Transverse vibrations of double-tapered cantilever beams with end support and with end mass

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The free vibrations of a double-tapered cantilever beam with (1) end support and (2) end mass have been investigated using the Bernoulli–Euler equation. The beam was tapered linearly in the horizontal and in the vertical planes simultaneously with the taper ratio in the horizontal plane equal to that in the vertical plane. A table is presented for the first case from which the fundamental frequency, second, third, fourth, and fifth harmonic can easily be obtained for various taper ratios. A chart, plotted from this table, shows the effect of taper ratio on the various harmonics. For the second case, a table and resulting charts show the effect of taper ratio and ratio of end mass to beam mass on the fundamental frequency and higher harmonics. Although previously presented, the case of the beam with free end is also included for purposes of comparison.

Subject Classification: 40.22.

INTRODUCTION

This analysis is a continuation of the work[1–3] started by the authors several years ago on the vibration of tapered cantilever beams. This type of beam tapered linearly in either the horizontal or the vertical plane is widely used for electrical contacts and for springs in electro-mechanical devices. Occasionally, however, the desired spring rate cannot be achieved by tapering the beam in only one plane so that it is necessary to resort to a taper in the horizontal and in the vertical plane simultaneously. Both the single-tapered and the double-tapered beams will require an end mass or an end support depending upon whether the beam is used for an electrical contact (normally open) or for a spring. Single-tapered cantilever beams with end mass and with end support have been treated by the authors.[1,2]

This paper deals with the vibration of double-tapered cantilever beams with end support and with end mass for the case where the taper ratio in the horizontal plane equals that in the vertical plane. Tables have been developed from which the fundamental frequency, second, third, fourth, and fifth harmonic can be obtained for various taper ratios and ratios of end mass to beam mass. Although the case of the double-tapered cantilever beam with free end has been presented in Ref. 3, it is included in this work for comparative purposes.

I. BEAM OF LINEARLY VARIABLE THICKNESS AND OF LINEARLY VARIABLE WIDTH

In Ref. 3 the differential equation of motion for a vibrating beam tapered in two planes as shown in Fig. 1 was developed from the Bernoulli–Euler equation

$$\frac{\partial^2}{\partial x^2}\left(\frac{EI\,\partial^2 y}{\partial x^2}\right) = -\left(\frac{\rho A}{g}\right)\frac{\partial^2 y}{\partial t^2}\ , \qquad (1)$$

where $\rho A/g$ is the mass per unit length (ρ weight density, A cross-sectional area, g gravitational constant), E the modulus of elasticity, and I the moment of inertia. A sustained free vibration at a frequency ω of $y(x,t) = z(x)\sin\omega t$ was assumed which gave the following:

$$\frac{d^4 z}{du^4} + \frac{2d^3 z}{du^3}\left[\frac{3(\alpha-1)}{1+(\alpha-1)u} + \frac{\beta-1}{1+(\beta-1)u}\right] + \frac{6d^2 z}{du^2} \times \left\{\frac{(\beta-1)(\alpha-1)}{[1+(\beta-1)u][1+(\alpha-1)u]} + \frac{(\alpha-1)^2}{[1+(\alpha-1)u]^2}\right\} = \frac{(lk)^4 z}{[1+(\alpha-1)u]^2}\ , \qquad (2)$$

where

$u = x/l$,

$\alpha = h_0/h_1$,

$\beta = b_0/b_1$,

$k^4 = 12\rho\omega^2/Egh_1^2$.

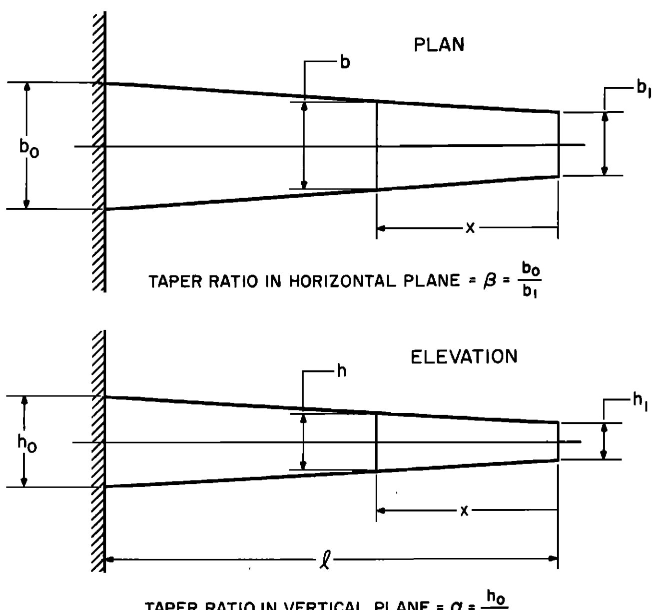

FIG. 1. Cantilever beam tapered linearly in horizontal and in vertical planes simultaneously.

