GENERAL NONLINEAR PLATE THEORY APPLIED TO A CIRCULAR PLATE WITH LARGE DEFLECTIONS

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NOMENCLATURE

$b_{\alpha\beta}$	curvature tensor
dA	element of surface area
ds	element of arc length
$E_{\alpha\beta}$	membrane strain tensor
$e_{\alpha\beta}$	linear part of membrane strain tensor
f^{α}	surface components of q^i
f	normal component of q^i
$g_{\alpha\beta}$	metric tensor
g	determinate of $g_{\alpha\beta}$
M^i	moment vector per unit undeformed length of middle surface
N^i	unit normal to surface
$n^{\alpha}x^i_{\|\alpha}$	unit surface vector normal to boundary
p	concentrated load
P	pressure loading per unit deformed area of surface
Q^i	force vector per unit undeformed length of middle surface
Q	component of Q^i normal to surface
q^i	loading force per unit undeformed area
T^{α}	components of Q^i in the surface
$t^{\alpha}x^i_{\|\alpha}$	unit surface vector tangent to boundary

U^i	displacement of the middle surface
u^α	surface components of displacement
w	normal component of displacement
x^i	position vector
$x^i_{\vert\alpha}$	surface base vectors
$\varepsilon_{\alpha\beta}$	alternating tensor
ξ^α	material point in middle surface of shell

I INTRODUCTION AND REVIEW OF THE LITERATURE

Practical inflated structures, which are used, for example, for satellite applications, quite often involve large displacements and strains. Therefore it is apparent that a general non-linear theory for large displacements and strains is required. Another reason for studying the general non-linear theory is to see where the linear theory and other approximate non-linear theories are no longer valid.

The exact non-linear equilibrium equations for thin plates and shells in deformed coordinates have been derived by several authors. Among them are Sanders (1963), Leonard (1961), Koiter (1966), Naghdi (1963), and Green and Zerna (1954). However the deformed coordinates or geometry are generally not the most useful. This is because the deformed geometry of a body is generally not prescribed or known and therefore an indirect or semi-inverse method must be used for solving problems involving large deformations.

A more direct solution could be obtained if the equilibrium equations were written in the underformed coordinates. A review of the literature shows that no one has written out explicitly the equilibrium equations in the undeformed coordinates for the combined membrane and bending problem for plates and shells.

Probably the first approach to the completely general non-linear shell problems was done by Synge and Chien (1941), (1943), (1944). In this theory large deformations were allowed but the strains had to be small. Their theory was fairly complicated since it included the effects of transverse shear. In Chien's work all the equations are written in terms of strains and the use of displacements is avoided.

Green and Zerna (1954) and Naghdi (1963) have derived the equilibrium equations in the deformed coordinates by beginning with the equations of three dimensional elasticity and integrating through the shell thickness.

Koiter (1966) has developed the same equations by using the principle of virtual work and the Kirchhoff-Love assumptions.

Leonard (1961) and Sanders (1963), working independently, have used essentially the same approach. They both begin with the equations of equilibrium as developed by Green and Zerna (1954). Using the principle of virtual work they then use these equations to derive consistent expressions for strains in terms of displacements which are justified by arguments presented by Koiter (1960). Then they define new "stress" and "moment" resultants to simplify the energy equations. The principle of virtual work is again applied to arrive at a more desirable form of the equilibrium equations. Leonard then emphasizes membrane theory with large strains and Sanders looks at the simplification of the equations for small strains.

Budiansky (1968) following the work of Sanders (1963), Leonard (1961), and Koiter (1966) used a variational approach to derive exact tensor equations of equilibrium for non-linear membrane shell theory. He also outlined a procedure for obtaining the non-linear membrane and bending equations for plates and shells in the undeformed coordinate system.

Following the procedure suggested by Budiansky (1968) the "first approximation" non-linear membrane and bending equations for an arbitrary plate will be developed in tensor notation. The term "first approximation" is used in the sense of Love's (1944) first approximation as interpreted by Koiter (1960); i.e., the contributions to the strain energy of deformation of stresses acting in the direction normal to the reference surface can be neglected compared to the contributions of bending and stretching. The Kirchhoff-Love assumptions and plane stress assumptions are equivalent and result in "first approximation" equations. To arrive at these equations we will start with the variational equation of equilibrium as presented by Budiansky for an arbitrary shell. When Budiansky's variational equation of equilibrium is applied to a plate, the equation becomes much simpler since the plate has no initial curvature. Variational calculus will then be used on this equation to obtain the exact tensor "first approximation" equilibrium equations for a thin plate.

The tensor equations will then be converted into physical equations for a rotationally symmetric plate with arbitrary symmetric loading using a method suggested by Frederick (1956).

The specific problem of a plate loaded with a concentrated load on a finite concentric rigid inclusion will be studied using perturbation type solutions and "matched asymptotic expansions" where they are needed. The equations will be studied for deflections of the orders

I $w \ll R$ and $u_r \ll R$,

(a) $w \ll h$

(b) $w = O(h)$

(c) $w \gg h$

II $w = O(R)$,

III $w \gg R$.

Tolefson (1967) has studied this same problem using the Reissner equations (1949), (1958) and the von Karman equations (1910). He obtains solutions to the von Karman equations in the form of perturbation expansions and compares them to the data obtained by Hart and Evans (1964) from the numerical integration of Reissner's equations. The results that we obtain will be compared with Tolefson's results. Our results will also be compared with results obtained by Caldwell (1968) from the numerical integration of Reissner's equations for the membrane case only.

The equilibrium equations that will be derived are independent of the relationships between the stress and moment resultants and the strains. However to solve a specific problem some constitutive relationship is necessary. Since it is not the purpose of this dissertation to study constitutive relations but rather the structure of the equilibrium equations, a simple linear relationship will be used throughout. That relationship is the one presented by Budiansky (1968). This relationship differs from Love's first approximation by terms which Koiter (1960) shows are of the same order as terms that would be neglected by a "first approximation" theory. This precise establishment by Koiter (1960) of the essential features of a "first approximation" shell theory and the careful delineation of the important from the unimportant differences between various theories has done much to remove the confusion which has long been an accepted part of the theory of thin shells.

The equations developed in this dissertation will be developed for a plate that is thin enough so that the state of stress is approximately "plane" in the sense that the contributions to the strain energy of deformation of stresses acting in the direction normal to the reference surface can be neglected. As pointed out by Koiter (1960) this implies that $(h/L)^2 \ll 1$ where L is the smallest "wave length" of deformation of the deformed reference surface and h is the plate thickness.

II PRELIMINARIES: NOTES ON TENSOR ANALYSIS APPLIED TO NONLINEAR SHELL THEORY

This chapter closely follows a paper by Budiansky (1968) and is concerned with certain preliminaries and selected results from the differential geometry of a surface. Also the development of specific relationships concerning nonlinear shell theory are included. These will then be used in chapter three to develop the exact "first approximation" tensor equilibrium equations for the nonlinear membrane and bending problem for an arbitrary thin flat plate.

Consider an arbitrary shell whose undeformed middle surface is prescribed by the equations

$$x^i = x^i(\xi^1 , \xi^2) \tag{2.1}$$

where ξ^1 and ξ^2 are general coordinates in the middle surface and x^1, x^2, x^3 are rectangular Cartesian coordinates. The displacement of the middle surface U^i is then defined by

$$U^i = \bar{x}^i - x^i \tag{2.2}$$

where barred quantities refer to the deformed state of the middle surface. If we define surface tensor components of displacement by u^α and normal displacements by w we have

$$U^i = u^\alpha x^i_{|\alpha} + wN^i \; . \tag{2.3}$$

In all the equations Latin indexes will denote three-dimensional

Cartesian components and Greek superscripts and subscripts denote contravariant and covariant surface tensor components. Slashes denote cocariant differentiation based on the geometry of the undeformed shell.

Using tensor analysis we know that (see Sokolnikoff (1966))

$$g_{\alpha\beta} = x^i_{|\alpha} x^i_{|\beta} \quad , \tag{2.4}$$

$$b_{\alpha\beta} = b_{\beta\alpha} = N^i_{|\alpha} x^i_{|\beta} \quad , \tag{2.5}$$

$$\varepsilon_{12} = -\varepsilon_{21} = \sqrt{g} \quad , \tag{2.6}$$

$$\varepsilon_{11} = \varepsilon_{22} = 0 \quad , \tag{2.7}$$

$$d\bar{A}/dA = \sqrt{\bar{g}/g} \quad , \tag{2.8}$$

$$\bar{n}_\alpha d\bar{s} = \sqrt{\bar{q}/g}\; n_\alpha ds \quad , \tag{2.9}$$

$$x^i_{|\alpha\beta} = -\, b_{\alpha\beta} N^i \quad , \tag{2.10}$$

and

$$N^i_{|\alpha} = b^\nu_\alpha x^i_{|\nu} \quad . \tag{2.11}$$

It then follows from equations (2.2), (2.10) and (2.11) that

$$\bar{x}^i_{|\alpha} = x^i_{|\alpha} + d^\nu_\alpha x^i_{|\nu} - \phi_\alpha N^i \tag{2.12}$$

where

$$d_{\nu\alpha} = u_{\nu|\alpha} + b_{\nu\alpha} w \tag{2.13}$$

and

$$\phi_\alpha = -w_{|\alpha} + b^\nu_\alpha u_\nu \quad . \tag{2.14}$$

Leonard (1961) has shown that

$$\bar{N}^i = \sqrt{\frac{g}{\bar{g}}}\left[(\phi^\nu + R^\nu)x^i_{|\nu} + (1 + d^\omega_\omega + H)N^i\right] \tag{2.15}$$

where

$$R_\nu = \phi_\nu d^\omega_\omega - \phi^\omega d_{\omega\nu} \tag{2.16}$$

and

$$H = \frac{1}{2}(d^\omega_\omega d^\rho_\rho - d_{\omega\rho}d^{\rho\omega}) \quad . \tag{2.17}$$

The membrane strain tensor $E_{\alpha\beta}$ is defined by

$$E_{\alpha\beta} = \frac{1}{2}(\bar{g}_{\alpha\beta} - g_{\alpha\beta}) \tag{2.18}$$

and can be written as

$$E_{\alpha\beta} = e_{\alpha\beta} + \frac{1}{2}(d^\nu_\alpha d^\nu_\beta + \phi_\alpha\phi_\beta) \tag{2.19}$$

where the linear part of $E_{\alpha\beta}$ is

$$e_{\alpha\beta} = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) + b_{\alpha\beta}w \quad . \tag{2.20}$$

Two types of surface loading are considered, a pressure p per unit deformed area acting in the direction of $\bar{N}^i$ and a loading of intensity q^i prescribed as a force per unit undeformed area written as

$$q^i = f^\alpha x^i_{|\alpha} + fN^i \quad . \tag{2.21}$$

The total force acting on the undeformed area is

$$p\sqrt{\frac{\bar{g}}{g}}\,\bar{N}^i \;+\; f^{\alpha}x^i_{|\alpha} \;+\; fN^i \quad . \tag{2.22}$$

In general on any edge there may exist a force vector Q^i per unit undeformed length of middle surface and a moment vector M^i per unit undeformed length. The force on any element of deformed arc length $d\bar{s}$ having unit normal $\bar{n}^i = \bar{n}^{\alpha}\bar{x}^i_{|\alpha}$ is given by

$$(N^{\alpha\beta}\bar{n}_{\alpha}\bar{x}^i_{|\beta} + Q\bar{N}^i)\,d\bar{s} \;=\; Q^i ds \tag{2.23}$$

where $N^{\alpha\beta}$ is the membrane stress-resultant and Q is the shear. The force vector Q^i could also be represented by

$$Q^i \;=\; T^{\alpha}x^i_{|\alpha} + QN^i \tag{2.24}$$

where T^{α} are the components in the surface and Q is perpendicular to the surface. The moment vector M^i is given by

$$\bar{g}^{\alpha\omega}\bar{\varepsilon}_{\beta\alpha}M^{\nu\beta}\bar{n}_{\nu}\bar{x}^i_{|\omega}d\bar{s} \;=\; M^i ds \tag{2.25}$$

where $\bar{g}^{\alpha\omega}$ is the inverse of $\bar{g}_{\alpha\omega}$ and constitutes the only exception in this paper to the convention that indexes are raised or lowered by means of the metric tensor of the undeformed shell. $M^{\alpha\beta}$ is the unsymmetrical stress couple tensor. One restriction on M^i is that in the absence of couple stresses $M^i\bar{N}^i = 0$.

For a more elaborate account of the material in this section the reader is referred to Budiansky (1968), Sokolnikoff (1966) and

Leonard (1961). It should be noted that the sign on $b_{\alpha\beta}$ given by equation (2.5) is opposite in sign to that usually given in books on tensors.

III DEVELOPMENT OF THE EQUATIONS

In this chapter the exact "first approximation" tensor equilibrium equations for the nonlinear membrane and bending problem for an arbitrary thin flat plate are developed. These equations will then be reduced to the equations needed to solve the problem of a rotationally symmetric thin plate with symmetric loading.

A. Exact Tensor Equations for Arbitrary Plates

Starting with the variational equation of equilibrium for an arbitrary shell the exact tensor equilibrium equations are developed for a plate. As a by product of the variational approach the boundary conditions are also developed.

The variational equation of equilibrium as presented by Budiansky (1968) for an arbitrary shell is

$$\int_A (\hat{n}^{\alpha\beta}\delta E_{\alpha\beta} + \bar{M}^{\alpha\beta}\delta\hat{K}_{\alpha\beta})\,dA = \int_{\bar{A}} p\bar{N}^i \delta U^i d\bar{A} + \int_A q^i \delta U^i dA$$

$$+ \int_C Q^i \delta U^i ds + \text{EVWM} \tag{3.1}$$

where A and C are the initial area and boundary of the shell middle surface, $\bar{A}$ is the deformed area, EVWM is the external virtual work done by the moments, and

$$\hat{n}^{\alpha\beta} = \tilde{n}^{\alpha\beta} + \frac{1}{2}(b^{\alpha}_{\nu}\bar{M}^{\nu\beta} + b^{\beta}_{\nu}\bar{M}^{\alpha\nu}) + g^{\alpha\beta}b_{\omega\nu}\bar{M}^{\omega\nu} , \tag{3.2}$$

$$\hat{K}_{\alpha\beta} = \tilde{K}_{\alpha\beta} - \frac{1}{2}(b^{\nu}_{\alpha}E_{\nu\beta} + b^{\nu}_{\beta}E_{\nu\alpha}) - b_{\alpha\beta}E^{\nu}_{\nu} , \tag{3.3}$$

$$\bar{M}^{\alpha\beta} = \frac{1}{2}(M^{\alpha\beta} + M^{\beta\alpha}) , \tag{3.4}$$

$$\tilde{n}^{\alpha\beta} = \sqrt{\frac{\bar{g}}{g}}\ \{N^{\alpha\beta} - \bar{b}_{\nu p}(\bar{g}^{\rho\beta}M^{\nu\alpha} + \bar{g}^{\alpha\beta}M^{\rho\nu})\} , \tag{3.5}$$

$$\tilde{K}_{\alpha\beta} = \sqrt{\frac{\bar{g}}{g}}\ \bar{b}_{\alpha\beta} - b_{\alpha\beta} , \tag{3.6}$$

and

$$\bar{b}_{\alpha\beta} = \sqrt{\frac{\bar{g}}{g}}\ \{(1 + e^{\omega}_{\omega} + H)(b_{\alpha\beta} + \phi_{\alpha|\beta} + b^{\nu}_{\beta}d_{\nu\alpha})$$

$$- (\phi^{\nu} + R^{\nu})(d_{\nu\alpha|\beta} - b_{\nu\beta}\phi_{\alpha})\} . \tag{3.7}$$

When the above variational equation of equilibrium is applied to a plate, the equation becomes much simpler since $b_{\alpha\beta} = 0$ for a plate. In this case

$$\hat{K}_{\alpha\beta} = \tilde{K}_{\alpha\beta} = \sqrt{\frac{\bar{g}}{g}}\ \bar{b}_{\alpha\beta} , \tag{3.8}$$

$$\hat{n}^{\alpha\beta} = \tilde{n}^{\alpha\beta} = \sqrt{\frac{\bar{g}}{g}}\ \{N^{\alpha\beta} - \bar{b}_{\nu\rho}(\bar{g}^{\rho\beta}M^{\nu\alpha} + \bar{g}^{\alpha\beta}M^{\rho\nu})\} , \tag{3.9}$$

$$\bar{M}^{\alpha\beta} = \frac{1}{2}(M^{\alpha\beta} + M^{\beta\alpha}) , \tag{3.10}$$

$$\bar{b}_{\alpha\beta} = \sqrt{\frac{g}{\bar{g}}}\ \{(1 + e^{\omega}_{\omega} + H)(\phi_{\alpha|\beta}) - (\phi_{\nu} + R^{\nu})(d_{\nu\alpha|\beta})\} . \tag{3.11}$$

By using variational calculus we arrive at the following equilibrium equations and boundary conditions:

Equilibrium Equations;

$$\left[(g_{\kappa\alpha} + d_{\kappa\alpha})\tilde{n}^{\alpha\beta}\right]\Big|_{\beta} - \left[\bar{M}^{\alpha\beta}\{[\delta^{\rho}_{\kappa} + (g^{\omega\lambda}\delta^{\rho}_{\kappa} - g^{\rho\omega}\delta^{\lambda}_{\kappa})u_{\omega|\lambda}]w_{|\alpha\beta}\right.$$

$$\left. + [g^{\rho\nu}w_{|\kappa} - g^{\lambda\nu}\delta^{\rho}_{\kappa}w_{|\lambda}]u_{\nu|\alpha\beta}\}\right]\Big|_{\rho} + [\bar{M}^{\alpha\beta}(\phi_{\kappa} + R_{\kappa})]\big|_{\beta\alpha}$$

$$+ p(\phi_{\kappa} + R_{\kappa}) + f_{\kappa} = 0 \tag{3.12}$$

and

$$(\tilde{n}^{\alpha\beta}w_{|\alpha})\big|_{\beta} - \{\bar{M}^{\alpha\beta}[- g^{\lambda\nu} + (g^{\phi\nu}g^{\lambda\omega} - g^{\lambda\nu}g^{\omega\phi})u_{\omega|\phi}]u_{\nu|\alpha\beta}\}\big|_{\lambda}$$

$$+ [\bar{M}^{\alpha\beta}(1 + e^{\omega}_{\omega} + H)]\big|_{\beta\alpha} + p(1 + e^{\omega}_{\omega} + H) + f = 0 \tag{3.13}$$

Boundary Conditions; prescribe

$$n_{\rho}\{\tilde{n}^{\phi\rho} + \tilde{n}^{\alpha\rho}g^{\phi\lambda}u_{\lambda|\alpha} - \bar{M}^{\alpha\beta}\{\left[g^{\phi\rho} + (g^{\omega\lambda}g^{\phi\rho} - g^{\rho\omega}g^{\phi\lambda})u_{\omega|\lambda}\right]w_{|\alpha\beta}$$

$$+ \left[g^{\rho\nu}g^{\omega\phi}w_{|\omega} - g^{\lambda\nu}g^{\phi\rho}w_{|\lambda}\right]u_{\nu|\alpha\beta}\} + \left[\bar{M}^{\rho\beta}(\phi^{\phi} + R^{\phi})\right]\Big|_{\beta}\} = T^{\phi} \tag{3.14}$$

or u_{ϕ} and

$$\{\tilde{n}^{\alpha\lambda} w_{|\alpha} - \bar{M}^{\alpha\beta} \left[-g^{\lambda\nu} + (g^{\phi\nu} g^{\lambda\omega} - g^{\lambda\nu} g^{\omega\phi}) u_{\omega|\phi} \right] u_{\nu|\alpha\beta}$$

$$+ \left[\bar{M}^{\lambda\beta} (1 + e^{\omega}_{\omega} + H) \right]_{|\beta} \} \, n_{\lambda} = Q \tag{3.15}$$

or w. There are also boundary conditions between the moments and slopes but these are difficult to formulate in general terms. They will be discussed in detail in the next section.

