

PHASE TRANSITIONS, RENORMALIZATION, AND SCALING

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Abstract

An introduction to the renormalization group (RG) theory of critical phenomena is presented, and a review of some of its most important results is given. After a brief description of the physical properties of systems near a second-order phase transition, and of the theoretical problems in providing an adequate mathematical treatment, the basic notions of universality and scale invariance are discussed. Then some of the most common microscopic models for systems with phase transitions are introduced, along with the Landau-Ginzburg-Wilson functional, which constitutes the corresponding effective field theory. For the latter both the mean-field approximation, where order parameter fluctuations are neglected entirely, as well as the Gaussian theory taking them into account only to the lowest order, are described. The RG concept is then introduced, and explained first in real space, by considering the one- and two-dimensional Ising model as examples. The emergence of scaling laws and universality is shown to be a consequence of the behavior of the RG flows near a critical fixed point. Specifically, the critical exponents are deduced from the RG flow behavior in the vicinity of the critical fixed point. Returning to the continuum model, the momentum shell elimination procedure in Fourier space (Wilson's RG scheme) is presented, and for the case of the $O(n)$ -symmetric ϕ^4 model it is demonstrated how the critical exponents may be systematically obtained within an ϵ expansion near the upper critical dimension. Some remarks on more advanced field-theoretical methods conclude the discussion of the RG concept. A list of important topics where scaling ideas and the RG approach have proven to be very successful finally illustrates that these methods provide a most powerful tool in modern statistical physics.

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INTRODUCTION

The fundamental laws of nature governing the properties of matter possess distinguished symmetry properties: They are invariant with respect to translation in space and time, to rotations and inversions. The states of matter that we observe, however, usually do not display the full symmetries of the underlying basic principles. Rather, the atoms or molecules of a specific substance can be found in certain condensed phases consisting of enormously large numbers (typically 10^{23}cm^{-3}) of constituents, and these phases may be classified according to some specific symmetry transformation under which the system remains invariant (a subgroup of the original symmetry group, see GROUP THEORY). In passing, we note that even the “fundamental” laws themselves, like the electroweak interaction and quantum chromodynamics, may have originated in the transition to a broken symmetry state of an even more fundamental and more symmetric basic law. In this article we shall be solely concerned with matter governed by the Schrödinger equation and electromagnetic forces.

The macroscopic properties of the phases of condensed matter, whose stability regimes are determined by the internal interactions of the particles involved, as well as by the external conditions (temperature, pressure, electric and magnetic fields, etc.) can be very different. Upon changing the external control parameters the system may undergo a phase transition to a new macroscopic phase, thus lowering its total free energy (see THERMODYNAMICS, EQUILIBRIUM). Frequently, the internal energy of the system as a function of the control variables is minimized by the change of its thermodynamic phase, but there are also important cases where the driving force towards a new equilibrium state is a gain of entropy.

The stability of a particular phase and the implications of phase transitions are of course of prominent importance for applications and technology. Moreover, these issues provide

a considerable intellectual challenge for their understanding in the framework of statistical mechanics (see STATISTICAL MECHANICS, CLASSICAL; STATISTICAL MECHANICS, QUANTUM). Accordingly, methods need to be developed in order to deal with a strongly interacting system containing a large number of particles, and near its stability boundary.

Fortunately it is not necessary to study all the microscopic degrees of freedom in detail; instead it suffices to investigate the statistical mechanics of a few macroscopic entities (which, however, usually still poses a formidable task), namely the order parameter characterizing the specific phase transition and distinguishing the phases involved and its correlation functions, as well as the dependence on the external fields serving as control parameters (see CATASTROPHE THEORY).

Furthermore, near a critical point in the phase diagram the fluctuations of the order parameter become increasingly strong and long-range correlations emerge in the system, although the interactions of the particles are genuinely short-ranged. Therefore any approximative approach which neglects these fluctuations entirely, or merely treats them in some perturbative manner, is at least questionable.

On the other hand, because of the almost entirely collective behavior in the vicinity of a critical point, the microscopic details of the system become irrelevant there, and its long-wavelength (and low-frequency) properties tend to display *universality*, i.e.: in some asymptotic regime they depend solely on the symmetry and number of components of the order parameter, the space dimension, and possibly on the long-range nature of the interactions. Historically, this emergence of universal behavior has provided one of the essential keys for a mathematical description and profound theoretical understanding of critical phenomena, in the framework of *renormalization group* (RG) theory. In turn, the RG approach has laid a solid foundation for the striking phenomena of scale invariance and univer-

ality near second-order phase transitions.

Originally, RG concepts were developed in the context of elementary particle quantum field theory. Within the theory of phase transitions and critical phenomena, scaling ideas had been developed before, but their sound justification was only provided when K.G. Wilson formulated a version of the RG approach designed to focus on the underlying structure of the nonlinear field theories involved, a procedure which also allowed for systematic and detailed calculations (Wilson and Kogut, 1974; Wilson, 1975). This considerable breakthrough has not only led to an enormous increase of the amount of knowledge and to a deeper understanding of the properties of condensed matter (see, e.g., Chaikin and Lubensky, 1994), but has in turn also provided important insights for the formulation and treatment of quantum field theories, from where it had originated (see UNIFIED FIELD THEORIES). For a thorough account of the historical development of the field, see CRITICAL PHENOMENA.

Remarkably, RG techniques, along with the ensuing notions of fixed points governing universal physical properties, relevant and irrelevant perturbations, the emphasis on the long-wavelength and low-frequency collective behavior, and the emergence of scaling laws have matured into a most important paradigm and tool well beyond the original realm of second-order phase transitions, and have found fruitful applications in a large variety of different fields of intensive current research (see Sec. 4).

1 PHASE TRANSITIONS

1.1 Phenomenology and Basic Concepts

At a phase transition, the free energy F (or some other appropriate thermodynamic potential, see STATISTICAL MECHANICS, CLASSICAL) is a non-analytic function of a control

parameter. According to a standard classification scheme introduced by P. Ehrenfest, a phase transition is called of n th order if at least one of the n th derivatives of F is discontinuous or singular at the transition, while all the $(n - 1)$ th derivatives are still continuous. In order to distinguish the different phases, a macroscopic *order parameter* ϕ is introduced, which is zero in one phase and finite in the other. If ϕ is given by the first derivative of the thermodynamic potential with respect to its conjugate external field h , its value will jump at a first-order transition; at a second-order transition, ϕ is to be a continuous function, but its derivatives will be singular or discontinuous.

To be specific, one may refer to a ferromagnetic material undergoing a second-order phase transition from the paramagnetic to the ferromagnetic phase at the critical (Curie) temperature T_c . The basic microscopic entities are the magnetic moments $\phi(x)$ at site x , also called local order parameter (in this article, x denotes a d -dimensional vector referring to a point either in a discrete lattice or a continuum). The macroscopic order parameter (the spontaneous magnetization) is then given by the thermal average $\phi = \langle \phi(x) \rangle$. In anisotropic crystals the local order parameter may assume just the two values $\phi(x) = \pm 1$ (referring to spin up and down, respectively). Accordingly, the (average) order parameter may point up or down. Whichever of the equivalent directions is chosen, the original discrete symmetry is spontaneously broken for $T < T_c$ (see Fig. 1a). In cubic crystals, the order parameter may be a three-dimensional vector $\phi(x)$, and point in an arbitrary direction. In this case, in the ordered phase a continuous (rotational) symmetry is broken.

The singularities in the vicinity of such continuous phase transitions generally assume the form of power laws, with certain *critical exponents* characterizing the transition. In case the temperature T serves as control parameter, with a second-order phase transition occurring at the critical temperature T_c , the most

important examples of such power laws, and associated exponents, are: the behavior of the order parameter near the critical point T_c [$\tau = (T - T_c)/T_c \ll 1$], see Fig. 1a,

$$\phi \propto (-\tau)^\beta \quad \text{for } T < T_c, \quad (1)$$

the divergence of the isothermal susceptibility on approaching the critical point (from both sides of the transition),

$$\chi = \left(\frac{\partial \phi}{\partial h} \right)_T \propto |\tau|^{-\gamma}, \quad (2)$$

the power law for the critical isotherm,

$$h \propto \phi^\delta \quad \text{for } T = T_c, \quad (3)$$

and the specific heat singularity in the vicinity of T_c ,

$$C_{h=0} \propto |\tau|^{-\alpha}. \quad (4)$$

Eq. (2) leads to a most remarkable behavior of the order parameter *correlation function*, defined as second cumulant (again, the brackets $\langle \dots \rangle$ denote the thermal average)

$$G(x - x') = \langle \phi(x)\phi(x') \rangle - \langle \phi \rangle^2, \quad (5)$$

(here translational symmetry has been assumed); namely from the fluctuation-response theorem in d dimensions,

$$\chi = \frac{1}{k_B T} \int d^d x G(x), \quad (6)$$

it becomes apparent that a diverging susceptibility implies correlations of infinite range. This feature can be described by two additional critical exponents η and ν , for the power-law behavior of the correlation function precisely at the critical point,

$$G(x) \propto \frac{1}{|x|^{d-2+\eta}} \quad \text{for } T = T_c, \quad (7)$$

and the temperature dependence of the *correlation length* (measuring the spatial extent of the correlations) in the vicinity of T_c

$$\xi \propto |\tau|^{-\nu}, \quad (8)$$

respectively. The growth of correlated regions upon approaching the critical point is illustrated in Fig. 2, where the spin up/down configurations are plotted for a two-dimensional Ising model at different temperatures.

It turns out that for temperatures in the vicinity of T_c , the so-called *asymptotic region*, Eq. (7) may be generalized to the scaling form

$$G(x, \tau) = \frac{1}{|x|^{d-2+\eta}} \hat{G}_\pm(|x|/\xi), \quad (9)$$

or, by introducing the Fourier transform of the correlation function, $G(k) = \int d^d x G(x) e^{-ikx}$,

$$G(k, \tau) = \frac{1}{|k|^{2-\eta}} \hat{g}_\pm(|k|\xi). \quad (10)$$

(The suffix “ \pm ” distinguishes between the cases $T > T_c$ and $T < T_c$.) The *scaling functions* $\hat{G}_\pm(y)$ and $\hat{g}_\pm(p)$ approach a constant value in the limits $y \rightarrow 0$ and $p \rightarrow \infty$, respectively (see Sec. 1.4).

Near a critical point, where the only relevant length scale is set by the correlation length ξ , the increasing size of correlated regions also causes a *critical slowing down* of the dynamics of the system. In fact, the characteristic frequency ω_c of the system is expected to obey a scaling law similar to Eq. (10), namely

$$\omega_c(k, \tau) = |k|^z \hat{\Omega}_\pm(|k|\xi), \quad (11)$$

with $\hat{\Omega}_\pm(p) \rightarrow \text{const.}$ for $p \rightarrow \infty$, such that at the critical temperature

$$\omega_c(k, 0) \propto |k|^z \quad \text{for } T = T_c, \quad (12)$$

with the *dynamic critical exponent* z (Hohenberg and Halperin, 1977; Enz, 1979). For example, in the case of purely relaxational dynamics ω_c may be identified with the inverse relaxation time t_0^{-1} ; thus by using (11) and $\hat{\Omega}_\pm(p) \rightarrow p^{-z}$ for $p \rightarrow 0$, the temperature dependence of the relaxation time is found to be $t_0 \propto \xi^z \propto |\tau|^{-z\nu}$.

In addition, certain universal features of the thermodynamic phases themselves can sometimes be characterized by power laws also.

Specifically, the cost in free energy Σ of creating a domain wall in a phase with broken symmetry is assumed to scale with its linear dimension L according to

$$\Sigma = \sigma L^\Theta, \quad (13)$$

where σ denotes the interfacial tension; a positive *stiffness exponent* Θ means that long-range order is stable in the phase under consideration, whereas for $\Theta < 0$ the system would spontaneously break up in domains (see, e.g., Fisher, 1993).

1.2 Examples

We have already referred to the phase transition from a paramagnetic to a ferromagnetic state, occurring at the Curie temperature T_c (in fact, the language used in this article is adopted from that particular physical realization). The order parameter is the macroscopic magnetization, χ is the magnetic susceptibility, and h the external magnetic field. A nonzero order parameter selects a particular direction; thus, depending on the Hamiltonian (see Sec. 2), the broken symmetry may be discrete (inversions) or continuous (rotations). At T_c , the order parameter sets in continuously (Fig. 1a), justifying the terminology “continuous transition”. In anticipation of the comparison with the liquid–gas transition, we note, however, that reverting the external magnetic field for $T < T_c$ leads to a discontinuous change of sign of the magnetization (Fig. 1b). The full phase diagram containing ϕ , h , and T , is depicted in Fig. 3a.

Probably the most familiar phase transitions are the solid–liquid and the liquid–gas transitions (Fig. 3b). For the latter, the density at the coexistence line minus the critical density may serve as an order parameter, which thus has a low- and high-density, i.e. a gas and liquid branch. The associated external field is the pressure P , the (isothermal) compressibility κ would have to be identified with the order parameter susceptibility, and the surface tension of the liquid defines

the stiffness. At the critical point, where the isotherm as a function of P and V has an inflection point, one may observe the critical power laws as stated in Eqs. (1)–(4) and (7), (8) above (e.g., the divergence of κ becomes apparent in the phenomenon of critical opalescence near T_c).

In everyday life, the liquid–gas transition is usually observed at constant pressure well below P_c . As the system is heated up, the density changes discontinuously as a function of the temperature. Consequently, the liquid–gas transition is frequently referred to as being of first order, and the critical point as the endpoint of the transition line where the distinction between liquid and gas disappears. The analogy between the liquid–gas and the ferromagnetic transitions may be exhibited vividly if one encloses a fluid of critical density ρ_c in a so-called Natterer tube. For $T > T_c$, there is a uniform fluid phase, which splits up into a liquid and a gas phase if the system is cooled down below T_c . These phases appear separated by a meniscus.

At the liquid–solid transition the translational symmetry is broken, and the order parameter may be chosen as the Fourier component of the density at a reciprocal lattice vector of the solid; this transition does not terminate at a critical point, however (Fig. 3b). Somewhat more subtle and diverse are structural transformations between different solid phases, where either suitable components of the displacement field, or of the strain tensor appear as order parameters (see PHASE TRANSITIONS, STRUCTURAL); and furthermore the various ordered phases of liquid crystals (see LIQUID CRYSTALS, STRUCTURE OF). In the former, the order parameter may be the polarization or staggered polarization at ferroelectric or antiferroelectric phase transitions, or some quantity characterizing the orientation of molecular groups. At elastic transitions it is a component of the strain tensor, specifying the deformation involved.

It should be noted that the phase diagrams of systems as different as ferromagnets and

fluids look remarkably similar near their respective critical points (Fig. 3). Again, there is a very rich variety of magnetically ordered systems (e.g., antiferromagnets, ferrimagnets, helical phases), the order parameters of which may be of a rather complicated structure (see MAGNETIC ORDERING IN SOLIDS). A powerful tool to study the various phases of solids is neutron scattering, which directly measures the Fourier transform of the correlation function (5). Moreover, time-dependent phenomena can be explored by inelastic scattering of neutrons and light, which renders the critical slowing down [Eq. (12)] of the collective excitations (see COLLECTIVE PHENOMENA IN SOLIDS) directly observable.

Finally, phase transitions to macroscopic quantum states, namely superfluidity and superconductivity, should be mentioned. The order parameter in these cases is a complex field Φ , which may be identified with a macroscopic wavefunction, and the broken symmetry is a gauge invariance with respect to the phase of Φ (see SUPERFLUIDITY: LIQUID HELIUM SYSTEMS, and SUPERCONDUCTIVITY, LOW TEMPERATURE). Nature also allows for most interesting interplays between several kinds of phase transitions, which can lead to so-called multicritical points, and fairly complicated phase diagrams (see, e.g., SUPERCONDUCTIVITY, HIGH TEMPERATURE).

1.3 Universality

Near critical points, the topologies of the phase diagrams of such diverse systems as a liquid-gas mixture and a ferromagnetic material look amazingly similar (Fig. 3). Moreover, both in computer simulations of simplified models, and in real experiments the critical exponents for the corresponding phase transitions turn out to be identical for a broad variety of physical systems, and to depend only on the number of components and symmetry properties of the order parameter, the space dimension, and possibly the character of the particle interaction, namely if it is long-

range (e.g.: Coulomb or dipolar forces). This surprising feature is usually termed as *universality*. The microscopic details of the strongly interacting many-particle system merely enter the prefactors of the ensuing power laws, and even ratios of such amplitudes are in fact universal numbers.

