

[11] The maximum-likelihood estimator of A for correlated noise is given by J. Capon, R. J. Greenfield, and R. J. Kolker, "Multidimensional Maximum-likelihood Processing of a Large Aperture Seismic Array," Proc. IEEE 55, 192–211 (1967).

[12] C. S. Clay, M. J. Hinich, and P. Shaman, "Error Analysis of Velocity and Direction Measurements of Plane Waves Using Thick Large Aperture Arrays," J. Acoust. Soc. Am. 53, 1161–1166 (1973).

[13] This is equivalent to steering a null at θ_J, and then estimating θ_0. See V. C. Anderson and P. Rudnick, "Reflection of a Coherent Arrival of an Array," J. Acoust. Soc. Am. 45, 406–410 (1969).

[14] In this paper, $[x]$ denotes the integer closest to x.

[15] If $\hat{\theta}_1$ and $\hat{\theta}_2$ are maximum-likelihood estimators of θ_0 computed from two independent samples, the $(\sigma_1^{-2}+\sigma_2^{-2})^{-1}(\sigma_1^{-2}\hat{\theta}_1+\sigma_2^{-2}\hat{\theta}_2)$ is maximum-likelihood for the combined sample. Its variance is $(\sigma_1^{-2}+\sigma_2^{-2})^{-1}$.

[16] C. R. Rao, *Linear Statistical Interference and Its Applications* (Wiley, New York, 1965), Chap. 4.