Attrition Models of the Ardennes Campaign

Ronald D. Fricker, Jr.
RAND, Santa Monica, California

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Abstract: This paper revisits the modeling by Bracken [3] of the Ardennes campaign of World War II using the Lanchester equations. It revises and extends that analysis in a number of ways: (1) It more accurately fits the model parameters using linear regression; (2) it considers the data from the entire campaign; and (3) it adds in air sortie data. In contrast to previous results, it concludes by showing that neither the Lanchester linear or Lanchester square laws fit the data. A new form of the Lanchester equations emerges with a physical interpretation. © 1998 John Wiley & Sons, Inc. Naval Research Logistics **45:** 1–22, 1998

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INTRODUCTION

Lanchester [15] proposed a set of differential equations as a model of warfare. The *Lanchester equations,* as they have come to be known, are used extensively in modeling warfare, yet little empirical model validation has been done because of the lack of data. Past empirical validation studies include the work of Engel [9] on the Iwo Jima campaign of World War II, Busse [2] on the Incheon–Seoul campaign of the Korean War, and Bracken [3] on the Ardennes Campaign of World War II. Various reanalyses of the Engel and Busse work have been conducted by Samz [18], Hartley [13], and Hartley and Helmbold [11].

The general form of the model considered here is

$$\dot{B}(t) = aR(t)^p B(t)^q,$$

$$\dot{R}(t) = bB(t)^p R(t)^q, \tag{1}$$

where B and R are the strengths of blue and red forces at time t, $\dot{B}$ and $\dot{R}$ are blue forces and red forces killed at time t, a and b are attrition parameters, p is the exponent parameter of the attacking force, and q is the exponent parameter of the defending force. The model begins with initial force sizes, $B(0)$ and $R(0)$, which are then incrementally decreased according to the relationship $B(t+1) = B(t) - \dot{B}(t)$ and $R(t+1) = R(t) - \dot{R}(t)$. In an equally matched battle where the ratio of the forces stays constant over time $B(t)/R(t) = \dot{B}(t)/\dot{R}(t)$, for all t. This is equivalent to the condition that $bB(t)^{p-q+1} = aR(t)^{p-q+1}$ for some p and q and all t.

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Two particular versions of the Lanchester equations have been of general interest. When $p = q = 1$ (or more generally when $q - p = 0$) force ratios remain equal if $a \times R = b \times B$, and this condition is thus called *Lanchester linear.* The interpretation is that a battle governed by this model is characterized as a collection of small engagements and was proposed by Lanchester [15] as a model for ancient warfare. Lanchester contrasted it with the condition $p = 1$, $q = 0$ (or $p - q = 1$), which is called *Lanchester square,* where the force ratios remain equal when $a \times R^2 = b \times B^2$. He theorized that this model fit modern warfare in which both sides are able to concentrate forces. A third version with $p = 0$, $q = 1$ (or $q - p = 1$) is called *Lanchester logarithmic.*

Hartley [12] proposed an even more specific Lanchestrian model which he based on an analysis of numerous historical battles:

$$\dot{B}(t) = aR^{0.4}B^{0.75},$$

$$\dot{R}(t) = bB^{0.4}R^{0.75}, \tag{2}$$

commenting that this "... particular homogeneous, mixed, linear-logarithmic law ... does provide a good approximation to the historical data." He maintained that the attrition parameters (a and b) are independent of force size and that the size of the attrited force dominates.

Bracken [3] introduced an additional parameter d to the standard Lanchester equation (1), which he called the *tactical parameter,* to account for a battle in which it is known that defense and offense switch during the course of the campaign. Using the notation BA and $\overline{BA}$ to indicate blue force attacking or red force attacking, Bracken's model can be represented as

$$\dot{B} = a\left(\frac{d^2 I\{\overline{BA}\} + I\{BA\}}{d}\right)R^p B^q,$$

$$\dot{R} = b\left(\frac{d^2 I\{BA\} + I\{\overline{BA}\}}{d}\right)B^p R^q. \tag{3}$$

$I\{\cdot\}$ is the indicator function defined as follows: $I\{x\} = 1$ if x is true and $I\{x\} = 0$ if x is false. The result of this addition is that, for example, if the blue force is defending and the tactical parameter $d < 1$, then blue gets an advantage multiplier of d, whereas the red force has a disadvantage multiplier of $1/d$.

Engel [9] concluded that the square law might fit the Iwo Jima data, but the data were incomplete on the Japanese side. Hartley [13] concluded that, depending on the assumptions made for the Japanese data, other laws could be made to fit. With complete daily data available for the Incheon–Seoul campaign, work by Busse [2], Hartley [12], and Hartley [14] proved inconclusive. The Ardennes data has complete daily tallies, but some of the German data were estimated. The estimation was based on extrapolation from existing records and was done by World War II historical experts [6]. For the Ardennes Campaign, Bracken [3] concluded that the Lanchester linear law fit the data.

Thus, the empirical evidence needed to validate Lanchester theory is sparse, and the results to date are somewhat inconclusive and conflicting. In spite of this, the Lanchester equations are commonly employed to explain observed or reported phenomena, such as the work of David [5], which models the biblical battles of Gibeah or the work of Franks and Partridge [10] modeling ant warfare.

This paper revisits the modeling by Bracken [3] of the Ardennes Campaign. It revises and extends that analysis in a number of ways: (1) It more accurately fits the model parameters using linear regression; (2) it considers the data from the entire campaign; and (3) it adds in air sortie data. In contrast to previous results, it concludes by showing that neither the linear, square, nor Hartley's form (2) of the Lanchester equations fit the data. A new form of the Lanchester equations emerges with a physical interpretation.

THE ARDENNES DATA

The Ardennes Campaign of World War II, more popularly known as the Battle of the Bulge, began on December 16, 1944. It was the final German offensive of the war—a desperate last gamble planned by Hitler and executed by Marshall von Rundstedt. German forces under the command of von Rundstedt launched a concentrated surprise attack against a thinly manned portion of the front held by the United States VIII Corps. In an effort to split the U.S. and British forces, the Germans planned for a decisive breakthrough at Ardennes to the River Meuse, followed by a swift advance to the port city of Antwerp. German forces attacked from 16 to 26 December. During this time the U.S. line sustained major German penetrations, but ultimately rallied to slow and then stop the German attack. Allied air forces, originally grounded by poor weather and visibility, began flying on December 23. The German advance was halted east of the Meuse on December 24 and Allied counter offensives began on December 25. On New Year's Day the Germans conducted one final air offensive, but by then they had lost the initiative. On January 16, 1945 the front was restored to its original position.

Detailed information on the Ardennes Campaign was compiled by Data Memory Systems, Inc. (DMSI) under contract to the U.S. Army Concepts Analysis Agency [6]. DMSI created an extensive electronic database from archives and libraries in the United States, Great Britain, and the Federal Republic of Germany, including detailed daily information on U.S., British, and German ground and air forces. One of the DMSI researchers, Trevor N. Dupuy, later took the results and consolidated them into a detailed historical account of the Ardennes Campaign: *Hitler's Last Gamble: The Battle of the Bulge, December 1994–January 1995* [8]. The Ardennes battle data used in this analysis is taken from the DMSI database; in particular, it includes the data presented by Bracken [3] augmented with new aircraft sortie data. By convention U.S. forces are labeled Blue and the German forces Red; capitalization will be used to distinguish between referrals to the Ardennes forces versus generic forces in a general discussion.

The Bracken data consists of daily tallies of manpower, tanks, APCs, and artillery available and killed on December 15, 1945 ("day 0") and on the ensuing 32 days of battle. Manpower data is furnished in two ways: (1) *Combat* manpower consists of the infantry, armor, and artillery personnel; (2) *total* manpower consists of all personnel, including logistics and support personnel. Manpower "killed" is defined as personnel killed, wounded, captured/missing in action, and those incapacitated by disease and nonbattle injuries. The new aircraft data consists of the daily number of sorties flown by each side in direct support of the ground forces. Unfortunately, unlike the manpower and equipment data, the air data did not list number of aircraft available or killed, only sorties flown. Also, as with the German casualty data, the German air data required some estimation. The DMSI documentation states, "... German Luftwaffe air operations in the Air Data Base are not as exact as those for the Allied operations because of the nature of the available German records and the methodology used to estimate daily German air data." [6, p. V-5]

The DMSI database included air data for operations that were targeted on detrained units (units not on a train), nonrail supported logistics, any stationary target such as a depot or positions that directly support combat operations, or missions that were of an unspecified purpose. It specifically excluded "Allied air operations against industrial and rail targets ... which were primarily of a strategic nature" [6, p. V-4]. The DMSI database also did not include interdiction missions against German ground units and supplies arriving by rail once they were detrained.

