NONLINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS IN PRODUCT SPACES

by

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CHAPTER I

INTRODUCTION AND NOTATION

1.1 Introduction

The problem of developing approximation techniques for the identification and optimal control of systems governed by retarded functional differential equations (RFDE) has received considerable attention during the past few years (see [2-11,16,26-28]). Although significant advances have been made for retarded systems, approximation methods are not well developed for hereditary systems of neutral type. Any scheme to solve optimal control or identification problems for such systems will require that at some point in the analysis the governing functional differential equation be approximated.

The primary goal of this work is to lay the foundation for the development of approximation methods for optimal control and identification of neutral functional differential equations (NFDE). Although the motivation for the work is to obtain efficient numerical methods for optimal control and identification we shall concentrate primarily on the development and analysis of approximating schemes for the underlying NFDE.

There are two extreme approaches of attacking this problem; A) Discretize the equations, cost functional, etc., and reduce the entire problem to a (large) non-linear programming problem, or B) Construct an abstract state space model and "solve" the corresponding infinite dimension problems.

In this paper we shall make use of a state space model and an "approximating system" approach to develop a general approximation scheme for hereditary systems of neutral type. This approach is similar in spirit to that used by many authors for retarded equations (see [2-11]). Other approach is discussed in [23] and [24].

In general, numerical methods for neutral equations are difficult to analyze (see [15], [30]). Moreover, it is not clear that one can construct high order "finite difference" schemes for such systems. The approach suggested by Tavernini [30] does not appear to work (see [20]). On the other hand, the "approximating system" approach requires that a reasonable state space model be developed.

In this paper we extend the recent results in [12] to non-linear systems. Using the product spaces $\mathbb{R}^n \times L_2$, we develop a state space model for non-linear neutral control systems analogous to those models which have been used for retarded equations. The resulting abstract evolution equation provides the basic structure for the constructions of approximating schemes.

The paper is organized as follows. In Chapter II, Section 2.2, we establish the basic existence and continuous dependence theory for classical and generalized solutions to the neutral F.D.E. with initial data in $\mathbb{R}^n \times L_p$. Then, in Section 2.3, we obtain a nonlinear abstract evolution equation (A.E.E.) in the space $Z = \mathbb{R}^n \times L_p$ which is equivalent to the N.F.D.E. These results are a generalization of results achieved in [12] and [21] for a non-homogeneous linear neutral F.D.E. In Chapter III we study spline approximations for N.F.D.E's. Our approach does not employ the Trotter-Kato approximation theorems used in other investigations (see [4]). We use the simpler and direct ideas developed in [3] for nonlinear retarded F.D.E.'s. In Section 3.2 we construct a general approximation scheme and then in Section 3.4 we show how spline based methods can be treated as a special case of the approximation ideas of Section 3.2. In Section 3.4 we illustrate the basic idea with an example.

1.2 Notation

The following standard notation will be used throughout the paper. $L_p(a,b;\mathbb{R}^\nu)$ will denote the normed space of $\mathbb{R}^\nu$-valued "functions" (we shall not distinguish between representatives and equivalence classes since the meaning will always be clear to the reader) defined on the interval $[a,b]$ whose components are integrable when raised to the $p^{\underline{th}}$ power. $C^k(a,b;\mathbb{R}^\nu)$, $k = 0, 1, 2, \ldots$, denotes the space of $\mathbb{R}^\nu$-valued continuous functions which have k continuous derivatives on $[a,b]$. The case $k = 0$ is the usual space of continuous functions which we denote simply by $C(a,b;\mathbb{R}^\nu)$.

$PC^k(a,b;\mathbb{R}^\nu)$, $k = 0, 1, \ldots,$ denotes the space of $\mathbb{R}^\nu$-valued continuous functions x which have k derivatives on $[a,b]$ and $x^{(k)}$ (k^{th}-derivative) is piecewise continuous on $[a,b]$. $W^{k,p}(a,b;\mathbb{R}^\nu)$ is the space of $\mathbb{R}^\nu$-valued absolutely continuous functions x with derivatives such that $x^{(j)}$ (j^{th}-derivative) is absolutely continuous for $j = 1, \ldots, k-1$ and $x^{(k)}$ belongs to $L_p(a,b;\mathbb{R}^\nu)$. If the space is clear from the context, then we shall write $L_p(a,b)$, $C^k(a,b)$, $C(a,b)$, $PC^k(a,b)$ and $W^{k,p}(a,b)$ for $L_p(a,b;\mathbb{R}^\nu)$, $C^k(a,b;\mathbb{R}^\nu)$, $C(a,b;\mathbb{R}^\nu)$, $PC^k(a,b;\mathbb{R}^\nu)$ and $W^{k,p}(a,b;\mathbb{R}^\nu)$, respectively, and in the special case where $a = -r$, $b = 0$ with $r > 0$ we shall simply write L_p, C^k, C, PC^k and $W^{k,p}$. We use the symbol $||\cdot||_X$ to denote the norm on the normed linear space X. However, when it is clear which norm is intended we shall only write $||\cdot||$. The space of bounded linear operators from X to the normed linear space Y will be represented by $\mathcal{B}(X,Y)$. For an operator A we use the notation $\mathcal{D}(A)$, $\mathcal{R}(A)$ for the domain and range of A, respectively. The state space for our considerations will be the Banach space $Z = \mathbb{R}^n \times L_p$ with norm $||(\eta,\phi)||_Z = (||\eta||^2_{\mathbb{R}^n} + ||\phi||^2_{L_p})^{\frac{1}{2}}$ for $(\eta,\phi)\varepsilon Z$. $C(a,b;Z)$ will denote the set of all continuous mappings from $[a,b]$ to Z. Finally for a function $x\colon [r,\alpha) \to \mathbb{R}^n$, $\alpha > 0$, the symbol x_t, $t\varepsilon[0,\alpha)$ denotes the function $[-r,0] \to \mathbb{R}^n$ defined by $x_t(\theta) = x(t+\theta)$, $\theta\varepsilon[-r,0]$. The set of $\mathbb{R}^n$-valued measurable functions on $[-r,0]$ will be denoted by M.

CHAPTER II

NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATION ON $\mathbb{R}^n \times L_p$

2.1 Preliminaries

We consider a nonlinear control system governed by the functional differential equation (F.D.E)

$$\frac{d}{dt} Dx_t = Lx_t + f(x(t),x_t,u(t)), \tag{2.1.1}$$

with initial data

$$Dx_0 = \eta, \quad x_0 = \phi \tag{2.1.2}$$

where $\eta \varepsilon \mathbb{R}^n$, $\phi \varepsilon L_p$, $u \varepsilon L_p(0,T)$, $1 < p < +\infty$ and $0 \leq t \leq T < +\infty$.

Throughout the remainder of the paper we shall assume that L and D are linear $\mathbb{R}^n$-valued operators and f is mapping from $\mathcal{D}(f) \subset \mathbb{R}^n \times M \times \mathbb{R}^m$ to $\mathbb{R}^n$ which satisfy the following hypothesis:

H_1) The domain of L, $\mathcal{D}(L)$, and the domain of D, $\mathcal{D}(D)$, are subspaces of the Lebesgue measurable $\mathbb{R}^n$-valued functions on $[-r,0]$ such that $W^{1,p} \subset \mathcal{D}(L)$ and $C \subset \mathcal{D}(D)$.

H_2) The restriction of L to $W^{1,p}$ is bounded, the restriction of D to C is bounded and D is atomic at zero.

H_3) i) For each $a\varepsilon[0,T]$, $x\varepsilon L_p(-r,a)$ and $u\varepsilon L_p(0,T)$ the mapping $t \to f(x(t),x_t,u(t))$ belongs to $L_1(0,a)$ and depends only on the equivalence class of x.

ii) There exist two nondecreasing continuous functions α and β such that

$$\int_0^t \| f(x(s),x_s,u(s)) - f(y(s),y_s,v(s))\| ds$$

$$\leq \alpha(t)\left[\int_{-r}^{t} \|x(s)-y(s)\|^p ds\right]^{1/p}$$

$$+ \beta(t)\left[\int_0^t \|u(s)-v(s)\|^p ds\right]^{1/p}$$

for all $t\varepsilon[0,T]$, x and y belonging to $L_p(-r,T)$ and u, v in $L_p(0,T)$.

If H_1 - H_3 hold and $x\,\varepsilon\, L_p(-r,T)$, then it follows that the function

$$y(t) = Dx_t \tag{2.1.3}$$

is defined almost everywhere on $[0,T]$ and $y\,\varepsilon\, L_p(0,T)$. This observation allows us to make the following definition.

Definition 2.1.1. A **(classical)** solution to the initial value problem (2.1.1)-(2.1.2) is a function $x:[-r,T] \to \mathbb{R}^n$ such that: i) $x \varepsilon W^{1,p}(-r,T)$; ii) $x_0(s) = \phi(s)$ for $-r \leq s < 0$; iii) $y(t) = Dx_t$ is absolutely continuous on $[0,T]$; iv) $y(0) = \eta$; and v) for almost all $t \varepsilon [0,T]$

$$\frac{d}{dt} y(t) = Lx_t + f(x(t),x_t,u(t)) \ . \tag{2.1.4}$$

If $H_1 - H_2$ hold, then standard representation theorems (see [1]) imply that there exist $n \times n$ matrix-valued functions F and G whose column vectors belong to L_q, $\frac{1}{p} + \frac{1}{q} = 1$, and such that if $\phi \ \varepsilon \ W^{1,p}$, then

$$L\phi = \int_{-r}^{0} \{F(s)\phi(s) + G(s)\dot{\phi}(s)\}ds \ . \tag{2.1.5}$$

Also, without loss of generality we may assume that there exists an $n \times n$ matrix-valued function H such that H is of bounded variation on $[-r,0]$, with $\lim\limits_{s \to 0^+} \underset{[-s,0]}{\{\mathrm{Var}(H)\}} = 0$ and such that if $\phi \ \varepsilon \ C$, then

$$D\phi = \phi(0) + \int_{-r}^{0} dH(s)\phi(s) \ . \tag{2.1.6}$$

We shall use the representations 2.1.5 - 2.1.6 throughout the remainder of the paper.

If x is a solution of 2.1.1 - 2.1.2 and y is defined by 2.1.3 , then the pair $(y(t),x(t))$ satisfies the equations

$$y(t) = \eta + \int_0^t [Lx_s]ds + \int_0^t f(x(s),x_s,u(s))ds \tag{2.1.7}$$

$$Dx_t = y(t) \ . \tag{2.1.8}$$

Using the representations 2.1.5 - 2.1.6 and changing the order of integration of the integral involving G, 2.1.7 - 2.1.8 can be written in the equivalent form

$$\begin{aligned} y(t) = \eta + \int_{-r}^{0} G(s)[x(t+s)-\phi(s)]ds + \int_0^t\int_{-r}^{0} F(s)x(\tau+s)dsd\tau \\ + \int_0^t f(x(s),x_s,u(s))ds \ , \end{aligned} \tag{2.1.9}$$

$$x(t) = y(t) - \int_{-r}^{0} dH(s)x(t + s). \tag{2.1.10}$$

Equations 2.1.9 - 2.1.10 may be used to extend the definition of a solution to 2.1.1 - 2.1.2 .

Definition 2.1.2. A generalized solution of the initial value problem 2.1.1 - 2.1.2 is a pair of functions $y(t) = y(t;\eta,\phi,u)$, $x(t) = x(t;\eta,\phi,u)$ such that: i) $y \in C[0,T]$ and $x \in L_p(-r,T)$; ii) $x_0(s) = \phi(s)$ a.e. on $[-r,0]$; and iii) the pair (y,x) satisfy the system of integral equations 2.1.9 - 2.1.10 on $[0,T]$.

Note that if $x \in L_p(-r,T)$ and y is defined by 2.1.9 , then the convolution theorem (see [18]) implies that $y \in C[0,T]$. Consequently there is no loss in generality in assuming that $y \in C[0,T]$ when defining a generalized solution.

2.2 Existence and continuous dependence results

In this section we establish the basic existence, uniqueness and continuous dependence results for generalized and classical solutions of the neutral functional differential equation (2.1.1) - (2.1.2) with initial data $(\eta,\phi) \in \mathbb{R}^n \times L_p$. The next three lemmas are needed in the proofs of the main theorems. Each lemma may be found (with a proof) in the cited reference.

Lemma 2.2.1. (See [22], Lemma 5.4.3) Let $T:X \to X$ be a mapping on a complete metric space X. If for some positive integer m the mapping T^m is a contraction, then T has a unique fixed point in X.

If $\mu(s)$ is a n X n matrix valued function defined on $[-r,0]$, then we extend μ to all of $\mathbb{R}$ by

$$\mu(s) = \begin{cases} \mu(0) & s \geq 0, \\ \mu(s) & -r < s < 0, \\ \mu(-r) & s \leq -r. \end{cases}$$

If $\phi: \mathbb{R} \to \mathbb{R}^n$ is continuous and μ is of bounded variation on $[-r,0]$, then

$$\int_{-\infty}^{+\infty} d\mu(s)\phi(s) = \int_{-r}^{0} d\mu(s)\phi(s).$$

Lemma 2.2.2. (See [12], Lemma 2.4) Let $\mu:[-r,0] \to \mathbb{R}^{n \times n}$ be of bounded variation on $[-r,0]$.

(1) If $x \in L_p(\mathbb{R})$, then

$$[\mu * x](t) = \int_{-\infty}^{+\infty} d\mu(s)x(t+s)$$

is finite a.e. on $\mathbb{R}$, $[\mu * x] \in L_p(\mathbb{R})$

and

$$\|\mu * x\|_{L_p(\mathbb{R})} \leq \operatorname*{Var}_{\mathbb{R}}(\mu) \|x\|_{L_p(\mathbb{R})}.$$

(2) If $x \in L_p(\mathbb{R}) \cap W^{1,p}(-r,a)$ for some $a > 0$, then $[\mu * x] \in W^{1,p}(0,a)$, $\frac{d}{dt}[\mu * x] = [\mu * \dot{x}]$ a.e. on $[0,a]$ and there exists a constant k (independent of a, r, μ and x) such that

$$\|\mu * x\|_{W^{1,p}(0,a)} \leq k \operatorname*{Var}_{\mathbb{R}}(\mu) \|x\|_{W^{1,p}(-r,a)} .$$

<u>Lemma 2.2.3.</u> (See [12], Lemma 2.5.) Let $L \in \mathcal{B}(W^{1,p}, \mathbb{R}^n)$ have the representation (2.1.5) and $D \in \mathcal{B}(C, \mathbb{R}^n)$ be atomic at $s = 0$ with the representation (2.1.6).

(1) If $-r < -\varepsilon < 0 < a$ and $z \in L_p(-r,a)$ satisfies $z(t) = 0$ a.e. on $[-r,-\varepsilon]$, then

$$\left\|\int_{-r}^{0} dH(s)z(\cdot+s) - \int_{-r}^{0} G(s)z(\cdot+s)ds\right\|_{L_p(0,a)}$$

$$\leq \left[\operatorname*{Var}_{[-\varepsilon,0]}(H) + \int_{-\varepsilon}^{0} \|G(s)\|ds\right] \|z\|_{L_p(-r,a)}.$$

(2) If $-r < -\varepsilon < 0 < a$ and $z \in W^{1,p}(-r,a)$ satisfies $z(t) = 0$ on $[-r,-\varepsilon]$, then

$$\left\| \int_{-r}^{0} dH(s)z(\cdot+s) - \int_{-r}^{0} G(s)z(\cdot+s)ds \right\|_{W^{1,p}(0,a)}$$

$$\leq k[\operatorname*{Var}_{[-\varepsilon,0]}(H) + \int_{-\varepsilon}^{0} \|G(s)\|ds]\,\| z \|_{W^{1,p}(-r,a)}$$

where k is the constant in Lemma 2.2.2.

The following theorem establishes the fundamental existence and uniqueness of generalized solutions. The proof is based on the fixed point result given in Lemma 2.2.1.

Theorem 2.2.1. Assume that conditions $H_1 - H_3$ hold. If $(\eta,\phi) \in \mathbb{R}^n \times L_p$ and $u \in L_p(0,T)$, then the initial value problem (2.1.1) - (2.1.2) has a unique generalized solution $y(t) = y(t;\eta,\phi,u)$, $x(t) = x(t;\eta,\phi,u)$ on $[0,T]$.