B. Exact Tensor and Physical Equations for Symmetric Circular Plates

For a rotationally symmetric circular plate $\xi^1 = r$, $\xi^2 = \theta$. It therefore follows that

$$x = x^1 = r \cos \theta \quad , \tag{3.16}$$

$$y = x^2 = r \sin \theta \quad , \tag{3.17}$$

and

$$z = x^3 = 0 \quad . \tag{3.18}$$

The metric is given by

$$g_{\alpha\beta} \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \tag{3.19}$$

and

$$g^{\alpha\beta} \quad \Rightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix} \tag{3.20}$$

Symmetry requires that

$$\frac{\partial}{\partial\theta}\ (\ ) = 0\ ,\quad u_2 = 0$$

and

$$\bar{M}^{12} = \bar{M}^{21} = \tilde{n}^{12} = \tilde{n}^{21} = 0\ . \qquad (3.21\ a\text{-}f)$$

Substituting the above into the equilibrium equations (3.12) and (3.13) we find that the tensor equilibrium equations for a rotationally symmetric circular plate are as follows:

r Equilibrium Equation (tensor notation)

$$\begin{aligned}
&\tilde{n}^{11}\{u_{1,11} + \frac{1}{r}(1 + u_{1,1})\} - \tilde{n}^{22}(r + u_1) + \tilde{n}^{11}_{,1}\ (1 + u_{1,1}) \\
&+ \bar{M}^{11}\{ - 2(1 + \frac{u_1}{r})w_{,111} - \frac{2}{r}(1 + u_{1,1})w_{,11} - \frac{2}{r}u_{1,11}w_{,1}\} \\
&+ \bar{M}^{22}\{2(1 + \frac{u_1}{r})w_{,1}\} + \bar{M}^{11}_{,1}\{- 3(1 + \frac{u_1}{r})w_{,11} - \frac{2}{r}(1 + u_{1,1})w_{,1}\} \\
&+ \bar{M}^{11}_{,11}\{-(1 + \frac{u_1}{r})w_{,1}\} + p\{ -(1 + \frac{u_1}{r})w_{,1}\} + f_1 = 0 \qquad (3.22)
\end{aligned}$$

and

z Equilibrium Equation (tensor notation)

$$\begin{aligned}
&\tilde{n}^{11}w_{,11} + \tilde{n}^{11}_{,1}w_{,1} + \frac{\tilde{n}^{11}}{r}w_{,1} + \bar{M}^{11}\{2(1 + \frac{u_1}{r})u_{1,111} + \frac{4}{r}(1 + u_{1,1})u_{1,11}\} \\
&+ \bar{M}^{11}_{,1}\{3(1 + \frac{u_1}{r})u_{1,11} + \frac{2}{r}(1 + u_{1,1})^2\} + \bar{M}^{22}\{-2(1 + \frac{u_1}{r})(1 + u_{1,1})\}
\end{aligned}$$

$$+ \bar{M}^{22}_{,1}\{-r(1 + \frac{u_1}{r})^2\} + \bar{M}^{11}_{,11}\{(1 + \frac{u_1}{r})(1 + u_{1,1})$$

$$+ p\{(1 + \frac{u_1}{r})(1 + u_{1,1}) + f = 0 \quad . \tag{3.23}$$

The equilibrium equation in the θ direction is identically satisfied. Commas in the above equation denote ordinary differentiation. From equations (3.14) and (3.15) we find the boundary conditions on the forces and displacements in tensor notation.

Boundary Conditions (tensor notation); prescribe

$$T^1 = \tilde{n}^{11}(1 + u_{1,1}) + \bar{M}^{11}\{-2(1 + \frac{u_1}{r})w_{,11} - \frac{1}{r}(1 + u_{1,1})w_{,1}\}$$

$$+ \bar{M}^{11}_{,1}\{-(1 + \frac{u_1}{r})w_{,1}\} \tag{3.24}$$

or u_1

and

$$Q = \tilde{n}^{11}w_{,1} + \bar{M}^{11}\{2(1 + \frac{u_1}{r})u_{1,11} + \frac{1}{r}(1 + u_{1,1})^2\}$$

$$+ \bar{M}^{22}\{-r(1 + \frac{u_1}{r})^2\} + \bar{M}^{11}_{,1}\{(1 + \frac{u_1}{r})(1 + u_{1,1})\} \tag{3.25}$$

or w.

In order to determine the boundary condition on the moment and rotation an expression for the rotation must first be determined. From Figure 1 it is seen that the slope of the plate or tan β is $\dfrac{w_{,1}}{1 + u_{1,1}}$.

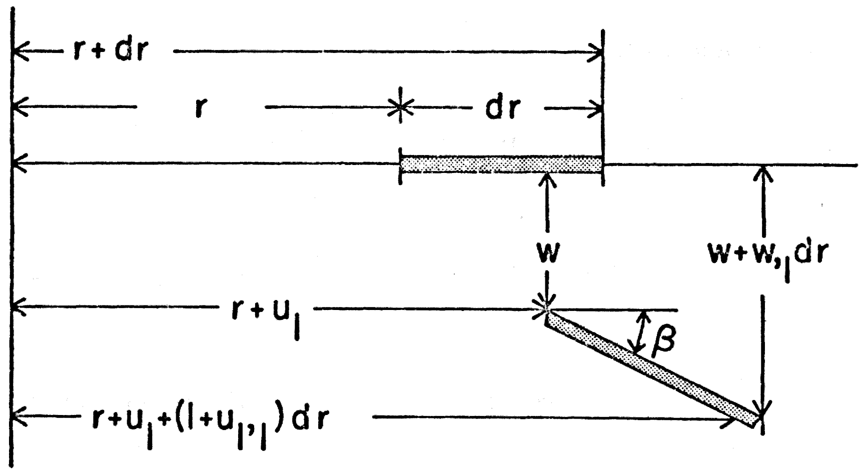


Figure 1. Plate element in deformed and undeformed positions

This means that

$$\beta = \tan^{-1} \frac{w_{,1}}{1 + u_{1,1}} \tag{3.26}$$

and the term EVWM in equation (3.1) is

$$\text{EVWM} = \int_C M\delta\beta \quad .$$

Using this we find that the boundary condition on the moment and slope is to specify either

$$M = M^{11}(1 + \frac{u_1}{r})\{(1 + u_{1,1})^2 + (w_{,1})^2\} \tag{3.27}$$

or $$\beta = \tan^{-1} \frac{w_{,1}}{1 + u_{1,1}} \quad . \tag{3.28}$$

To convert these equations into physical equations we use the relations given by Frederick (1956). From these we see that

$$w = w \qquad \tilde{n}_r = \tilde{n}^{11} \qquad \bar{M}_r = \bar{M}^{11}$$

$$u_r = u_1 \qquad \tilde{n}_\theta = r^2\tilde{n}^{22} \qquad \bar{M}_\theta = r^2\bar{M}^{22} \tag{3.29 a-f}$$

The equations in physical variables are

r Equilibrium Equation (physical variables)

$$\begin{aligned}
&\tilde{n}_r\{u_r'' + \frac{1}{r}(1 + u_r') + \tilde{n}_\theta\{- \frac{1}{r}(1 + \frac{u_r}{r}) + \tilde{n}_r'(1 + u_r')\} \\
&+ \bar{M}_r\{-2(1 + \frac{u_r}{r})w''' - \frac{2}{r}(1 + u_r')w'' - \frac{2}{r}u_r''w'\} \\
&+ \bar{M}_\theta\{\frac{2}{r^2}(1 + \frac{u_r}{r})w'\} + \bar{M}_r'\{-3(1 + \frac{u_r}{r})w'' - \frac{2}{r}(1 + u_r')w'\} \\
&+ \bar{M}_r''\{-(1 + \frac{u_r}{r})w'\} + p\{-(1 + \frac{u_r}{r})w'\} + f_r = 0
\end{aligned} \tag{3.30}$$

and

z Equilibrium Equation (physical variables)

$$\begin{aligned}
&\tilde{n}_r(w'' + \frac{w'}{r}) + \tilde{n}_r'w' + \bar{M}_r\{2(1 + \frac{u_r}{r})u_r''' + \frac{4}{r}(1 + u_r')u_r''\} \\
&+ \bar{M}_\theta\{\frac{-2}{r^2}(1 + \frac{u_r}{r})(u_r' - \frac{u_r}{r}) + \bar{M}' - \frac{1}{r}(1 + \frac{u_r}{r})^2\}
\end{aligned}$$

$$+ \bar{M}_r'\{3(1 + \frac{u_r}{r})u_r'' + \frac{2}{r}(1 + u_r')^2\} + \bar{M}_r''\{(1 + \frac{u_r}{r})(1 + u_r')\}$$

$$+ p\{(1 + \frac{u_r}{r})(1 + u_r')\} + f \quad = \quad 0 \tag{3.31}$$

Boundary Conditions (physical variables); prescribe

$$T_r \quad = \quad \tilde{n}_r(1 + u_r') + \bar{M}_r\{-2(1 + \frac{u_r}{r})w'' - \frac{1}{r}(1 + u_r')w'\}$$

$$+ \bar{M}_r'\{-(1 + \frac{u_r}{r})w'\} \tag{3.32}$$

or u_r

and

$$Q \quad = \quad \tilde{n}_r w' + \bar{M}_r\{2(1 + \frac{u_r}{r})u_r'' + \frac{1}{r}(1 + u_r')^2\}$$

$$+ \bar{M}_\theta\{-\frac{1}{r}(1 + \frac{u_r}{r})^2\} + \bar{M}_r'\{(1 + \frac{u_r}{r})(1 + u_r')\} \tag{3.33}$$

or w

and

$$M \quad = \quad M_r(1 + \frac{u_r}{r})\{(1 + u_r')^2 + (w')^2\} \tag{3.34}$$

or

$$\beta \quad = \quad \tan^{-1} \frac{w'}{1 + u_r'} \quad . \tag{3.35}$$

Primes in the above equations represent ordinary differentiation with respect to r.

For the case of the circular plate it happens that $\frac{dQ}{dr} + \frac{Q}{r} = 0$ is the equilibrium equation in the z direction where

$$Q = Q_1 + Q_2 \tag{3.36}$$

where

$$Q_1 = \frac{\int_0^r 2\pi r f(r)\,dr}{2\pi r} \tag{3.37}$$

and

$$Q_2 = \frac{\int_0^{\bar{r}} 2\pi \bar{r} p(\bar{r})\,d\bar{r}}{2\pi r} \quad . \tag{3.38}$$

In equation (3.37) $f = f(r)$ and in equation (3.38) $p = p\ (\bar{r})$ where $\bar{r} = r + u_r$. Thus we see that the equation (3.33) for Q may be used instead of the equilibrium equation (3.31) in the z direction.

If we now multiply equation (3.30) by $(1 + u_r')$ and equation (3.31) by w' and add we obtain the following equation which does not contain a $\bar{M}_r''$ term.

$$\tilde{n}_r\{u_r''(1 + u_r') + \frac{1}{r}[(1 + u_r')^2 + (w')^2] + w''w'\}$$

$$+ \tilde{n}_\theta\{- \frac{1}{r}(1 + \frac{u_r}{r})(1 + u_r') + \tilde{n}_r'\{(1 + u_r')^2 + (w')^2\}$$

$$+ \bar{M}_r\{- 2(1 + \frac{u_r}{r})(1 + u_r')w''' - \frac{2}{r}(1 + u_r')^2 w''$$

$$+ \frac{2}{r}(1 + u_r')u_r''w' + w(1 + \frac{u_r}{r})u_r'''w'\} + \bar{M}_r'\{-3w''(1 + \frac{u_r}{r})(1 + u_r')$$

$$+ 3u_r''(1 + \frac{u_r}{r})w'\} + \bar{M}_\theta'\{- \frac{1}{r}(1 + \frac{u_r}{r})^2 w'\} + f_r(1 + u_r')$$

$$+ fw' + M_\theta\{\frac{2}{r}(1 + \frac{u_r}{r})^2 w'\} = 0 \tag{3.39}$$

We can now use equations (3.39) and (3.33) as the equilibrium equations. These equations are easier to work with in the general case since they are of lower order than equations (3.30) and (3.31). They may be further simplified by making the following substitutions:

$$\bar{r} = r + u_r \tag{3.40a}$$

$$y = \tan\beta = \frac{w'}{1 + u_r'} \tag{3.40b}$$

If we substitute (3.40a,b) into (3.39) and (3.33) we obtain the following equilibrium equations.

<u>Equilibrium Equations</u>

$$\tilde{n}_r\{\bar{r}''\bar{r}' + \bar{r}'y(\bar{r}''y + \bar{r}'y') + \frac{(\bar{r}')^2}{r}(1 + y^2)\}$$

$$+ \tilde{n}_\theta - \frac{\bar{r}\bar{r}'}{r^2}\} + n_r'\{(\bar{r}')^2(1 + y^2)\} + M_\theta\{\frac{2}{r^4}\bar{r}^2 y\bar{r}'\}$$

$$+ \bar{M}_r\{- \frac{2\bar{r}'\bar{r}}{r^2}(2\bar{r}''y' + \bar{r}'y'')\} + \bar{M}_r'\{- \frac{3\bar{r}(\bar{r}')^2 y'}{r}\}$$

$$+ \bar{M}_\theta'\{- \frac{y\bar{r}'\bar{r}^2}{r^3}\} + f_r\bar{r}' + fy\bar{r}' = 0 \tag{3.41}$$

and

$$Q = \tilde{n}_r y\bar{r}' + \bar{M}_r\{\frac{2\bar{r}\bar{r}''}{r} + \frac{(\bar{r}')^2}{r}\} + \bar{M}_\theta\{-\frac{\bar{r}^2}{r^3}\} + M'_r\{\frac{\bar{r}\bar{r}'}{r}\} \quad . \tag{3.42}$$

It should be noted that by making these substitutions (3.40 a,b) the $\bar{r}'''$ or u_r''' term has been eliminated from equation (3.39). This is important since it reduces the order of the equations.

Substituting equations (3.40 a,b) into equations (3.32) and (3.34) we obtain the boundary conditions in the new variables.

Boundary Conditions: prescribe

$$T_r = \tilde{n}_r\bar{r}' + \bar{M}_r\{-\frac{2\bar{r}}{r}(y\bar{r}'' + y'\bar{r}') - \frac{y(\bar{r}')^2}{r}\} + M'_r\{-\frac{\bar{r}}{r}yr'\} \tag{3.43}$$

or $\bar{r}$

and

$$M = M_r\frac{\bar{r}}{r}(\bar{r}')^2(1 + y^2) \tag{3.44}$$

$$\text{or } y = \tan\beta \tag{3.45}$$

IV CONSTITUTIVE EQUATIONS

The equilibrium equations and boundary conditions developed in chapter three are exact. However, the theory is not complete since in addition to the equilibrium equations and boundary conditions we need a relationship between stress and moment resultants and the strains.

Since $E_{\alpha\beta}$ and $\tilde{K}_{\alpha\beta}$ are suitable measures of the deformed shell, we are able to postulate an elastic strain energy per unit area, I, that depends only on these tensors. For an elastic material it follows from the First Law of Thermodynamics that

$$\tilde{n}^{\alpha\beta} = \frac{1}{2}\left[\frac{\partial I}{\partial E_{\alpha\beta}} + \frac{\partial I}{\partial E_{\beta\alpha}}\right] \tag{4.1}$$

and

$$\bar{M}^{\alpha\beta} = \frac{1}{2}\left[\frac{\partial I}{\partial \tilde{K}_{\alpha\beta}} + \frac{\partial I}{\partial \tilde{K}_{\beta\alpha}}\right] . \tag{4.2}$$

Since it is not the purpose of this paper to study constitutive relations but rather the structure of the equilibrium equations a simple linear relationship will be used throughout. It is therefore postulated that the material has a strain energy density such that

$$\tilde{n}^{\alpha\beta} = \frac{Eh}{(1-\nu^2)}\left[(1-\nu)E^{\alpha\beta} + \nu E^{\nu}_{\nu} g^{\alpha\beta}\right] . \tag{4.3}$$

and

$$\bar{M}^{\alpha\beta} = \frac{Eh^3}{12(1-\nu^2)} \left[(1-\nu)\tilde{K}^{\alpha\beta} + \nu \tilde{K}^{\nu}_{\nu} g^{\alpha\beta} \right] . \tag{4.4}$$

For the circular plate these constitutive equations in physical variables are

$$\tilde{n}_r = \frac{Eh}{(1-\nu^2)} \left\{ \frac{(w')^2}{2} + u_r' + \frac{(u_r')^2}{2} + \frac{\nu u_r}{r} + \frac{\nu u_r^2}{2r^2} \right\} , \tag{4.5}$$

$$\tilde{n}_\theta = \frac{Eh}{(1-\nu^2)} \left\{ \frac{\nu (w')^2}{2} + \nu u_r' + \frac{\nu (u_r')^2}{2} + \frac{u_r}{r} + \frac{u_r^2}{2r^2} \right\} , \tag{4.6}$$

$$\bar{M}_r = -\frac{Eh^3}{12(1-\nu^2)} \left\{ w''(1+u_r')(1+\frac{u_r}{r}) - u_r''(1+\frac{u_r}{r})w' \right.$$

$$\left. + \nu \left[\frac{w'}{r}(1+\frac{u_r}{r})^2 \right] \right\} , \tag{4.7}$$

and

$$\bar{M}_\theta = -\frac{Eh^3}{12(1-\nu^2)} \left\{ \nu \left[w''(1+u_r')(1+\frac{u_r}{r}) - u_r''(1+\frac{u_r}{r})w' \right] \right.$$

$$\left. + \frac{w'}{r}(1+\frac{u_r}{r})^2 \right\} . \tag{4.8}$$

If we substitute equations (3.40a,b) into equations (4.5) - (4.8), we find that the order of equations (4.7) and (4.8) is reduced since the