From the documentation: "For the purposes of the [database], boundaries were established to define the Ardennes area within which all tactical air sorties that occurred ... These boundaries are:

- on the east, the Rhine and Mosele Rivers.
- on the north, an east–west line running from Mulheim (F5064) to Mechelen (K5864).
- on the west, the Meuse River.
- on the south, a west–east line running from Flize (O8626) to Wasserbillig (Q1125)."

However, it goes on to say: "Essentially, any German air operation over the Ardennes Campaign area was included in the [database]." Thus it must be noted that the German air operations may have been overcounted (for the purposes of this study) in relation to the Allies.

The 33,048 air sorties recorded in the DMSI database were classified into categories for which total sorties flown per day for each side were recorded. The following categories were included in Table 3: attack, armed reconnaissance, bombing, patrol, immediate support, support. The following categories were excluded: aerial resupply, escort, pathfinding, paradrop (supply of ground units by parachute), photo recon, scramble, and weather recon.

A review of Lanchester [15] shows that the Lanchester equations are predicated on fixed initial force sizes for red and blue from which casualties are then incrementally subtracted. That is, $\dot{B}$ and $\dot{R}$ represent decrements in the force size, so that $B(t+1) = B(t) - \dot{B}(t)$, for example. Clearly the Bracken data are not in this format, with the daily force sizes (R and B) reflecting the effects of both previous casualties and incremental reinforcements.

One could theorize that structuring the data as shown in Tables 1 and 2 is reasonable on the grounds that each day comprises an independent battle within the larger campaign. Then under the assumption that the attrition and exponent parameters are constant for all 32 days (because they reflect fixed capabilities of the overall forces), one might reasonably choose to model each day as an independent observation from some fixed form of Lanchester's equations. The idea would be that casualties occur according to the fixed Lanchester equations using the previous day's force size, but the overall force size for the current day also depends on the transfer of troops in or out of the fighting force.

An alternate way to structure the data is to estimate initial force sizes that reflect all of the troops that eventually fought in the campaign and then subtract the casualty attrition from this total on a daily basis. Consider resource X, for example, where $X(0)$ is the initial quantity of the resource on day 0, as listed in Tables 1 and 2. Assume that when $X(t+1) > X(t) - \dot{X}(t)$ reinforcements of resource X were added on day t, $X_r(t)$, so that

Table 1. Ardennes battle manpower data for the Allied (Blue) and German (Red) forces.

Day (i)	Manpower							
	Blue				Red			
	Available: $M_b(i)$		Killed: $\dot{M}_b(i)$		Available: $M_r(i)$		Killed: $\dot{M}_r(i)$	
	Combat	Total	Combat	Total	Combat	Total	Combat	Total
0	351005	632105	458	1468	0	0	0	0
1	349247	630557	1589	3062	360716	575838	2191	5590
2	347915	628985	2383	5712	356818	571301	2423	5559
3	358321	640969	2085	5093	353529	568508	2015	4711
4	366495	807140	2175	12101	350750	565173	1993	4332
5	387342	834136	1389	5334	356278	572181	1985	4351
6	403289	859906	1174	3197	354297	570711	2084	4582
7	410817	874600	1905	4815	361684	581177	2046	4531
8	412811	877247	1548	3730	359353	579660	2468	5351
9	426360	895976	1608	3857	362904	584610	2685	5609
10	432094	907490	1527	3635	359750	580731	2538	5563
11	451316	933045	2320	5411	362611	584551	2504	5526
12	451724	948024	1376	3596	361023	583610	2544	5751
13	451291	928230	1277	3435	356892	578737	2121	4511
14	461189	941188	1005	2934	349900	568768	1682	3900
15	465334	946424	1042	2743	346100	564548	1844	4076
16	467620	948226	1159	3022	343134	560993	1550	3635
17	467801	948379	1004	2773	340875	558214	1788	3898
18	474562	956144	832	2631	338278	555741	1724	3821
19	474192	955821	1831	3580	334356	550854	1752	3892
20	481704	965135	2259	4899	328069	544031	2054	4283
21	480952	964928	1639	4093	321195	534885	1709	3767
22	478593	962193	1228	3388	322830	536481	1946	4169
23	475732	959776	1868	4627	324376	540896	1865	4076
24	475685	959011	1276	3928	322337	538328	1676	3756
25	475155	958799	1379	3725	320612	536719	1434	3466
26	472749	956330	1643	4002	319143	534764	1696	3732
27	472535	956090	1281	3502	319259	533256	1536	3967
28	468127	952030	1083	3590	317406	530919	1167	3199
29	467646	952210	1681	4189	316217	528237	1579	4026
30	466072	950879	1597	4277	314858	526387	1504	3866
31	464643	949508	2098	4477	313074	524150	1425	3744
32	455218	937500	1483	3600	310347	521038	1213	3219

$$X(t+1) = X(t) - \dot{X}(t) + X_r(t), \qquad \text{for } t = 0, \ldots, 31.$$

If the data reflected only the simple daily additions of reinforcements, then the new initial quantity of resource X, $\tilde{X}(0)$, could simply be defined as

$$\tilde{X}(0) = X(0) + \sum_{t=0}^{31} X_r(t). \tag{4}$$

But the data are more complicated than that, with various resources sometimes temporarily decreasing over time; for example, sometimes $X(t+1) < X(t) - \dot{X}(t)$ for one or more time periods, as if some of the resources were removed from battle and held as ‘‘local

Table 2. Ardennes battle equipment data for tanks, APCs, and artillery of the Allied (Blue) and German (Red) forces.

	Equipment											
	Blue						Red					
	Available			Killed			Available			Killed		
Day (i)	Tank $T_b(i)$	APC $A_b(i)$	Art. $Y_b(i)$	Tank $\dot{T}_b(i)$	APC $\dot{A}_b(i)$	Art. $\dot{Y}_b(i)$	Tank $T_r(i)$	APC $A_r(i)$	Art. $Y_r(i)$	Tank $\dot{T}_r(i)$	APC $\dot{A}_r(i)$	Art. $\dot{Y}_r(i)$
0	2853	6103	3006	1	0	0	0	0	36	0	0	0
1	2863	6019	2972	12	33	15	747	2046	4789	10	5	6
2	2867	5970	2963	43	46	9	663	2041	4791	7	20	41
3	2840	5908	2950	60	18	6	639	2021	4768	13	12	27
4	2808	6004	3103	64	37	14	669	2009	4727	21	18	19
5	3965	7274	3531	33	11	33	619	1984	4786	11	22	24
6	4082	7295	3609	10	6	10	595	1952	4773	21	19	30
7	4109	7507	3772	15	13	2	615	2065	4858	5	16	16
8	4086	7533	3772	36	6	10	645	2034	4845	24	34	35
9	4062	7486	3847	48	18	5	596	1970	4885	22	20	94
10	4265	8105	3931	24	2	4	544	1875	4750	28	36	59
11	4520	8552	4063	20	2	4	483	1800	4779	14	31	34
12	4511	8629	4093	19	3	3	466	1731	4661	13	31	23
13	4526	8536	4004	18	3	2	450	1659	4638	7	22	31
14	4541	8552	4065	16	2	0	433	1595	4415	7	15	19
15	4516	8565	4087	20	0	0	428	1542	4321	21	7	26
16	4610	8554	4086	10	2	1	403	1532	4314	5	14	17
17	4695	8615	4077	14	0	0	413	1523	4283	9	8	36
18	4701	8593	4087	24	0	0	419	1516	4246	6	10	14
19	4710	8462	4088	26	0	1	431	1451	4242	2	10	35
20	4728	8578	4150	22	0	3	428	1441	4110	12	12	28
21	4686	8564	4153	13	2	0	394	1419	4016	2	6	22
22	4719	8502	4144	13	0	0	396	1409	4014	2	5	23
23	4684	8375	4133	12	0	1	400	1403	3981	8	16	26
24	4703	8418	4131	9	0	0	407	1364	3971	0	2	23
25	4743	8446	4128	7	0	0	398	1360	3944	7	3	21
26	4761	8476	4131	5	0	0	407	1358	3925	2	9	33
27	4745	8348	4090	7	1	0	407	1349	3916	2	8	32
28	4717	8459	4108	2	1	0	393	1341	3895	0	3	13
29	4699	8454	4106	6	0	0	418	1335	3854	13	17	20
30	4678	8374	4081	16	0	2	410	1322	3867	5	6	19
31	4662	8436	4092	11	0	0	434	1318	3856	3	4	14
32	4628	8363	4080	20	0	9	432	1309	3824	2	3	7

reserves.'' Usually these resources later appear again in the data. Thus, without accounting for these local reserves, (4) would over count the total forces.