Proof. Let $0 < a < r$ and define the operator $U = U(\eta,\phi,u)$ on $L_p(-r,a)$ by

$$[Ux](t) = \begin{cases} \phi(t), & -r \leq t < 0, \\ \eta - \int_{-r}^{0} G(s)\phi(s)ds + \int_{0}^{t}\int_{-r}^{0} F(s)x(\tau+s)dsd\tau & \\ \quad + \int_{-r}^{0} G(s)x(t+s)ds - \int_{-r}^{0} dH(s)x(t+s) & \\ \quad + \int_{0}^{t} f(x(s),x_s,u(s))ds, & 0 \leq t \leq a. \end{cases} \tag{2.2.1}$$

Define μ_G and μ_F by

$$\mu_G(s) = \begin{cases} -\int_{-r}^{0} G(\tau)d\tau \ , & s \le -r \ , \\ -\int_{s}^{0} G(\tau)d\tau \ , & -r \le s \le 0, \\ 0 \ , & 0 \le s \ , \end{cases} \qquad (2.2.2)$$

and

$$\mu_F(s) = \begin{cases} -\int_{-r}^{0} F(\tau)d\tau \ , & s \le -r \ , \\ -\int_{-s}^{0} F(\tau)d\tau \ , & -r \le s < 0 \ , \\ 0 \ , & 0 \le s \ , \end{cases} \qquad (2.2.3)$$

respectively. Given $x \varepsilon L_p(-r,T)$ we shall extend x to all of $\mathbb{R}$ by defining $x(s) \equiv 0$ if $s < -r$ or $s > T$ and without loss of generality assume that $a \le T$.

If $x \varepsilon L_p(-r,a)$, then conditions H_1-H_3 and Lemma 2.2.2 - (1) imply that $Ux \varepsilon L_p(-r,a)$. Consequently, U defined by (2.2.1) maps $L_p(-r,a)$, into $L_p(-r,a)$. If $x,y \varepsilon L_p(-r,a)$, then

$$[Ux - Uy](t) = \begin{cases} 0 \ , & -r \le t < 0 \ , \\ \sum_{i=1}^{4} E_i(t) \ , & 0 \le t \le a \ , \end{cases} \qquad (2.2.4)$$

where

$$E_1(t) = \int_0^t \int_{-r}^0 F(s)[x(\tau + s) - y(\tau + s)]dsd\tau \ , \qquad (2.2.5)$$

$$E_2(t) = \int_{-r}^0 G(s)[x(t + s) - y(t + s)]ds \ , \qquad (2.2.6)$$

$$E_3(t) = -\int_{-r}^0 dH(s)[x(t + s) - y(t + s)]ds \ , \qquad (2.2.7)$$

and

$$E_4(t) = \int_0^t [f(x(s),x_s,u(s)) - f(y(s),y_s,u(s))]ds \ . \qquad (2.2.8)$$

Condition H_3 implies that

$$\begin{aligned} \|E_4(\cdot)\|^p_{L_p(0,a)} &\le \int_0^a [\int_0^t \|f(x(s),x_s,u(s))-f(y(s),y_s,u(s))\|ds]^p dt \\ &\le [\int_0^a \alpha^p(t)dt]\,\|x - y\|^p_{L_p(-r,a)} \ . \end{aligned} \qquad (2.2.9)$$

Lemma 2.2.2-(1) can be applied to H, μ_G and μ_F to yield

$$\|E_3(\cdot)\|^p_{L_p(0,a)} \le [\operatorname*{Var}_{[-r,0]}(H)]^p \|x - y\|^p_{L_p(-r,a)} \ , \qquad (2.2.10)$$

$$\begin{aligned} \|E_2(\cdot)\|^p_{L_p(0,a)} &\le [\operatorname*{Var}_{\mathbb{R}}(\mu_G)]^p \|x - y\|^p_{L_p(-r,a)} \\ &= [\int_{-r}^0 \|G(s)\| ds]^p \|x - y\|^p_{L_p(-r,a)} \ , \end{aligned} \qquad (2.2.11)$$

and

$$\|E_1(\cdot)\|^p_{L_p(0,a)} = \|\int_0^t [\mu_{F^*}(x-y)](\tau)d\tau\|^p_{L_p(0,a)}$$

$$\leq \int_0^a [\int_0^t \| [\mu_{F^*}(x-y)](\tau)\|^p d\tau]\,dt \tag{2.2.12}$$

$$\leq a\,\|\mu_{F^*}(x-y)\|^p_{L_p(-r,a)}$$

$$\leq a\,[\int_{-r}^0 \|F(s)\|ds]^p \|x-y\|^p_{L_p(-r,a)}\,.$$

Define the function $M_1(z)$ by

$$M_1(z) = [\int_0^z \alpha^p(t)dt]^{1/p} + \operatorname*{Var}_{[-z,0]}(H) + \int_{-z}^0 \|G(s)\|ds + |z|^{1/p} \int_{-r}^0 \|F(s)\|ds\,. \tag{2.2.13}$$

Since $0 < a < r$, the inequalities 2.2.10 - 2.2.13 combine to yield

$$\|Ux - Uy\|_{L_p(-r,a)} = \|Ux - Uy\|_{L_p(0,a)} \leq M_1(r)\,\|x-y\|_{L_p(-r,a)}. \tag{2.2.14}$$

Consider the operator U^2. If $x,y \in L_p(-r,a)$, then let $\hat{x} = Ux$, $\hat{y} = Uy$ and note that since $[Ux](t) = [Uy](t) = \phi(t)$ a.e. on $[-r,0]$, the function $\hat{z} = \hat{x} - \hat{y}$ is zero a.e. on $[-r,0]$. Let

$$[U^2x-U^2y](t) = [U\hat{x}-U\hat{y}](t) = \begin{cases} 0 & , \quad -r \leq t < 0 \\ \sum_{i=1}^{4} \hat{E}_i(t) & , \quad 0 \leq t \leq a\,, \end{cases} \tag{2.2.15}$$

where $\hat{E}_i(t)$ are defined by 2.2.5 - 2.2.8 with x and y replaced by $\hat{x}$ and $\hat{y}$, respectively. Since $[\hat{x} - \hat{y}](t) = 0$ for a.e. $t \in [-r,0]$, Lemma 2.2.3-(1) can be applied (with $\varepsilon = a$) to obtain the inequality

$$\|\hat{E}_2(\cdot) + \hat{E}_3(\cdot)\|_{L_p(0,a)} = \left\| \int_{-r}^{0} dH(s)\hat{z}(\cdot+s) - \int_{-r}^{0} G(s)\hat{z}(\cdot+s)ds \right\| \tag{2.2.16}$$
$$\leq \left[\operatorname*{Var}_{[-a,0]}(H) + \int_{-a}^{0} \|G(s)\|\, ds \right] \|\hat{z}\|_{L_p(-r,a)} .$$

Consequently, 2.2.16 along with 2.2.9 and 2.2.12 combine to yield

$$\|U\hat{\hat{x}} - U\hat{\hat{y}}\|_{L_p(-r,a)} = \|U\hat{x} - U\hat{y}\|_{L_p(-r,a)}$$
$$\leq M_1(a)\|\hat{x} - \hat{y}\|_{L_p(-r,a)} . \tag{2.2.17}$$

Inequalities 2.2.17 and 2.2.16 imply that if $x, y \in L_p(-r,a)$, then

$$\|U^2x - U^2y\|_{L_p(-r,a)} \leq M_1(a)M_1(r)\|x - y\|_{L_p(-r,a)} . \tag{2.2.18}$$

Since $M_1(a) \to 0$ as $a \to 0$, it follows that for sufficiently small a, the operator U^2 is a contraction on $L_p(-r,a)$ with Lipschitz constant $M_1(a)M_1(r) < 1$. By Lemma 2.2.1, the operator U has a unique fixed point $x \in L_p(-r,a)$. Moreover, if we define $y:[0,a] \to \mathbb{R}^n$ by

$$y(t) = \eta + \int_{-r}^{0} G(s)[x(t+s) - x(s)]ds$$
$$+ \int_{0}^{t}\int_{-r}^{0} F(s)x(\tau+s)dsd\tau \qquad (2.2.19)$$
$$+ \int_{0}^{t} f(x(s),x_s,u(s))ds\ ,$$

then $y \in C[0,a]$ and the pair (y,x) defines a generalized solution to the initial value problem 2.1.1 - 2.1.2 on the interval $[0,a]$.

In order to obtain a solution on $[a,2a]$, let $(\eta^1,\phi^1) = (y(a),x_a)$ and let $x^1 \varepsilon L_p(-r,a)$ be the unique fixed point of $U(\eta^1,\phi^1,u)$ on $L_p(-r,a)$. Define $x^1 \varepsilon L_p(-r,2a)$ by

$$x^1(t) = \begin{cases} x(t) & , \quad -r \leq t \leq a\ , \\ x^1(t-a) & , \quad a \leq t \leq 2a, \end{cases}$$

and define $y^1(t)$ by (2.2.19) with x replaced by x^1. Clearly, (y^1,x^1) is a generalized solution defined on $[0,2a]$. This process can be continued a finite number of times to obtain a solution on $[0,T]$.∎

For each $(\eta,\phi) \ \varepsilon\ \mathbb{R}^n \times L_p$ and $u \ \varepsilon\ L_p(0,T)$, we have established the existence of a unique generalized solution to 2.1.1 - 2.1.2 $y(t) = y(t;\eta,\phi,u)$ and $x(t) = x(t;\eta,\phi,u)$ with $y \ \varepsilon\ C[0,T]$ and $x \ \varepsilon\ L_p(-r,T)$. Let $\Phi: \mathbb{R}^n \times L_p(-r,0) \times L_p(0,T) \to C[0,T] \times L_p(-r,T)$ be defined by

$$\Phi(\eta,\phi,u) = (y(\cdot;\eta,\phi,u),\ x(\cdot;\eta,\phi,u)) \qquad (2.2.20)$$

where (y,x) is the unique generalized solution to (2.1.1) - (2.2.2).

Theorem 2.2.2. If conditions H_1-H_3 hold, then the mapping Φ defined by (2.2.20) is continuous.

Proof. First we establish that the mapping $(\eta,\phi,u) \to x(\cdot;\eta,\phi,u)$ is continuous. Let Γ denote the product space $\mathbb{R}^n \times L_p(-r,0) \times L_p(0,T)$ and $\gamma = (\eta,\phi,u)$ denotes an element of Γ. We shall use the norm $\|\gamma\|_\Gamma \overset{\text{def}}{=} \|\eta\| + \|\phi\|_{L_p} + \|u\|_{L_p(0,T)}$ on Γ. Since $x(\cdot;\gamma)$ is the unique fixed point of $U^2(\gamma):L_p(-r,a) \to L_p(-r,a)$ and the family $\mathcal{U} = \{U^2(\gamma)/\gamma \in \Gamma\}$ defines a uniform contraction on $L_p(-r,a)$, if $\mathcal{U}$ is continuous in γ, then by an application of Theorem 3.3 in [17] it follows that the mapping $\gamma \to x(\cdot;\gamma)$ from Γ into $L_p(-r,a)$ is continuous.

We show that $\mathcal{U}$ is continuous in γ. If $\gamma = (\eta,\phi,u)$ and $\lambda = (\xi,\psi,v)$ belong to Γ, then for $x \in L_p(-r,a)$

$$[U(\gamma)x - U(\lambda)x](t) = \phi(t) - \psi(t), \quad \text{a.e. on } [-r,0),$$

and

$$[U(\gamma)x - U(\lambda)x](t) = \eta-\xi-\int_{-r}^{0} G(s)[\phi(s)-\psi(s)]ds$$
$$+\int_0^t [f(x(s),x_s,u(s))-f(x(s),x_s,v(s))]ds,$$
$$\text{a.e. on } [0,a]\ .$$

Consequently, defining $g(t,u,v) = \int_0^t [f(x(s),x_s,u(s))-f(x(s),x_s,v(s))]ds$ it follows that

$$\begin{aligned}\|U(\gamma)x-U(\lambda)x\|_{L_p(-r,a)} &\leq \|\phi-\psi\|_{L_p} + \|\eta-\xi\|_{L_p(0,a)} \\ &+ \left\| \int_{-r}^{0} G(s)[\phi(s)-\psi(s)]ds \right\|_{L_p(0,a)} \\ &+ \|g(t,u,v)\|_{L_p(0,a)} .\end{aligned} \tag{2.2.21}$$

Observe that

$$\|\phi-\psi\|_{L_p} + \|\eta-\xi\|_{L_p(0,a)} \leq (1 + a^{1/p}) \|\gamma-\lambda\|_{\Gamma} \tag{2.2.22}$$

and

$$\left\|\int_{-r}^{0} G(s)[\phi(s)-\psi(s)]ds\right\| \leq \|G\|_{L_q} \|\phi-\psi\|_{L_p} \leq \|G\|_{L_q} \|\gamma-\lambda\|_{\Gamma} . \tag{2.2.23}$$

Condition H_3 ii) implies that

$$\begin{aligned}\|g(t,u,v)\|_{L_p(0,a)} &\leq [\int_0^a \beta^p(t) \int_{-r}^{t} \|u(s)-v(s)\|^p \, ds \, dt]^{1/p} \\ &\leq [\int_0^a \beta^p(t)dt]^{1/p} \|u-v\|_{L_p(0,T)} \\ &\leq [\int_0^a \beta^p(t)dt]^{1/p} \|\gamma-\lambda\|_{\Gamma} .\end{aligned} \tag{2.2.24}$$

Combining 2.2.21 through 2.2.24 we have that for all $\gamma,\lambda \in \Gamma$

$$\|U(\gamma)x - U(\lambda)x\|_{L_p(-r,a)} \leq \Delta\|\gamma-\lambda\|_\Gamma \tag{2.2.25}$$

where

$$\Delta = 1 + a^{1/p} + \|G\|_{L_q} + [\int_0^a \beta^p(t)dt]^{1/p} . \tag{2.2.26}$$

Using the identity

$$\begin{aligned} U^2(\gamma)x - U^2(\lambda)x &= U(\gamma)[U(\gamma)x] - U(\lambda)[U(\lambda)x] \\ &= U(\gamma)\{U(\gamma)x - U(\lambda)x\} \\ &\quad + \{U(\gamma) - U(\lambda)\}U(\lambda)x \end{aligned}$$

and inequalities 2.2.14 and 2.2.25, it follows that

$$\|U^2(\gamma)x - U^2(\lambda)x\|_{L_p(-r,a)} \leq [M_1(r) + 1]\Delta\|\gamma-\lambda\|_\Gamma \tag{2.2.27}$$

and hence the family $\mathcal{U}$ is continuous in γ. If $x(\cdot;\gamma)$ denotes the fixed point of $U(\gamma)$, then $x(\cdot;\gamma)$ is a fixed point of $U^2(\gamma)$ and is continuous in γ. Moreover, it follows that

$$\|x(\cdot;\gamma) - x(\cdot;\lambda)\|_{L_p(-r,a)} \leq [M_1(r) + 1]\Delta\|\gamma-\lambda\|_\Gamma . \tag{2.2.28}$$