$\bar{r}''$ or u_r'' term is eliminated. The constitutive equations in the new variables are

$$\tilde{n}_r = \frac{Eh}{2(1-\nu^2)} \{(\bar{r}')^2(1 + y^2) - 1 + \nu \left[- 1 + \frac{\bar{r}^2}{r^2} \right]\} \quad , \tag{4.9}$$

$$\tilde{n}_\theta = \frac{Eh}{2(1-\nu^2)} \{\nu \left[(\bar{r}')^2(1 + y^2) - 1 \right] - 1 + \frac{\bar{r}^2}{r^2}\} \quad , \tag{4.10}$$

$$\bar{M}_r = \frac{- Eh^3}{12(1-\nu^2)r} \{(\bar{r}')^2 y'\bar{r} + \nu \frac{\bar{r}^2}{r^2} \bar{r}'y\} \quad , \tag{4.11}$$

and

$$\bar{M}_\theta = \frac{- Eh^3}{12(1-\nu^2)r} \{\nu(\bar{r}')^2 y'\bar{r} + \frac{\bar{r}^2}{r^2} \bar{r}'y\} \quad . \tag{4.12}$$

It should be remembered that $\tilde{n}_r$ and $\tilde{n}_\theta$ are not true stress resultants. They are related to the true stress resultants by equation (3.5). Using the following relations obtained from equation (3.7)

$$\bar{b}_{11} = \sqrt{\frac{g}{\bar{g}}} (1 + \frac{u_r}{r})\{- w''(1 + u_r') + w'u_r''\} \quad , \tag{4.13}$$

and

$$\bar{b}_{22} = \sqrt{\frac{g}{\bar{g}}} (1 + \frac{u_r}{r})\{-w'(r + u_r)\} \quad , \tag{4.14}$$

and those obtained from (2.4)

$$\bar{g}_{11} = (1 + u_r')^2 + (w')^2 \quad , \tag{4.15}$$

and

$$\bar{g}_{22} = r^2 (1 + \frac{u_r}{r})^2 \quad , \tag{4.16}$$

we can obtain expressions for N_r and N_θ using equation (3.5)

$$N_r = \frac{\tilde{n}_r}{\{(1+u_r')^2 + (w')^2\}^{1/2} (1 + \frac{u_r}{r})} - \frac{1}{\{(1+u_r')^2 + (w')^2\}^{3/2}} \Big[\{2w''(1+u_r') - 2w'u_r''\} M_r + \frac{w'}{r}(1 + \frac{u_r}{r}) M_\theta \Big] \tag{4.17}$$

and

$$N_\theta = \frac{\tilde{n}_\theta}{\{(1+u_r')^2 + (w')^2\}^{1/2} (1 + \frac{u_r}{r})} - \frac{1}{\{(1+u_r')^2 + (w')^2\}^{1/2} (1 + \frac{u_r}{r})^2} \Big[\frac{2w'}{r}(1 + \frac{u_r}{r}) M_\theta + \{(1+u_r')w'' - w'u_r''\} M_r \Big] \tag{4.18}$$

As shown by Koiter (1960) the moment terms in equations (4.17) and (4.18) are terms of the same order as those that are neglected in a first approximation theory. Koiter shows that if

$$(h/L)^2 << 1 \quad , \tag{4.19}$$

then the moment terms in equations (4.17) and (4.18) are small compared to the other terms. The L in equation (4.19) is the smallest "wave length" of deformation of the deformed reference surface. It is defined by Koiter (1960) by

$$\left|\frac{d\varepsilon_1}{ds}\right| \quad , \quad \left|\frac{d\varepsilon_2}{ds}\right| \quad = \quad O(\frac{\varepsilon}{L})$$

(4.20 a-b)

$$\left|\frac{dK_1}{ds}\right| \quad , \quad \left|\frac{dK_2}{ds}\right| \quad = \quad O(\frac{K}{L})$$

where ε_1, ε_2 are the principal strains, K_1, K_2 are the principal curvatures, ε and K are the maximum principal strain and curvature respectively and ds is any element on the middle surface. Thus we see that if the moment terms in equations (4.17) and (4.18) become important then shear deformations would have to be taken into account. Koiter (1960) shows that a refinement of Love's approximation is meaningless, unless the effects of transverse shear and normal stresses are taken into account at the same time. This is discussed further in chapter six.

V ANALYSIS OF CIRCULAR MEMBRANES

In this chapter we will analyze two circular membranes. One of them is loaded only with a transverse load. The other is loaded with a transverse load plus a small torque. To solve these problems we will use equations (3.12) and (3.13) and retain only the membrane terms. These equations written in tensor components with $p=f_1=f_2=f=0$ are

$$\frac{\tilde{n}^{11}}{r}(1 + u_{1,1} + ru_{1,11}) - \tilde{n}^{22}(r + u_1) + \tilde{n}^{11}_{,1}(1 + u_{1,1}) + \tilde{n}^{12}\left(-\frac{2u_{2,1}}{r} + \frac{u_2}{r^2}\right) - \tilde{n}^{12}_{,1}\frac{u_2}{r} = 0 \quad , \tag{5.1}$$

$$\frac{\tilde{n}^{11}}{r}\left(ru_{2,11} - u_{2,1} + \frac{u_2}{r}\right) + \tilde{n}^{11}_{,1}\left(u_{2,1} - \frac{u_2}{r}\right) - \tilde{n}^{22}u_2 + \tilde{n}^{12}(3r + u_1 + 2ru_{1,1}) + \tilde{n}^{12}_{,1}(r^2 + ru_1) = 0 \quad , \tag{5.2}$$

and

$$\tilde{n}^{11}\left(w_{,11} + \frac{w_{,1}}{r}\right) + \tilde{n}^{11}_{,1}w_{,1} = 0 \quad . \tag{5.3}$$

From equation (4.3) we find that the constitutive equations in tensor components are

$$\tilde{n}^{11} = \frac{Eh}{2(1 - \nu^2)} \{2u_{1,1} + (u_{1,1})^2 + \frac{(u_{2,1})^2}{r^2} - \frac{2u_2}{r^3} u_{2,1}$$

$$+ \frac{(u_2)^2}{r^4} + (w_{,1})^2 + \nu \left[\frac{2u_1}{r} + \frac{(u_2)^2}{r^4} + \frac{(u_1)^2}{r^2} \right] \} \quad , \tag{5.4}$$

$$\tilde{n}^{22} = \frac{Eh}{2(1 - \nu^2)} \frac{1}{r^2} \{\frac{2u_1}{r} + \frac{(u_2)^2}{r^4} + \frac{(u_1)^2}{r^2}$$

$$+ \nu \left[2u_{1,1} + (u_{1,1})^2 + \frac{(u_{2,1})^2}{r^2} - \frac{2u_2 u_{2,1}}{r^3} + \frac{(u_2)^2}{r^4} + (w_{,1})^2 \right] \} \ , \tag{5.5}$$

and

$$\tilde{n}^{12} = \frac{Eh}{2(1 - \nu^2)} \frac{(1 - \nu)}{r} \{ - \frac{2u_2}{r^2} + \frac{u_{2,1}}{r} - \frac{u_{1,1}}{r^2} u_2$$

$$+ \frac{u_{2,1}}{r^2} u_1 - \frac{u_2 u_1}{r^3} \} \quad . \tag{5.6}$$

Note that in equations (5.1) - (5.6) the $\tilde{n}^{12}$ and u_2 terms have not been neglected as they were in chapter three. The reason for this is that in the torsion problem they will not be zero even though it is still an axisymmetric problem.

A. Membrane Loaded with Transverse Load

The problem we are interested in here is that of a circular membrane with a central rigid inclusion of finite radius. The membrane is deformed due to a load, P, which acts normal to the plane of the rigid inclusion at its center. The outer edge of the plate is simply supported such that it undergoes no displacement but is free to rotate. The inner edge is free to displace and rotate. The boundary conditions for the problem are

$$w = 0 \quad \text{at} \quad r = a \quad , \tag{5.7}$$

and

$$u_1 = 0 \quad \text{at} \quad r = a \quad \text{and} \quad r = r_i \quad , \tag{5.8}$$

where a is the outside radius of the plate and r_i is the radius of the rigid inclusion.

For this problem $u_2 = \tilde{n}^{12} = 0$ and equation (5.2) is satisfied identically. From chapter three we know that a first integral to equation (5.3) is

$$\tilde{n}^{11} w_{,1} = Q = \frac{P}{2\pi r} \tag{5.9}$$

If the constitutive equations (5.4) and (5.5) are substituted into equations (5.1) and (5.9) and the resulting equations are divided by $\frac{Eh}{(1 - \nu^2)}$ we obtain the following

$$\left\{\frac{(w_{,1})^2}{2} + u_{1,1} + \frac{(u_{1,1})^2}{2} + \frac{\nu u_1}{r} + \frac{\nu (u_1)^2}{2r^2}\right\}\left\{u_{1,11} + \frac{1}{r}(1 + u_{1,1})\right\}$$

$$+\left\{\frac{\nu (w_{,1})^2}{2} + u_{1,1} + \frac{\nu (u_{1,1})^2}{2} + \frac{u_1}{r} + \frac{(u_1)^2}{2r^2}\right\}\left\{- \frac{1}{r}(1 + \frac{u_1}{r})\right\}$$

$$+ \{\frac{(w_{,1})^2}{2} + u_{1\ 1} + \frac{(u_{1,1})^2}{2} + \frac{\nu u_1}{r} + \frac{\nu (u_1)^2}{2r^2}\}_{,1} (1 + u_{1,1}) = 0 \ , \quad (5.10)$$

and

$$\{\frac{(w_{,1})^2}{2} + u_{1,1} + \frac{(u_{1,1})^2}{2} + \frac{\nu u_1}{r} + \frac{\nu (u_1)^2}{2r^2}\} w_{,1}$$

$$= \frac{P}{2\pi} \frac{(1 - \nu^2)}{Eh} \frac{1}{r} \quad . \qquad (5.11)$$

A solution to equations (5.10) and (5.11) for $\nu = 1/3$ is

$$u_1 = 0 \qquad (5.12)$$

and

$$w = \frac{3}{2} \left[\frac{P(1 - \nu^2)}{Eh\pi} \right]^{1/3} (r^{2/3} + C) \quad . \qquad (5.13)$$

This solution was found by Tolefson (1967) for the von Karman equations and by Armstrong (1969) who found that equations (5.1-6) reduce to the von Karman equations when $u_2 = u_1 = 0$.

Equation (5.12) satisfies boundary conditions (5.8) identically. Equation (5.13) satisfies boundary condition (5.7) if $C = - a^{2/3}$. Thus the solution to our problem for $\nu = 1/3$ is

$$u_1 = 0 \quad , \qquad (5.14)$$

and

$$w = \frac{3}{2}\left[\frac{P(1-\nu^2)}{Eh\pi}\right]^{1/3}(r^{2/3} - a^{2/3}) \quad . \tag{5.15}$$

Solutions for other values of Poisson's ratio could be found using a numerical technique. This has been done by Armstrong (1969) and his results show that equations (5.10) and (5.11) are not very sensitive to slight variations in ν away from 1/3.

In chapter six we will obtain a solution to the membrane equations for any value of ν for the case where the deflections are large compared to the outside radius.

Caldwell (1968) has numerically integrated the Reissner (1949) equations for this same problem and has obtained some very interesting results for large displacements (see Figure 2). The most reasonable explanation of the discrepency between the two curves is that Reissner's assumption of small strains is violated when dw/dr becomes large.

B. Membrane Loaded With Transverse Load Plus a Small Torque

Once the exact solution to the problem of the membrane loaded with a transverse load is obtained, solutions to other problems which are perturbations of the original problem may be solved. The one of interest in this section is that of a circular membrane with a central rigid inclusion of finite radius. The membrane is loaded with a transverse load, P, which acts normal to the plane of the rigid inclusion at its center, and a small torque which acts about this normal.

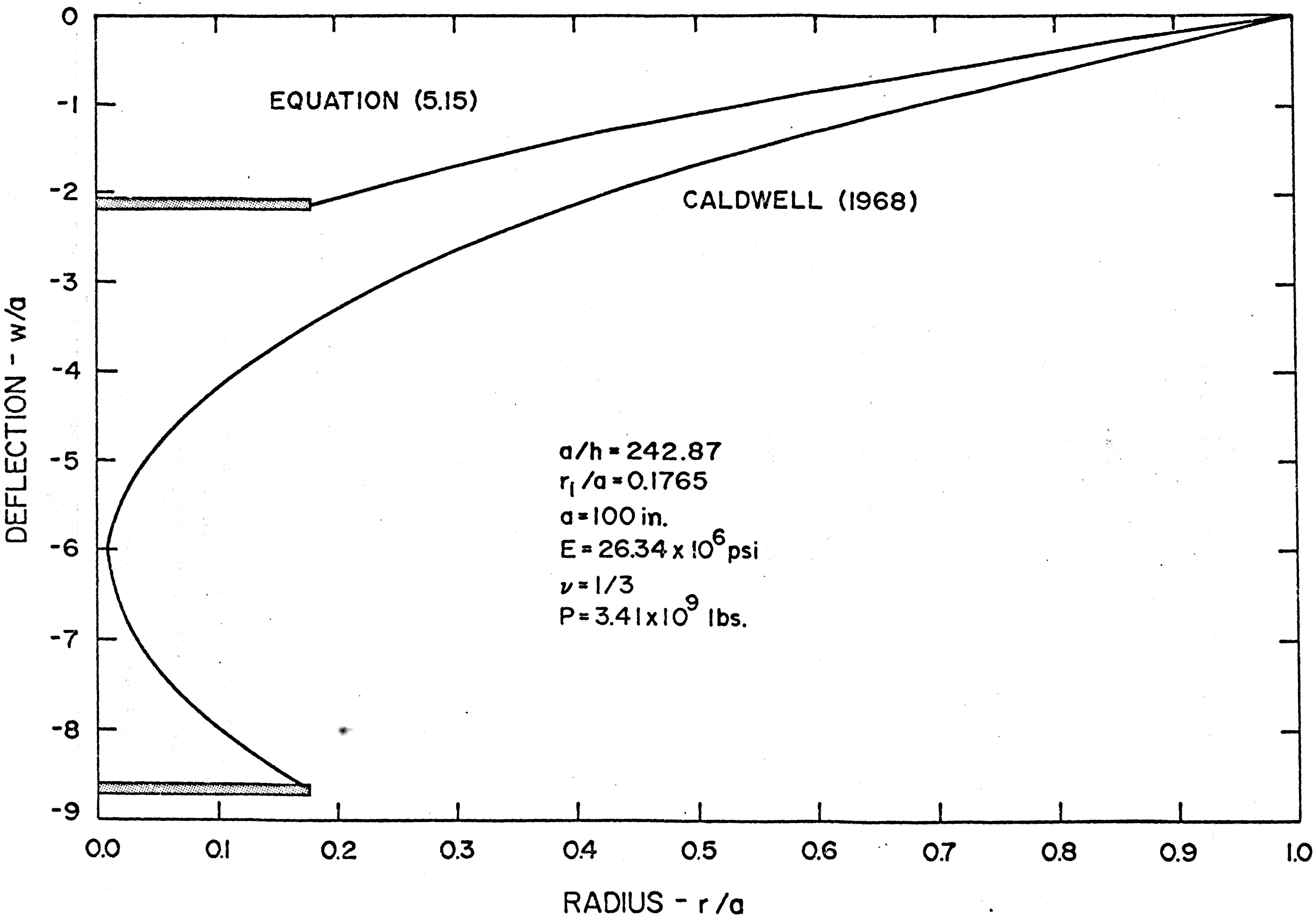


Figure 2. Deflection profile for large deflections

The outer edge of the plate is simply supported such that it undergoes no displacement but its slope is free to change. The inner edge is free to displace in both the z, and θ directions and its slope may also change. Thus the boundary conditions are

$$w = 0 \quad \text{at} \quad r = a \;, \tag{5.16}$$

$$u_2 = 0 \quad \text{at} \quad r = a \;, \tag{5.17}$$

$$u_1 = 0 \quad \text{at} \quad r = a \quad \text{and} \quad r = r_i \;, \tag{5.18}$$

where a is the outside radius of the plate and r_i is the radius of the rigid inclusion. Also either the applied torque or the u_2 displacement at some point other than r = a must be specified.

To solve this problem we will use $\nu = 1/3$ and assume a solution

$$w = \frac{3}{2} C_1 r^{2/3} + C_2 + C_1 w_0(r) \tag{5.19}$$

$$u_1 = u_1(r) \tag{5.20}$$

$$u_2 = u_2(r) \tag{5.21}$$

where $$C_1 = \left[\frac{P(1 - \nu^2)}{Eh\pi}\right]^{1/3}$$ and

w_0, u_1 and u_2 are small compared to unity. If the constitutive equations (5.4-6) are substituted into the equilibrium equations (5.1-3) and equations (5.19-21) are substituted into the resulting equations, we obtain after dividing by $\dfrac{Eh}{2(1 - \nu^2)}$ the following equations after higher order terms have been neglected

$$2C_1^2\, r^{-1/3} w_{0,11} + \frac{2}{3} C_1^2\, r^{-4/3} w_{0,1} + (2 + C_1^2\, r^{-2/3}) u_{1,11}$$

$$\left(\frac{2}{r} + \frac{C_1^2}{3}\, r^{-5/3}\right) u_{1,1} - \left(\frac{2}{r^2} + \frac{C_1^2}{3}\, r^{-8/3}\right) u_1 = 0 \quad , \tag{5.22}$$

$$C_1^2\, r^{-2/3}\left(u_{2,11} - \frac{5}{3}\,\frac{u_{2,1}}{r} + \frac{4}{3}\,\frac{u_2}{r^2}\right) + \frac{2}{3}\left(u_{2,11} - \frac{u_{2,1}}{r}\right) = 0 \quad , \tag{5.23}$$

and

$$-\frac{2}{9}\, u_1 r^{-7/3} + 2u_{1,1} r^{-4/3} + 2u_{1,11} r^{-1/3} + 3w_{0,11}\, C_1^2 r^{-2/3} = 0 \quad . \tag{5.24}$$

If $u_1 = w_0 = 0$, equations (5.22) and (5.24) are identically satisfied and we are left with one second-order linear equation, (5.23), to solve for u_2. An infinitessimal rigid body rotation would be represented by

$$u_2 = Ar^2 \tag{5.25}$$

where A is an arbitrary constant. (Note $u_2 = ru_\theta$, Frederick (1956)). Equation (5.25) satisfies (5.23) since it is a rigid body rotation.