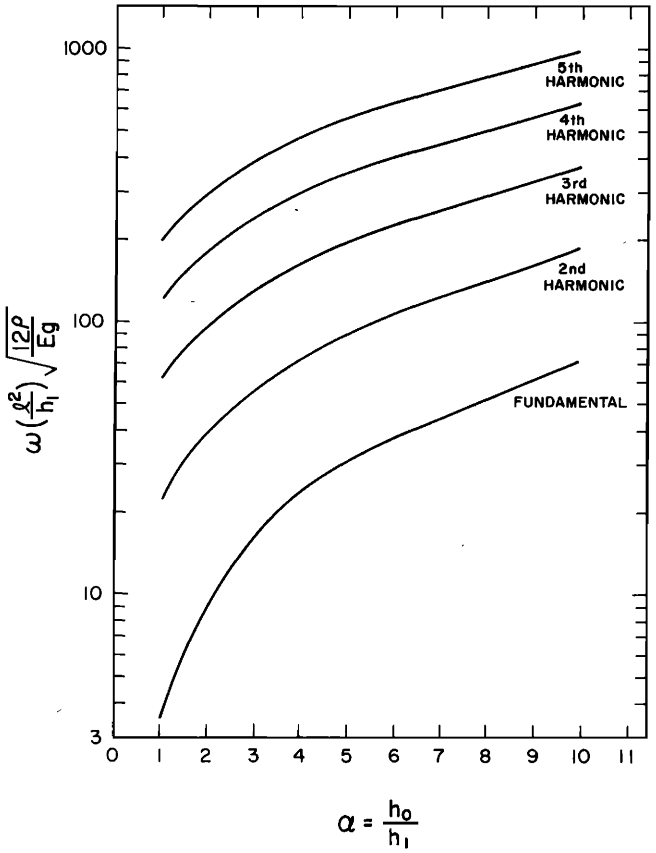

FIG. 2. Frequencies for double-tapered cantilever beam with free end with $\beta=\alpha$.

A formal solution for this equation could not be obtained, and it was solved by numerical integration to give values of (lk) for various taper ratios of α and β.

A solution to Eq. 2 can be obtained by considering the special case where the taper ratios α and β are equal. The resulting differential equation can then be solved in terms of Bessel functions. If $\beta=\alpha$ is substituted in Eq. 2, the following equation results:

$$\frac{d^4z}{du^4}+\frac{8d^3z}{du^3}\left[\frac{(\alpha-1)}{1+(\alpha-1)u}\right]+\frac{12d^2z}{du^2}\left[\frac{(\alpha-1)}{1+(\alpha-1)u}\right]^2=\frac{(lk)^4z}{[1+(\alpha-1)u]^2}. \quad (3)$$

Equation 3 may be placed in a more recognizable form if $\phi=1+(\alpha-1)u$; this substitution yields

$$\phi^4\frac{d^4z}{d\phi^4}+8\phi^3\frac{d^3z}{d\phi^3}+12\phi^2\frac{d^2z}{d\phi^2}=\left(\frac{lk}{\alpha-1}\right)^4\phi^2z. \quad (4)$$

Siddall and Isackson[4] list the steps to put Eq. 4 into operator notation for which Watson[5] gives the solution as

$$z=\frac{1}{\phi}\left[AJ_2\left(\frac{2lk}{\alpha-1}\sqrt{\phi}\right)+BY_2\left(\frac{2lk}{\alpha-1}\sqrt{\phi}\right)+CI_2\left(\frac{2lk}{\alpha-1}\sqrt{\phi}\right)+DK_2\left(\frac{2lk}{\alpha-1}\sqrt{\phi}\right)\right], \quad (5)$$

where J_2 and Y_2 are Bessel functions of the first and second kind and I_2 and K_2 are modified Bessel functions of the first and second kind.

TABLE I. Factor $(lk)^2$ $\beta=\alpha$, free end.

α	Fundamental frequency	Second harmonic	Third harmonic	Fourth harmonic	Fifth harmonic
1.0	3.51602	22.03449	61.69721	120.9019	199.8595
1.2	4.54997	25.46665	68.98686	133.9884	220.6668
1.4	5.64828	28.87848	76.14960	146.7374	240.8424
1.6	6.80221	32.28233	83.21828	159.2265	260.5263
1.8	8.00478	35.68615	90.21556	171.5096	279.8154
2.0	9.25030	39.09523	97.15780	183.6255	298.7798
2.5	12.5226	47.6622	114.3459	213.3596	345.0885
3.0	15.9785	56.3146	131.3862	242.5213	390.2255
3.5	19.5781	65.0655	148.3511	271.2901	434.5191
4.0	23.2923	73.9188	165.2854	299.7799	478.1801
5.0	30.9820	91.9273	199.1682	356.2088	564.1394
10.0	72.0487	186.802	371.238	635.049	981.657

A. Beam with free end

For a beam with a free end at $x=0$, the boundary conditions are

at $x=0$ or $u=0$, $d^2z/du^2=0$ and $d^3z/du^3=0$,
at $x=l$ or $u=1$, $dz/du=0$ and $z=0$.

With these boundary conditions, the solution becomes that of a double-tapered cantilever beam which is truncated and tapers from the fixed end only. Imposing the above boundary conditions on the general solution in Eq. 5 gives the following determinantal equation for obtaining the natural frequencies of the beam:

$$\begin{vmatrix} J_2(\Theta\sqrt{\alpha}) & Y_2(\Theta\sqrt{\alpha}) & I_2(\Theta\sqrt{\alpha}) & K_2(\Theta\sqrt{\alpha}) \\ J_3(\Theta\sqrt{\alpha}) & Y_3(\Theta\sqrt{\alpha}) & -I_3(\Theta\sqrt{\alpha}) & K_3(\Theta\sqrt{\alpha}) \\ J_4(\Theta) & Y_4(\Theta) & I_4(\Theta) & K_4(\Theta) \\ J_5(\Theta) & Y_5(\Theta) & -I_5(\Theta) & K_4(\Theta) \end{vmatrix}=0. \quad (6)$$

The various values of $\Theta=[2lk/(\alpha-1)]$ were found. To compare with values tabulated in Ref. 3, Table I was developed giving values of $(lk)^2$ corresponding to fundamental, second harmonic, third harmonic, fourth harmonic, and fifth harmonic frequencies for the free-end case with $\beta=\alpha$. Rearranging the equation $k^4=12\rho\omega^2/Egh_1^2$ to obtain the term $(lk)^2$, it follows that

$$\omega(l^2/h_1)(12\rho/Eg)^{1/2}=(lk)^2. \quad (7)$$

TABLE II. Factor $(lk)^2$ $\beta=\alpha$, end support.

