RECOVERY OF INTERBLOCK INFORMATION

by

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1. INTRODUCTION

The problem of the recovery of inter-block information in incomplete block designs was recognized by Yates (10) in 1939. The idea is to combine the two independent intra- and inter-block estimates in order to increase the accuracy of our estimation of the treatment effects, and consequently to recover some of the efficiency that was lost by reducing the number of plots in a block from v, the number of treatments, to some number $k < v$.

The best linear combination of the intra-block and inter-block estimates is:

$$\frac{\text{Intra-variance} \times \text{Inter-estimate} + \text{Inter-variance} \times \text{Intra-estimate}}{\text{Intra-variance} + \text{Inter-variance}} .$$

However this combined estimate is merely theoretical since there is, in practice, no knowledge about the exact inter- and intra-variances. A reasonable way of overcoming this difficulty is to use a random weight which can be computed from the data of our experiment.

Yates suggests in (10) a rather complicated expression for the combined estimate in the B.I.B. designs. Later in 1960, V. Seshadri (8) has shown that Yates' combined estimate is unbiased and that it is uniformly better (better in the sense of less variance) than the intra-block estimator only when $\frac{\sigma_b^2}{\sigma^2}$ is smaller than $\frac{r}{\lambda v}$. Although no specific results were available about the case when $\frac{\sigma_b^2}{\sigma^2} > \frac{r}{\lambda v}$, nevertheless Seshadri's work indicates clearly that Yates' combined estimator is not uniformly better than the intra-block estimate when $\frac{\sigma_b^2}{\sigma^2} > \frac{r}{\lambda v}$.

In (7) Rao, C.R., has established what we shall call the conventional method of combining estimates in any incomplete block design. This method was adopted by Bose, R.C., in (1) and is the method pursued in practice almost everywhere.

Graybill in (4) suggests a combined estimator which is uniformly better than the intra-estimate provided severe restrictions are placed on the size of the experiment.

Seshadri, V., in (9) combines two estimates x and u in the balanced case in such a way that the combined estimate is uniformly better than both x and u provided that $v > 5$. It should be noted that he misnames x and u the inter- and intra-treatment estimates, respectively; in fact, his x and u are special linear functions of the observations as they were previously defined in (3) by Graybill and Weeks. This article, however, is suggestive and indicates that something can be done which permits an effective utilization of the idea of the recovery of inter-block information even when v is small.

In general, one can say that there is, so far, no practical solution to the problem without severe restrictions on the size of the experiment, and no solution at all for a clear and precise answer to the question of how much is recovered. In fact, regardless of how large the experiment is, the experimenter applying the methodology available to him now cannot be sure that he is really improving the accuracy of his estimation.

In this dissertation, after giving a brief critique of the conventional method of recovering the inter-block information, a practical solution which avoids the handicaps of the conventional method in B.I.B. designs will be discussed. This practical solution is extended to the P.B.I.B. designs with two associate classes. Finally, an exhaustive enumeration of the amount of recovery achieved or a lower bound of it is given.

2. THE CONVENTIONAL METHOD OF COMBINING ESTIMATES IN INCOMPLETE BLOCK DESIGNS

The most widely used method is that of Rao (7), where the intra- and inter-normal equations are combined in the same way that two independent estimates are usually combined, that is by first weighing inversely by the variance of each and then obtaining a least squares joint solution. Bose (1) gives the resulting combined intra- and inter-block estimate T_i of treatment t_i as

$$r\{w'+w(k-1)\}T_i = (k-d_2)P_i + (d_1-d_2)S_1(P_i) \quad , \quad (2.1)$$

where

$$w = \frac{1}{\sigma^2} \quad , \quad (2.2)$$

$$w' = \frac{1}{\sigma^2+k\sigma_b^2} \quad , \quad (2.3)$$

$$P_i = w\mathbf{Q}_i + w'\mathbf{Q}_i' \quad , \quad (2.4)$$

$\mathbf{Q}_i$ and $\mathbf{Q}_i'$ are the adjusted yields in the intra- and inter-analysis respectively,

$$d_j = \frac{c_j\Delta+r\lambda_j\mathbf{W}}{\Delta+rH\mathbf{W}+r^2\mathbf{W}^2} \quad , \qquad (j = 1,2) \qquad (2.5)$$

and,

$$\mathbf{W} = \frac{w'}{w-w'} \quad . \quad (2.6)$$

r, k, λ_1, λ_2, c_1, c_2, Δ, and H are defined in reference (1), and they will be defined explicitly in the sequel.

In practice, estimates for σ^2, σ_b^2 are usually obtained from the analysis of variance table, namely:

$$\hat{\sigma}^2 = S_e^2$$

and

$$\hat{\sigma}_b^2 = \frac{b-1}{N-v}(S_b^2 - S_e^2) \quad ,$$

where S_e^2 is the error mean square and S_b^2 is the blocks adjusted mean square. Since $\hat{\sigma}^2$ and $\hat{\sigma}_b^2$ obtained from the analysis of variance table are consistent estimates for σ^2 and σ_b^2, these estimates would be good enough to represent the unknown parameters σ^2 and σ_b^2 if estimated from a large experiment.

The questions now arise: (a) How reliable is the substitution of these estimates into a complicated expression as (1)? (b) If $\hat{\sigma}$ and $\hat{\sigma}_b^2$ are estimated from a large experiment, do the estimates $\hat{w} = \frac{1}{\hat{\sigma}^2}$ and $\hat{w}' = \frac{1}{\hat{\sigma}+k\hat{\sigma}_b^2}$ possess the same reliability? (c) Are $\hat{W} = \frac{\hat{w}}{\hat{w}-\hat{w}'}$ and $\hat{d}_j = \frac{c_j\Delta + r\lambda_j\hat{W}}{\Delta + rH\hat{W} + r^2\hat{W}^2}$ $(j=1,2)$, consistent estimators? One way of answering these questions is to find the variance of T_i after substituting in (1), w, w', d_1, d_2 by their estimates. A casual glance at this expression is enough to show that this is not mathematically feasible. It should also be noted that by taking $P_i = w\,Q_i + w'Q_i'$, a new source of sampling variation is being introduced into the adjusted yield, and this is of special significance for small v.

To show that at least, theoretically, the combined estimator obtained by the Rao method is better than the best linear combination of the intra-block and inter-block estimates, we will compare the theoretical variances of each.

The theoretical variance of T_i will be shown later to be:

$$V(T_i) = \frac{k(v-1)-n_1d_1-n_2d_2}{vr[w(k-1)+w']} \quad . \tag{2.7}$$

Letting $[w(k-1)+w'] = \eta$ and substituting (2.5) into (2.7), one obtains

$$V(T_i) = \frac{k(v-1)}{vr\eta} - \frac{n_1}{vr\eta}\,\frac{c_1\Delta+r\lambda_1W}{\Delta+rHW+r^2W^2} - \frac{n_2}{vr\eta}\,\frac{c_2\Delta+r\lambda_2W}{\Delta+rHW+r^2W^2} \quad . \tag{2.8}$$

By substituting (2.6) into (2.8) and simplifying

$$V(T_i) = \frac{k(v-1)}{vr\eta} - \Big\{\frac{n_1}{vr\eta}\,\frac{c_1\Delta(w-w')^2+r\lambda_1w'(w-w')}{\Delta(w-w')^2+rHw'(w-w')+r^2w'^2} + \frac{n_2}{vr\eta}\,\frac{c_2\Delta(w-w')^2+r\lambda_2w'(w-w')}{\Delta(w-w')^2+rHw'(w-w')+r^2w'^2}\Big\} \quad . \tag{2.9}$$

Now by collecting like terms, (2.9) may be written as

$$V(T_i) = \frac{k(v-1)}{vr\eta} - \frac{(n_1c_1+n_2c_2)\Delta(w-w')^2+rw'(w-w')(n_1\lambda_1+n_2\lambda_2)}{vr\eta[\Delta(w-w')^2+rHw'(w-w')+r^2w'^2]} . \tag{2.10}$$

Then by letting $(n_1c_1+n_2c_2) = \varphi$, $u = \frac{w'}{w}$, and since $(n_1\lambda_1+n_2\lambda_2) = r(k-1) = a$, (2.10) may be written as

$$V(T_i) = \frac{k(v-1)}{vr\eta} - \frac{\varphi\Delta(1-u)^2+rau(1-u)}{vr\eta[\Delta(1-u)^2+rHu(1-u)+r^2u^2]} . \tag{2.11}$$

Since

$$u = \frac{w'}{w} = \frac{\sigma^2}{\sigma^2+k\sigma_b^2} = \frac{1}{1+k\frac{\sigma_b^2}{\sigma^2}} = \frac{1}{1+kR} ,$$

the following inequalities are obtained

$$1 < R < \infty \quad \text{and} \quad 0 < u < \frac{1}{1+k} .$$

Now by substituting for η, after some manipulation (2.11) becomes

$$vwV(T_i) = \frac{k(v-1)}{r(k-1+u)} - \frac{(\varphi\Delta-ra)u^2-(2\varphi\Delta-ra)u+\varphi\Delta}{r(k-1+u)[(\Delta+r^2-rH)u^2+(rH-2\Delta)u+\Delta]} ,$$

$$= \frac{k(v-1)[\Delta+r^2-rH)u^2+(rH-2\Delta)u+\Delta]-(\varphi\Delta-ra)u^2+(2\varphi\Delta-ra)u-\varphi\Delta}{r(k-1+u)[(\Delta+r^2-rH)u^2+(rH-2\Delta)u+\Delta]} . \tag{2.12}$$

Finally, letting $A = \Delta+r^2-rH$,

$$vwV(T_i) = \frac{[Ak(v-1)-\varphi\Delta+ra]u^2+[k(v-1)(rH-2\Delta+2\varphi\Delta-ra]u+k(v-1)\Delta-\varphi\Delta}{r(k-1+u)[Au^2+(rH-2A)u+\Delta]} . \tag{2.13}$$

Now if the intra-estimate $\hat{t}_i$ and the inter-estimate $\hat{t}'_i$ of t_i are combined, the best linear theoretical estimate is

$$\hat{\tau}_i = \frac{\hat{t}_i V(\hat{t}'_i) + \hat{t}'_i V(\hat{t}_i)}{V(\hat{t}'_i) + V(\hat{t}_i)} \tag{2.14}$$

and

$$V(\hat{\tau}_i) = \frac{V(\hat{t}_i) V(\hat{t}'_i)}{V(\hat{t}'_i) + V(\hat{t}_i)} \quad . \tag{2.15}$$

It will be shown later that:

$$V(\hat{t}_i) = \frac{k(v-1) - n_1 c_1 - n_2 c_2}{avw} \quad , \tag{2.16}$$

$$V(\hat{t}'_i) = \frac{k(v-1) - n_1 c_1 - n_2 c_2}{rvw'} \quad , \tag{2.17}$$

where

$$c'_j = \frac{\Delta c_j - r\lambda_j}{\Delta - rH + r^2} = \frac{\Delta c_j - r\lambda_j}{A} \quad . \tag{2.18}$$

Substituting (2.16) and (2.17) into (2.15) one obtains

$$V(\hat{\tau}_i) = \frac{\left[\dfrac{k(v-1) - n_1 c'_1 - n_2 c'_2}{rvw'}\right] \left[\dfrac{k(v-1) - n_1 c_1 - n_2 c_2}{avw}\right]}{\left[\dfrac{k(v-1) - n_1 c'_1 - n_2 c'_2}{rvw'}\right] + \left[\dfrac{k(v-1) - n_1 c_1 - n_2 c_2}{avw}\right]} \quad . \tag{2.19}$$

Now from (2.18)

$$n_1c_1' + n_2c_2' = n_1 \frac{c_1\Delta - r\lambda_1}{A} + n_2 \frac{c_2\Delta - r\lambda_2}{A} = \frac{\Delta}{A}(n_1c_1+n_2c_2) - \frac{r}{A}(n_1\lambda_1+n_2\lambda_2)$$

$$= \frac{\Delta\varphi - ra}{A} \quad . \tag{2.20}$$

Substituting (2.20) into (2.19),

$$V(\hat{\tau}_i) = \frac{[k(v-1) - \frac{\Delta\varphi - ra}{A}][k(v-1)-\varphi]}{avw[k(v-1) - \frac{\Delta\varphi ra}{A}]+rvw'[k(v-1)-\varphi]} ,$$

$$= \frac{[Ak(v-1)-\Delta\varphi+ra][k(v-1)-\varphi]}{avw[Ak(v-1)-\Delta\varphi+ra]+rvw'A[k(v-1)-\varphi]} \quad . \tag{2.21}$$

Then

$$vwV(\hat{\tau}_i) = \frac{[Ak(v-1)-\Delta\varphi+ra][k(v-1)-\varphi]}{rA[k(v-1)-\varphi]u+a[Ak(v-1)-\Delta\varphi+ra]} \quad . \tag{2.22}$$

A numerical study of the difference $y = vw[var(T_i) - Var(\hat{\tau}_i)]$ has been conducted for the 68 Regular Group divisible experiments listed in reference (1). The range of u from 0 to $\frac{1}{1+k}$ is divided into ten equal intervals and eleven numerical values have been computed for the difference y, and for the percentage difference $S = y/[v \cdot V(\hat{\tau}_i)]$.

The results show that y is always negative, and the difference in absolute value increases as the ratio

$\frac{\sigma_b^2}{\sigma^2}$ gets closer to one. The difference is zero when $\frac{\sigma_b^2}{\sigma^2} \to \infty$. Although the percentage difference is very small, this shows that the Rao method is theoretically better than obtaining, first, a separate intra- and inter-estimate, and then combining. The results for $S = \frac{1}{\sigma^2}$[percentage gain due to Rao method] appear in Table 1.

In brief, one can say that, theoretically, the Rao method has some desirable properties. It constitutes the best way, known so far, for getting a linear combined estimate; it is, relatively, simple to apply; and it is very general. But, the formula it produces contains a rather complicated function of unknown parameters which have to be estimated. Thus, the desirable theoretical properties of Rao's method are not likely to stand up in practice. Many valid questions arise about its reliability in practice, and it appears that the only amnesty it has is that it is very difficult to show mathematically if it is good or not.

3. THEOREM 1

In this section a theorem will be proved concerning a new method of combining two independent unbiased estimates. It will also be shown how much improvement one can hope to get by combining two independent estimates. Finally, it will be shown how much of the improvement this new method utilizes.

Consider the n independent parameters $\tau_1, \tau_2, \ldots, \tau_n$ and suppose that for each τ_i there exist two independent unbiased estimates U_i and X_i, where $U_i \cap N(\tau_i, \rho\sigma^2)$ and $X_i \cap N(\tau_i, \rho'\sigma'^2)$. Suppose also that independently of the X_i's and U_i's, there exist two unbiased estimates s^2 and s'^2 for σ^2, σ'^2 respectively, where $s^2 \cap \chi^2 \frac{\sigma^2}{f}$ and $s'^2 \cap \chi^2 \frac{\sigma'^2}{f'}$, then:

(1) the combined estimate:

$$\hat{\tau}_i = U_i + \frac{\theta(m-2)s^2}{\sum_{j=1}^{m} (X_j - U_j)^2} (X_i - U_i) , \quad m \leq n , \qquad (3.1)$$

is unbiased and uniformly better than U_i, and

(2) the combined estimate:

$$\hat{\tau}_i = X_i + \frac{\theta'(m-2)s'^2}{\sum_{j=1}^{m}(U_j-X_j)^2}(U_i-X_i)\ , \qquad m \leq n\ , \tag{3.2}$$

is unbiased and uniformly better than X_i. θ and θ' are constants.

Proof: Since $U_1, U_2, \ldots, U_n;\ X_1, X_2, \ldots, X_n;\ s^2, s'^2$ are independent, and $E(U_i) = E(X_i) = \tau_i$, $i = 1, 2, \ldots, n$,

$$E(\hat{\tau}_i) = \tau_i + \theta(m-2)E(s^2)\cdot E\,\frac{X_i-U_i}{\sum_{j=1}^{m}(X_j-U_j)^2}\quad . \tag{3.3}$$

Let $X_i-U_i = z_i$, then $z_i \frown N(0,\ V(U_i)+V(X_i))$. The joint distribution of $z_1, z_2, \ldots, z_m$ is of the form:

$$f(z_1),\ f(z_2),\ \ldots,\ f(z_m) = A\ e^{-B\sum_{j=1}^{m} z_j^2}\ ,\ -\infty < z_i < +\infty, \tag{3.4}$$

an even function of $z_1, z_2, \ldots, z_m$; the expectation, therefore, of any odd function of $z_1, z_2, \ldots, z_m$ is zero. If z_i belongs to the subset $z_1, z_2, \ldots, z_m$, then

$$E\,\frac{z_i}{\sum_{j=1}^{m} z_j^2} = 0 \quad .$$

Hence $E(\hat{\tau}_i) = \tau_i$ and $\hat{\tau}_i$ is the unbiased estimate for τ_i.

Now

$$V(\hat{\tau}_i) = E(\hat{\tau}_i - \tau_i)^2 = E\left[(U_i - \tau_i) + \frac{\theta(m-2)s^2}{\sum_{j=1}^{m}(X_j - U_j)^2}(X_i - U_i)\right]^2 \tag{3.5}$$

$$= E(U_i - \tau_i)^2 + \theta^2(m-2)^2 \cdot E\left[\frac{s^2(X_i - U_i)}{\sum_{j=1}^{m}(X_j - U_j)^2}\right]^2 + 2\theta(m-2) \cdot E\left[\frac{s^2(X_i - U_i)(U_i - \tau_i)}{\sum_{j=1}^{m}(U_j - X_j)^2}\right]$$

$$= V(U_i) + \theta^2(m-2)^2 \cdot E(s^4) \cdot E\frac{z_i^2}{(\sum_{j=1}^{m} z_j^2)^2} + 2\theta(m-2) \cdot E(s^2) \cdot E\frac{z_i(U_i - \tau_i)}{\sum_{j=1}^{m} z_j^2},$$

where

$$E(s^4) = \left(\frac{\sigma^2}{f}\right)^2 E(\chi_f^2)^2 = \frac{\sigma^4 f(f+2)}{f^2} = \frac{\sigma^4(f+2)}{f}, \tag{3.6}$$

and

$$E\frac{z_i^2}{(\sum_{j=1}^{m} z_j^2)^2} = \frac{1}{m} E\left[\frac{z_1^2 + z_2^2 + \ldots + z_m^2}{(\sum_{j=1}^{m} z_j^2)^2}\right], \tag{3.7}$$

because of the symmetry of the z_i's; that is,

$$E\frac{z_i^2}{(\sum_{j=1}^{m} z_j^2)^2} = E\frac{z_k^2}{(\sum_{j=1}^{m} z_j^2)^2}, \quad k = 1, 2, \ldots, m.$$

Hence:

$$E \frac{z_i^2}{\left(\sum_{j=1}^{m} z_j^2\right)^2} = \frac{1}{m} \cdot E \frac{\sum_{j=1}^{m} z_j^2}{\left(\sum_{j=1}^{m} z_j^2\right)^2} = \frac{1}{m} \cdot E \frac{1}{\sum_{j=1}^{m} z_j^2} ,$$

$$= \frac{1}{m} \cdot E \frac{1}{(V(U_i)+V(X_i))\chi_m^2} = \frac{1}{m(m-2)(V(U_i)+V(X_i))} . \quad (3.8)$$

Then

$$E \frac{z_i(U_i-\tau_i)}{\sum_{j=1}^{m} z_j^2} = E_{z_1,z_2,\ldots,z_m}\left[E_{U_i|z_1,z_2,\ldots,z_m} \frac{z_i(U_i-\tau_i)}{\sum_{j=1}^{m} z_j^2}\right] ,$$

$$= E_{z_1,z_2,\ldots,z_m}\left[\frac{z_i}{\sum_{j=1}^{m} z_j^2} \cdot E_{U_i|z_1,z_2,\ldots,z_m}(U_i-\tau_i)\right] .$$

(3.9)

Consider now the multivariate normal vector $[U_i,\ z_1,z_2,\ldots,z_i,\ldots,z_m]$; its mean vector is $[\tau_i,\ 0,\ 0,\ \ldots,\ 0]$ and the variance covariance matrix is:

$$
\underset{(m+1)\times(m+1)}{\Sigma} = \left[\begin{array}{c|cccccccc}
V(U_i) & , & 0, & 0, & \ldots, & -V(U_i), & 0, & \ldots & 0 \\
\hline
0, & V(U_i)+V(X_i), & 0, & \ldots, & & 0, & 0, & \ldots & 0 \\
\hdashline
-V(U_i) & , & 0, & \ldots, & & V(U_i)+V(X_i), & 0 & \ldots & 0 \\
\hdashline
0, & 0 & & \ldots & & & , & & V(U_i)+V(X_i)
\end{array}\right] = \left[\begin{array}{c|c}
\Sigma_{11} & \Sigma_{12} \\
\hline
\Sigma_{21} & \Sigma_{22}
\end{array}\right]
$$

Since the conditional mean of $U_i | z_1, z_2, \ldots, z_m$ is:

$$\tau_i + \Sigma_{12}\Sigma_{22}^{-1}\underline{Z} = \tau_i + (0,0,\ldots,-V(U_i),0,\ldots,0)[(V(U_i)+V(X_i))I_m]^{-1}\cdot\underline{Z}$$

$$= \tau_i + (0,0,\ldots,\frac{-V(U_i)}{V(U_i)+V(X_i)},0,\ldots,0)\underline{Z} = \tau_i - \frac{V(U_i)\cdot z_i}{V(U_i)+V(X_i)} ,$$

then

$$E_{U_i|z_1,z_2,\ldots,z_m}(U_i-\tau_i) = -\frac{V(U_i)\cdot z_i}{V(U_i)+V(X_i)} . \qquad (3.10)$$

Substituting (3.10) into (3.9) one obtains:

$$E\,\frac{z_i(U_i-\tau_i)}{\sum_{j=1}^{m} z_j^2} = E_{z_1\ldots z_m}\left[\frac{-V(U_i)}{V(U_i)+V(X_i)}\,\frac{z_i^2}{\sum_{j=1}^{m} z_j^2}\right] ,$$

$$= \frac{-V(U_i)}{V(U_i)+V(X_i)}\,E_{z_1\ldots z_m}\frac{z_i^2}{\sum_{j=1}^{m} z_j^2} = \frac{-V(U_i)}{m(V(U_i)+V(X_i))}\,E\left[\frac{z_1^2+z_2^2+\ldots+z_m^2}{\sum_{j=1}^{m} z_j^2}\right] ,$$

$$= \frac{-V(U_i)}{m(V(U_i)+V(X_i))} , \qquad (3.11)$$

making use of the symmetry of $z_1, z_2, \ldots, z_n$. Substituting (3.6), (3.8), and (3.11) into (3.5), one obtains

$$V(\hat{\tau}_i) = V(U_i) + \frac{\theta^2\sigma^4(m-2)^2(f+2)}{fm(m-2)(V(U_i)+V(X_i))} - 2\,\frac{\theta\sigma^2 V(U_i)(m-2)}{m[V(U_i)+V(X_i)]}\,,$$

$$= V(U_i) + \frac{\theta^2(m-2)(f+2)[V(U_i)]^2}{\rho^2 fm[V(U_i) + V(X_i)]} - 2\,\frac{\theta(m-2)[V(U_i)]^2}{\rho m[V(U_i)+V(X_i)]}\,,$$

$$= V(U_i) + \frac{\theta(m-2)[V(U_i)]^2}{m\rho[V(U_i)+V(X_i)]}\left[\frac{\theta(f+2)}{\rho f} - 2\right] .$$

The value of θ which makes the second term above the most negative is $\theta=\frac{\rho f}{f+2}$, or $\frac{\theta}{\rho}=\frac{f}{f+2}$, substituting for $\frac{\theta}{\rho}$ one obtains:

$$V(\hat{\tau}_i) = V(U_i)-(1-\tfrac{2}{m})(1-\tfrac{2}{f+2})\frac{[V(U_i)]^2}{V(U_i)+V(X_i)} \le V(U_i), \quad \text{provided } m\geq 2. \tag{3.12}$$

The proof of the second part follows similarly.