Returning to 2.2.19 , if $y(t;\gamma)$ and $y(t;\lambda)$ are defined with $x(\cdot;\gamma)$ and $x(\cdot;\lambda)$ as above then setting $g(s,\gamma,u,v) = f(x(s;\gamma),x_s(\cdot;\gamma),u(s)) - f(x(s;\gamma),x_s(\cdot;\gamma),v(s))$ it follows that

$$\begin{aligned}
\|y(t;\gamma) - y(t;\lambda)\| &\le \|\eta-\xi\| + \Big\|\int_{-r}^{0} G(s)[x(t+s;\gamma) - x(t+s\ ;\lambda)]ds\Big\| \\
&+ \int_{-r}^{0} \|G(s)\|\ \|x(s;\gamma) - x(s,\lambda)\|\,ds \\
&+ \int_{0}^{t} \Big\|\int_{-r}^{0} F(s)[x(\tau+s;\gamma) - x(\tau+s;\lambda)]ds\Big\|\,d\tau \\
&+ \int_{0}^{t} \|g(s,\gamma,u,v)\|ds \\
&\le \|\gamma-\lambda\|_{\Gamma} + \sup_{0\le t\le a} \|G(\cdot)*[x(\cdot;\gamma)-x(\cdot;\lambda)]\| \\
&+ \|G\|_{L_q}\ \|x(\cdot;\gamma) - x(\cdot;\lambda)\|_{L_p(-r,a)} \\
&+ \int_{0}^{t} \sup_{0\le\tau\le t} \|F(\cdot)*[x(\cdot;\gamma) - x(\cdot;\lambda)]\|\,d\tau \\
&+ \alpha(t)\Big[\int_{-r}^{t} \|x(s;\gamma)-x(s;\lambda)\|^{p} d\tau\Big]^{1/p} \\
&+ \beta(t)\Big[\int_{0}^{t} \|u(s) - v(s)\|^{p} ds\Big]^{1/p} \\
&\le \|\gamma-\lambda\|_{\Gamma} + 2\|G\|_{L_q} \|x(\cdot;\gamma) - x(\cdot;\lambda)\|_{L_p(-r,a)}
\end{aligned}$$

$$+ \; T \, \|F\|_{L_q} \|x(\cdot;\gamma) - x(\cdot;\lambda)\|_{L_p(-r,a)}$$

$$+ \; \alpha(t) \| x(\cdot;\gamma) - x(\cdot;\gamma)\|_{L_p(-r,a)} \tag{2.2.29}$$

$$+ \; \beta(t) \| u - v \|_{L_p(0,a)}.$$

Inequality 2.2.28 combined with 2.2.29 yield the existence of a continuous function $k(\cdot)$ (independent of a) such that

$$\|y(t;\gamma) - y(t;\lambda)\| \leq k(t) \, \|\gamma-\lambda\|_{\Gamma}$$

for all $t \in [0,a]$. As in the proof of the existence theorem above, this process can be continued to obtain the continuity of the mapping Φ from Γ into $C[0,T] \times L_p(-r,T)$. ■

In order to obtain regularity of the solutions we shall make use of an additional hypothesis on f

H_4) i) For each $a \in [0,T]$, $x \in W^{1,p}(-r,a)$ and $u \in L_p(0,a)$ the mapping $t \to \int_0^t f(x(s),x_s,u(s))ds$ belongs to $W^{1,p}(0,a)$.

ii) If $0<a<r$, there exists a continuous, nonnegative and nondecreasing function γ such that $\gamma(0) = 0$ and

$$\left\|\int_0^t [f(x(s),x_s,u(s))-f(y(s),y_s,u(s))]ds\right\|_{W^{1,p}(0,a)} \leq \gamma(a) \, \| x-y \|_{W^{1,p}(0,a)}$$

for all $x,y \in W^{1,p}(-r,a)$ such that $x(s) = y(s)$ for $s \in [-r,0]$ and $u \in L_p(0,a)$.

In Chapter III we shall give a broad family of functions which satisfies this hypothesis.

Theorem 2.2.3. Assume H_1-H_4 hold. If ϕ belongs to $W^{1,p}$ and $\eta = D\phi$, then the initial value problem 2.1.1 - 2.1.2 has a unique (classical) solution $x(t) = x(t;\eta,\phi,u)$ on $[-r,T]$.

Proof. For $0 < a < r$ let $U^* = U^*(\eta,\phi,u)$ be the operator on $W^{1,p}(0,a)$ defined by

$$(U^*x)(t) = (U\tilde{x})(t) \qquad 0 \leq t \leq a$$

where U is given by 2.2.1 and $\tilde{x}:[-r,a] \to \mathbb{R}^n$ is defined by

$$\tilde{x}(s) = \begin{cases} \phi(s) & \text{for} \quad s \in [-r,0) \,, \\ x(s) & \text{for} \quad s \in [0,a] \,. \end{cases}$$

Define μ_G and μ_F as in 2.2.2 and 2.2.3 respectively. Also if $x \in W^{1,p}(0,a)$ we shall extend $\tilde{x}$ to all of $\mathbb{R}$ by defining $\tilde{x}(s) \equiv 0$ if $s < -r$ or $s > a$ and without loss of generality assume that $a \leq T$.

If $x \in W^{1,p}(0,a)$, then conditions H_1-H_4 and Lemma 2.2.2 - (2) imply that $U^*x \in W^{1,p}(0,a)$. Consequently, U^* is an operator that maps $W^{1,p}(0,a)$ **into** $W^{1,p}(0,a)$. Also, if $x,y \in W^{1,p}(0,a)$, we have that

$$[U^*x - U^*y](t) = \sum_{i=1}^{4} E_i(t) \quad , \qquad 0 \leq t \leq a \qquad (2.2.30)$$

where $E_i(t)$, $i = 1, 2, 3, 4$, are defined by 2.2.5 through 2.2.8 , replacing x and y by $\tilde{x}$ and $\tilde{y}$, respectively. Condition H_4 implies that

$$\|E_4(\cdot)\|_{W^{1,p}(0,a)} \leq \gamma(a) \; \|x-y\|_{W^{1,p}(0,a)} . \qquad (2.2.31)$$

Lemma 2.2.3-(2) can be applied to μ and μ_G to yield

$$\|E_2(\cdot)+E_3(\cdot)\|_{W^{1,p}(0,a)} \leq k[\underset{[-a,0]}{\mathrm{Var}}(H) + \int_{-a}^{0} \|G(s)\| \, ds] \cdot \|x-y\|_{W^{1,p}(0,a)} \qquad (2.2.32)$$

and Lemma 2.2.2-(2) applied to μ_F yields

$$\|E_1(\cdot)\|_{W^{1,p}(0,a)} \leq (a^{1/p}+1) \quad (\int_{-a}^{0} \|F(s)\| \, ds) \cdot \|x-y\|_{W^{1,p}(0,a)} . \qquad (2.2.33)$$

Define the function $N_1(z)$ by

$$N_1(z) = \gamma(z) + \underset{[-z,0]}{\mathrm{Var}}\,(H) + \int_{-z}^{0} \|G(s)\|\, ds$$
$$+ (z^{1/p}+1) \cdot \int_{-z}^{0} \|F(s)\|\, ds\,. \tag{2.2.34}$$

Since $0 < a < r$, it follows from the inequalities 2.2.31 - 2.2.33 that

$$\|U^*x - U^*y\|_{W^{1,p}(0,a)} \le N_1(a) \quad \|x-y\|_{W^{1,p}(0,a)}\,.$$

Note that $N_1(a) \to 0$ as $a \to 0$, and hence for sufficiently small a the operator U^* is a contraction on $W^{1,p}(0,a)$. Consequently, it has a unique fixed point $x \in W^{1,p}(0,a)$. Moreover, $\tilde{x}$ satisfies
i) $\tilde{x} \in W^{1,p}(-r,a)$; ii) $\tilde{x}_0(s) = \phi(s)$ for $-r \le s < 0$, and if we define $y(t)$ by

$$y(t) = \eta + \int_{-r}^{0} G(s)[\tilde{x}(t+s)-\tilde{x}(s)]ds + \int_0^t\int_{-r}^{0} F(s)\tilde{x}(\tau+s)\, ds d\tau$$
$$+ \int_0^t f(\tilde{x}(s),\, \tilde{x}_s, u(s))ds\,, \tag{2.2.35}$$

then y satisfies iii), iv) and v) in definition 2.1.1. Therefore , $\tilde{x}$ is a solution of 2.1.1 - 2.1.2 defined on $[-r,a]$. This solution can be extended to the whole interval $[-r,T]$ using the same argument given in the proof of Theorem 2.2.1. ∎

If L and D are of the following special type:

$$D\phi = \phi(0) - \sum_{j=1}^{\nu} B_j\phi(-r_j) - \int_{-r}^{0} B(s)\phi(s)ds \;,$$

$$L\phi = \sum_{j=0}^{\nu} A_j\phi(-r_j) + \int_{-r}^{0} A(s)\phi(s)ds \;, \qquad (2.2.36)$$

where $0 = r_0 < \ldots < r_\nu = r$ and $A(\cdot)$, $B(\cdot)$ are in $L_q(-r,0;\mathbb{R}^{n\times n})$, then the following result is a direct consequence of the above proposition (the case $f(x(t),x_t,u(t)) = u(t)$ was established by Kappel in [21]).

Corollary 2.2.1. Assume H_3-H_4 hold. If $\phi \in W^{1,p}$ and $\eta = D\,\phi$, then x is in $W^{1,p}$ and x is the unique solution of

$$\dot{x}(t) = \sum_{j=1}^{\nu} B_j\dot{x}(t-r_j) + \int_{-r}^{0} B(s)\dot{x}(t+s)ds + L(x_t)+f(x(t),x_t,u(t)),$$

$$x_0 = \phi \quad , \qquad D\phi = \eta.$$

The conditions H_1-H_4 are global in the sense that the functions α, β, and γ are independent of the "size of x". However, one can easily modify these assumptions to obtain local existence and continuous dependence results. In particular, condition H_3 may be replaced by the following condition.

LH_3) In addition to H_3i) we assume that for any bounded set M in $L_p(-r,t_1)\times L_p(0,t_1)$ there exist two nondecreasing continuous functions

α_M and β_M such that

$$\int_0^t \|f(x(s),x_s,u(s))-f(y(s),y_s, v(s))\| \, ds$$

$$\leq \alpha_M(t)[\int_0^t \|x(s)-y(s)\|^p ds]^{1/p}$$

$$+ \beta_M(t)[\int_0^t \|u(s)-v(s)\|^p ds]^{1/p} ,$$

for all (x,u), $(y,v) \in M$ and $0 \leq t \leq t_1$.

With this hypothesis one can establish the following local results similar to Theorem 2.2.1.

Proposition 2.2.1. Suppose that conditions H_1,H_2 and LH_3 hold. Given $\delta > 0$, there exists a constant $a(\delta) > 0$ such that if $(\eta,\phi) \in \mathbb{R}^n \times L_p$ and $\|(\eta,\phi)\| \leq \delta$, then the problem 2.1.1 - 2.1.2 has a unique generalized solution on $[-r,a(\delta)]$.

Proof. For $0 < a < r$, let B^1 be the set

$$B^1 = \{\psi \in L_p(0,a) / \|\psi\|_{L_p(0,a)} \leq 1\}$$

and for each $\psi \in B^1$, define $\tilde{\psi}$ by

$$\tilde{\psi}(s) = \begin{cases} \phi(s) & \text{for} \quad s \varepsilon [-r,0] \\ \psi(s) & \text{for} \quad s \varepsilon [0,a] \end{cases} .$$

The operator U^* is defined on B^1 by

$$(U^*x)(t) = (U\tilde{x})(t) \qquad \forall t \varepsilon [0,a],$$

where U is given by 2.2.1. Since $U\tilde{x} \varepsilon L_p(-r,a)$, it follows that $U^*x \varepsilon L_p(0,a)$ and

$$||U^*x||_{L_p(0,a)} \leq ||U^*x - U^*\theta||_{L_p(0,a)} + ||U^*\theta||_{L_p(0,a)} , \qquad (2.2.37)$$

where $\theta \varepsilon B^1$ is the zero function.

Since $\tilde{x}(s) = \tilde{\theta}(s) = \phi(s)$ on $[-r,0)$, as in Theorem 2.2.1 we obtain

$$||U\tilde{x}-U\tilde{\theta}||_{L_p(0,a)} \leq M_1(a) \, ||\tilde{x} - \tilde{\theta}||_{L_p(-r,a)} \leq M_1(a). \qquad (2.2.38)$$

Therefore,

$$||U^*x||_{L_p(0,a)} \leq M_1(a) + ||U^*\theta||_{L_p(0,a)} = P_1(a). \qquad (2.2.39)$$

On the other hand,

$$\|U^*x-U^*y\|_{L_p(0,a)} \le \|U\tilde{x}-U\tilde{y}\|_{L_p(0,a)}$$

$$\le M_1(a)\,\|\tilde{x}-\tilde{y}\|_{L_p(-r,a)} = P_2(a)\|x-y\|_{L_p(0,a)} \ . \tag{2.2.40}$$

If $i=1,2$, then $\lim_{a\to 0^+} P_i(a)=0$ and hence there is a value $a > 0$ such that U^* is a contraction from B^1 to B^1. Thus the operator U^* has a unique fixed point $x \in B^1$. Moreover, if we define $y:[0,a] \to \mathbb{R}^n$ by 2.2.19 we have that $y \in C(0,a)$ and the pair $(y,\tilde{x})$ defines a generalized solution to the initial value problem 2.1.1 - 2.1.2 on the interval $[0,a]$. ■

As indicated by the previous result, it is clearly possible to improve upon the "global" existence and continuous dependence results (i.e. Theorems 2.2.1 - 2.2.3). In particular, one can establish a local existence result (such as Proposition 2.2.1) and then ask if these local solutions can be extended to the interval $[0,T]$. Since our primary concern is the development of approximation schemes for neutral equations, we shall not take time to investigate these types of questions. Consequently, we shall be interested in approximating a solution on the fixed interval $[0,T]$ on which we assume the solution exists.

2.3 An Equivalence Theorem

In this section we establish an equivalence between generalized solutions to NFDES and the mild solutions to a corresponding abstract evolution equation.

Definition 2.3.1. For each $0 \leq t \leq T$ define the operator $S(t)$ from Z to Z by the relation

$$S(t)(\eta,\phi) = (y(t),x_t) \quad , \quad (\eta,\phi)\varepsilon Z$$

where $(y(t), x(t))$ is the unique generalized solution of the initial value problem 2.1.1 - 2.1.2 for f identically zero.

The following lemma establishes the properties of the family $\{S(t);0\leq t\leq T\}$ that will be needed later. It is stated without proof. For a proof of 1) and 2) see Burns-Herdman-Stech [12] and for a proof of 3) see Hille and Phillips [19].

Lemma 2.3.1. The family of operators $\{S(t);0\leq t\leq T\}$ defined above satisfies the following properties:

1) It is a C_o-semigroup of bounded linear operators on Z.

2) If $\mathcal{A}$ denotes its infinitesimal generator, then $\mathcal{D}(\mathcal{A}) = \{(\eta,\phi)\varepsilon Z/\phi\varepsilon W^{1,p}$ and $D\phi = \eta\}$ and $\mathcal{A}(\eta,\phi) = (L\phi,\dot{\phi})$ for all $(\eta,\phi)\varepsilon\mathcal{D}(\mathcal{A})$.

3) There exist constants $M,\beta > 0$ such that

$$\|S(t)\| \leq M\, e^{\beta t} \ , \quad 0 \leq t \leq T \ .$$

Define the mapping $F:\mathcal{D}(A) \times \mathbb{R}^m \to Z$ by

$$F((D\phi,\phi),u) = (f(\phi(0),\phi,u),0). \tag{2.3.1}$$

Suppose that $\phi \varepsilon W^{1,p}$ and $D\phi = \eta$ and let $(y(t),x(t))$ be the unique solution to 2.1.1 - 2.1.2 . If $z(t) = (y(t),x_t) = (Dx_t,x_t)$, then it follows from Theorem 2.2.3 that $z(t) \varepsilon \mathcal{D}(A)$ and

$$z'(t) = (Lx_t, \frac{\partial}{\partial s} x(t+s)) + (f(x(t),x_t\ ,u(t))\ ,0)$$

$$= Az(t) + F((y(t),x_t),u(t)) \tag{2.3.2}$$

$$= Az(t) + F(z(t),u(t)).$$

Moreover, if one defines $G(s) = S(t-s)z(s)$ then $G(s)$ is continuously differentiable for $0 < s < t$ (see [25], page 110) and

$$G'(s) = -S(t-s)Az(s) + S(t-s)z'(s) = S(t-s)F(z(s),u(s)).$$

Integrating $G'(s)$ from 0 to t we get

$$z(t) = S(t)z_0 + \int_0^t S(t-s)F(z(s),u(s))ds, \tag{2.3.3}$$

where $z_0 = z(0) = (\eta,\phi)$.

Therefore, $z(t) = (y(t),x_t)$ is a solution of the abstract evolution equation 2.3.3 . Later, see Lemma 2.3.2 below, we shall prove that it is the unique solution of that equation.