α	Fundamental frequency	Second harmonic	Third harmonic	Fourth harmonic	Fifth harmonic
1.0	15.4182	49.9649	104.248	178.270	272.031
1.2	17.5615	55.4779	115.077	196.337	299.264
1.4	19.6505	60.8528	125.577	213.801	325.540
1.6	21.6980	66.1184	135.817	230.790	351.061
1.8	23.7128	71.2951	145.847	247.391	375.967
2.0	25.7010	76.3976	155.702	263.671	400.361
2.5	30.5821	88.8974	179.735	303.257	459.572
3.0	35.3696	101.117	203.108	341.627	516.841
3.5	40.0904	113.129	225.989	379.089	572.656
4.0	44.7611	124.980	248.488	415.841	627.331
5.0	53.9936	148.315	292.615	487.727	734.077
6.0	63.1233	171.289	335.881	558.001	838.218
10.0	99.0859	261.086	503.841	829.474	1239.11

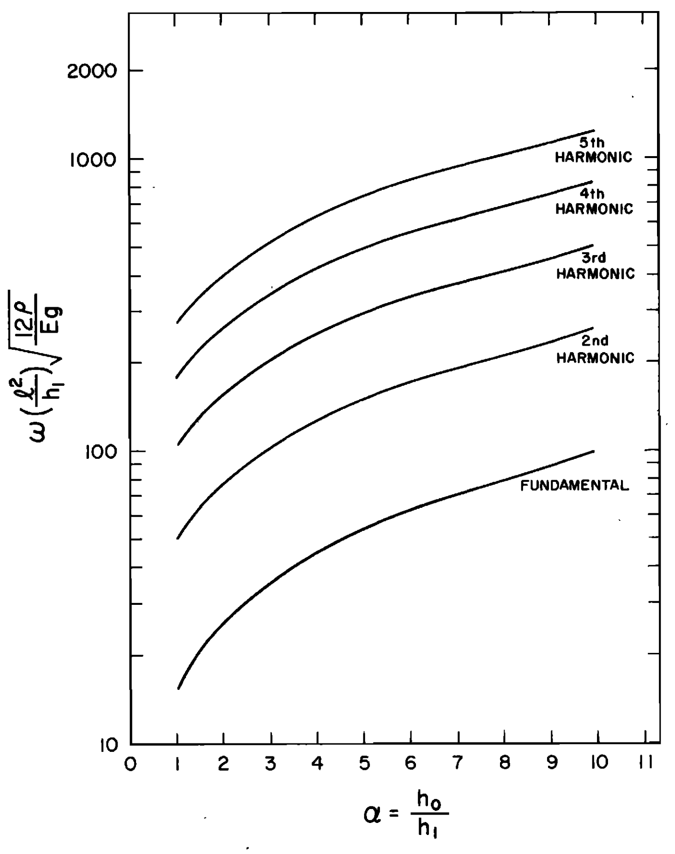

FIG. 3. Frequencies for double-tapered cantilever beam with end support with $\beta=\alpha$.

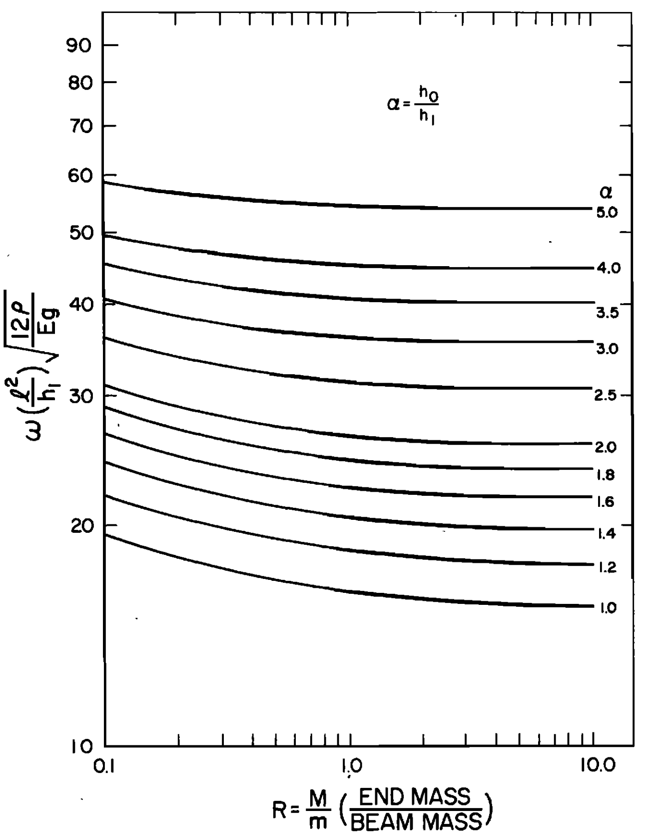

FIG. 5. Second-harmonic frequency for double-tapered cantilever beam with end mass with $\beta=\alpha$.