3.1 Remarks

(a) The best linear unbiased combined estimate $\hat{\tau}_i$ of τ_i is:

$$\hat{\tau}_i = \frac{U_i V(X_i)+X_i V(U_i)}{V(U_i)+V(X_i)}$$

with the minimum variance:

$$V(\hat{\tau}_i) = \frac{V(U_i)\cdot V(X_i)}{V(U_i)+V(X_i)} = V(U_i) - \frac{[V(U_i)]^2}{V(U_i)+V(X_i)} . \tag{3.1.1}$$

This shows that the most improvement one can hope to get by combining the two independent estimates of τ_i, namely U_i and X_i is the quantity $\frac{[V(U_i)]^2}{V(U_i)+V(X_i)}$. It should be noticed that the above suggested estimate recovers $\frac{(m-2)f}{m(f+2)}$ of the most improvement possible $\frac{[V(U_i)]^2}{V(U_i)+V(X_i)}$. The fraction

$\frac{(m-2)f}{m(f+2)} = (1-\frac{2}{m})(1-\frac{2}{f+2}) \to 1$ when m and f are large.

It is obvious now that the larger m is, the better the new estimate will be. This means that we should take m = n whenever it is possible.

(b) The variance of the combined estimate τ_i' is:

$$V(\tau_i') = V(X_i) - \frac{(m-2)f'[V(X_i)]^2}{(f'+2)m[V(U_i)+V(X_i)]} . \qquad (3.1.2)$$

Subtracting (3.1.2) from (3.12),

$$V(\tau_i) - V(\tau_i') = [V(U_i)-V(X_i)] + \frac{m-2}{m[V(U_i)+V(X_i)]} \left\{ \frac{f'[V(X_i)]^2}{f'+2} - \frac{f[V(U_i)]^2}{f+2} \right\} ,.$$

Simplifying the R.H.S., the sign of $V(\hat{\tau}_i) - V(\hat{\tau}_i')$ is the same as the sign of:

$$m(f'+2)(f+2)[V^2(U_i)-V^2(X_i)]+(m-2)[f'(f+2)V^2(X_i)-f(f'+2)V^2(U_i)]$$

$$= 2V^2(U_i)[mf'+ff'+2f+2m]-2V^2(X_i)[mf+ff'+2f'+2m]$$

$$= 2(m+f)(f'+2)V^2(U_i)-2(m+f')(f+2)V^2(X_i) \quad .$$

Thus:

$$V(\hat{\tau}_i) > V(\hat{\tau}_i'), \quad \text{if } \frac{V^2(U_i)}{V^2(X_i)} > \frac{(f'+m)(f+2)}{(f+m)(f'+2)} \quad \text{and}$$

$$V(\hat{\tau}_i) < V(\hat{\tau}_i'), \quad \text{if } \frac{V^2(U_i)}{V^2(X_i)} < \frac{(f'+m)(f+2)}{(f+m)(f'+2)} \quad .$$

3.2 Use of Theorem 1

This theorem could be utilized for obtaining better estimates for a number n of independent comparisons between the treatments in any incomplete block design, provided that each comparison has the same variance, and the number of such comparisons is more than 2.

It could also be used to combine the estimates of the same treatment from two independent similar experiments, provided that the interaction over time and location is negligible.

Also, if one is in doubt about the homogeneity assumption when a randomized block design has been utilized, the blocks could be divided into two homogeneous groups, and then obtain estimates for the v-1 comparisons among treatments within each group. Then combine these estimates by theorem 1. The gain in the accuracy of this estimate increases as the heterogeneity between the two groups increases. Again, the block by treatment interaction must be negligible.

4. LEMMA 1

If two independent random variables X and Y have density functions:

$$f(X) = \frac{1}{\Gamma(\alpha_1+1)\beta_1^{\alpha_1+1}} x^{\alpha_1} e^{-x/\beta_1} dx \qquad x > 0,\ \beta_1 > 0,$$

$$f(Y) = \frac{1}{\Gamma(\alpha_2+1)\beta_2^{\alpha_2+1}} y^{\alpha_2} e^{-y/\beta_2} dy \qquad y > 0,\ \beta_2 > 0. \tag{4.1}$$

Then:

$$E\left(\frac{1}{X+Y}\right)^r = \frac{\Gamma(\alpha_1+\alpha_2+2-r)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} \int_0^1 \frac{u^{\alpha_1}(1-u)^{\alpha_2}}{[\beta_1 u+\beta_2(1-u)]^r} du \quad . \tag{4.2}$$

<u>Proof</u>:

$$E\left(\frac{1}{X+Y}\right)^r = Ct. \int_0^\infty\int_0^\infty \frac{1}{(x+y)^r} x^{\alpha_1} e^{-x/\beta_1} y^{\alpha_2} e^{-y/\beta_2} dxdy \quad , \tag{4.3}$$

where $\qquad Ct. = \dfrac{1}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\beta_1^{\alpha_1+1}\beta_2^{\alpha_2+1}} \quad .$

Let $\dfrac{x}{\beta_1} = z_1$, $\dfrac{y}{\beta_2} = z_2$, $dx = \beta_1 dz_1$, $dy = \beta_2 dz_2$,

then

$$E\left(\frac{1}{x+y}\right)^r = \text{Ct.} \int_0^\infty \int_0^\infty \frac{1}{(\beta_1 z_1+\beta_2 z_2)^r} z_1^{\alpha_1} e^{-z_1} z_2^{\alpha_2} e^{-z_2} \beta_1^{\alpha_1+1} \beta_2^{\alpha_2+1} \cdot dz_1 dz_2 \ . \quad (4.4)$$

Now let $u = \dfrac{z_1}{z_1+z_2}$, $z_2 = z_2$, then:

$$E\left(\frac{1}{x+y}\right)^r = \text{Ct.} \ \beta_1^{\alpha_1+1} \beta_2^{\alpha_2+1} \int_0^\infty \int_0^1 \frac{1}{\left(\frac{\beta_1 u z_2}{1-u} + \beta_2 z_2\right)^r} \left(\frac{u z_2}{1-u}\right)^{\alpha_1} e^{-\frac{u z_2}{1-u}} \cdot z_2^{\alpha_2} e^{-z_2} \frac{z_2}{(1-u)^2} \, du dz_2 \quad ,$$

$$= \text{Ct.} \ \beta_1^{\alpha_1+1} \beta_2^{\alpha_2+1} \int_0^\infty \int_0^1 \frac{(1-u)^r}{[\beta_1 u z_2+\beta_2 z_2(1-u)]^r} \left(\frac{u}{1-u}\right)^{\alpha_1} z_2^{\alpha_1+\alpha_2+1} \cdot e^{-z_2} e^{-\frac{u z_2}{1-u}} \frac{1}{(1-u)^2} \, du dz_2 \quad . \quad (4.5)$$

Let $z_2 = (1-u)v$, $u = u$, then:

$$E\left(\frac{1}{x+y}\right)^r = \text{Ct. } \beta_1^{\alpha_1+1} \beta_2^{\alpha_2+1} \int_0^\infty\int_0^1 \frac{(1-u)^r}{[\beta_1 u(1-u)v+\beta_2 v(1-u)^2]^r}$$

$$\cdot \left(\frac{u}{1-u}\right)^{\alpha_1} (1-u)^{\alpha_1+\alpha_2+1} v^{\alpha_1+\alpha_2+1} e^{-v(1-u)} e^{-uv} \frac{du}{(1-u)^2}(1-u)\,dv,$$

$$= \text{Ct. } \beta_1^{\alpha_1+1} \beta_2^{\alpha_2+1} \int_0^\infty\int_0^1 \frac{(1-u)^r}{v^r(1-u)^r[\beta_1 u+\beta_2(1-u)]^r} u^{\alpha_1} (1-u)^{\alpha_2}$$

$$\cdot v^{\alpha_1+\alpha_2+1} e^{-v} \, du \, dv \quad ,$$

$$= \text{Ct. } \beta_1^{\alpha_1+1} \beta_2^{\alpha_2+1} \int_0^\infty v^{\alpha_1+\alpha_2-r+1} e^{-v} \, dv \int_0^1 \frac{1}{[\beta_1 u+\beta_2(1-u)]^r}$$

$$\cdot u^{\alpha_1} (1-u)^{\alpha_2} \, du \quad ,$$

$$= \text{Ct. } \beta_1^{\alpha_1+1} \beta_2^{\alpha_2+1} \Gamma(\alpha_1+\alpha_2+2-r)\int_0^1 \frac{1}{[\beta_1 u+\beta_2(1-u)]^r} u^{\alpha_1} (1-u)^{\alpha_2} \, du \; ,$$

$$= \frac{\Gamma(\alpha_1+\alpha_2+2-r)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} \int_0^1 \frac{u^{\alpha_1} (1-u)^{\alpha_2}}{[\beta_1 u+\beta_2(1-u)]^r} \, du \quad . \tag{4.6}$$

4.1 <u>Corollary</u>

Since $\beta_2 > 0$, $0 \leq u \leq 1$, and assuming $\beta_1 > \beta_2$, then:

$$\beta_1 u < \beta_1 u - \beta_2 u + \beta_2 < \beta_1$$

or (4.1.1)

$$\beta_2 < \beta_1 u - \beta_2 u + \beta_2 < \beta_1 \quad .$$

Then

$$\frac{\Gamma(\alpha_1+\alpha_2+2-r)\beta(\alpha_1+1,\alpha_2+1)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\beta_1^r} < E(\frac{1}{X+Y})^r < \frac{\Gamma(\alpha_1+\alpha_2+2-r)\beta(\alpha_1+1-r,\alpha_2+1)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\beta_1^r}$$

and

$$\frac{\Gamma(\alpha_1+\alpha_2+2-r)\beta(\alpha_1+1,\alpha_2+1)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\beta_1^r} < E(\frac{1}{X+Y})^r < \frac{\Gamma(\alpha_1+\alpha_2+2-r)\beta(\alpha_1+1,\alpha_2+1)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\beta_2^r} ,$$

which can be written respectively as:

$$\frac{\Gamma(\alpha_1+\alpha_2+2-r)}{\Gamma(\alpha_1+\alpha_2+2)\beta_1^r} < E(\frac{1}{X+Y})^r < \frac{\Gamma(\alpha_1+1-r)}{\beta_1^r\ \Gamma(\alpha_1+1)} , \qquad (4.1.2)$$

and

$$\frac{\Gamma(\alpha_1+\alpha_2+2-r)}{\Gamma(\alpha_1+\alpha_2+2)\beta_1^r} < E(\frac{1}{X+Y})^r < \frac{\Gamma(\alpha_1+\alpha_2+2-r)}{\Gamma(\alpha_1+\alpha_2+2)\beta_2^r} . \qquad (4.1.3)$$

5. LEMMA 2

Let z_1, z_2, ..., z_n have the joint multivariate normal distribution, with mean vector $\underline{\mu}' = \underline{0}$ and variance covariance matrix Σ with the diagonal element (a) and the element (b) otherwise. Then $\sum_{i=1}^{n} z_i^2$ can be considered as equivalent to the sum of two independent, Gamma distributed, random variables.

<u>Proof</u>: The characteristic function of $\sum_{i=1}^{n} z_i^2$ is:

$$\Phi_\theta(\Sigma z_i^2) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{+\infty} \frac{|\Sigma|^{-\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} e^{\theta \underline{Z}'\underline{Z}} e^{-\frac{1}{2}\underline{Z}'\Sigma^{-1}\underline{Z}} dz_1 \ldots dz_n ,$$

$$= \frac{|\Sigma|^{-\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-\frac{1}{2}[\underline{Z}'\Sigma^{-1}\underline{Z} - 2\theta \underline{Z}'\underline{Z}]} d\underline{Z} ,$$

$$= \frac{|\Sigma|^{-\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-\frac{1}{2}\underline{Z}'[\Sigma^{-1} - 2\theta I]\underline{Z}} d\underline{Z} , \tag{5.1}$$

$$= \frac{(2\pi)^{\frac{n}{2}}}{|\Sigma^{-1} - 2\theta I|^{\frac{1}{2}}} \cdot \frac{|\Sigma|^{-\frac{1}{2}}}{(2\pi)^{\frac{n}{2}}} = \frac{|\Sigma^{-1}|^{\frac{1}{2}}}{|\Sigma^{-1} - 2\theta I|^{\frac{1}{2}}} = \frac{1}{|I - 2\theta\Sigma|^{\frac{1}{2}}} .$$

Since the inverse of an (nxn) two element matrix, a along the diagonal and b otherwise is an (nxn) two element matrix, d along the diagonal and f off diagonal where:

$$d = \frac{-[(n-2)b+a]}{b^2(n-1)-a[(n-2)b+a]} \quad , \tag{5.2}$$

and

$$f = \frac{b}{b^2(n-1)-a[(n-2)b+a]} \quad . \tag{5.3}$$

It is known also that the determinant of such a matrix is:

$$\Delta = (a-b)^{n-1}(a-b+nb) \quad . \tag{5.4}$$

Substituting (5.4) into (5.1),

$$|I-2\theta\Sigma|^{\frac{1}{2}} = [1-2(a-b)\theta]^{\frac{n-1}{2}} [1-2(a+nb-b)\theta]^{\frac{1}{2}} \quad ,$$

and then

$$\Phi_\theta(\sum_{i=1}^{n} z_i^2) = \frac{1}{[1-2(a-b)\theta]^{\frac{n-1}{2}}} \frac{1}{[1-2(a+nb-b)\theta]^{\frac{1}{2}}} \quad .$$

This means that $\sum_{i=1}^{n} z_i^2$ can be considered as the sum of two independent random variables X and Y, where:

$$f(X) = \frac{1}{[2(a-b)]^{\frac{n-1}{2}}\,\Gamma(\frac{n-1}{2})} x^{\frac{n-3}{2}} e^{-x/2(a-b)} dx \quad , \tag{5.5}$$

$$x > 0, \quad a-b > 0 \quad ,$$

and

$$f(Y) = \frac{1}{2^{\frac{1}{2}}(a+nb-b)^{\frac{1}{2}}\Gamma(\frac{1}{2})} y^{-\frac{1}{2}} e^{-y/2(a+nb-b)} dy \quad , \qquad y > 0, \quad a+nb-b > 0 \quad . \tag{5.6}$$

5.1 <u>Corollary 1</u>

Using Lemma 2, one can write:

$$E(\frac{1}{\Sigma z_i^2}) = E(\frac{1}{X+Y}) = E[\frac{1}{x}(1+\frac{y}{x})^{-1}] < E\,\frac{1}{x} = \frac{1}{(n-3)(a-b)} \quad ,$$

i.e.,

$$E(\frac{1}{\Sigma z_i^2}) < \frac{1}{(n-3)(a-b)} \quad . \tag{5.7}$$

5.2 <u>Corollary 2</u>

Applying the results of Lemma 1,

$$\alpha_1 = \frac{n-3}{2} , \qquad \beta_1 = 2(a-b) ,$$
$$\alpha_2 = \frac{-1}{2} , \qquad \beta_2 = 2(a+nb-b) . \tag{5.2.1}$$

Substituting (5.2.1) into (4.6),

$$E(\frac{1}{\Sigma z_i^2})^r = E(\frac{1}{X+Y})^r = \frac{\Gamma(\frac{n}{2} - r)}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2})} \int_0^1 \frac{u^{\frac{n-3}{2}} (1-u)^{-\frac{1}{2}}}{[-2nbu+2(a+nb-b)]^r} du \quad . \tag{5.2.2}$$

6. SOME RELATIONS IN B.I.B. DESIGNS

Let $\hat{t}_i$ denote the intra-block estimate of the i-th treatment effect and let $\hat{t}_i'$ denote the corresponding inter-block estimate, then

$$V(\hat{t}_i) = V_1 = \frac{k(v-1)}{\lambda v^2}\sigma^2 \quad , \tag{6.1}$$

$$V(\hat{t}_i') = V_2 = \frac{k(v-1)}{v(r-\lambda)}\sigma'^2 \quad , \tag{6.2}$$

$$\text{cov}(\hat{t}_i, \hat{t}_j) = C_1 = \frac{-k}{\lambda v^2}\sigma^2 \qquad \text{for all } i \neq j \quad , \tag{6.3}$$

and

$$\text{cov}(\hat{t}_i', \hat{t}_j') = C_2 = \frac{-k}{v(r-\lambda)}\sigma'^2 \qquad \text{for all } i \neq j \quad . \tag{6.4}$$

The following relations hold:

$$C_1V_2 - V_1C_2 = 0 \quad , \tag{6.5}$$

$$V_1 + (v-1)C_1 = 0 \quad , \tag{6.6}$$

and

$$V_2 + (v-1)C_2 = 0 \quad . \tag{6.7}$$

Then

$$V - C = -\,vC \quad , \tag{6.8}$$

where $V = V_1 + V_2$, $C = C_1 + C_2$,

and

$$\frac{V_1 - C_1}{V_1} = \frac{v}{v - 1} \quad . \tag{6.9}$$

The intra-estimate of the i-th treatment is given in (6) as:

$$\hat{t}_i = \frac{Q_i}{rE} \quad , \tag{6.10}$$

where Q_i is the adjusted treatment total, and the efficiency,

$$E = \frac{(k-1)v}{k(v-1)} \quad . \tag{6.11}$$

The inter-estimate of the i-th treatment is given in (11) as:

$$\hat{t}'_i = \frac{Q'_i}{r(1-E)} \quad , \tag{6.12}$$

where

$$Q'_i = Y_{i.} - Q_i - ry_{..} \quad , \tag{6.13}$$

$Y_{i.}$ is the total yield of the i-th treatment, and $y_{..}$ is the overall mean.

7. THEOREM 2

In B.I.B. designs, the combined estimate:

$$\hat{\tau}_i = \hat{t}_i + \frac{k(v-4)s^2}{\lambda v(f+2)\sum_{i=1}^{v-1}(\hat{t}_i'-\hat{t}_i)^2}(\hat{t}_i'-\hat{t}_i) , \tag{7.1}$$

is unbiased for the i-th treatment effect t_i, and uniformly better than intra-block estimate provided $v > 4$, where s^2 and f are the error sum of squares and the error degrees of freedom, respectively.

<u>Proof</u>: Consider first the estimate:

$$\hat{\tau}_i = \hat{t}_i + \frac{\theta B s^2}{\sum_{j=1}^{n}(\hat{t}_j'-\hat{t}_j)^2}(\hat{t}_i'-\hat{t}_i) , \quad n < v , \tag{7.2}$$

where $\theta = \frac{k(v-1)}{\lambda v^2}$, the coefficient of σ^2 in $\mathrm{var}(\hat{t}_i)$, s^2 is the error mean square, and B is to be determined later.