If one homogeneous solution to a general second-order linear differential equation is known the second can always be found, at least in terms of integrals. Using (5.25) as the known solution to (5.23) we are able to obtain the second homogeneous solution and thus the complete solution which is

$$u_2 = A\,\{-\frac{r^{2/3}}{12C_1^2} + \frac{r^{4/3}}{9C_1^4} + \frac{2r^2}{27C_1^6}\,\ell n(\frac{r^{2/3}}{3C_1^2 + 2r^{2/3}})\} + Br^2 \quad . \tag{5.26}$$

To satisfy boundary condition (5.17) B must be given by

$$B = -A\{\frac{-1}{12C_1^2a^{4/3}} + \frac{1}{9C_1^4a^{2/3}} + \frac{2}{27C_1^6}\,\ell n(\frac{a^{2/3}}{3C_1^2 + 2a^{2/3}})\} \quad . \tag{5.27}$$

The resisting torque is given by

$$T = \int_C \frac{rn^{1\beta}\bar{x}^i|_\beta \bar{x}^i|_2 n_1}{||\bar{x}^i|_2||}\, rd\theta \tag{5.28}$$

where $\bar{x}^i|_\beta$ is given by equation (2.12). Thus

$$\bar{x}^1|_1 = \cos\theta + (\frac{u_2}{r} - u_{2,1})\sin\theta \quad ,$$

$$\bar{x}^2|_1 = \sin\theta + (u_{2,1} - \frac{u_2}{r})\cos\theta \quad ,$$

$$\bar{x}^3|_1 = -w_{,1} \quad , \tag{5.29 a-f}$$

$$\bar{x}^1|_2 = -r\sin\theta - \frac{u_2}{r}\cos\theta \quad ,$$

$$\bar{x}^2|_2 = r\cos\theta - \frac{u_2}{r}\sin\theta \quad ,$$

$$\bar{x}^3\big|_2 = 0 \quad ,$$

and

$$\|\bar{x}^i\big|_2\| = \left[r^2 + \frac{u_2^2}{r^2}\right]^{1/2} \approx r \tag{5.30}$$

if $u_2 << r^2$.

Substituting equations (5.29) and (5.30) into (5.28), integrating and neglecting u_2 compared to r^2 we find

$$T = 2\pi r\{n^{11}(u_{2,1} - \frac{2u_2}{r}) + n^{12}r^2\} \quad . \tag{5.31}$$

Now substitute (5.4), (5.6), (5.19), and (5.26) into (5.31) with $w_0 = u_1 = 0$. We find that

$$T = \frac{\pi EhA}{18(1 - \nu^2)} \tag{5.32}$$

where the A in equations (5.26), (5.27) and (5.32) is determined by specifying the applied torque or the u_2 displacement at some value of r.

Since $u_\theta = u_2/r$ and $\theta = u_\theta/r$ we can find an expression for θ by dividing equation (5.26) by r^2

$$\theta = A\{\frac{-1}{12C_1^2r^{4/3}} + \frac{1}{9C_1^4r^{2/3}} + \frac{2}{27C_1^6}\ \ell n(\frac{r^{2/3}}{3C_1^2 + 2r^{2/3}})\} + B \tag{5.33}$$

The stiffness K is found by dividing the applied torque by θ. It must be remembered that the applied torque is opposite in sign to the resisting torque T. Thus

$$K = -\frac{T}{\theta} \tag{5.34}$$

where T is given by (5.32) and θ by (5.33).

If $C_1 = 0$ the solution to equation (5.23) is

$$u_2 = -A + Br^2 \quad , \tag{5.35}$$

and then after applying boundary condition (5.17) we find that

$$\theta = -A\left(\frac{1}{r^2} - \frac{1}{a^2}\right) \quad , \tag{5.36}$$

$$T = \frac{2}{3}\frac{\pi Eh}{(1 - \nu^2)}A \quad , \tag{5.37}$$

and

$$K = \frac{2}{3}\frac{\pi Eh}{(1 - \nu^2)}\frac{a^2r^2}{(a^2 - r^2)} \quad . \tag{5.38}$$

If we assume $C_1 >> 1$ and neglect the second term in equation (5.23) we obtain a solution which may be written as

$$u_2 = -\frac{Ar^{2/3}}{12C_1^2} + Br^2 \tag{5.39}$$

and then after applying boundary condition (5.17) we find that

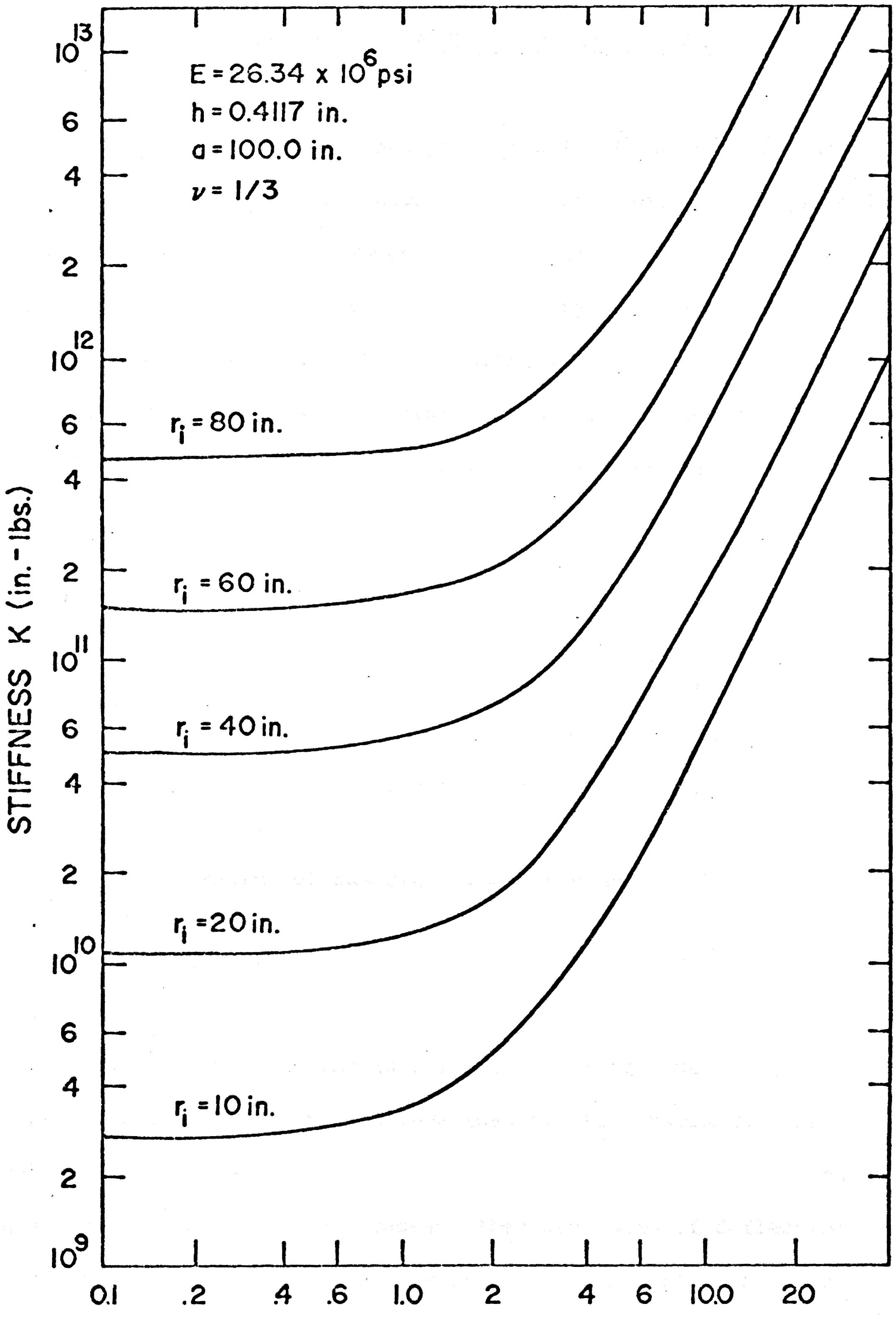
$$\theta = -\frac{A}{12C_1^2}\left(\frac{1}{r^{4/3}} - \frac{1}{a^{4/3}}\right) \tag{5.40}$$

$$T = \frac{\pi EhA}{18(1 - \nu^2)} \quad , \tag{5.41}$$

and

$$K = \frac{2\pi EhA}{3(1 - \nu^2)C_1^2} \frac{(ra)^{4/3}}{(a^{4/3} - r^{4/3})} \tag{5.42}$$

See Figure 3 for a plot of the stiffness for different values of r_i and C_1.



$C_1 = [P(1 - \nu^2)/(Eh\pi)]^{1/3}$

Figure 3. Stiffness vs C_1 for torsion problem

VI ANALYSIS OF A CIRCULAR PLATE PROBLEM

In this chapter we will analyze the deflections of a circular plate with a central rigid inclusion of finite radius. The plate is deformed due to a load, P, which acts normal to the plane of the rigid inclusion at its center. The outer edge of the plate is considered to be clamped such that it undergoes no rotation or displacement. The inner edge is also clamped such that it undergoes no rotation but is free to displace downward. The boundary conditions for this plate are

$$w = 0 \qquad \text{at } r = a \quad , \tag{6.1}$$

$$y = \frac{w'}{1 + u_r'} = 0 \qquad \text{at } r = a \quad \text{and } r = r_i \quad , \tag{6.2 a,b}$$

$$u_r = 0 \qquad \text{at } r = a \quad \text{and } r = r_i \quad , \tag{6.3 a,b}$$

where r_i is the radius of the rigid inclusion and

$$Q = Q_1 = \frac{P}{2\pi r} \quad . \tag{6.4}$$

Starting with equations (3.30) and (3.33) that we developed in chapter three an order of magnitude analysis is performed. This analysis will be used to determine the first approximations to these equations for three separate cases. They are ratio of deflection to outside radius small compared to unity, of order unity and large compared to unity. These approximate equations are further simplified by using the method of matched asymptotic expansions.

A. Order of Magnitude Analysis

For the order of magnitude analysis equilibrium equations (3.30) and (3.33) are used since their membrane terms are less complicated. In order to perform the order of magnitude analysis the equations must first be non-dimensionalized. This is accomplished by setting,

$$\tilde{n}_r = Nn_r \quad , \quad \tilde{n}_\theta = Nn_\theta \quad , \qquad (6.5\ a,b)$$

$$\bar{M}_r = MM_r \quad , \quad \tilde{M}_\theta = MM_\theta \quad , \qquad (6.6\ a,b)$$

$$r = aR \quad , \quad u_r = bU \quad , \quad w = cW \qquad (6.7\ a,b,c)$$

where n_r, n_θ, M_r, M_θ, R, U, and W are new dimensionless variables. N, M, b, and c are parametric quantities to be determined in the course of the analysis and a is the outer radius of the plate. Substituting (6.5 a) - (6.7 c) into equations (3.30), (3.33), and (4.5) - (4.8) we find

Equilibrium Equations (non-dimensional)

in r direction

$$\frac{N}{a}\{\frac{n_r}{R} - \frac{n_\theta}{R} + n_r'\} + \frac{N}{a}(\frac{b}{a})\{n_r(U'' + \frac{U'}{R}) - \frac{n_\theta U}{R^2} + n_r'U'\}$$

$$+ \frac{M}{a^2}(\frac{c}{a})\{M_r(-2W''' - \frac{2W''}{R}) + \frac{2M_\theta W'}{R} + M_r'(-3W'' - \frac{2W'}{R})$$

$$- M_r''W'\} + \frac{M}{a^2}(\frac{c}{a})(\frac{b}{a})\{M_r(-\frac{2W'''U}{R} - \frac{2W''U'}{R} - \frac{2W'U''}{R})$$

$$+ \frac{2M_\theta W'U}{R^3} + M_r'(-\frac{3W''U}{R} - \frac{2W'U}{R}) + M_r''(-\frac{W'U}{R})\} = 0 \quad , \qquad (6.8)$$

and

<u>in z direction</u>

$$N(\frac{c}{a})\ n_r W' + \frac{M}{a}\{\frac{M_r}{R} - \frac{M_\theta}{R} + M_r'\} + \frac{M}{a}(\frac{b}{a})\{2M_r U'' + \frac{2M_r U'}{R} - \frac{2M_\theta U}{R^2}$$

$$+ M_r'(U' + \frac{U}{R})\} + \frac{M}{a}(\frac{b}{a})^2\{M_r(\frac{2U''U}{R} + \frac{U'U'}{R}) - \frac{M_\theta U^2}{R^3}$$

$$+ \frac{M_r'U'U}{R}\} = \frac{P}{2\pi aR} \qquad (6.9)$$

<u>Constitutive Equations</u>

$$M_r = \frac{Eh^3c}{12(1-\nu^2)Ma^2}\{-W'' - \frac{\nu W'}{R} + (\frac{b}{a})(-W''U' - \frac{W''U}{R}$$

$$+ W'U'' - \frac{2\nu W'U}{R^2}) + (\frac{b}{a})^2 (- \frac{W''U'U}{R} + \frac{W'U''U}{R} - \frac{\nu W'U^2}{R^3}) \} \quad , \tag{6.10}$$

$$M_\theta = \frac{Eh^3c}{12(1 - \nu^2)Ma^2} \{ - \nu W'' - \frac{W'}{R} + (\frac{b}{a}) (- \nu W''U' - \frac{\nu W''U}{R}$$

$$+ \nu W'U'' - \frac{2W'U}{R^2}) + (\frac{b}{a})^2 (- \frac{\nu W''U'U}{R} + \frac{\nu W'U''U}{R} - \frac{W'U^2}{R^3}) \} \quad , \tag{6.11}$$

$$n_r = \frac{Eh}{(1 - \nu^2)N} \{ (\frac{b}{a}) (U' + \frac{\nu U}{R}) + (\frac{b}{a})^2 (\frac{U'U'}{2} + \frac{\nu UU}{2R^2})$$

$$+ (\frac{c}{a})^2 \frac{W'W'}{2} \} \quad , \tag{6.12}$$

and

$$n_\theta = \frac{Eh}{(1 - \nu^2)N} \{ (\frac{b}{a}) (\nu U' + \frac{U}{R}) + (\frac{b}{a})^2 (\frac{\nu U'U'}{2} + \frac{UU}{2R^2})$$

$$+ (\frac{c}{a})^2 \frac{\nu W'W'}{2} \} \quad . \tag{6.13}$$

1. Ratio of deflections to outside radius small compared to unity

Assuming that c and b are picked so that W and U are of order unity and that the ratio of the deflection to the outside radius is small compared to unity implies that $\frac{c}{a} \ll 1$ and $\frac{b}{a} \ll 1$. Under these conditions the first order terms of the equilibrium equations become

$$\frac{N}{a}\left\{\frac{n_r}{R} - \frac{n_\theta}{R} + n_r'\right\} + \frac{M}{a^2}\left(\frac{c}{a}\right)\left\{M_r\left(- 2W''' - \frac{2W''}{R}\right)\right.$$

$$\left. + \frac{2M_\theta W'}{R^2} + M_r'\left(- 3W'' - \frac{2W'}{R}\right) - M_r''W'\right\} = 0 \qquad (6.14)$$

and

$$N\left(\frac{c}{a}\right)n_r W' + \frac{M}{a}\left\{\frac{M_r}{R} - \frac{M_\theta}{R} + M_r'\right\} = \frac{P}{2\pi aR} \quad . \qquad (6.15)$$

The first order terms of the constitutive equations become

$$n_r = \frac{Eh}{(1 - \nu^2)N}\left\{\left(\frac{b}{a}\right)\left(U' + \frac{\nu U}{R}\right) + \left(\frac{c}{a}\right)^2\frac{(W')^2}{2}\right\} \quad , \qquad (6.16)$$

$$n_\theta = \frac{Eh}{(1 - \nu^2)N}\left\{\left(\frac{b}{a}\right)\left(\nu U' + \frac{U}{R}\right) + \left(\frac{c}{a}\right)^2\frac{\nu(W')^2}{2}\right\} \quad , \qquad (6.17)$$

$$M_r = \frac{Eh^3}{12(1 - \nu^2)Ma}\left(\frac{c}{a}\right)\left\{ - W'' - \frac{\nu W'}{R}\right\} \quad , \qquad (6.18)$$

and

$$M_\theta = \frac{Eh^3}{12(1 - \nu^2)Ma}\left(\frac{c}{a}\right)\left\{ - \nu W'' - \frac{W'}{R}\right\} \quad . \qquad (6.19)$$

An analysis of equations (6.14) - (6.19) shows that (b/a) and $(c/a)^2$ must be of the same order of magnitude. If $(b/a) \ll (c/a)^2$ then U does not enter into the equations and we have two equations in one unknown, W, which are not compatible. If $(b/a) \gg (c/a)^2$ then n_r and n_θ do not depend on W and also the resulting equations do not reduce to the classical linear theory. Since (b/a) and $(c/a)^2$ are of the same order of magnitude we will set them equal. Equations (6.16) and (6.17) then become

$$n_r = \frac{Eh}{(1 - \nu^2)N} \left(\frac{c}{a}\right)^2 \left\{ \left[U' + \frac{(W')^2}{2} \right] + \frac{\nu U}{R} \right\} \quad , \tag{6.20}$$

$$n_\theta = \frac{Eh}{(1 - \nu^2)N} \left(\frac{c}{a}\right)^2 \left\{ \frac{U}{R} + \nu \left[U' + \frac{(W')^2}{2} \right] \right. \quad . \tag{6.21}$$

This implies that the strains are

$$e_r = U' + \frac{(W')^2}{2} = \frac{N}{Eh} \left(\frac{a}{c}\right)^2 \{n_r - \nu n_\theta\} \quad , \tag{6.22}$$

and

$$e_\theta = \frac{U}{R} = \frac{N}{Eh} \left(\frac{a}{c}\right)^2 \{n_\theta - \nu n_r\} \quad . \tag{6.23}$$

From equations (6.22) and (6.23) we see that

$$\frac{e_r}{R} - e'_\theta - \frac{e_\theta}{R} = \frac{(W')^2}{2R} = \frac{N}{Eh}\left(\frac{a}{c}\right)^2\left\{\frac{n_r}{R} - \frac{\nu n_\theta}{R} - n'_\theta\right.$$

$$\left. + \nu n'_r - \frac{n_\theta}{R} + \frac{\nu n_r}{R}\right\} \quad . \tag{6.24}$$

From equation (6.24) we see that if n_r, n_θ, R and W are of order unity N must be of order $Eh(c/a)^2$. For convenience let

$$N = Eh\left(\frac{c}{a}\right)^2 \tag{6.25}$$

Now substitute equations (6.18), (6.19) and (6.25) into equation (6.14) and then divide by $\frac{Eh}{a}\left(\frac{c}{a}\right)^2$. The resulting equation is

$$\frac{n_r}{R} - \frac{n_\theta}{R} + n'_R + \frac{\tau^2 f(W)}{(1 - \nu^2)} = 0 \tag{6.26}$$

where

$$\tau^2 = \frac{1}{12}\left(\frac{h}{a}\right)^2 \quad , \tag{6.27}$$

and

$$f(W) = - W'' + \frac{\nu W'}{R}\left(2W''' + \frac{2W''}{R}\right) - 2\left(\nu W'' + \frac{W'}{R}\right)\frac{W'}{R^2}$$

$$+ \left(W'' + \frac{\nu W'}{R}\right)'\left(3W'' + \frac{2W'}{R}\right) + \left(W'' + \frac{\nu W'}{R}\right)''W' \quad . \tag{6.28}$$

Substituting equations (6.18), (6.19) and (6.25) into equation (6.15) and dividing by $\frac{Eh}{12(1-\nu^2)} \left(\frac{h}{a}\right)^2 \left(\frac{c}{a}\right)$ we find

$$\lambda^2 n_r W' - W''' - \frac{W''}{R} + \frac{W'}{R^2} = \frac{P}{2\pi} \frac{12(1-\nu^2)}{Ehc} \left(\frac{a}{h}\right)^2 \frac{1}{R} \tag{6.29}$$

where $\lambda^2 = 12(1-\nu^2)\left(\frac{c}{h}\right)^2$. (6.30)

For a thin plate $\tau^2 << 1$ and equation (6.26) becomes

$$\frac{n_r}{R} - \frac{n_\theta}{R} + n_r' = 0 \quad . \tag{6.31}$$

If we let

$$\psi = Rn_r \quad \text{and} \quad \psi' = n_\theta \tag{6.32 a,b}$$

equation (6.31) is satisfied identically.