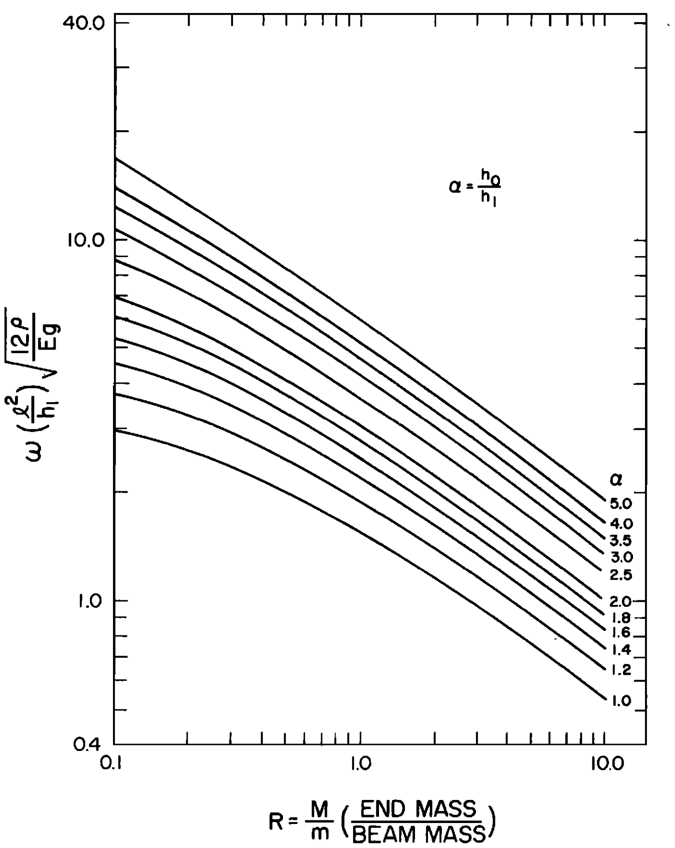

FIG. 4. Fundamental frequency for double-tapered cantilever beam with end mass with $\beta=\alpha$.

Using this equation and the values of $(lk)^2$ from Table I, curves were plotted of $\omega(l^2/h_1)(12\rho/Eg)^{1/2}$ vs α for the five harmonics as shown in Fig. 2.

B. Beam with end support

For a beam with end support at $x=0$, the boundary conditions are:

at $x=0$ or $u=0$, $d^2z/du^2=0$ and $z=0$;
at $x=l$ or $u=1$, $dz/du=0$ and $z=0$.

Imposing these boundary conditions gives the following determinantal equation for obtaining the natural frequencies of the beam with end support:

$$\begin{vmatrix} J_2(\Theta\sqrt{\alpha}) & Y_2(\Theta\sqrt{\alpha}) & I_2(\Theta\sqrt{\alpha}) & K_2(\Theta\sqrt{\alpha}) \\ J_3(\Theta\sqrt{\alpha}) & Y_3(\Theta\sqrt{\alpha}) & -I_3(\Theta\sqrt{\alpha}) & K_3(\Theta\sqrt{\alpha}) \\ J_4(\Theta) & Y_4(\Theta) & I_4(\Theta) & K_4(\Theta) \\ J_2(\Theta) & Y_2(\Theta) & I_2(\Theta) & K_2(\Theta) \end{vmatrix} = 0. \quad (8)$$

Table II was developed to give values of $(lk)^2$ corresponding to the fundamental, second, third, fourth, and fifth harmonic frequencies for the end support case with $\beta=\alpha$. From Table II, curves were plotted of $\omega(l^2/h_1)\cdot(12\rho/Eg)^{1/2}$ vs α for the five harmonics as shown in Fig. 3.

C. Beam with end mass

For the case of a concentrated mass M located at $x=0$, the boundary conditions are:

TABLE III. Factor $(lk)^2 \beta = \alpha$, end mass.