A clue for this most remarkable fact may be found in the divergence of the correlation length [Eq. (8)] at a continuous phase transition. Upon approaching T_c , ξ will asymptotically become the only relevant length scale in the system, dominating over all microscopic scales. Even though the phase transition itself is caused by some (usually short-range) interactions of the microscopic constituents, the ensuing large fluctuations will markedly reduce all dependences on the degrees of freedom acting on short length scales, like the underlying crystal structure, the interaction range (as long as it is short), etc. Therefore, the system's thermodynamic properties near T_c should be entirely determined by global features such as its overall symmetry and large-scale properties (this is why long-range interactions, though possibly weak, may become important). In this regime, the system will behave strongly collectively, and one expects universal laws to apply for its long-wavelength and low-frequency excitations.

1.4 Scaling Laws

Another striking experimental feature near a critical point is the apparent *scale invariance* of physical properties (Fig. 4 illustrates this spectacular effect for the specific heat of Helium 4 near the normal-superfluid transition). Again, this may be related to the diverging correlation length. The appropriate mathematical description for this phenomenon of scale invariance is that the free energy and the correlation functions assume the form of generalized homogeneous functions. In the case that the only relevant control parameters are the temperature and the external field, and using the definitions of the critical exponents

(also called critical indices), one is thus led to the formulation of the *scaling hypotheses* for the singular part of the free energy

$$F_s = A_{\pm} |\tau|^{2-\alpha} \hat{f}_{\pm}(h/|\tau|^{\beta\delta}) \quad , \quad (14)$$

and the correlation function, Eq.(9) [or Eq.(10)] (Fisher, 1974). In (14), the scaling function \hat{f}_{\pm} approaches a constant value in the limit $h \rightarrow 0$.

There are various ways to make Eq. (14) plausible. Firstly, it can be considered as a generalization of the mean-field results [see Sec. 2.3.1] to general critical exponents. Secondly, one may start from the observation that singularities are present only at $\tau = 0$ and $h = 0$. To what extent these become apparent, depends on the separation from the critical point, τ , and on the value of $h/|\tau|^{\beta\delta}$, i.e.: the ratio of the external field to the field equivalent of τ , namely $h_{\tau} = \phi^{\delta} = |\tau|^{\beta\delta}$. As long as $h \ll h_{\tau}$, the system is essentially in zero-field, and $F_s \approx A_{\pm} |\tau|^{2-\alpha} \hat{f}_{\pm}(0) \propto |\tau|^{2-\alpha}$. On the other hand, when τ gets so small that $|\tau| \leq h^{1/\beta\delta}$, the system is manifestly in a finite field. Any further decrease of τ will then not lead to any changes: F_s remains at the value it acquired for $|\tau| = h^{1/\beta\delta}$, i.e. $A_{\pm} h^{(2-\alpha)/\beta\delta} \hat{f}_{\pm}(1)$; that is to say, in the limit $\tau \rightarrow 0$ one must have $\hat{f}_{\pm}(y) \rightarrow y^{(2-\alpha)/\beta\delta}$, in order to cancel the singular dependence of F_s on τ .

The scaling forms (14) and (9), (10) incorporate the power laws Eqs.(1)–(4), (7), (8); moreover by taking suitable derivatives with respect to τ and h , several *scaling relations* connecting the different critical indices may readily be obtained, namely

$$\alpha = 2 - \beta(\delta + 1) \quad , \quad (15)$$

$$\gamma = \beta(\delta - 1) = \nu(2 - \eta) \quad . \quad (16)$$

Relations of the type (14) and (9) are called *scaling laws*, because they are invariant under the following scale transformations:

$$x \rightarrow x/b, \quad \xi \rightarrow \xi/b, \quad (17)$$

$$\begin{aligned} \tau &\rightarrow \tau b^{1/\nu}, & h &\rightarrow h b^{\beta\delta/\nu}, \\ F_s &\rightarrow F_s b^{(2-\alpha)/\nu}, & G &\rightarrow G b^{(d-2+\eta)/\nu}. \end{aligned} \quad (17)$$

If, in addition, one assumes that underlying these scale transformations there is a microscopic elimination procedure mapping the original system with lattice constant a and N sites to a new one with the same lattice constant a , but a reduced number Nb^{-d} degrees of freedom, one has

$$\frac{F_s(\tau, h)}{N} = b^{-d} \frac{F_s(\tau b^{1/\nu}, h b^{\beta\delta/\nu})}{Nb^{-d}} \quad , \quad (18)$$

which in turn implies the *hyperscaling relation*

$$2 - \alpha = d\nu \quad (19)$$

containing the space dimension d . Eqs. (15), (16), and (19) leave only *two independent static critical exponents* characterizing the transition.

For dynamic critical phenomena, one arrives at a *dynamic scaling hypothesis* (Hohenberg and Halperin, 1977), originally proposed in the context of the λ transition of Helium 4, by generalizing Eq. (10) to

$$G(k, \omega) = \frac{1}{|k|^{2-\eta}} \hat{g}_{\pm}[|k|\xi, \omega/\omega_c(k)] \quad (20)$$

for the frequency-dependent correlation function. Scaling laws of the form (14), (9), and (20) can be derived by the renormalization group theory (see Sec. 3.2.4).

2 MATHEMATICAL REPRESENTATIONS

2.1 Microscopic Hamiltonians

In view of the expected universality of critical phenomena, simplified models containing the essential physics, namely the symmetry properties of the order parameter and the nature of the interactions, have provided a useful means for theoretical studies. E.g., the *Ising*

Hamiltonian describes a scalar order parameter, $\phi = \langle \sigma_i \rangle$, where the spin variables $\sigma_i = \pm 1$ are defined on a d -dimensional lattice, interacting with each other via a pair interaction J_{ij} , and subject to an external field with local strength h_i ,

$$H[\sigma_i] = -\frac{1}{2} \sum_{i,j} J_{ij} \sigma_i \sigma_j - \sum_i h_i \sigma_i \quad (21)$$

Positive exchange interaction J amounts to a ferromagnetic coupling, tending to align the spins, while an antiferromagnetic $J < 0$ favors alternating spin orientation. Its range is often restricted to nearest-neighbor interaction, but could also be taken as long-range. In the simplest case, the nonzero J_{ij} are chosen as constant, but in order to describe disordered systems they may as well be drawn from a certain random distribution (*random-bond* Ising model). Similarly, for most applications the external field is assumed to be uniform, $h_i = h$, but may in certain circumstances also be considered a random variable (*random-field* Ising model). Once the Hamiltonian is specified, in principle all thermodynamic properties of the model can of course be derived from the partition sum (see STATISTICAL MECHANICS, CLASSICAL)

$$Z = e^{-F/k_B T} = \text{Tr} \left(e^{-H/k_B T} \right) \quad (22)$$

$$= \sum_{\{\sigma_i = \pm 1\}} e^{-H[\sigma_i]/k_B T} \quad (23)$$

and the definition of thermodynamic averages

$$\langle A \rangle = \frac{1}{Z} \text{Tr} \left(e^{-H/k_B T} A \right) \quad (24)$$

The short-range Ising model without disorder can be shown to display a second-order finite-temperature phase transition for all dimensions $d \geq 2$, whereupon the discrete symmetry $\sigma_i \leftrightarrow -\sigma_i$ is broken (see CRITICAL PHENOMENA). The statistical mechanics of lattice gases as well as binary liquid mixtures may be mapped onto Ising-type models; this

once more illustrates the intimate analogies of phase transitions in very different systems.

Similarly, a three-component vector order parameter is incorporated in the anisotropic *Heisenberg* model,

$$H = -\frac{1}{2} \sum_{i,j} \left[J_{ij}^{\perp} (S_i^x S_j^x + S_i^y S_j^y) + J_{ij}^{\parallel} S_i^z S_j^z \right] - \sum_i h_i S_i^z \quad (25)$$

(we use the notation S_i^{α} , $\alpha = x, y, z$ for the three-component spin variables here; of course, this may be readily generalized to an n -component order parameter), where the symmetry-breaking field h_i is taken along the z direction. For nearest-neighbor interactions, in the cases $J^{\parallel} > J^{\perp} > 0$ and $J^{\perp} > J^{\parallel} > 0$ the Hamiltonian (25) describes an uniaxial ferromagnet or planar ferromagnet, respectively; while in the special situation $J_{ij}^{\parallel} = J_{ij}^{\perp}$ Eq. (25) yields the isotropic Heisenberg model, and for $J_{ij}^{\parallel} = 0$ the xy model, the latter being appropriate for the description of superfluid Helium 4, for example (see Fisher, 1967; Ma 1976). As a consequence of the continuous rotational symmetry broken in the ordered phase, the Heisenberg and xy models may have long-range order only for $d > 2$ dimensions. The two-dimensional xy model displays another very interesting scenario, namely the Berezinskii-Kosterlitz-Thouless transition (Nelson, 1983): There appears a phase, although without long-range order, in which the correlation function decays as a power law. The system then disorders at higher temperatures via the creation and unbinding of pairs of topological defects (vortex-antivortex pairs of spin excitations). The relatively simple models introduced above have become of paradigmatic importance in the statistical mechanics of phase transitions. One should keep in mind, however, that in more complicated physical situations more elaborate models may be required.

2.2 Ginzburg–Landau Functional

Near a continuous phase transition, predominantly long-wavelength properties will be of significance, due to the divergence of the correlation length according to Eq. (8). Therefore a continuum model for just the long-wavelength degrees of freedom should be sufficient and appropriate, at least for the description of the universal features. We remark that this concept of an *effective (field) theory* is of course defined on a certain energy scale, which in this context is set by the transition temperature. On different scales, the corresponding effective models may differ considerably, and one may in fact imagine them to be generated by a sequence of symmetry-breaking phase transitions occurring at successively lower energies.

Such an effective model for the long-wavelength degrees of freedom may either be derived from a microscopic Hamiltonian, such as (21) or (25), by mapping the partition function on a continuum problem and performing a gradient expansion; or merely be written down by carefully exploiting symmetry arguments. Let us denote the n -component vector order parameter by ϕ . For the case of a second-order phase transition breaking an $O(n)$ symmetry, the effective free energy, also called Hamiltonian in the context of critical phenomena, which retains only the lowest orders in the gradients and nonlinear terms, is the *Ginzburg–Landau free energy functional* (in d dimensions)

$$\mathcal{F}[\phi] = \int d^d x \left[\frac{r}{2} \phi^2 + \frac{u}{4} (\phi^2)^2 + \frac{1}{2} (\nabla \phi)^2 - \mathbf{h} \phi \right], \quad (26)$$

where $r = a(T - T_c)$, and a, u are positive coefficients, while \mathbf{h} is the (n -component) external field. The coefficient of the gradient term in (26) was assumed to be positive, assuring that spatially uniform configurations be energetically preferred to inhomogeneous ones, and has then been absorbed in a redefinition

of the field variable ϕ . For $n = 1$ the Hamiltonian (26) has the discrete symmetry of the Ising model (21) (i.e., for vanishing h), with a scalar order parameter, while for $n = 2$ and $n = 3$ the continuous symmetries of the xy model and the isotropic Heisenberg model are recovered.

Again, the thermodynamics can be retrieved from the partition function, which mathematically assumes the form of a functional integral of the probability distribution $\mathcal{P} \propto \exp(-\mathcal{F}/k_B T)$ over the vector field $\phi(x)$

$$Z = Z_0(T) \int \mathcal{D}[\phi(x)] e^{-\mathcal{F}[\phi]/k_B T} \quad (27)$$

Here the term Z_0 contains those contributions that are not explicitly stated in the Ginzburg–Landau functional (26), and which should be insignificant near the transition. Fig. 5 depicts the probability distribution \mathcal{P} in the case of a uniform one-component order parameter ϕ , containing the first two terms of the integrand in (26), for $r > 0$, and $r < 0$, respectively (the phase transition occurs at $r = 0$). We note that a very convenient way to calculate correlation functions uses derivatives with respect to the components of \mathbf{h} , e.g.:

$$G_{\alpha\beta}(x - x') = \frac{(k_B T)^2 \partial^2 \ln Z[\mathbf{h}]}{\partial h_\alpha(x) \partial h_\beta(x')} \quad (28)$$

2.3 Landau–Ginzburg Approximation

2.3.1 Mean-Field Theory. Evaluating the functional integral (27) is a highly nontrivial task, and in general one is forced to resort to some approximation. Because of the exponential in the integrand, one might expect the probability distribution to be sharply peaked, and the thermodynamics of the system thus to be dominated by the most probable configuration, which is given by the solution to the stationarity equation

$$\begin{aligned} 0 &= \frac{\delta \mathcal{F}}{\delta \phi(x)} \\ &= [r + u \phi(x)^2 - \nabla^2] \phi(x) - \mathbf{h}(x) \end{aligned} \quad (29)$$

In order to determine the homogeneous order parameter $[\nabla\phi(x) = 0]$ as a function of T , one must evaluate the Landau-Ginzburg equation (29) for $\mathbf{h} = h\mathbf{e}_n \rightarrow 0$ (note the order of limits here: *first* take the thermodynamic limit, and only *then* let the symmetry-breaking field vanish; with the reverse procedure there would be no nonzero order parameter for symmetry reasons, because one would have to average over the degenerate ground states). This yields

$$\phi = \begin{cases} 0 & T \geq T_c \\ \pm\sqrt{|r|/u}\mathbf{e}_n & T < T_c \end{cases}, \quad (30)$$

and inserting (30) into Eq. (26) one arrives at the following approximative result for the singular part of the free energy F_s ,

$$F_s[\phi] = \begin{cases} 0 & T \geq T_c \\ -r^2 L^d / 4u & T < T_c \end{cases} \quad (31)$$

(here L^d is the volume of a cube in d -dimensional space). By comparison with (1) and (4), one thus immediately finds the critical exponents $\beta = 1/2$, and $\alpha = 0$; more precisely, the specific heat displays a discontinuity at the transition temperature T_c , the height of which is

$$\Delta C_{h=0} = T_c a^2 L^d / 2u. \quad (32)$$

Similarly, for finite external field Eq. (29) leads to the equation of state in terms of the order parameter component ϕ_n parallel to the symmetry-breaking field $\mathbf{h} = h\mathbf{e}_n$

$$h = (r + u\phi_n^2)\phi_n, \quad (33)$$

from which the exponent for the critical isotherm is readily obtained to be $\delta = 3$. Finally, differentiating (33) with respect to h yields the isothermal susceptibility

$$\chi = \begin{cases} 1/r & = C_+(T - T_c)^{-1} & T > T_c \\ 1/2|r| & = C_-(T_c - T)^{-1} & T < T_c \end{cases}, \quad (34)$$

and hence the critical exponent $\gamma = 1$ (on both sides of the transition), as well as the also universal amplitude ratio $C_+/C_- = 2$. The above

results for the critical indices coincide with those of the mean-field approximation for the Ising and Heisenberg models, and with the exponents found for the van-der-Waals gas near its critical point. [In the mean-field approximation for the Ising model (21), for instance, the exchange field $\sum_j J_{ij}\sigma_j$ acting on spin σ_i is replaced by its average $\sum_j J_{ij}\langle\sigma_j\rangle$.]

We note that if u in Eq. (26) became negative, which may happen as some external parameter, e.g. the pressure, is varied, additional terms would have to be included in the Ginzburg-Landau functional for stability reasons. In the case of an underlying $O(n)$ symmetry, the leading such nonlinearity would be $\propto (\phi^2)^3$, and the ensuing phase transition of first order. Its transition temperature $T_1 \neq T_c$ is to be defined by the condition that the free energies of the distinct coexisting phases be equal. In the special case that u vanishes, T_c becomes a tricritical point, displaying most interesting and complex scaling behavior.