It should be noted here that n_r and n_θ are actually the true stres resultants since all other terms in equations (4.17) and (4.18) are negligible for the case where $b/a = (c/a)^2 << 1$.

If we now also let

$$\beta = W' \tag{6.33}$$

and substitute (6.32 a,b) and (6.33) into equations (6.29) and (6.24) we obtain the following

$$\frac{d^2\beta}{dR^2} + \frac{1}{R}\frac{d\beta}{dR} - \frac{1}{R^2}\beta - \frac{\lambda^2\psi\beta}{R} = -\frac{P}{2\pi}\frac{12(1-\nu^2)}{Ehc}\left(\frac{a^2}{h}\right)\frac{1}{R} \tag{6.34}$$

and

$$\frac{d^2\psi}{dR^2} + \frac{1}{R}\frac{d\psi}{dR} - \frac{1}{R^2}\psi + \frac{\beta^2}{2R} = 0 \quad . \tag{6.35}$$

Equations (6.34) and (6.35) are the von Karman (1910) equations in dimensionless variables. Thus we see that the exact equilibrium equations reduce to von Karman's for the case where the assumptions used in obtaining von Karman's equations are applicable.

Equations (6.34) and (6.35) may be further analyzed by considering the ratio of the deflections to the thickness small compared to unity, of order unity, and large compared to unity, but the ratio of deflections to outside radius still small compared to unity. This further analysis will not be done in great detail. For further details see Tolefson (1967).

a. Ratio of deflections to thickness small compared to unity

In this case $c/h << 1$ and therefore $\lambda^2 << 1$. In order for the load P to appear in the first approximation, $\frac{P}{2\pi}\frac{12(1-\nu^2)}{Eh}\left(\frac{a^2}{h}\right)\frac{1}{c}$ must be of order unity. Therefore we let this quantity equal to some constant which we choose to be 4 so that we will be consistant with Tolefson (1967). Thus

$$\frac{P}{2\pi}\,\frac{12(1 - \nu^2)}{Eh}\,\left(\frac{a}{h}\right)\,\frac{1}{c} = 4 \tag{6.36}$$

and

$$c = \frac{P12(1 - \nu^2)}{8\pi Eh}\,\left(\frac{a}{h}\right)^2 \quad . \tag{6.37}$$

Since $c/h \ll 1$, we see from equation (6.37) that for case 1a to apply P must satisfy

$$P \ll \frac{8\pi Eh^2}{12(1 - \nu^2)}\,\left(\frac{h}{a}\right)^2 \tag{6.38}$$

b. Ratio of deflections to thickness of order unity

In this case $c/h = O(1)$ and for convenience we let $c = h$. Then

$$\lambda^2 = 12(1 - \nu^2) \tag{6.39}$$

and P must satisfy

$$P = O\left[\frac{8\pi Eh^2}{12(1 - \nu^2)}\,\left(\frac{h}{a}\right)^2\right] \quad . \tag{6.40}$$

c. Ratio of deflections to thickness large compared to unity

In this case $c/h >> 1$ and therefore $\lambda^2 >> 1$. Equation (6.34) then may be rewritten as

$$\frac{1}{\lambda^2}\left[\beta'' + \frac{1}{R}\beta' - \frac{1}{R^2}\beta\right] - \frac{\psi\beta}{R} = - \frac{P12(1-\nu^2)}{2\pi Eh\lambda^2}\left(\frac{a}{h}\right)^2 \frac{1}{c}\frac{1}{R} \tag{6.41}$$

To guarantee that the load enters into the first approximation set

$$\frac{P12(1-\nu^2)}{2\pi Eh\lambda^2}\left(\frac{a}{h}\right)^2 \frac{1}{c} = 4 \tag{6.42}$$

again to be consistant with Tolefson (1967). From equation (6.42) and (6.30) we see that

$$c = a\left(\frac{P}{8Eha\pi}\right)^{1/3} \quad . \tag{6.43}$$

Since $c/h >> 1$ and $c/a << 1$, we see from equation (6.43) that for case 1c to apply P must satisfy

$$\left(\frac{h}{a}\right)^3 8\pi aEh << P << 8\pi aEh \quad . \tag{6.44}$$

For cases 1 a,b,c we have found what the value of the parametric quantity c must be. Since $b/a = (c/a)^2$ we can determine b for all three cases. From equation (6.25) we can determine N. M can be determined from equation (6.18) and (6.19). Since M_r, M_θ, W and R are all of order unity, M must be of order $\frac{Eh^3}{12(1 - \nu^2)} \frac{c}{a^2}$ and for convenience we will set

$$M = \frac{Eh^3}{12(1 - \nu^2)} \frac{c}{a^2} \quad . \tag{6.45}$$

This then completes the order of magnitude analysis for ratio of deflections to outside radius small compared to unity.

In this section we have shown that the general nonlinear plate equations reduce to the von Karman equations when the ratio of deflections to outside radius is small compared to unity. Since the equations reduce to the von Karman equations for small deflections we know that the general nonlinear equations will reduce to the linear theory for infinitesimal deflections since the von Karman equations do.

2. Ratio of deflections to outside radius of order unity

For this case $c/a = O(1)$. So for convenience we will let $c = a$. We still however must decide the order of magnitude of b/a. An analysis of equations (6.8) to (6.13) shows that $b/a \geqq O(1)$ otherwise U would not enter into the equations and we would have two equations in one unknown, W, which would not be compatible. If $b/a >> 1$, then we can not obtain the membrane solution we got in chapter five. Thus b/a must be of order unity and for convenience we will set it equal to one. If equations (6.10) to (6.13) are substituted into (6.8) and (6.9) with $c = b = a$ and the resulting equations are divided by $\dfrac{Eh}{(1 - \nu^2)a}$ and $\dfrac{Eh}{(1 - \nu^2)}$ respectively we obtain the following

$$A_1\{U'' + \frac{1}{R}(1 + U')\} + A_2\{ - \frac{1}{R}(1 + \frac{U}{R})\} + A_1'(1 + U')$$

$$+ \tau^2 \Bigg[A_3\{ - 2(1 + \frac{U}{R})W''' - \frac{2}{R}(1 + U')W'' - \frac{2}{R}U''W'\}$$

$$+ A_4\{ \frac{2}{R^2}(1 + \frac{U}{R})W'\} + A_3'\{ - 3(1 + \frac{U}{R})W'' - \frac{2}{R}(1 + U')W'\}$$

$$+ A_3''\{ - (1 + \frac{U}{R})W'\} \Bigg] = 0 \quad , \tag{6.46}$$

and

$$A_1 W' + \tau^2 \left[A_3 \{2(1 + \frac{U}{R})U'' + \frac{1}{R} (1 + U')^2\} + A_4 \{ - \frac{1}{R} (1 + \frac{U}{R})^2 \} \right.$$

$$\left. + A_3' \{ (1 + \frac{U}{R}) (1 + U') \} \right] = \frac{P}{2\pi a} \frac{(1 - \nu^2)}{Eh} \frac{1}{R} \quad , \tag{6.47}$$

where

$$A_1 = \frac{(W')^2}{2} + U' + \frac{(U')^2}{2} + \frac{\nu U}{R} + \frac{\nu U^2}{2R^2} \quad , \tag{6.48}$$

$$A_2 = \frac{\nu (W')^2}{2} + \nu U' + \frac{\nu (U')^2}{2} + \frac{U}{R} + \frac{U^2}{2R^2} \quad , \tag{6.49}$$

$$A_3 = - \{ W''(1 + U')(1 + \frac{U}{R}) - U''(1 + \frac{U}{R})W'$$

$$+ \nu \left[\frac{W'}{R} (1 + \frac{U}{R})^2 \right] \} \quad , \tag{6.50}$$

$$A_4 = - \{ \nu \left[W''(1 + U')(1 + \frac{U}{R}) - U''(1 + \frac{U}{R})W' \right]$$

$$+ \frac{W'}{R} (1 + \frac{U}{R})^2 \} \quad , \tag{6.51}$$

and

$$\tau^2 = \frac{1}{12}\left(\frac{h}{a}\right)^2 \quad . \tag{6.52}$$

In order for the load to appear in the first approximation $\frac{P}{2\pi a}\frac{(1 - \nu^2)}{Eh}$ must be of order 1. Therefore for these equations to apply $P = O\left[\frac{\pi a E h}{(1 - \nu^2)}\right]$.

3. Ratio of deflections to outside radius large compared with unity

For this case $(c/a) >> 1$. An analysis of equation (6.8) to (6.13) shows that $b/a \gtrsim O(1)$ otherwise U would not enter into the equations and we would have two equations in one unknown, W, which would not be compatible. This leaves four other possibilities for the order of magnitude of b/a. They are

$$(1) \quad b/a = O(1) \quad , \tag{6.53}$$

$$(2) \quad O(1) << b/a << c/a \quad , \tag{6.54}$$

$$(3) \quad b/a = c/a \quad , \tag{6.55}$$

and

$$(4) \quad b/a >> c/a \quad . \tag{6.56}$$

Case (1) is actually the case that applies and therefore will be presented last.

Case 2 $\quad O(1) << b/a << c/a$

For this case the first order terms in the equilibrium equations are

$$\frac{N}{a}\frac{b}{a}\left[n_r(U'' + \frac{U'}{R}) - \frac{n_\theta U}{R^2} + n_r' U'\right]$$

$$+ \frac{M}{a^2}\frac{c}{a}\frac{b}{a}\{M_r\left[-\frac{2W'''U}{R} - \frac{2W''U'}{R} - \frac{2W'U''}{R}\right]$$

$$+ \frac{2M_\theta W'U}{R^3} + M_r'\left[-\frac{3W''U}{R} - \frac{2W'U'}{R}\right] - M_r''\frac{W'U}{R}\} = 0 \qquad (6.57)$$

and

$$N\frac{c}{a}n_r W' + \frac{M}{a}(\frac{b}{a})^2\{M_r(\frac{2U''U}{R} + \frac{U'U'}{R})$$

$$- M_\theta\frac{U^2}{R^3} + \frac{M'U'U}{R}\} = \frac{P}{2\pi aR} \quad . \qquad (6.58)$$

The first order terms in the constitutive equations are

$$n_r = \frac{Eh}{(1 - \nu^2)N}(\frac{c}{a})^2\frac{W'W'}{2} \quad , \qquad (6.59)$$

$$n_\theta = \frac{Eh}{(1 - \nu^2)N} (\frac{c}{a})^2 \frac{\nu W'W'}{2} \quad , \tag{6.60}$$

$$M_r = \frac{Eh^3}{12(1 - \nu^2)Ma} \frac{c}{a} (\frac{b}{a})^2 \{ - \frac{W''U'U}{R} + \frac{W'U''U}{R} - \frac{\nu W'U^2}{R^3}\} \quad , \tag{6.61}$$

and

$$M_\theta = \frac{Eh^3}{12(1 - \nu^2)Ma} \frac{c}{a} (\frac{b}{a})^2 \{ - \frac{\nu W''U'U}{R} + \frac{\nu W'U''U}{R} - \frac{W'U^2}{R^3}\} \quad . \tag{6.62}$$

A first order solution to equations (6.57) to (6.62) is

$$U = 0 \quad , \tag{6.63}$$

and

$$W' = \left[\frac{P(1 - \nu^2)}{\pi aEh} \right]^{1/3} (\frac{a}{c}) R^{-1/3} \quad . \tag{6.64}$$

The above solution would be for any value of ν and from chapter five we know that this solution should only be good for $\nu = 1/3$.

Case 3 $b/a = c/a$

For this case the first order terms in the equilibrium equation are the same as for case 2, that is equations (6.57) and (6.58). The first order terms in the moment resultant constitutive equations are also the same as for case 2, equations (6.61) and (6.62). The first order terms in the equations for n_r and n_θ are

$$n_r = \frac{Eh}{2(1 - \nu^2)N} \left(\frac{c}{a}\right)^2 \left\{U'U' + \frac{\nu U^2}{R^2} + W'W'\right\} \tag{6.65}$$

and

$$n_\theta = \frac{Eh}{2(1 - \nu^2)N} \left(\frac{c}{a}\right)^2 \left\{\nu U'U' + \frac{U^2}{R^2} + \nu W'W'\right\} \tag{6.66}$$

The first order solution for case 3 is the same as for case 2 and thus is not valid.

<u>Case 4 b/a >> c/a</u>

Again for this case the equilibrium equations and the equations for M_r and M_θ are the same as for cases 2 and 3. The first order terms in the equations for n_r and n_θ are

$$n_r = \frac{Eh}{2(1 - \nu^2)N} \left(\frac{b}{a}\right)^2 \left[U'U' + \frac{\nu U^2}{R^2}\right] \tag{6.67}$$

and

$$n_\theta = \frac{Eh}{2(1 - \nu^2)N} \left(\frac{b}{a}\right)^2 \left[\nu U'U' + \frac{U^2}{R^2}\right] . \tag{6.68}$$

If $\nu = 1/3$ we should be able to get the solution given in chapter five. However this solution does not solve the equations for case 4. Thus the only choice left is for $b/a = O(1)$. Therefore we will let $b = a$. If equations (6.10) to (6.13) are substituted into (6.8) and (6.9) with $b = a$ and the resulting equations are divided by $\frac{Eh}{(1-\nu^2)a}\left(\frac{c}{a}\right)^2$ and $\frac{Eh}{(1-\nu^2)}\left(\frac{c}{a}\right)^3$ respectively we obtain the following

$$\frac{(W')^2}{2R} - \frac{\nu(W')^2}{2R} + W'W'' + \frac{(W')^2}{2}\left(U'' + \frac{U'}{R}\right) - \frac{\nu(W')^2U}{R^2} + W'W''U'$$

$$+ \delta^2\left[A_5\{U'' + \frac{1}{R}(1 + U')\} + A_6\{-\frac{1}{R}(1 + \frac{U}{R})\} + A_5'(1 + U')\right]$$

$$+ \tau^2\left[A_3\{-2(1 + \frac{U}{R})W''' - \frac{2}{R}(1 + U')W'' - \frac{2}{R}U''W'\}\right.$$

$$+ A_4\{\frac{2}{R^2}(1 + \frac{U}{R})W'\} + A_3'\{-3(1 + \frac{U}{R})W'' - \frac{2}{R}(1 + U')W'\}$$

$$\left. + A_3''\{-(1 + \frac{U}{R})W'\}\right] = 0 \quad , \tag{6.69}$$

and

$$\frac{(W')^3}{2} + \delta^2 \left[W'\{U' + \frac{\nu U}{R} + \frac{(U')^2}{2} + \frac{\nu U^2}{2R^2} \} \right]$$

$$+ \tau^2 \delta^2 \left[A_3 \{2(1 + \frac{U}{R})U'' + \frac{1}{R} (1 + U')^2\} \right.$$

$$\left. + A_4 \{ - \frac{1}{R} (1 + \frac{U}{R})^2 \} + A_3' (1 + \frac{U}{R})(1 + U')\} \right]$$

$$= \frac{P}{2\pi a} \frac{(1 - \nu^2)}{Eh} \delta^3 \frac{1}{R} \tag{6.70}$$

where A_3, A_4 and τ are defined by equations (6.50) - (6.52) respectively and

$$A_5 = U' + \frac{(U')^2}{2} + \frac{\nu U}{R} + \frac{\nu U^2}{2R^2} \quad , \tag{6.71}$$

$$A_6 = \nu U' + \frac{\nu (U')^2}{2} + \frac{U}{R} + \frac{U^2}{2R^2} \quad , \tag{6.72}$$

and

$$\delta = \frac{a}{c} \quad . \tag{6.73}$$

In order for the load to appear in the first approximation the coefficient of the right hand side of equation (6.70) must be of order unity. For convenience we will set it equal to 1/2. This means that

$$\delta^3 = \frac{\pi aEh}{P(1 - \nu^2)} \tag{6.74}$$

or that

$$c = a\left\{\frac{\pi aEh}{P(1 - \nu^2)}\right\}^{-1/3} \tag{6.75}$$

For these equations to apply, c/a must be large compared to unity. This means that $P \gg \dfrac{\pi aEh}{(1 - \nu^2)}$.

It should be noted that the arguments used in this section to show that b/a was of order unity depended on the existence of an exact solution to the membrane problem. The arguments are valid; however, an exact solution might not be available for comparison for a general plate problem. Thus it might be necessary to compare the second or third approximations in order to determine the order of magnitude of U.

B. Membrane Solutions

In this section we will find perturbation type solutions to the approximate equations for cases 2 and 3 as discussed in the previous section. Case 1 will not be discussed further since it was solved by Tolefson (1967).

First we will solve equations (6.46) and (6.47) for case 2 and equations (6.69) and (6.70) for case 3 by means of a straightforward perturbation scheme. It is found that the higher derivatives in the equations vanish in the first approximations and therefore two boundary conditions must be abandoned in the first approximations. This implies that the straightforward or membrane expansions, as we shall call them, must be restricted to the interior region of the plate away from its edges.

The edge zones of the plate, where boundary-layers exist, must be investigated by transforming the equations into a system of boundary-layer coordinates. Separate perturbation expansions are obtained in the vicinity of each edge of the plate. We shall refer to these as boundary-layer expansions and they will be discussed further in section C of this chapter.

1. Ratio of deflections to outside radius of order unity

We assume that W and U can be expanded in the following form

$$W = W_0 + \varepsilon W_1 + \varepsilon^2 W_2 + \ldots \quad , \tag{6.76}$$

$$U = U_0 + \varepsilon U_1 + \varepsilon^2 U_2 + \ldots \tag{6.77}$$

where ε is a small parameter and the coefficients of ε and its powers are of order unity and independent of ε. The appropriate choice for ε is not obvious at this point. However in the course of the boundary-layer analysis in section C of this chapter, it is found that the boundary-layer expansions should be in powers of τ. Let us assume that the same perturbation parameter is also appropriate for (6.76) and (6.77). Later we verify that this is the correct choice for ε by virtue of the fact that the boundary-layer and membrane expansions match.