Now

$$E(\hat{\tau}_i) = t_i + \theta B \sigma^2 E\left(\frac{z_i}{\sum_{j=1}^{n} z_j^2}\right) , \tag{7.3}$$

where $z_i = \hat{t}_i' - \hat{t}_i$. Assuming $\hat{t}_i \frown N(t_i, V_1)$, and $\hat{t}_i' \frown N(t_i, V_2)$ for $i = 1,2,\ldots,n$, the $z_i \frown N(0, V)$, where $V = V_1 + V_2$,

then

$$\mathrm{cov}(z_i,z_j) = E(z_iz_j) = E[(\hat{t}_i'-t_i)-(\hat{t}_i-t_i)][(\hat{t}_j'-t_j)-(\hat{t}_j-t_j)]$$

$$= E(\hat{t}_i'-t_i)(\hat{t}_j'-t_j) + E(\hat{t}_i-t_i)(\hat{t}_j-t_j) = \mathrm{cov}(\hat{t}_i',\hat{t}_j') + \mathrm{cov}(\hat{t}_i,\hat{t}_j)$$

$$= C_2 + C_1 = C \quad . \tag{7.4}$$

Now $z_1,z_2,\ldots,z_n$ have the multivariate normal distribution with mean vector $\underline{\mu} = \underline{0}$, and variance covariance matrix $\underset{n\times n}{\Sigma}$ with the element V along the diagonal and the element C otherwise. Now one can write the density function as:

$$f(z_1,z_2,\ldots,z_n) = \mathrm{Ct.}\ e^{-\frac{1}{2}\underline{Z}'\Sigma^{-1}\underline{Z}}\ d\underline{Z} \quad , \quad -\infty < z_i < +\infty \ .$$

This is an even function and the expectation of any odd function of z_i's in zero. Since $\frac{z_i}{\sum_{j=1}^{n} z_j^2}$ is an odd function, its expectation is zero. Going back to (7.3) one can write $E(\hat{\tau}_i) = t_i$. Hence

$$V(\hat{\tau}_i) = E[(\hat{t}_i-t_i) + \frac{\theta B s^2}{\sum_{j=1}^{n} z_j^2} z_i]^2 \quad , \tag{7.5}$$

$$= E(\hat{t}_i-t_i)^2 + \theta^2B^2E(s^4)\cdot E\left(\frac{z_i}{\sum_{j=1}^{n} z_j^2}\right)^2 + 2\theta B E(s^2)\cdot E\,\frac{z_i(\hat{t}_i-t_i)}{\sum_{j=1}^{n} z_j^2} \ .$$

Since

$$s^2 \frown \frac{\sigma^2}{f}\chi^2,\ \mathbf{E}(s^4) = \frac{\sigma^4}{f^2}\mathbf{E}(\chi^2)^2 = \frac{f(f+2)\sigma^4}{f^2} = \frac{f+2}{f}\sigma^4 \quad ,$$

(7.5) may be written as

$$V(\hat{\tau}_i) = V_1 + \frac{\theta^2 B^2(f+2)\sigma^4}{f}\mathbf{E}\frac{z_i^2}{(\sum_{j=1}^{n} z_j^2)^2} + 2\theta B\sigma^2\mathbf{E}\frac{z_i(\hat{t}_i - t_i)}{\sum_{j=1}^{n} z_j^2} \ . \tag{7.6}$$

By symmetry of the z_j's, we can write:

$$\mathbf{E}\frac{z_1^2}{z_1^2+z_2^2+\ldots+z_n^2} = \mathbf{E}\frac{z_2^2}{z_1^2+z_2^2+\ldots+z_n^2} = \ldots = \mathbf{E}\frac{z_n^2}{z_1^2+z_2^2+\ldots+z_n^2} \quad ,$$

i.e.,

$$\mathbf{E}\frac{z_i^2}{(\sum_{j=1}^{n} z_j^2)^2} = \frac{1}{n}\mathbf{E}\frac{\sum_{j=1}^{n} z_j^2}{(\sum_{j=1}^{n} z_j^2)^2} = \frac{1}{n}\mathbf{E}\frac{1}{\sum_{j=1}^{n} z_j^2} \quad . \tag{7.7}$$

Now,

$$\mathbf{E}\frac{z_i(\hat{t}_i - t_i)}{\sum_{j=1}^{n} z_j^2} = \mathbf{E}_{z_1\ldots z_n}\left(\frac{z_i}{\sum_{j=1}^{n} z_j^2}\right)\cdot\mathbf{E}_{\hat{t}_i|z_1,z_2,\ldots,z_n}(\hat{t}_i - t_i) \ . \tag{7.8}$$

Consider now the multivariate vector

$[\hat{t}_i,\ z_i,\ z_1,\ z_2,\ \ldots,\ z_{i-1},\ z_{i+1},\ \ldots,\ z_n]$ with mean vector

$[t_i, 0, 0, \ldots, 0]$ and variance covariance matrix:

$$\underset{(n+1)(n+1)}{\Sigma_1} = \left[\begin{array}{c|cccc} V_1 & -V_1 & -C_1 & \cdots & -C_1 \\ \hline -V_1 & & & & \\ -C_1 & & & & \\ \vdots & & & \Sigma & \\ -C_1 & & & & \end{array}\right] = \left[\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma \end{array}\right] ,$$

where Σ is as defined before. Now

$$E_{\hat{t}_i|z_1,z_2,\ldots,z_n}(\hat{t}_i) = t_i + \Sigma_{12}\Sigma^{-1}\underline{\mathbf{z}} \quad , \tag{7.9}$$

where Σ^{-1} is a two-element matrix with diagonal elements d and off-diagonal elements f, where:

$$d = \frac{-[(n-2)C+V]}{C^2(n-1)-V[(n-2)C+V]} \quad ,$$

and (7.10)

$$f = \frac{C}{C^2(n-1)-V[(n-2)C+V]} \quad .$$

Now

$$\Sigma_{12}\Sigma^{-1} = (-V_1, -C_1, -C_1, \ldots, -C_1)\begin{bmatrix} d & f & f & \cdots & f \\ f & d & f & \cdots & f \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ f & f & & \cdots & d \end{bmatrix} =$$

$$[(-V_1d-(n-1)C_1f),(-C_1d-(n-2)C_1f-V_1f),(-C_1d-(n-2)C_1f-V_1f),$$

$$, \ldots, \text{the same} \ldots] \quad .$$

But

$$-V_1 d - (n-1)C_1 f = \frac{V_1[(n-2)C+V]-(n-1)C_1 C}{C^2(n-1)-V[(n-2)C+V]} \quad ,$$

$$= \frac{V_1(n-1)C-C_1(n-1)C+V_1(V-C)}{(n-1)C^2-(n-1)VC-V(V-C)} = \frac{V_1(V-C)-(n-1)C(V_1-C_1)}{(n-1)C(C-V)+V(-V+C)} \quad ,$$

$$= \frac{(n-1)C(V_1-C_1)+V_1V-V_1(C_1+C_2)}{(n-1)C(C-V)+V(C-V)} \quad , \qquad (\text{since } C = C_1 + C_2)$$

$$= \frac{(n-1)C(V_1-C_1)+V_1V-V_1C_1-V_2C_1}{(C-V)[(n-1)C+V]} \quad , \tag{7.11}$$

using (6.5). Thus

$$-V_1 d - (n-1)C_1 f = \frac{(n-1)C(V_1-C_1)+V(V_1-C_1)}{(C-V)[(n-1)C+V]} \quad , \qquad (V = V_1+V_2)$$

$$= \frac{V_1-C_1}{C-V} \quad . \tag{7.12}$$

To compute the remaining elements in the nxl row $\Sigma_{12}\Sigma^{-1}$, one needs to compute only

$$-C_1 d - (n-2)C_1 f - V_1 f = -C_1[d+(n-2)f] - V_1 f$$

$$= \frac{-C_1\{-[(n-2)C+V]+(n-2)C\}-V_1 C}{(n-1)C^2-V[(n-2)C+V]} \quad ,$$

$$= \frac{VC_1-V_1C}{(n-1)C^2-V[(n-2)C+V]} = \frac{V_1C_1+V_2C_1-V_1C_1-V_1C_2}{(n-1)C^2-V[(n-2)C+V]}$$

$$= \frac{V_2C_1-V_1C_2}{(n-1)C^2-V[(n-2)C+V]} = 0 \quad ,$$

i.e.,

$$-C_1 d - (n-2)C_1 f - V_1 f = 0 \quad . \tag{7.13}$$

Now (7.9) can be written

$$E_{\hat{t}_i | z_1, z_2, \ldots, z_n}(\hat{t}_i - t_i) = \Sigma_{12}\Sigma^{-1}\underline{Z} = \left(\frac{V_1 - C_1}{C-V}, 0, 0, \ldots, 0\right)\begin{pmatrix} z_i \\ z_1 \\ \vdots \\ z_{i-1} \\ z_{i+1} \\ \vdots \\ z_n \end{pmatrix}$$

$$= \frac{V_1 - C_1}{C-V} z_i = -\frac{V_1 - C_1}{V-C} z_i \quad . \tag{7.14}$$

Substituting (7.7) and (7.14) into (7.6),

$$V(\hat{\tau}_i) = V_1 + \frac{V_1^2 B^2 (f+2)}{nf} E \frac{1}{\sum_{j=1}^{n} z_j^2} - 2V_1 BE \left[\frac{V_1 - C_1}{V-C} \cdot \frac{z_i^2}{\sum_{j=1}^{n} z_j^2}\right] \quad . \tag{7.15}$$

It should be noted that $\theta\sigma^2 = V_1$. Utilizing again the symmetry property, $E \dfrac{z_i^2}{\sum_{j=1}^{n} z_j^2} = \dfrac{1}{n}$, hence:

$$V(\hat{\tau}_i) = V_1 + \frac{V_1^2 B^2 (f+2)}{nf} E \frac{1}{\sum_{j=1}^{n} z_j^2} - 2 \frac{V_1 B(V_1 - C_1)}{n(V-C)} \quad . \tag{7.16}$$

Using (5.7) with a = V, b = C,

$$E \frac{1}{\sum_{j=1}^{n} z_j^2} < \frac{1}{(n-3)(V-C)} \quad . \tag{7.17}$$

Now one can write

$$V(\hat{\tau}_i) < V_1 + \frac{V_1^2 B^2 (f+2)}{nf(n-3)(V-C)} - 2 \frac{V_1 B(V_1 - C_1)}{n(V-C)} \quad . \tag{7.18}$$

We would like the quantity $\frac{V_1^2 B^2 (f+2)}{n(n-3)f(V-C)} - 2 \frac{V_1 B(V_1 - C_1)}{n(V-C)}$ to be negative. First we notice by (6.8) that

$V-C = -vC = -v(C_1+C_2) = \frac{k}{\lambda v} \sigma^2 + \frac{k}{r-\lambda} \sigma'^2$ which is always positive; if, in addition $n > 3$, then the above quadratic in B is negative whenever

$$0 < B < \frac{2f(V_1 - C_1)(n-3)}{V_1 (f+2)} \quad .$$

The value of B which makes it the most negative is

$$B = \frac{f(V_1 - C_1)(n-3)}{V_1 (f+2)} \quad .$$

From (6.9)

$$B = \frac{fv(n-3)}{(v-1)(f+2)} \quad . \tag{7.19}$$

Now substituting B in (7.18),

$$V(\hat{\tau}_i) < V_1 + \frac{V_1^2}{V-C}\left[\frac{f+2}{nf(n-3)} \frac{v^2f^2(n-3)^2}{(v-1)^2(f+2)^2} - \frac{2v}{n(v-1)} \cdot \frac{vf(n-3)}{(v-1)(f+2)}\right]$$

$$= V_1 + \frac{V_1^2}{V-C}\left[\frac{-v^2f(n-3)}{n(v-1)^2(f+2)}\right] \quad ,$$

i.e.,

$$V(\hat{\tau}_i) < V_1 - \frac{v^2f(n-3)}{n(v-1)^2(f+2)} \frac{V_1^2}{V-C} \quad . \tag{7.20}$$

It is obvious that one should take n as large as possible, but since n is strictly less than v, take n = v-1. It should be noted that one cannot take n = v, for then the matrix Σ is singular. Hence

$$V(\hat{\tau}_i) < V_1 - \frac{v^2f(v-4)}{(v-1)^3(f+2)} \frac{V_1^2}{V-C} \quad . \tag{7.21}$$

As was mentioned previously, the quantity $\frac{V_1^2}{V_1+V_2}$ represents the utmost improvement possible over the intra-block variance. Now one can write $\frac{V_1^2}{V-C} = \frac{V_1^2}{V_1+V_2} + \frac{V_1^2C}{V(V-C)}$

$= \frac{V_1^2}{V}[1 + \frac{C}{V-C}] = \frac{V_1^2}{V}(1 + \frac{C}{-vC})$, using (6.8).

Then $\frac{V_1^2}{V-C} = \frac{V_1^2}{V_1+V_2}\,\frac{v-1}{v}$, and substituting into (7.21),

$$V(\hat{\tau}_i) < V_1 - \frac{fv(v-4)}{(v-1)^2(f+2)} \cdot \frac{V_1^2}{V_1+V_2} \quad . \tag{7.22}$$

The upper bound of the variance of the proposed estimate is uniformly better than the intra-block variance for any B.I.B. design when $v > 4$. Substituting $n = v-1$ into (7.19) one obtains $B = \frac{fv(v-4)}{(f+2)(v-1)}$, and then from (7.2),

$$\hat{\tau}_i = \hat{t}_i + \frac{\theta v(v-4)fs^2}{(f+2)(v-1)\sum_{j=1}^{v-1}(\hat{t}'_j-\hat{t}_j)^2}(\hat{t}'_i-\hat{t}_i) \quad ,$$

and recalling that $\theta = \frac{k(v-1)}{\lambda v^2}$ or $\frac{\theta v}{v-1} = \frac{k}{\lambda v}$,

$$\hat{\tau}_i = \hat{t}_i + \frac{k(v-4)S^2}{\lambda v(f+2)\sum_{j=1}^{v-1}(\hat{t}'_j-\hat{t}_j)^2}(\hat{t}'_i-\hat{t}_i) \quad , \tag{7.23}$$

where $S^2 = fs^2$ is the error sum of squares in the intra-analysis. From (7.22) it is seen that the ratio of improvement or the recovery ratio is:

$$D_1 = \frac{fv(v-4)}{(v-1)^2(f+2)} \quad . \tag{7.24}$$

7.1 Exact Results for Theorem 2

Using lemma 2 one can get a fairly exact value for $E\,\frac{1}{\Sigma z_i^2}$. Here a = V, b = C, n = v-1, so,

$$E\left(\frac{1}{\sum_{i=1}^{v-1} z_i^2}\right) = \frac{\Gamma(\frac{v-3}{2})}{\Gamma(\frac{v-1}{2})\,\Gamma(\frac{1}{2})} \int_0^1 \frac{u^{\frac{v-4}{2}}(1-u)^{-\frac{1}{2}}}{2(v-1)C(1-u)+2(V-C)}\,du \quad ,$$

but since V-C = -vC, one obtains

$$E\left(\frac{1}{\sum_{i=1}^{v-1} z_i^2}\right) = \frac{2}{(v-3)\,\beta(\frac{v-2}{2},\frac{1}{2})} \int_0^1 \frac{u^{\frac{v-4}{2}}(1-u)^{-\frac{1}{2}}}{2(v-1)C(1-u)-2vC}\,du \quad ,$$

$$= \frac{1}{-C(v-3)\,\beta(\frac{v-2}{2},\frac{1}{2})} \int_0^1 \frac{u^{\frac{v-4}{2}}(1-u)^{-\frac{1}{2}}}{1+(v-1)u}\,du \quad .$$

The problem now is to evaluate the quantity,

$$F = \frac{1}{\beta(\frac{v-2}{2},\frac{1}{2})} \int_0^1 \frac{u^{\frac{v-4}{2}}(1-u)^{-\frac{1}{2}}}{1+(v-1)u}\,du \quad ,$$

for different values of v. Later a method of computing this quantity will be given. Now, one may write

$$E\left(\frac{1}{\sum_{i=1}^{v-1} z_i^2}\right) = \frac{-F}{C(v-3)} \quad . \tag{7.1.1}$$

Substituting (7.1.1) into (7.16), and taking $n = v-1$, one obtains

$$V(\hat{\tau}_i) = V_1 + \frac{V_1^2 B^2 (f+2)F}{-Cf(v-1)(v-3)} - \frac{2V_1 B(V_1 - C_1)}{(v-1)(V-C)} \quad ,$$

but $V-C = -vc$ and $\frac{V_1 - C_1}{v} = \frac{V_1}{v-1}$ or $V_1 - C_1 = \frac{vV_1}{v-1}$, $\frac{V_1 - C_1}{V-C}$ $= \frac{V_1}{-C(v-1)}$,

$$V(\hat{\tau}_i) = V_1 + \frac{V_1^2 B}{-(v-1)C}\left[\frac{B(f+2)F}{f(v-3)} - \frac{2}{v-1}\right] \quad . \tag{7.1.2}$$

Since C is negative, one needs

$$0 < B < \frac{2f(v-3)}{F(v-1)(f+2)} \quad .$$

The required B is

$$B = \frac{f(v-3)}{F(v-1)(f+2)} \quad . \tag{7.1.3}$$

Substituting (7.1.3) into (7.1.2), one may write

$$V(\hat{\tau}_i) = V_1 + \frac{V_1^2 B}{-C(v-1)}\left[\frac{-1}{v-1}\right] = V_1 - \frac{V_1^2 B}{-C(v-1)^2} \; , \tag{7.1.4}$$

since $\frac{V_1^2}{V-C} = \frac{V_1^2}{-vC} = \frac{V_1^2}{V_1 + V_2} \cdot \frac{v-1}{v}$, then (7.1.4) becomes

$$V(\hat{\tau}_i) = V_1 - \frac{B}{v-1} \; \frac{V_1^2}{V_1 + V_2} \quad . \tag{7.1.5}$$

The combined estimate in (7.2) becomes, after substituting, the value of θ:

$$\hat{\tau}_i = \hat{t}_i + \frac{\theta B s^2}{\sum_{j=1}^{v-1} (\hat{t}_j' - \hat{t}_j)^2} (\hat{t}_i' - \hat{t}_i) \quad ,$$

$$= \hat{t}_i + \frac{k(v-1)Bs^2}{\lambda v^2 \sum_{j=1}^{v-1} (\hat{t}_j' - \hat{t}_j)^2} (\hat{t}_i' - \hat{t}_i) \quad . \qquad (7.1.6)$$

From (7.1.5) the recovery ratio is

$$D_2 = B/(v-1) \quad . \qquad (7.1.7)$$

The estimate in (7.1.6) can be applied whenever $v > 3$. The values of B and D_2 appear in Table II for the 58 B.I.B. designs listed in Reference (2).

7.1.1* Method for computing F

The hypergeometric series,

$$F(a,b;\ c;\ z) = \sum_{n=0}^{\infty} \frac{(a)_n\ (b)_n}{n!\ (c)_n} z^n \quad , \qquad (7.1.8)$$

converges to the definite integral [see Reference (12)]:

$$I = \frac{1}{\beta(b,c-b)} \int_0^1 u^{b-1}\ (1-u)^{c-b-1}\ (1-uz)^{-a}\ du \quad . \qquad (7.1.9)$$

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*I am indebted to Dr. L. R. Shenton who introduced me to this method.

Now one may write:

$$F = \frac{1}{\beta(\frac{v-2}{2}, \frac{1}{2})} \int_0^1 \frac{u^{\frac{v-4}{2}} (1-u)^{-\frac{1}{2}}}{1+(v-1)u} du \quad . \qquad (7.1.10)$$

Comparing (7.1.9) and (7.1.10), we get:

$$b-1 = \frac{v-4}{2} , \quad c-b-1 = -\frac{1}{2} , \quad z = 1-v , \quad a = 1 ,$$

or

$$b = \frac{v-2}{2} , \quad c = \frac{v-1}{2} , \quad z = 1-v , \quad a = 1 .$$

One needs to evaluate, therefore, the quantity,

$$F(1, \frac{v-2}{2}; \frac{v-1}{2}; 1-v) = F(\frac{v-2}{2}, 1; \frac{v-1}{2}; 1-v) \quad , \qquad (7.1.11)$$

due to the symmetry of a and b in the hypergeometric form above. Now $F(b,1;c;z)$ can be put in the form of Gauss's continued fraction as follows [see Reference (13)]:

$$F(b,1;\ c;\ z) = \cfrac{1}{1 - \cfrac{h_1 z}{1 - \cfrac{(1-h_1)h_2 z}{1 - \cfrac{(1-h_2)h_3 z}{1 - \cfrac{(1-h_3)h_4 z}{1 - \cdots}}}}} \quad , \qquad (7.1.12)$$

where $$h_{2p-1} = \frac{b+p-1}{c+2p-2} , \quad h_{2p} = \frac{p}{c+2p-1} , \quad p = 1,2,3,\ldots \quad . \qquad (7.1.13)$$

The convergence is very rapid and it is far easier to compute than the usual Simpson rule, in this case. It is to be noted that the terms required to assure an accuracy up to the sixth decimal place, range from 4 terms for $v = 91$ to 14 terms for $v = 4$.