Let $C(0,a;Z)$ denote the space of continuous Z-valued functions defined on $[0,a]$ (with $0 < a \leq T$) and let $\mathcal{D} \subseteq C(0,a;Z)$ be defined by

$$\mathcal{D} = \{z\varepsilon C(0,a;Z)/z(t) = (Dx_t,x_t) \text{ for some } x\varepsilon W^{1,p}(-r,a)\}.$$

For each $u\varepsilon L_p(0,a)$ define the operator $\mathcal{F}(\cdot,u):\mathcal{D} \to C(0,a;Z)$ by

$$[\mathcal{F}(z,u)](t) = \int_0^t S(t-s)F(z(s), u(s))ds \ .$$

We shall need the following strengthened form of H_3

SH_3) In addition to H_3-i we assume that there exists a nondecreasing continuous function α_1 such that the inequality

$$\left[\int_0^t \|f(x(s),x_s,u(s)) - f(y(s),y_s,u(s))\|^p ds\right]^{1/p} \leq \alpha_1(t) \left[\int_{-r}^t \|x(s)-y(s)\|^p ds\right]^{1/p}$$

holds for all $t\varepsilon[0,T]$, x and y belonging to $W^{1,p}(-r,T)$ and u in $L_p(0,T)$.

<u>Proposition 2.3.1</u>. 1) For each $z\varepsilon\mathcal{D}$ and $u\varepsilon L_p(0,a)$, $\mathcal{F}(z,u)$ belongs to

the closure of $\mathcal{D}$ in $C(0,a;Z)$. 2) If SH_3 holds, then there exists a continuous function $\Gamma(a)$ such that $\Gamma(a) \to 0$ as $a \to 0$ and

$$\| F(z,u) - F(w,u) \|_{C(0,a;Z)} \leq \Gamma(a) \cdot \|z-w\|_{C(0,a;Z)}$$

for z, $w \varepsilon \mathcal{D}$ and $u \varepsilon L_p(0,a)$.

<u>Proof of 1)</u>. Let $\lambda(N)$ be a sequence of real numbers such that $\lambda(N) \to +\infty$ as $N \to +\infty$. For N sufficiently large the resolvent operator $R_{\lambda(N)} = (\lambda(N) - A)^{-1}$ exists and we may define F^N on $\mathcal{D}(F)$ by

$$[F^N(z,u)](t) = \int_0^t S(t-s)\lambda(N)R_{\lambda(N)}F(z(s),u(s))ds \ .$$

Since $R_{\lambda(N)} F(z(s),u(s))$ belongs to $\mathcal{D}(A)$ for each $s \varepsilon [0,a]$ it follows that

$$S(t-s)\lambda(N)R_{\lambda(N)}F(z(s),u(s)) \varepsilon \mathcal{D}$$

and a simple limiting argument shows that

$$F^N(z,u) \varepsilon \bar{\mathcal{D}} \ .$$

On the other hand, the inequality

$$\begin{aligned}
&\|F^N(z,u)(t)-F(z,u)(t)\|_Z \\
&\quad \le \int_0^t M\, e^{\beta(t-s)} \|(\lambda(N)R_{\lambda(N)}-I)F(z(s),u(s))\|_Z ds \\
&\quad \le M\, e^{\beta a} \int_0^a \|(\lambda(N)R_{\lambda(N)}-I)F(z(s),u(s))\|_Z ds
\end{aligned}$$

yields the estimate

$$\begin{aligned}
&\|F^N(z,u) - F(z,u)\|_{C(0,a;Z)} \\
&\quad \le M\, e^{\beta a} \int_0^a \|(\lambda(N)R_{\lambda(N)}-I)F(z(s),u(s))\|_Z ds \ .
\end{aligned}$$

The dominated convergence theorem and the strong convergence of $\lambda(N)R_{\lambda(N)}$ to I imply that $F^N(z,u)$ converges to $F(z,u)$ in $C(0,a;Z)$. Since $F^N(z,u)\varepsilon\bar{\mathcal{D}}$, it follows that $F(z,u)\varepsilon\bar{\mathcal{D}}$.

<u>Proof of 2)</u>. Let $z(t) = (Dx_t,x_t)$, $w(t) = (Dy_t,y_t)$ where $x,y\varepsilon W^{1,p}(-r,a)$ and $u\varepsilon L_p(0,a)$. We may assume without loss in generality that $0 < a < r$. For each $t\varepsilon[0,a]$ we have that

$$\begin{aligned}
&\|[F(z,u)](t) - [F(w,u)](t)\|_Z \\
&\quad \le \int_0^t \|S(t-s)\| \ \|F(z(s),u(s))-F(w(s),u(s))\|_Z ds \\
&\quad \le M\, e^{\beta t} \int_0^t e^{-\beta s} \|f(x(s),x_s,u(s)) \\
&\qquad - f(y(s),y_s,u(s))\| ds \ .
\end{aligned} \tag{2.3.4}$$

Applying Holder's inequality to the integral we obtain

$$\| [F(z,u)](t) - [F(w,u)](t) \|$$

$$\leq Me^{\beta a} \left[\frac{1-e^{-q\beta a}}{q\beta} \right]^{1/q} \cdot$$

$$[\int_0^t \| f(x(s),x_s,u(s))-f(y(s),y_s u(s))\|^p ds]^{1/p}.$$

An application of SH_3 yields

$$\| [F(z,u](t) - [F(w,u)](t) \|_Z$$

$$\leq Me^{\beta a} \left[\frac{1-e^{-q\beta a}}{q\beta} \right]^{1/q} \alpha_1(a)$$

$$\cdot \; [\int_{-r}^a \|x(s)-y(s)\|^p ds]^{1/p}. \tag{2.3.5}$$

However, from the relation

$$\int_{-r}^a \|x(s)-y(s)\|^p ds = \int_{-r}^0 \|x(s)-y(s)\|^p ds$$

$$+ \int_0^a \|x(s)-y(s)\|^p ds$$

$$= \|x_0-y_0\|_{L_p}^p + \int_{-a}^0 \|x(s+a)-y(s+a)\|^p ds$$

$$\leq \|x_0-y_0\|_{L_p}^p + \|x_a-y_a\|_{L_p}^p$$

and Minkowski's inequality it follows that

$$[\int_{-r}^{a} \|x(s)-y(s)\|^p ds]^{1/p} \leq \|x_0-y_0\|_{L_p} + \|x_a-y_a\|_{L_p}$$

$$\leq 2 \sup_{0\leq t\leq a} \|(Dx_t,x_t)$$

$$- (Dy_t,y_t)\|_Z \qquad (2.3.6)$$

$$\leq 2\|z-w\|_{C(0,a;Z)}.$$

If we define

$$\Gamma(a) = Me^{\beta a}\left[\frac{1-e^{-q\beta a}}{q\beta}\right]^{1/q} 2 \quad \alpha_1(a) \quad ,$$

then 2.3.5 and 2.3.6 imply that

$$\|F(z,u)-F(w,u)\|_{C(0,a;Z)} \leq \Gamma(a)\,\|z-w\|_{C(0,a;Z)}$$

and this completes the proof. ■

The above proposition implies that F is a continuous operator from $\mathcal{D}$ to $\bar{\mathcal{D}}$. Since $\bar{\mathcal{D}}$ is complete we can extend F continuously to $\bar{\mathcal{D}}$. This extension will be called $\bar{F}$ and is defined for $z\varepsilon\bar{\mathcal{D}}$ by

$$\bar{F}(z,u) = \lim F(z^n,u)$$

where z^n is a sequence in $\mathcal{D}$ that converges to z. The next result is a direct consequence of Proposition 2.3.1 and the definition of $\bar{F}$.

Corollary 2.3.1. If SH_3 holds, then there exists a nonnegative and nondecreasing function $\Gamma(a)$ such that $\Gamma(a) \to 0$ as $a \to 0$ and

$$||\bar{F}(z,u) - \bar{F}(w,u)||_{C(0,a;Z)} \leq \Gamma(a) \cdot ||z-w||_{C(0,a;Z)}$$

for each z, $w\varepsilon\bar{\mathcal{D}}$ and $u\varepsilon L_p(0,a)$.

Proposition 2.3.2. For each $(\eta,\phi)\varepsilon Z$ and $u\varepsilon L_p(0,T)$ the abstract equation

$$z(t) = S(t)(\eta,\phi) + \bar{F}(z,u)(t) , \qquad 0 \leq t \leq T \qquad (2.3.7)$$

has a unique solution in $\bar{\mathcal{D}}$ which depends continuously on (η,ϕ).

Proof. Let a be such that $\Gamma(a) < 1$ and $\mathcal{H}:\bar{\mathcal{D}} \to \bar{\mathcal{D}}$ be the operator defined by

$$[\mathcal{H}(z)](t) = S(t)(\eta,\phi) + \bar{\mathcal{F}}(z,u)(t) \; .$$

By Corollary 2.3.1 $\bar{\mathcal{F}}$ is a contraction on $\bar{\mathcal{D}}$, thus $\mathcal{H}$ is also a contraction on $\bar{\mathcal{D}}$, and by the contraction mapping theorem (see [17], page 7) it has a unique fixed point that depends continuously on (η,ϕ). This fixed point is the unique solution for the Equation 2.3.7 on [0,a]. A simple step argument allows us to extend this solution to the whole interval [0,T]. ■

If ϕ belongs to $W^{1,p}$, $D\phi = \eta \in \mathbb{R}^n$ and $x(t)$ is the corresponding classical solution of 2.1.1-2.1.2, we have already shown that $z(t) = (y(t),x_t)$ belongs to $\mathcal{D}$ and satisfies 2.3.3. As a consequence of Proposition 2.3.2 we have the following result.

<u>Lemma 2.3.2</u>. Let $\eta \in \mathbb{R}^n, \phi \in W^{1,p}$ and $D\phi = \eta$. If $x(t)$ is the corresponding classical solution of 2.1.1 - 2.1.2, then $z(t) = (y(t),x_t)$ is the unique solution of the equation

$$z(t) = S(t)\, z_0 + \int_0^t S(t-s)F(z(s),u(s))ds$$

where $z_0 = (\eta,\phi)$.

Theorem 2.3.1. Let $\eta \varepsilon \mathbb{R}^n$, $\phi \varepsilon L_p$. If $(y(t),x(t))$ is the generalized solution of 2.1.1 - 2.1.2 then $z(t) = (y(t),x_t)$ is the unique solution of the equation

$$z(t) = S(t)z_0 + \bar{F}(z,u)(t),$$

where $z_0 = (\eta,\phi)$.

Proof. If $\phi \varepsilon L_p$ there exists a sequence $\phi^n \varepsilon W^{1,p}$ such that $\phi^n \to \phi$ in L_p and $D\phi^N = \eta^N \to \eta$. If, for each n, $(y^n(t),x^n(t))$ is the corresponding solution of 2.1.1 - 2.1.2, by Lemma 2.3.2 $z^n(t) = (y^n(t),x_t^n)$ satisfies

$$z^n(t) = S(t)(\eta^n,\phi^n) + F(z^n,u)(t) .$$

By proposition 2.3.2 the solution of this equation converges to the solution of 2.3.7. On the other hand, by Theorem 2.2.2 $y^n(t) = y(t;\eta^n,\phi^n,u)$, $x^n(t) = x(t;\eta^n,\phi^n,u)$ converge to $y(t) = y(t;\eta,\phi,u)$, $x(t) = x(t;\eta,\phi,u)$ in $C(0,T) \times L_p(-r,T)$. Consequently, $(y^n(t),x_t^n)$ converges (uniformly) to $(y(t),x_t)$ and hence $z(t) = (y(t),x_t)$ satisfies

$$z(t) = S(t)(\eta,\phi) + \bar{F}(z,u)(t) . \quad \blacksquare$$

CHAPTER III

APPROXIMATION OF SOLUTIONS

3.1 Preliminaries

Our primary concern in this chapter is the construction of approximate solutions to the nonlinear FDE initial value problem defined by

$$\frac{d}{dt}Dx_t = Lx_t + h(x(t),x(t-r_1),\ldots,x(t-r_\nu),x_t,u(t)), \qquad (3.1.1)$$

$$Dx_0 = \eta \;, \qquad x_0 = \phi \;, \qquad (3.1.2)$$

where L and D are linear operators, h is a mapping $\mathbb{R}^{n(\nu+1)} \times L_2 \times \mathbb{R}^m \to \mathbb{R}^n$; $u \in L_2(0,T)$, $\eta \in \mathbb{R}^n$, $\phi \in W^{1,2}$, $0 \leq t \leq T < +\infty$ and $0 < r_1 < \cdots \cdots < r_\nu = r$.

We shall make use of the following hypothesis throughout this chapter:

$\hat{H}_1$) L and D have the form

$$L\phi = \sum_{j=0}^{\nu} A_j\phi(-r_j) + \int_{-r}^{0} A(s)\phi(s)ds$$

$$D\phi = \phi(0) - \sum_{j=0}^{\nu} B_j\phi(-r_j) - \int_{-r}^{0} B(s)\phi(s)ds \;,$$

where A_j and B_j are $n \times n$ constant matrices, $A(\cdot)$ belongs to $L_2(-r,0;\mathbb{R}^{n\times n})$ and $B(\cdot)$ is C^1.

$\hat{H}_2$) There exists a constant K such that

$$\| h(x_0,x_1,\ldots,x_\nu,\phi,u) - h(y_0,y_1,\ldots,y_\nu,\psi,v)\|$$

$$\leq K\{\sum_{j=0}^{\nu} \| x_j-y_j \|_{\mathbb{R}^n} + \|\phi-\psi\|_{L_2} + \|u-v\|_{\mathbb{R}^m}\}$$

for all $(x_0,x_1,\ldots,x_\nu,\phi,u),(y_0,y_1,\ldots,y_\nu,\psi,v)$ $\varepsilon\, \mathbb{R}^{n(\nu+1)} \times L_2 \times \mathbb{R}^m$.

$\hat{H}_3$) h is continuously differentiable.

Remark 3.1.1. Corresponding to the right-hand side of equation 3.1.1 we define the function $H:C \times \mathbb{R}^m \to \mathbb{R}^n$ by

$$H(\phi,u) = L\phi + h(\phi(0),\phi(-r_1),\ldots,\phi(-r_\nu),\phi,u) .$$

It is easy to verify the existence of a constant K' such that

$$\|H(\phi,u)-H(\psi,v)\| \leq K'\{\sum_{j=0}^{\nu} \|\phi(-r_j)-\psi(-r_j)\|_{\mathbb{R}^n}$$

$$+ \|\phi-\psi\|_{L_2} + \|u-v\|_{\mathbb{R}^m}\}$$

for all $\phi,\psi\varepsilon C$ and $u,v\varepsilon\, \mathbb{R}^m$.

Remark 3.1.2. From $\hat{H}_1$ it is clear that the operators L and D satisfy hypothesis H_1 and H_2 imposed in 2.1. Now, we shall show that the mapping $f:\mathcal{D}(f) \equiv \mathbb{R}^n \times C \times \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$f(x,\phi,u) = h(x,\phi(-r_1), \ldots, \phi(-r_\nu),\phi,u) \tag{3.1.3}$$

satisfies hypothesis H_3, H_4 and SH_3 for $p = 2$.

To prove the first part of H_3 and H_4 it is enough to show that the mapping $t \to f(x(t),x_t,u(t))$ belongs to $L_2(0,T)$ for all $x \epsilon L_2(-r,T)$ and $u \epsilon L_2(0,T)$. This is easily established using the inequality

$$\int_0^T \|f(x(s),x_s,u(s))\|^2 ds \leq 2 \int_0^T \| f(x(s),x_s,u(s))-f(0,0,0)\|^2 ds$$

$$+ 2 \int_0^T \|f(0,0,0)\|^2 ds$$

and hypothesis $\hat{H}_2$.