Because of this phenomenon, the following algorithm was used to estimate the original total for each resource. It works by sequentially stepping through each resource from day 0 to day 32, accounting for any local reserves (X_{lr}) or the addition of reinforcements (X_r) as they may result, while first using local reserves for any force increase before assuming that reinforcements were added. For resource X:

1. Set $X_r = X_{lr} = 0$
2. Let $t = 1$:

Table 3. Sorties flown by Red and Blue in direct support of ground forces.

Air Sorties					
Day	Allied S_b	German S_r	Day	Allied S_b	German S_r
0	15	0	17	1295	8
1	48	108	18	1195	188
2	760	249	19	971	0
3	1341	195	20	24	100
4	477	225	21	1507	47
5	0	0	22	463	0
6	64	19	23	456	0
7	119	32	24	576	45
8	1413	234	25	0	0
9	2143	254	26	257	77
10	1754	129	27	153	0
11	1686	203	28	47	0
12	886	178	29	707	65
13	727	15	30	617	155
14	831	95	31	210	0
15	952	19	32	394	29
16	718	175			

- If $X(t+1) > X(t) - \dot{X}(t)$ and $X_{lr} = 0$, then $X_r = X_r + [X(t+1) - X(t) + \dot{X}(t)]$.
- Else, if $X(t+1) > X(t) - \dot{X}(t)$ and $X_{lr} \geq X(t+1) - X(t) + \dot{X}(t)$, then $X_{lr} = X_{lr} - [X(t+1) - X(t) + \dot{X}(t)]$.
- Else, if $X(t+1) > X(t) - \dot{X}(t)$ and $0 < X_{lr} < X(t+1) - X(t) + \dot{X}(t)$, then $X_r = X_r + [X(t+1) - X(t) + \dot{X}(t)] - X_{lr}$, $X_{lr} = 0$
- Else, if $X(t+1) < X(t) - \dot{X}(t)$, then $X_{lr} = X_{lr} + [X(t) - \dot{X}(t) - X(t+1)]$.

3. If $t < 31$, increment t and go to step #2; else $\tilde{X}(0) = X(0) + X_r$.

Then the new daily resources $\tilde{X}(t+1)$ are calculated as $\tilde{X}(t+1) = \tilde{X}(t) - \dot{X}(t)$, $t = 0, \ldots, 31$. Tables 4 and 5 reflect the revised data.

How close are the initial (day 0) manpower estimates to historical accounts? Astor stated that the Ardennes combat forces consisted of ''600,000 American soldiers and perhaps 50,000 British against the Third Reich's 550,000'' [1, p. xi]. MacDonald, an official historian of World War II for the U.S. Army, wrote: ''Among 600,000 Americans eventually involved in the fighting . . . casualties totaled 81,000. Among 55,000 British . . . casualties totaled 1,400. The Germans, employing close to 500,000 men . . . lost at least 100,000 killed, wounded, and captured'' [16, p. 618]. These two accounts are quite close. Note that the historian's estimates of Allied combat manpower (650,000 to 655,000) and their estimates of German combat manpower (500,000 to 550,000) lie between the day 0 combat and total manpower totals of Table 4. The differences between the historian's estimates and the Table 4 combat manpower totals can be attributed to variations in classifying troops as either combat or logistics. MacDonald's count of Allied and German casualties also fall in between the combat and total casualties counts reported in Table 4. Thus the day 0 totals of the revised data in Table 4 are reasonable when compared to historical accounts.

Table 4. The reformatted Ardennes battle manpower data for the Allied (Blue) and German (Red) forces.

	Reformatted manpower							
	Blue				Red			
	Available: $M_b(i)$		Killed: $\dot{M}_b(i)$		Available: $M_r(i)$		Killed: $\dot{M}_r(i)$	
Day (i)	Combat	Total	Combat	Total	Combat	Total	Combat	Total
0	513514	1075857	458	1468	385955	656278	0	0
1	513056	1074389	1589	3062	385955	656278	2191	5590
2	511467	1071327	2383	5712	383764	650688	2423	5559
3	509084	1065615	2085	5093	381341	645129	2015	4711
4	506999	1060522	2175	12101	379326	640418	1993	4332
5	504824	1048421	1389	5334	377333	636086	1985	4351
6	503435	1043087	1174	3197	375348	631735	2084	4582
7	502261	1039890	1905	4815	373264	627153	2046	4531
8	500356	1035075	1548	3730	371218	622622	2468	5351
9	498808	1031345	1608	3857	368750	617271	2685	5609
10	497200	1027488	1527	3635	366065	611662	2538	5563
11	495673	1023853	2320	5411	363527	606099	2504	5526
12	493353	1018442	1376	3596	361023	600573	2544	5751
13	491977	1014846	1277	3435	358479	594822	2121	4511
14	490700	1011411	1005	2934	356358	590311	1682	3900
15	489695	1008477	1042	2743	354676	586411	1844	4076
16	488653	1005734	1159	3022	352832	582335	1550	3635
17	487494	1002712	1004	2773	351282	578700	1788	3898
18	486490	999939	832	2631	349494	574802	1724	3821
19	485658	997308	1831	3580	347770	570981	1752	3892
20	483827	993728	2259	4899	346018	567089	2054	4283
21	481568	988829	1639	4093	343964	562806	1709	3767
22	479929	984736	1228	3388	342255	559039	1946	4169
23	478701	981348	1868	4627	340309	554870	1865	4076
24	476833	976721	1276	3928	338444	550794	1676	3756
25	475557	972793	1379	3725	336768	547038	1434	3466
26	474178	969068	1643	4002	335334	543572	1696	3732
27	472535	965066	1281	3502	333638	539840	1536	3967
28	471254	961564	1083	3590	332102	535873	1167	3199
29	470171	957974	1681	4189	330935	532674	1579	4026
30	468490	953785	1597	4277	329356	528648	1504	3866
31	466893	949508	2098	4477	327852	524782	1425	3744
32	464795	945031	1483	3600	326427	521038	1213	3219

FITTING THE LANCHESTER EQUATIONS

Bracken [3] reduced the data of Tables 1 and 2 to single measures of force strength by aggregating the resource data with tanks, APCs, artillery, and manpower weighted by 20, 5, 40, and 1, respectively. These weights were derived from standard U.S. Army Concepts Analysis Agency practices. The study applied (3) to force strength based on combat forces and total forces (here referred to as Bracken Model 1 and Bracken Model 2, respectively). It also considered the standard Lanchester models of (1)—that is, models without the tactical parameter—for force strength calculated for combat forces and total forces (referred to as Bracken Model 3 and Bracken Model 4, respectively). Bracken chose the best model parameters by searching over a grid in the $\{a, b, p, q, d\}$ space for the minimum sum of

Table 5. The reformatted Ardennes battle equipment data for tanks, APCs, and artillery of the Allied (Blue) and German (Red) forces.