If we substitute (6.76) and (6.77) into equations (6.46) and (6.47) we find that the first approximations to (6.46) and (6.47) are

$$\begin{aligned}
&\left\{\frac{(W_0')^2}{2} + U' + \frac{(U_0')^2}{2} + \frac{\nu U_0}{R} + \frac{\nu U_0^2}{2R^2}\right\}\left\{U_0'' + \frac{1}{R}(1 + U_0')\right\} \\
&+ \left\{\frac{\nu(W_0')^2}{2} + \nu U_0' + \frac{\nu(U_0')^2}{2} + \frac{U_0}{R} + \frac{U_0^2}{2R^2}\right\}\left\{-\frac{1}{R}\left(1 + \frac{U_0}{R}\right)\right\} \\
&+ \left\{\frac{(W_0')^2}{2} + U_0' + \frac{(U_0')^2}{2} + \frac{\nu U_0}{R} + \frac{\nu U_0^2}{2R^2}\right\}'\{1 + U_0'\} = 0 \qquad (6.78)
\end{aligned}$$

and

$$\left\{\frac{(W_0')^2}{2} + U_0' + \frac{(U_0')^2}{2} + \frac{\nu U_0}{R} + \frac{\nu U_0^2}{2R^2}\right\}W' = \frac{P}{2\pi a}\,\frac{(1 - \nu^2)}{Eh}\,\frac{1}{R}\,. \qquad (6.79)$$

Note that these equations are the same as equations (5.10) and (5.11) with the exception that equations (6.78) and (6.79) are dimensionless. Thus we see that the solution to (6.78) and (6.79) is

$$W_0 = \frac{3}{2}\left[\frac{P}{2\pi a}\,\frac{(1-\nu^2)}{Eh}\right]^{1/3} R^{2/3} + C_1 \tag{6.80}$$

and

$$U_0 = 0 \tag{6.81}$$

if $\nu = 1/3$. As in chapter five equations (6.78) and (6.79) can be solved numerically for other values of ν.

2. Ratio of deflections to outside radius large compared to unity

Again we assume that W and U can be expanded as in equations (6.76) and (6.77). Here again ε is a small parameter and the coefficients of ε and its powers are of order unity and independent of ε. They may however be functions of δ. The appropriate choice of ε is $\tau\delta$ as will be shown in section C.

If we substitute (6.76) and (6.77) into (6.69) and (6.70) using (6.74) we find that the first approximations to (6.69) and (6.70) are

$$\frac{(W_0')^2}{2R} - \frac{\nu(W_0')^2}{2R} + W_0'W_0'' + \frac{(W_0')^2}{2}\left(U_0'' + \frac{U_0'}{R}\right)$$

$$- \frac{\nu(W_0')^2U_0}{R^2} + W_0'W_0''U_0' = 0 \tag{6.82}$$

and

$$(W_0')^3 = \frac{1}{R} \quad . \tag{6.83}$$

Thus solving (6.83) for W we find

$$W_0 = \frac{3R^{2/3}}{2} + C_1 \tag{6.84}$$

Substituting (6.84) into (6.82) we obtain

$$3RU_0'' + 4U_0' - \frac{3\nu U_0}{R} = 3\nu - 1 \quad . \tag{6.85}$$

The solution to (6.85) is

$$U_0 = C_2 R^{n_1} + C_3 R^{n_3} + \left(\frac{3\nu - 1}{4 - 3\nu} \right) R \quad , \tag{6.86}$$

where

$$n_1 = \frac{-1 + \sqrt{1 + 36\nu}}{6} \quad , \tag{6.87}$$

and

$$n_2 = \frac{-1 - \sqrt{1 + 36\nu}}{6} \quad . \tag{6.88}$$

Thus for very large deflections we are able to obtain a closed form solution for any value of ν.

From matching with the boundary-layer solution we find that $U = 0$ at $r = a$ and at $r = r_i$. For this to happen C_2 and C_3 in equation (6.86) must be given by

$$C_2 = - C_3 - \left[\frac{3\nu - 1}{4 - 3\nu}\right] \tag{6.89}$$

and

$$C_3 = \frac{-\left[\frac{r_i}{a} - \left(\frac{r_i}{a}\right)^{n_1}\right]\left[\frac{3\nu - 1}{4 - 3\nu}\right]}{\left[\left(\frac{r_i}{a}\right)^{n_2} - \left(\frac{r_i}{a}\right)^{n_1}\right]} . \tag{6.90}$$

Equations (6.84) and (6.86) can be compared with numerical results obtained by Armstrong (1969) for the case where $c/a = 151$ and $\nu = 0.0$ and for $\nu = 0.5$. In both cases the maximum transverse displacement is the same as that given by equation (6.84) and the maximum U displacement differed by about 3.5%. Figure 4 shows a comparison of Armstrong's results for u_r with those given by equation 6.86.

C. Boundary Layer Solutions

In the previous section we found that when a straight forward perturbation scheme was used the higher derivatives vanished in the

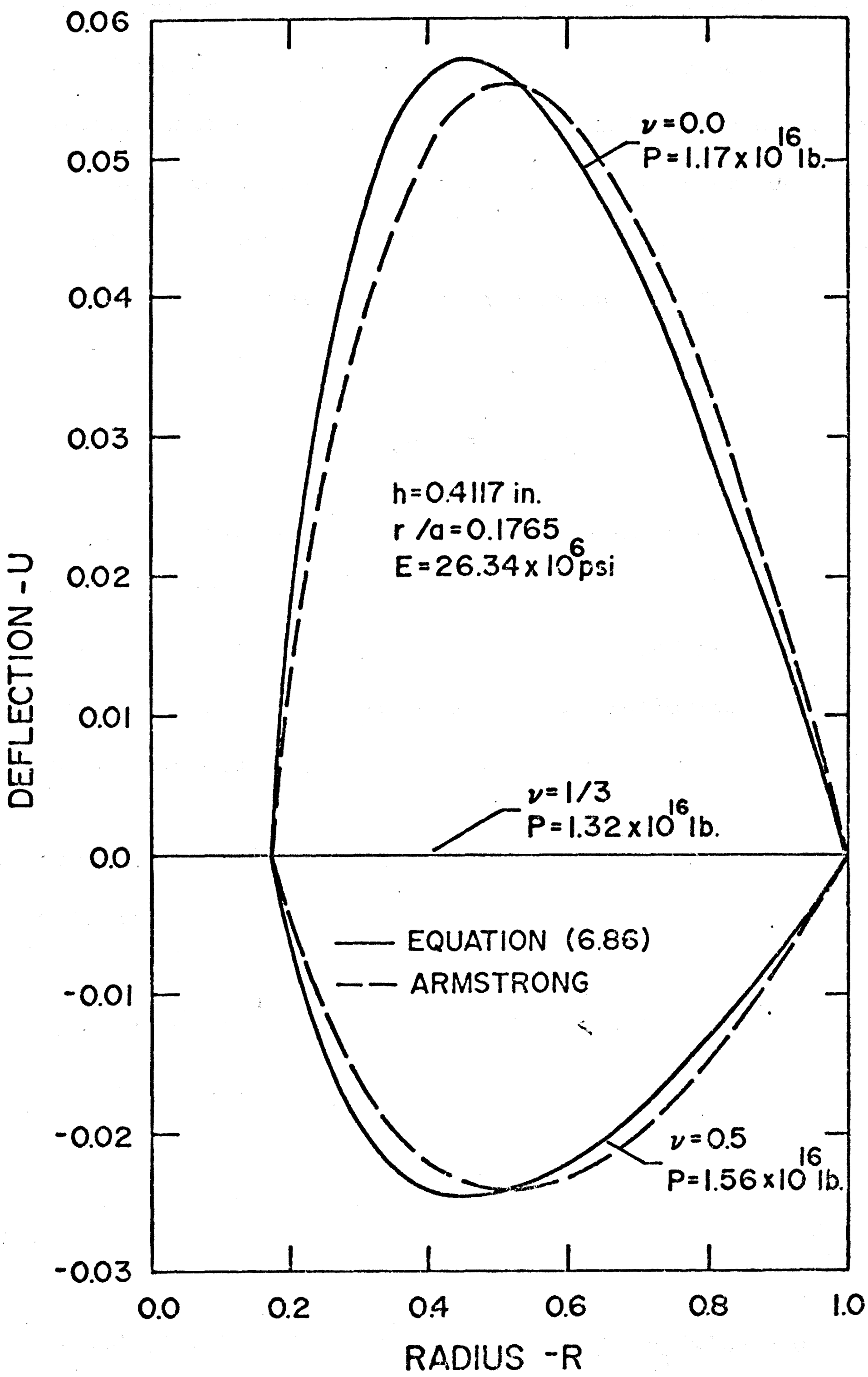


Figure 4. U deflection vs radius for large deflections

first approximation. In this section the equations valid in the vicinity of the plate edge will be developed and solved.

Assuming that the boundary-layers are confined to narrow zones at the edges of the plate let us introduce a new outer edge boundary-layer coordinate η which remains of order unity as the widths of the boundary-layers shrink to zero with decreasing ε. To convert the full equations (6.8-13) to boundary-layer equations we let

$$R = 1 - \varepsilon\eta \quad , \; W = D\tilde{w} \quad , \quad U = B\tilde{u} \quad , \qquad \text{(6.91 a-c)}$$

where ε, D, and B are to be determined and $\tilde{w}$ and $\tilde{u}$ are the boundary-layer variables.

If we substitute (6.91 a-c) into (6.8-13) and neglect the higher order terms in ε in like terms we obtain the following equilibrium and constitutive equations.

<u>Equilibrium Equations (boundary-layer)</u>

<u>in r direction</u>

$$-\frac{N}{\varepsilon} n_r' + \frac{NB}{\varepsilon^2}\{n_r\tilde{u}'' + n_r'\tilde{u}'\} + \frac{M}{a}\frac{c}{a}\frac{D}{\varepsilon^3}\{2M_r\tilde{w}'''$$

$$+ M_r''\tilde{w}' + 3M_r'\tilde{w}''\} + \frac{M}{a}\frac{c}{a}\frac{BD}{\varepsilon^3}\{M_r\left[2\tilde{w}'''\tilde{u}\right.$$

$$\left. + 2\tilde{w}''\tilde{u}' + 2\tilde{w}'\tilde{u}''\right] + M_r'\left[3\tilde{w}''\tilde{u} + 2\tilde{w}'\tilde{u}'\right] + M_r''\tilde{w}'\tilde{u}\} = 0 \qquad \text{(6.92)}$$

and

<u>*in z direction*</u>

$$- \frac{ND}{\varepsilon} \frac{c}{a} n_r \tilde{w}' - \frac{M}{a\varepsilon} M_r' + \frac{M}{a} \frac{B}{\varepsilon^2} \{2M_r \tilde{u}'' + M_r' \tilde{u}'\}$$

$$+ \frac{M}{a} \frac{B^2}{\varepsilon^2} \{M_r (2\tilde{u}''\tilde{u} + \tilde{u}'\tilde{u}') + M_r' \tilde{u}'\tilde{u}\} = \frac{P}{2\pi a} \quad , \tag{6.93}$$

<u>Constitutive Equations</u>

$$M_r = \frac{Eh^3}{12(1 - \nu^2)} \frac{cD}{Ma^2} \{ - \frac{\tilde{w}''}{\varepsilon^2} + \frac{B}{\varepsilon^3} \left[\tilde{w}''\tilde{u}' - \tilde{w}'\tilde{u}''\right]$$

$$+ \frac{B^2}{\varepsilon^3} \left[\tilde{w}''\tilde{u}'\tilde{u} - \tilde{w}'\tilde{u}''\tilde{u}\right] \} \quad , \tag{6.94}$$

$$M_\theta = \nu M_r \quad , \tag{6.95}$$

$$n_r = \frac{Eh}{(1 - \nu^2)N} \{- \frac{B}{\varepsilon} \tilde{u}' + \frac{B^2}{\varepsilon^2} \frac{\tilde{u}'\tilde{u}'}{2} + D^2 (\frac{c^2}{a}) \frac{w'w'}{2\varepsilon^2}\} \quad , \tag{6.96}$$

and

$$n_\theta = \nu n_r \quad . \tag{6.97}$$

Primes in these equations represent ordinary differential with respect to η.

Another equation that will be useful is the one for T_r, equation (3.32). In boundary-layer coordinates it is

$$T_r = Nn_r(1 - \frac{B}{\varepsilon}\tilde{u}') + \frac{M}{a}\frac{c}{a}\frac{D}{\varepsilon^2}\{- 2M_r\tilde{w}'' - M_r'\tilde{w}'\}$$

$$+ \frac{M}{a}\frac{c}{a}\frac{DB}{\varepsilon^2}\{- 2M_r\tilde{w}''\tilde{u} - M_r\tilde{w}'\tilde{u}' - M_r'\tilde{w}'\tilde{u}'\} \quad . \tag{6.98}$$

1. Determination of constants and matching conditions

If equation (6.98) is differentiated considering T_r to be a constant and compared with (6.92) it is found to differ only in the term whose coefficient is $\frac{McDB}{aa\varepsilon^2}$. This means that if B is small compared to unity equation (6.98) is an integral of equation (6.92) in the first approximation. This also indicates that T_r is a constant in the boundary layer. Since the magnitude of u_r should decrease when the boundary layer thickness decreases it is reasonable to assume that B is not large compared to unity or not of order unity and thus that T_r is a constant in the first approximation. We still however must determine what the values of B and D are. From equations (6.93) and (6.98) we see that if we choose $B = D = \varepsilon$ no further terms would be eliminated from the equations and we have what is known as the least degenerate case. Thus even if we have made a mistake in the choice for B and D our answers will be correct since we still have all the terms in the equations. We can also see from the matching conditions that ε is the correct choice for B and D. In the membrane we have

$$U(R) = U_0 + \varepsilon U_1 + \ldots\ldots ,$$

$$U'(R) = U'_0 + \varepsilon U'_1 + \ldots\ldots ,$$

$$W(R) = W_0 + \varepsilon W_1 + \ldots\ldots , \qquad (6.99\ \text{a-c})$$

$$W'(R) = W'_0 + \varepsilon W'_1 + \ldots\ldots ,$$

$$\text{slope} = \frac{W'}{1 + U'} ,$$

and in the boundary layer we have

$$B\tilde{u}(\eta(R)) = B\tilde{u}_0 + \varepsilon B\tilde{u}_1 + \ldots\ldots ,$$

$$-\frac{B\tilde{u}'(\eta(R))}{\varepsilon} = -\frac{B\tilde{u}'_0}{\varepsilon} - B\tilde{u}' + \ldots\ldots ,$$

$$D\tilde{w}(\eta(R)) = D\tilde{w}_0 + \varepsilon D\tilde{w}_1 + \ldots\ldots , \qquad (6.100\ \text{a-e})$$

$$-\frac{D\tilde{w}'(\eta(R))}{\varepsilon} = -\frac{D}{\varepsilon}\tilde{w}'_0 - D\tilde{w}'_1 + \ldots\ldots ,$$

and

$$\text{slope} = \frac{-\frac{D}{\varepsilon}\tilde{w}'}{1 - \frac{B}{\varepsilon}\tilde{u}'} \quad ,$$

where $$\eta = \frac{1 - R}{\varepsilon} \tag{6.101}$$

and primes indicate differentiation with respect to R.

Since the slopes and the deflections must match it is obvious that $B = D = \varepsilon$ and thus that

$$\begin{aligned}
U_0\Big|_{R\to 1} &= 0 \quad , \\
W_0\Big|_{R\to 1} &= 0 \quad , \\
U_1\Big|_{R\to 1} &= \tilde{u}_0\Big|_{\eta\to\infty} \quad , \\
W_1\Big|_{R\to 1} &= \tilde{w}_0\Big|_{\eta\to\infty} \quad , \\
U'_0\Big|_{R\to 1} &= -\tilde{u}'_0\Big|_{\eta\to\infty} = 0 \quad ,
\end{aligned} \tag{6.102 a-f}$$

and

$$W'_0\Big|_{R\to 1} = -\tilde{w}'_0\Big|_{\eta\to\infty} = 1 \quad .$$

2. Development of boundary layer equations

If we now substitute equations (6.94) and (6.96) into (6.93) and (6.98) using the fact that $B = D = \varepsilon$ and neglecting higher order terms we obtain the following

$$\frac{P}{2\pi a}\,\frac{(1-\nu^2)}{Eh} = -\frac{c}{a}\tilde{w}'\left(-\tilde{u}' + \frac{\tilde{u}'\tilde{u}'}{2} + \left(\frac{c}{a}\right)^2\frac{\tilde{w}'\tilde{w}'}{2}\right)$$

$$+\frac{\tau^2}{\varepsilon^2}\left(\frac{c}{a}\right)\Big\{2u''\Big[-\tilde{w}''(1-\tilde{u}') - \tilde{w}'\tilde{u}''\Big]$$

$$-(1-\tilde{u}')\Big[-\tilde{w}'''(1-\tilde{u}') - \tilde{w}'\tilde{u}'''\Big]\Big\} \quad , \tag{6.103}$$

and

$$\frac{T_r(1-\nu^2)}{Eh} = (1-\tilde{u}')\left(-\tilde{u}' + \frac{\tilde{u}'\tilde{u}'}{2} + \left(\frac{c}{a}\right)^2\frac{\tilde{w}'\tilde{w}'}{2}\right)$$

$$+\frac{\tau^2}{\varepsilon^2}\left(\frac{c}{a}\right)^2\Big\{-2\tilde{w}''\Big[-\tilde{w}''(1-\tilde{u}') - \tilde{w}'\tilde{u}''\Big]$$

$$-\tilde{w}'\Big[-\tilde{w}'''(1-\tilde{u}') - \tilde{w}'\tilde{u}'''\Big]\Big\} \quad . \tag{6.104}$$

From the membrane solution and equation (6.74) we see that if $\nu = 1/3$ the first approximation to U is zero in the membrane. Then the left hand side of equation (6.103) is $1/(2\delta^3)$ and similarly the left hand side of (6.104) is $1/(2\delta^2)$. If we make a substitution similar to that made in chapter three,

$$\tilde{w}' = yx \tag{6.105}$$

and

$$\tilde{u}' = 1 - x \quad , \tag{6.106}$$

we obtain

$$\frac{1}{2\delta^3} = -\frac{yx}{2\delta}\{-1 + x^2(1 + \frac{y^2}{\delta^2})\}$$

$$+ \frac{\tau^2}{\varepsilon^2\delta}\{y''x^3 + 4y'x'x^2\} \tag{6.107}$$

and

$$\frac{1}{2\delta^2} = \frac{x}{2}\{-1 + x^2(1 + \frac{y^2}{\delta^2})\}$$

$$+ \frac{\tau^2}{\varepsilon^2\delta^2}\{x^3y''y + 4x^2x'y'y + 2x^3(y')^2\} \tag{6.108}$$

If we now multiply (6.107) by y/δ, subtract from (6.108) and then multiply by $2\delta^4$ we obtain

$$\delta^2 - y = x(\delta^2 + y^2)\{-\delta^2 + x^2(\delta^2 + y^2)\}$$

$$+ (\frac{\tau\delta}{\varepsilon})^2 4(y')^2x^3 \quad . \tag{6.109}$$

If we multiply (6.108) by y/δ, add to (6.107) and then multiply by $2\varepsilon^2\delta^3/\tau^2$ we obtain

$$\left(\frac{\varepsilon}{\tau\delta}\right)^2 \delta^2(1 + y) = 2y''x^3(\delta^2 + y^2)$$

$$+ 8y'x'x^2(\delta^2 + y^2) + 4(y')^2x^3y \quad . \tag{6.110}$$

From equation (6.109) we see that ε should be of the order $\tau\delta$. For convenience we will let

$$\varepsilon = \tau\delta \quad . \tag{6.111}$$

Equation (6.109) can be solved for (y'). This can be differentiated and then y' and y'' can be substituted into (6.110) which can then be solved for x'. Thus we find that

$$y' = -\frac{1}{2}\left[\frac{1}{x^3}\{\delta^2 - y - x(\delta^2 + y^2)(-\delta^2\right.$$

$$\left. + x^2(\delta^2 + y^2))\}\right]^{1/2} \tag{6.112}$$

and

$$x' = \frac{y'\{5 - 6xy\delta^2 + 8x^3y(\delta^2 + y^2)\}}{\{\frac{5}{x}(\delta^2 - y) + 6(\delta^2 + y^2)\delta^2 - 8x^2(\delta^2 + y^2)^2\}} \tag{6.113}$$

The variable η can be eliminated from equation (6.112) and (6.113) by dividing one by the other and we obtain an equation which can be integrated analytically. That equation is

$$\frac{dx}{dy} = \frac{5 - 6xy\delta^2 + 8x^3y(\delta^2 + y^2)}{\{\frac{5}{x}(\delta^2 - y) + 6(\delta^2 + y^2)\delta^2 - 8x^2(\delta^2 + y^2)^2\}} \tag{6.114}$$

which is a perfect differential whose integral is

$$5x(\delta^2 - y) + 3x^2\delta^2(\delta^2 + y^2) - 2x^4(\delta^2 + y^2)^2 = \text{constant} . \tag{6.115}$$

The constant is determined from the matching condition which is that at $y = -1$, $x = 1$. From this we find that the constant is $(\delta^2 + 1)(\delta^2 + 3)$ and thus the integral to (6.114) is

$$5x(\delta^2 - y) + 3x^2\delta^2(\delta^2 + y^2) - 2x^4(\delta^2 + y^2)^2$$

$$= (\delta^2 + 1)(\delta^2 + 3) \quad . \tag{6.116}$$

One nice point about equation (6.116) is that it is a solution to (6.114) which is independent of the boundary conditions. Thus it could be used for other problems besides the one with the fixed edge condition that we are interested in.