For each $x,y \epsilon L_2(-r,T)$ and $u,v \epsilon L_2(0,T)$, $\hat{H}_2$ implies that

$$\int_0^t \|f(x(s),x_s,u(s))-f(y(s),y_s,v(s))\| ds$$

$$\leq \int_0^t \|h(x(s),x(s-r_1),\ldots,x(s-r_\nu),x_s,u(s))$$

$$- h(y(s),y(s-r_1),\ldots,y(s-r_\nu),y_s,v(s))\| ds$$

$$\leq K \int_0^t \{ \sum_{j=0}^{\nu} \| x(s-r_j) - y(s-r_j) \|_{\mathbb{R}^n}$$

$$+ \| x_s - y_s \|_{L_2} + \| u(s) - v(s) \|_{\mathbb{R}^m} \} ds$$

$$\leq K \{ \sum_{j=0}^{\nu} t^{\frac{1}{2}} [\int_0^t \| x(s-r_j) - y(s-r_j) \|^2 ds]^{\frac{1}{2}}$$

$$+ \int_0^t [\int_{-r}^0 \| x_s(\theta) - y_s(\theta) \|^2 \, d\theta]^{\frac{1}{2}} ds$$

$$+ t^{\frac{1}{2}} [\int_0^t \| u(s) - v(s) \|^2 ds]^{\frac{1}{2}} \}$$

$$= K \{ \sum_{j=0}^{\nu} t^{\frac{1}{2}} [\int_{-r_j}^{t-r_j} \| x(s') - y(s') \|^2 ds']^{\frac{1}{2}}$$

$$+ \int_0^t [\int_{s-r}^s \| x(s') - y(s') \|^2 \, ds']^{\frac{1}{2}} ds$$

$$+ t^{\frac{1}{2}} [\int_0^t \| u(s) - v(s) \|^2 \, ds]^{\frac{1}{2}} \}$$

$$\leq K \, t^{\frac{1}{2}} \{ (\nu+1) + t^{\frac{1}{2}} \} [\int_{-r}^t \| x(s) - y(s) \|^2 \, ds]^{\frac{1}{2}}$$

$$+ K \, t^{\frac{1}{2}} [\int_0^t \| u(s) - v(s) \|^2 \, ds]^{\frac{1}{2}}.$$

Consequently, H_2-ii holds with $\alpha(t) = Kt^{\frac{1}{2}}\{(\nu+1) + t^{\frac{1}{2}}\}$ and $\beta(t) = Kt^{\frac{1}{2}}$.

Another application of $\hat{H}_2$ yields

$$\|f(x(t),x_t,u(t)) - f(y(t),y_t,u(t))\|_{L_2(0,a)}$$

$$\leq \|h(x(t),x(t-r_1),\dots,x(t-r_\nu),x_t,u(t))$$

$$- h(y(t),y(t-r_1),\dots,y(t-r_\nu),y_t,u(t))\|_{L_2(0,a)}$$

$$\leq K\{\sum_{j=0}^{\nu} \|x(t-r_j)-y(t-r_j)\|_{L_2(0,a)}$$

$$+\| \|x_t - y_t\|_{L_2} \|_{L_2(0,a)}\} . \qquad (3.1.4)$$

Define the linear operators $L^j : C \to \mathbb{R}^n$, $j = 0, 1, \dots, \nu$, by

$$L^j(\phi) = \phi(-r_j) .$$

Clearly, for each $\phi \varepsilon W^{1,2}$

$$L^j(\phi) = \int_{-r}^{0} [F^j(s)\phi(s) + G^j(s)\dot{\phi}(s)]ds ,$$

where F^j and G^j, $j = 0, 1, \dots, \nu$, are $n \times n$ matrix functions such that $F^j(s) = G^j(s) = 0$ if $-r \leq s < -r_j$; $F^j(s) = (1/r_j) \cdot I_n$ and $G^j(s) = (s/r_j) \cdot I_n$ if $-r_j \leq s \leq 0$ with I_n the $n \times n$ identity matrix. Changing the order of integration of the integral involving G and assuming that $x(s) = y(s)$ for $-r \leq s < 0$ it follows that

$$\int_0^t L^j(x_s-y_s)ds = \int_{-a}^0 G^j(s)[x(t+s)-y(t+s)]ds$$

$$+\int_0^t\int_{-a}^0 F^j(s)[x(z+s)-y(z+s)]dsdz$$

$$= E_1(t) + E_2(t) \ , \qquad (3.1.5)$$

where the functions $E_i(t)$, $i = 1,2$ are defined by the relations 2.2.5 and 2.2.6 with $F = F^j$, $G = G^j$ and $r = a$. As in the proof of Theorem 2.2.3 we obtain the estimates

$$\|E_1(\cdot)\|_{W^{1,2}(0,a)} \leq [a^{\frac{1}{2}} + 1][\int_{-a}^0 \|F^j(s)\| ds]$$

$$\cdot \|x - y\|_{W^{1,2}(0,a)} \ , \qquad (3.1.6)$$

and

$$\|E_2(\cdot)\|_{W^{1,2}(0,a)} \leq k \ [\int_{-a}^0 \|G^j(s)\| ds]$$

$$\cdot \|x - y\|_{W^{1,2}(0,a)} \ , \qquad (3.1.7)$$

which in turn yield the inequality

$$\|\int_0^t[x(s-r_j)-y(s-r_j)]ds\|_{W^{1,2}(0,a)}$$

$$\leq \gamma_{1_j}(a) \ \|x - y\|_{W^{1,2}(0,a)} \ , \qquad (3.1.8)$$

where

$$\gamma_{1_j}(a) = [a^{\frac{1}{2}} + 1]\cdot[\int_{-a}^{0} \|F^j(s)\| \, ds]$$

$$+ k \int_{-a}^{0} \|G^j(s)\| \, ds \quad . \tag{3.1.9}$$

Therefore,

$$\|x(t-r_j)-y(t-r_j)\|_{L_2(0,a)} \leq \gamma_{1_j}(a)\cdot \|x - y\|_{W^{1,2}(0,a)} \tag{3.1.10}$$

for all $j = 0, 1, \ldots, \nu$. On the other hand we have

$$\| \|x_t-y_t\|_{L_2} \|^2_{L_2(0,a)} = \int_0^a \|x_t-y_t\|^2_{L_2} dt$$

$$= \int_0^a [\int_{-r}^{0} \|x(t+s)-y(t+s)\|^2 ds] dt$$

$$= \int_0^a \int_{t-r}^{t} \|x(s')-y(s')\|^2 ds' dt$$

$$\leq a \cdot \int_0^a \|x(s')-y(s')\|^2 ds' = a\cdot\|x-y\|^2_{L_2(0,a)}$$

$$\leq a \cdot \|x-y\|^2_{W^{1,2}(0,a)} . \tag{3.1.11}$$

Combining 3.1.4-3.1.10 with 3.1.11 we obtain

$$\|f(x(t),x_t,u(t))-f(y(t),y_t,u(t)\|_{L_2(0,a)}$$

$$\leq \gamma_1(a) \cdot \|x-y\|_{W^{1,2}(0,a)} \qquad (3.1.12)$$

where

$$\gamma_1(a) = K\{\sum_{j=0}^{\nu} \gamma_{1_j}(a) + a^{\frac{1}{2}}\} .$$

Moreover,

$$\|\int_0^t[f(x(s),x_s,u(s)-f(y(s),y_s,u(s))]ds\|^2_{L_2(0,a)}$$

$$= \int_0^a \|\int_0^t[f(x(s),x_s,u(s))-f(y(s),y_s,u(s))]ds\|^2 dt$$

$$\leq \int_0^a\int_0^a \|f(x(s),x_s,u(s))-f(y(s),y_s,u(s))\|^2 ds\, dt$$

$$= a \|f(x(t),x_t,u(t))-f(y(t),y_t,u(t))\|^2_{L_2(0,a)}$$

$$\leq a\gamma_1^2(a) \|x-y\|^2_{W^{1,2}(0,a)} . \qquad (3.1.13)$$

Therefore, we have

$$\|\int_0^t[f(x(s),x_s,u(s))-f(y(s),y_s,u(s))]ds\|_{W^{1,2}(0,a)}$$

$$\leq \gamma(a)\,||x-y||_{W^{1,2}(0,a)}$$

where,

$$\gamma(a) = (1 + a^{\frac{1}{2}})$$

is a continuous, nondecreasing and nonnegative function of a such that $\gamma(0) = 0$. This proves that condition H_4 holds.

Finally, we show that condition SH_3 is satisfied. Assume that $x,y\varepsilon L_2(-r,T)$ and $u\varepsilon L_2(0,T)$, from $\hat{H}_2$ it follows that

$$[\int_0^t ||f(x(s),x_s,u(s))-f(y(s),y_s,u(s))||^2 ds]^{\frac{1}{2}}$$

$$\leq [\int_0^t K^2\{\sum_{j=0}^{\nu} ||x(s-r_j)-y(s-r_j)||_{\mathbb{R}^n} + ||x_s-y_s||_{L_2}\}^2 ds]^{\frac{1}{2}}$$

$$\leq K\{\sum_{j=0}^{\nu}[\int_0^t ||x(s-r_j)-y(s-r_j)||^2 ds]^{\frac{1}{2}}$$

$$+ [\int_0^t\int_{-r}^0 ||x_s(\theta)-y_s(\theta)||^2 d\theta ds]^{\frac{1}{2}}\}$$

$$= K\{\sum_{j=0}^{\nu}[\int_{-r_j}^{t-r} ||x(s')-y(s')||^2 ds']^{\frac{1}{2}}$$

$$+ [\int_0^t\int_{s-r}^s ||x(s')-y(s')||^2 ds' ds]^{\frac{1}{2}}\}$$

$$\leq K(\nu+1+t^{\frac{1}{2}})\cdot [\int_{-r}^t ||x(s)-y(s)||^2 ds]^{\frac{1}{2}} ,$$

and hence SH_3 is satisfied with $\alpha_1(t) = K(\nu + 1 + t^{\frac{1}{2}})$.

Since H1, H2, H3 and H4 hold, if $(D\phi,\phi)\varepsilon\mathcal{D}(A)$ then the initial value problem 3.1.1 - 3.1.2 has a unique classical solution $x(t)$ on $[-r,T]$. If $z(t) = (Dx_t,x_t)$ for all $t\varepsilon[0,T]$, then

$$z'(t) = (Lx_t+h(x(t),x(t-r_1),\ldots,x(t-r_\nu),x_t,u(t)),\frac{\partial}{\partial t}x(t+s))$$

$$= (H(x_t,u(t)),\ \frac{\partial}{\partial t}x(t+s)) \ .$$

For each $u\,\varepsilon\mathbb{R}^m$, define the nonlinear operator $A(u):\mathcal{D}(A)\subseteq Z \to Z$ by

$$\mathcal{D}(A) = \{(D\phi,\phi)/\phi\varepsilon W^{1,2}\}$$

and

$$A(u)z = (H(\phi,u),\dot{\phi}) \ .$$

Therefore, $z(t) = (Dx_t,x_t)$ satisfies the abstract differential equation

$$z'(t) = A(u(t))z(t) \ , \qquad 0 \le t \le T$$

$$\text{(3.1.14)}$$

$$z(0) = z_0\varepsilon\mathcal{D}(A) \quad .$$

Since in this case, $z(t)$ is strongly absolutely continuous and for each $u\varepsilon L_2(0,T)$ the function $A(u(t))z(t)$ is integrable on $[0,T]$, it follows that

$$z(t) = z_0 + \int_0^t A(u(\sigma))z(\sigma)d\sigma \quad , \; 0 \le t \le T \; . \qquad (3.1.15)$$

In order to prove equivalence between the nonlinear abstract equation 3.1.15 and the nonlinear initial value problem 3.1.1-3.1.2 we present two lemmas that will also be used in the next section. For a proof of the first one see [13 , p. 100]. The proof of the second lemma requires that we consider Z with the equivalent norm $\| \; \|_g$ introduced by Kappel [21]. This norm is defined by

$$\|(\eta,\phi)\|_g^2 = \|\eta\|_{\mathbb{R}^n}^2 + \int_{-r}^0 \|\phi(s)\|^2 g(s)ds$$

where $g = g_1 + g_2$, with

$$g_1(s) = \nu - j + 1 \quad \text{for} \quad s \in (-r_j, -r_{j-1}) \; ,$$

$$g_2(s) = \alpha_j(s + r_{j-1}) \quad \text{for} \quad s \in (-r_j, -r_{j-1}) \; ,$$

and

$$\alpha_j \le -\frac{\nu(\nu+2)}{r_j - r_{j-1}} |B_j|^2 \; , \quad j = 1, 2, \ldots \nu \; .$$

Moreover, one has the relationship

$$\|(\eta,\phi)\|_Z \le \|(\eta,\phi)\|_g \le (\nu + \alpha\rho_0)^{\frac{1}{2}} \|(\eta,\phi)\|_Z \; ,$$

with $\alpha = \max\{|\alpha_j|;\ j = 1, \ldots, \nu\}$ and $\rho_0 = \max\{r_j - r_{j-1};\ j = 1, \ldots, \nu\}$.
The space Z supplied with the norm $\|\ \|_g$ will be denoted by Z_g.

Lemma 3.1.1. If X is a Hilbert space and $x:[a,b] \to X$ is given by

$$x(t) = x(a) + \int_a^t y(\sigma)d\sigma\ , \qquad a \le t \le b\ ,$$

then

$$\|x(t)\|^2 = \|x(a)\|^2 + 2\int_0^t \langle x(\sigma),\ y(\sigma)\rangle d\sigma\ .$$

Lemma 3.1.2. For each $u \in \mathbb{R}^m$ and $z, w \in \mathcal{D}(A)$ the operator $A(u)$ defined above satisfies the following dissipative inequality

$$\langle A(u)z - A(u)w,\ z - w\rangle_g \le \omega \|z-w\|_g^2$$

where ω is an appropriate constant independent of u.

Proof. (The proof is a generalization of the proof given by Kappel [21] for linear operators).

For all $z = (\eta,\phi)$ and $w = (\xi,\psi)$ in the $\mathcal{D}(A)$ and $u \in \mathbb{R}^m$ we have that

$$\langle A(u)z - A(u)w,\ z-w\rangle_g = (\eta-\xi)^T \cdot [H(\phi,u) - H(\psi,u)]$$

$$+ \int_{-r}^0 (\dot\phi - \dot\psi)^T(s)(\phi-\psi)(s)g(s)ds = I + II. \qquad (3.1.16)$$

Concentrating on I, it follows that

$$I = (\eta-\xi)^T\cdot[H(\phi,u)-H(\psi,u)] \leq ||\eta-\xi|| \ ||H(\phi,u)-H(\psi,u)||$$

$$\leq ||\eta-\xi|| \ K'\{||\phi(0)-\psi(0)|| + \sum_{j=1}^{\nu} ||\phi(-r_j)-\psi(-r_j)||$$

$$+||\phi-\psi||_{L_2}\} . \tag{3.1.17}$$

Since $\eta-\xi = D(\phi-\psi)$, the special form of D yields

$$\phi(0)-\psi(0) = \eta-\xi + \sum_{j=1}^{\nu} B_j(\phi(-r_j)-\psi(-r_j))$$

$$+ \int_{-r}^{0} B(s)(\phi(s)-\psi(s))ds . \tag{3.1.18}$$

Substituting 3.1.18 in 3.1.17 we obtain

$$I \leq K' ||\eta-\xi||^2 + K' || \eta - \xi|| \sum_{j=1}^{\nu} (1+ ||B_j||) ||\phi(-r_j)-\psi(-r_j)||$$

$$+ K'(1+ ||B||_2) ||\phi-\psi||_2|| \eta-\xi||$$

$$\leq \omega_1 || (\eta-\xi,\phi-\psi) ||_g^2 + \frac{1}{2} \sum_{j=1}^{\nu} ||\phi(-r_j)-\psi(-r_j)||^2 \tag{3.1.19}$$

where,

$$\omega_1 = K'\{1 + \frac{1}{2} K' \sum_{j=1}^{\nu} (1+ \|B_j\|)^2 + \frac{1}{2} + \frac{1}{2} \|B\|_2\} .$$

On the other hand, (see Kappel [21]) we have that

$$II = \frac{1}{2} \int_{-r}^{0} (\frac{d}{ds} \|\phi(s)-\psi(s)\|^2) g(s) \, ds$$

$$\leq \omega_2 \ \|(\eta-\xi,\phi-\psi)\|_g^2 - \frac{1}{2} \sum_{j=1}^{\nu} \|\phi(-r_j)-\phi(-r_j)\|^2 \qquad (3.1.20)$$

where

$$\omega_2 = \frac{\nu(\nu+2)}{2} (1+ \|B\|_2^2) + \frac{\alpha}{2} .$$

Finally, combining 3.1.16, 3.1.19 and 3.1.20 we obtain the result

$$\langle A(u)z - A(u)w, \ z - w\rangle_g \leq \omega \|z - w\|_g^2$$

with $\omega = \omega_1 + \omega_2$.

<u>Theorem 3.1.1</u>. Let $\eta \in \mathbb{R}^n$, $\phi \in W^{1,2}$ and $D\phi = \eta$. If x is the solution of 3.1.1-3.1.2 on [-r,T] corresponding to (η,ϕ) and $z(t) = (Dx_t,x_t)$, we have that z(t) is the unique solution of the equation 3.1.15 on the finite interval [0,T].