	Reformatted equipment											
	Blue						Red					
	Available			Killed			Available			Killed		
Day (i)	Tank $T_b(i)$	APC $A_b(i)$	Art. $Y_b(i)$	Tank $\dot{T}_b(i)$	APC $\dot{A}_b(i)$	Art. $\dot{Y}_b(i)$	Tank $T_r(i)$	APC $A_r(i)$	Art. $Y_r(i)$	Tank $\dot{T}_r(i)$	APC $\dot{A}_r(i)$	Art. $\dot{Y}_r(i)$
0	5350	8821	4275	1	0	0	747	2161	5130	0	0	0
1	5349	8821	4275	12	33	15	747	2161	5130	10	5	6
2	5337	8788	4260	43	46	9	737	2156	5124	7	20	41
3	5294	8742	4251	60	18	6	730	2136	5083	13	12	27
4	5234	8724	4245	64	37	14	717	2124	5056	21	18	19
5	5170	8687	4231	33	11	33	696	2106	5037	11	22	24
6	5137	8676	4198	10	6	10	685	2084	5013	21	19	30
7	5127	8670	4188	15	13	2	664	2065	4983	5	16	16
8	5112	8657	4186	36	6	10	659	2049	4967	24	34	35
9	5076	8651	4176	48	18	5	635	2015	4932	22	20	94
10	5028	8633	4171	24	2	4	613	1995	4838	28	36	59
11	5004	8631	4167	20	2	4	585	1959	4779	14	31	34
12	4984	8629	4163	19	3	3	571	1928	4745	13	31	23
13	4965	8626	4160	18	3	2	558	1897	4722	7	22	31
14	4947	8623	4158	16	2	0	551	1875	4691	7	15	19
15	4931	8621	4158	20	0	0	544	1860	4672	21	7	26
16	4911	8621	4158	10	2	1	523	1853	4646	5	14	17
17	4901	8619	4157	14	0	0	518	1839	4629	9	8	36
18	4887	8619	4157	24	0	0	509	1831	4593	6	10	14
19	4863	8619	4157	26	0	1	503	1821	4579	2	10	35
20	4837	8619	4156	22	0	3	501	1811	4544	12	12	28
21	4815	8619	4153	13	2	0	489	1799	4516	2	6	22
22	4802	8617	4153	13	0	0	487	1793	4494	2	5	23
23	4789	8617	4153	12	0	1	485	1788	4471	8	16	26
24	4777	8617	4152	9	0	0	477	1772	4445	0	2	23
25	4768	8617	4152	7	0	0	477	1770	4422	7	3	21
26	4761	8617	4152	5	0	0	470	1767	4401	2	9	33
27	4756	8617	4152	7	1	0	468	1758	4368	2	8	32
28	4749	8616	4152	2	1	0	466	1750	4336	0	3	13
29	4747	8615	4152	6	0	0	466	1747	4323	13	17	20
30	4741	8615	4152	16	0	2	453	1730	4303	5	6	19
31	4725	8615	4150	11	0	0	448	1724	4284	3	4	14
32	4714	8615	4150	20	0	9	445	1720	4270	2	3	7

square residuals (SSR). The search considered $0 \leq p, q \leq 2.0$, $0.6 \leq d \leq 1.4$, and $4 \times 10^{-9} \leq a, b \leq 1.2 \times 10^{-8}$. As Bracken stated: "This does not guarantee that an optimal fit will be found. However, it does guarantee that the identified parameters are optimal over the options made available."

Instead of this brute force approach, linear regression applied to logarithmically transformed Lanchester equations can be used to estimate the model parameters. Willard [19] used this technique to estimate the parameters a and p for equations of the form

$$\frac{\dot{B}}{\dot{R}} = a\left(\frac{R}{B}\right)^p.$$

Here linear regression is used to solve for the parameters a, b, p, and q that minimize the SSR. Consider the basic Lanchester equations of (1) logarithmically transformed:

$$\log(\dot{B}) = \log(a) + p\log(R) + q\log(B),$$
$$\log(\dot{R}) = \log(b) + p\log(B) + q\log(R), \tag{5}$$

where B, R, $\dot{B}$, $\dot{R}$, a, and b may be scalars or $1 \times N$ vectors corresponding to N time periods. All four parameters (a, b, p, and q) may be estimated using linear regression which fits separate intercepts for the red and blue data; the linear model is of the form

$$y_i = \theta + \alpha_1 I\{i \le N\} + \alpha_2 I\{i > N\} + px_{1,i} + qx_{2,i} + \epsilon, \qquad \text{for } i = 1, \ldots, 2N, \tag{6}$$

where $\theta + \alpha_1 = \log(a)$ and $\theta + \alpha_2 = \log(b)$. The tactical parameter can also be incorporated in the linear model: let $y_i = \log(\dot{B}_i/f(d))$ for $i = 1 \cdots, N$, and $y_i = \log(\dot{R}_{i-N}/f(d))$ for $i = N + 1 \cdots, 2N$, where $f(d)$ is the tactical parameter function of (3),

$$f(d) = \left(\frac{d^2 I\{\overline{BA}\} + I\{BA\}}{d}\right). \tag{7}$$

The linear regression methodology will not explicitly solve for the optimal value of $f(d)$, but the minimum SSR is convex as a function of $f(d)$, so that it can be iteratively solved for to any desired level of accuracy.

Using a transformation that converts a nonlinear model to a linear one is a standard regression technique. See, for example, Draper and Smith [7]. In fact, the Lanchester equations (1) are actually a specific form of the "Cobb-Douglas" production function, used in various business and economic applications, to which linear regression is routinely applied to the transformed function (see Press [17]). Implicit in this approach is that the error term is multiplicative and has a log-normal distribution, an assumption that will be carried through here since it is not addressed in Lanchester's formulation. Advantages of using linear regression include that the sum of squared residuals is minimized (under linearity and normality assumptions) and that standard statistical techniques can be used to judge the significance of the parameters and the fit of the model.

RESULTS OF THE ANALYSIS

Bracken [3] analyzed only 10 days of the data in Tables 1 and 2 (days 1–10). This section revisits those results and compares them to new results obtained via the application on linear regression to better fit the model parameters. It then fits models for the full 32 days of the Ardennes Campaign, both with and without air power.

The Bracken Results Revisited

Using the first 10 days of the original data in Tables 1 and 2, Bracken concluded that the best fitting model, in terms of minimizing the SSR was Lanchester linear. In particular, Bracken found that the data for combat forces with tactical parameter (Bracken Model 1) were exactly Lanchester linear and the other three models were close to Lanchester linear.

Table 6. Comparison of Bracken results to new results, where the new "a" results fit the attrition parameters a and b using linear regression with the exponent parameters constrained $p, q > 0$ and the "b" results fit all the parameters a, b, p and q using linear regression, for days 1–10 of the battle.

Model	Type of forces	Parameters $\hat{d}$	$\hat{p}$	$\hat{q}$	$\hat{a}$	$\hat{b}$	Sum of squared residuals (SSR)
Bracken Model 1	Combat	0.80	1.0	1.0	8×10^{-9}	1×10^{-8}	1.63×10^7
New 1a	Combat	0.83	0.3	0.0	54.59	72.05	1.38×10^7
New 1b	Combat	0.88	0.43	−0.50	8,164.1	10,152.3	1.37×10^7
Bracken Model 3	Combat	1.0	1.3	0.7	8×10^{-9}	1×10^{-8}	2.08×10^7
New 3a	Combat	1.0	2.0	0.0	9.0×10^{-9}	1.0×10^{-8}	1.66×10^7
New 3b	Combat	1.0	1.64	−1.72	9,189.4	9,020.0	1.48×10^7
Bracken Model 2	Total	0.80	0.8	1.2	8×10^{-9}	8×10^{-9}	9.38×10^7
New 2a	Total	0.83	0.0	0.5	6.268	7.464	7.54×10^7
New 2b	Total	0.69	−1.81	1.57	117,793.0	306,573.4	6.22×10^7
Bracken Model 4	Total	1.0	1.2	0.8	8×10^{-9}	8×10^{-9}	1.19×10^8
New 4a	Total	1.0	0.4	0.0	27.410	26.914	1.02×10^8
New 4b	Total	1.0	0.32	−0.57	210,015.4	181,429.1	9.98×10^7

Table 6 shows that fitting the attrition parameters by linear regression improves the fit of the models as measured by sum of squared residuals. The "New _ a" models fit the attrition parameters a and b using linear regression with the exponent parameters constrained so that $p, q > 0$. (The p and q parameters were found by searching over the values $0 \leq p, q \leq 2.5$ in increments of 0.1.) The "New _ b" models fit all the parameters a, b, p, and q using linear regression, so that p and q (as well as a and b) can assume any value in $\mathbb{R}$. For the models with the tactical parameter, Red forces were assumed to be attacking on days 1–6 and Blue on the remaining days. While the historical record leaves this open to interpretation, it is consistent with Bracken [3].

It is important to note that blind application of linear regression to choose the best parameters sidesteps the issue of whether the resulting parameter values are meaningful in a physical sense. In particular, the original Lanchester equations (1) were formulated based on a physical interpretation of combat that casualties were a function of the product of the opposing force strengths. Yet the unconstrained models (the "b" models) of Table 6 give the best fits (in terms of minimizing the SSR) with either p or q negative. A negative exponent parameter in the transformed equations of (5) means that the logarithm of a force's casualties decreases as one of the force strengths increases. While this physically does not make much sense, their inclusion in Table 6 demonstrates the pattern that the SSR for the semiconstrained "a" models is consistently less than Bracken's model results but more than the unconstrained "b" model results, as should be expected.