α	R	Fundamental frequency	Second harmonic	Third harmonic	Fourth harmonic	Fifth harmonic
1.0	.0	3.51602	22.0345	61.6972	120.902	199.860
	.1	2.96784	19.3558	55.5182	110.708	185.346
	.2	2.61275	18.2078	53.5586	108.192	182.431
	.4	2.16799	17.1763	52.0632	106.457	180.544
	.6	1.89246	16.7007	51.4451	105.781	179.834
	.8	1.70064	16.4274	51.1080	105.421	179.461
	1.0	1.55730	16.2501	50.8958	105.198	179.232
	2.0	1.15820	15.8609	50.4476	104.735	178.760
	5.0	.756937	15.6024	50.1623	104.446	178.468
	10.0	.541375	15.5115	50.0644	104.347	178.369
1.2	.0	4.54997	25.4666	68.9869	133.988	220.667
	.1	3.75054	21.9023	61.1597	121.434	203.130
	.2	3.25843	20.5336	59.0259	118.835	200.220
	.4	2.66608	19.3789	57.4964	117.138	198.423
	.6	2.31036	18.8692	56.8871	116.496	197.763
	.8	2.06697	18.5825	56.5601	116.159	197.420
	1.0	1.88708	18.3988	56.3562	115.951	197.210
	2.0	1.39383	18.0022	55.9301	115.523	196.780
	5.0	.90655	17.7434	55.6620	115.257	196.516
	10.0	.64725	17.6534	55.5705	115.167	196.427
1.4	.0	5.64828	28.8785	76.1496	146.737	240.842
	.1	4.54374	24.3118	66.5588	131.742	220.249
	.2	3.89751	22.7418	64.3072	129.126	217.404
	.4	3.14824	21.4913	62.7768	127.492	215.711
	.6	2.71077	20.9598	62.1850	126.887	215.100
	.8	2.41594	20.6662	61.8715	126.573	214.785
	1.0	2.20010	20.4800	61.6773	126.380	214.593
	2.0	1.61576	20.0832	61.2750	125.985	214.202
	5.0	1.04678	19.8282	61.0241	125.741	213.963
	10.0	.74633	19.7401	60.9389	125.659	213.882
1.6	.0	6.80221	32.2823	83.2183	159.227	260.526
	.1	5.33828	26.6067	71.7742	141.743	236.870
	.2	4.52367	24.8590	69.4525	139.154	234.125
	.4	3.61163	23.5373	67.9436	137.595	232.540
	.6	3.09243	22.9936	67.3741	137.028	231.976
	.8	2.74709	22.6977	67.0753	136.735	231.687
	1.0	2.49632	22.5117	66.8913	136.556	231.511
	2.0	1.82461	22.1195	66.5125	136.192	231.154
	5.0	1.17831	21.8703	66.2779	135.968	230.936
	10.0	.83915	21.7848	66.1985	135.893	230.863
1.8	.0	8.00478	35.6862	90.2156	171.510	279.815
	.1	6.12761	28.8055	76.8494	151.509	253.106
	.2	5.13319	26.9054	74.4959	148.979	250.481
	.4	4.05537	25.5335	73.0228	147.498	249.000
	.6	3.45551	24.9849	72.4776	146.968	248.478
	.8	3.06105	24.6901	72.1938	146.695	248.212
	1.0	2.77658	24.5061	72.0199	146.529	248.051
	2.0	2.02142	24.1213	71.6634	146.192	247.724
	5.0	1.30198	23.8792	71.4439	145.986	247.525
	10.0	.92637	23.7965	71.3697	145.917	247.458
2.0	.0	9.25030	39.0952	97.1578	183.626	298.780
	.1	6.90715	30.9239	81.8171	161.094	269.036
	.2	5.72408	28.8964	79.4612	158.641	266.537
	.4	4.47965	27.4920	78.0323	157.239	265.154
	.6	3.80088	26.9438	77.5118	156.742	264.671
	.8	3.35894	26.6522	77.2427	156.488	264.425
	1.0	3.04210	26.4714	77.0782	156.334	264.277
	2.0	2.20735	26.0959	76.7426	156.021	263.976
	5.0	1.41865	25.8613	76.5367	155.831	263.794
	10.0	1.00862	25.7815	76.4674	155.767	263.733
2.5	.0	12.52258	47.6622	114.3459	213.360	345.088
	.1	8.79472	35.9508	93.9102	184.480	307.844
	.2	7.11589	33.7041	91.6299	182.253	305.657
	.4	5.46099	32.2738	90.3256	181.032	304.485
	.6	4.59487	31.7411	89.8638	180.609	304.082
	.8	4.04183	31.4633	89.6277	180.394	303.878
	1.0	3.64982	31.2928	89.4842	180.263	303.755
	2.0	2.63178	30.9433	89.1934	180.001	303.507
	5.0	1.68463	30.7280	89.0164	179.842	303.358
	10.0	1.19606	30.6553	88.9570	179.788	303.308
3.0	.0	15.9785	56.3146	131.386	242.521	390.225
	.1	10.5776	40.7106	105.707	207.291	345.583
	.2	8.38981	38.3575	103.559	205.290	343.670
	.4	6.34346	36.9508	102.377	204.222	342.665
	.6	5.30525	36.4437	101.966	203.856	342.323
	.8	4.65153	36.1827	101.757	203.671	342.151
	1.0	4.19184	36.0237	101.631	203.559	342.047
	2.0	3.00974	35.7003	101.376	203.334	341.838
	5.0	1.92138	35.5028	101.221	203.199	341.712
	10.0	1.36289	35.4363	101.169	203.153	341.670
3.5	.0	19.5781	65.0655	148.351	271.290	434.519
	.1	12.2504	45.3044	117.324	229.703	382.541
	.2	9.55820	42.9227	115.324	227.905	380.856
	.4	7.14473	41.5643	114.252	226.962	379.983
	.6	5.94889	41.0857	113.883	226.641	379.687
	.8	5.20359	40.8416	113.697	226.479	379.539
	1.0	4.68250	40.6936	113.585	226.382	379.449
	2.0	3.35188	40.3942	113.358	226.186	379.269
	5.0	2.13578	40.2124	113.221	226.068	379.161
	10.0	1.51400	40.1515	113.175	226.028	379.125
4.0	.0	23.2923	73.9188	165.285	299.780	478.180
	.1	13.8168	49.7958	128.823	251.811	418.887
	.2	10.6353	47.4353	126.967	250.189	417.392
	.4	7.87962	46.1360	125.990	249.348	416.623
	.6	6.53887	45.6857	125.657	249.064	416.364
	.8	5.70965	45.4575	125.489	248.921	416.234
	1.0	5.13237	45.3196	125.388	248.834	416.155
	2.0	3.66580	45.0417	125.185	248.661	415.998
	5.0	2.33262	44.8736	125.062	248.557	415.904
	10.0	1.65278	44.8174	125.021	248.522	415.872
5.0	.0	30.9820	91.9273	199.168	356.209	564.139
	.1	16.6644	58.6146	151.581	295.344	490.174
	.2	12.5670	56.3745	149.980	293.999	488.964
	.4	9.19450	55.1988	149.155	293.312	488.349
	.6	7.59518	54.8000	148.877	293.081	488.142
	.8	6.61634	54.5994	148.737	292.965	488.038
	1.0	5.93881	54.4788	148.653	292.895	487.976
	2.0	4.22934	54.2366	148.484	292.755	487.852
	5.0	2.68636	54.0909	148.382	292.671	487.777
	10.0	1.90227	54.0423	148.348	292.643	487.752