2.3.2 Gaussian Fluctuations. In keeping the most probable state only, fluctuations have been neglected entirely so far. However, the divergence of the susceptibility, Eq.(34), implies that fluctuations must indeed become very strong near the critical point. The next step is therefore to check whether the above results are altered by including fluctuations to lowest order. Thus we introduce the deviations $\psi(x)$ from the most probable state $\phi(x)$, given by the solution to Eq. (29), by writing the order parameter field as $\phi(x) + \psi(x)$, and expand the Ginzburg-Landau functional (26) with respect to $\psi(x)$, keeping terms up to second order only. This is most conveniently done in Fourier space, where we define the Fourier transform of $\psi(x)$ as

$$\psi(x) = L^{-d/2} \sum_{k \in B} \psi_k e^{ikx}; \quad (35)$$

here B denotes the Brillouin zone for the allowed values of the k vectors. The partition

function (27) then becomes

$$Z = Z_0(T) \int \prod_k d\psi_k e^{-\mathcal{F}_G[\psi]/k_B T} \quad (36)$$

Beginning with the situation above T_c , for $h = 0$, the Fourier expansion renders the Ginzburg–Landau functional diagonal, $\mathcal{F}_0 = \sum_k (r + k^2) \psi_k \psi_{-k} / 2$. The functional integrals in the partition function thus reduce to simple Gaussian integrals, yielding for the free energy in *Gaussian approximation*

$$F_s(T, h = 0) = -k_B T \frac{n}{2} \sum_k \ln \frac{2\pi k_B T}{r + k^2}, \quad (37)$$

and by transforming the discrete sum over the Brillouin zone into an integral, the most singular part of the specific heat follows as

$$C_s = k_B L^d \frac{n}{2} (Ta)^2 \int \frac{d^d k}{(2\pi)^d} \frac{1}{(r + k^2)^2} \quad (38)$$

The result of evaluating the integral is

$$C_s = \begin{cases} A_+(T - T_c)^{-(4-d)/2} & d < 4 \\ \propto \ln(T - T_c) & d = 4 \\ B - A(T - T_c)^{(d-4)/2} & d > 4 \end{cases} \quad (39)$$

hence while for $d > 4$ the specific heat shows a (non-analytical) cusp at T_c , it diverges for $d \leq 4$, with the exponent $\alpha = (4 - d)/2$; here an exponent “0” is to be understood as a logarithmic singularity. Neglecting deviations from the most probable state therefore turns out to be entirely unjustifiable in the latter case, where the thermodynamics is obviously dominated by the order parameter fluctuations.

Similarly, in the ordered phase one finds

$$C_s = TL^d \frac{a^2}{2u} + A_-(T_c - T)^{-(4-d)/2} \quad (40)$$

for $d < 4$; the critical exponent α is therefore the same above and below the transition. The universal (i.e.: depending only on

d and n) amplitude ratio for the specific heat is $A_+/A_- = n/2^{d/2}$, where the factor n stems from the fact that the $n - 1$ transverse fluctuations (with respect to the direction of the spontaneous order parameter) do not contribute to the specific heat in the ordered phase.

With the help of the result (40) the *Ginzburg–Landau criterion* (see Als–Nielsen and Birgeneau, 1977) may be formulated, stating that fluctuations are negligible as long as the first term ΔC dominates over the second contribution; introducing $\xi_0 = (aT_c)^{-1/2}$ this means that the Landau–Ginzburg approximation gives essentially correct answers, provided

$$|\tau|^{(4-d)/2} \gg \frac{\mathcal{N}}{(\xi_0 L)^d} \Delta C \quad (41)$$

where \mathcal{N} is a numerical factor. Therefore, for $d > d_c = 4$ fluctuations are not very important [$\alpha_H < 0$, Eq. (39)], and mean-field type of approaches yield reliable results. Below this *upper critical dimension*, however, such approximations will break down in the vicinity of T_c . Depending on the value of ξ_0 , the non-classical scaling behavior (with exponents different from the mean-field indices) will be important only very near to the critical point, or already noticeable even far away from it.

For example, for magnets typically $\xi_0 \approx 1\text{\AA}$, and fluctuations become important for $|\tau| < \tau_{GL} \approx 0.01$. For the normal–superfluid transition of Helium 4, $\xi_0 \approx 4\text{\AA}$, and $\tau_{GL} \approx 0.3$, which renders this system a prominent candidate for the study of critical phenomena (also, by varying the pressure one can explore a line of critical points, and thus examine universality issues in detail). On the other hand, for conventional (low- T_c) superconductors ξ_0 is of the order 1000\AA , and $\tau_{GL} \approx 10^{-10} - 10^{-14}$, which means that the Ginzburg–Landau theory is an excellent approximation there. This is in remarkable contrast with the high- T_c materials, where $\xi_0 \approx 1\text{\AA}$, and fluctuations play a dominant role; in fact, they account for an enormously rich phase diagram of the cuprate based superconductors (see Blatter, Feigel’man, Geshkenbein,

Larkin, and Vinokur, 1994).

For $|\tau| < \tau_{GL}$ it does not at all suffice to add to the results of the Ginzburg–Landau approximation the Gaussian corrections, since in this temperature range higher-order nonlinear contributions will be even more infrared-divergent and eventually exceed those from the harmonic term. This constitutes of course a serious breakdown of the standard perturbational procedure, and signals that an entirely new theoretical approach is required in the case $|\tau| < \tau_{GL}$.

For finite external field the partition function [Eq. (36)] becomes above T_c

$$Z[\mathbf{h}] = \exp \left[\frac{1}{2k_B T} \sum_k \frac{\mathbf{h}_k \mathbf{h}_{-k}}{r + k^2} \right], \quad (42)$$

and the correlation function can be obtained by differentiating this generating functional [see Eq. (28)]. Above T_c , the result in k space is

$$\langle \psi_k^\alpha \psi_{-k}^\beta \rangle = \frac{k_B T}{r + k^2} \delta_{kk'} \delta^{\alpha\beta}; \quad (43)$$

consequently, for $r = 0$ one finds in real space

$$G(x) \propto \frac{1}{|x|^{d-2}}, \quad (44)$$

and thus the critical exponent defined in (7) is $\eta = 0$ in the Landau–Ginzburg approximation. Also, the inverse of r defines the square of a characteristic length, which we identify with the correlation length, $\xi_+ = r^{-1/2}$, and hence $\nu = 1/2$. In three dimensions the Fourier transform of (43) is the *Ornstein–Zernicke correlation function*

$$G(x) = \frac{k_B T}{4\pi|x|} e^{-|x|/\xi_+}. \quad (45)$$

In the ordered phase, one has to distinguish between the longitudinal and transverse correlation functions, with respect to the spontaneous order parameter, which are still diagonal in the indices α, β , and for the $O(n)$ symmetric model (26) in Gaussian approximation read

$$G_{\parallel}(k) = \frac{k_B T}{\xi_-^{-2} + k^2}, \quad (46)$$

and

$$G_{\perp}(k) = \frac{k_B T}{k^2}, \quad (47)$$

respectively. Hence on this level $G_{\parallel}(k)$ is of the form (43), with a different correlation length $\xi_- = (2|r|)^{-1/2}$, however. Note that the amplitude ratio for the correlation lengths above and below the transition is universal again, with the mean-field value $\sqrt{2}$. Remarkably, the transverse correlation function diverges for $k \rightarrow 0$, which means that the range for the transverse correlations is infinite in the entire low-temperature phase. This can be shown to be a rigorous consequence of the breaking of the continuous rotational symmetry. For $n \geq 2$ and $d \leq 4$ the large transverse fluctuations (47) via nonlinear effects imply a longitudinal correlation function surpassing the result obtained within the harmonic approximation (46), which has to be replaced by

$$G_{\parallel}(k) \propto \frac{1}{k^{4-d}}. \quad (48)$$

Remarkably, Eq. (48) is a universal power law characterizing not a phase transition singularity, but an *entire ordered phase*. Yet the longitudinal correlation length ξ_- still retains some physical meaning; namely, it provides the scale on which the crossover from the critical behavior (7) to the asymptotic form (48) occurs.

In the Ising case ($n = 1$), Eq. (46) is qualitatively correct outside the critical region, and ξ_- sets the characteristic length scale in the ordered phase. For instance, a domain wall configuration $\phi_K(x)$ connecting the two degenerate ground states $\phi_{\pm} = \pm\sqrt{|r|/u}$, and centered at x_0 , extends over the range ξ_- ,

$$\phi_K(x) = \phi_{\pm} \tanh \frac{x - x_0}{2\xi_-}, \quad (49)$$

as can be seen by solving the Landau–Ginzburg differential equation (29) for $h = 0$, allowing for a spatially varying order parameter with the appropriate boundary conditions.

3 RENORMALIZATION GROUP

3.1 Introductory Remarks

The term “renormalization” of a theory stands for a certain reparametrization, with the goal that the resulting renormalized theory be more tractable than the original one. Historically, renormalization was invented by P. Stückelberg and R.P. Feynman, in order to cure quantum field theories (e.g., quantum electrodynamics) from divergences (see UNIFIED FIELD THEORIES). Instead of the bare parameters (masses and coupling constants) the Lagrangian was expressed in terms of physical masses and coupling coefficients, such that ultraviolet-divergent (UV) virtual processes were systematically absorbed in the relationship between the bare and physical quantities, thus rendering the renormalized theory finite. The renormalization procedure is not unique, the renormalized quantities may for instance depend on a cutoff length scale up to which certain virtual processes are taken into account. The *renormalization group* (RG) theory studies the dependence on this specific length scale, also called *flow parameter*. The name renormalization group derives from the fact that two successive RG transformations lead to a third such transformation.

In the context of critical phenomena, where one needs to explain the long-distance (or, in Fourier space, small-wavenumber) regime, it is natural to implement the renormalization scheme by performing some kind of elimination of the short-wavelength fluctuations. Such a partial evaluation of the partition function will be much easier to achieve than a calculation of the complete partition sum, and also permits the systematic implementation of approximative methods. As a result of the elimination step, the remaining degrees of freedom will experience modified *effective* interactions.

Quite generally, various benefits can be expected from such a RG procedure: (i) The new

coupling coefficients might be smaller. By repeating the renormalization process, one may thus arrive at an effectively free theory with no interactions. (ii) The successively iterated coupling coefficients, also called the *parameter flow*, may have a finite *fixed point*, at which the system does not change any more under subsequent RG transformations. Since the elimination of some degrees of freedom is accompanied by a change of the underlying lattice spacing, or length scale, one may anticipate that the fixed points are under certain circumstances related to critical points. Furthermore, one may hope that the properties of the flow near these fixed points will yield information about the universal physical quantities in the vicinity of the critical point. Below, we shall find scenario (i) in the case of the one-dimensional Ising model, and (ii) for the Ising model in two dimensions, as well as for the Ginzburg–Landau–Wilson effective field theory (ϕ^4 model).

3.2 Real Space Renormalization Group

The idea of any RG approach in statistical mechanics is to exploit the scale invariance near a critical point in parameter space. In real space RG transformations, which refers to procedures working in direct space (as opposed to Fourier space) one eliminates specific degrees of freedom that are defined on a lattice, thus performing a partial partition sum. Then the lattice constant of the ensuing system is readjusted, and also the internal variables are renormalized, in order to achieve a new Hamiltonian whose form resembles the original one. By comparison, effective couplings are defined, which thus become scale-dependent. The fundamental paradigm is that certain fixed points in parameter space exist, at which these running couplings remain invariant. At these fixed points the theory is manifestly scale-invariant, and the critical exponents can be obtained by studying the linearized flow equations. As this concept of the

RG may seem rather abstract at first sight, the method shall be explained by applying a simplified version to the ferromagnetic Ising model (21) in $d = 1$ and $d = 2$ dimensions (Niemeijer and van Leeuwen, 1976). Then a more general discussion, including an explanation of the emergence of scaling laws, will be given. A continuous field-theoretical formulation for the RG in momentum space shall be presented in Sec. 3.3.

3.2.1 1d Ising Model. For a one-dimensional Ising chain with constant ferromagnetic nearest-neighbor exchange coupling $J_{ij} = J > 0$ in zero external field the Hamiltonian (21) becomes $H = -J \sum_i \sigma_i \sigma_{i+1}$, and introducing the abbreviation $K = J/k_B T$ the partition function (23) reads (for N spins, and periodic boundary conditions $\sigma_{N+1} = \sigma_1$)

$$Z_N = \sum_{\{\sigma_i = \pm 1\}} \exp \left[K \sum_i \sigma_i \sigma_{i+1} \right] . \quad (50)$$

A partial evaluation of (50) may be performed by summing over every other spin in the chain, marked by a “x” in Fig. 6. The sum over the possible orientations of spin i , for instance, yields,

$$\begin{aligned} \sum_{\sigma_i = \pm 1} e^{K \sigma_i (\sigma_{i-1} + \sigma_{i+1})} &= \\ &= \exp [2g' + K' \sigma_{i-1} \sigma_{i+1}] , \end{aligned} \quad (51)$$

for the result can only depend on whether the adjacent spins σ_{i-1} and σ_{i+1} are oriented parallel or antiparallel to each other. By comparing the results for these two configurations, $\sigma_{i-1} \sigma_{i+1} = \pm 1$, one readily obtains the two recursion relations

$$K' = \frac{1}{2} \ln \cosh 2K , \quad (52)$$

$$g' = \frac{1}{2} (\ln 2 + K') . \quad (53)$$

Thus, via decimation of half of the spins in the chain one arrives at another Ising Hamiltonian, whose nearest-neighbor coupling is K' ;

in addition, a spin-independent contribution g' to the free energy is generated. If this transformation is repeated, the *exact* recursion relations (52) and (53) quite generally describe how the effective couplings $K^{(k)}$ and $g^{(k)}$ in the k th step result from the corresponding values in the $(k-1)$ th elimination step. In Fig. 7 the ensuing flow for the coupling K , starting with some initial value K_0 , is depicted for several steps of the procedure. Clearly, the fixed point equation $K^* = (1/2) \ln \cosh 2K^*$ has two solutions: $K^* = 0$, which (for the original fixed J) corresponds to infinite temperature, or vanishing effective interaction; and $K^* = \infty$, the $T = 0$ fixed point. By starting at some initial value $K_0 > 0$ the parameter flow always drives the system into the $K^* = 0$ fixed point [by expanding Eq. (52) around $K^* = 0$ this fixed point is seen to be stable]. Physically, this means that there can be no ordered state at any finite temperature $T > 0$, because asymptotically the ratio of exchange coupling and temperature vanishes, and the system becomes completely disordered.

At $T = 0$, however, the situation is rather special. Expanding (52) near $1/K^* = 0$ yields $1/K' = 1/K + \ln b/(2K^2)$, where $b = 2$ is just the factor rescaling all lengths. It now turns out to be useful to introduce $\delta = \ln b$, such that after ℓ steps $b^\ell = e^{\delta \ell}$, and then take the limit $\delta \rightarrow 0$; thereby the discrete RG steps are mapped onto a continuous flow with the flow parameter ℓ . The resulting differential flow equation for the now scale-dependent coupling $K(\ell)$ reads to lowest order

$$\frac{dK^{-1}(\ell)}{d\ell} = \frac{1}{2} [K^{-1}(\ell)]^2 , \quad (54)$$

which is readily integrated to $K(1) - K(\ell) = \ell/2$. From this the temperature dependence of the correlation length ξ can be determined by applying the (“matching”) condition that $K(b = \xi)$ be a constant of order 1. This gives

$$\xi \propto \exp(2J/k_B T) , \quad (55)$$

i.e.: the correlation length diverges exponentially at the *marginal* critical point $T = 0$, a

result which may also be readily obtained by more elementary methods.

Using the above recursion relations, the partition function may be evaluated:

$$Z_N = e^{Ng(K')} Z_{N/2}(K') = \dots = \quad (56)$$

$$= \exp \left[N \sum_{k=1}^n \frac{g(K^{(k)})}{2^{k-1}} + \ln Z_{N/2^n}(K^{(n)}) \right].$$

The factor 2 in Eq. (56) can be interpreted as stemming from a readjustment of the overall scale, as the number of degrees of freedom is being diminished by the elimination procedure. The reduced free energy (in units of $k_B T$) thus becomes $f = -(1/N) \ln Z_N(K)$. As is apparent from Fig. 7, $K^{(k)}$ rapidly becomes very small, and perturbative methods may be employed. Assuming $K^{(n)} \approx 0$, one arrives at the approximative result

$$f^{(n)}(K) = - \sum_{k=1}^n \frac{g(K^{(k)})}{2^{k-1}} - \frac{1}{2^n} \ln 2 \quad , \quad (57)$$

for in a non-interacting spin 1/2 system the free energy per spin is $-\ln 2$. Fig. 8 shows the rapid approach towards the exact result, $f(K) = -\ln(2 \cosh K)$, as the number of elimination steps is increased.