Focusing now only on the "a" models and Bracken's models, note the range of values taken by the estimated attrition parameters ($\hat{a}$ and $\hat{b}$). In contrast to Bracken's attrition parameters, which were restricted to a range of small values ($4 \times 10^{-9} \leq a, b \leq 1.2 \times 10^{-8}$), the new "a" models' parameters take on a wide range of values. Only model "3a" has attrition parameters within the constrained range of Bracken. For the other 3 models (''1a,'' ''2a,'' and ''4a''), the attrition parameters assumed values much larger in the new models than in the Bracken models and this caused the exponent parameters in the new models to be smaller than the Bracken exponent parameters. The larger attrition parameters can be interpreted to mean that the new models find the forces to be much more lethal. Yet the smaller exponent parameters, in particular because they are less than 1, can be interpreted to mean that the greater lethality is concentrated on a smaller fraction of the opposing force.

Because the new attrition parameters in the ''a'' models drastically affect the exponent parameters (p and q), none of the resulting new models are Lanchester linear. Models ''1b'' and ''4b'' are close to Lanchester square (with $\hat{p} - \hat{q}$ values of 0.9336 and 0.89, respectively), yet these were previously rejected for a lack of physical interpretation. The rest of the models are neither linear, square, nor logarithmic. Thus, if the criteria in model selection is minimizing the SSR, the only possible conclusion is that the data from the first 10 days of the Ardennes Campaign do not fit either of the basic Lanchester's models. They also do not fit the Lanchester logarithmic nor the Hartley models.

Results for the Complete Data

Before fitting the Lanchester equations, one should consider the appropriateness of doing so. Note that the multiplicative Lanchester equations are inherently linear, as the logarithmically transformed equations (5) show. Thus $\log(\dot{B})$ and $\log(\dot{R})$ should show a linear relationship when plotted against $\log(B)$ and $\log(R)$. Depending on the distribution of the error term, the linearity will be more or less visible because of the transformation. For example, if the multiplicative error term in the Lanchester equations (1) is log-normal, then the transformed error term will be distributed normal. If there is evidence of nonlinearity, then the applicability of the model is called into question.

As Figure 1 shows, the original Ardennes data of Tables 1 and 2 contains nonlinear behavior that clearly cannot be explained by an error term. The figure plots the aggregated force strengths using total manpower for the full 32 days of the campaign versus casualties. The plots based on combat manpower showed similar trends. Thus, the various nonlinearities present make it inappropriate to fit the Lanchester equations to the data as originally formatted.

As shown in Figure 2, the reformatted data of Tables 4 and 5 look much more linear, so fitting linear models to the logarithmically transformed data using linear regression is better justified. The models considered were only those with the tactical parameter, as the inclusion of the parameter in the model clearly improves the fit by accounting for the known attacker/defender change point.

Two linear models were fit to the data: one based on combat manpower and the other based on total manpower. Aggregated force strength was computed using the same weights and in the same manner as previously described. Using standard linear regression techniques, the estimated exponent parameter $\hat{p}$ was found to be statistically insignificant in both models. The final models were:

Combat Manpower

$$\dot{B} = 4.7 \times 10^{-27} f(0.8093) B^5,$$
$$\dot{R} = 3.1 \times 10^{-26} f(0.8093) R^5. \qquad (8)$$

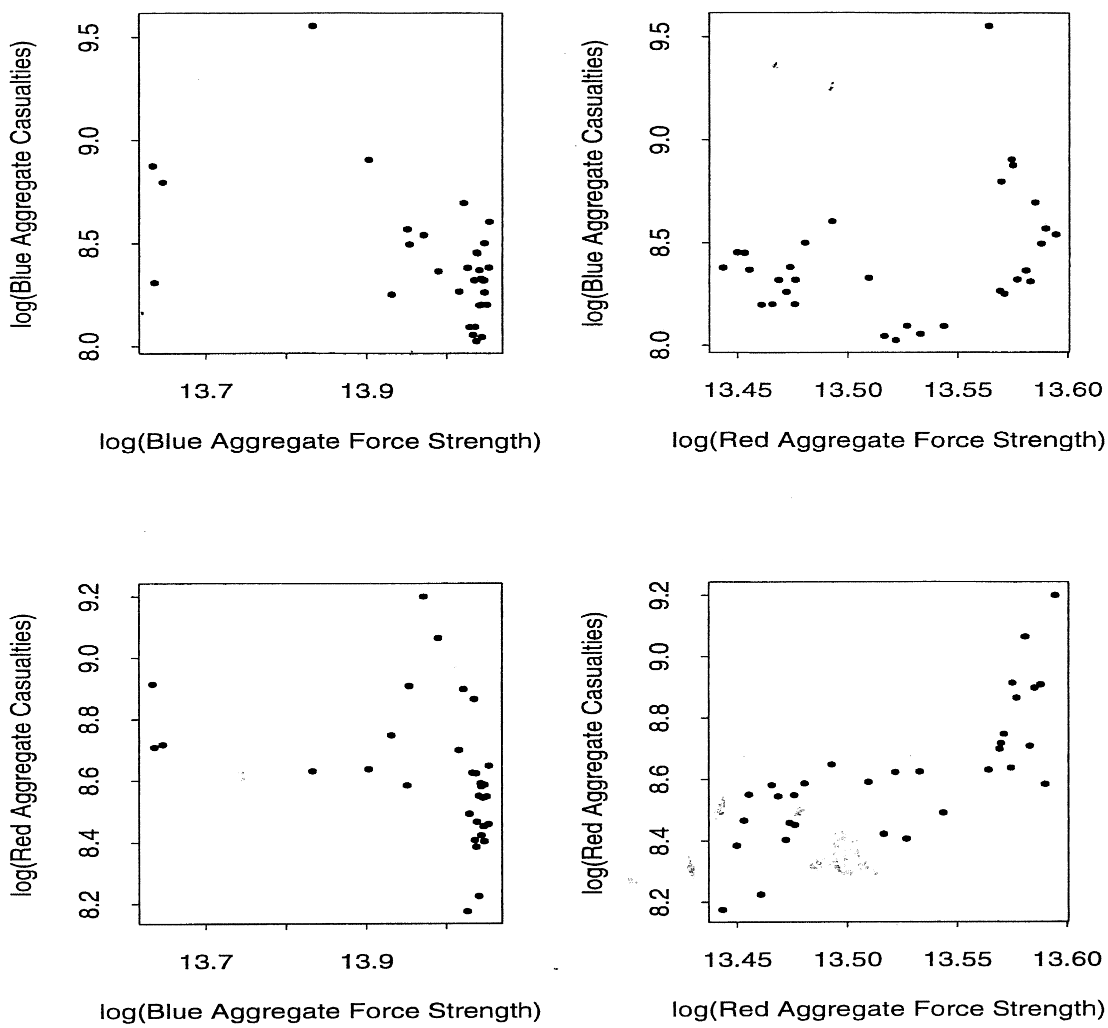

Figure 1. Plot of the logarithmically transformed force strengths (based on total manpower) versus casualties from the originally formatted data of Tables 1 and 2.

Total Manpower

$$\dot{B} = 1.7 \times 10^{-16} f(0.824) B^{3.2},$$
$$\dot{R} = 8.0 \times 10^{-16} f(0.824) R^{3.2}. \qquad (9)$$

Figure 3 shows the residuals of the two models. The top two plots are for the model based on combat manpower, and the bottom two for the model based on total manpower. The Blue combat residuals look nicely random. The other three plots also look somewhat random, with the Red residuals seeming to show a pattern that is probably attributable to the estimation scheme employed by DMSI, and each of the three has one or more large outliers. In particular, the Blue total manpower plot has a large positive outlier on day 4, which corresponds to the point in the campaign when German troops, after swift advancement resulting in significant penetration of the U.S. line, became mired in intense fighting with Allied troops that were beginning to hold ground. Note that the vast majority of the Allied casualties occurred with the noncombat troops, emphasizing German penetration into the