7.2 Application of Theorem 1 in B.I.B. Designs

Consider the vxv matrix

$$M_1 = \begin{bmatrix} 1 & 1 & 1 & & \cdots & & 1 \\ 1 & -1 & & & & & \\ 1 & 1 & -2 & & & & \\ 1 & 1 & 1 & -3 & & & \\ \vdots & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & \cdots & (2-v) & \\ 1 & 1 & 1 & 1 & \cdots & 1 & (1-v) \end{bmatrix}, \quad (7.2.1)$$

and let M be the orthogonal matrix we get by normalizing M_1. Consider the two vectors of comparisons:

$$\underline{U} = \begin{bmatrix} 0 \\ U_1 \\ U_2 \\ \vdots \\ U_{v-1} \end{bmatrix} = M \begin{bmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \vdots \\ \hat{t}_v \end{bmatrix} = M \underline{\hat{t}} \quad , \qquad (7.2.2)$$

and

$$\underline{X} = \begin{bmatrix} 0 \\ X_1 \\ X_2 \\ \vdots \\ X_{v-1} \end{bmatrix} = M \begin{bmatrix} \hat{t}'_1 \\ \hat{t}'_2 \\ \vdots \\ \hat{t}'_v \end{bmatrix} = M\, \underline{\hat{t}}' \quad . \qquad (7.2.3)$$

In general,

$$V(U_m) = V[\frac{1}{\sqrt{m(m+1)}} (\hat{t}_1+\hat{t}_2+\ldots+\hat{t}_m-m\hat{t}_{m+1})] \quad ,$$

$$= \frac{1}{m(m+1)}[m(m+1)V_1+m(m-1)C_1-2m(mC_1)] = V_1-C_1 \quad . \qquad (7.2.4)$$

This result is independent of m, that is, all U_i's, where $i = 1,2,\ldots,v-1$, have the same variance, namely V_1-C_1. Similarly, $V(X_i) = V'_1-C'_1$ for $i = 1,2,\ldots,v-1$. Therefore, theorem 1 is applicable for combining the two comparisons U_i and X_i, and their combined estimate is:

$$\hat{\tau}_i = U_i + \frac{\theta(v-3)s^2}{\sum_{j=1}^{v-1}(X_j-U_j)^2}(X_i-U_i) \quad , \quad (\theta=\frac{\rho f}{f+2})$$

but $V_1-C_1 = \frac{k}{\lambda v}\sigma^2$, hence $\rho = \frac{k}{\lambda v}$, and

$$\hat{\tau}_i = U_i + \frac{fk(v-3)s^2}{(f+2)\lambda v\sum_{j=1}^{v-1}(X_j-U_j)^2}(X_i-U_i) \quad . \qquad (7.2.5)$$

It should be noted that

$$\sum_{j=1}^{v-1}(X_j-U_j)^2 = (\underline{X}-\underline{U})'(\underline{X}-\underline{U}) = (\hat{\underline{t}}'-\hat{\underline{t}})'M'M(\hat{\underline{t}}'-\hat{\underline{t}})$$

$$= (\hat{\underline{t}}'-\hat{\underline{t}})'(\hat{\underline{t}}'-\hat{\underline{t}}) = \sum_{j=1}^{v}(\hat{t}_j'-\hat{t}_j)^2. \quad \text{Let } \hat{\underline{\tau}} = \begin{bmatrix} \hat{\tau}_1 \\ \hat{\tau}_2 \\ \vdots \\ \hat{\tau}_{v-1} \end{bmatrix} = M \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_{v-1} \end{bmatrix} = M\underline{T},$$

then (7.2.5) can be written in vector notation as

$$M\underline{T} = M\hat{\underline{t}} + \frac{f\,k(v-3)s^2}{(f+2)\lambda v\sum_{j=1}^{v}(\hat{t}_j'-\hat{t}_j)^2} M(\hat{\underline{t}}'-\hat{\underline{t}}) \quad . \tag{7.2.6}$$

Since $M'M = I$, one can write

$$T_i = \hat{t}_i + \frac{fk(v-3)s^2}{(f+2)\lambda v\sum_{j=1}^{v}(\hat{t}_j'-\hat{t}_j)^2}(\hat{t}_i'-\hat{t}_i) \;, \quad i=1,2,3...v. \tag{7.2.7}$$

This is applicable when $v > 3$, and the recovery ratio here is

$$D_3 = \frac{(v-3)\,f}{(v-1)(f+2)} \quad . \tag{7.2.8}$$

The results for D_1 as given in (7.2.4), and D_3 above appear in Table II. The results show that D_2 and D_3 are always better than D_1; the three ratios approach practically the same value as v becomes relatively large. D_2 and D_3

are practically the same, but D_3 is,

, always better than D_2.

7.3 Procedure for Recovery of Inter-block Information in B.I.B. Designs

Compute:

(1) $Y_{i.} = \sum_{j=1}^{b} \delta_{ij} y_{ij}$, $i=1,2,3...v$. (Total yield of i-th treatment).

(2) $Y_{.j} = \sum_{i=1}^{v} \delta_{ij} y_{ij}$, $j=1,2,...,b$. (Total yield of j-th block.)

(3) $Y_{..} = \sum_{ij} \delta_{ij} y_{ij}$. (Grand total.)

(4) $Q_i = Y_{i.} - \frac{1}{k}$ (Sum of block totals in which treatment i occurs.),

$i = 1,2,...,v$. (Adjusted treatment totals.)

(5) $\hat{t}_i = \frac{Q_i}{rE}$, $i=1,2,...,v$. (The intra-estimate, $E = \frac{(k-1)v}{k(v-1)}$.)

(6) $SST = \sum_{i=1}^{v} \hat{t}_i Q_i$. (Treatment sum of squares.)

(7) $s^2 = \frac{1}{f}\,(\sum_{ij} \delta_{ij} y^2_{ij} - SST - \frac{1}{k} \sum_{j=1}^{b} Y^2_j)$. (Error mean square, $f = bk-b-v+1$.)

(8) $Q_i + Q'_i = Y_{i.} - \frac{1}{v} Y_{..}$, $i=1,2,\ldots,v$. (Sum of intra- and inter-adjusted totals.)

(9) $\hat{t}'_i - \hat{t}_i = \frac{Q'_i}{r(1-E)} - \frac{Q_i}{rE} = \frac{E(Q'_i + Q_i)-Q_i}{rE(1-E)}$, $i=1,2,\ldots,v$.

(10) $\sum_{i=1}^{v} (\hat{t}_i - \hat{t}_i)^2$.

(11) $J = \frac{fk(v-3)s^2}{(f+2)\lambda v \sum_{i=1}^{v}(\hat{t}'_i - \hat{t}_i)^2}$

(12) The combined estimate $T_i = \hat{t}_i + J(\hat{t}_i - \hat{t}_i)$, $i=1,2,\ldots,v$.

This estimate recovers the ratio D_3 of the utmost possible recovery, and D_3 appears in Table II.

7.4 Worked Example

An experiment with 6 treatments in blocks of 2 will be worked out. The data is taken from Reference (2), page 444; the treatment estimates obtained in (2) by applying the conventional method are to be compared with the results of the new method.

Rep. I

				Block totals
(1)	7	(2)	17	24
(3)	26	(4)	25	51
(5)	33	(6)	29	62

Rep. II

				Block totals
(1)	17	(3)	27	44
(2)	23	(5)	27	50
(4)	29	(6)	30	59

Rep. III

				Block totals
(1)	10	(4)	25	35
(2)	26	(6)	37	63
(3)	24	(5)	26	50

Rep. IV

				Block totals
(1)	25	(5)	40	65
(2)	25	(4)	34	59
(3)	34	(6)	32	66

Rep. V

				Block totals
(1)	11	(6)	27	38
(2)	24	(3)	21	45
(4)	26	(5)	32	58

In this experiment $v = 6$, $k = 2$, $r = 5$, $b = 15$, $\lambda = 1$, $E = .6$.

The following table is to be set up.

Treat. No.	$Y_{i.}$	Q_i	$\hat{t}_i$	$Q_i+Q'_i$	$\hat{t}'_i-\hat{t}_i$	$\hat{\tau}_i$	T_i
1	70	-33.0	-11.00	-58.17	-1.59	-11.31	-11.2
2	115	- 5.5	- 1.83	-13.17	-2.00	- 2.23	- 2.1
3	132	4.0	1.33	3.83	-1.42	1.05	1.1
4	139	8.0	2.67	10.83	-1.25	2.42	2.5
5	158	15.5	5.17	29.83	2.00	5.57	5.5
6	155	11.0	3.67	26.83	4.25	4.52	4.4
		0	0	0	0	0	

For Treatment 1, for instance, the following steps should be carried out.

Step 1: Treatment total $= Y_{1.} = 7 + 17 + \ldots + 11 = 70$.

Step 2: $Q_1 = Y_{1i} - \frac{1}{2}$ (sum of block totals in which Treatment 1 appears),

$$= 70 - \frac{1}{2}(24 + 44 + \ldots + 38) = -33 .$$

Step 3: $\hat{t}_1 = \frac{Q_1}{rE} = \frac{-33}{5(.6)} = -11$,

where E = Efficiency of the design $= \frac{(k-1)v}{k(v-1)} = \frac{(2-1)\cdot 6}{2\cdot 5} = .6.$

Step 4: $Q_1 + Q'_1 = Y_{1.} - \frac{1}{v}(Y_{..}) = 70 - \frac{769}{6} = -58.17$,

where $Y_{..}$ = Grand Total $= 24 + 51 + \ldots + 58 = 769$.

Step 5: $\hat{t}_1' - \hat{t}_1 = \dfrac{Q_1'}{r(1-E)} - \dfrac{Q_1}{rE} = \dfrac{E(Q_1+Q_1')-Q_1}{rE(1-E)}$,

$$= \frac{.6(-58.17)+33}{5(.6)(.4)} = -1.59 \quad .$$

After computing the same quantities for all treatments, one should compute the following quantities:

Sum squares of blocks (unadjusted)

$$= \frac{1}{2}(24^2 + 51^2 + \ldots + 58^2) - \frac{(769)^2}{30} \quad ,$$

$$= 1051.5 \quad .$$

Sum squares of treatments (adjusted)

$$= \sum_{i=1}^{v} \hat{t}_i Q_i = \hat{t}_1 Q_1 + \ldots + \hat{t}_6 Q_6 \quad ,$$

$$= 520.2 \quad .$$

Error mean square $= \dfrac{1}{rv-v-b+1}$(Total sum of squares - Sum squares of treatments adjusted - Sum squares of blocks unadjusted) ,

$$= \frac{1}{10}(1649 - 520.2 - 1051.5) = 7.73 \quad .$$

$$\sum_{i=1}^{v} (\hat{t}_i'-\hat{t}_i)^2 = (\hat{t}_1'-\hat{t}_1)^2 + (\hat{t}_2'-\hat{t}_2)^2 + \ldots + (\hat{t}_6'-\hat{t}_6)^2$$

$$= (-1.59)^2 + (-2)^2 + \ldots + (4.25)^2 = 32.2 \quad .$$

$$J = \frac{fk(v-3)s^2}{(f+2)\lambda v \sum\limits_{i=1}^{v} (t_i' - t_i)^2} = \frac{10\cdot 2\cdot 3\cdot(7\cdot 73)}{12\cdot 1\cdot 6\cdot(32\cdot 2)} =$$

now the combined estimate τ_i is:

$$\hat{\tau}_i = \hat{t}_i + J(\hat{t}_i' - \hat{t}_i)$$

For the first treatment, for instance:

$$\hat{\tau}_1 = -11 + (.20005)(-1.59) = -11.31$$

8. P.B.I.B. WITH TWO ASSOCIATE CLASSES

8.1 Definitions and Useful Relations

An incomplete block design is said to be partially balanced with two associate classes if it satisfies the following requirements:

(i) The experimental material is divided into b blocks of k units each, different treatments being applied to the units in the same block.

(ii) There are $v(>k)$ treatments each of which occurs in r blocks.

(iii) There can be established a relation of association between any two treatments satisfying the following requirements:

(a) Two treatments are either first associates or second associates.

(b) Each treatment has exactly n_i i-th associates $(i = 1,2)$.

(c) Given any two treatments, which are i-th associates, the number of treatments common to the j-th associate of the first and the k-th associate of the second is p^i_{jk} and is independent of the pair of treatments we start with. Also $p^i_{jk} = p^i_{kj}$ $(i,j,k = 1,2)$.

(iv) Two treatments which are i-th associates occur together in exactly λ_i blocks (i = 1,2).

The numbers v, r, k, b, n_1, n_2, λ_1, and λ_2 are called parameters of the first kind, whereas the numbers p^i_{jk} (i,j,k = 1,2) are called the parameters of the second kind.

The following relations between the parameters are known to hold:

$$
\begin{aligned}
&vr = bk\ , && n_1+n_2 = v-1\ , \\
&n_1\lambda_1+n_2\lambda_2 = r(k-1)\ , && p^1_{11}+p^1_{12} = n_1-1\ , \\
&p^1_{21}+p^1_{22} = n_2\ , && p^2_{11}+p^2_{12} = n_1\ , \qquad (8.1.1) \\
&p^2_{21}+p^2_{22} = n_2-1\ , && n_1p^1_{12} = n_2p^2_{11}\ , \\
&n_1p^1_{22} = n_2p^2_{12}\ .
\end{aligned}
$$

In the analysis of such designs, T_i is defined as the total of the observations for the i-th treatment, B_j as the sum of the k observations from the j-th block. Q_i denotes the adjusted yield for the i-th treatment obtained by subtracting from T_i, the sum of the block averages for those blocks in which the i-th treatment occurs. Also, $S_1(Q_i)$ denotes the sum of the adjusted yields for all the first associates of the i-th treatment, and G denotes the total of all N observations.

R. C. Bose defines the constants Δ, H, c_1, c_2 by the relations:

$$K^2\Delta = (a+\lambda_1)(a+\lambda_2)+(\lambda_1-\lambda_2)[a(f-g)+f\lambda_2-g\lambda_1] , \tag{8.1.2}$$

$$KH = (2a+\lambda_1+\lambda_2)+(f-g)(\lambda_1-\lambda_2) , \tag{8.1.3}$$

$$K\Delta c_1 = \lambda_1(a+\lambda_2)+(\lambda_1-\lambda_2)(f\lambda_2-g\lambda_1) , \tag{8.1.4}$$

$$K\Delta c_2 = \lambda_2(a+\lambda_1)+(\lambda_1-\lambda_2)(f\lambda_2-g\lambda_1) , \tag{8.1.5}$$

where,

$$a = r(k-1) , \quad f = p^1_{12} , \quad g = p^2_{12} . \tag{8.1.6}$$

In the intra-block analysis, the best linear estimate $\hat{t}_i$ of the treatment effect t_i is given by,

$$\hat{t}_i = \frac{k-c_2}{a} Q_i + \frac{c_1-c_2}{a} S_1(Q_i) . \tag{8.1.7}$$

The variance of the intra-estimate of the difference between two treatment effects is given by:

$$V(\hat{t}_i-\hat{t}_u) = \frac{2(k-c_j)}{a} \sigma^2 , \tag{8.1.8}$$

where the treatments i and u are j-th associates (j=1,2).

In the inter-block analysis where only the block totals are used, M. Zelen (11) gives the best linear estimate $\hat{t}_i'$ of the treatment effect t_i as:

$$\hat{t}_i' = \frac{k-c_2'}{r} Q_i' + \frac{c_1'-c_2'}{r} S_1(Q_i') \quad , \qquad (8.1.9)$$

where:

$$Q_i' = T_i - Q_i - \frac{rG}{N} \quad , \qquad (8.1.10)$$

and

$$c_j' = \frac{c_j\Delta - r\lambda_j}{\Delta - rH + r^2} \quad , \qquad (j=1,2) \; . \qquad (8.1.11)$$

In this case,

$$V(\hat{t}_i' - \hat{t}_u') = \frac{2(k-c_j')}{r} \sigma'^2 \quad , \qquad (8.1.12)$$

where the treatments i and u are j-th associates (j=1,2), and where

$$\sigma'^2 = \sigma^2 + k\sigma_b^2 \quad . \qquad (8.1.13)$$

σ^2 is the error variance in the intra-model, and σ_b^2 is the variance of the block effect in the random or inter-model.

8.2 <u>Variances and Covariances</u>

From the restriction $\sum_{i=1}^{v} \hat{t}_i = 0$, the $V(\sum_{i=1}^{v} \hat{t}_i) = 0$, or:

$$v\, V(\hat{t}_i) + v(v-1)\, \mathrm{Cov}(\hat{t}_i, \hat{t}_j) = 0 \quad . \qquad (8.2.1)$$

Since every treatment has n_1 first associates and n_2 second associates, and $n_1+n_2 = v-1$, then:

$$v\,V(\hat{t}_i) + vn_1\,\mathrm{Cov}(\hat{t}_i,\hat{t}_{i1}) + vn_2\,\mathrm{Cov}(\hat{t}_i,\hat{t}_{i2}) = 0 \quad ,$$

or (8.2.2)

$$V(\hat{t}_i) + n_1\,\mathrm{Cov}(\hat{t}_i,\hat{t}_{i1}) + n_2\,\mathrm{Cov}(\hat{t}_i,\hat{t}_{i2}) = 0 \quad ,$$

where $\hat{t}_{ij}$ is the j-th associate of $\hat{t}_i$, $(j=1,2)$. From (8.1.8) for $j = 1,2$:

$$V(\hat{t}_i) + V(\hat{t}_{i1}) - 2\,\mathrm{Cov}(\hat{t}_i,\hat{t}_{i1}) = \frac{2(k-c_1)}{a}\,\sigma^2 \quad ,$$

and (8.2.3)

$$V(\hat{t}_i) + V(\hat{t}_{i2}) - 2\,\mathrm{Cov}(\hat{t}_i,\hat{t}_{i2}) = \frac{2(k-c_2)}{a}\,\sigma^2 \quad .$$

Let $V(\hat{t}_i) = V_1$ for $i = 1, 2, \ldots, v$, $\mathrm{Cov}(\hat{t}_i,\hat{t}_{i1}) = C_1$, and $\mathrm{Cov}(\hat{t}_i,\hat{t}_{i2}) = C_2$, then (8.2.3) may be written as:

$$V_1 + n_1C_1 + n_2C_2 = 0 \quad ,$$

$$2V_1 - 2C_1 = \frac{2(k-c_1)}{a}\,\sigma^2 \quad ,$$

and

$$2V_1 - 2C_2 = \frac{2(k-c_2)}{a}\,\sigma^2 \quad ,$$

or

$$V_1 + n_1C_1 + n_2C_2 = 0 \quad , \tag{8.2.4}$$

$$V_1 - C_1 = \frac{k-c_1}{a}\,\sigma^2 \quad , \tag{8.2.5}$$

$$V_1 - C_2 = \frac{k-c_2}{a}\,\sigma^2 \quad . \tag{8.2.6}$$

By subtracting (8.2.6) from (8.2.5) and (8.2.5) from (8.2.4), one obtains:

$$C_2 - C_1 = \frac{c_2-c_1}{a}\,\sigma^2 \quad , \tag{8.2.7}$$

and

$$(n_1+1)C_1 + n_2C_2 = \frac{c_1-k}{a}\,\sigma^2 \quad . \tag{8.2.8}$$

Remembering that $n_1+n_2+1 = v$, (8.2.7) and (8.2.8) may be written as:

$$C_1 = \frac{c_1(n_2+1)-n_2c_2-k}{av}\,\sigma^2 \quad , \tag{8.2.9}$$

and

$$C_2 = \frac{c_2(n_1+1)-n_1c_1-k}{av}\,\sigma^2 \quad . \tag{8.2.10}$$

Substituting (8.2.9) into (8.2.5), one obtains:

$$\frac{1}{\sigma^2}V_1 = \frac{k-c_1}{a} + \frac{c_1(n_2+1)-n_2c_2-k}{av} \quad ,$$

$$= \frac{vk-vc_1+c_1(n_2+1)-n_2c_2-k}{av} = \frac{k(v-1)-(n_1+n_2+1)c_1+(n_2+1)c_1-n_2c_2}{av} ,$$

or

$$V_1 = \frac{k(v-1)-n_1c_1-n_2c_2}{av}\,\sigma^2 . \tag{8.2.11}$$

Since $\sum_{i=1}^{v} \hat{t}_i' = 0$, where $\hat{t}_i'$'s are the inter-estimates for treatments, one obtains, in exactly the same way:

$$V_1' = \frac{k(v-1)-n_1c_1'-n_2c_2'}{vr}\,\sigma'^2 \quad , \qquad (8.2.12)$$

$$C_1' = \frac{c_1'(n_2+1)-n_2c_2'-k}{vr}\,\sigma'^2 \quad , \qquad (8.2.13)$$

$$C_2' = \frac{(n_1+1)c_2'-n_1c_1'-k}{vr}\,\sigma'^2 \quad , \qquad (8.2.14)$$

where $V_1' = V(\hat{t}_i')$, $C_1' = Cov(\hat{t}_i',\hat{t}_{i1}')$, $C_2' = Cov(\hat{t}_i',\hat{t}_{i2}')$, and $\hat{t}_{ij}'$ is the j-th associate of $\hat{t}_i'$, $(j=1,2)$. Let

$$V = V_1 + V_1' \quad , \qquad (8.2.15)$$

$$C = C_1 + C_1' \quad , \qquad (8.2.16)$$

and

$$C' = C_2 + C_2' \quad . \qquad (8.2.17)$$

In the combined analysis of Rao, the variance of the difference between two treatments is given in Reference (1) as:

$$V(T_i-T_j) = \frac{2(k-d_j)}{r[w'+w(k-1)]} \quad , \quad (j=1,2) \quad . \qquad (8.2.18)$$

Also, here

$$\sum_{i=1}^{v} T_i = 0 \ , \quad V[\sum_{i=1}^{v} T_i] = 0 \ , \quad \text{or as in (8.2.2)},$$

$$V(T_i) + n_1\, Cov(T_i,T_{i1}) + n_2\, Cov(T_i,T_{i2}) = 0 \quad . \qquad (8.2.19)$$

(8.2.19), together with the two equations resulted from (8.2.18) for j=1,2, give a system of three equations of three unknowns. The work is exactly parallel to that of the intra-estimates, and

$$V(T_i) = \frac{k(v-1)-n_1d_1-n_2d_2}{vr[w(k-1)+w']} \quad , \tag{8.2.20}$$

as it was mentioned in (2.7).