3.2.2 2d Ising Model. More interesting is the case of the two-dimensional Ising model, which is known to have a phase transition at a finite temperature $T_c > 0$ (see CRITICAL PHENOMENA). Here a quadratic lattice with nearest-neighbor ferromagnetic interactions will be considered (Fig. 9). In order to find the recursion relations in this case, one may perform the partial partition sum over those spins marked with a "x" in Fig. 9, which leads to a new square lattice with a lattice constant larger by a factor of $\sqrt{2}$ than that of the original one. A typical sum is

$$\sum_{\sigma=\pm 1} e^{K\sigma(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} = \quad (58)$$

$$= e^{A'+K'(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_1\sigma_4)} \times$$

$$\times e^{L'(\sigma_1\sigma_3+\sigma_2\sigma_4)+M'\sigma_1\sigma_2\sigma_3\sigma_4} \quad ;$$

thus some new interactions, namely a next-nearest-neighbor coupling L' and a four-spin interaction M' are *generated* by the transformation. In the vicinity of the actual critical value $K_c = J/k_B T_c = 0.4406$ one finds $M' \ll L' \leq K'$. Therefore, M' shall henceforth be neglected here (a more thorough perturbational analysis is given in Niemeijer and van Leeuwen, 1973). A' , K' , and L' can readily be expressed by K . If these relations are expanded to second order in K , and if one realizes that a next-nearest-neighbor interaction L in the original Hamiltonian appears as a contribution to the nearest-neighbor-interaction in the primed Hamiltonian, one obtains

$$K' = 2K^2 + L \quad , \quad (59)$$

$$L' = K^2 \quad . \quad (60)$$

A simple physical picture behind these approximate recursion relations is that they just represent the numbers of paths between the remaining spins, weighted by the product of the bonds involved.

It should be clear, however, that with each step of the elimination new couplings appear, as opposed to the one-dimensional case. Of course, restricting the flow to the subspace (K, L) will not produce quantitatively satisfactory results, but still the structure and typical features of such recursion relations are preserved.

The fixed points of the set of equations (59) and (60) are (i) $K^* = L^* = 0$, (ii) $K^* = L^* = \infty$, and (iii) $K_c^* = 1/3$, $L_c^* = 1/9$ (Fig. 10). As in the 1d Ising model, fixed points (i) and (ii) refer to infinite temperature (disordered phase) and $T = 0$, respectively. Yet now the latter describes an entire (ordered) phase. Furthermore, there is an additional nontrivial finite fixed point, which governs the second-order transition separating these two phases. For, at each step in the procedure the effective lengths are reduced by a factor $b = \sqrt{2}$, which applies to the correlation length also, and therefore at a fixed point either $\xi = 0$ (which is true for the $T = 0$ and $T = \infty$ fixed points), or $\xi = \infty$ (for finite values of K^* and

L^*), which implies scale-invariance. Therefore the fixed point (K_c^*, L_c^*) with diverging correlation length describes the critical state. Even more, actually all points lying on a trajectory leading into the fixed point represent critical behavior, because on these paths the correlation length is infinite as well.

In order to determine the critical behavior of the model, the recursion relations (59), (60) are linearized near the point (K_c^*, L_c^*) in parameter space; with $\delta K_\ell = K_\ell - K_c^*$ and $\delta L_\ell = L_\ell - L_c^*$ in the ℓ th step of the procedure, one finds

$$\begin{pmatrix} \delta K_\ell \\ \delta L_\ell \end{pmatrix} = \begin{pmatrix} 4/3 & 1 \\ 2/3 & 0 \end{pmatrix} \begin{pmatrix} \delta K_{\ell-1} \\ \delta L_{\ell-1} \end{pmatrix} . \quad (61)$$

The eigenvalues of the transformation matrix are $\lambda_{1,2} = (2 \pm \sqrt{10})/3$, and the corresponding eigenvectors $\mathbf{e}_{1,2} = (3, \pm\sqrt{10}-2)$. Thus, if the original couplings are expanded in the basis of eigenvectors,

$$\begin{pmatrix} K_0 \\ L_0 \end{pmatrix} = \begin{pmatrix} K_c^* \\ L_c^* \end{pmatrix} + c_1 \mathbf{e}_1 + c_2 \mathbf{e}_2 , \quad (62)$$

after ℓ steps one arrives at

$$\begin{pmatrix} K_\ell \\ L_\ell \end{pmatrix} = \begin{pmatrix} K_c^* \\ L_c^* \end{pmatrix} + \lambda_1^\ell c_1 \mathbf{e}_1 + \lambda_2^\ell c_2 \mathbf{e}_2 . \quad (63)$$

As is depicted in the flow diagram (Fig. 10), if $c_1 \neq 0$, the flow will in general run away from the critical fixed point (K_c^*, L_c^*) , which is thus unstable towards the *relevant perturbation* $c_1 \mathbf{e}_1$, which is characterized by $|\lambda_1| > 1$. On the other hand, if $c_1 = 0$, the flow will always tend towards this fixed point, because $|\lambda_2| < 1$ (\mathbf{e}_2 is called the *irrelevant* direction; had we included more coupling parameters initially, additional irrelevant directions would have emerged). The original nearest-neighbor model had $L_0 = 0$. Inserting this in Eq. (62) for $c_1 = 0$ one finds the critical value of K (within the linear approximation), for which the decimation transformation will lead into the critical fixed point, to be

$$K_c = \frac{1}{3} \left(1 + \frac{1}{\sqrt{10} + 2} \right) \approx 0.3979 , \quad (64)$$

which should be compared to the mean-field and exact values, $K_c = 0.25$ and $K_c = 0.4406$, respectively. Note that $K - K_c = -J(T - T_c)/k_B T T_c$, and hence $c_1 \neq 0$ whenever $T \neq T_c$. In this case, the flow will tend towards either the zero temperature (for $T < T_c$) or the disordered fixed point (for $T > T_c$).

After a sufficient number of steps λ_2^ℓ will be minute, and

$$(K - K_c)_\ell = b^{y_1 \ell} (K - K_c) , \quad (65)$$

where $b = \sqrt{2}$, and $y_1 = 2 \ln \lambda_1 / \ln 2 \approx 1.566$. As also $\xi_\ell = b^{-\ell} \xi$, and the definition of the critical exponent ν for the correlation length translates to $\xi \propto |K - K_c|^{-\nu}$, one finds

$$\nu = 1/y_1 \approx 0.638 , \quad (66)$$

which is already larger than the mean-field result $\nu = 1/2$, but still far off the exact value $\nu = 1$.

Similarly, an external field h may be included in the scheme. The approximate recursion relation for h turns out to be

$$h' = (1 + K)h , \quad (67)$$

with the fixed point $h^* = 0$, and the eigenvalue for the linearized recursion relation is $\lambda_h = 1 + K_c^* = 4/3$. Hence the external field constitutes a second relevant variable besides (essentially) $T - T_c$. Notice that the flows for K and h decouple. The two relevant fields τ and h precisely correspond to the two independent critical exponents of the model.

In addition, the universal properties of the phases, as the stiffness exponent Θ in Eq. (13), can be determined in a similar way. For the stiffness, the most convenient way is to study the change in free energy of a finite system as the boundary conditions are changed, e.g. from periodic to antiperiodic (see Fisher, 1993). For an Ising system of linear dimension $L = b$ in d dimensions, the domain wall energy at $T = 0$ turns out to be

$$\Sigma \approx 2Jb^{d-1} , \quad (68)$$

and hence $\Theta = d - 1$. Thus a rescaling of the length by b is equivalent to a recursion relation for the temperature

$$T' = b^{-\Theta} T \quad (69)$$

for the temperature, which shows that the stiffness exponent may be identified with the negative eigenvalue $\lambda_T = -\Theta$ at the zero-temperature fixed point ($T^* = 0$). Eq. (68) demonstrates that the ordered phase is stable for $d \geq 2$ in the Ising model. For Heisenberg models with continuous rotational symmetry, on the other hand, one finds $\Theta = d - 2$, and no long-range order is possible in two dimensions. Note that mean-field theory fails to predict such a *lower critical dimension*, at and below which fluctuations destroy the formation of spatially homogeneous long-range order.

3.2.3 General Real Space RG Transformations. A general real space RG transformation maps a certain spin system $\{\sigma\}$, defined on a lattice, and with Hamiltonian $H\{\sigma\}$, onto a new spin system $\{\sigma'\}$ with new Hamiltonian $H'\{\sigma'\}$ and by a factor of $N'/N = b^{-d}$ less degrees of freedom. It may be represented by a transformation $\tau\{\sigma', \sigma\}$, such that ($k_B T$ has been set to unity here)

$$e^{-F-H'\{\sigma'\}} = \sum_{\{\sigma\}} \tau\{\sigma', \sigma\} e^{-H\{\sigma\}} \quad , \quad (70)$$

with the conditions

$$\sum_{\{\sigma'\}} H'\{\sigma'\} = 0 \quad , \quad (71)$$

and

$$\sum_{\{\sigma'\}} \tau\{\sigma', \sigma\} = 1 \quad , \quad (72)$$

which guarantee that $e^{-F} \text{Tr}_{\{\sigma'\}} e^{-H'\{\sigma'\}} = \text{Tr}_{\{\sigma\}} e^{-H\{\sigma\}}$.

Important examples for such transformations are decimation transformations, and linear as well as nonlinear blockspin transformations. Several types of approximations, like the cumulant expansion, cluster and finite-lattice approximations have been applied to

various kinds of lattices (quadratic, triangular, etc.), where a large number of interaction coefficients have been included. In two dimensions, critical exponents and thermodynamic properties as a whole have thereby been obtained to an impressive degree of accuracy (Niemeijer and van Leeuwen, 1976). A combination of numerical Monte-Carlo methods and real-space RG transformations leads to the *Monte-Carlo Renormalization Group* (Swendsen, 1984). Apart from that, however, the more versatile momentum space methods (see Sec. 3.3 below) have been far more useful in higher dimensions ($d > 2$).

3.2.4 Emergence of Scaling Laws.

While the decimation procedure employing only a few parameters does not provide quantitatively satisfactory results, and also is inappropriate for the study of correlation functions, it still elucidates the general structure of RG transformations R , whereupon the original Hamiltonian H is mapped onto a new one, $H' = R(H)$ (this includes the rescaling of lengths in the problem, or for the degrees of freedom: $N' = Nb^{-d}$ in d dimensions). Near the fixed-point Hamiltonian $H^* = R(H^*)$ one may expand $R(H^* + \delta H) \approx H^* + L\delta H$, which yields the linearized recursion relation $\delta H' = L\delta H$. The eigenoperators $\delta H_1, \delta H_2, \dots$ for this linear transformation are then determined by the eigenvalue equations

$$L\delta H_i = \lambda_i \delta H_i \quad . \quad (73)$$

A given Hamiltonian H , which does not differ too much from the fixed-point Hamiltonian, may be expanded according to

$$H = H^* + \tau \delta H_\tau + h \delta H_h + \sum_{j \geq 3} c_j \delta H_j \quad , \quad (74)$$

where δH_τ and δH_h denote the two relevant perturbations, $|\lambda_\tau| = b^{y_\tau} > 1$, $|\lambda_h| = b^{y_h} > 1$, connected with the temperature variable $T - T_c$ and the external field h , while $|\lambda_j| < 1$ (and hence $y_j < 0$) for $j \geq 3$ (in the Ising case, $\delta H_h = \sum_i \sigma_i$). The coefficients τ , h , and c_j are called *scaling fields*.

After ℓ steps, the free energy per spin, $f(\{c_j\}) = F_N(\{c_j\})/N$, transforms according to

$$f(\tau, h, c_3, \dots) = b^{-d\ell} f(\tau b^{y_\tau \ell}, h b^{y_h \ell}, c_3 b^{y_3 \ell}, \dots) \quad (75)$$

in the linear approximation. (Here we have omitted an additive term, which does not affect the following derivation of the scaling law, but would of course be important for the evaluation of the total free energy.)

The scale parameter ℓ may now be chosen conveniently, for example $|\tau| b^{y_\tau \ell} = 1$, which sets the first argument of f to ± 1 ; this yields

$$f(\tau, h, \dots) = |\tau|^{d/y_\tau} \hat{f}_\pm(h |\tau|^{-y_h/y_\tau}, c_3 |\tau|^{y_3/y_\tau}, \dots), \quad (76)$$

where $\hat{f}_\pm(x, y, \dots) = f(\pm 1, x, y, \dots)$. Close to T_c , the dependence on the irrelevant fields c_3, \dots can be neglected, and (76) then precisely assumes the scaling form (14), with the exponents

$$\beta\delta = y_h/y_\tau, \quad (77)$$

and

$$2 - \alpha = d/y_\tau. \quad (78)$$

Thus the scaling law (14) has been derived within the RG theory for fixed points with precisely one relevant field, aside from the external magnetic field and the irrelevant operators. Furthermore, the dependence of the irrelevant fields c_3, \dots yields *corrections to scaling*, which may have to be taken into account in situations not yet close enough to T_c to be in the fully asymptotic regime.

In order to relate y_τ to the exponent ν , one has to recall that ℓ iterations reduce the correlation length to $\xi' = b^{-\ell} \xi$, which implies $(\tau b^{y_\tau \ell})^{-\nu} = b^{-\ell} \tau^{-\nu}$, and thus

$$\nu = 1/y_\tau \quad (79)$$

[see Eq. (66) for the 2d Ising model]. Thus, with the basic assumption that there exists a fixed point Hamiltonian with two relevant

operators, the scaling form of the free energy could be derived, and moreover, a prescription for the calculation of the critical exponents was provided. A similar procedure may be followed for the correlation function (9). At this point the renormalization of the spin variables, $\sigma' = b^\zeta \sigma$, becomes important; it turns out that the choice

$$\zeta = \frac{d-2+\eta}{2} \quad (80)$$

ensures the validity of Eq. (7) at the critical point.

We add some remarks on the generic structure of the flow diagram in the neighborhood of a critical fixed point (Fig. 11). In the multidimensional space of coupling coefficients, there will be one specific direction which leads away from the fixed point (the relevant direction). The other eigenvectors of the linearized RG transformation span the *critical hypersurface*. Further away from the fixed point the hypersurface is no longer a plane but acquires some curvature. From any point on the critical hypersurface the trajectories lead into the critical fixed point. If the starting point is close to but not at the critical hypersurface, the corresponding trajectory will first run parallel to the hypersurface until the relevant part will be magnified sufficiently, so that finally the trajectory leaves the vicinity of the critical hypersurface, and turns off to either the low- or high-temperature fixed point. For a particular physical system (a ferromagnet, a binary mixture, etc.), the parameters will depend on the temperature (the point-dashed curve in Fig. 11). The temperature for which this curve intersects the critical hypersurface constitutes its transition temperature T_c .

From the above discussion the universality property becomes obvious, too. All systems belonging to a particular part of parameter space, namely the region of attraction of a specific fixed point, are governed by the same scaling behavior, and are described by identical power laws in the vicinity of the critical hypersurface of the fixed point.

3.3 Momentum Shell Renormalization Group

3.3.1 Wilson's RG Scheme. The RG method shall now be applied to the continuum model (26), called *Landau-Ginzburg-Wilson functional* in this context. A very intuitive approach was proposed by K.G. Wilson (Wilson and Kogut, 1974; Ma, 1976). Basically, in momentum space the trace over the large k degrees of freedom is performed, and thus recursion relations for the Ginzburg-Landau coefficients are obtained. Because the short-wavelength fluctuations are not expected to contribute essentially, the Brillouin zone may be approximated simply by a d -dimensional sphere with radius (cutoff) Λ . The momentum shell RG transformation then consists of the following steps: (i) Performing the trace over all Fourier components ϕ_k with $\Lambda/b < |k| < \Lambda$ (Fig. 12) eliminates these short-wavelength modes. By means of a (ii) scale transformation

$$k' = bk \quad , \quad (81)$$

$$\phi' = b^\zeta \phi \quad , \quad (82)$$

and therefore

$$\phi'_{k'} = b^{\zeta-d} \phi_k \quad , \quad (83)$$

the ensuing effective Hamiltonian is brought to a form resembling the original model, by which then effective scale-dependent coupling parameters are defined. By successive application of this RG transformation (which constitutes a semigroup, as there is no inverse element) the supposedly universal features of the long-wavelength regime are disclosed. The fixed points of the transformation will, as above, correspond to the distinct thermodynamic phases and the phase transitions between them. The eigenvalues of the linearized RG flow equations near the critical fixed point finally yield the critical exponents [see (79), (77), and (78)].

Although a perturbational approach (with respect to u) is not at all justifiable in the critical region, it appears entirely sound in the

region far from the critical point, where the fluctuations are negligible. The important observation now is that the RG flow connects these very different regimes, and thus results of the perturbation expansion obtained in the noncritical region can be transported to the vicinity of T_c , keeping track of the nonanalytic singularities on the way in a consistent, controlled and reliable manner. Perturbative methods may also be employed in the elimination of the short-wavelength degrees of freedom [step (i)].