<u>Proof</u>. We have already established that $z(t) = (Dx_t,x_t)$ satisfies

3.1.15. The only thing that remains to be shown is that 3.1.15 has a unique solution. Suppose $z(t)$ and $w(t)$ are two solutions of 3.1.15 and let $\Delta(t) = z(t) - w(t)$. Clearly we have that

$$\Delta(t) = \int_0^t \{A(u(\sigma))z(\sigma) - A(u(\sigma))w(\sigma)\}d\sigma \ .$$

It follows from Lemma 3.1.1 that

$$\|\Delta(t)\|_g^2 \leq 2 \int_0^t <A(u(\sigma))z(\sigma)-A(u(\sigma))w(\sigma),z(\sigma)-w(\sigma)>_g d\sigma$$

$$\leq 2 \int_0^t \omega \|\Delta(\sigma)\|_g^2 \, d\sigma \ ,$$

and an application of Gronwall's inequality yields $\Delta(t) \equiv 0$. Consequently, 3.1.15 has a unique solution.

<u>Remark 3.1.3</u>. Let z_0, w_0 be in $\mathcal{D}(A)$ and u,v be in $L_2(0,T)$. Assume that $z(t;z_0,u) = (Dx_\sigma,x_\sigma)$ and $w(t;w_0,v) = (Dy_\sigma,y_\sigma)$ are the corresponding solutions of 3.1.15. If we define $\Delta(t) = z(t) - w(t)$, then

$$\Delta(t) = z_0 - w_0 + \int_0^t [A(u(\sigma))z(\sigma) - A(u(\sigma))w(\sigma)]d\sigma \ .$$

Applying Lemma 3.1.1 we obtain

$$\|\Delta(t)\|_g^2 = \|z_0-w_0\|_g^2 + 2\int_0^t \langle A(u(\sigma))z(\sigma)-A(v(\sigma))w(\sigma),\ \Delta(\sigma)\rangle_g\, d\sigma$$

$$= \|z_0-w_0\|_g^2 + 2\int_0^t \langle A(u(\sigma))z(\sigma)-A(v(\sigma))z(\sigma),\ \Delta(\sigma)\rangle_g\, d\sigma$$

$$+\ 2\int_0^t \langle A(v(\sigma))z(\sigma)-A(v(\sigma))w(\sigma),\ \Delta(\sigma)\rangle_g\, d\sigma$$

$$\leq \|z_0-w_0\|_g^2 + \int_0^t \|A(u(\sigma))z(\sigma)-A(v(\sigma))z(\sigma)\|_g^2\, d\sigma$$

$$+ \int_0^t \|\Delta(\sigma)\|_g^2\, d\sigma + 2\omega\int_0^t \|\Delta(\sigma)\|_g^2\, d\sigma$$

$$= \|z_0-w_0\|_g^2 + \int_0^t \|H(x_\sigma,u(\sigma))-H(x_\sigma,v(\sigma))\|^2\, d\sigma$$

$$+ (2\omega+1) \int_0^t \|\Delta(\sigma)\|_g^2\, d\sigma$$

$$\leq \|z_0-w_0\|_g^2 + K'^2 \int_0^t \|u(\sigma)-v(\sigma)\|^2\, d\sigma$$

$$+ (2\omega+1)\int_0^t\|\Delta(\sigma)\|_g^2\, d\sigma\ .$$

Gronwall's inequality implies that

$$\|\Delta(\sigma)\|_g^2 \leq \{\|z_0-w_0\|_g^2+K'^2 \cdot \|u-v\|^2_{L_2(0,T)}\} \cdot e^{(2\omega+1)T}\ ,$$

which establishes continuous dependence of the solutions of 2.1.15 on z_0 and u in the Z-norm and L_2-norm, respectively.

3.2 A General Approximation Scheme

In this section we present a general approximation scheme for the nonlinear F.D.E. initial value problem 3.1.1-3.1.2. This scheme will be a generalization of the spline based scheme developed by Banks [3] for retarded equations.

Definition 3.2.1. We call $\{Z^N, P^N, A^N(u)\}$, $N = 1, 2, 3, \ldots$ an approximation scheme for the Cauchy problem 3.1.1-3.1.2 if

i) $\{Z^N\}$ is a sequence of subspaces of Z_g such that $Z^N \subset \mathcal{D}(A)$ and $\dim(Z^N) = k_N < +\infty$, $N = 1,2,3,\ldots$.

ii) P^N is the sequence of orthogonal projections $Z_g \to Z^N$ and

iii) $\{A^N(u)\}$ is the sequence of operators $Z \to Z^N$ defined by $A^N(u) = P^N A(u) P^N$, $N = 1,2,3,\ldots$.

Remark 3.2.1. To be consistent with our notation we have written $A^N(u) = P^N A(u) P^N$, $u \in \mathbb{R}^m$. At times we shall not explicitly write out the dependence of A on u when u is not relevant.

Remark 3.2.2. As it can be seen from the following estimate, the approximating operators A^N satisfy the same dissipative inequality as A;

$$\langle A^N z - A^N w, z-w\rangle_g = \langle P^N A P^N z - P^N A P^N w, z-w\rangle_g$$

$$= \langle AP^N z - AP^N w,\ P^N z - P^N w\rangle_g$$

$$\leq \omega \, \|P^N z - P^N w\|_g^2 \leq \omega \, \|z-w\|_g^2 .$$

Now we consider the approximating equations in Z^N given by

$$z^N(t) = P^N z_0 + \int_0^t A^N(u(\sigma)) z^N(\sigma) d\sigma \qquad (3.2.1)$$

which, because Z^N is finite dimensional, are equivalent to

$$\dot{z}^N(t) = A^N(u(t)) z^N(t),$$
$$z^N(0) = P^N z_0 \ . \qquad (3.2.2)$$

Equation 3.2.2 is a nonlinear ordinary differential equation in a Euclidean space. Select a basis $\hat{e}_1^N, \ldots, \hat{e}_{k_N}^N$ for Z^N, where $\hat{e}_j^N = (De_j^N, e_j^N)$ with $e_j^N \varepsilon W^{1,2}$, $j = 1, 2, \ldots, k_N$, and define the $n \times k_N$ matrix function E^N by

$$E^N = (e_1^N, \ldots, e_{k_N}^N)$$

and let

$$\hat{E}^N = (D(E^N), E^N) \ .$$

If $w^N(t)$ is defined by $E^N w^N(t) = z^N(t)$ where $z^N(t;\eta,\psi,u)$ is the solution of 3.2.2, then (see Kappel [21]) one can show that w^N is the solution of the k_N dimensional system

$$\dot{w}^N(t) = H^N(w^N(t),u(t)),$$

with initial data (3.2.3)

$$w^N(0) = w_0^N .$$

where

$$H^N(\alpha^N,u) = (Q^N)^{-1}[D(E^N)^T H(E^N\alpha^N,u)$$

$$+ \int_{-r}^{0} E^N(\sigma)^T \dot{E}^N(\sigma)\alpha^N g(\sigma)d\sigma],$$

$$w_0^N = (Q^N)^{-1}[D(E^N)^T\eta + \int_{-r}^{0} E^N(\sigma)^T \phi(\sigma)g(\sigma)d\sigma] ,$$

and

$$Q^N = D(E^N)^T D(E^N) + \int_{-r}^{0} E^N(\sigma)^T E^N(\sigma)g(\sigma)d\sigma .$$

From the definition of H and the hypothesis on h, it is clear that $H^N(\alpha^N,u)$ satisfies a global Lipschitz condition in α^N on the interval [0,T] with Lipschitz constant independent of u. Existence, uniqueness and continuous dependence of solutions of the O.D.E. 3.2.3 follows easily and hence 3.2.1 has a unique solution on the finite interval [0,T].

In order to prove that the solutions of 3.2.1 converge to the solution of 3.1.15 (see Theorem 3.2.1 below) we establish a sequence of technical lemmas and definitions. As in [29] we use $PC^{2,\infty}$ to denote the set of all functions ϕ such that $\phi \varepsilon W^{1,2}$ and $\dot{\phi}$ is piecewise continuously differentiable with $\ddot{\phi}$ bounded.

<u>Definition 3.2.2</u> We say that an approximation scheme $\{Z^N, P^N, A^N(u)\}$ satisfies property P1 if and only if for every $z \varepsilon Z$,

$$\lim_{N\to\infty} P^N z = z \; .$$

<u>Definition 3.2.3.</u> We say that an approximation scheme $\{Z^N, P^N, A^N(u)\}$ satisfies property P2 if and only if for each $u \varepsilon \mathbb{R}^m$

$$\lim A(u) P^N z = A(u) z$$

for all $z \varepsilon PC^{2,\infty} = \{(D\phi, \phi)/\phi \varepsilon PC^{2,\infty}\}$.

<u>Remark 3.2.3.</u> Observe that P2 is equivalent to

$$\lim_{N\to\infty} H(\phi^N, u) = H(\phi, u)$$

for all $u \varepsilon \mathbb{R}^m$ and

$$\lim_{N\to\infty} \dot{\phi}^N = \dot{\phi}$$

for all $\phi \varepsilon PC^{2,\infty}$ with ϕ^N defined by $P^N(D\phi,\phi) = (D\phi^N,\phi^N)$.

Lemma 3.2.1. Assume $\hat{H}_2$ and let $\{Z^N,P^N,A^N(u)\}$ be an approximating scheme that satisfies P1 and P2. Then, $A^N(u)z$ converges to $A(u)z$ for each $z\varepsilon\mathcal{PC}^{2,\infty}$ and $u\varepsilon\mathbb{R}^m$.

Proof. For each $z\varepsilon\mathcal{PC}^{2,\infty}$ and $u\varepsilon\mathbb{R}^m$ we have that

$$\begin{aligned}||A^Nz - Az||_Z &= ||P^NAP^Nz - Az||_Z \\ &\leq ||P^NAP^Nz - P^NAz||_Z + ||P^NAz - Az||_Z \\ &\leq ||AP^Nz - Az||_Z + ||P^NAz - Az||_Z.\end{aligned}$$

The lemma follows from this estimate and properties P1 and P2.

Lemma 3.2.2. Let $\mathcal{H} = \{(z,u)/z = (\eta,\phi),\phi\varepsilon C^2,D\phi = \eta,\ u\varepsilon C^1$ and $D\dot{\phi} = H(\phi,u(0))\}$. Assume $\hat{H}_2$ and $\hat{H}_3$. If $(z_0,u)\varepsilon\mathcal{H}$ and $x(t;z_0,u)$ is the corresponding solution of 3.1.1 - 3.1.2, then, $z(\sigma) = (Dx_\sigma,x_\sigma)\varepsilon\mathcal{PC}^{2,\infty}$ for each $\sigma\varepsilon[0,T]$.

Proof. We recall from Theorem 2.2.3 that for $z_0 = (\eta,\phi)\varepsilon\mathcal{D}(A)$ and $u\varepsilon L_2(0,T)$ the solution $x(t;\eta,\phi,u)$ of the initial value problem 3.1.1-3.1.2 belongs to $W^{1,2}(-r,T)$ and satisfies

$$x(t) = \begin{cases} \phi(t) , & -r \le t < 0 , \\ (U^*x)(t) , & 0 \le t \le T , \end{cases}$$

where

$$(U^*x)(t) = \eta - \int_{-r}^{0} G(s)\phi(s)ds + \int_0^t \int_{-r}^{0} F(s)\tilde{x}(\tau+s)ds d\tau$$

$$+ \int_{-r}^{0} G(s)\tilde{x}(t+s)ds - \int_{-r}^{0} dH(s)\tilde{x}(t+s)$$

$$+ \int_0^t h(\tilde{x}(s),\tilde{x}(s-r_1),\ldots,\tilde{x}(s-r_\nu),\tilde{x}_s,u(s))ds$$

with

$$\tilde{x}(t) = \begin{cases} \phi(t) , & -r \le t < 0 \\ x(t) , & 0 \le t \le T . \end{cases}$$

Given the special form of the operator D (hypothesis $\hat{H}_1$) we have that

$$\dot{x}(t) = \begin{cases} \dot{\phi}(t) , & -r \le t \le 0, \\ Lx_t + h(x(t),x(t-r_1),\ldots,x(t-r_\nu),x_t,u(t)) & \\ + \sum_{j=1}^{\nu} B_j \dot{x}(t-r_j) + \int_{-r}^{0} B(s)\, \dot{x}(t+s)ds, & 0 \le t \le T. \end{cases}$$

Clearly $\dot{x}$ is continuous on $[-r,0)$ and $\dot{x}(0^-) = \dot{\phi}(0)$. On the other hand, for all $0 \le t \le r_1$ we have that

$$\dot{x}(t) = Lx_t + h(x(t),\phi(t-r_1),\ldots,\phi(t-r_\nu),x_t,u(t))$$

$$+ \sum_{j=1}^{\nu} B_j\ \dot{\phi}(t-r_j) + \int_{t-r}^{0} B(u-t)\dot{\phi}(u)du + \int_0^t B(u-t)\dot{x}(u)du\ .$$

Therefore, if $\phi\varepsilon C^2$, $\dot{x}$ is clearly continuous on $(0,r_1]$. Moreover, since x is uniformly continuous on $[-r,T]$ it is easy to establish that x_t converges to ϕ uniformly as t goes to zero. Therefore, the continuity of L implies that Lx_t converges to $L\phi$ and

$$\dot{x}(0^+) = L\phi + h(\phi(0),\phi(-r_1),\ldots,\phi(-r_\nu),\phi,u(0))$$

$$+ \sum_{j=1}^{\nu} B_j\dot{\phi}(-r_j) + \int_{-r}^{0} B(s)\dot{\phi}(s)ds\ .$$

Since $D\dot{\phi} = H(\phi,u(0))$, it follows that $\dot{x}(0^-) = \dot{x}(0^+)$ and $\dot{x}$ is continuous on $[-r,r_1]$. By a step argument we can establish that x is continuous on $[-r,T]$.

It is also clear that $\dot{x}$ is differentiable on $[-r,0]$ and on $(0,r_1]$ with

$$\ddot{x}(t) = \begin{cases} \ddot{\phi}(t) & , r \le t \le 0\ , \\ L\dot{x}_t + \frac{d}{dt}\, h(x(t),\phi(t-r_1),\ldots,\phi(t-r_\nu),x_t,u(t)) & \\ + \sum_{j=1}^{\nu} B_j\ddot{\phi}(t-r_j) + \frac{d}{dt}\int_{t-r}^{t} B(u-t)\ \dot{x}(u)du, & 0 \le t \le r_1. \end{cases}$$

Since $\phi\varepsilon C^2$, h is continuously differentiable and B is C^1, $\ddot{x}$ is PC and bounded on $[-r,r_1]$. Again, a step argument can be used to show that x is in $PC^{2,\infty}[-r,T]$.

Definition 3.2.4. Let $\{Z^N,P^N,A^N(u)\}$ be an approximating scheme that satisfies P2, and for $x\varepsilon PC^{2,\infty}(-r,T)$ let $z(\sigma) = (Dx_\sigma,x_\sigma)$. We say that $\{Z^N,P^N,A^N(u)\}$ satisfies P3 if and only if there exist a function $k(\sigma)\varepsilon L_1(0,T)$ and a sequence $\varepsilon_N \to 0$ such that

$$||A(u(\sigma))P^Nz(\sigma) - A(u(\sigma))z(\sigma)|| \leq k(\sigma)\varepsilon_N$$

for $0 \leq \sigma \leq T$. We note that in general, k may depend on u and x.

Lemma 3.2.3. Assume $\hat{H}_2$ and $\hat{H}_3$. Let $\{Z^N,P^N,A^N(u)\}$ be an approximating scheme that satisfies P1, P2 and P3. If $((\eta,\phi),u)\varepsilon H$ and z^N and z are the corresponding solutions of 3.2.1 and 3.1.15, $z^N(t)$ converges to z(t) uniformly in t on [0,T].