8.3 Sign of the Quantity C-C'

By subtracting (8.2.10) from (8.2.9),

$$C_1-C_2 = \frac{c_1(n_2+1)-n_2c_2-k-(n_1+1)c_2+n_1c_1+k}{av}\sigma^2 \quad ,$$

$$= \frac{c_1(n_1+n_2+1)-c_2(n_1+n_2+1)}{av} = \frac{c_1-c_2}{a}\sigma^2 \quad ,$$

$$= \frac{k\Delta c_1-k\Delta c_2}{k\Delta a} \quad , \text{ by (8.1.4) and (8.1.5),}$$

$$= \frac{a(\lambda_1-\lambda_2)}{k\Delta a}\sigma^2 \quad ,$$

$$= \frac{\lambda_1-\lambda_2}{k\Delta}\sigma^2 \quad . \tag{8.3.1}$$

Similarly, by subtracting (8.2.14) from (8.2.13),

$$C_1'-C_2' = \frac{c_1'(n_1+n_2+1)-c_2'(n_1+n_2+1)}{vr}\sigma'^2 \quad ,$$

$$= \frac{c_1'-c_2'}{r}\sigma'^2 \quad , \tag{8.3.2}$$

where c_1', c_2', σ'^2 are as defined in (8.1.11) and (8.1.13). Substituting for c_1', c_2', (8.3.2) may be written,

$$C_1'-C_2' = \frac{\Delta(c_1-c_2)-r(\lambda_1-\lambda_2)}{r(\Delta-rH+r^2)}\,\sigma'^2 \quad . \tag{8.3.4}$$

Now by adding (8.3.4) and (8.3.1) one obtains:

$$C-C' = (C_1+C_1')-(C_2+C_2') = \frac{\lambda_1-\lambda_2}{k\Delta}\,\sigma^2 + \frac{\Delta(c_1-c_2)-r(\lambda_1-\lambda_2)}{r(\Delta-rH+r^2)}\,\sigma'^2 \quad ,$$

$$= \frac{\lambda_1-\lambda_2}{k\Delta}\,\sigma^2 + \frac{[\Delta a(\lambda_1-\lambda_2)/k\Delta]-r(\lambda_1-\lambda_2)}{r(\Delta-rH+r^2)}\,\sigma'^2 \quad ,$$

$$= \frac{\lambda_1-\lambda_2}{k\Delta}\,\sigma^2 + \frac{a(\lambda_1-\lambda_2)-kr(\lambda_1-\lambda_2)}{rk(\Delta-rH+r^2)}\,\sigma'^2 \quad ,$$

and since $a = rk-r$,

$$C-C' = \frac{\lambda_1-\lambda_2}{k\Delta}\,\sigma^2 - \frac{\lambda_1-\lambda_2}{k(\Delta-rH+r^2)}\,\sigma'^2 \quad . \tag{8.3.5}$$

Substituting for σ'^2 from (8.1.13),

$$C-C' = \frac{\lambda_1-\lambda_2}{k}\Big[\frac{1}{\Delta} - \frac{1}{\Delta-rH+r^2}\Big]\sigma^2 - \frac{\lambda_1-\lambda_2}{\Delta-rH+r^2}\,\sigma_b^2 \quad ,$$

$$= (\lambda_1-\lambda_2)\Big[\frac{\Delta-rH+r^2-\Delta}{k\Delta(\Delta-rH+r^2)}\,\sigma^2 - \frac{1}{\Delta-rH+r^2}\,\sigma_b^2\Big] \quad ,$$

$$= (\lambda_1-\lambda_2)\Big[\frac{-r(H-r)}{k\Delta(\Delta-rH+r^2)}\,\sigma^2 - \frac{1}{\Delta-rH+r^2}\,\sigma^2\Big] \quad . \tag{8.3.6}$$

Now if

$$\Delta > 0, \quad \Delta-rH+r^2 > 0, \quad H > r \;, \tag{8.3.7}$$

then (8.3.6) is of the form:

$$C-C' = (\lambda_1-\lambda_2)[\text{negative quantity}] \quad ,$$

i.e.,

$$\begin{aligned} C-C' &> 0 \quad \text{if} \quad \lambda_1 < \lambda_2 \quad , \\ C-C' &< 0 \quad \text{if} \quad \lambda_1 > \lambda_2 \quad . \end{aligned} \tag{8.3.8}$$

It is to be noticed that the above conditions in (8.3.7) are satisfied in every design listed in R.C. Bose "Tables of p.b.i.b. designs with two associate classes", Reference (1).

8.4 Theorem 3

As a generalization to Theorem 1, consider the t independent parameters $\tau_1, \tau_2, \ldots, \tau_t$ and suppose that for each τ_i there exist two independent unbiased estimates U_i and X_i, where $U_i \frown N(\tau_i, V_i = \theta_i\sigma^2)$ and $X_i \frown N(\tau_i, V_i' > V_i)$. Suppose also that independently of the X_i's, U_i's there exists an unbiased estimate s^2 for σ^2 where $s^2 \frown \chi^2\frac{\sigma^2}{f}$, then we can determine a known constant B so that the unbiased combined estimate,

$$\hat{\tau}_i = U_i + \frac{\theta_i Bs^2}{\sum\limits_{\substack{j=1 \\ j\neq i}}^{t} (X_j-U_j)^2} (X_i-U_i) \quad , \tag{8.4.1}$$

has a variance less than $V(U_i)$ whenever $t > 5$.

<u>Proof</u>:

$$E(\hat{\tau}_i) = \tau_i + \theta_i B\sigma^2 E \frac{1}{\sum\limits_{j\neq i}^{t}(X_j-U_j)^2} \cdot E(X_i-U_i) = \tau_i \quad . \qquad (8.4.2)$$

$$V(\hat{\tau}_i) = V(U_i) + \theta_i^2 B^2 E(s^4)\cdot E \frac{z_i^2}{\left(\sum\limits_{\substack{j=1\\ j\neq i}}^{t} z_j^2\right)^2} + 2\theta_i BE(s^2)\cdot E \frac{z_i(U_i-\tau_i)}{\sum\limits_{\substack{j=1\\ j\neq i}}^{t} z_j^2} ,$$

where $z_j = X_j-U_j$, $z_i = X_i-U_i$.

$$V(\hat{\tau}_i) = V_i + \frac{\theta_i^2\sigma^4 B^2(f+2)}{f} E(z_i^2)\cdot E \frac{1}{\left(\sum\limits_{j\neq i}^{t} z_j^2\right)^2}$$

$$+ 2\theta_i\sigma^2 BE \frac{1}{\sum\limits_{j\neq i}^{t} z_j^2} \cdot E[z_i(U_i-\tau_i)] \quad , \qquad (8.4.3)$$

but

$$E(z_i^2) = E[(X_i-\tau_i)-(U_i-\tau_i)]^2 = E(X_i-\tau_i)^2+E(U_i-\tau_i)^2 = V_i+V_i' ,$$

substituting into (8.4.3),

$$V(\hat{\tau}_i) = V_i + \frac{V_i^2 B^2(f+2)(V_i+V_i')}{f} E \frac{1}{\left(\sum\limits_{j\neq i}^{t} z_j^2\right)^2}$$

$$+ 2V_i BE \frac{1}{\sum\limits_{j\neq i}^{t} z_j^2} E[(X_i-U_i)(U_i-\tau_i)] \quad . \qquad (8.4.4)$$

Let

$$\Phi_i = V_i + V_i' = V(z_i), \qquad i=1,2,3\ldots t \quad , \tag{8.4.5}$$

then

$$V(\hat{\tau}_i) = V_i + \frac{V_i^2 B^2 (f+2)\Phi_i}{f} \, E \frac{1}{\left(\sum_{j\neq i}^{t} z_j^2\right)^2} + 2V_i BE\left(\frac{1}{\sum_{j\neq i}^{t} z_j^2}\right)(-V_i) \quad ,$$

$$= V_i + \frac{V_i^2 B (f+2)\Phi_i}{f} \, E \frac{1}{\left(\sum_{j\neq i}^{t} z_j^2\right)^2} - 2V_i^2 BE \frac{1}{\sum_{j\neq i}^{t} z_j^2} \quad . \tag{8.4.6}$$

It should be noticed that $z_j \frown N(0, \Phi_j)$. Let $z_j = \sqrt{\Phi_j} \cdot Y_j$,

then, $Y_j \frown N(0, 1)$, $\sum_{j\neq i}^{t} Y_j^2 \frown \chi^2_{(t-1)}$, and,

$$\sum_{j\neq i}^{t} z_j^2 = \sum_{j\neq i}^{t} \Phi_j Y_j^2 \quad . \tag{8.4.7}$$

Now one can write:

$$E \frac{1}{\left(\sum_{j\neq i}^{t} z_j^2\right)^2} \leq E \frac{1}{\Phi^2_{min.}\left(\sum_{j\neq i}^{t} Y_j^2\right)^2} = \frac{1}{\Phi^2_{min.}} \, E\left(\frac{1}{\chi^2_{(t-1)}}\right)^2$$

$$= \frac{1}{\Phi^2_{min.}(t-3)(t-5)} \quad , \tag{8.4.8}$$

and

$$E \frac{1}{\sum_{j\neq i}^{t} Z_j^2} \geq E \frac{1}{\Phi_{max.} \sum_{j\neq i}^{t} Y_j^2} = \frac{1}{\Phi_{max.}} \cdot E \frac{1}{\chi^2_{(t-1)}} = \frac{1}{\Phi_{max.}(t-3)} \quad . \tag{8.4.9}$$

Using (8.4.8) and (8.4.9), we can write (8.4.6) as:

$$V(\hat{\tau}_i) < V_i + \frac{V_i^2 B^2 (f+2)\Phi_i}{f\Phi^2_{min.}(t-3)(t-5)} - \frac{2V_i^2 B}{\Phi_{max.}(t-3)} \quad . \tag{8.4.10}$$

In (8.4.10) we want the quadratic in B, in the last two terms to be negative, i.e.,

$$\frac{V_i^2 B}{t-3}\left[\frac{B(f+2)\Phi_i}{f\Phi^2_{min.}(t-5)} - \frac{2}{\Phi_{max.}}\right] < 0 \quad ,$$

or

$$0 < B < \frac{2f\Phi^2_{min.}(t-5)}{(f+2)\Phi_i \cdot \Phi_{max.}} \quad , \qquad t > 5 \ .$$

$$B_{opt.} = \frac{f\Phi^2_{min.}(t-5)}{(f+2)\Phi_i \cdot \Phi_{max.}} \quad , \tag{8.4.11}$$

where $B_{opt.}$ is the value of B which minimizes the above quadratic.

Since $\Phi_i \leq \Phi_{max.}$, one can write:

$$B = \frac{f(t-5)}{f+2} \cdot \left(\frac{\Phi_{min.}}{\Phi_{max.}}\right)^2 \quad , \tag{8.4.12}$$

as an admissible B, i.e., within the above range of B which makes $V(\hat{\tau}_i) < V(U_i)$.

In the incomplete block designs where both inter- and intra-block estimates are available and independent, the variance is of the form $\alpha_1\sigma^2 + \alpha_2\sigma_b^2$. Let $\frac{\sigma_b^2}{\sigma^2} = R$, and knowing that σ_b^2 should never be less than σ^2 in any reasonable incomplete block design,

$$\frac{\Phi_{min.}}{\Phi_{max.}} = \frac{\alpha_1\sigma^2+\alpha_2\sigma_b^2}{\gamma_1\sigma^2+\gamma_2\sigma_b^2} = \frac{\alpha_1+\alpha_2 R}{\gamma_1+\gamma_2 R} = F(R) \quad , \tag{8.4.13}$$

where α_1, α_2, γ_1, γ_2 are known in terms of the parameters of the design.

$F(R)$ is a hyperbolic function; it is monotonically decreasing if $\alpha_2\gamma_1 < \alpha_1\gamma_2$, and monotonically increasing if $\alpha_2\gamma_1 > \alpha_1\gamma_2$.

Now if $\alpha_2\gamma_1 < \alpha_1\gamma_2$, then

$$F(\infty) < F(R) < F(1) \quad , \tag{8.4.14}$$

and from (8.4.13),

$$\frac{\Phi_{min.}}{\Phi_{max.}} = F(R) > F(\infty) = \frac{\alpha_2}{\gamma_2} \quad . \tag{8.4.15}$$

In this case,

$$B = \frac{f(t-5)}{f+2} \; \frac{\alpha_2}{\gamma_2} \quad . \tag{8.4.16}$$

If $\alpha_2\gamma_1 > \gamma_2\alpha_1$, then,

$$F(1) < F(R) < F(\infty) \quad , \tag{8.4.17}$$

and from (8.4.13),

$$\frac{\Phi_{\min.}}{\Phi_{\max.}} = F(R) > F(1) = \frac{\alpha_1+\alpha_2}{\gamma_1+\gamma_2} \quad . \qquad (8.4.18)$$

In this case

$$B = \frac{f(t-5)}{f+2} \; \frac{\alpha_1+\alpha_2}{\gamma_1+\gamma_2} \quad . \qquad (8.4.19)$$

In both cases B can be computed from the parameters of our design.

8.5 Special Case

When there are only two different variances, let t_1 of the z's have variance Φ_1 and let the remaining $t-t_1$ have variance Φ_2, then (8.4.7) becomes:

$$\sum_{j\neq i}^{t} z_j^2 = \Phi_1\chi^2(t_1-1)+\Phi_2\chi^2(t-t_1) = \Phi_1\chi^2(\nu_1)+\Phi_2\chi^2(\nu_2) \quad ,$$

$$\text{when} \quad V(z_i) = \Phi_1 \quad , \qquad (8.5.1)$$

$$\sum_{j\neq i}^{t} z_j^2 = \Phi_1\chi^2(t_1)+\Phi_2\chi^2(t-t_1-1) = \Phi_1\chi^2(\nu_1)+\Phi_2\chi^2(\nu_2) \quad ,$$

$$\text{when} \quad V(z_i) = \Phi_2 \quad . \qquad (8.5.2)$$

It should be noticed that ν_1 and ν_2 assume different values in the two cases, but $\nu_1+\nu_2 = t-1$ always.

In both cases let us assume that

$$\sum_{\substack{j=1\\ j\neq i}}^{h} z_j^2 = \Phi_1\chi^2_{\nu_1} + \Phi_2\chi^2_{\nu_2} = X + Y \quad , \tag{8.5.3}$$

where

$$f(X) = \frac{1}{\Gamma(\tfrac12\nu_1)(2\Phi_1)^{\frac12\nu_1}}\, x^{\frac12\nu_1-1}\, e^{-x/2\Phi_1}\, dx \;, \qquad x > 0, \quad \Phi_1 > 0 \;, \tag{8.5.4}$$

and

$$f(Y) = \frac{1}{\Gamma(\tfrac12\nu_1)(2\Phi_2)^{\frac12\nu_2}}\, y^{\frac12\nu_2-1}\, e^{-y/2\Phi_2}\, dy \;, \qquad y > 0, \quad \Phi_2 > 0 \;. \tag{8.5.5}$$

Applying lemma 1, we have

$$\beta_1 = 2\Phi_1,\ \alpha_1 = \tfrac12\nu_1-1,\ \beta_2 = 2\Phi_2,\ \alpha_2 = \tfrac12\nu_2-1 \;,$$

thus substituting into (4.6) we get:

$$E\left(\frac{1}{X+Y}\right)^r = \frac{\Gamma(\tfrac12\overline{\nu_1+\nu_2}-r)}{\Gamma(\tfrac12\nu_1)\Gamma(\tfrac12\nu_2)} \int_0^1 \frac{u^{\frac12\nu_1-1}\,(1-u)^{\frac12\nu_2-1}}{[2\Phi_1 u+2\Phi_2(1-u)]^r}\, du \quad ,$$

$$= \frac{\Gamma(\tfrac12\overline{\nu_1+\nu_2}-r)}{2^r\Gamma(\tfrac12\nu_1)\Gamma(\tfrac12\nu_2)\Phi_1^{\,r}} \int_0^1 \frac{u^{\frac12\nu_1-1}\,(1-u)^{\frac12\nu_2-1}}{\left[u + \frac{\Phi_2}{\Phi_1}(1-u)\right]^r}\, du \quad . \tag{8.5.6}$$

Suppose that one is able to find a lower bound L and an upper bound P for the ratio $\frac{\Phi_2}{\Phi_1}$ which are relatively not far from each other, then one can write:

$$E\left(\frac{1}{X+Y}\right)^2 \leq \frac{\Gamma(\tfrac{1}{2}\overline{\nu_1+\nu_2}-2)}{4\Gamma(\tfrac{1}{2}\nu_1)\Gamma(\tfrac{1}{2}\nu_2)\Phi_1^2}\int_0^1 \frac{u^{\frac{1}{2}\nu_1-1}(1-u)^{\frac{1}{2}\nu_2-1}}{[u+L(1-u)]^2}\,du \quad ,$$

$$= \frac{\Gamma(\tfrac{1}{2}\overline{\nu_1+\nu_2}-2)}{4\Gamma(\tfrac{1}{2}\nu_1)\Gamma(\tfrac{1}{2}\nu_2)\Phi_1^2}\,G_2 = \frac{G_2}{(t-3)(t-5)\beta(\tfrac{1}{2}\nu_1,\ \tfrac{1}{2}\nu_2)\Phi_1^2} \quad , \qquad (8.5.7)$$

and

$$E\,\frac{1}{X+Y} \geq \frac{\Gamma(\tfrac{1}{2}\overline{\nu_1+\nu_2}-1)}{2\Gamma(\tfrac{1}{2}\nu_1)\Gamma(\tfrac{1}{2}\nu_2)\Phi_1}\int_0^1 \frac{u^{\frac{1}{2}\nu_1-1}(1-u)^{\frac{1}{2}\nu_2-1}}{u+P(1-u)}\,du \quad ,$$

$$= \frac{\Gamma(\tfrac{1}{2}\overline{\nu_1+\nu_2}-1)}{2\Gamma(\tfrac{1}{2}\nu_1)\Gamma(\tfrac{1}{2}\nu_2)\Phi_1}\,G_1 = \frac{G_1}{(t-3)\beta(\tfrac{1}{2}\nu_1,\ \tfrac{1}{2}\nu_2)\Phi_1} \quad . \qquad (8.5.8)$$

Accounting for (8.5.3), and substituting (8.5.7) and (8.5.8) into (8.4.6):

$$V(\hat{\tau}_i) < V_i + \frac{V_i^2B^2(f+2)\Phi_iG_2}{f(t-3)(t-5)\beta(\tfrac{1}{2}\nu_1,\ \tfrac{1}{2}\nu_2)\Phi_1^2} - \frac{2V_i^2BG_1}{(t-3)\beta(\tfrac{1}{2}\nu_1,\ \tfrac{1}{2}\nu_2)\Phi_1} \quad ,$$

$$= V_i + \frac{V_i^2B}{(t-3)\beta(\tfrac{1}{2}\nu_1,\ \tfrac{1}{2}\nu_2)\Phi_1}\left[\frac{B(f+2)\Phi_iG_2}{f(t-5)\Phi_1} - 2G_1\right] \quad . \qquad (8.5.9)$$

Assuming $t > 5$, the second term is negative when

$$0 < B < \frac{2f(t-5)G_1\Phi_1}{(f+2)\Phi_iG_2} \quad . \qquad (8.5.10)$$

The value $B_{opt.}$ which makes it the most negative is:

$$B_{\text{opt.}} = \frac{f\Phi_1 G_1(t-5)}{\Phi_i G_2(f+2)} \quad . \tag{8.5.11}$$

Since Φ_i is either Φ_1 or Φ_2 in this special case, we have

$$B_{\text{opt.}} = \frac{fG_1(t-5)}{(f+2)G_2} \quad , \quad \text{when} \quad \Phi_i = \Phi_1 \quad . \tag{8.5.12}$$

However, (8.5.11) depends on the variances Φ_1, Φ_2 when $\Phi_i = \Phi_2$; to avoid this, we go back to (8.5.6) and take Φ_2 outside the integral instead of Φ_1 to get:

$$E\left(\frac{1}{X+Y}\right)^r = \frac{\Gamma(\tfrac{1}{2}\overline{\nu_1+\nu_2}-r)}{2^r\Gamma(\tfrac{1}{2}\nu_1)\Gamma(\tfrac{1}{2}\nu_2)\Phi_2^r}\int_0^1 \frac{u^{\frac{1}{2}\nu_1-1}(1-u)^{\frac{1}{2}\nu_2-1}}{\left[\frac{\Phi_1}{\Phi_2}u+(1-u)\right]^r}\,du \,. \tag{8.5.13}$$

The lower and upper bounds for $\frac{\Phi_2}{\Phi_1}$ are $\frac{1}{P}$ and $\frac{1}{L}$ respectively, hence using (8.5.13):

$$E\left(\frac{1}{X+Y}\right)^2 < \frac{\Gamma(\tfrac{1}{2}\overline{\nu_1+\nu_2}-2)}{4\Phi_2^2\Gamma(\tfrac{1}{2}\nu_1)\Gamma(\tfrac{1}{2}\nu_2)}\int_0^1 \frac{u^{\frac{1}{2}\nu_1-1}(1-u)^{\frac{1}{2}\nu_2-1}}{\left[\frac{1}{P}u+(1-u)\right]^r}\,du \quad ,$$

$$= \frac{\Gamma(\tfrac{1}{2}\overline{\nu_1+\nu_2}-2)G_2'}{4\Phi_2^2\Gamma(\tfrac{1}{2}\nu_1)\Gamma(\tfrac{1}{2}\nu_2)} = \frac{G_2'}{(t-3)(t-5)\beta(\tfrac{1}{2}\nu_1,\ \tfrac{1}{2}\nu_2)\Phi_2^2} \quad . \tag{8.5.14}$$

$$E\left(\frac{1}{X+Y}\right) > \frac{\Gamma(\tfrac{1}{2}\overline{\nu_1+\nu_2}-1)}{2\Phi_2\Gamma(\tfrac{1}{2}\nu_1)\Gamma(\tfrac{1}{2}\nu_2)}\int_0^1 \frac{u^{\frac{1}{2}\nu_1-1}(1-u)^{\frac{1}{2}\nu_2-1}}{\frac{1}{L}u+(1-u)}\,du \quad ,$$

$$= \frac{\Gamma(\tfrac{1}{2}\overline{\nu_1+\nu_2}-1)G_1'}{2\Gamma(\tfrac{1}{2}\nu_1)\Gamma(\tfrac{1}{2}\nu_2)\Phi_2} = \frac{G_1'}{(t-3)\beta(\tfrac{1}{2}\nu_1,\ \tfrac{1}{2}\nu_2)\Phi_2} \quad . \tag{8.5.15}$$

Substituting in (8.5.7) for the case, where $\Phi_i = \Phi_2$:

$$V(\hat{\tau}_i) < V_i + \frac{V_i^2B^2(f+2)\Phi_2G_2'}{f(t-3)(t-5)\beta(\tfrac{1}{2}\nu_1,\ \tfrac{1}{2}\nu_2)\Phi_2^2} - \frac{2V_i^2BG_1'}{(t-3)\beta(\tfrac{1}{2}\nu_1,\ \tfrac{1}{2}\nu_2)\Phi_2}\ ,$$

$$= V_i + \frac{V_i^2B}{(t-3)\beta(\tfrac{1}{2}\nu_1,\ \tfrac{1}{2}\nu_2)\Phi_2}\left[\frac{B(f+2)G_2'}{f(t-5)} - 2G_1'\right] \quad . \qquad (8.5.16)$$