at $x=0$, $\partial^2 y/\partial x^2 = 0$ and $\dfrac{\partial}{\partial x}\left(EI\dfrac{\partial^2 y}{\partial x^2}\right) = -V = \dfrac{-M\partial^2 y}{\partial t^2}$;

at $x=l$, $\partial y/\partial x = 0$ and $y=0$.

For the assumed free vibration $y(x,t) = z(x)\sin\omega t$, the boundary conditions at $x=0$ transform into

$$d^2z/dx^2 = 0$$

and

$$d^3z/dx^3 = M\omega^2 z_0/EI_0\,,$$

where

$$I_0 = \tfrac{1}{12} b_1 h_1^3\,.$$

Therefore,

$$d^3z/dx^3 = 12M\omega^2 z_0/Eb_1h_1^3$$

$$= \left(\frac{12\rho\omega^2}{Egh_1^2}\right)\left(\frac{Mz_0g}{\rho b_1 h_1}\right)\,.$$

The volume of the beam is $(l/3)b_1h_1(\alpha^2+\alpha+1)$ so that the mass of the beam m can be expressed as

$$m = \left(\frac{\rho}{g}\right)\left[\left(\frac{l}{3}\right)(b_1h_1)\,(\alpha^2+\alpha+1)\right]\,.$$

Therefore,

$$\frac{d^3z}{dx^3}=\left(\frac{12\rho\omega^2}{Egh_1^2}\right)\frac{M[\frac{1}{3}l(\alpha^2+\alpha+1)]}{(\rho/g)\,b_1h_1[\frac{1}{3}l(\alpha^2+\alpha+1)]}\,z_0$$

$$=k^4(M/m)\,(l/3)\,(\alpha^2+\alpha+1)\,z_0\,.$$

Substituting $\phi=1+[(\alpha-1)/l]\,x$,

$$\left(\frac{d^3z}{d\phi^3}\right)_{\phi=1}=k^4\left(\frac{M}{m}\right)\left(\frac{l^4}{3}\right)\left[\frac{\alpha^2+\alpha+1}{(\alpha-1)^3}\right]\,(z)_{\phi=1}\,.$$

The boundary conditions at $x=0$ or $\phi=1$ are therefore

$$\frac{d^2z}{d\phi^2}=0 \tag{9a}$$

and

$$\frac{d^3z}{d\phi^3}=\left(\frac{M}{m}\right)\left[\frac{(lk)^4}{3}\right]\left[\frac{\alpha^2+\alpha+1}{(\alpha-1)^3}\right]z\,.$$

The boundary conditions at $x=l$ or $\phi=\alpha$ become

$$z=0 \tag{9b}$$

and

$$\frac{dz}{d\phi}=0.$$

Imposing the boundary conditions given by Eqs. 9 on Eq. 5 gives the following determinantal equation for obtaining the natural frequencies of the beam with an end mass M:

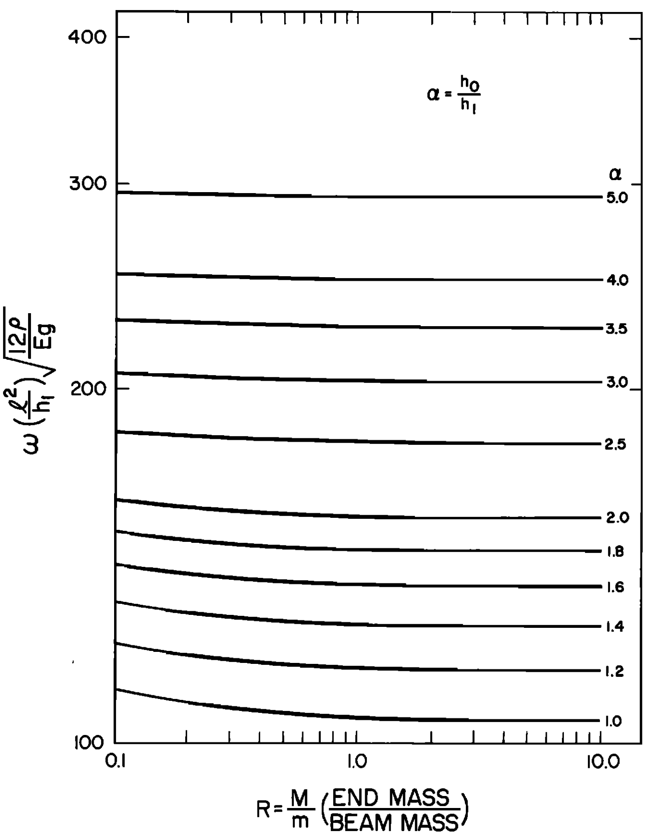

FIG. 7. Fourth-harmonic frequency for double-tapered cantilever beam with end mass with $\beta=\alpha$.