Proof. Defining $\Delta^N(t) = z^N(t) - z(t)$, we have that

$$\Delta^N(t) = (P^N-I)z_0 + \int_0^t\{A^N(u(\sigma))\,z^N(\sigma)-A(u(\sigma))z(\sigma)\}d\sigma \qquad (3.2.4)$$

As in Banks [3], applying Lemma 3.1.1 and using the dissipative inequality satisfied by A^N (Remark 3.2.2) we obtain

$$\|\Delta^N(t)\|_g^2 \le \|(P^N - I)z_0\|_g^2$$

$$+ \int_0^t \|[A^N(u(\sigma)) - A(u(\sigma))]\, z(\sigma)\|_g^2 \, d\sigma$$

$$+ (2\omega + 1) \int_0^t \|\Delta^N(\sigma)\|_g^2 \, d\sigma \;.$$

An application of Gronwall's inequality produces

$$\|\Delta^N(t)\|_g^2 \le \{E_1(N) + E_2(N)\} \cdot e^{(2\omega + 1)T}$$

where,

$$E_1(N) = \|P^N z_0 - z_0\|_g^2$$

and

$$E_2(N) = \int_0^T \|A^N(u(\sigma))z(\sigma) - A(u(\sigma))z(\sigma)\|_g^2 \, d\sigma \;.$$

P1 implies that $E_1(N) \to 0$. From Lemma 3.2.1 it follows that $A^N(u(\sigma))z(\sigma) \to A(u(\sigma))z(\sigma)$ for each $\sigma\varepsilon[0,T]$. Since

$$\|A^N(u(\sigma))z(\sigma)\|_g = \|P^N A(u(\sigma))P^N z(\sigma)\|_g \le \|A(u(\sigma))P^N z(\sigma)\|_g$$

$$\le \|A(u(\sigma))z^N(\sigma) - A(u(\sigma))z(\sigma)\|_g$$

$$+ \|A(u(\sigma))z(\sigma)\|_g$$

P3 implies that the convergence is dominated in σ on [0,T]. Thus, $\mathcal{E}_2(N) \to 0$, concluding the proof of the lemma.

Lemma 3.2.4. Let F be a mapping from C to $\mathbb{R}^n$ which satisfies the following property: there exists a constant $k > 0$ such that

$$\|F(\phi) - F(\psi)\| \leq k\{\sum_{j=0}^{\nu} \|\phi(-r_j)-\psi(-r_j)\| + \|\phi-\psi\|_{L_2}\} \quad \text{for all } \phi,\psi\varepsilon C \tag{3.2.5}$$

Then, the set $C_0^2 = \{(\eta,\phi)/\phi\varepsilon C^2, D\phi = \eta \text{ and } D\dot{\phi} = F(\phi)\}$ is dense in Z.

Proof. We define the operator B by

$$B(\eta,\phi) = -(2k\phi(0),\dot{\phi}) \ , \quad \mathcal{D}(B) = C_0^2 \ . \tag{3.2.6}$$

We note that this operator is nonlinear because of its domain. Let $\mathcal{C}^1 = \{(\eta,\phi)/\phi\varepsilon C^1 \text{ and } D\phi = \eta\}$ and for reasons that will become clear in the proof choose λ_0 such that

$$\nu \cdot e^{-r_1/\lambda_0} + (\lambda_0/2)^{\frac{1}{2}} < (\tfrac{1}{2}). \tag{3.2.7}$$

For each $(\eta,\phi)\varepsilon\mathcal{C}^1$ and $0 < \lambda < \lambda_0$ we wish to find $(\eta_\lambda,\phi_\lambda)\varepsilon\mathcal{D}(B)$ such that

$$(I + \lambda B)(\eta_\lambda,\phi_\lambda) = (\eta,\phi) \ . \tag{3.2.8}$$

If$(\eta_\lambda,\phi_\lambda)$ is a solution of 3.2.8 it has to satisfy, thus

$$\eta_\lambda - 2k\lambda\phi_\lambda(0) = \eta \tag{3.2.9}$$

and

$$\phi_\lambda(x) - \lambda\dot{\phi}_\lambda(x) = \phi(x), \quad -r \leq x \leq 0 \tag{3.2.10}$$

thus by 3.2.10 ϕ_λ must be of the form

$$\phi_\lambda(x) = \phi_\lambda(0)e^{x/\lambda} + \frac{1}{\lambda}\int_x^0 \exp[(x-\mu)/\lambda]\ \phi(\mu)d\mu \ . \tag{3.2.11}$$

Clearly for each $\phi_\lambda(0)$, ϕ_λ belongs to C^2. Moreover, $(\eta_\lambda,\phi_\lambda)$ must satisfy

$$D\phi_\lambda = \eta_\lambda , \tag{3.2.12}$$

and

$$D\dot{\phi}_\lambda = F(\phi_\lambda). \tag{3.2.13}$$

Using 3.2.9-3.2.10,3.2.12 and 3.2.13 we obtain

$$F(\phi_\lambda) = D\dot{\phi}_\lambda = D(\frac{1}{\lambda}(\phi_\lambda - \phi)) = \frac{1}{\lambda}\ D\phi_\lambda - \frac{1}{\lambda}\ D\phi$$

$$= \frac{1}{\lambda}\eta_\lambda - \frac{1}{\lambda}\eta = 2k\phi_\lambda(0) \ . \tag{3.2.14}$$

Therefore, 3.2.11 and 3.2.14 imply that $\phi_\lambda(0)$ must satisfy

$$\phi_\lambda(0) = \frac{1}{2k} F(\phi_\lambda(0) \cdot e^{x/\lambda} + \frac{1}{\lambda} \int_x^0 \exp[(x-\mu)/\lambda]\phi(\mu)d\mu) \ , \qquad (3.2.15)$$

and hence it has to be a fixed point of the operator $T_\lambda : \mathbb{R}^n \to \mathbb{R}$ defined by

$$T_\lambda \theta = \frac{1}{2k} F(\theta \cdot e^{x/\lambda} + \frac{1}{\lambda} \int_x^0 \exp[(x-\mu)/\lambda]\phi(\mu)d\mu). \qquad (3.2.16)$$

To show that T_λ has a (unique) fixed point it suffices to prove that T_λ is a contraction. From 3.2.16, 3.2.5 and 3.2.7 we obtain

$$\begin{aligned} ||T_\lambda\theta_1 - T_\lambda\theta_2|| &\leq \frac{1}{2}\{||\theta_1-\theta_2|| + \sum_{j=1}^{\nu} ||\theta_1-\theta_2|| \ e^{-r_j/\lambda} \\ &\quad + ||\theta_1-\theta_2|| \ || e^{x/\lambda}||_2\} \\ &\leq \frac{1}{2}\{1 + \nu \cdot e^{-r_1/\lambda} + \frac{\lambda}{2}\} \ ||\theta_1-\theta_2|| \\ &\leq \frac{1}{2}\{1 + \nu \cdot e^{-r_1/\lambda_0} + \frac{\lambda_0}{2}\} \ ||\theta_1-\theta_2|| \\ &\leq \frac{3}{4} \ ||\theta_1-\theta_2||. \end{aligned} \qquad (3.2.17)$$

Setting $\phi_\lambda(0) = \theta_\lambda$, where θ_λ is the unique fixed point of T_λ we obtain the $(\eta_\lambda, \phi_\lambda) \varepsilon \mathcal{D}(B)$ that satisfies (3.2.8). Therefore,

$$\|\phi_\lambda(0)\| = \|T_\lambda \phi_\lambda(0)\| \leq \|T_\lambda \phi_\lambda(0) - T_\lambda(0)\| + \|T_\lambda(0)\|$$

$$\leq \frac{3}{4}\|\phi_\lambda(0)\| + \frac{1}{2k}\|F(\frac{1}{\lambda}\int_x^0 \exp[(x-\mu)/\lambda]\phi(\mu)d\mu)\|$$

and hence

$$\|\phi_\lambda(0)\| \leq \frac{2}{k}\|F(\frac{1}{\lambda}\int_x^0 \exp[(x-\mu)/\lambda]\phi(\mu)d\mu) - F(0)\| + \frac{2}{k}\|F(0)\|$$

$$\leq 2\{\sum_{j=1}^{\nu}\|\frac{1}{\lambda}\int_{-r_j}^0 \exp[(-r_j-\mu)/\lambda]\phi(\mu)d\mu\|_{\mathbb{R}^n}$$

$$+ \|\frac{1}{\lambda}\int_x^0 \exp[(x-\mu)/\lambda]\phi(\mu)d\mu\|_{L_2} + \frac{2}{k}\|F(0)\|$$

$$\leq 2(\nu + r^{\frac{1}{2}})\ \|\phi\|_\infty + \frac{2}{k}\|F(0)\| = M.$$

Thus $\|\phi_\lambda(0)\| \leq M$ for all $0 < \lambda < \lambda_0$ which implies that $\eta_\lambda = \eta + 2k\lambda\phi_\lambda(0)$ converges to η as $\lambda \to 0$.

On the other hand, by 3.2.11,

$$\phi_\lambda(x) - \phi(x) = e^{x/\lambda}(\phi_\lambda(0) - \phi(x)) + \frac{1}{\lambda}\int_x^0 \exp[(x-\mu)/\lambda]\,(\phi(\mu)-\phi(x))d\mu \tag{3.2.18}$$

which implies that

$$\|\phi_\lambda(x)-\phi(x)\|_2 \le \|e^{x/\lambda}\cdot(\phi_\lambda(0) - \phi(x))\|_2$$
$$+ \|\frac{1}{\lambda} \int_x^0 \exp[(x-\mu)/\lambda](\phi(\mu) - \phi(x))d\mu\|_2 \tag{3.2.19}$$

In addition, we have that

$$\|e^{x/\lambda} \cdot (\phi_\lambda(0) - \phi(x))\|_2 \le (M + \|\phi\|_\infty) \cdot \|e^{x/\lambda}\|_2$$
$$\le (M + \|\phi\|_\infty)\sqrt{(\lambda/2)} \tag{3.2.20}$$

and

$$\|\frac{1}{\lambda}\int_x^0 \exp[(x - \mu)/\lambda](\phi(\mu) - \phi(x))d\mu\|_2$$
$$\le r^{\frac{1}{2}} \cdot \|h_\lambda(x)\|_\infty , \tag{3.2.21}$$

where

$$h_\lambda(x) = \frac{1}{\lambda} \int_x^0 \exp[(x - \mu)/\lambda](\phi(\mu) - \phi(x))d\mu .$$

It is easy to prove, see Brewer [14, p. 182] that $\|h_\lambda\|_\infty$ goes to zero as $\lambda \to 0$. Consequently, 3.2.19, 3.2.20 and 3.2.21 imply that

$\phi_\lambda \to \phi$ in L_2. Therefore, $(\eta_\lambda,\phi_\lambda) \to (\eta,\phi)$ as $\lambda \to 0$. This establishes that $C^1 \subset \overline{C_0^2} \subset Z$. The fact that C^1 is dense in Z (see Kappel [21]) completes the proof.

Lemma 3.2.5. The set H defined in Lemma 3.2.2 is dense in $Z \times L_2(0,T)$.

Proof. Let $(z,u) \varepsilon Z \times L_2(0,T)$. It is clear that there exists a sequence $u^n \varepsilon C^1$ such that $u^n(0) = 0$, $n = 1, 2, 3, \ldots\ldots$ and u^n converges to u in $L_2(0,T)$. Let $F(\phi) = H(\phi,0)$. By Lemma 3.2.4 there exists a sequence z^n such that $z^n = (\eta^n,\phi^n)$, $\phi^n \varepsilon C^2$, $D\phi^n = \eta^n$, $D\dot{\phi}^n = F(\phi^n) = H(\phi^n,0) = H(\phi^n,u^n(0))$ and $z^n \to z$ in Z. Thus, (z^n,u^n) is a sequence in H that converges to (z,u) in $Z \times L_2(0,T)$.

Lemmas 3.2.3 and 3.2.4 combined with the continuous dependence of solutions of 3.1.15 and 3.2.1 yield the following result.

Theorem 3.2.1. Assume $\hat{H}_1$, $\hat{H}_2$ and $\hat{H}_3$. Let $\{Z^N,P^N,A^N\}$ be an approximating scheme satisfying P1, P2 and P3. If $(z_0,u) \varepsilon \mathcal{D}(A) \times L_2(0,T)$ with $z^N(t)$ and $z(t)$ the corresponding solutions of 3.2.1 and 3.1.15, respectively, then $z^N(t)$ converges to $z(t)$ uniformly in t on $[0,T]$.

3.3 Spline approximation

In this section we show that the general scheme given in

Section 3 can be realized by choosing the Z^N as certain spaces of spline functions. The presentation and the arguments used are similar and in some cases identical, to the ones used by Kappel [21] and Banks and Kappel [10].

Corresponding to the partition $t_j^N = -j\ r/n$, $j = 0,1,\ldots,N$, of $[-r,0]$ we define Z_s^N to be the space of all elements $(D\phi,\phi)$, where ϕ is a first order, a cubic or a cubic Hermite spline with knots at t_j^N, respectively. Clearly, $Z_s^N \subset \mathcal{D}(A)$ and Z_s^N is finite dimensional. Let P_s^N be the orthogonal projection $Z_g \to Z_s^N$ and $A_s^N = P_s^N\ A\ P_s^N$, $N = 1,2,3,\ldots$.

For $\hat{\phi} = (D\phi,\phi)\varepsilon PC^{2,\infty}$ we denote $P_s^N\hat{\phi}$ by $\hat{\phi}^N = (D\phi^N,\phi^N)$. Moreover, let ϕ_I^N be the interpolating spline function which satisfies

$$\phi_I^N(t_j^N) = \phi(t_j^N)\ , \quad j = 0,\ 1,\ \ldots\ ,\ N\ .$$

In case of cubic splines we take the standard boundary conditions $\dot{\phi}_I^N(0) = \dot{\phi}(0)$, $\dot{\phi}_I^N(-r) = \dot{\phi}(-r)$, in case of cubic Hermite splines we demand in addition that $\dot{\phi}_j^N(t_j^N) = \dot{\phi}(t_j^N)$, $j = 0,\ 1,\ \ldots,\ N$ (see [10]).

Using the well known convergence properties of interpolating splines we have the estimates

$$\|\phi_I^N - \phi\|_{L_2} \le M_{s1}\ (r/N)^2\ \|\ddot{\phi}\|_{L_2} \tag{3.3.1}$$

$$\|\dot{\phi}_I^N - \dot{\phi}\|_{L_2} \le M_{s2}\ (r/N)\ \|\ddot{\phi}\|_{L_2} \tag{3.3.2}$$

and

$$||\phi_I^N - \phi||_\infty \leq M_{s3}\,(r/N)^2 \cdot ||\ddot{\phi}||_\infty, \qquad (3.3.3)$$

for all $\phi \varepsilon PC^{2,\infty}$, where M_{s1}, M_{s2} and M_{s3} are constants independent of N and different for each type of spline (cf. Schultz [29], Th. 2.5, 3.4 and 4.5 for 3.3.1 and 3.3.2; Th. 2.6, 3.7 and 4.8 for 3.3.3).

We now show that the approximation scheme $\{Z_s^N, P_s^N, A_s^N\}$ satisfies the conditions required in Theorem 3.2.1 for any of the splines considered above.

<u>Theorem 3.3.1</u>. The spline approximation scheme $\{Z_s^N, P_s^N, A_s^N\}$ satisfies properties P1, P2 and P3.