Assuming $t > 5$, the second term is negative when

$$0 < B < \frac{2f(t-5)G_1'}{(f+2)G_2'}\ , \qquad (8.5.17)$$

and

$$B_{opt.} = \frac{f(t-5)G_1'}{(f+2)G_2'} \quad . \qquad (8.5.18)$$

This is independent of the Φ's, and (8.5.18) will be used when $\Phi_i = \Phi_2$. Substituting (8.5.12) into (8.5.9), we get for the case $\Phi_i = \Phi_1$:

$$V(\hat{\tau}_i) < V_i + \frac{V_i^2fG_1(t-5)}{(f+2)G_2(t-3)\beta(\tfrac{1}{2}\nu_1,\ \tfrac{1}{2}\nu_2)\Phi_1}[G_1-2G_1]\ ,$$

or

$$V(\hat{\tau}) < V_i - \frac{f(t-5)G_1^2}{(f+2)G_2(t-3)\beta(\tfrac{1}{2}\nu_1,\ \tfrac{1}{2}\nu_2)} \cdot \frac{V_i^2}{\Phi_1} \quad . \qquad (8.5.19)$$

But $\dfrac{V_i^2}{\Phi_1} = \dfrac{V_i^2}{V_i+V_i'}$ is the utmost possible recovery. Thus the recovery ratio is at least:

$$D = \frac{f(t-5)G_1^2}{(f+2)G_2(t-3)\beta(\frac{1}{2}\nu_1, \frac{1}{2}\nu_2)} . \qquad (8.5.20)$$

Similarly substituting (8.5.18) into (8.5.16), we get for the case $\Phi_i = \Phi_2$:

$$V(\hat{\tau}_i) < V_i + \frac{V_i^2 f(t-5)G_1'}{(t-3)\beta(\frac{1}{2}\nu_1, \frac{1}{2}\nu_2)\Phi_2(f+2)G_2'}[G_1'-2G_1'] ,$$

$$= V_i - \frac{f(t-5)G_1'^2}{(f+2)G_2'(t-3)\beta(\frac{1}{2}\nu_1, \frac{1}{2}\nu_2)} \cdot \frac{V_i^2}{\Phi_2} . \qquad (8.5.21)$$

Again $\frac{V_i^2}{\Phi_2} = \frac{V_i^2}{V_i+V_i'}$ is the utmost possible recovery, and the recovery ratio is at least:

$$D = \frac{f(t-5)G_1'^2}{(f+2)G_2'(t-3)\beta(\frac{1}{2}\nu_1, \frac{1}{2}\nu_2)} . \qquad (8.5.22)$$

8.6 Recovery of Inter-block Information in Group Divisible Designs

In this case $v = mn$, and the treatments can be divided into m groups of n treatments each, such that any two treatments of the same group are first associates, while two treatments of different groups are second associates. The association scheme can be displayed by placing the treatments in a rectangular of m rows and n columns, where each row of n treatments constitutes a group. Clearly:

$$n_1 = n-1 \quad , \qquad n_2 = n(m-1) \quad . \tag{8.6.1}$$

Further:

$$(n-1)\lambda_1 + n(m-1)\lambda_2 = r(k-1) \quad , \tag{8.6.2}$$

or

$$rk - \lambda_2 v = r-\lambda_1 + n(\lambda_1-\lambda_2) \quad . \tag{8.6.3}$$

Also:

$$P_1 = \begin{pmatrix} n-2 & 0 \\ 0 & n(m-1) \end{pmatrix} , \quad P_2 = \begin{pmatrix} 0 & n-1 \\ n-1 & n(m-2) \end{pmatrix} \quad . \tag{8.6.4}$$

Bose and Connor have shown that the following inequalities hold in group divisible (G.D.) designs:

$$r \geq \lambda_1 \quad , \qquad rk - \lambda_2 v \geq 0 \quad . \tag{8.6.5}$$

They have divided the G.D. into three subclasses:

(i) Singular (S) if $r = \lambda_1$,

(ii) Semi-regular (SR) if $r > \lambda_1$ and $rk - \lambda_2 v = 0$,

(iii) Regular (R) if $r > \lambda_1$ and $rk - \lambda_2 v > 0$.

8.6.1 <u>Recovery in Regular Group Divisibles</u>

In regular G.D. designs we define the mn x mn matrix M_1 as follows:

$$
M_1 = \begin{array}{r|c|c|c|c|c|c|}
 & 1 & 2 & 3 & & m-1 & m \\
 & 1\;\;2\;\;3\dots\;\;n & 1\;\;2\;\;3\dots\;\;n & 1\;\;2\;\;3\dots\;n & \dots & 1\;\;2\;\;3\;\;\dots\;\;n & 1\;\;2\;\;3\dots\;n \\
\hline
1 & 1\;\;-1 & & & & & \\
2 & 1\;\;1\;\;-2 & & & & & \\
\cdot & \cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot & & & & & \\
n-1 & 1\;\;1\;\;1\dots(-n+1) & & & & & \\
\hline
n & & 1\;\;-1 & & & & \\
\cdot & & 1\;\;1\;\;-2 & & & & \\
\cdot & & \cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot & & & & \\
2n-2 & & 1\;\;1\;\;1\dots(-n+1) & & & & \\
\hline
\cdot & & & & & & \\
\cdot & \cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot & \cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot & \cdot\;\;\cdot\;\;\cdot\;\;\cdot & \cdot & \cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot & \cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot \\
\cdot & & & & & & \\
\hline
\cdot & & & & & 1\;\;-1 & \\
\cdot & & & & & 1\;\;1\;\;-2 & \\
\cdot & & & & & \cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot & \\
\cdot & & & & & 1\;\;1\;\;1\;\;\dots\;\;(-n+1) & \\
\hline
\cdot & & & & & & 1\;\;-1 \\
\cdot & & & & & & 1\;\;1\;\;-2 \\
\cdot & & & & & & \cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot \\
m(n-1) & & & & & & 1\;\;1\;\;1\dots(-n+1) \\
\hline
m(n-1)+1 & 1\;\;1\;\;1\dots\;\;1 & -1\;\;-1\;\;-1\dots\;\;-1 & & & & \\
\cdot & 1\;\;1\;\;1\dots\;\;1 & 1\;\;1\;\;1\dots\;\;1 & -2\;\;-2\;\;-2\dots-2 & & & \\
 & \cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot & \cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot & \cdot\;\;\cdot\;\;\cdot\;\;\cdot & \cdot & \cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot & \cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot \\
mn-2 & 1\;\;1\;\;1\dots\;\;1 & 1\;\;1\;\;1\dots\;\;1 & 1\;\;1\;\;1\dots\;1 & \dots & (-m+2)\;(-m+2)\dots(-m+2) & \\
mn-1 & 1\;\;1\;\;1\dots\;\;1 & 1\;\;1\;\;1\dots\;\;1 & 1\;\;1\;\;1\dots\;1 & \dots & 1\;\;1\;\;1\;\;\dots\;\;1 & (-m+1)\dots(-m+1) \\
mn & 1\;\;1\;\;1\dots\;\;1 & 1\;\;1\;\;1\dots\;\;1 & 1\;\;1\;\;1\dots\;1 & \dots & 1\;\;1\;\;1\;\;\dots\;\;1 & 1\;\;\dots\;\;1 \\
\hline
\end{array}
$$

Define the following contrasts between the mn intra-block estimates of the treatments $\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_v$:

$$\underline{U} = \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_{nm-1} \\ 0 \end{pmatrix} = M \begin{pmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \vdots \\ \hat{t}_{nm} \end{pmatrix} = M\underline{\hat{t}} \quad , \qquad (8.6.6)$$

where M is the same as M_1 after normalizing its rows.

Define also the similar contrasts between the inter-block estimates of the treatments, namely $\hat{t}'_1$, $\hat{t}'_2$, ..., $\hat{t}'_v$:

$$\underline{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_{nm-1} \\ 0 \end{pmatrix} = M\underline{\hat{t}}' = M \begin{pmatrix} \hat{t}'_1 \\ \vdots \\ \hat{t}'_{nm} \end{pmatrix} \qquad .(8.6.7)$$

The problem now is to combine $\underline{U}$ and $\underline{X}$ to get new estimates $\underline{\hat{\tau}} = \begin{pmatrix} \hat{\tau}_1 \\ \vdots \\ \hat{\tau}_{nm} \end{pmatrix}$. It should be noted that $U_{nm} = 0$, $X_{nm} = 0$, and the corresponding combined estimate is assumed to be zero, i.e., $\hat{\tau}_{nm} = 0$.

For the variance-covariance matrix of both $\underline{U}$ and $\underline{X}$, it is noticed first that U_i is uncorrelated with U_j for $i \neq j$; also X_i is uncorrelated with X_j for $i \neq j$, and the U_i's and X_j's are independent of each other for every i, j, by virtue of the well known fact that the inter-block estimates are independent from the intra-block estimates.

It should also be noted that in the vector $\hat{\underline{t}}$ or $\hat{\underline{t}}'$ we are grouping the treatments according to group divisible association plan, i.e., the first n are the first group (or row) in the plan, the next n treatments are the second row in the association plan, etc.; moreover, it should be noted that the contrasts included in the matrix M are of two kinds, within group contrasts and among group contrasts. It has been shown in (7.2.4) that all within contrasts have exactly the same variance, namely $V_1 - C_1$, where V_1 is the variance of the intra-treatment estimate and C_1 is the covariance between two first associate intra-estimates. For the variance of the contrasts among groups, let us take a general one, U_{mn-2}, say. Then

$$V(U_{mn-2}) = \frac{1}{n(m-2)(m-1)}[n(m-2)V_1+n(m-2)^2V_1+(m-2)n(n-1)C_1$$

$$+ (m-2)^2(n-1)nC_1+(m-2)(m-3)n^2C_2-2(m-2)n^2(m-2)C_2] \quad ,$$

$$= \frac{1}{n(m-2)(m-1)}\{n(m-2)(m-1)V_1+(n-1)C_1[n(m-2)+n(m-2)^2]$$

$$+ n^2(m-2)C_2[(m-3)-2(m-2)]\} = V_1+(n-1)C_1-nC_2 \quad ,$$

$$= (V_1-C_1) + n(C_1-C_2) \quad , \tag{8.6.8}$$

where C_2 is the covariance between two second associate intra-estimates. Now $V(U_{mn-2})$ is independent of m, which means that all among contrasts have the same variance.

Similarly, one can deduce the variances of the X's, it is $V_1'-C_1'$ for within contrasts and $V_1'-C_1'+n(C_1'-C_2')$ for among contrasts, where V_1' is the variance of the inter-treatment estimate, and C_1', C_2' are the covariances between two, first or second associate inter-estimates respectively.

Consider now the vector:

$$\underline{Z} = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{mn-1} \end{pmatrix} = \underline{X} - \underline{U} \quad , \tag{8.6.9}$$

z_i and z_j are independent for $i \neq j$.

For within group z's, one has

$$V(z_i) = V(X_i) + V(U_i) = (V_1-C_1) + (V_1'-C_1') \quad ,$$

$$= (V_1+V_1') - (C_1+C_1') = V - C \quad . \qquad (8.6.10)$$

For among group z's, one has

$$V(z_i) = V_1-C_1 + n(C_1-C_2) + V_1'-C_1' + n(C_1'-C_2') \quad ,$$

$$= V_1+V_1' - (C_1+C_1') + n[(C_1+C_1') - (C_2+C_2')] \quad ,$$

$$= V-C + n(C-C') \quad , \qquad (8.6.11)$$

where V, C, C' are defined in (8.2.15) through (8.2.17).

From (8.3.8), it is seen that

$$V-C > (V-C) + n(C-C') \quad , \quad \text{if} \quad \lambda_1 > \lambda_2 \, ,$$

and (8.6.12)

$$V-C < (V-C) + n(C-C') \quad , \quad \text{if} \quad \lambda_1 < \lambda_2 \, .$$

This will give rise to two divisions of the problem:

(i) Regular G.D. with $\lambda_1 > \lambda_2$,

(ii) Regular G.D. with $\lambda_1 < \lambda_2$.

8.6.1.1 Study of the Ratio (V-C)/[V-C+n(C-C')]

Subtracting (8.2.9) from (8.2.11),

$$V_1-C_1 = \left[\frac{k(v-1)-n_1c_1-n_2c_2}{av} - \frac{c_1(n_2+1)-n_2c_2-k}{av}\right]\sigma^2 \quad ,$$

$$= \frac{kv-c_1(n_1+n_2+1)}{av}\,\sigma^2 = \frac{k-c_1}{a}\,\sigma^2 \quad . \qquad (8.6.14)$$

Subtracting (8.2.13) from (8.2.12),

$$V_1'-C_1' = \left(\frac{k(v-1)-n_1c_1'-n_2c_2'}{vr} - \frac{c_1'(n_2+1)-n_2c_2'-k}{vr}\right)\sigma'^2 \quad ,$$

$$= \frac{kv-c_1'(n_1+n_2+1)}{vr}\,\sigma'^2 = \frac{k-c_1'}{r}\,\sigma'^2 \quad . \qquad (8.6.15)$$

Adding (8.6.14) and (8.6.15),

$$V-C = V_1-C_1 + V_1'-C_1' = \frac{k-c_1}{a}\,\sigma^2 + \frac{k-c_1'}{r}\,\sigma'^2 \quad . \qquad (8.6.16)$$

By (8.3.5) one can write:

$$V-C + n(C-C') = \left[\frac{k-c_1}{a} + \frac{n(\lambda_1-\lambda_2)}{k\Delta}\right]\sigma^2 + \left[\frac{k-c_1'}{r} - \frac{n(\lambda_1-\lambda_2)}{k(\Delta-rH+r^2)}\right]\sigma'^2 \quad . \qquad (8.6.17)$$

From (8.6.4) one obtains $f = p^1_{12} = 0$, $g = p^2_{12} = n-1$, and substituting in (8.1.2),

$$k^2\Delta = (a+\lambda_1)(a+\lambda_2) + (\lambda_1-\lambda_2)(-a-\lambda_1)(n-1) \quad ,$$

$$= (a+\lambda_1)[a+\lambda_2-(n-1)(\lambda_1-\lambda_2)] \quad ,$$

$$= (a+\lambda_1)[a+\lambda_1-n(\lambda_1-\lambda_2)] \quad ,$$

and using (8.6.3), one can write

$$k^2\Delta = (a+\lambda_1)\lambda_2 v \quad . \qquad (8.6.18)$$

Substituting f and g in (8.1.3),

$$kH = (2a+\lambda_1+\lambda_2) - n(\lambda_1-\lambda_2) + \lambda_1-\lambda_2 \quad ,$$

$$= a+\lambda_1 + [a+\lambda_1-n(\lambda_1-\lambda_2)] \quad ,$$

and using (8.6.3), one obtains

$$kH = a + \lambda_1 + \lambda_2 v \quad . \tag{8.6.19}$$

Substituting f and g in (8.1.4),

$$k\Delta c_1 = \lambda_1(a+\lambda_2) - (n-1)\lambda_1(\lambda_1-\lambda_2) \quad ,$$

$$= \lambda_1[a+\lambda_2-n(\lambda_1-\lambda_2)+\lambda_1-\lambda_2] \quad ,$$

$$= \lambda_1[a+\lambda_1-n(\lambda_1-\lambda_2)] \quad ,$$

by (8.6.3):

$$k\Delta c_1 = \lambda_1\lambda_2 v \quad . \tag{8.6.20}$$

Subtracting (8.1.5) from (8.1.4),

$$k\Delta(c_1-c_2) = \lambda_1(a+\lambda_2)-\lambda_2(a+\lambda_1) = a(\lambda_1-\lambda_2) \; ,$$

or

$$c_1-c_2 = \frac{ak(\lambda_1-\lambda_2)}{k^2\Delta} \quad ,$$

and using (8.6.18), one obtains:

$$c_1-c_2 = \frac{ka(\lambda_1-\lambda_2)}{\lambda_2 v(a+\lambda_1)} \quad . \tag{8.6.21}$$

Now using (8.6.18) and (8.6.20),

$$\frac{k-c_1}{a} = \frac{k^2\Delta-k\Delta c_1}{ak\Delta} = \frac{a\lambda_2 v}{ak\Delta} = \frac{\lambda_2 vk}{k^2\Delta} = \frac{\lambda_2 vk}{\lambda_2 v(a+\lambda_1)} \quad ,$$

$$= \frac{k}{a+\lambda_1} \quad . \tag{8.6.22}$$

(8.6.22) indicates that in regular G.D., $k-c_1 > 0$ or $k > c_1$. Let $A = \Delta - rH + r^2$, then

$$kA = k\Delta - rkH + r^2k \quad ,$$

and using (8.6.18) and (8.6.19), one obtains

$$kA = \frac{\lambda_2 v}{k}(a+\lambda_1) - r(a+\lambda_1+\lambda_2 v) + kr^2 \quad ,$$

$$= \frac{1}{k}[\lambda_2 v(rk-r+\lambda_1)-rk(rk-r+\lambda_1+\lambda_2 v)+k^2r^2] \quad ,$$

$$= \frac{1}{k}(-r\lambda_2 v+\lambda_1\lambda_2 v+r^2k-rk\lambda_1) = \frac{(r-\lambda_1)(rk-\lambda_2 v)}{k} \quad . \tag{8.6.23}$$

Since $r > \lambda_1$, $rk > \lambda_2 v$ in regular G.D., one can say that $A = \Delta - rH + r^2 > 0$ always in this class of designs.

Also, from (8.1.11),

$$\frac{k-c_1'}{r} = \frac{k}{r} - \frac{c_1\Delta - r\lambda_1}{rA} \quad ,$$

and using (8.6.20) and (8.6.23), one obtains

$$\frac{k-c_1'}{r} = \frac{k}{r} - \frac{\lambda_1\lambda_2 v - kr\lambda_1}{\frac{r}{k}(r-\lambda_1)(rk-\lambda_2 v)} \quad ,$$

$$= \frac{k(r-\lambda_1)(rk-\lambda_2 v)+k\lambda_1(rk-\lambda_2 v)}{r(r-\lambda_1)(rk-\lambda_2 v)} = \frac{k}{r-\lambda_1} \quad . \tag{8.6.24}$$

Substituting (8.6.22) and (8.6.24) into (8.6.16), one can write

$$V-C = \frac{k}{a+\lambda_1}\sigma^2 + \frac{k}{r-\lambda_1}\sigma'^2 \quad . \tag{8.6.25}$$

From (8.6.18),

$$\frac{n(\lambda_1-\lambda_2)}{k\Delta} = \frac{nk(\lambda_1-\lambda_2)}{k^2\Delta} = \frac{nk(\lambda_1-\lambda_2)}{\lambda_2 v(a+\lambda_1)} \quad ,$$

and using (8.6.3), one obtains

$$\frac{n(\lambda_1-\lambda_2)}{k\Delta} = \frac{k(a+\lambda_1-\lambda_2 v)}{\lambda_2 v(a+\lambda_1)} = \frac{k}{\lambda_2 v} - \frac{k}{a+\lambda_1} \quad . \qquad (8.6.26)$$

From (8.6.23), one can write

$$\frac{n(\lambda_1-\lambda_2)}{kA} = \frac{nk(\lambda_1-\lambda_2)}{nk^2A} = \frac{nk(\lambda_1-\lambda_2)}{(r-\lambda_1)(rk-\lambda_2 v)} \quad . \qquad (8.6.27)$$

Substituting (8.6.22), (8.6.24), (8.6.26), and (8.6.27) into (8.6.17),

$$V-C + n(C-C') = \left(\frac{k}{a+\lambda_1} + \frac{k}{\lambda_2 v} - \frac{k}{a+\lambda_1}\right)\sigma^2 + \left[\frac{k}{r-\lambda_1} - \frac{nk(\lambda_1-\lambda_2)}{(r-\lambda_1)(rk-\lambda_2 v)}\right]\sigma'^2,$$

$$= \frac{k}{\lambda_2 v}\,\sigma^2 + \frac{k[rk-\lambda_2 v-n(\lambda_1-\lambda_2)]}{(r-\lambda_1)(rk-\lambda_2 v)}\,\sigma'^2 \quad ,$$

and by using (8.6.3), one obtains

$$V-C + n(C-C') = \frac{k}{\lambda_2 v}\,\sigma^2 + \frac{k(r-\lambda_1)}{(r-\lambda_1)(rk-\lambda_2 v)}\,\sigma'^2 \quad ,$$

$$= \frac{k}{\lambda_2 v}\,\sigma^2 + \frac{k}{rk-\lambda_2 v}\,\sigma'^2 \quad . \qquad (8.6.28)$$

From (8.6.25) and (8.6.28), one can write:

$$\frac{V-C}{V-C+n(C-C')} = \frac{\frac{k}{a+\lambda_1}\,\sigma^2 + \frac{k}{r-\lambda_1}\,\sigma'^2}{\frac{k}{\lambda_2 v}\,\sigma^2 + \frac{k}{rk-\lambda_2 v}\,\sigma'^2} \quad . \tag{8.6.29}$$

Substituting (8.1.13) into (8.6.29),

$$\frac{V-C}{V-C+n(C-C')} = \frac{\left(\frac{1}{a+\lambda_1} + \frac{1}{r-\lambda_1}\right)\sigma^2 + \frac{k}{r-\lambda_1}\,\sigma_b^2}{\left(\frac{1}{\lambda_2 v} + \frac{1}{rk-\lambda_2 v}\right)\sigma^2 + \frac{k}{rk-\lambda_2 v}\,\sigma_b^2} \quad ,$$

$$= \frac{\frac{rk}{(r-\lambda_1)(a+\lambda_1)}\,\sigma^2 + \frac{k}{r-\lambda_1}\,\sigma_b^2}{\frac{rk}{\lambda_2 v(rk-\lambda_2 v)}\,\sigma^2 + \frac{k}{rk-\lambda_2 v}\,\sigma_b^2} \quad .$$