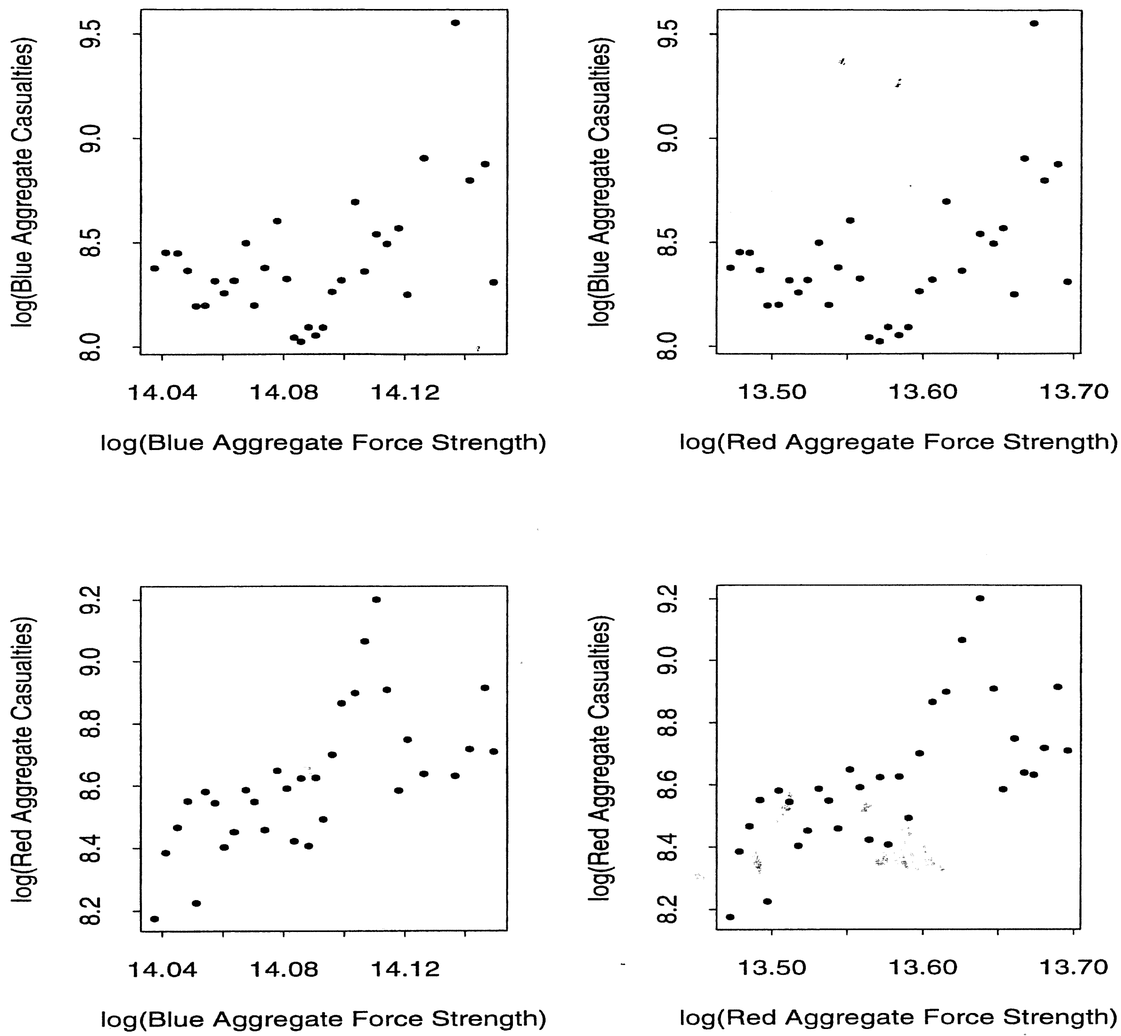

Figure 2. Plot of the logarithmically transformed force strengths (based on total manpower) versus casualties from the reformatted data of Tables 4 and 5.

rear echelon. On both of the Red plots, the largest outliers occur on days 9 and 10. These days correspond to a shift in the weather that allowed the Allied air force, grounded until this point, to enter the battle. With its overwhelming superiority, the Allied air power was able to effectively attack the German artillery, tanks, and APCs. Thus, while both models seem to fit well in general, there are some features of the campaign, corresponding to the outliers, that are not accounted for in the current models.

The models of (8) and (9) are similar to the Lanchester logarithmic formulation, and it is natural to ask whether the $\hat{q}$ exponent values are statistically different from $q = 1$. That is, do the data fail to reject the null hypothesis that the true value of q is 1, so that the observed values of $\hat{q}$ are simply a result of random variation? To check this, it is a simple matter to construct a confidence interval C so that $\mathbb{P}\{q \in C\} = 0.95$ or, to be even more stringent, 0.99. Such a confidence interval is defined as $C = [\hat{q} - t(n-2, 1-\alpha/2)*\text{s.e.}(\hat{q}), \hat{q} + t(n-2, 1-\alpha/2)*\text{s.e.}(\hat{q})]$, where $t(n-2, 1-\alpha/2)$ is the $1-\alpha/2$ quantile of the

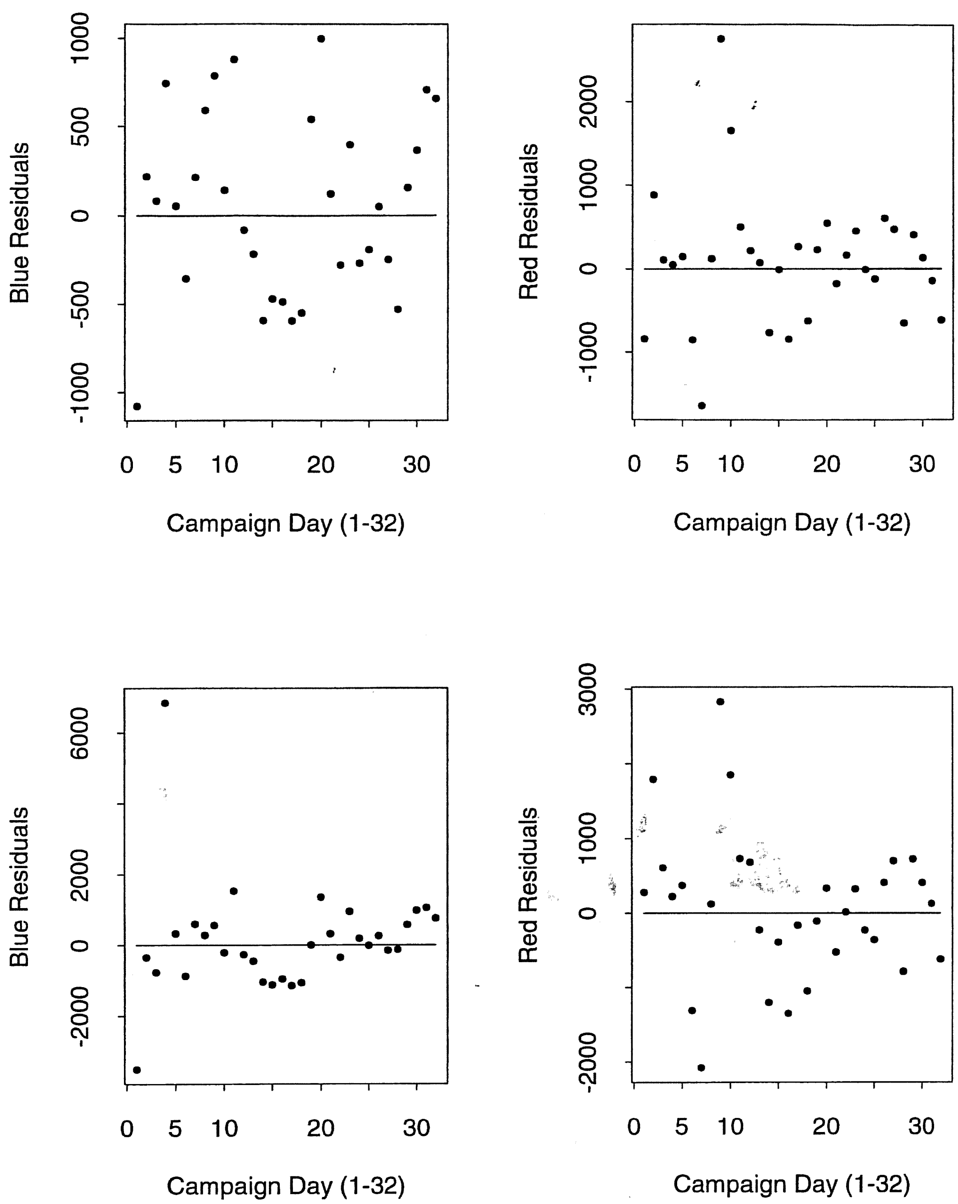

Figure 3. Residuals from the models of the reformatted data. The top two plots are for the model based on combat manpower and the bottom two for the model based on total manpower.

t distribution with $n - 2$ degrees of freedom and s.e.($\hat{q}$) is the standard error of the estimated parameter. See Draper and Smith [7] for details. The null hypothesis is rejected if $1 \notin C$.

For the combat manpower model, a 99% confidence interval is $C = [5.0 - (2.66)(0.68),\ 5.0 + (2.66)(0.68)] = [3.19, 6.81]$. For the total manpower model, a 99% confidence interval is $C = [1.90, 4.46]$. Thus, in both cases the Lanchester logarithmic model is rejected.