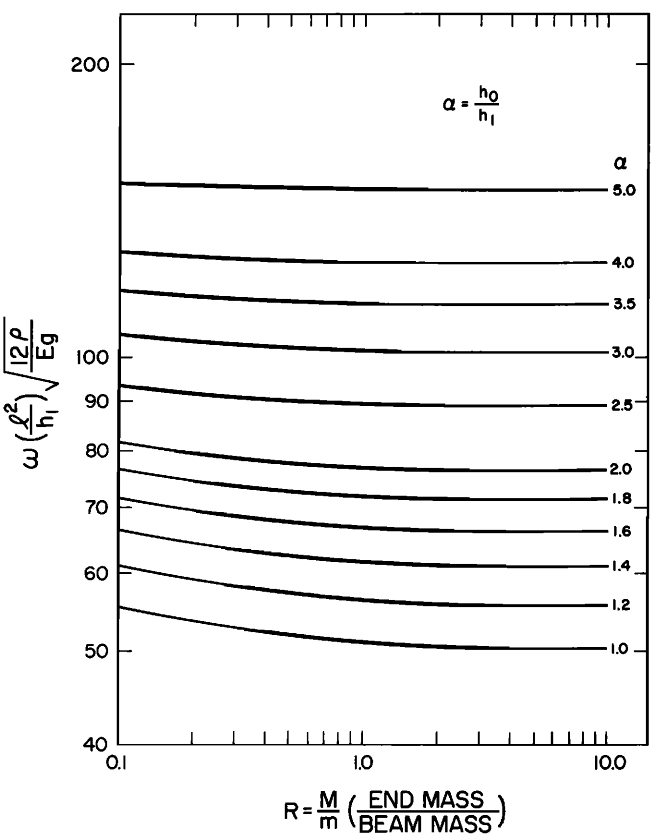

FIG. 6. Third-harmonic frequency for double-tapered cantilever beam with end mass with $\beta=\alpha$.

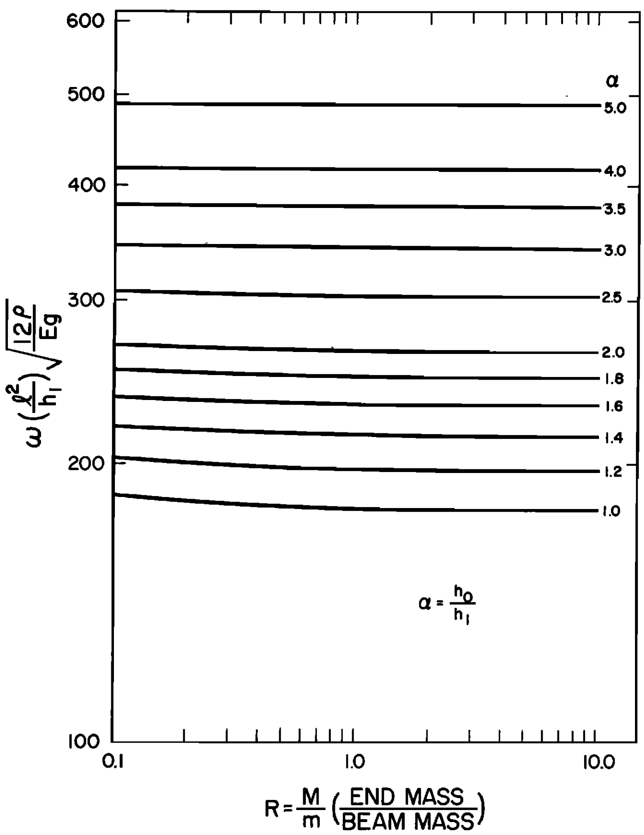

FIG. 8. Fifth-harmonic frequency for double-tapered cantilever beam with end mass with $\beta=\alpha$.