Let $\frac{\sigma_b^2}{\sigma^2} = R > 1$, then,

$$\frac{V-C}{V-C+n(C-C')} = \frac{\frac{r}{(r-\lambda_1)(a+\lambda_1)} + \frac{1}{r-\lambda_1}\,R}{\frac{r}{\lambda_2 v(rk-\lambda_2 v)} + \frac{1}{rk-\lambda_2 v}\,R} \quad ,$$

$$= \frac{\alpha_1+\beta_1 R}{\alpha_2+\beta_2 R} = F(R) \quad , \tag{8.6.30}$$

where

$$\alpha_1 = \frac{r}{(r-\lambda_1)(a+\lambda_1)} \quad , \qquad \beta_1 = \frac{1}{r-\lambda_1} \quad ,$$

$$\alpha_2 = \frac{r}{\lambda_2 v(rk-\lambda_2 v)} \quad , \qquad \beta_2 = \frac{1}{rk-\lambda_2 v} \quad . \tag{8.6.31}$$

$$\frac{dF(R)}{dR} = \frac{\beta_1(\alpha_2+\beta_2 R)-\beta_2(\alpha_1+\beta_1 R)}{(\alpha_2+\beta_2 R)^2} ,$$

$$= \frac{\beta_1\alpha_2-\beta_2\alpha_1}{(\alpha_2+\beta_2 R)^2} .$$

Substituting from (8.6.31), one can write,

$$\beta_1\alpha_2-\beta_2\alpha_1 = \frac{r}{\lambda_2 v(r-\lambda_1)(rk-\lambda_2 v)} - \frac{r}{(r-\lambda_1)(a+\lambda_1)(rk-\lambda_2 v)} ,$$

$$= \frac{r(a+\lambda_1)-r\lambda_2 v}{\lambda_2 v(r-\lambda_1)(a+\lambda_1)(rk-\lambda_2 v)} = \frac{r(a+\lambda_1-\lambda_2 v)}{\lambda_2 v(r-\lambda_1)(a+\lambda_1)(rk-\lambda_2 v)} ,$$

by (8.6.3),

$$= \frac{nr(\lambda_1-\lambda_2)}{\lambda_2 v(r-\lambda_1)(a+\lambda_1)(rk-\lambda_2 v)} . \tag{8.6.32}$$

Since $r-\lambda_1 > 0$, $rk-\lambda_2 > 0$ in regular G.D., one can say that

$$\frac{dF(R)}{dR} \text{ is } \begin{cases} > 0 & \text{if } \lambda_1 > \lambda_2 \\ < 0 & \text{if } \lambda_1 < \lambda_2 \end{cases} . \tag{8.6.33}$$

Thus, $F(R)$ is monotonically increasing if $\lambda_1 > \lambda_2$, and monotonically decreasing if $\lambda_1 < \lambda_2$. This means that

$$\begin{aligned} F(1) < F(R) < F(\infty) \quad & \text{when } \lambda_1 > \lambda_2 , \\ F(\infty) < F(R) < F(1) \quad & \text{when } \lambda_1 < \lambda_2 . \end{aligned} \tag{8.6.34}$$

Substituting from (8.6.31),

$$F(1) = \frac{\alpha_1+\beta_1}{\alpha_2+\beta_2} = \frac{\dfrac{r}{(r-\lambda_1)(a+\lambda_1)} + \dfrac{1}{(r-\lambda_1)}}{\dfrac{r}{\lambda_2 v(rk-\lambda_2 v)} + \dfrac{1}{rk-\lambda_2 v}} = \frac{\lambda_2 v(rk-\lambda_2 v)(rk+\lambda_1)}{(r-\lambda_1)(a+\lambda_1)(r+\lambda_2 v)} , \tag{8.6.35}$$

and

$$F(\infty) = \frac{\beta_1}{\beta_2} = \frac{rk-\lambda_2 v}{r-\lambda_1} . \tag{8.6.36}$$

For the inverse ratio $G(R) = \frac{1}{F(R)} = \frac{V-C+n(C-C')}{V-C}$, one has

$$G(1) = \frac{1}{F(1)} = \frac{(r-\lambda_1)(a+\lambda_1)(r+\lambda_2 v)}{\lambda_2 v(rk-\lambda_2 v)(rk+\lambda_1)} , \tag{8.6.37}$$

and

$$G(\infty) = \frac{1}{F(\infty)} = \frac{r-\lambda_1}{rk-\lambda_2 v} . \tag{8.6.38}$$

Also,

$$\begin{aligned} &G(\infty) < G(R) < G(1) \quad \text{if} \quad \lambda_1 > \lambda_2 , \\ &G(1) < G(R) < G(\infty) \quad \text{if} \quad \lambda_1 < \lambda_2 . \end{aligned} \tag{8.6.39}$$

8.6.1.2 <u>Application of Special Case of Theorem 3</u>

It should be noticed that the results of the special case are exactly what one desires here, where $\Phi_1 = V-C$, $\Phi_2 = V-C + n(C-C')$. Let:

$$L = \frac{r-\lambda_1}{rk-\lambda_2 v}, \qquad P = \frac{(r-\lambda_1)(a+\lambda_1)(r+\lambda_2 v)}{\lambda_2 v(rk-\lambda_2 v)(rk+\lambda_1)} . \tag{8.6.40}$$

Then for <u>within</u> comparisons with variance $\Phi_1 = V-C$, one has $\nu_1 = m(n-1)-1$, $\nu_2 = m-1$. Thus for:

(i) <u>$\lambda_1 > \lambda_2$:</u>

$$L < \frac{\Phi_2}{\Phi_1} < P \quad , \tag{8.6.41}$$

and G_2 in (8.5.7) becomes:

$$G_2 = \int_0^1 \frac{u^{\frac{1}{2}(v-m-3)}\,(1-u)^{\frac{1}{2}(m-3)}}{[u+L(1-u)]^2}\,du \quad . \tag{8.6.42}$$

Also G_1 in (8.5.8) becomes:

$$G_1 = \int_0^1 \frac{u^{\frac{1}{2}(v-m-3)}\,(1-u)^{\frac{1}{2}(m-3)}}{u + P(1-u)}\,du \quad . \tag{8.6.43}$$

And for:

(ii) <u>$\lambda_1 < \lambda_2$:</u>

$$P < \frac{\Phi_2}{\Phi_1} < L \quad ;$$

G_2 and G_1 in (8.5.7) and (8.5.8) become:

$$G_2 = \int_0^1 \frac{u^{\frac{1}{2}(v-m-3)}\,(1-u)^{\frac{1}{2}(m-3)}}{[u+P(1-u)]^2}\,du \quad , \tag{8.6.44}$$

and

$$G_1 = \int_0^1 \frac{u^{\frac{1}{2}(v-m-3)} \; (1-u)^{\frac{1}{2}(m-3)}}{u + L(1-u)} \, du \quad . \tag{8.6.45}$$

For among comparisons with variance $\Phi_2 = V-C + n(C-C')$, one has $\nu_1 = m(n-1)$, $\nu_2 = m-2$, and for:

(i) $\lambda_1 > \lambda_2$:

$$\frac{1}{P} < \frac{\Phi_1}{\Phi_2} < \frac{1}{L} \quad ; \tag{8.6.46}$$

G_2' in (8.5.14) becomes:

$$G_2' = \int_0^1 \frac{u^{\frac{1}{2}(v-m-2)} \; (1-u)^{\frac{1}{2}(m-4)}}{[\frac{1}{P} u + (1-u)]^2} \, du \quad ,$$

$$= P^2 \int_0^1 \frac{u^{\frac{1}{2}(v-m-2)} \; (1-u)^{\frac{1}{2}(m-4)}}{[u + P(1-u)]^2} \, du \quad . \tag{8.6.47}$$

G_1' in (8.5.15) becomes:

$$G_1' = \int_0^1 \frac{u^{\frac{1}{2}(v-m-2)} \; (1-u)^{\frac{1}{2}(m-4)}}{\frac{1}{L} u + (1-u)} \, du \quad ,$$

$$= L \int_0^1 \frac{u^{\frac{1}{2}(v-m-2)} \; (1-u)^{\frac{1}{2}(m-4)}}{u + L(1-u)} \, du \quad . \tag{8.6.48}$$

And for:

(ii) $\underline{\lambda_1 < \lambda_2}$:

$$\frac{1}{L} < \frac{\Phi_1}{\Phi_2} < \frac{1}{P} \quad ; \tag{8.6.49}$$

G_2' and G_1' become, in this case:

$$G_2' = \int_0^1 \frac{u^{\frac{1}{2}(v-m-2)} \ (1-u)^{\frac{1}{2}(m-4)}}{[\frac{1}{L}\, u + (1-u)]^2}\, du \quad ,$$

$$= L^2 \int_0^1 \frac{u^{\frac{1}{2}(v-m-2)} \ (1-u)^{\frac{1}{2}(m-4)}}{[u + L(1-u)]^2}\, du \quad , \tag{8.6.50}$$

and

$$G_1' = \int_0^1 \frac{u^{\frac{1}{2}(v-m-2)} \ (1-u)^{\frac{1}{2}(m-4)}}{\frac{1}{P}\, u + (1-u)}\, du \quad ,$$

$$= P \int_0^1 \frac{u^{\frac{1}{2}(v-m-2)} \ (1-u)^{\frac{1}{2}(m-4)}}{u + P(1-u)}\, du \quad . \tag{8.6.51}$$

One notices that the integrals for the within comparisons can be evaluated for $m \geq 2$, but for among comparisons one must have $m > 2$. The combining constant B and a conservative lower bound of the recovery ratio D are given in (8.5.12) and (8.5.20) for the within comparisons; and for the among comparisons B and D are given in (8.5.18) and (8.5.22), respectively. The required integrals have been evaluated and the corresponding values for B and D have been computed for 62 regular G.D. designs that appear in Bose's "Tables for P.B.I.B". The results are listed in Tables III and IV for the within and among comparisons respectively. The combined estimate, as given in (8.4.1) is

$$\hat{\tau}_i = U_i + \frac{\theta_i Bs^2}{\sum\limits_{\substack{j=1 \\ j\neq i}}^{v-1} (X_j - U_j)^2} (X_i - U_i) \quad , \qquad (8.6.52)$$

where θ_i is the coefficient of σ^2 in $V(U_i)$.

For within comparisons, one has, using (8.2.5) and (8.6.22):

$$V(U_i) = V_1 - C_1 = \frac{k-c_1}{a} \sigma^2 = \frac{k}{a+\lambda_1} \sigma^2 \quad , \qquad (8.6.53)$$

hence $\theta_i = \frac{k}{a+\lambda_1}$ and the combined estimate in (8.6.52) becomes:

$$\hat{\tau}_i = U_i + \frac{kBs^2}{(a+\lambda_1) \sum\limits_{\substack{j=1 \\ j\neq i}}^{v-1} (X_j - U_j)^2} (X_i - U_i) \; . \qquad (8.6.54)$$

For among comparisons, one has, using (8.6.8), (8.2.5), (8.3.1), (8.6.22), and (8.6.26):

$$V(U_i) = V_1 - C_1 + n(C_1 - C_2) = \left(\frac{k-c_1}{a} + \frac{n(\lambda_1-\lambda_2)}{k\Delta}\right)\sigma^2 \quad ,$$

$$= \left(\frac{k}{a+\lambda_1} + \frac{k}{\lambda_2 v} - \frac{k}{a+\lambda_1}\right)\sigma^2 = \frac{k}{\lambda_2 v}\,\sigma^2 \quad , \qquad (8.6.55)$$

hence $\theta_i = \frac{k}{\lambda_2 v}$ and the combined estimate in (8.6.52) becomes:

$$\hat{\tau}_i = U_i + \frac{kBs^2}{\lambda_2 v \sum_{\substack{j=1 \\ j\neq i}}^{v-1} (X_j - U_j)^2} (X_i - U_i) \quad . \qquad (8.6.56)$$

The B's in (8.6.54) and (8.6.56) are the combining constants to be obtained from Tables III and IV, respectively.

8.6.2 <u>Analysis for Singular and Semi-regular G.D. Designs</u>

The formula (8.1.9) of M. Zelen for the inter-block estimate is not applicable in the two subclasses, singular G.D., and semi-regular G.D. designs because the quantity $\Delta - rH + r^2 = 0$ for both. An inter-analysis for those subclasses will now be given.

It is known that in the inter-block analysis it is assumed that the block effects b_s are random variables, uncorrelated with each other and with the plot errors ε_{ijs}'s, having mean zero and (unknown) variance σ_b^2. Let the total of blocks be denoted by B_s, $s = 1,2,\ldots,b$, and the effect of the treatment in the i-th row and j-th column in the association scheme rectangle by t_{ij}, where $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, $mn = v$. One can then express the yield y_{ijs} as:

$$y_{ijs} = \mu + t_{ij} + b_s + \varepsilon_{ijs} \quad , \qquad (8.6.57)$$

where y_{ijs} is defined only when the treatment (ij) occurs in the block s. The total of the s-th block is:

$$B_s = k\mu + \sum_{i=1}^{m}\sum_{j=1}^{n}\delta_{ijs}t_{ij} + kb_s + \sum_{i=1}^{m}\sum_{j=1}^{n}\delta_{ijs}\varepsilon_{ijs} \quad , \qquad (8.6.58)$$

where

$$\delta_{ijs} = \begin{cases} 1 & \text{if (ij)-th treatment occurs in s-th block,} \\ 0 & \text{otherwise .} \end{cases}$$

Here $E(B_s) = k\mu + \sum\sum_{ij} \delta_{ijs}t_{ij}$ and $V(B_s) = k(\sigma^2 + k\sigma_b^2)$. One obtains the normal equations by minimizing the quantity,

$$\sum_{s=1}^{b}(B_s - k\mu - \sum_{i}\sum_{j}\delta_{ijs}t_{ij})^2 - \theta\sum_{ij}t_{ij} \quad . \qquad (8.6.59)$$

The normal equations, therefore, are:

$$-2k\sum_{s=1}^{b}(B_s - k\hat{\mu} - \sum_{ij}\delta_{ijs}\hat{t}'_{ij}) = 0 \quad , \qquad (8.6.60)$$

$$-\sum_{s=1}^{b}\delta_{ijs}(B_s - k\hat{\mu} - \sum_{i'j'}\delta_{i'j's}\hat{t}'_{i'j'}) - \theta = 0 \quad . \qquad (8.6.61)$$

Noticing that $\sum_{s=1}^{b} \delta_{ijs} = r$, $\sum_{i=1}^{m} \sum_{j=1}^{n} \delta_{ijs} = k$, and simplifying (8.6.60), and (8.6.61), one can write

$$bk^2\hat{\mu} + k\sum_{s=1}^{b} \sum_{ij} \delta_{ijs}\hat{t}'_{ij} = kG \quad , \tag{8.6.62}$$

$$kr\hat{\mu} + \sum_{i'j'} (\sum_{s} \delta_{ijs}\delta_{i'j's})\hat{t}'_{i'j'} = \sum_{s=1}^{b} \delta_{ijs}B_s + \theta \ . \tag{8.6.63}$$

Let $\sum\sum\sum_{sij} \delta_{ijs}\hat{t}'_{ij} = 0$, then (8.6.62) gives $\hat{\mu} = \frac{G}{bk}$, and substituting in (8.6.63) one obtains:

$$\sum_{i'j'} (\sum_{s} \delta_{ijs}\delta_{i'j's})\hat{t}'_{i'j'} = B_{ij.} - \frac{rG}{b} + \theta = Q'_{ij} + \theta \quad , \tag{8.6.64}$$

where $B_{ij.} = \sum_{s=1}^{b} \delta_{ijs}B_s$ = the sum of block totals in which treatment (ij) occurs. If ij = i'j' then

$\sum_{s} \delta_{ijs}\delta_{i'j's} = \sum_{s} \delta^2_{ijs} = \sum_{s} \delta_{ijs} = r$, thus (8.6.64) becomes

$$r\hat{t}'_{ij} + \sum_{i'j'\neq ij} (\sum_{s} \delta_{ijs}\delta_{i'j's})\hat{t}'_{i'j'} = Q'_{ij} + \theta \quad ,$$

or

$$r\hat{t}'_{ij} + \sum_{\substack{j'\neq j \\ i'=i}} (\sum_{s} \delta_{ijs}\delta_{ij's})\hat{t}'_{ij'} + \sum_{i'\neq i} \sum_{j'} (\sum_{s} \delta_{ijs}\delta_{i'j's})\hat{t}'_{i'j'} = Q'_{ij} + \theta \ , \tag{8.6.65}$$

but,

$$\sum_s \delta_{ijs}\delta_{ij's} = \text{the number of blocks in which treatment}$$

(ij) and treatment (ij'), where $j \neq j'$ occur together $= \lambda_1$.

Also,

$$\sum_s \delta_{ijs}\delta_{i'j's} = \text{the number of blocks in which treatment}$$

(ij) and treatment (i'j') occur together, where $i \neq i' = \lambda_2$.

Hence (8.6.65) becomes:

$$r\hat{t}'_{ij} + \sum_{j'=j} \lambda_1 \hat{t}'_{ij'} + \sum_{i' \neq i} \sum_{j'} \lambda_2 \hat{t}'_{i'j'} = Q'_{ij} + \theta \ . \qquad (8.6.66)$$

Summing over all treatments one obtains:

$$r\text{(sum of all treatments)} + n_1\lambda_1\text{(sum of all treatments)}$$
$$+ n_2\lambda_2\text{(sum of all treatments)} = 0 + v\theta \ , \qquad (8.6.67)$$

i.e., $v\theta = 0$ or $\theta = 0$. Using again the assumption that the sum of all treatments = 0, one can write:

$$\sum_{i' \neq i} \sum_{j'} \hat{t}'_{i'j'} = - \sum_{j'} \hat{t}'_{ij'} \ , \qquad (8.6.68)$$

i.e., the sum of all treatments except the i-th row = - the sum of the i-th row. Hence (8.6.66) takes the following form for all treatments in the i-th row:

$$r\hat{t}'_{ij} + \lambda_1 \sum_{j' \neq j} \hat{t}'_{ij'} - \lambda_2 \sum_{j'} \hat{t}'_{ij'} = Q'_{ij} \ , \qquad j = 1,2,\ldots,n \ ,$$

or

$$(r-\lambda_2)\hat{t}'_{ij} + (\lambda_1-\lambda_2)\sum_{j'\neq j} \hat{t}'_{ij'} = Q'_{ij} , \quad j = 1,2,\ldots, n . \tag{8.6.69}$$

Now there are m equations similar to (8.6.69) for $i = 1,2,3\ldots m$. Equation (8.6.69) becomes in matrix notation, for $j = 1, 2, \ldots, n$:

$$\begin{bmatrix} r-\lambda_2 & \lambda_1-\lambda_2 & \cdots & \lambda_1-\lambda_2 \\ \lambda_1-\lambda_2 & r-\lambda_2 & \cdots & \lambda_1-\lambda_2 \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_1-\lambda_2 & \lambda_1-\lambda_2 & \cdots & r-\lambda_2 \end{bmatrix} \begin{bmatrix} \hat{t}'_{i1} \\ \hat{t}'_{i2} \\ \vdots \\ \hat{t}'_{in} \end{bmatrix} = \begin{bmatrix} Q'_{i1} \\ Q'_{i2} \\ \vdots \\ Q'_{in} \end{bmatrix} . \tag{8.6.70}$$

8.6.2.1 Inter-estimates and Recovery of Inter-information in Singular G.D.

In this case $r = \lambda_1$ and the coefficient matrix in (8.6.70) is of rank one. Out of each group or row of treatments, one can in fact obtain an inter-estimate of one selected treatment and impose arbitrary values for the remaining n-1 treatments. This is natural, due to the fact that one can construct singular G.D. designs by stretching every treatment in a B.I.B. design to become a group of n treatments or a row in the association scheme. Assume that n-1 treatments in each row have the estimate zero. Then one can write

$$\hat{t}'_{ij} = \frac{Q'_{ij}}{\lambda_1-\lambda_2} \qquad i = 1,2,3\ldots m \quad , \qquad (8.6.71)$$

and $\hat{t}'_{ij'} = 0$ for $j' = 1,2,\ldots, j-1, j+1, j+2, \ldots, n$.

In this way one has inter-estimates for m treatments, one from each row. It should be noted that these estimates must sum to zero, and that each treatment is a second associate of all the remaining m-1 treatments. It is obvious that one can inter-estimate m-1 comparisons; we shall choose the m-1 among comparisons defined in Section 8.6.1 to be inter- and intra-estimated. They have the same variance, as it was shown in (8.6.8), so that theorem 1 is applicable. From formula (3.1), where m is m-1, one obtains:

$$\hat{\tau}_i = U_i + \frac{\theta(m-3)s^2}{\sum_{j=1}^{m-1}(X_j-U_j)^2}(X_i-U_i) \quad . \qquad (8.6.72)$$

This combined estimate is applicable for m 3. $\theta=\frac{\rho f}{f+2}$, ρ is the coefficient of σ^2 in $V(U_i) = V_1-C_1+n(C_1-C_2) = \frac{k}{\lambda_2 v}\sigma^2$, as was shown before in (8.6.55). Hence $\rho = \frac{k}{\lambda_2 v}$, and then

(8.6.72) becomes:

$$\hat{\tau}_i = U_i + \frac{fk(m-3)s^2}{(f+2)\lambda_2 v \sum_{j=1}^{m-1} (X_j - U_j)^2} (X_i - U_i) \ . \qquad (8.6.73)$$

The recovery ratio, from theorem 1, is then:

$$D = \frac{f\ (m-3)}{(2+f)(m-1)} \quad , \qquad (8.6.74)$$

where $f = rv - v - b + 1$.

The choice of the among comparisons is justified here by the fact that the efficiency E of any comparison between first associate treatments is unity, as R. C. Bose's tables for P.B.I.B. indicate.