The Addition of Air Power

Missing from the previous models is air force data. Whether by chance or by design, the German's attack date coincided with poor weather so that the numerically superior Allied air force was effectively grounded. Once the weather cleared Allied air power decisively affected the campaign. Per Churchill [4], the Allied air force began flying on December 23, or day 9. A quick visual inspection of German casualties in Tables 1 and 2 shows dramatic increases in all categories. It is thus reasonable to conclude that air power was a key component in the campaign and should be included in the modeling.

The air data provided in the DMSI database are in a slightly different format from the manpower and equipment data, where, instead of aircraft available and aircraft killed, only the number of sorties flown per day is recorded. Simply weighting the sorties and adding them into the aggregate force strength of Table 5 on a daily basis can be justified on the grounds that, unlike the men and equipment, aircraft are able to fly into and out of the battle zone as a sort of ''instant asset'' to the ground forces. And, particularly in this campaign, where the weather restricted the availability of air forces for a period of time, applying the total number of planes (or sorties) available for the rest of the campaign to a day's aggregate force strength—regardless of whether they could fly into the battle zone—is not a good measure of overall force strength for that day.

Thus the air sortie data, with each sortie weighted at 30,[1] is added to the aggregate force strength on a daily basis. For example, the aggregate force strength for Red on day i was revised to

$$F_r(i) = M_r(i) + A_r(i) \times 5 + T_r(i) \times 20 + Y_r(i) \times 40 + S_r(i) \times 30,$$

where the appropriate numbers for day i are taken from Tables 3, 4, and 5. The relationship between force casualties and the revised force strengths remain roughly linear and are very similar to the plots of Figure 2.

Similar to models 8 and 9 in the previous section, the estimated exponent parameter $\hat{p}$ is statistically insignificant in both models and the estimated exponent $\hat{q}$ is large. The final models are:

Combat Manpower

$$\dot{B} = 2.7 \times 10^{-24} f(0.7971) B^{4.6},$$
$$\dot{R} = 1.6 \times 10^{-23} f(0.7971) R^{4.6}. \tag{10}$$

Total Manpower

$$\dot{B} = 1.3 \times 10^{-15} f(0.8197) B^{3},$$
$$\dot{R} = 5.6 \times 10^{-15} f(0.8197) R^{3}. \tag{11}$$

Also similar to models 8 and 9, the Lanchester logarithmic model is rejected. Thus, even with the addition of the air sortie data, the final models do not fit any of the Lanchester models: neither linear, square, nor logarithmic.

[1] Other weights could have been used based on the various accepted force scoring methodologies. The weights used in this paper were chosen to maintain consistency with Bracken [3].

DISCUSSION

The Lanchester equations can be interpreted probabilistically in the following way. Assume that each soldier on a battle field[2] has a probability p_k of being killed on a given day, and assume that all the soldiers are independent of each other. Then for a blue force size of B soldiers the expected number of casualties on that day is $\mathbb{E}(\dot{B}) = p_k B$, since $\dot{B}$ has a binomial distribution with ''success'' probability p_k and B number of observations.

Realistically, the probability of kill p_k is a function of many things, such as the opposing force's size (B or R), equipment (E), terrain (T), leadership (L), preparedness (P), and many other (O_i) factors; write $p_k = f(X, E, T, L, P, O_1, O_2, \cdots)$, where X represents B or R. For the Lanchester linear model,

$$\dot{B} = aRB,$$

$$\dot{R} = bBR,$$

the probability of kill is implicitly defined as $p_{k,b} \triangleq aR$ for a blue soldier and $p_{k,r} \triangleq bB$ for a red soldier. The interpretation is that the probability of being killed is a function of a constant, representing the lethality of a soldier, times the number of soldiers in the opposing force. In this definition Lanchester has simplified $p_k = f(X, E, T, L, P, O_1, O_2, \cdots)$ to $p_k = f(X, c) = cX$, where all the factors except force size are represented by a constant c and a simple multiplicative relationship is assumed.

This analysis carried the simplification one step further and fixed $p_k = c$ after it was demonstrated that the opponent's force size term was statistically insignificant. This should not be troubling because the opponent's force size is accounted for in the magnitude of the attrition parameters. The interpretation is that the probability of kill p_k is essentially constant *over the range of the opponent force sizes given in the data.* This is not to say that p_k would not change if the opponent's size were drastically increased or decreased (or, for that matter, if the opponent's equipment were improved/removed, or if the terrain changed to favor/disfavor the opponent, etc.). It simply says that, *for the given data,* the change in opponent force size did not significantly affect p_k.

Using this approach, the Lanchester logarithmic model has the interpretation that the attrition parameter represents the opponent's probability of killing a soldier and that this probability of kill is constant for a particular range of opponent force sizes. Such a model would not have occurred to Lanchester since in the warfare he considered that an opponent's strength was a strong function of his force size. That is, Lanchester formulated the linear form to model ''ancient'' warfare characterized by hand-to-hand combat. He formulated the square form to model ''modern'' warfare characterized by rifle carrying troops (cf. [15], Chap. 5). In both of these models, increased firepower is directly and strongly related to force size.

The firepower of an opponent in current warfare is less of a function of force size than in Lanchester's time. Depending on the type of combat and the aggregation weighting scheme, this is more or less true, of course. But the Gulf War is a good recent example illustrating this fact, in which the resultant Iraqi casualties were more a function of the number of Iraqis in the combat zone than of the Allied force size. That is, it is quite likely

[2] In contrast to using force scores, the discussion here will be phrased in terms of soldiers to make it more intuitive and concrete.

that the number of Iraqi casualties would have stayed about the same whether the Allied force had been reduced by 25%, say, or doubled. Another way of saying this is, given the Allied force size, Iraqi casualties were simply a function of how many Iraqi "targets" existed: If there had been a larger Iraqi force there would have been more casualties and from a lesser force there would have been fewer, *given the same sized opposing force.*

The findings in this work are consistent with the results of other researchers working with data from other battles: There is little or no empirical evidence to support either particular form of the equations that Lanchester advocated. See Hartley [12], for example. Indeed, the empirical results here support a significantly different model in which a force's casualties are simply a function of the size of one's own force and the enemy's lethality. Taken to the extreme this seems absurd, in the sense that the model would indicate that casualties continue to occur in one's own force even when the opponent's size is reduced to zero.

Yet this is not an absurd case at all. Peacetime military forces still experience casualties, though at a significantly reduced rate, where the opponents are accidents and nature. For the general model

$$\dot{B} = aB^q,$$

$$\dot{R} = bR^q, \tag{12}$$

the opponent's lethality is accounted for in the attrition parameter (a or b) and the exponent parameter in some way characterizes the fraction of a force that is exposed to the opponent. This interpretation fits with the results of the Ardennes data, where for the models with and without the air sortie data the following occurs:

- The exponent parameter for the models based on combat manpower are higher than those based on total manpower, reflecting the fact that the combat troops were more exposed to the lethality of the opponent.
- The attrition parameters for the Red forces are higher than the Blue's, indicating that the Allied forces were superior (more lethal) than the German forces.

To apply these models in the peacetime case, a force's attrition parameter would be drastically reduced, essentially reflecting the accident rate, and the exponent parameter would be set to indicate that proportion of the forces exposed to such accidents.

Of course, if this model holds in warfare the question becomes: How can one employ it to the advantage? Recall that for the original Lanchester equations force ratios stay equal over time if $bB^{p-q+1} = aR^{p-q+1}$. In the case of the new model (12) this is equivalent to the condition $bB^{1-q} = aR^{1-q}$. In the hypothetical case of equal lethality ($a = b$) and, similar to the total manpower models, if $q = 3$, then force ratios stay equal if $B^2 = R^2$. This means that if blue starts out with a superior force strength to red, and both commit all of their forces at the beginning of the conflict, then blue will eventually "win" (in the sense that blue's force strength will remain greater than red's throughout the conflict). But if red is able to withhold some of its force for a period of time *while maintaining lethality parity with blue,* then it is possible for red to inject them at a later time and achieve a superior force strength.