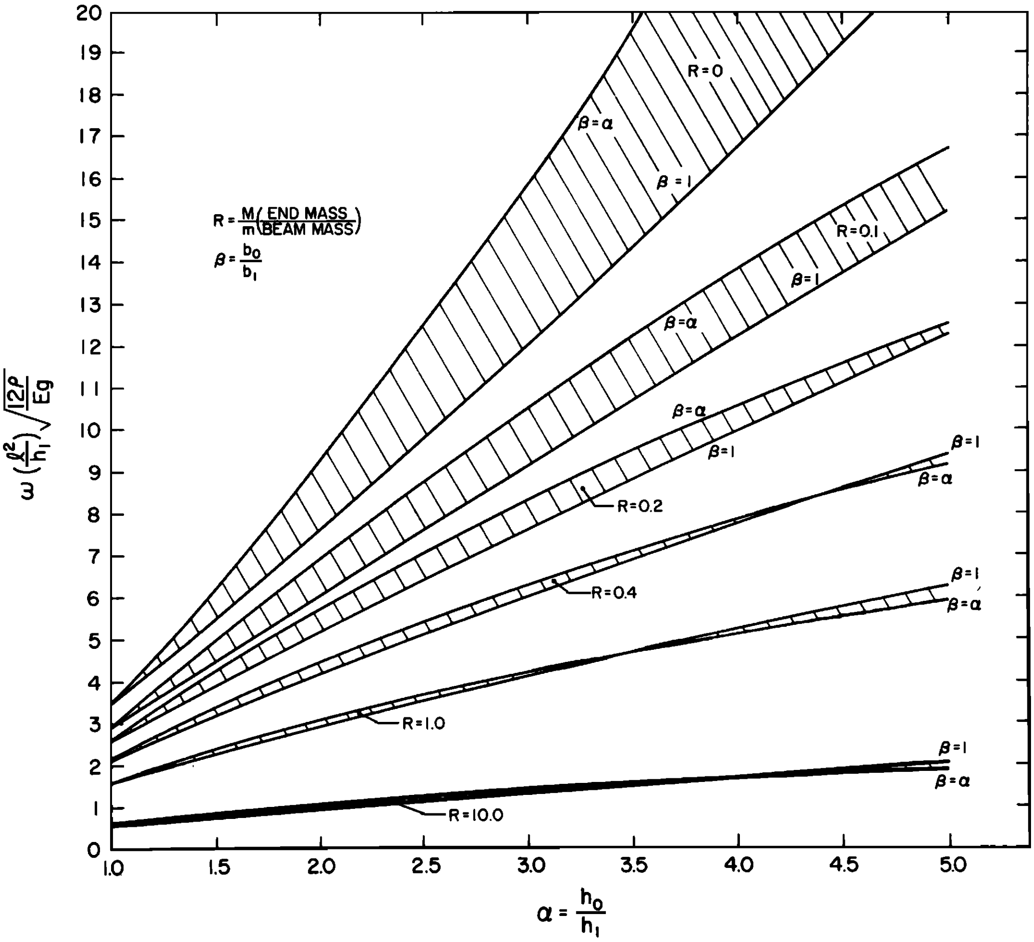

FIG. 9. Fundamental frequency for double-tapered cantilever beam with end mass with $\beta=\alpha$ and $\beta=1$ for several values of R.

$$\begin{vmatrix} J_2(\Theta\sqrt{\alpha}) & Y_2(\Theta\sqrt{\alpha}) & I_2(\Theta\sqrt{\alpha}) & K_2(\Theta\sqrt{\alpha}) \\ J_3(\Theta\sqrt{\alpha}) & Y_3(\Theta\sqrt{\alpha}) & -I_3(\Theta\sqrt{\alpha}) & K_3(\Theta\sqrt{\alpha}) \\ J_4(\Theta) & Y_4(\Theta) & I_4(\Theta) & K_4(\Theta) \\ A & B & C & D \end{vmatrix} = 0, \qquad (10)$$

where

$$A=\left[\frac{M}{m}\right]\left[\frac{lk}{3}\right](\alpha^2+\alpha+1)\,J_2(\Theta)+J_5(\Theta),$$

$$B=\left[\frac{M}{m}\right]\left[\frac{lk}{3}\right](\alpha^2+\alpha+1)\,Y_2(\Theta)+Y_5(\Theta),$$

$$C=\left[\frac{M}{m}\right]\left[\frac{lk}{3}\right](\alpha^2+\alpha+1)\,I_2(\Theta)-I_5(\Theta),$$

$$D=\left[\frac{M}{m}\right]\left[\frac{lk}{3}\right](\alpha^2+\alpha+1)\,K_2(\Theta)+K_5(\Theta).$$

Table III was developed to give values of $(lk)^2$ corresponding to the fundamental, second, third, fourth, and fifth harmonic frequencies for a beam with end mass with $\beta=\alpha$. In this table R is the ratio M/m of the mass of the concentrated load to that of the beam.

Figures 4–8 show curves of $\omega(l^2/h_1)(12\rho/Eg)^{1/2}$ vs R for the five harmonics plotted from the data in Table III. It is interesting to note that the curves of $\omega(l^2/h_1)(12\rho/Eg)^{1/2}$ vs R, after the fundamental frequency (Fig. 4), are almost independent of R. This is especially true for the higher harmonics and higher values of α.

Figure 9 shows a plot of $\omega(l^2/h_1)(12\rho/Eg)^{1/2}$ vs α for the fundamental frequency for $\beta=\alpha$ and $\beta=1$ for several values of R. The values for $\beta=1$ were taken from the previously solved case given in Ref. 1. It can be seen that as the value of R increases, the spread of the two curves decreases for a particular R. This is particularly significant for the values of $R>0.4$. In these cases the value of the fundamental frequency is approximately the same whether the beam has a double taper or whether it tapers only in the vertical plane ($\beta=1$).

[1] H. H. Mabie and C. B. Rogers, "Transverse Vibrations of Tapered Cantilever Beams With End Loads," J. Acoust. Soc. Am. 36, 463–469 (1964).
[2] H. H. Mabie and C. B. Rogers, "Transverse Vibrations of Tapered Cantilever Beams With End Support," J. Acoust. Soc. Am. 44, 1739–1741 (1968).
[3] H. H. Mabie and C. B. Rogers, "Transverse Vibrations of Double-Tapered Cantilever Beams," J. Acoust. Soc. Am. 51, 1771–1774 (1972).
[4] J. N. Siddall and G. Isackson, "Approximate Analytical Methods for Determining Natural Modes and Frequencies of Vibration," MIT Rep. ONR Proj. NR-035-259, pp. 141–146 (Jan. 1951).
[5] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge U. P., Cambridge, England, 1952).