3.3.2 Gaussian Model. This concept shall now first be applied to the Gaussian model, where $u = 0$ (see Sec. 2.3.2),

$$\mathcal{F}_0[\phi_k] = \int_{|k| < \Lambda} \frac{r + k^2}{2} |\phi_k|^2 \quad (84)$$

[$\int_k = \int d^d k / (2\pi)^d$]. As (84) is diagonal in the Fourier modes, elimination of the large k components simply produces constants; the form of the effective Hamiltonian is then unaltered, provided

$$\zeta = \frac{d-2}{2} \quad , \text{ i.e. : } \eta = 0 \quad , \quad (85)$$

$$r' = b^2 r \quad . \quad (86)$$

The fixed points of Eq. (86) are $r^* = \pm\infty$ corresponding to the high- and low-temperature phases, respectively, and the critical fixed point $r^* = 0$. The eigenvalue in the relevant temperature direction at this critical fixed point is obviously $y_T = 2$, and therefore the exponent $\nu = 1/2$ is recovered, as in mean-field theory.

3.3.3 Perturbation Theory and ϵ Expansion. The nonlinear “interaction” term

$$\mathcal{F}_{\text{int}}[\phi_k] = \frac{u}{4} \int_{|k_i| < \Lambda} \phi_{k_1} \phi_{k_2} \phi_{k_3} \phi_{-k_1-k_2-k_3} \quad (87)$$

can now be treated perturbationally by expanding the exponential in Eq. (27) with respect to u . Separating the field variables into their parts in the outer and inner momentum

shell, respectively, $\phi_k = \phi_{k<} + \phi_{k>}$, with $|k_{<}| < \Lambda/b$ and $\Lambda/b < |k_{>}| < \Lambda$, to first order in u one arrives at terms of the following (symbolically written) forms (henceforth $k_B T$ shall be set to 1): (i) $u \int \phi_{<}^4 e^{-\mathcal{F}_0}$ just needs to be reexponentiated, for these degrees of freedom are not eliminated; (ii) any terms with odd numbers of $\phi_{<}$ or $\phi_{>}$ like $u \int \phi_{<}^3 \phi_{>} e^{-\mathcal{F}_0}$ vanish; (iii) $u \int \phi_{>}^4 e^{-\mathcal{F}_0}$ yields a constant, contributing to the free energy, and finally $u \int \phi_{<}^2 \phi_{>}^2 e^{-\mathcal{F}_0}$, for which the Gaussian integral over the $\phi_{>}$ may be performed by using Eq. (43) for the propagator $\langle \phi_{k>}^\alpha \phi_{-k>}^\beta \rangle_0$, an average calculated with the statistical weight $e^{-\mathcal{F}_0}$. Quite generally, *Wick's theorem* states that expressions of the form $\langle \prod_i^m \phi_{k_i>} \rangle_0$ factorize into a sum of products of all possible pairs $\langle \phi_{k>} \phi_{-k>} \rangle_0$, if m is even, and vanish otherwise. Especially for treating higher orders in perturbation theory, *Feynman diagrams*, with lines symbolizing the propagators, and *interaction vertices* standing for the nonlinear coupling u , provide a very helpful representation for the large number of contributions to be summed in the perturbation series.

With these means, the two-point function, $\langle \phi_{k<} \phi_{-k<} \rangle$ and the similarly defined four-point function can be evaluated. Using Eq. (80), the following recursions for r and u are then obtained to first nontrivial order (“*1-loop*”, a notation stemming from the graphical representation):

$$r' = b^2 [r + (n+2)A(r)u] \quad , \quad (88)$$

$$u' = b^{4-d} u [1 - (n+8)C(r)u] \quad , \quad (89)$$

where A and C denote the integrals $A(r) = K_d \int_{\Lambda/b}^{\Lambda} (k^{d-1}/r + k^2) dk = K_d [\Lambda^{d-2}(1 - b^{2-d})/(d-2) - r\Lambda^{d-4}(1 - b^{4-d})/(d-4)] + O(r^2)$, $C(r) = K_d \int_{\Lambda/b}^{\Lambda} [k^{d-1}/(r + k^2)^2] dk = K_d \Lambda^{d-4}(1 - b^{4-d})/(d-4) + O(r)$, with $K_d = 1/2^{d-1} \pi^{d/2} \Gamma(d/2)$, and the factors containing the number of components n of the order parameter field originate from the combinatorics of counting the equivalent ways for “contracting” the fields $\phi_{k>}$, i.e., performing the inte-

grals over the large momenta. Note that again Eq. (85) is valid here.

By linearizing Eqs. (88) and (89) at the Gaussian fixed point $r^* = 0$, $u^* = 0$, one immediately finds the eigenvalues $y_r = 2$ and $y_u = 4 - d$. Thus for $d > d_c = 4$, the nonlinearity $\propto u$ turns out to be irrelevant, and the mean-field exponents are valid, as has already been anticipated in Sec. 2.3.2. However, for $d < 4$ the fluctuations become relevant, and any initial $u \neq 0$ grows under renormalization. In order to obtain the scaling behavior in this case, we thus have to search for a nontrivial, finite fixed point. This is most readily done by introducing the differential flow again, via $b^\ell = e^{\delta\ell}$, and $\delta \rightarrow 0$, whereby the number of RG steps effectively becomes a continuous variable, and studying the ensuing differential recursion relations

$$\frac{dr(\ell)}{d\ell} = 2r(\ell) + (n+2)u(\ell)K_d\Lambda^{d-2} - (n+2)r(\ell)u(\ell)K_d\Lambda^{d-4} \quad , \quad (90)$$

$$\frac{du(\ell)}{d\ell} = (4-d)u(\ell) - (n+8)u(\ell)^2 K_d \Lambda^{d-4} \quad . \quad (91)$$

Namely, a fixed point is defined by the condition that $dr/d\ell = 0 = du/d\ell$. In Fig. 13 the flow of $u(\ell)$ according to Eq. (91) is depicted; for any initial value $u_0 \neq 0$ one finds that asymptotically, i.e.: for $\ell \rightarrow \infty$, the nontrivial fixed point

$$u_c^* K_d = \frac{\epsilon}{n+8} \Lambda^\epsilon \quad , \quad \epsilon = 4 - d \quad (92)$$

is approached, which therefore describes scale-invariant behavior and should govern the universal critical properties of the model. Here, too, the RG procedure generates new interactions; for instance, terms $\propto \phi^6$ and $\nabla^2 \phi^4$ etc., which in successive steps couple back into the recursion relations for r and u . However, it turns out that up to order ϵ^3 these terms may be disregarded.

The original assumption that u be small, justifying a perturbation expansion, now

means that the effective expansion parameter here is the deviation from the upper critical dimension ϵ . Inserting this in Eq. (90) and collecting the terms of $O(\epsilon)$, one finds

$$r_c^* = -\frac{n+2}{2}u_c^*K_d\Lambda^{d-2} = -\frac{(n+2)\epsilon}{2(n+8)}\Lambda^2 \quad ; \quad (93)$$

the physical interpretation for this result is that the fluctuations lead to a downward shift of the transition temperature. Finally, with $\tau = r - r_c^*$, the differential flow equation

$$\frac{d\tau(\ell)}{d\ell} = \tau(\ell)[2 - (n+2)uK_d\Lambda^{d-4}] \quad (94)$$

yields the eigenvalue $y_\tau = 2 - (n+2)\epsilon/(n+8)$ near the critical fixed point (92). To one-loop order, $O(\epsilon)$, one therefore finds the critical exponent

$$\nu = \frac{1}{2} + \frac{n+2}{4(n+8)}\epsilon + O(\epsilon^2) \quad . \quad (95)$$

Using $\eta = O(\epsilon^2)$ and the scaling relations (15), (16), and (19), one arrives at the following results [note the remarkable difference to the result (39), obtained within the Gaussian approximation]

$$\alpha = \frac{4-n}{2(n+8)}\epsilon + O(\epsilon^2) \quad , \quad (96)$$

$$\beta = \frac{1}{2} - \frac{3}{2(n+8)}\epsilon + O(\epsilon^2) \quad , \quad (97)$$

$$\gamma = 1 + \frac{n+2}{2(n+8)}\epsilon + O(\epsilon^2) \quad , \quad (98)$$

$$\delta = 3 + \epsilon + O(\epsilon^2) \quad (99)$$

to first order in the expansion parameter $\epsilon = 4 - d$. The first nontrivial contribution to the exponent η turns out to appear at two-loop order,

$$\eta = \frac{n+2}{2(n+8)^2}\epsilon^2 + O(\epsilon^3) \quad . \quad (100)$$

The universality of these results becomes manifest by the fact that they solely depend on

the space dimension d and the number of order parameter components n , and not on the original “microscopic” Ginzburg–Landau parameter. (In passing, we note that for long-range power-law interactions, $\propto |x|^{-(d+\sigma)}$, the critical exponents acquire an additional dependence on the parameter σ .) At the upper critical dimension, $d_c = 4$, the solution of Eq. (91) becomes an inverse power law instead of an exponential, which leads to logarithmic corrections to the mean-field exponents.

Differential recursion relations of the form (90), (91) also constitute the basis for the treatment of more subtle issues, like the calculation of the scaling functions, or even *crossover phenomena*, within the RG approach. For instance, an anisotropic perturbation in the n -component Heisenberg model favoring m directions leads to a crossover from the $O(n)$ Heisenberg fixed point to an m -component fixed point (see Amit, 1984, Chap.5-3). The instability of the former is described by a crossover exponent. For a small anisotropic perturbation, the RG flow trajectory still comes very close to this unstable fixed point, which implies that further away from T_c the system behaves like an n -component system, before it is finally dominated by the anisotropic critical behavior. The crossover from one RG fixed point to another may be illustrated (and measured) by introducing *effective exponents*, defined as logarithmic derivatives of appropriate physical quantities. Other important perturbations which have been treated in the framework of RG theory are so-called cubic terms reflecting the underlying crystal structure, and contributing a fourth power of the Cartesian components of ϕ , as well as the dipolar interaction which alters the harmonic part of the theory.

For systems characterized by certain spatial anisotropies, the upper critical dimension may be shifted from $d_c = 4$ to lower values due to restricted space for fluctuations. E.g., for uniaxial dipolar ferromagnets $d_c = 3$, as well as for tricritical points, where $u = 0$ and a sixth-order term in ϕ is required. In the case of elas-

tic structural phase transitions, for instance, one finds $d_c = 5/2$ or $d_c = 3$, for one- or two-dimensional soft sector, respectively (Schwabl, 1985).

3.3.4 More Sophisticated Field-Theoretical Methods. Once one has to go beyond the first order in the perturbation expansion, Wilson's k shell renormalization scheme, despite its very intuitively appealing features, is not the best choice for practical calculations. The technical reason is that the integrals in Fourier space involve nested momenta, and become rather tedious with a finite cutoff Λ . Thus it appears more convenient to use a field-theoretical renormalization scheme, where $\Lambda \rightarrow \infty$ (Amit, 1984). Yet this leads to additional ultraviolet (UV) divergences in the integrals for $d \geq d_c$. Observing that at the critical dimension d_c both the ultraviolet and infrared (IR) singularities appear combined in logarithmic form [$\propto \ln(\Lambda^2/r)$], the idea is to treat the UV divergences with the methods originally developed in quantum field theory, and through this diversion arrive at the correct scaling equations for the opposite, IR limit.

Formally, this is done by taking advantage of the fact that the original unrenormalized theory does not depend on an arbitrarily chosen renormalization point; as a consequence one is led to the *Callan-Symanzik* or *RG equation*. This is a partial differential equation, which, when solved via the method of characteristics, leads to a set of ordinary differential equations that resemble the differential flow equations in Wilson's scheme. This procedure may be pursued provided the theory is renormalizable, which for the ϕ^4 theory is the case for $d \leq d_c$. However, this also means that irrelevant operators cannot, at least without considerable precautions, be included in the scheme. The field-theoretical treatments are therefore confined to the critical manifold in parameter space; yet, for the universal properties of the system under consideration, they yield fully equivalent results as compared to the momentum shell approach.

ϵ expansions have been carried out up to

seventh order; the resulting series is, however, only asymptotically convergent (quite obviously, the convergence radius of the perturbation series in u must be zero, as $u < 0$ corresponds to an unstable theory). Yet, by a combination of these fairly high-order results with the divergent asymptotic behavior and Borel-type resummation techniques, the critical exponents can be obtained with impressive accuracy, see Table 1 (Le Guillou and Zinn-Justin, 1985).

Near the lower critical dimension of the $O(n)$ ($n \geq 2$) symmetric Heisenberg model, $d = 2$ (for the Ising case, $n = 1$, the lower critical dimension is $d = 1$, see Sec. 3.2.1), the ϵ expansion does not provide very accurate results, as might be expected. Near two dimensions, the investigation of the $O(n)$ -symmetric non-linear σ model by means of a $2 + \epsilon$ expansion, with a nontrivial fixed point of order ϵ corresponding to T_c , has proven to be more valuable (see Amit, 1984).

Besides the ϵ expansion, an expansion in terms of powers of $1/n$ is possible, where the limit $n \rightarrow \infty$ corresponds to the exactly solvable spherical model (Ma, 1976a). While this approach may help to clarify certain general issues, it is usually less accurate numerically, though, for the small values of n that are of practical interest.

4 SCALING LAWS IN STATISTICAL PHYSICS

4.1 Phase Transitions and Properties of Phases

Phenomenological Ginzburg-Landau expansions, scaling approaches and the language and methods of RG methods have become fundamental tools in the study of phase transitions and of other universal properties of condensed-matter phases, in a large variety of systems. An (incomplete) list in-

cludes structural phase transitions of various kinds, namely distortive, ferroelectric, order-disorder, and elastic transitions (see PHASE TRANSITIONS, STRUCTURAL, COLLECTIVE PHENOMENA IN SOLIDS), liquid crystals (see LIQUID CRYSTALS, STRUCTURE OF), magnetic systems (see MAGNETIC ORDERING IN SOLIDS), magnetic impurities in a lattice (*Kondo problems*; see Wilson, 1975), superfluidity (see SUPERFLUIDITY: LIQUID HELIUM SYSTEMS), and superconductivity (see SUPERCONDUCTIVITY, LOW TEMPERATURE); here specifically the numerous phases of type II superconductors subject to an external magnetic field have aroused considerable interest lately (see SUPERCONDUCTIVITY, HIGH TEMPERATURE, and Blatter, Feigel'man, Geshkenbein, Larkin, and Vinokur, 1994). Complex scaling and crossover behavior may ensue when either anisotropies or additional long-range interactions are included.

Finite-size scaling studies how the asymptotic power laws of the thermodynamic limit are approached as the system size grows, and is thus an important tool for numerical studies of continuous phase transitions (Privman, 1990). Furthermore, the fact that all real systems are finite leads to additional interesting phenomena. At surfaces, new critical exponents may come into play (Diehl, 1986). Also, interfacial structures may change drastically as external parameters are varied; an important example is the wetting transition of liquid drops on a substrate (Dietrich, 1988).

Scaling and RG concepts have also been applied to the theory of symmetry breaking in the fundamental field theories of matter (see UNIFIED FIELD THEORIES), which also have important implications for cosmology (see GALAXIES AND COSMOLOGY). (We note that in elementary particle physics a Compton wavelength $\lambda_c = \hbar c/m$ may be assigned to a particle of rest mass m . For large momenta $\hbar k \gg mc$, the theory becomes scale-invariant, and the propagator behaves as $k^{-2+\eta}$. The correlation length ξ thus corresponds to the Compton wavelength or inverse mass.)

4.2 Disordered Systems

In many physical systems, disorder plays an important role. The basic question therefore is, under which circumstances can quenched randomness affect the nature of a phase transition and its properties. Let the disorder induce a relevant perturbation $g(x)$ in the effective Hamiltonian, e.g., changing the harmonic part of the Landau-Ginzburg-Wilson functional (*random T_c*), the scaling dimension of which is $[g] \propto L^{\Delta-d}$, with $\Delta \leq d$. The *Harris criterion* then states that the character of the pure transition may be considerably affected by the disorder, provided $\Delta < d/2$ (Itzykson and Drouffe, 1989). If defects of this kind are correlated only over short distances, this criterion says that the critical behavior is changed if the specific-heat exponent of the pure system is positive ($\alpha > 0$). On the other hand, in the case of *random fields*, both the upper and lower critical dimension can be changed as compared to the pure case. The *Imry-Ma argument* (see Nattermann and Villain, 1988) compares the interfacial energy $\propto L^\Theta$ [Eq. (13)] with the gain in bulk free energy $\propto L^{d/2}$ by splitting into domains of size L (the mean-square order parameter fluctuations due to the random field add up to a contribution $\propto L^d$ for each domain). Thus, for $\Theta \leq d/2$, the system will always break into domains and no long-range order can emerge. For Ising systems ($\Theta = d-1$), this will be true for $d \leq 2$, while in the case of continuous symmetry ($\Theta = d-2$) the lower critical dimension becomes $d = 4$.