3. Analysis for small deflections

For the case where δ is large compared to unity (i.e. small deflections) a perturbation type solution can be found to equation (6.114). This solution is more useful than (6.116) since it expresses x explicitly as a function of y. Since the deflections are small, $\tilde{u}'$ is small compared to unity and we can expand x in a power series as

$$x = 1 + \rho^2 x_2 + \rho^4 x_4 + \rho^6 x_6 + 0(\rho^8) \tag{6.117}$$

where $\rho^2 = 1/\delta^2 << 1$. (6.118)

After we have found x we then can substitute it into equation (6.112) and find y′ as a function of y. The reason we are using four terms in x is that we need this many terms in x to obtain two terms in the expression for y′.

If equation (6.117) is substituted into (6.114) we obtain

$$\begin{aligned} \frac{dx}{dy} = \rho^2 \frac{dx_2}{dy} + \rho^4 \frac{dx_4}{dy} + \rho^6 \frac{dx_6}{dy} &= -\rho^2 y \\ &- \frac{\rho^4}{2}(-2y^3 + 2yx_2 + 5y + 5) \\ &+ \rho^6(-yx_4 + y^3x_2 + 20yx_2 + 20x_2 - y^5 \\ &+ 15y^3 + 15y^2 - \frac{25}{4}y - \frac{25}{4}) + 0(\rho^8) \quad . \end{aligned} \tag{6.119}$$

Thus $$\frac{dx_2}{dy} = - y \tag{6.120}$$

or $$x_2 = - \frac{y^2}{2} + C \quad . \tag{6.121}$$

Since x = 1 at y = - 1 we see that at y = - 1, $x_2 = 0$. Thus C = 1/2 and

$$x_2 = \frac{1 - y^2}{2} \tag{6.122}$$

Substituting (6.122) into the second term of (6.119) we see that

$$\frac{dx_4}{dy} = - \frac{1}{2} (- 3y^3 + 6y +5) \tag{6.123}$$

or $$x_4 = - \frac{1}{8} (- 3y^4 + 12y^2 + 20y + C) \quad . \tag{6.124}$$

Using the same matching condition we see that at y = - 1, $x_4 = 0$, thus C = 11 and

$$x_4 = - \frac{1}{8} (- 3y^4 + 12y^2 + 20y + 11) \quad . \tag{6.125}$$

If we now substitute (6.122) and (6.125) into the third term of (6.119) we see that

$$\frac{dx_6}{dy} = \frac{1}{8} (- 15y^5 + 56y^3 + 60y^2 + 41y + 30) \tag{6.126}$$

or

$$x_6 = \frac{1}{16} (- 5y^6 + 28y^4 + 40y^3 + 41y^2 + 60y + C) \quad . \tag{6.127}$$

where C is again determined from the matching condition that at $y = -1$, $x_6 = 0$, thus $C = 36$ and

$$x_6 = \frac{1}{16} (- 5y^6 + 28y^4 + 40y^3 + 41y^2 + 60y + 36) \quad . \tag{6.128}$$

If equations (6.122), (6.125) and (6.128) are substituted into (6.117) and then into (6.112) we find, neglecting higher order terms, that

$$\frac{dy}{d\eta} = - \frac{1}{\sqrt{2}} \{(1 + y) + \frac{\rho^2}{8} (1 - y^2)(- 7 - 5y)\} \quad . \tag{6.129}$$

If we let

$$y = y_0 + \rho^2 y_2 + O(\rho^4) \tag{6.130}$$

equation (6.129) becomes

$$\frac{dy}{d\eta} = \frac{dy_0}{d\eta} + \rho^2 \frac{dy_2}{d\eta} = - \frac{1}{\sqrt{2}} \left[1 + y_0 + \frac{\rho^2}{8} \{8y_2 + (1 - y_0^2)(- 7 - 5y_0)\} \right] \quad . \tag{6.131}$$

Thus we find

$$\frac{dy_0}{d\eta} = -\frac{(1 + y_0)}{\sqrt{2}} \tag{6.132}$$

or

$$y_0 = e^{-\eta/\sqrt{2} + C} - 1 \quad . \tag{6.133}$$

From the boundary condition that $y_0 = 0$ at $\eta = 0$ we see that $C = 0$ and

$$y_0 = e^{-\eta/\sqrt{2}} - 1 \quad . \tag{6.134}$$

Substituting (6.134) into (6.131) we find

$$\frac{dy_2}{d\eta} = -\frac{y_2}{\sqrt{2}} - \frac{1}{8\sqrt{2}}\left(5e^{-3\eta/\sqrt{2}} - 8e^{-2\eta/\sqrt{2}} - 4e^{-\eta/\sqrt{2}}\right) \tag{6.135}$$

which has a solution

$$y_2 = -\frac{1}{16}\left(- 5e^{-3\eta/\sqrt{2}} + 16e^{-2\eta/\sqrt{2}}\right.$$

$$\left. - 4\sqrt{2}\,\eta e^{-\eta/\sqrt{2}} + Ce^{-\eta/\sqrt{2}} \right) \quad . \tag{6.136}$$

From the boundary condition that $y_2 = 0$ at $\eta = 0$ we see that $C = -11$ and

$$y_2 = -\frac{1}{16}\left(-5e^{-3\eta/\sqrt{2}} + 16e^{-2\eta/\sqrt{2}} - 4\sqrt{2}\,\eta e^{-\eta/\sqrt{2}} - 11e^{-\eta/\sqrt{2}} \right) . \qquad (6.137)$$

This then solves the problem for the case when δ is large. The value of y is determined from equation (6.130) with y_0 and y_2 being determined by (6.134) and (6.137) respectively. After y is determined x can be determined from equation (6.117) with x_2, x_4 and x_6 being determined by (6.122), (6.125) and (6.128) respectively.

Equation (6.133) agrees with Tolefson's (1967) first approximation to the von Karmam equations. This is as it should be since we have shown that our equations reduce to the von Karman equations for small deflections. One thing that should be noted here is that in order to obtain equation (6.133) we needed the first three terms in equation (6.117) for x. This is one more term that one would expect to need. This indicates that if the slope is found from the von Karman equations and then the u_r displacement is determined knowing the slope, that the u_r displacement will be more accurate than would ordinarily be expected.

4. Analysis for large deflections

Since, as was pointed out in chapter four, the theory presented in this dissertation is no longer valid when the moment term begins to predominate in equation (4.17), we should examine equation (4.17) before we proceed to look at the case of very large deflections. To do this we will write equation (4.17) in boundary layer coordinates and look at the point where the maximum bending occurs which is at $\eta = 0$. For our particular problem with the fixed boundary $\tilde{u} = \tilde{w}' = 0$ at $\eta = 0$ and thus the equation for the true stress resultant at the boundary is

$$N_r = \frac{Eh}{(1 - \nu^2)} \left\{ \frac{-\tilde{u}' + (\tilde{u}')^2/2}{(1 - \tilde{u}')} + \frac{2(\tilde{w}'')^2}{\delta^4(1 - \tilde{u}')} \right\} \quad . \tag{6.138}$$

We see from equation (6.138) that when δ becomes small the $\tilde{w}''$ term or moment term does indeed predominate and thus a better plate theory would have to be used for this problem.

We will however proceed to solve our equations for small δ realizing now that problems might be encountered and any solution that we do obtain would not be accurate. To do this we again will obtain a perturbation type solution as we did for large δ so that x will be an explicit function of y. We therefore let

$$x = x_0 + \delta^2 x_2 + O(\delta^4) \tag{6.139}$$

and after we substitute into equation (6.114) we find that

$$\frac{dx_0}{dy} + \delta^2 \frac{dx_2}{dy} = - \frac{1}{5y/x_0 + 8y^4x_0^2} \Bigg[5 + 8y^3x_0^3$$

$$+ \delta^2\{-6yx_0 + 8yx_0^3 + 24y^3x_0^2x_2\} \Bigg] \Bigg[1$$

$$+ \frac{\delta^2}{5y/x_0 + 8y^4x_0^2} \{5/x_0 + 5yx_2/x_0^2 + 6y^2$$

$$- 16y^2x_0^2 - 16y^4x_0x_2\} \Bigg] \quad . \tag{6.140}$$

Thus we see that

$$\frac{dx_0}{dy} = - x_0/y \tag{6.141}$$

which has the solution

$$x_0 = C/y. \tag{6.142}$$

From the matching condition we know that at $y = - 1$, $x = 1$, thus $x_0 = 1$ and $C = - 1$ and then

$$x_0 = - 1/y \quad . \tag{6.143}$$

Substituting (6.143) into (6.140) we find that

$$\frac{dx_2}{dy} = - \frac{x_2}{y} - \frac{8}{3y^4} - \frac{5}{3y^3} \tag{6.144}$$

which has the solution

$$x_2 = \frac{1}{3} \left(\frac{4}{y^3} + \frac{5}{y^2} + \frac{C}{y} \right) \quad . \tag{6.145}$$

From the matching condition we know that $x_2 = 0$ at $y = -1$, therefore $C = -1$ and

$$x_2 = \frac{1}{3} \left(\frac{4}{y^3} + \frac{5}{y^2} - \frac{1}{y} \right) \quad . \tag{6.146}$$

If we substitute equations (6.143) and (6.146) into (6.139) we see that

$$x = \frac{1}{y} \left\{ -1 + \frac{\delta^2}{3} \left(-1 + \frac{5}{y} + \frac{4}{y^2} \right) \right\} \quad . \tag{6.147}$$

From equation (6.147) we see that we have a singular perturbation problem since the second term is more singular than the first in the region near $y = 0$. So we now let

$$y = \delta \bar{y} \tag{6.148}$$

and

$$x = \bar{x}/\delta \quad . \tag{6.149}$$

We now substitute (6.148) and (6.149) into (6.114). The first order term then is

$$\frac{d\bar{x}}{d\bar{y}} = \frac{5\bar{x} + 8x^4\bar{y}(1 + \bar{y}^2)}{-5\bar{y} - 8\bar{x}^3(1 + \bar{y}^2)^2} \tag{6.150}$$

which has the solution

$$5\bar{y}\bar{x} + 2\bar{x}^4(1 + \bar{y}^2)^2 = C \quad . \tag{6.151}$$

The constant is determined by matching equation (6.151) with (6.143). The first order terms of equation (6.151) expanded in the y,x variables is

$$5yx + 2x^4y^4 = C \quad . \tag{6.152}$$

The first term of equation (6.143) expanded in the $\bar{y},\bar{x}$ variables is

$$\bar{x}\bar{y} = -1 \quad . \tag{6.153}$$

Thus we see that $\bar{x}\bar{y} = xy = -1$ and $C = -3$ and then

$$5\bar{y}\bar{x} + 2\bar{x}^4(1 + \bar{y}^2)^2 = -3 \quad . \tag{6.154}$$

If we now look at the boundary, where $\bar{y} = y = 0$, we find that equation (6.154) becomes

$$2\bar{x}^4 = -3 \quad . \tag{6.155}$$

Thus we see that there is no real solution for $\bar{x}$ or x when δ is small compared to unity.

To find out just how small δ could be made and still be able to have y = 0 at the boundary we analyzed the integral to equation (6.114) which is equation (6.116) at the point where y = 0. Equation (6.116) at y = 0 is

$$(- 2x^4 + 3x^2 - 1)\delta^4 + (5x - 4)\delta^2 - 3 = 0 \quad . \tag{6.156}$$

This equation can be solved for real values of δ only when x satisfies

$$1/\sqrt{2} < x < 1.054 \quad . \tag{6.157}$$

For these values of x, δ ranges between 1.725 and ∞. $\delta = \infty$ corresponds to zero deflection at a value of x = 1 or $\tilde{u}' = 0$. Thus we see that there is not a solution to equation (6.114) for $\delta < 1.725$, which will satisfy the boundary condition that y = 0 at the boundary. A plot of δ versus x is shown in Figure 5.

Figure 6 shows a plot of x versus y for different values of δ that was obtained by integrating equation (6.114) numerically using as initial condition x = 1 at y = - 1. As can be seen from the plot the lowest value of δ which would satisfy the condition that y = 0 at the boundary is $\delta = 1.725$.

Equations (6.112) and (6.113) could not be integrated analytically so they were integrated numerically. The initial condition for y was y = 0 (no slope) at $\eta = 0$. The initial condition for x was determined

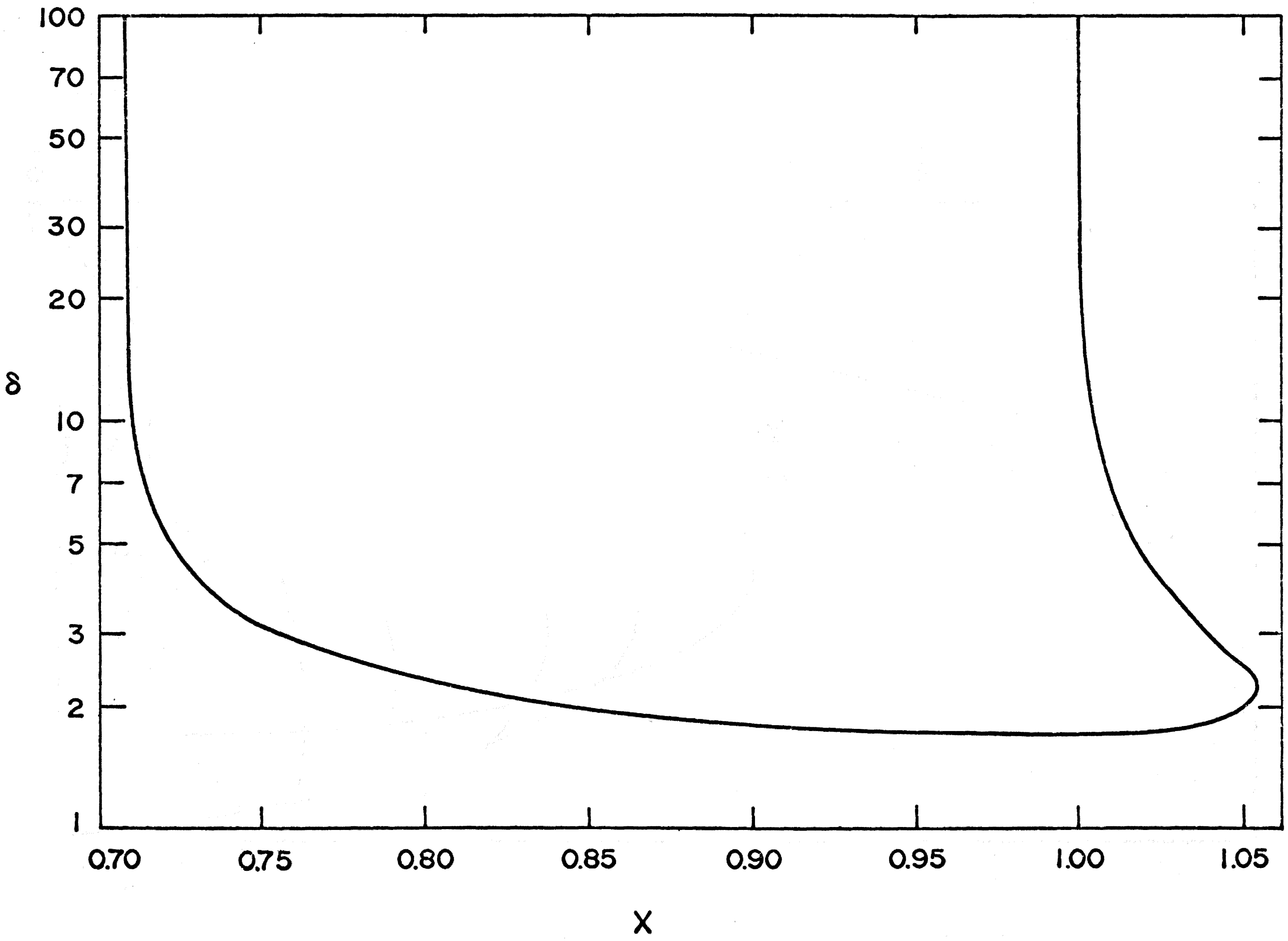


Figure 5. Relationship between δ and x defined by equation (6.156)