For example, consider blue and red forces with initial force strengths $B(0) = 100$ and $R(0) = 90$, and let the lethality (attrition parameters) of the two forces be equal at $a = b$

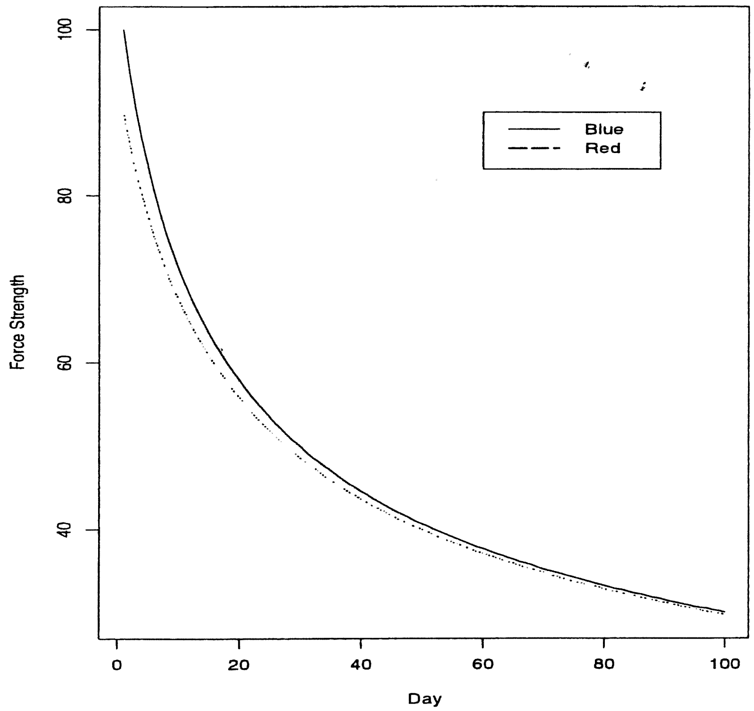

Figure 4. Blue and red force sizes over the course of a hypothetical battle. In this battle $q = 3$ (with $p = 0$), initial force strengths were $B(0) = 100$ and $R(0) = 90$, and the attrition parameters were equal, $a = b = 5 \times 10^{-6}$.

$= 5 \times 10^{-6}$. Then as Figure 4 shows, over the course of a 100-day campaign, blue maintains the greater force strength. Yet, as Figure 5 shows, if red withholds half of its force strength until day 50 (assuming that the remaining force is capable of maintaining $a = 5 \times 10^{-6}$), then red can gain the upper hand in force strength.

This result can even hold for a force of inferior strength and lethality which would lose in a direct confrontation of all of its forces versus all of the opponent's forces. Figure 6 shows that for the initial force sizes $B(0) = 100$ and $R(0) = 90$, with red's forces less lethal than blue's, $a = 3 \times 10^{-6}$ versus $b = 5 \times 10^{-6}$, red can still achieve a greater force strength than blue.

CONCLUSION

Lanchester proposed two basic models for warfare and justified them based on a discussion of modern and ancient warfare methods. They have gained some prominence and are now routinely being used as warfare models. These results show that Lanchester's basic models (linear and square) *do not hold* when fit to known data from an actual battle; this is a clear counterexample that they are not universally applicable. In particular, Bracken's conclusion [3] that the Lanchester linear law fit the Ardennes Campaign (based on part of the Ardennes data and resulting from a search over a subset of the parameter space) does not hold. Using Bracken's criteria for choosing the best model, one can base rejection of the linear model on the fact that other models with smaller sum of squared residuals are shown. Rejection can also be based on the fact that the estimated exponent parameter $\hat{p}$ is found to be statistically insignificant.

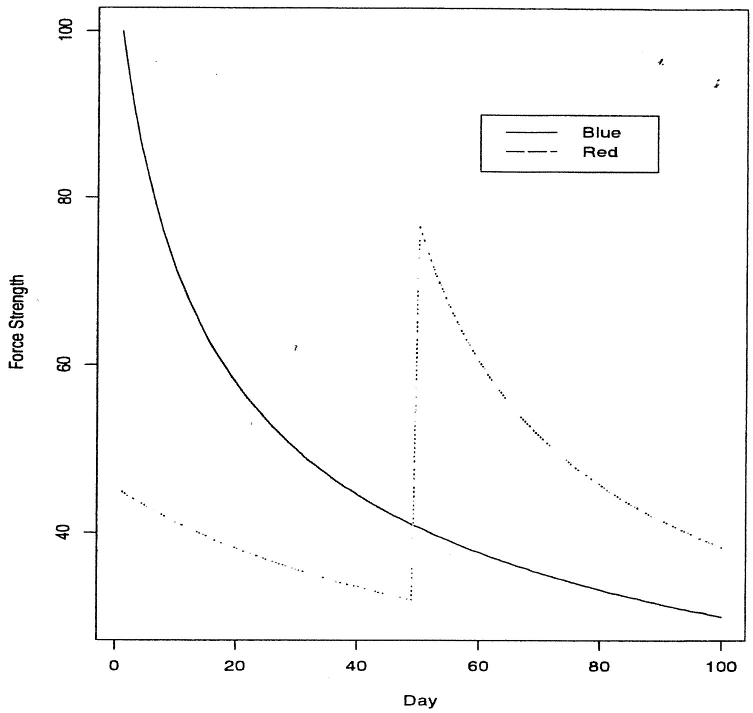

Figure 5. This hypothetical battle is equivalent with the exception that red only committed one-half of its force strength initially using the second half as reinforcements midway through the campaign.

Similarly, Hartley's particular model (2) which was proposed as universally applicable is rejected for the Ardennes Campaign. This result holds whether one models the nonhomogeneous Ardennes Campaign force as shown in this paper or whether one only considers the homogeneous personnel data. Yet Hartley's general conclusions are not wholly inconsistent with the results presented here. Specifically, he concludes that the size of the attrited force is the dominant factor in computing attrition; the final models here do not contradict this, but instead show a stronger dominance.

The result of this analysis is a different form of the Lanchester equations for the Ardennes Campaign in which the exponent parameter of the attacking force is zero: $p = 0$. This results in a relationship in which a force's casualties are a function of the size (or force strength) fielded and the enemy's lethality. Such a model applies to two forces in combat or one force in peacetime. Whether the model generalizes to other campaigns and battles remains to be investigated.

It must be noted that the Ardennes Campaign as analyzed here was a "battle" on a scale perhaps not envisioned by Lanchester, ultimately involving approximately 1.7 million people. In effect it was an aggregation of many smaller battles, and that aggregation may be masking warfare behavior at the unit level similar to one of Lanchester's basic models or Hartley's model. This is an area deserving of further research.

Yet the model derived in this analysis is consistent with the Ardennes Campaign and its outcome. In this battle the Allied forces had the higher attrition parameter (so that they were more lethal), and they were able to continue to inject forces over the course of the campaign. Via the previous discussion these two factors worked to the Allies advantage in ensuring that they would have the greater force strength in the long term. This is indeed what happened.

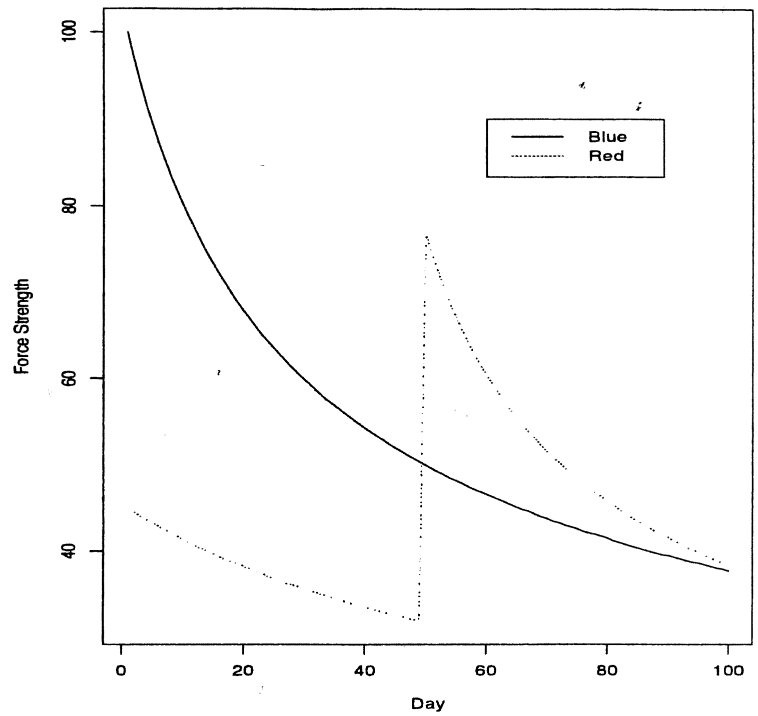

Figure 6. Using the same strategy of withholding some of the force, red can still maintain a superior force strength in spite of having a smaller attrition parameter ($a = 3 \times 10^{-6}$, $b = 5 \times 10^{-6}$).

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