An example for such disorder-dominated systems are *spin glasses*, e.g.: an Ising spin glass with bond disorder, where the couplings J_{ij} in (21) are drawn from some stochastic distribution. Similarly, disorder can be introduced in Heisenberg models (25). As model systems for phase transitions in highly disordered materials, considerable amount of effort has been devoted to the examination of such spin glasses (Binder and Young, 1986). In addition to the thermal average, all physical quantities have to be averaged over all possi-

ble configurations of the quenched disorder. A frequently used method in analytical calculations is the *replica trick*, which replaces the computation of the mean free energy by the average of the n th power of the partition function, and then taking both the derivative with respect to n and the limit $n \rightarrow 0$:

$$\overline{F} = - \lim_{n \rightarrow 0} \frac{\partial}{\partial n} \overline{Z^n} \quad (101)$$

Extensions of these studies have led to the investigation of manifolds in random media, a direct application of which may be found in biological membranes, or flux lines in high- T_c superconductors, and the design of neural networks (see NEURAL NETWORKS).

Disorder may also play a prominent role in the quantum regime ($T \rightarrow 0$). While near a phase transition with finite T_c quantum fluctuations are of minor importance (although quantum effects may provide the fundamental mechanism for the transition itself, as in magnets, superfluids and superconductors), at $T = 0$ the interplay between disorder and quantum mechanics can lead to very interesting novel phases with unusual phase transitions. Examples for such quantum critical phenomena (see Continentino, 1994) are electron localization (see ELECTRON STATES, LOCALIZED), metal-insulator transitions (see METAL-INSULATOR TRANSITIONS; Belitz and Kirkpatrick, 1994), Helium 4 on disordered substrates (disordered bosons), and quantum antiferromagnets, which are of relevance in the microscopic theory of high-temperature superconducting materials and as one phase of solid Helium 3. For a Fermi system, the interesting singularities and the universal behavior are expected in the vicinity of the Fermi surface, instead of the point $k = 0$ in Fourier space, and therefore the RG approach has to be modified accordingly (Shankar, 1994).

The mathematical theory of percolation also has immediate applications for the structure of certain disordered systems, and their transport properties (see FRACTAL GEOMETRY; Bunde and Havlin, 1991). The percolation

threshold may be understood as a critical point, and the scaling behavior in its vicinity can be obtained by mapping onto the so-called Potts model (Wu, 1982), and with the help of certain continuum models of the same universality class.

4.3 Polymers, Membranes

An important ingredient to the physics of macromolecules like polymers is the interplay and competition of energetic and entropic optimization. De Gennes has demonstrated that the partition function of a polymer, or the statistics of a self-avoiding random walk, can be mapped to the $n \rightarrow 0$ limit of the n component Heisenberg model (see Table 1). As a consequence, scaling and RG approaches to the static and dynamic properties of polymers have been successfully employed (see POLYMERS, STRUCTURE OF; De Gennes, 1979; Freed, 1987). The inverse separation from the critical point $|\tau|^{-1}$ formally corresponds to the degree of polymerization (chain length) N of a long flexible polymer, and, e.g., in a dilute solution in a good solvent the average end-to-end distance of the polymer scales as $R_e \propto N^\nu$, where $\nu \approx 0.588$ (in three dimensions) is the $n = 0$ result for the correlation length exponent of the $O(n)$ model. Naturally, similar attempts have been used for biological systems like muscular fibres and membranes (see MEMBRANE BIOPHYSICS).

4.4 Critical Dynamics and Non-Equilibrium Phenomena

In Sec. 1.4, the extension of the above scaling concepts to dynamical phenomena was mentioned. The order parameter dynamics displays critical slowing down near a second-order phase transition, described by the critical exponent z [Eq. (11)]. One very powerful and common mathematical description of critical dynamics rests on *Langevin-type equations of motion* (see STOCHASTIC PROCESSES) for the order parameter fluctuations and the con-

served quantities, which constitute the *slow modes* of the system (Hohenberg and Halperin, 1977; Enz, 1979). For, as a consequence of the continuity equation related to a conservation law, the corresponding density shows diffusive behavior (inverse relaxation time $t_0^{-1} \propto k^2$). In a phase with broken continuous symmetry, in addition the fluctuations transverse to the order parameter have to be included, for their static susceptibility is also singular for $k \rightarrow 0$ [Eq. (47)], and thus the corresponding mode has frequency $\propto \chi^{-1/2}$, and damping coefficient $\propto \chi^{-1}$. Collecting all these slow modes in a multi-component field $\Phi(x, t)$, the equations of motion assume the general form

$$\frac{\partial \Phi^\alpha(x, t)}{\partial t} = K[\Phi] - \Gamma^\alpha \frac{\delta \mathcal{F}[\Phi]}{\delta \Phi^\alpha(x, t)} + \zeta^\alpha(x, t). \quad (102)$$

Here, $K[\Phi]$ comprises the reversible forces, as obtained from the microscopic equations of motion. The second term provides a damping term, which after some small perturbation drives the system back into the equilibrium state, where $\delta \mathcal{F}/\delta \Phi = 0$; the damping coefficients Γ^α are constants for the order parameter fluctuations, if the latter is not conserved, and take the form $\lambda \nabla^2$ for each conserved quantity. Finally, all “fast” modes have been subsumed in the stochastic forces $\zeta^\alpha(x, t)$, whose mean is taken to be zero, $\langle \zeta^\alpha(x, t) \rangle = 0$, and whose second moments are, for simplicity, assumed to be drawn from a Gaussian distribution with a variance corresponding to “white noise”

$$\begin{aligned} \langle \zeta^\alpha(x, t) \zeta^\beta(x', t') \rangle &= \\ &= 2k_B T \Gamma^\alpha \delta^{\alpha\beta} \delta(x - x') \delta(t - t') \end{aligned} \quad (103)$$

The *fluctuation-dissipation theorem* (*Einstein relation*) (103) assures that for $t \rightarrow \infty$ the probability distribution is given by the equilibrium expression $\exp(-\mathcal{F}/k_B T)$.

The set of equations (102) with (103) can then be very elegantly mapped onto a dynamic functional, and the ensuing field theory be treated with precisely those perturbation expansions and RG methods explained

above for the statics. In some cases, the underlying symmetries and/or conservation laws lead to so-called *Ward identities*, which yield a connection between the dynamic exponent z and already known static indices, prominently η . E.g., for a model merely consisting of a conserved order parameter field (model B according to the classification of Hohenberg and Halperin, 1977), the result is $z = 4 - \eta$; for the isotropic ferromagnet (model J) one finds $z = (d + 2 - \eta)/2$, and for planar ferromagnets as well as isotropic antiferromagnets (models E and G, respectively), $z = d/2$. In Table 2 a list of dynamical systems, with the corresponding value of z , and the physical quantities for which the critical slowing down can be observed in experiment, is presented. Note that although the range of the dynamical critical exponent is limited to $3/2 \leq z \leq 4$ in three dimensions, the ensuing behavior of the transport coefficients can yet be as different as the diverging magnetization diffusion $D_M \propto |\tau|^{-1/3}$ for an isotropic antiferromagnet, and the vanishing order parameter relaxation rate $\Gamma \propto |\tau|$ in the same system, or the vanishing magnetization diffusion constant $D_M \propto |\tau|^{1/3}$ for an isotropic Heisenberg ferromagnet.

An alternative route to finding these exponents, and probably superior for the issue of precisely calculating the scaling functions, is the *mode-coupling* approach, which at least in some cases may be represented as a self-consistent one-loop theory for the generalized (i.e.: wavenumber- and frequency-dependent) transport coefficients (Frey and Schwabl, 1995).

Another physical situation, where equations of motion of the type (102) and a RG approach have been used, is the problem of quenching a system from above T_c into the low-temperature phase, and observing how the ordered domains grow with time (Bray, 1993).

Upon relaxing the condition (103), the same methods may also be applied to phase transitions in non-equilibrium systems. A prototypical example might be the Burgers-Kardar-

Parisi–Zhang equation, which is supposed to describe the kinetic roughening of growing surfaces (Family and Vicsek, 1991), but is also intimately connected to the problem of directed lines in random environment, and to the hydrodynamics of randomly stirred fluids. Related issues are the depinning transitions of lines and interfaces, and the dynamics of charge–density waves. The behavior of interfaces in disordered systems in general poses a number of most interesting issues and requires certain new conceptual approaches (see Forgacs, Lipowsky, and Nieuwenhuizen, 1991).

The properties of systems very far from equilibrium, well within the scope of scaling approaches and RG methods, turn out to be remarkably rich. On the one hand, there is the vast field of non–equilibrium pattern formation (Cross and Hohenberg, 1994). On the other hand, self–similar and self–affine structures emerge in many dynamical processes (see FRACTAL GEOMETRY), which are hence good candidates for the application of the above methods. The ubiquity of fractals and $1/f$ noise has led to the paradigm of *self-organized criticality (SOC)* (reviewed in by Bak and Creutz, 1994). SOC refers to the tendency of large dissipative systems to drive themselves into a self–similar critical state without the special adjustment of parameters. SOC has been suggested for avalanches in sand piles, earth quake statistics, evolution, spreading of diseases, forest fires, and many others (Bak and Creutz, 1994). In the so–called forest fire model one finds SOC if a certain double separation of time scales holds (which represents still a large region in parameter space), while in other regions of the parameter space spiral structures emerge (Drossel and Schwabl, 1994).

At any rate, scaling ideas and the language of the renormalization group, including such concepts as effective Hamiltonians, relevant and irrelevant perturbations, fixed points, diverging correlation lengths, self–similarity and power laws, universal long–wavelength and low–frequency behavior of collective modes, as

well its powerful mathematical tools, have provided considerable new insight in the physics of many important research areas, and it seems likely that more interesting applications will be discovered in the future.

GLOSSARY

Asymptotic regime: Region near the critical point where the relevant perturbations dominate, leading to pure power laws.

Corrections to scaling: Contributions of irrelevant operators to thermodynamic quantities.

Correlation length: Length scale for the spatial decay of correlations, usually a function of the external control parameters (temperature, magnetic field, etc.).

Critical exponents: Exponents describing the power laws at continuous phase transitions.

Critical slowing down: Decrease of the order parameter decay or oscillation rate ω_c , characterized by the dynamic exponent z , cf. Eq. (11).

Critical surface: Hypersurface in parameter space with the property that all RG trajectories starting on it lead into the critical fixed point.

Crossover: Change in scaling behavior from power laws characterized by a certain unstable fixed point, to different power laws governed by the asymptotically stable fixed point.

Effective theory: Mathematical model describing the long-wavelength and low-frequency behavior of a physical system, at a certain energy scale.

Fixed point: Point in the space of coupling coefficients which remains invariant under the RG transformation.

Lower critical dimension: Borderline dimension, below which a certain model or physical system does not have a phase transition leading to long-range order.

Order parameter: Certain macroscopic quantity which may be used to characterize different phases.

Relevant and irrelevant perturbations: Eigenoperators of the linearized RG transformation which grow resp. decrease under successive RG steps.

Renormalization group (RG) transformation: Operation by which a Hamiltonian

is transformed into a new one, with a reduced number of degrees of freedom, accompanied by a subsequent scale transformation.

Scaling law: Mathematical expression of the fact that at a critical point the free energy and correlation functions are scale-invariant, i.e. are generalized homogeneous functions.

Scaling relations: Equations which relate certain critical indices, thus reducing the number of independent exponents.

Universality: The fact that critical properties depend only on global properties as the dimension, the symmetry and number of components of the order parameter.

Upper critical dimension: Borderline dimension, above which the critical exponents of a certain model or physical system are correctly given by their mean-field values. Below the upper critical dimension, fluctuations are relevant and affect the scaling behavior.

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Further Reading

General

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Figure Captions

FIG. 1. (a) The order parameter ϕ of a ferromagnet for zero external field $h = 0$, as a function of the temperature T . ϕ sets in continuously at T_c , and has at least two equivalent orientations. (b) Phase diagram (external field h vs. temperature T) of a ferromagnet.

FIG. 2. Correlated clusters in the two-dimensional Ising model (the black and white points correspond to the two possible states at each site i in the lattice) for two different temperatures: (a) far away from T_c in the disordered phase, and (b) near the critical point.

FIG. 3. Comparison of the phase diagrams of a ferromagnet (a), and of a one-component gas/liquid/solid (b).

FIG. 4. Temperature dependence of the specific heat of Helium 4 at the normal-superfluid phase transition, shown for successively reduced scales in the temperature variable (from: Buckingham and Fairbank, 1961, p. 86).

FIG. 5. Probability distribution $\mathcal{P} \propto \exp(-\mathcal{F}/k_B T)$ for the Landau part of the free energy \mathcal{F} , Eq. (26), as function of a single-component order parameter ϕ , for (a) $T > T_c$, and (b) $T < T_c$.

FIG. 6. Decimation transformation for the one-dimensional Ising chain with nearest-neighbor interaction.

FIG. 7. Recursion relation for the nearest-neighbor coupling of the Ising chain (full; the dashed line marks the line $K' = K$). The iteration steps are indicated by the arrows.

FIG. 8. Reduced free energy f for the nearest-neighbor Ising chain. The $f^{(k)}$ denote the approximation to the exact result f as obtained in the k th step of the RG procedure.

FIG. 9. Ising model on a quadratic lattice. Even if originally there were only nearest-neighbor interactions K present, by applying the decimation procedure next-nearest neighbor couplings L are generated.

FIG. 10. Flow diagram for the two-dimensional Ising model on a square lattice near its critical point. (Only the results of every other recursion step are shown.) The inset shows the behavior in the vicinity of the critical fixed point, which is described by Eq. (63).

FIG. 11. Critical hypersurface. A trajectory in the critical hypersurface is shown dashed, and a trajectory starting close to the critical hypersurface is displayed as a full curve. The coupling coefficient of a particular physical system as a function of a control parameter is indicated by the long-dashed line.

FIG. 12. Momentum space RG: The partial trace is performed over Fourier components ϕ_k with momenta in the shell $\Lambda/b < |k| < \Lambda$.

FIG. 13. Flow of the effective coupling $u(\ell)$, as determined from the right-hand-side of Eq. (91). Both for initial values $u_0 > u_c^*$ and $0 < u_0 < u_c^*$, one has $u(\ell) \rightarrow u_c^*$ for $\ell \rightarrow \infty$.

Tables

		γ	ν	β	η
$d = 2$	$n = 0$	1.39 ± 0.04	0.76 ± 0.03	0.065 ± 0.015	0.21 ± 0.05
	$n = 1$	1.73 ± 0.06	0.99 ± 0.04	0.120 ± 0.015	0.26 ± 0.05
2d Ising	(exact)	1.75	1	0.125	0.25
$d = 3$	$n = 0$	1.160 ± 0.004	0.5885 ± 0.0025	0.3025 ± 0.0025	0.031 ± 0.003
	$n = 1$	1.239 ± 0.004	0.6305 ± 0.0025	0.3265 ± 0.0025	0.037 ± 0.003
	$n = 2$	1.315 ± 0.007	0.671 ± 0.005	0.3485 ± 0.0035	0.040 ± 0.003
	$n = 3$	1.390 ± 0.010	0.710 ± 0.007	0.368 ± 0.004	0.040 ± 0.003

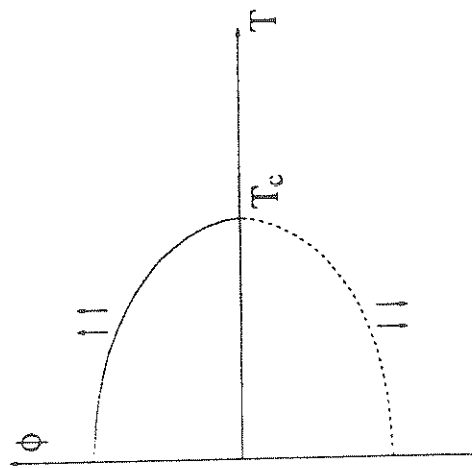
Table 1. Best estimates for the static critical exponents γ , ν , β , and δ for the $O(n)$ -symmetric ϕ^4 model in $d = 2$ and $d = 3$ dimensions, as obtained from large-order ϵ expansion in combination with Borel resummation techniques (from: Le Guillou and Zinn-Justin, 1985). For comparison, the exact Onsager results for the 2d Ising model are also listed. The limit $n = 0$ describes the statistical mechanics of polymers.