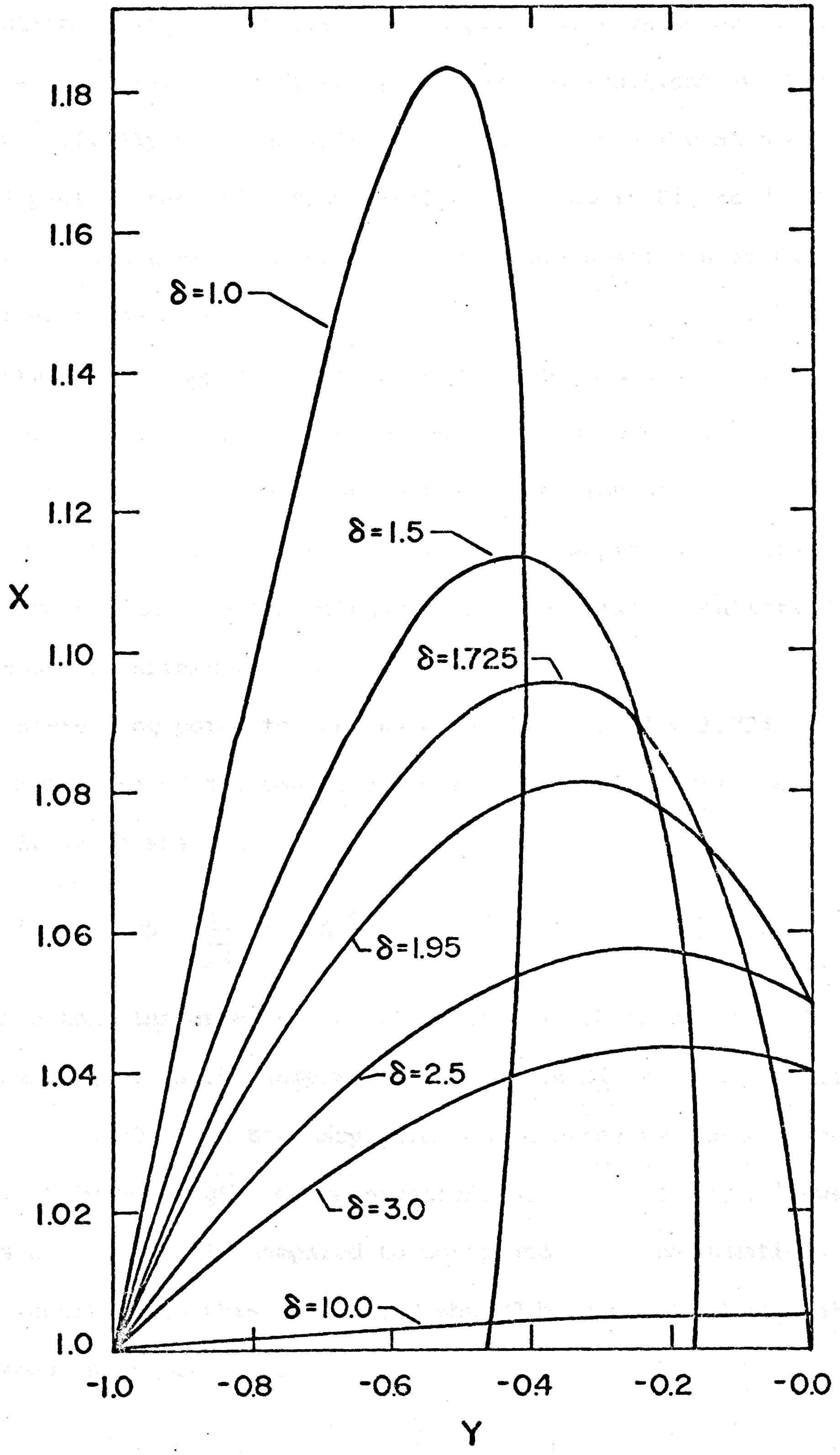


Figure 6. Relationship between boundary layer variables

from equation (6.156) or Figure 5 for a particular value of δ. The value of $\tilde{w}$ and $\tilde{u}$ were then found by integrating equations (6.105) and (6.106) numerically using as initial condition $\tilde{w} = \tilde{u} = 0$ at $\eta = 0$. A typical plot of the deflection profile is shown in Figure 7. Points are shown in both their undeformed and deformed positions so that the $\tilde{u}$ displacement can be seen.

In Figure 5 it appears that there are two values of x that would satisfy the equations for a given value of δ. However, if a value of x less than 0.99 is used as an initial condition at $\eta = 0$, the matching conditions will not be satisfied as η approaches infinity. If x is larger than 0.99 the matching condition will be satisfied as η approaches infinity.

An interesting point to make here is that when $\delta = 1.725$, ε which is a measure of the boundary layer thickness is very small. When $\delta = 1.725$ we see that

$$\varepsilon = \delta\tau = 1.725 \frac{h}{\sqrt{12a}} \simeq 0.5 \frac{h}{a} \tag{6.158}$$

Thus we see that the order of magnitude of ε which is the thickness of the boundary layer is the same as the thickness of the plate. Writing equation (4.20 b) in boundary layer coordinates we can show that the smallest "wave length" of deformation, L, is of order ε. Thus $(h/L)^2$ is not negligible compared to unity and shear deformations would be important in this region and should be taken into account as discussed in chapter four.

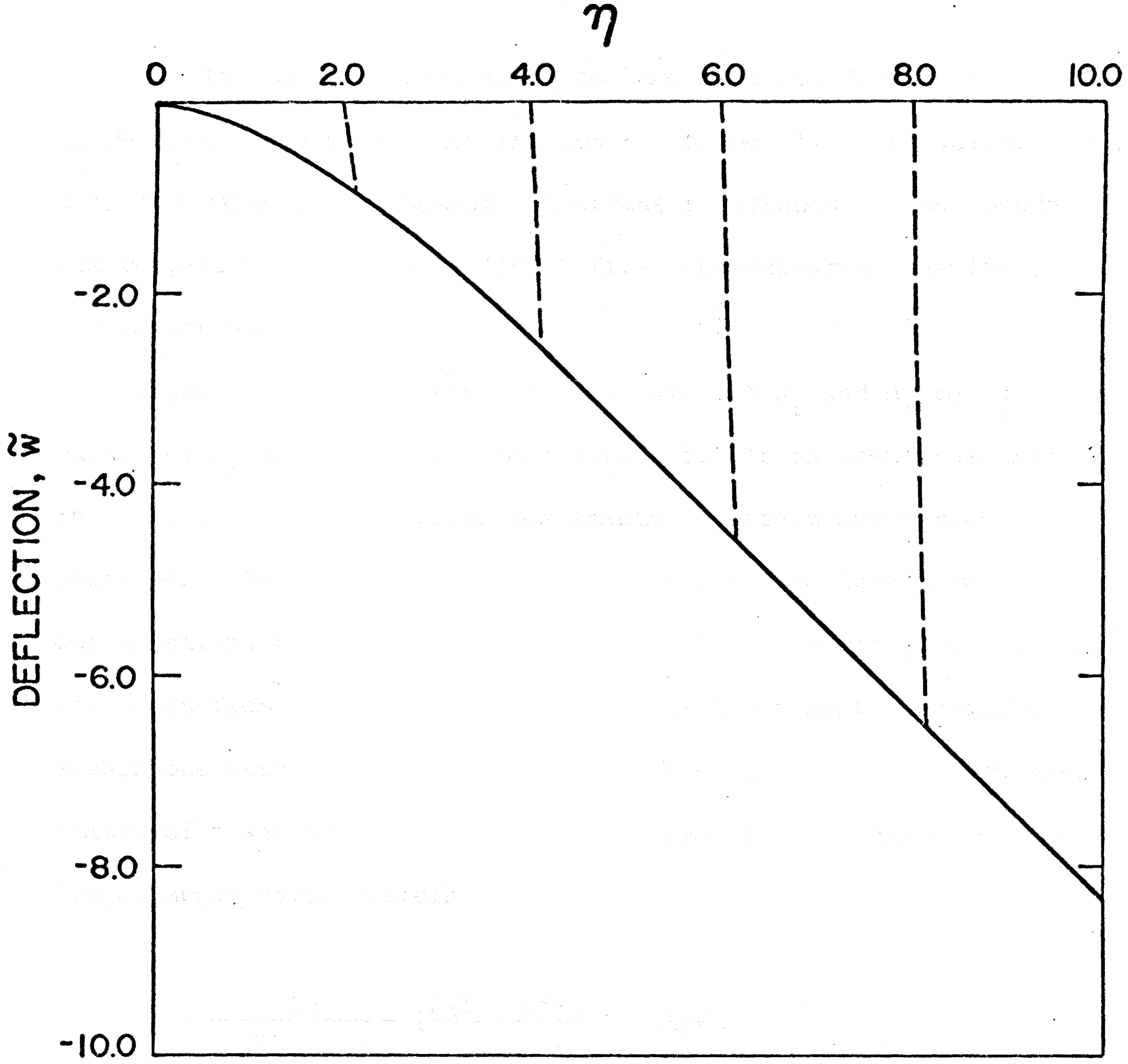


Figure 7. Deflection profile vs η in boundary layer variables for δ = 2.27

5. Comparison with von Karman equations

It would be interesting to know just how far the plate could be deflected and still have the von Karman equations be valid. To do this the value of the moment and stress resultants at the boundary are compared to Tolefson's (1967) first approximation results to the von Karman equations.

Figure 8 shows the ratio of our value of N_r and M_r to Tolefson's values of N_r and M_r respectively versus δ. It is seen from Figure 8 that the ratio of the stress resultants is within one percent of unity for $\delta>10$. This is as it should be since we have shown that our equations reduce to von Karman's for small deflections. It is also seen from Figure 8 that the ratio of the moment resultants is within one percent of unity for $\delta>4$. This agreement for such small values of δ was unexpected since von Karman's equations should no longer apply here. Tolefson's expression for the moment is

$$M_r = \frac{Eh^2}{\sqrt{12}\,(1 - \nu^2)} \left\{\frac{P(1 - \nu^2)}{\pi aEh}\right\}^{2/3} (1/\sqrt{2}) \tag{6.159}$$

and our expression is

$$M_r = \frac{Eh^2}{\sqrt{12}\,(1 - \nu^2)} \left\{\frac{P(1 - \nu^2)}{\pi aEh}\right\}^{2/3} (-y'x^2) \quad . \tag{6.160}$$

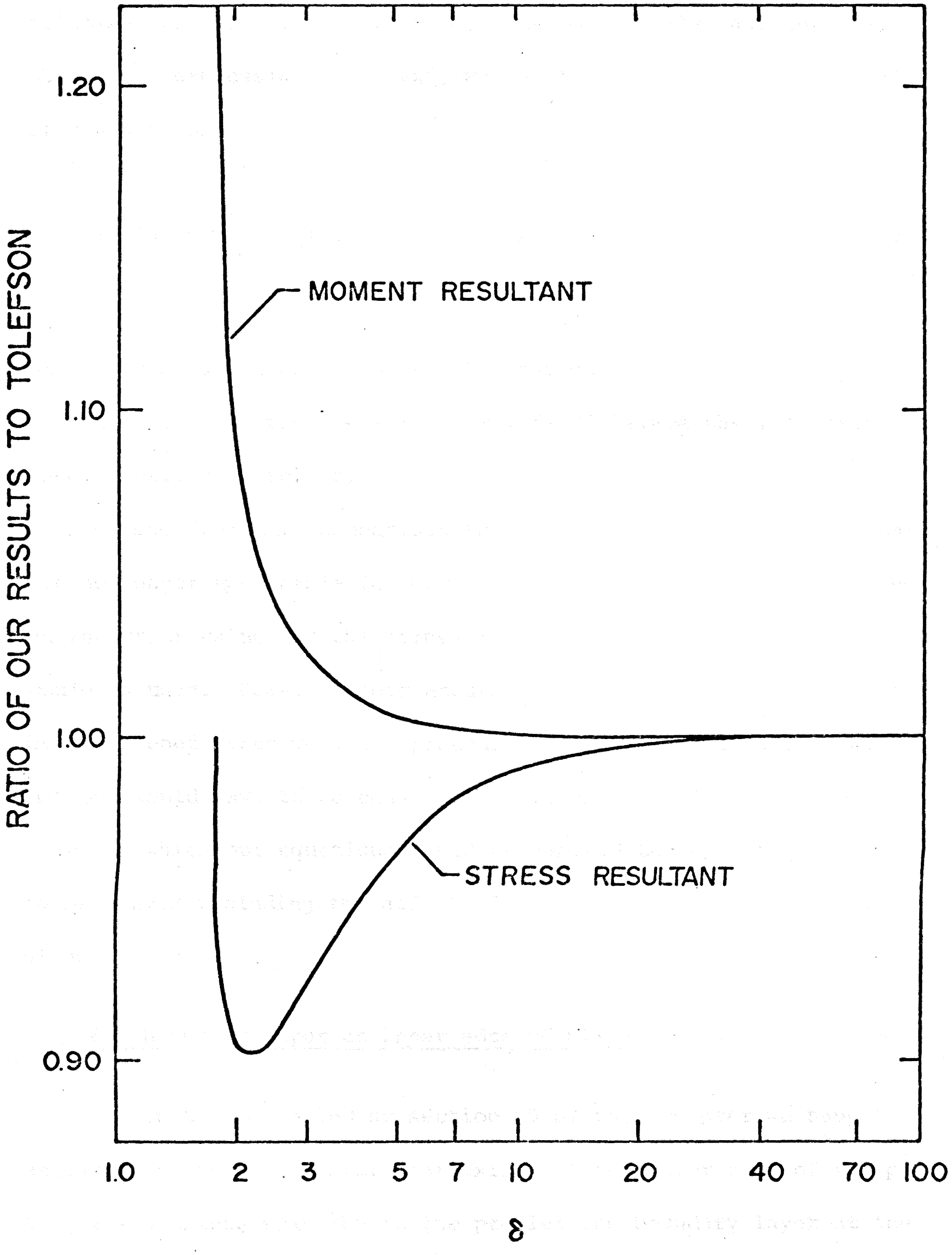


Figure 8. Comparison of Tolefson's results with ours

These appear to be different but if we look at the perturbation type solutions we obtained for large δ, equations (6.117) and (6.130), and obtain the expression for $- yx^2$, we see that when $- yx^2$ is evaluated at $y = 0$ it is

$$- y'x^2 = \frac{1}{\sqrt{2}} \{1 + 0.125/\delta^2 + 0(1/\delta^4)\} \quad . \tag{6.161}$$

We see from equations (6.159 - 161) that our expression for M_r is indeed almost exactly the same as Tolefson's since the $1/\delta^2$ term has such a small coefficient.

We see from this comparison that for $\delta<10$ the von Karman equations are no longer applicable for our particular problem since they predict an incorrect value for the stress resultant. For $\delta<10$ our equations could be used. However their accuracy in this region would have to be questioned since we are approaching the point where shear deformations would have to be considered. In order to find the limiting point at which our equations could be applied the problem would have to be solved including the effect of transverse shear and normal stresses.

6. Boundary layer at inner edge of plate

Up to and including section C5 of this chapter we have analyzed the boundary layer that exists at the outer edge of the plate. To get a complete solution to the problem the boundary layer at the inner edge of the plate would also have to be solved. The procedure would be exactly the same as for the outer edge of the plate. We would first let

$$R = r_i/a + \varepsilon\xi \quad , \tag{6.162 a}$$

$$W = \bar{D}\bar{w} \quad , \tag{6.162 b}$$

and

$$U = \bar{B}\bar{u} \quad , \tag{6.162 c}$$

where ε, $\bar{D}$ and $\bar{B}$ are to be determined and $\bar{w}$ and $\bar{u}$ are the boundary layer variables at the inner edge of the plate. Equations (6.162 a-c) would then be substituted into equations (6.8-13) and equations similar to (6.92-97) would be obtained. This particular part of the problem will not be worked since the steps to be followed are exactly the same as those followed for the outer edge of the plate.

VII DISCUSSION OF RESULTS AND CONCLUSIONS

In this dissertation the general nonlinear first approximation thin plate tensor equations in undeformed coordinates, valid for large strains, rotations and displacements, have been derived on the basis of the single assumption of plane stress. These equations were then reduced to the exact tensor and physical equations for symmetrical circular plates.

An order of magnitude analysis of the equations derived shows that they reduce to the classical linear equations when the ratio of the deflections to the thickness is small when compared to unity. The analysis also shows that the equations reduce to the von Karman (1910) equations when the ratio of the deflections to the outside radius is small compared to unity. However the equations do not reduce to the Reissner (1949) equations when the ratio of the deflections to the outside radius is equal to or greater than unity (i.e., large slopes). This is shown quite graphically in Figure 2 for the large deflections of a circular membrane.

For the problem of the clamped circular plate loaded with a concentrated load on a central rigid inclusion it was shown that the solutions corresponded very closely to those obtained from the von Karman equations. This correspondence was good even up to the point where our results begin to be questionable. This was a coincidence since the von Karman equations have the same membrane solution as our equations and part of that solution is $u_r = 0$. Thus the problem we chose to analyze did not point out the differences that we would expect to see in a problem which had comparitively large values of u_r in the membrane solution. A problem which would probably show the difference between our equations and von Karman's without approaching the limit of the first approximation theory is that of a circular plate loaded with a uniform pressure. The reason this particular problem was not worked is that the membrane solution is not readily available as it is for the problem we worked.

Perhaps the most important result we have obtained is finding the order of magnitude of the limiting value of deflection that would be allowed under the assumption of plane stress for our particular problem. We have shown that when the deflection approaches the order of magnitude of the radius, the boundary layer approaches the order of magnitude of the thickness and thus the first approximation theory is no longer valid. If results are desired for larger deflections a more detailed theory, one including the energy contributed by transverse shear and transverse normal stress, must be derived. This more detailed theory would then also point out where our results would no longer be valid.

Finally another important result is that we have obtained the first approximation solution to the problem of large deformations of a circular membrane loaded with a concentrated load on a central rigid inclusion for any value of Poisson's ratio. Also we have obtained the solution to the problem of the circular membrane loaded with a concentrated load plus a small torque.

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GENERAL NONLINEAR PLATE THEORY APPLIED

TO A CIRCULAR PLATE WITH LARGE DEFLECTIONS

by George Junkin II

ABSTRACT

The general nonlinear first approximation thin plate tensor equations in undeformed coordinates valid for large strains, rotations and displacements are developed based on the single assumption of plane stress. These equations are then reduced to the exact tensor and physical component equations for symmetrical circular plates.

An order of magnitude analysis is performed on the resulting equations which shows that they reduce to the classical linear equations for very small deflections and to the von Karman equations for moderate deflections. However, the equations do not reduce to the Reissner equations for large deflections.

The solution to the problem of a clamped circular plate loaded with a concentrated load on a central rigid inclusion was obtained and agreed with the solution of von Karman's equations for moderate deflections.

Perhaps the most important result is that of finding the order of magnitude of the limiting value of deflection that would be allowed

under the assumption of plane stress for this particular problem. It is shown that when the deflection approaches the order of magnitude of the radius, the boundary layer approaches the order of magnitude of the thickness and thus a first approximation theory is no longer valid.

Two membrane problems are also solved. The first is that of a circular membrane deformed by a load which acts normal to the plane of a central rigid inclusion. A closed form solution is obtained for this problem when Poisson's ratio is equal to 1/3. An approximate solution is obtained for any value of Poisson's ratio for the case where the deflections are very large. The second problem is the same as the first with the addition of a small torque about a normal to the rigid inclusion. An approximate solution is obtained to this problem.