<u>Proof</u>. If $\phi \varepsilon P, C^{2,\infty}$, then $||\ddot{\phi}||_{L_2} \leq r^{1/2} ||\ddot{\phi}||_\infty$ and

$$\begin{aligned} ||D(\phi_I^N - \phi)|| &\leq \sum_{j=1}^{\nu} ||\phi_I^N(-r_j) - \phi(-r_j)|| \\ &\quad + \int_{-r}^{0} ||B(s)|| \cdot ||\phi_I^N(s) - \phi(s)|| ds \\ &\leq \nu ||\phi_I^N - \phi||_\infty + ||B||_\infty \cdot r^{1/2} \cdot ||\phi_I^N - \phi||_{L_2} \\ &\leq \nu\, M_{s3}(r/N)^2 \cdot ||\ddot{\phi}||_\infty \\ &\quad + ||B||_\infty \cdot r^{1/2} \cdot M_{s1}(r/N)^2 \cdot ||\ddot{\phi}||_{L_2} \end{aligned}$$

$$\leq \mathcal{O}(1/N^2) \cdot ||\ddot{\phi}||_\infty \ . \tag{3.3.4}$$

Since $||\hat{\phi}^N - \hat{\phi}||_g = \min\{||\hat{\psi} - \hat{\phi}||_g / \hat{\psi}\varepsilon Z_s^N\}$ we obtain immediately that

$$||\hat{\phi}^N - \hat{\phi}||_Z \leq ||\hat{\phi}^N - \hat{\phi}||_g \leq ||\hat{\phi}_I^N - \hat{\phi}||_g \leq (\nu + \alpha\rho_0)^{\frac{1}{2}} ||\hat{\phi}_I^N - \phi||_Z$$

$$\leq (\nu + \alpha\rho_0)^{\frac{1}{2}} \cdot \{||D(\phi_I^N - \phi)||_{\mathbb{R}^n}$$

$$+ ||\phi_I^N - \phi||_{L_2}\} \leq \mathcal{O}(1/N^2) \cdot ||\ddot{\phi}||_\infty \ . \tag{3.3.5}$$

This proves that $\lim_{N\to\infty} P^N\hat{\phi} = \hat{\phi}$ for all $\hat{\phi}\varepsilon PC^{2,\infty}$. Since $PC^{2,\infty}$ is dense in Z, P1 follows. On the other hand ,

$$\begin{aligned} ||\dot{\phi}^N - \dot{\phi}||_{L_2} &\leq ||\dot{\phi}_I^N - \dot{\phi}||_{L_2} + ||\dot{\phi}_I^N - \dot{\phi}^N||_{L_2} \\ &\leq ||\dot{\phi}_I^N - \dot{\phi}||_{L_2} + M_s \cdot \frac{N}{r} \cdot ||\phi_I^N - \phi^N||_{L_2} \ , \end{aligned} \tag{3.3.6}$$

where for the second term on the right-hand side we have used the Schmidt inequality on each of the intervals $[t_j^N, t_{j-1}^N]$, $j = 1,2,\ldots,N$ (cf. Schultz[29], Th. 1.5). We note that M_s is a constant independent of N but different for each type of spline. The inequality

$$||\phi_I^N - \phi^N||_{L_2} \leq ||\phi_I^N - \phi||_{L_2} + ||\phi^N - \phi||_{L_2}$$

$$\leq ||\phi_I^N - \phi||_{L_2} + ||\hat{\phi}^N - \hat{\phi}||_Z , \qquad (3.3.7)$$

along with 3.3.1, 3.3.2 and 3.3.5, yield the estimate

$$||\dot{\phi}^N - \dot{\phi}||_{L_2} \leq \mathcal{O}(1/N) \cdot ||\ddot{\phi}||_\infty. \qquad (3.3.8)$$

We now verify that $H(\phi^N,u) \to H(\phi,u)$. Since $||\phi^N - \phi||_{W^{1,2}}$ $\leq M ||\ddot{\phi}||_\infty$, there exists a number $\tau = \tau(N)\varepsilon[-r,0]$ such that

$$||\phi^N(\tau) - \phi(\tau)|| \leq ||\phi^N - \phi||_{L_2} r^{-1/2}.$$

Thus, for each $\theta\varepsilon[-r,0]$ we have that

$$\begin{aligned} ||\phi^N(\theta) - \phi(\theta)|| &\leq ||\phi^N(\tau) - \phi(\tau)|| \\ &+ ||\textstyle\int_\tau^\theta \dot{\phi}^N(s) - \dot{\phi}(s)]ds|| \\ &\leq ||\phi^N(\tau) - \phi(\tau)|| + \textstyle\int_{-r}^0 ||\dot{\phi}^N(s) - \dot{\phi}(s)||ds \\ &\leq r^{-1/2} ||\phi^N - \phi||_{L_2} + r^{1/2} ||\dot{\phi}^N - \dot{\phi}||_{L_2} . \end{aligned}$$

This estimate together with 3.3.1 and 3.3.2 imply that

$$||\phi^N(\theta) - \phi(\theta)|| \leq \mathcal{O}(1/N) \cdot ||\ddot{\phi}||_\infty . \qquad (3.3.9)$$

Therefore, the condition

$$||H(\phi^N,u) - H(\phi,u)|| \leq K'\{\sum_{j=0}^{\nu}||\phi^N(-r_j)-\phi(-r_j)|| + ||\phi^N-\phi||_{L_2}\}$$

combined with 3.3.9 and 3.3.1 yield

$$||H(\phi^N,u) - H(\phi,u)|| \leq \mathcal{O}(1/N) \cdot ||\ddot{\phi}||_\infty \tag{3.3.10}$$

for all $u \in \mathbb{R}^m$ and $\phi \in PC^{2,\infty}$. Conditions 3.3.8 and 3.3.10 imply that for $z \in PC^{2,\infty}$ and $z^N = P_s^N z$

$$\lim_{N\to\infty} \mathcal{A}(u)z^N = \mathcal{A}(u)z$$

for $u \in \mathbb{R}^m$. Consequently, P2 holds. Moreover, if $u \in L_2(0,T)$ and $x \in PC^{2,\infty}(-r,T)$, then

$$||A(u(\sigma))z^N(\sigma)-A(u(\sigma))z(\sigma)|| \leq ||H(x_\sigma^N,u(\sigma))-H(x_\sigma,u(\sigma))||_{\mathbb{R}^n}$$

$$+ ||\dot{x}_\sigma^N - \dot{x}_\sigma||_{L_2} \leq \mathcal{O}(1/N) \cdot ||\ddot{x}_\sigma||_\infty$$

$$\leq \mathcal{O}(1/N) \sup\{||\ddot{x}(s)||;-r\leq s\leq\sigma\} ,$$

and P3 holds with $k(\sigma) = \sup\{||\ddot{x}(s)||; -r \leq s \leq \sigma\}$.

3.4 An Example: Piecewise Linear Splines.

To illustrate the above ideas we consider the following equation

$$\frac{d}{dt}[x(t) + Bx(t-r)] = A_0x(t)+A_1x(t-r)+Cu(t), \tag{3.4.1}$$

where A_0, A_1, B and C are $n \times n$ constant matrices. In this case the linear operators L and D are given by

$$L\phi = A_0\phi(0) + A_1\phi(-r)$$

and

$$D\phi = \phi(0) + B\phi(-r) ,$$

and the nonlinear term reduces to Cu(t). For each N = 1, 2, 3,... let

$$Z_1^N = \text{Span}\,\{{}^{1}\hat{e}^{0}_N,\ldots.{}^{n}\hat{e}^{0}_N \,;\, {}^{1}\hat{e}^{1}_N,\ \ldots,\ {}^{n}\hat{e}^{1}_N;\ldots.\,;{}^{1}\hat{e}^{N}_N,\ldots,{}^{n}\hat{e}^{N}_N\},$$

with

$${}^{j}\hat{e}^{k}_N = (De^k_N(\cdot)e_j, e^k_N(\cdot)e_j), \quad j = 1,2,\ldots,n \ \text{ and } k = 0,1,\ldots,N,$$

where e_j is the unit vector in R^n, i.e., $e_j = (0,\ldots,1,\ldots,0)$, and $e^k(\cdot)$ is the first order spline with knot at $-\frac{kr}{N}$. Using the notation introduced in Section 3.2 we have that

$$E^N = (e_N^0(\cdot)e_1,\ldots,e_N^0(\cdot)e_n \vdots \ldots\ldots\ldots \vdots e_N^N(\cdot)e_1,\ldots,e_N^N(\cdot)e_n)$$

$$= (e_N^0(\cdot)I \vdots \ldots\ldots\ldots \vdots e_N^N(\cdot)I) \qquad (3.4.2)$$

and

$$D(E^N) = (De_N^0(\cdot)e_1,\ldots,De_N^0(\cdot)e_n \vdots \ldots\ldots \vdots De_N^N(\cdot)e_1,\ldots,De_N^N(\cdot)e_n).$$

Since, for each $j = 1,2,\ldots,n$ and $k = 1,2,\ldots,N-1$

$$De_N^0(\cdot)e_j = e_N^0(0)e_j + Be_N^0(-r)e_j = e_j,$$

$$De_N^k(\cdot)e_j = e_N^k(0)e_j + Be_N^k(-r)e_j = Be_j \, ,$$

and

$$De_N^N(\cdot)e_j = e_N^N(0)e_j + Be_N^N(-r)e_j = Be_j \, ,$$

it follows that

$$D(E^N) = (e_1,\ldots,e_n \vdots 0 \ldots\ldots\ldots 0 \vdots Be_1,\ldots,Be_n)$$

$$= (I \vdots 0 \vdots \ldots\ldots\ldots \vdots 0 \vdots B) \qquad (3.4.3)$$

Therefore,

$$D(E^N)^T D(E^N) = \begin{bmatrix} I & 0 & \cdots & B \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B^T & 0 & \cdots & B^T B \end{bmatrix} .$$

On the other hand, since for this particular case $g(\sigma) = 1 + \alpha\sigma$, we obtain

$$\int_{-r}^{0} e_0^N(\theta)^2 g(\theta)d\theta = [(1/3)-(\alpha/12)(r/N)](r/N),$$

$$\int_{-r}^{0} e_k^N(\theta)^2 g(\theta)d\theta = [(2/3)-(2\alpha k/3)(r/N)](r/N), \; k=1,2,3,\ldots,N-1,$$

$$\int_{-r}^{0} e_N^N(\theta)^2 g(\theta)d\theta = [(1/3)-((4N-1)\alpha/12)(r/N](r/N),$$

and

$$\int_{-r}^{0} e_k^N(\theta) \; e_{k+1}^N(\theta)g(\theta)d\theta = [(1/6)-((1+2k)\alpha/12)(r/N)](r/N), \; k=0,1,\ldots,N-1.$$

The matrix Q^N is given by

$$
\begin{bmatrix}
(1+\frac{1}{3}\frac{r}{N}-\frac{\alpha}{12}\frac{r^2}{N^2})I & (\frac{1}{6}\frac{r}{N}-\frac{\alpha}{12}\frac{r^2}{N^2})I & 0 & \cdots & B \\
(\frac{1}{6}\frac{r}{N}-\frac{\alpha}{12}\frac{r^2}{N^2})I & (\frac{2}{3}\frac{r}{N}-\frac{4\alpha}{3}\frac{r^2}{N^2})I & (\frac{1}{6}\frac{r}{N}-\frac{\alpha}{4}\frac{r^2}{N^2})I & \cdots & 0 \\
0 & (\frac{1}{6}\frac{r}{N}-\frac{\alpha}{4}\frac{r^2}{N^2})I & \ddots & \ddots & \vdots \\
0 & 0 & \ddots & \ddots & \\
\vdots & \vdots & & & \\
B^T & 0 & \cdots & & B^TB+(\frac{1}{3}\frac{r}{N}-\frac{(4N-1)\alpha}{12}\frac{r^2}{N^2})I
\end{bmatrix}.
$$

On the other hand

$$\int_{-r}^{0} e_0^N(\theta)\dot{e}_0^N(\theta)g(\theta)d\theta = 1 - (\alpha/6)(r/N) ,$$

$$\int_{-r}^{0} e_k^N(\theta)\dot{e}_k^N(\theta)g(\theta)d\theta = (\alpha/3)(r/N), \ K = 1,2,\ldots,N-1 ,$$

$$\int_{-r}^{0} e_N^N(\theta)\dot{e}_N^N(\theta)g(\theta)d\theta = (\alpha(1-3N)/6)(r/N) ,$$

and for $k = 0, 1, 2, \ldots, N - 1$,

$$\int_{-r}^{0} e_k^N(\theta)\dot{e}_{k+1}^N(\theta)g(\theta)d\theta = (1/2)+(\alpha(1+3k)/6)(r/N).$$

while

$$\int_{-r}^{0} e_{k+1}^N(\theta)\dot{e}_k^N(\theta)g(\theta)d\theta = 1/2 \quad .$$

Let H_1^N be the matrix $H_1^N = \int_{-r}^{0} E^N(\theta)^T\dot{E}^N(\theta)g(\theta)\, d\theta$ given by

$$
\begin{bmatrix}
(1-\frac{\alpha}{6}\frac{r}{N})I & (-\frac{1}{2}+\frac{\alpha}{6}\frac{r}{N})I & 0 & \cdots & 0 \\
(\frac{1}{2}-\frac{\alpha}{6}\frac{r}{N})I & (-\frac{\alpha}{3}\frac{r}{N})I & (-\frac{1}{2}+\frac{2\alpha}{3}\frac{r}{N})I & \cdots & 0 \\
0 & (\frac{1}{2}-\frac{2\alpha}{3}\frac{r}{N})I & (-\frac{\alpha}{3}\frac{r}{N})I & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \\
0 & 0 & \cdots & & (-1-\frac{\alpha(1-3N)}{6}\frac{r}{N})I
\end{bmatrix},
$$

and observe that

$$H(E^N\alpha^N,u) = (Le_N^0(\cdot),\ \ldots\ldots,\ Le_N^N(\cdot))\alpha^N + Cu \ .$$

Consequently, we find that

$$D(E^N)^T \cdot H(E^N\alpha^N,u) = H_2^N \cdot \alpha^N + H_3^N \cdot u$$

where H_2^N and H_3^N are defined by

$$
H_2^N = \begin{bmatrix}
A_0 & 0 & \cdots & A_1 \\
0 & 0 & \cdots & \\
\vdots & \vdots & \ddots & \vdots \\
B^T A_0 & 0 & \cdots & B^T A_1
\end{bmatrix}
$$

and

$$
H_3^N = \begin{bmatrix}
C \\
0 \\
\vdots \\
B^T C
\end{bmatrix} .
$$

Therefore, $w^N(t)$ satisfies the ordinary differential equation in $\mathbb{R}^{n\times(N+1)}$ given by

$$
\dot{w}^N(t) = F^N w^N(t) + G^N u(t),
$$

where,

$$
F^N = (Q^N)^{-1}[H_1^N + H_2^N]
$$

and

$$G^N = (Q^N)^{-1} H_3^N .$$

3.5 Conclusions and future work

We have developed a general state space model and a general approximation scheme for certain control systems governed by non-linear neutral functional differential equations. In Chapter II we investigated general existence and uniqueness questions and developed a non-linear variation of parameters formula. Section 3.1 contains the equivalent abstract integral equation that provides the basis for the general approximation scheme. The hypothesis on the non-linear terms are almost identical to the conditions imposed by Banks [3] in his study of retarded systems, except that we also allow the control to enter as a non-linear term. Moreover, the approximation scheme is based on a direct approximation of the abstract integral equation and is similar to the approach employed by Banks. Although the global Lipschitz conditions were assumed here, the results are still valid if one assumes that the conditions hold locally. Since the primary objective of the paper was the development of approximation techniques and since the local assumptions lead to a considerable increase in technicalities (see for example, Proposition 2.2.1, we have chosen (as did Banks [3]) to resrict attention to the global assumptions.

The spline based approximations discussed in Sections 3.3 and 3.4 fall within the general framework. These schemes have been used to approximate retarded systems and some numerical tests were made on neutral equations. However, except for Kappel's work [21] there has been little analysis of the convergence of these schemes for neutral systems.

The analysis in this paper should be viewed as a basis for analyzing approximation techniques for optimal control and identification of neutral control systems. In order to use these ideas for such problems, it will be necessary to more closely examine the uniformity of convergence with respect to control inputs and parameters. This is a problem that requires additional analysis. Moreover, numerical experiments should be conducted in order to test the theoretical convergence of the scheme. We plan to conduct this work in the near future.

There are a number of other approaches to approximating abstract evolution equations. Trotter-Kato semigroup approximations [4] and finite difference methods [26-28] are two that have been successfully applied to retarded equations. For neutral systems in product spaces, these methods are more complicated by the fact that the D operator is unbounded. Therefore, this too is another area that should be investigated.

As a final observation we note that the general approximation scheme leads to approximations of the "state" (Dx_t, x_t) of the neutral system. In particular, since the solution $x(t) = x_t(0)$ is obtained through the (unbounded) observation operator

$$C(Dx_t, x_t) = x_t(0) \ ,$$

direct approximation of the solution may require additional convergence analysis (i.e. uniform convergence of $x_t^N \to x_t$, see Kappel [21]). Although this is a point that may be important for parameter estimation, it is of less concern if one is interested in optimal control via state feedback. These problems as well as the previous questions will be investigated in future work.

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NONLINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL

EQUATIONS IN PRODUCT SPACES

by

Jose M. Amillo-Gil

(ABSTRACT)

Control systems governed by nonlinear neutral functional differential equations are formulated as abstract evolution equations in product spaces. At this point existence and uniqueness of solutions are studied.

This formulation is used to develop a general approximation scheme for those systems. Convergence of this scheme is analyzed. It is also shown how spline based approximating methods fall within this general framework. An illustrative example is presented.