System	Model	Dynamic exponent	Physical quantity
Anisotropic magnet	A	$z = 2 + c\eta$, $c = O(1)$	Relaxation rate $\Gamma \propto \tau ^{z\nu} \approx \tau ^{4/3}$
Uniaxial ferromagnet	B	$z = 4 - \eta$	Magnetization diffusion $D_M \propto \tau ^{(z-2)\nu} \approx \tau ^{4/3}$
Structural phase transitions	C	$z = 2 + \alpha/\nu$	Relaxation rate $\Gamma \propto \tau ^{z\nu} \approx \tau ^{4/3}$
Planar ferromagnet	E	$z = d/2$	Magnetization diffusion $D_M \propto \tau ^{z\nu} \approx \tau $
Superfluid Helium 4	F	$z = \frac{1}{2}(d + \tilde{\alpha})$, $\tilde{\alpha} = \max(\alpha, 0)$	Thermal conductivity $\lambda_T \propto \tau ^{(z-2)\nu} \approx \tau ^{-1/3}$ (+ logarithmic corrections)
Isotropic antiferromagnet	G	$z = d/2$	Order parameter relaxation $\Gamma_N \propto \tau ^{z\nu} \approx \tau $
Isotropic Heisenberg ferromagnet	J	$z = \frac{1}{2}(d + 2 - \eta)$	Magnetization diffusion $D_M \propto \tau ^{(z-2)\nu} \approx \tau ^{-1/3}$ $D_M \propto \tau ^{(z-2)\nu} \approx \tau ^{1/3}$

Table 2. Dynamic critical exponents z in d dimensions, for various dynamical systems, along with a list of the physical quantities, and their temperature dependence in the vicinity of T_c , for which the ensuing critical slowing down may be observed experimentally (classification according to Hohenberg and Halperin, 1977). The approximate numerical results are for the three-dimensional case, using $\nu \approx 2/3$, $\eta \approx 0$, and $\alpha \approx 0$.

Fig. 1

(a)



(b)

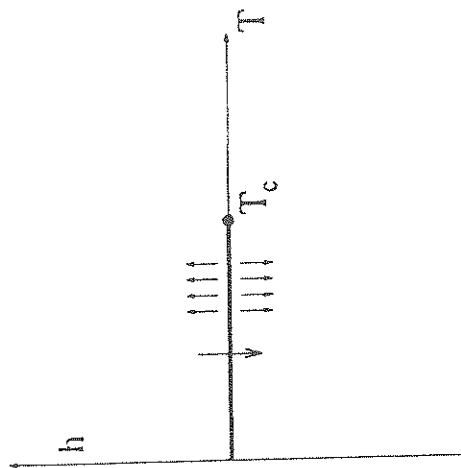
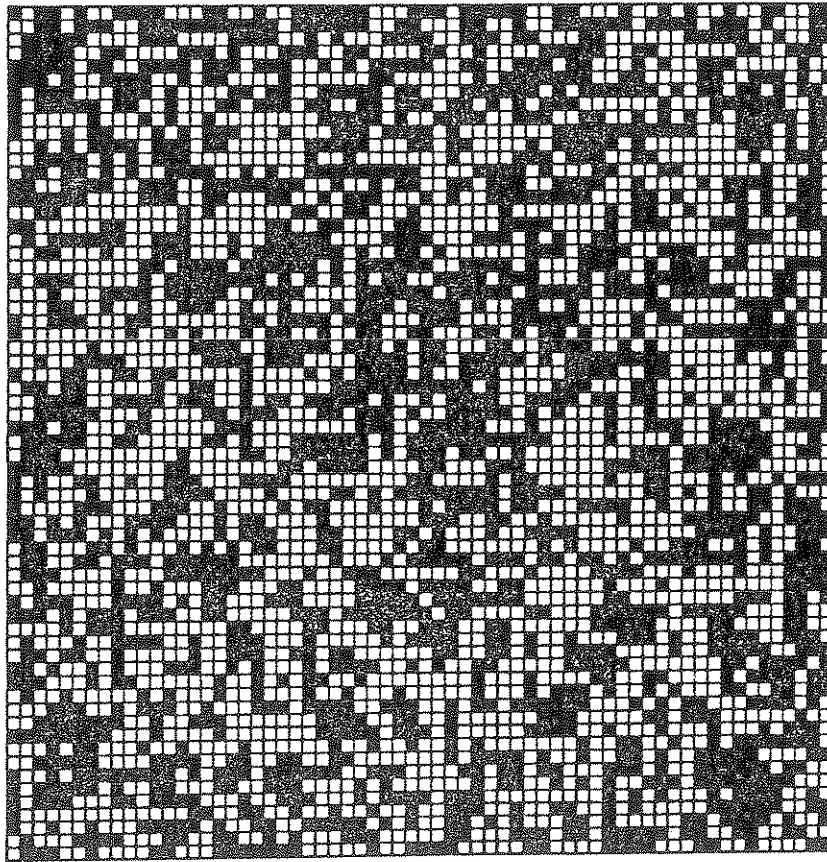


Fig 2

(a)



(b)

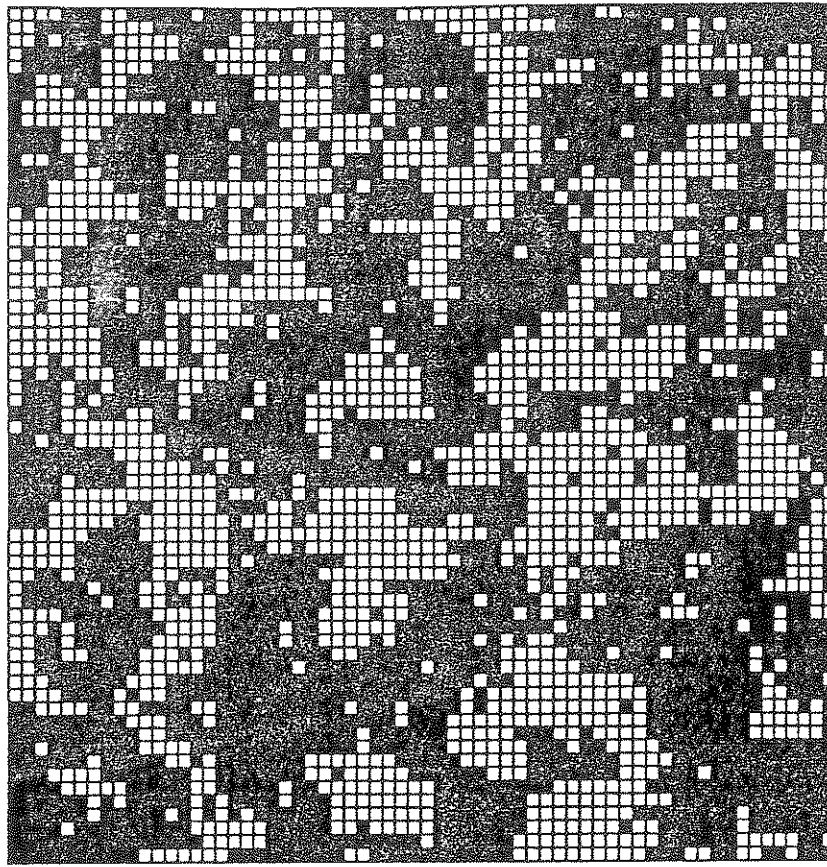


Fig 5

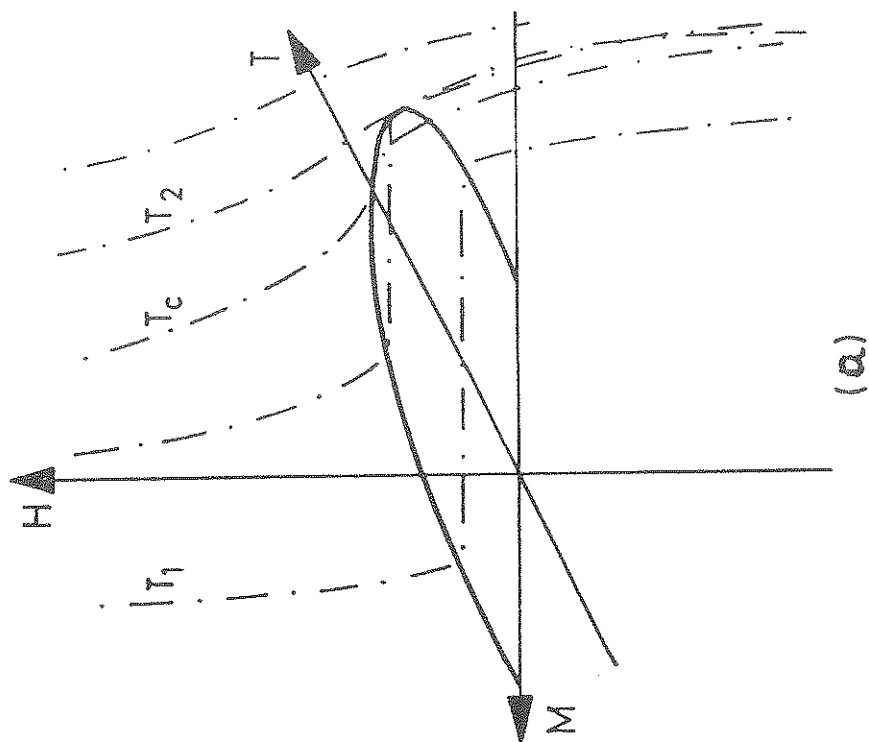
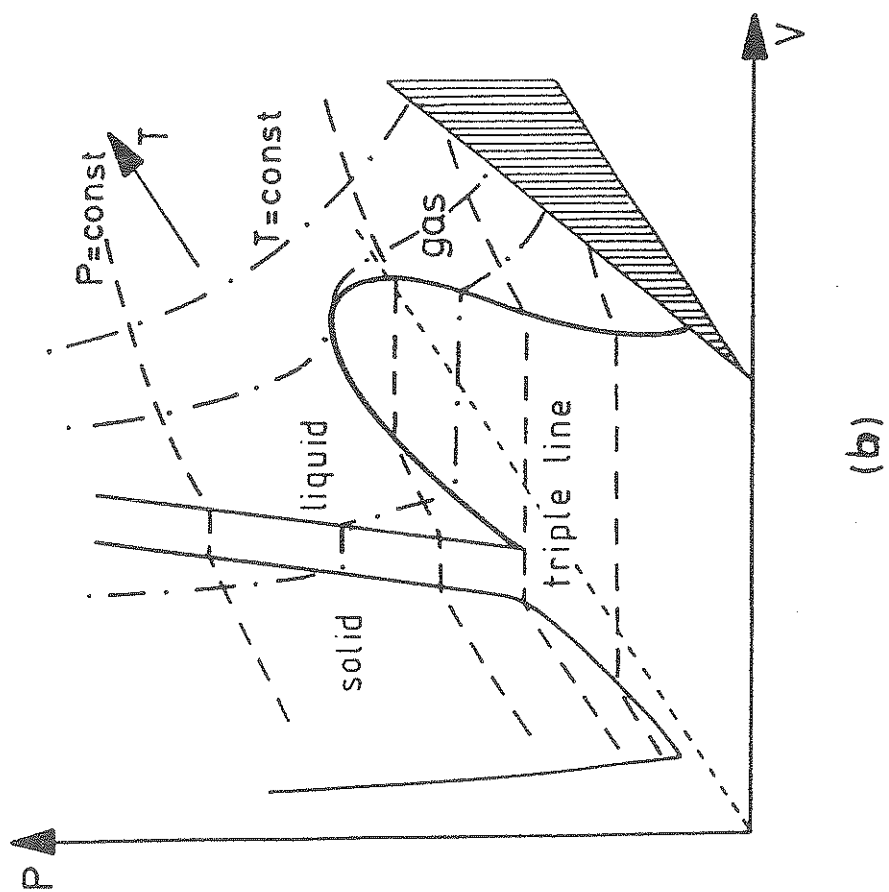


Fig 4

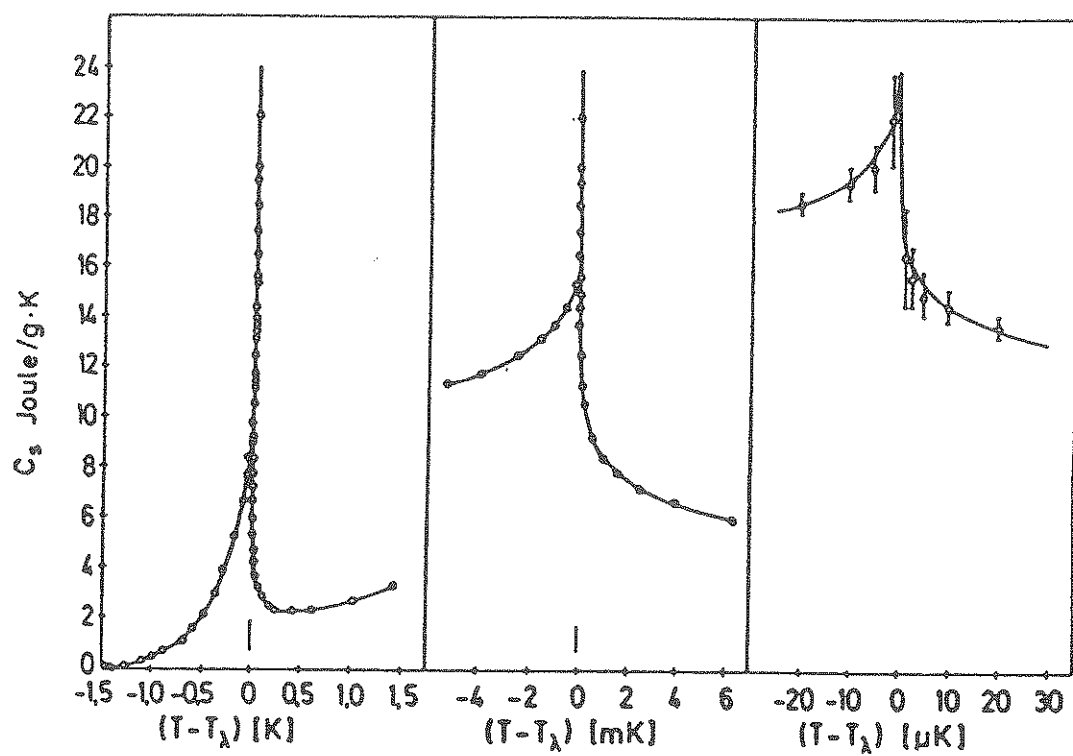


Fig 5

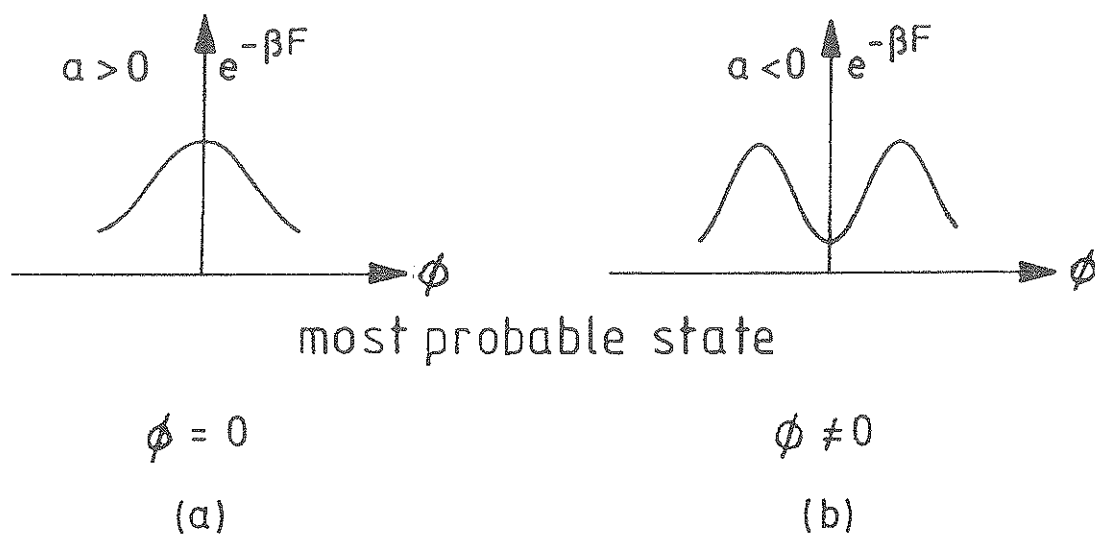


Fig. 6

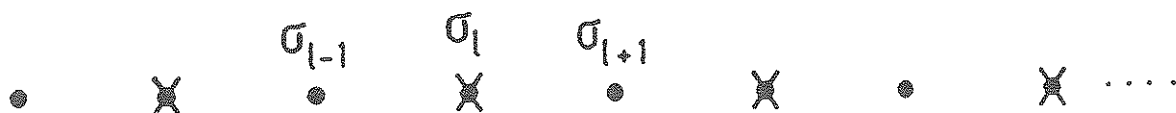


Fig. 7

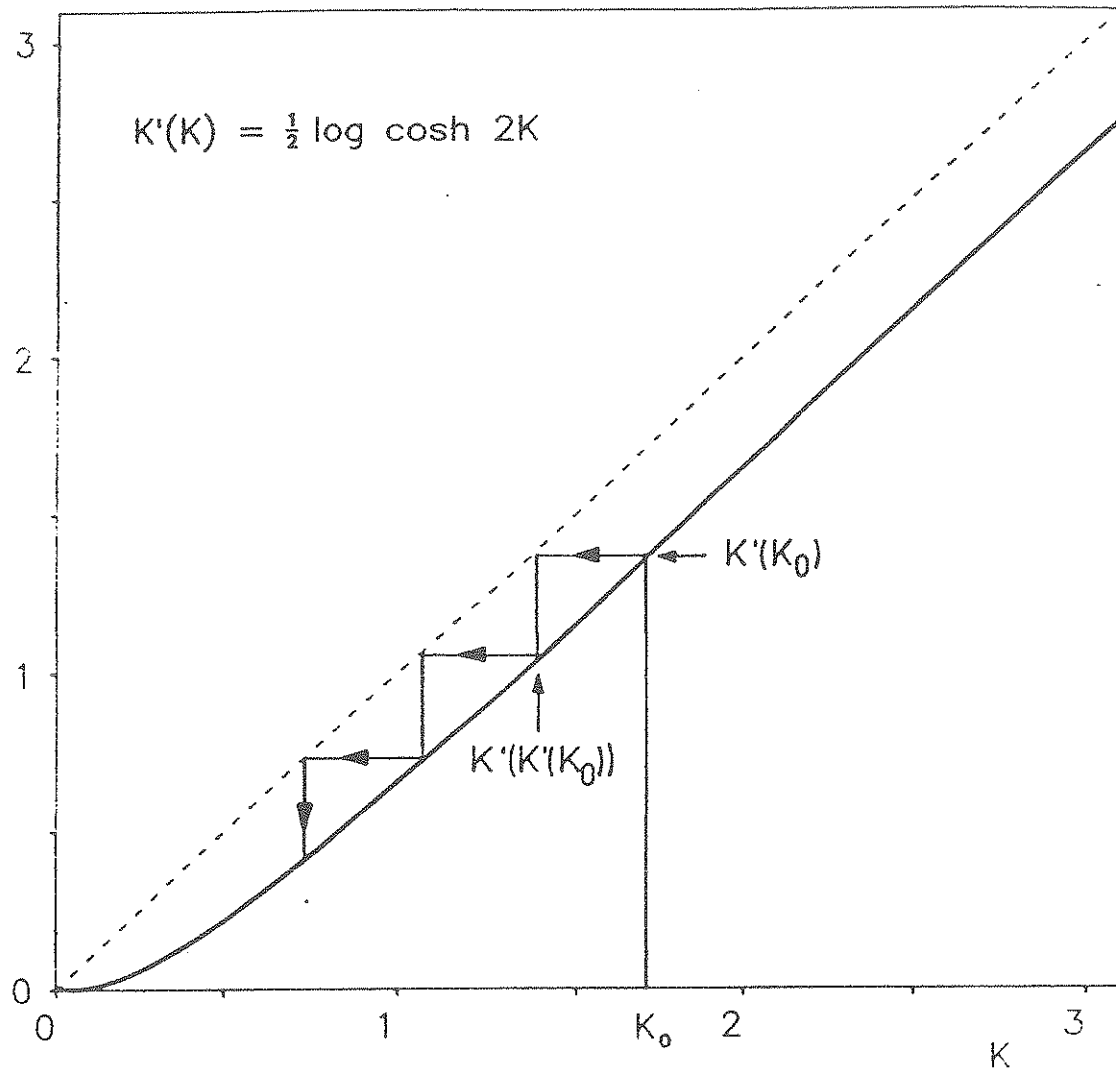


Fig. 8

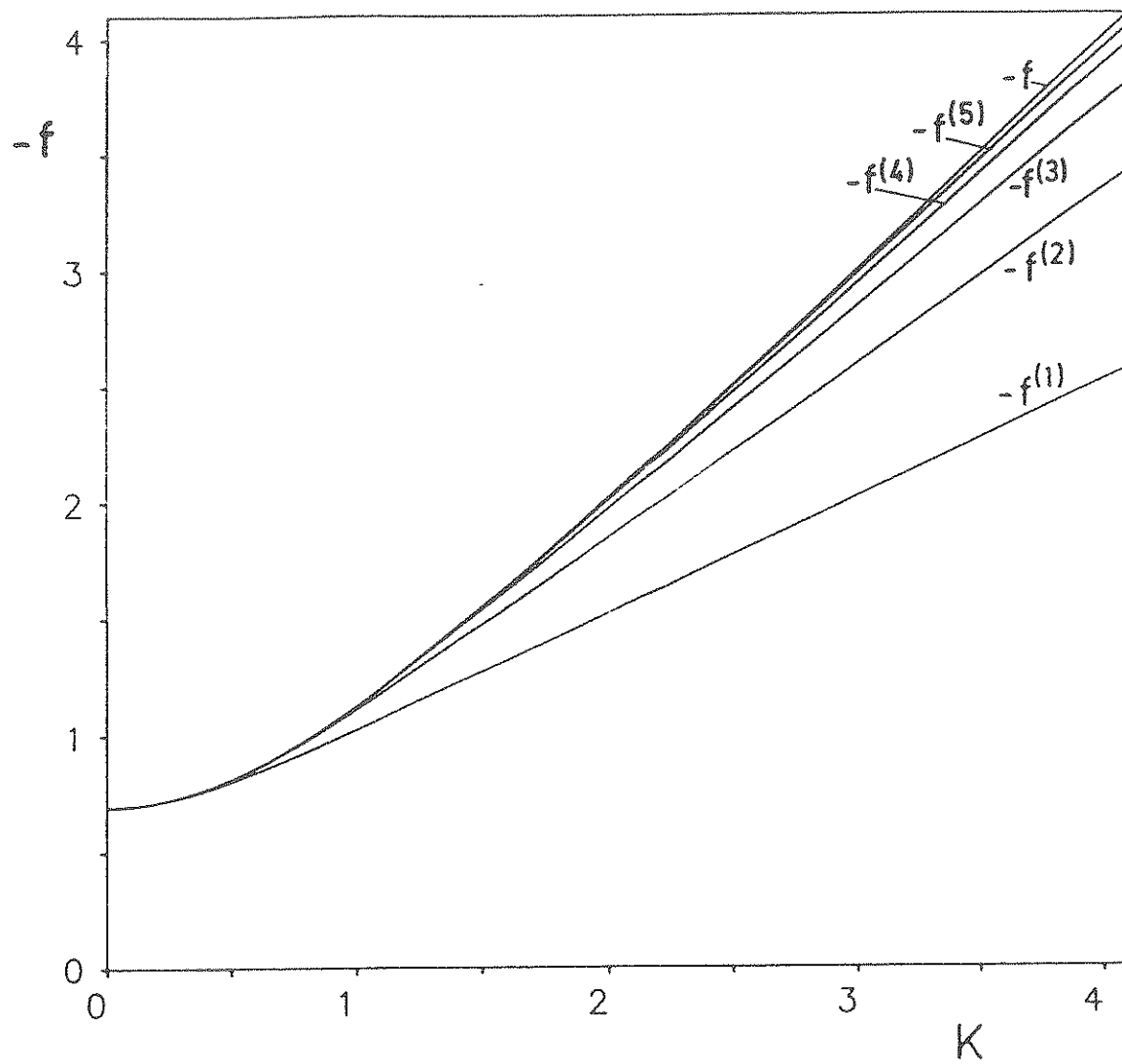


Fig. 3

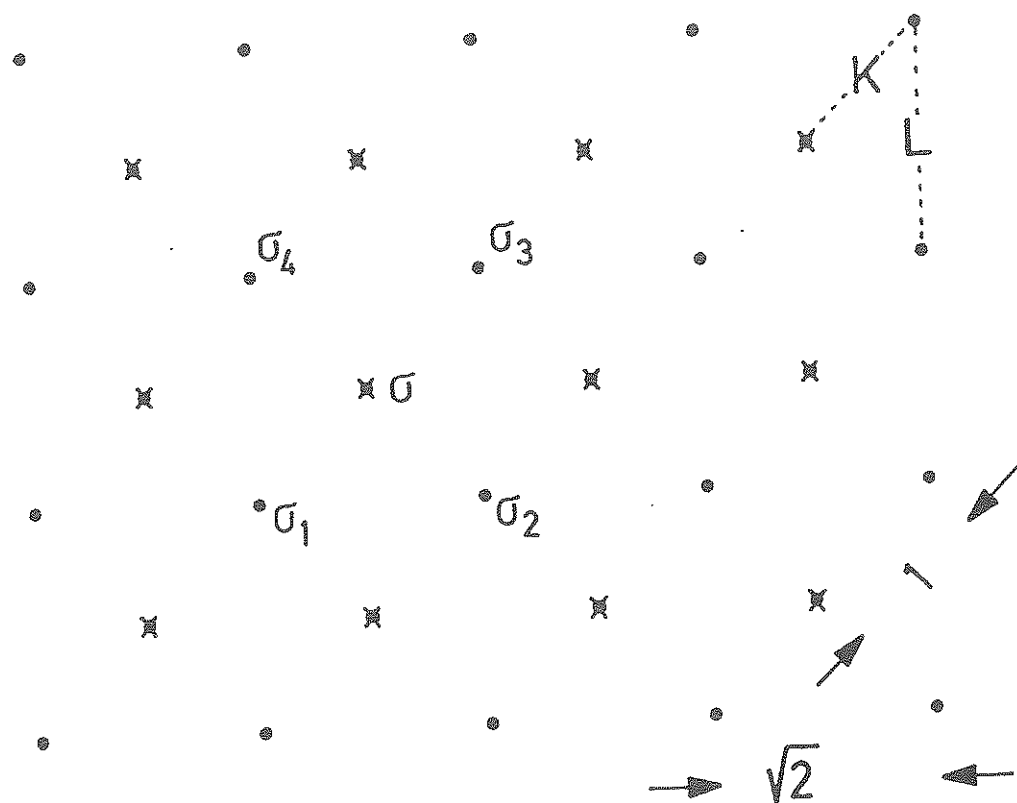


Fig. 15

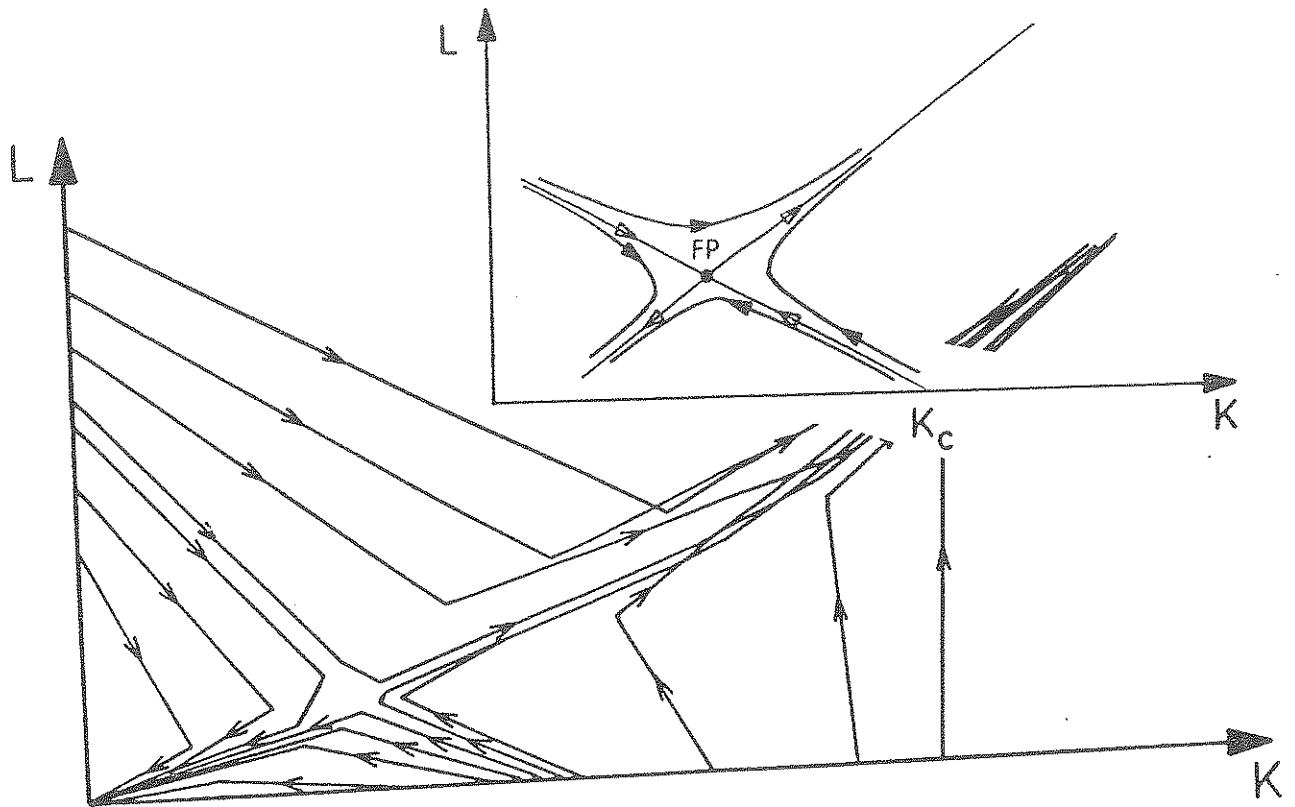


Fig. 11

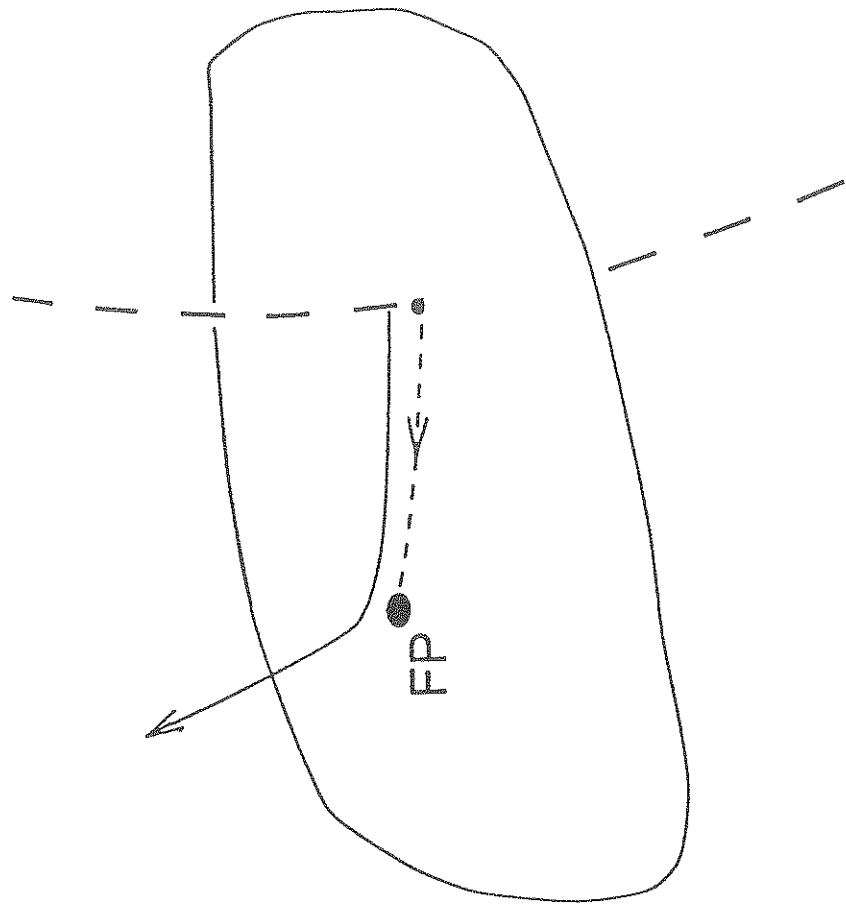


Fig. 12

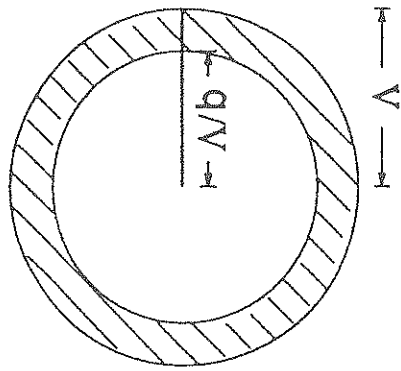


Fig. 13