8.6.2.2 Inter-estimates and Recovery of Inter-block Information in Semi-Regular G.D.

In this case $rk = \lambda_2 v$, and by (8.6.3) one can write:

$$r - \lambda_1 + n(\lambda_1 - \lambda_2) = 0 \ ,$$

or

$$(r-\lambda_1) + \lambda_1 - \lambda_2 + n_1(\lambda_1 - \lambda_2) = 0 \ , \quad \text{where } n_1 = n-1 \ ,$$

or

$$r - \lambda_2 = -n_1(\lambda_1 - \lambda_2) \quad . \qquad (8.6.75)$$

The matrix equation in (8.6.70) becomes:

$$\begin{bmatrix} -n_1(\lambda_1-\lambda_2) & \lambda_1-\lambda_2 & \cdots & \lambda_1-\lambda_2 \\ \lambda_1-\lambda_2 & -n_1(\lambda_1-\lambda_2) & \cdots & \lambda_1-\lambda_2 \\ \\ \lambda_1-\lambda_2 & \lambda_1-\lambda_2 & \cdots & -n_1(\lambda_1-\lambda_2) \end{bmatrix} \begin{bmatrix} \hat{t}'_{i1} \\ \hat{t}'_{i2} \\ \vdots \\ \hat{t}'_{in} \end{bmatrix} = \begin{bmatrix} Q'_{i1} \\ Q'_{i2} \\ \vdots \\ Q'_{in} \end{bmatrix} . \tag{8.6.76}$$

The coefficient matrix in (8.6.76) is of rank n-1, i.e., out of each group or row of treatments, one can in fact inter-estimate n-1 treatments and impose arbitrary value for the remaining one. Applying the restriction $\sum_{j=1}^{n} \hat{t}'_{ij} = 0$, and using (8.6.75) one can write (8.6.69) as:

$$-(n-1)(\lambda_1-\lambda_2)\hat{t}'_{ij} + (\lambda_1-\lambda_2)\sum_{j'\neq j}\hat{t}'_{ij'} = Q'_{ij} \quad , \quad \text{or}$$

$$-n(\lambda_1-\lambda_2)\hat{t}'_{ij} + (\lambda_1-\lambda_2)\sum_{j=1}^{n}\hat{t}'_{ij} = Q'_{ij} \quad , \quad \text{or}$$

$$-n(\lambda_1-\lambda_2)\hat{t}'_{ij} = Q'_{ij} \quad , \quad \text{i.e.,}$$

$$\hat{t}'_{ij} = \frac{Q'_{ij}}{n(\lambda_2-\lambda_1)} = \frac{Q'_{ij}}{r-\lambda_1} \quad , \tag{8.6.77}$$

$i = 1,2,\ldots,m; \quad j = 1,2,3\ldots n-1$.

It is obvious that one can inter-estimate n-1 comparisons within each row of treatments; in all one can inter-estimate m(n-1) comparisons. We shall choose the m(n-1) within comparisons defined in Section 8.6.1 to be inter- and intra-estimated. They have the same variance, namely V_1-C_1 for intra-estimate and $V_1'-C_1'$ for inter-estimate, so that theorem 1 is applicable. This gives the following combined estimate:

$$\hat{\tau}_i = U_i + \frac{\theta(mn-m-2)s^2}{\sum_{j=1}^{m(n-1)} (X_j-U_j)^2} (X_i-U_i) \quad . \tag{8.6.78}$$

This combined estimate is applicable for $mn-m-2 = v-m-2 > 0$ or $v-m > 2$. $\theta = \frac{\rho f}{f+2}$, ρ is the coefficient of σ^2 in $V(U_i) = V_1-C_1 = \frac{k-c_1}{a}\sigma^2 = \frac{k}{a+\lambda_1}\sigma^2$, as it was shown in (8.6.53). Hence $\rho = \frac{k}{a+\lambda_1}$, and then (8.6.78) becomes

$$\hat{\tau}_i = U_i + \frac{k(v-m-2)s^2}{(a+\lambda_1)\sum_{j=1}^{v-m} (X_j-U_j)^2} (X_i-U_i) \; \frac{f}{f+2} \quad . \tag{8.6.79}$$

The recovery ratio is,

$$D = \frac{f \;\; (v-m-2)}{(2+f)(v-m)} \quad , \tag{8.6.80}$$

where $f = rv-v-b+1$.

The choice of the within comparisons is justified by the fact that the efficiency E_1 of a comparison between first associate treatments is always less than E_2, the efficiency of a comparison between second associate treatments, as R. C. Bose's tables for P.B.I.B. indicate.

8.7 Recovery of Inter-block Information in LS Type

In this case $v = n^2$, and the treatments can be arranged in an nxn square association scheme so that two treatments are first associates if they occur together in the same row or in the same column, and they are second associates otherwise. Such a design will be said to belong to the sub-type L_2 of the Latin Square type design denoted by LS. We also have designs with n^2 treatments belonging to the sub-type L_3 of the Latin Square type design. In this case it is possible to form an nxn square array and to impose a Latin Square with n letters on this array, so that any two treatments are first associates if they occur in the same row or column of the array or correspond to the same letter, and are second associates otherwise.

The following relations hold for the sub-type L_i of LS designs (i=2,3).

$$n_1 = i(n-1) \ , \qquad n_2 = (n-1)(n-i+1) \ , \tag{8.7.1}$$

$$P_1 = \begin{pmatrix} i^2-3i+n & (i-1)(n-i+1) \\ (i-1)(n-i+1) & (n-i)(n-i+1) \end{pmatrix} ,$$

$$P_2 = \begin{pmatrix} i(i-1) & i(n-i) \\ i(n-1) & (n-i)^2+i-2 \end{pmatrix} , \tag{8.7.2}$$

Consider now the following $n^2 \times n^2$ matrix M_1,

$$
M_1 = \begin{array}{r l}
 & \begin{array}{cccc cccc c cccc} 1 & 2 & \cdots & n & n+1 & n+2 & \cdots & 2n & \cdots & n(n-1)+1 & n(n-1)+2 & \cdots & n^2 \end{array} \\
\begin{array}{r} 1 \\ 2 \\ \vdots \\ n-1 \\ \\ \\ \\ \\ \vdots \\ \\ \\ \\ n(n-1) \\ n(n-1)+1 \\ \vdots \\ n^2-1 \\ n^2 \end{array} &
\left[\begin{array}{cccc|cccc|c|cccc}
1 & -1 & & & & & & & & & & & \\
1 & 1 & -2 & & & & & & & & & & \\
\cdot & \cdot & \cdot & \cdot & & & & & & & & & \\
1 & 1 & \cdots & -n+1 & & & & & & & & & \\ \hline
 & & & & 1 & -1 & & & & & & & \\
 & & & & 1 & 1 & -2 & & & & & & \\
 & & & & \cdot & \cdot & \cdot & \cdot & & & & & \\
 & & & & 1 & 1 & \cdots & -n+1 & & & & & \\ \hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
 & & & & & & & & & 1 & -1 & & \\
 & & & & & & & & & 1 & 1 & -2 & \\
 & & & & & & & & & \cdot & \cdot & \cdot & \cdot \\
 & & & & & & & & & 1 & 1 & \cdots & -n+1 \\ \hline
1 & 1 & \cdots & 1 & -1 & -1 & \cdots & -1 & & & & & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & -n+1 & -n+1 & \cdots & -n+1 \\
1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1
\end{array} \right]
\end{array}
$$

Let M be the matrix M_1 after normalizing its rows, then M is orthogonal. Define:

$$\underline{U} = \begin{bmatrix} U_1 \\ \vdots \\ U_{n(n-1)} \\ \cdots\cdots \\ U_{n(n-1)+1} \\ \vdots \\ U_{n^2-1} \\ 0 \end{bmatrix} = M \begin{bmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \vdots \\ \hat{t}_{n^2} \end{bmatrix} = M\,\underline{\hat{t}} \quad . \tag{8.7.3}$$

Also define

$$\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_{n(n-1)} \\ \cdots\cdots \\ X_{n(n-1)+1} \\ \vdots \\ X_{n^2-1} \\ 0 \end{bmatrix} = M \begin{bmatrix} \hat{t}'_1 \\ \hat{t}'_2 \\ \vdots \\ \hat{t}'_{n^2} \end{bmatrix} = M\,\underline{\hat{t}}' \quad , \tag{8.7.4}$$

where $\hat{t}_i$ and $\hat{t}'_i$ are the intra- and inter-estimates of the i-th treatment effect, respectively. The treatments in $\underline{\hat{t}}$ or $\underline{\hat{t}}'$ are arranged according to the rows of the association scheme, i.e., the first n are the treatments of the first row, the next n treatments are those of the second row, etc.

Since the treatments in the same group (or row) are first associates, we have in similarity with (7.2.4) that:

$$V(U_j) = V_1 - C_1 \ , \quad \text{for } j = 1,2,\ldots,n(n-1) \ . \tag{8.7.5}$$

Thus all within comparisons have the same variance, namely $V_1 - C_1$. For the among comparisons, let us compute for L_i sub-type (i=2,3):

$$V(U_{n(n-1)+m}) = V\{\frac{1}{\sqrt{nm(m+1)}}[(\hat{t}_1+\ldots+\hat{t}_n)+(\hat{t}_{n+1}+\ldots+\hat{t}_{2n})$$

$$+ \ldots - m(t_{nm+1}+\ldots+t_{n(m+1)})]\} \ ,$$

$$= \frac{1}{nm(m+1)}\{nm(m+1)V_1 + n(n-1)mC_1 + m(m-1)[n(C_1(i-1)+C_2(n-i+1))]$$

$$+ n(n-1)m^2C_1 - 2m[n((i-1)mC_1 + m(n-i+1)C_2)]\} \ ,$$

$$= V_1 + \frac{1}{nm(m+1)} C_1[n(n-1)m + n(i-1)m(m-1) + n(n-1)m^2 - 2nm^2(i-1)]$$

$$+ \frac{1}{nm(m+1)} C_2[nm(m-1)(n-i+1) - 2m^2n(n-i+1)] \ ,$$

$$= V_1 + \frac{nmC_1}{nm(m+1)}[n-1+(i-1)(m-1)+m(n-1)-2m(i-1)]$$

$$+ \frac{C_2nm(n-i+1)}{nm(m+1)}(m-1-2m) \ ,$$

$$= V_1 + \frac{C_1}{m+1}[(n-1)(m+1)+(i-1)(m-1-2m)] - C_2(n-i+1) \ ,$$

$$= V_1 + C_1(n-1-i+1) - C_2(n-i+1) \ ,$$

$$= V_1 - C_1 + (n-i+1)(C_1-C_2) \ . \tag{8.7.6}$$

This is independent of m, which indicates that all among comparisons have the same variance, namely:

$$V(U_j) = V_1 - C_1 + (n-i+1)(C_1 - C_2) \ , \qquad (8.7.7)$$

$$\text{where } j = n(n-1)+1, \ \ldots, \ n^2-1 \ ,$$

and (i=2,3) according to the design being of L_2 or L_3 sub-types.

The inter-variances are similarly:

$$V(X_j) = V_1' - C_1' \ , \quad i = 1,2,\ldots,n(n-1) \quad , \qquad (8.7.8)$$

$$V(X_j) = V_1' - C_1' + (n-i+1)(C_1' - C_2') \ , \qquad (8.7.9)$$

$$j = n(n-1)+1,\ldots,n^2-1 \ ,$$

and (i=2,3).

Let:

$$\underline{Z} = \underline{X} - \underline{U} \quad . \qquad (8.7.10)$$

Then,

$$V(Z_j) = V(X_j) + V(U_j) \quad ,$$

$$= V_1' - C_1' + V_1 - C_1 = V - C \ , \quad j = 1,2\ldots n(n-1) \ , \qquad (8.7.11)$$

and

$$V(Z_j) = V_1' - C_1' + (n-i+1)(C_1' - C_2') + V_1 - C_1 + (n-i+1)(C_1 - C_2) \ ,$$

$$= V - C + (n-i+1)(C - C') \ , \quad j = n(n-1)+1,\ldots,n^2-1 \quad , \qquad (8.7.12)$$

where V, C, C' are as defined in (8.2.15) through (8.2.17).

As it was shown in (8.6.16), one can write in this case:

$$V-C = \frac{k-c_1}{a}\,\sigma^2 + \frac{k-c_1'}{r}\,\sigma'^2 \quad . \tag{8.7.12}$$

By (8.3.5) we can write:

$$\begin{aligned} V-C + (n-i+1)(C-C') = {} & \left[\frac{k-c_1}{a} + \frac{(n-i+1)(\lambda_1-\lambda_2)}{k\Delta}\right]\sigma^2 \\ & + \left[\frac{k-c_1'}{r} + \frac{(n-i+1)(\lambda_1-\lambda_2)}{k(\Delta-rH+r^2)}\right]\sigma'^2 \quad . \end{aligned} \tag{8.7.13}$$

One notices that the special case of theorem 3 is applicable here where $\Phi_1 = V-C$, $\Phi_2 = V-C + (n-i+1)(C-C')$, and

$$\frac{\Phi_1}{\Phi_2} = \frac{\alpha_1\sigma^2+\alpha_2\sigma'^2}{(\alpha_1+\beta_1)\sigma^2+(\alpha_2+\beta_2)\sigma'^2} \quad . \tag{8.7.14}$$

Substituting for σ'^2 from (8.1.13), one obtains

$$\frac{\Phi_1}{\Phi_2} = \frac{(\alpha_1+\alpha_2)\sigma^2+k\alpha_2\sigma_b^2}{(\alpha_1+\beta_1+\alpha_2+\beta_2)\sigma^2+k(\alpha_2+\beta_2)\sigma_b^2} \quad , \tag{8.7.15}$$

where:

$$\begin{aligned} \alpha_1 &= \frac{k-c_1}{a} \;, & \alpha_2 &= \frac{k-c_1'}{r} \;, \\ \beta_1 &= \frac{(n-i+1)(\lambda_1-\lambda_2)}{k\Delta} \;, & \beta_2 &= \frac{-(n-i+1)(\lambda_1-\lambda_2)}{k(\Delta-rH+r^2)} \quad . \end{aligned} \tag{8.7.16}$$

Remembering that $\frac{\sigma_b^2}{\sigma^2} = R > 1$, one can write (8.7.15) as:

$$\frac{\Phi_1}{\Phi_2} = F(R) = \frac{(\alpha_1+\alpha_2)+k\alpha_2 R}{(\alpha_1+\beta_1+\alpha_2+\beta_2)+k(\alpha_2+\beta_2)R} \quad , \qquad (8.7.17)$$

$$\frac{dF(R)}{dR} = \frac{k(\beta_1\alpha_2-\alpha_1\beta_2)}{[(\alpha_1+\beta_1+\alpha_2+\beta_2)+k(\alpha_2+\beta_2)R]^2} \quad . \qquad (8.7.18)$$

For the LS type designs listed in Bose's tables α_1, α_2, Δ, and $(\Delta-rH+r^2)$ are all positive, hence the sign of $\frac{dF(R)}{dR}$ is the same as the sign of $(\beta_1\alpha_2-\alpha_1\beta_2)$.

Substituting β_1, α_1, β_2, α_2 from (8.7.16), one obtains

$$\begin{aligned}\beta_1\alpha_2 - \alpha_1\beta_2 &= (\lambda_1-\lambda_2)\left[\frac{(n-i+1)\alpha_2}{k\Delta} + \frac{(n-i+1)\alpha_1}{k(\Delta-rH+r^2)}\right] \\ &= (\lambda_1-\lambda_2) \times \text{(positive quantity)} \quad .\end{aligned} \qquad (8.7.19)$$

Thus, the sign of $\frac{dF(R)}{dR}$ is the same as that of $\lambda_1-\lambda_2$, i.e.,

$F(R)$ is monotonically increasing if $\lambda_1 > \lambda_2$,

$F(R)$ is monotonically decreasing if $\lambda_1 < \lambda_2$.

Consequently one can write:

$$\begin{aligned}&F(1) < F(R) < F(\infty) \quad \text{when} \quad \lambda_1 > \lambda_2 \quad , \\ &F(\infty) < F(R) < F(1) \quad \text{when} \quad \lambda_1 < \lambda_2 \quad .\end{aligned} \qquad (8.7.19)$$

For the inverse ratio $\frac{\Phi_2}{\Phi_1} = G(R) = \frac{1}{F(R)}$, one has:

$$G(\infty) < G(R) < G(1) \quad \text{if} \quad \lambda_1 > \lambda_2 \ ,$$
$$G(1) < G(R) < G(\infty) \quad \text{if} \quad \lambda_1 < \lambda_2 \ . \tag{8.7.20}$$

Then let

$$L = G(1) = \frac{1}{F(1)} = 1 + \frac{\beta_1+(k+1)\beta_2}{\alpha_1+(k+1)\alpha_2} \ , \tag{8.7.21}$$

and

$$P = G(\infty) = \frac{1}{F(\infty)} = 1 + \frac{\beta_2}{\alpha_2} \ . \tag{8.7.22}$$

8.7.1 Within Comparisons

Since $\nu_1 = v-n-1$, $\nu_2 = n-1$, then:

(i) $\underline{\lambda_1 > \lambda_2}$:

$$P < \frac{\Phi_2}{\Phi_1} < L \quad , \quad \text{using (8.7.20).}$$

G_2 in (8.5.7) becomes:

$$G_2 = \int_0^1 \frac{u^{\frac{1}{2}(v-n-3)}\ (1-u)^{\frac{1}{2}(n-3)}}{[u + P(1-u)]^2}\, du \ . \tag{8.7.23}$$

Also G_1 in (8.5.8) becomes:

$$G_1 = \int_0^1 \frac{u^{\frac{1}{2}(v-n-3)}\ (1-u)^{\frac{1}{2}(n-3)}}{u + L(1-u)}\, du \ . \tag{8.7.24}$$

and

(ii) $\underline{\lambda_1 < \lambda_2}$:

$$L < \frac{\Phi_2}{\Phi_1} < P \quad , \qquad \text{using (8.7.20)} \ ,$$

and

$$G_2 = \int_0^1 \frac{u^{\frac{1}{2}(v-n-3)} \, (1-u)^{\frac{1}{2}(n-3)}}{[u + L(1-u)]^2} \, du \quad , \qquad (8.7.25)$$

$$G_1 = \int_0^1 \frac{u^{\frac{1}{2}(v-n-3)} \, (1-u)^{\frac{1}{2}(n-3)}}{u + P(1-u)} \, du \quad . \qquad (8.7.26)$$

The above integrals can be evaluated for $n \geq 2$. The combining constant B and a conservative lower bound D of the recovery ratio are given in (8.5.12) and (8.5.20). These integrals have been evaluated and B and D has been computed for all designs listed in reference (1), except in LS6 through LS9 where the inter-model is singular. It should be noted that LS16 through LS20 are of L_3 sub-type of LS type designs, and the remaining designs are of L_2 sub-type. The results are listed in Table V.

The combined estimate, as given in (8.4.1), is:

$$\hat{\tau}_i = U_i + \frac{\theta_i B s^2}{\sum_{\substack{j=1 \\ j \neq i}}^{v-1} (X_j - U_j)^2} (X_i - U_i) \quad , \qquad i = 1,2,\ldots,n(n-1) \quad . \qquad (8.7.27)$$

By (8.2.5),

$$V(U_i) = V_1 - C_1 = \frac{k-c_1}{a}\sigma^2 \quad , \quad \text{so} \quad \theta_i = \frac{k-c_1}{a} \ ,$$

and then (8.7.27) becomes,

$$\hat{\tau}_i = U_i + \frac{(k-c_1)Bs^2}{a\sum\limits_{\substack{j=1\\ j\neq i}}^{v-1}(X_j-U_j)^2}(X_i-U_i) \ , \quad i = 1,2,\ldots,n(n-1) \quad . \tag{8.7.28}$$

8.7.2 Among Comparisons

Here $\nu_1 = v-n$, $\nu_2 = n-2$, and for:

(i) $\lambda_1 > \lambda_2$:

$$\frac{1}{L} < \frac{\Phi_1}{\Phi_2} < \frac{1}{P} \quad , \qquad \text{using (8.7.19).}$$

G_2' in (8.5.14) becomes:

$$G_2' = \int_0^1 \frac{u^{\frac{1}{2}(v-m-2)}\,(1-u)^{\frac{1}{2}(n-4)}}{[\frac{1}{L}\,u + (1-u)]^2}\,du \quad ,$$

$$= L^2\int_0^1 \frac{u^{\frac{1}{2}(v-n-2)}\,(1-u)^{\frac{1}{2}(n-4)}}{[u + L(1-u)]^2}\,du \quad . \tag{8.7.29}$$

G_1' in (8.5.15) becomes:

$$G_1' = \int_0^1 \frac{u^{\frac{1}{2}(v-n-2)} \; (1-u)^{\frac{1}{2}(n-4)}}{\frac{1}{P}\, u + (1-u)} \, du \quad ,$$

$$= P \int_0^1 \frac{u^{\frac{1}{2}(v-n-2)} \; (1-u)^{\frac{1}{2}(n-4)}}{u + P(1-u)} \, du \quad . \qquad (8.7.30)$$

(ii) $\underline{\lambda_1 < \lambda_2}$:

$$\frac{1}{P} < \frac{\Phi_1}{\Phi_2} < \frac{1}{L} \quad , \text{ using (8.7.19), hence}$$

$$G_2' = \int_0^1 \frac{u^{\frac{1}{2}(v-n-2)} \; (1-u)^{\frac{1}{2}(n-4)}}{[\frac{1}{P}\, u + (1-u)]^2} \, du \quad ,$$

$$= P^2 \int_0^1 \frac{u^{\frac{1}{2}(v-n-2)} \; (1-u)^{\frac{1}{2}(n-4)}}{[u + P(1-u)]^2} \, du \quad , \qquad (8.7.31)$$

and

$$G_1' = \int_0^1 \frac{u^{\frac{1}{2}(v-n-2)} \; (1-u)^{\frac{1}{2}(n-4)}}{\frac{1}{L}\, u + (1-u)} \, du \quad ,$$

$$= L \int_0^1 \frac{u^{\frac{1}{2}(v-n-2)} \; (1-u)^{\frac{1}{2}(n-4)}}{u + L(1-u)} \, du \quad . \qquad (8.7.32)$$

For the convergence of the above integrals $n > 2$. The combining constant B and a conservative lower bound D of the recovery ratio are given in (8.5.18) and (8.5.22) respectively. The above integrals along with B and D are evaluated and listed in Table VI for the same set of designs mentioned in Section 8.7.1.

The combined estimate is

$$\hat{\tau}_i = U_i + \frac{\theta_i Bs^2}{\sum_{\substack{j=1 \\ j\neq i}}^{v-1} (X_j - U_j)^2} (X_i - U_i) \quad , \quad i = n(n-1)+1, \ldots, n^2-1 \ . \tag{8.7.33}$$

By (8.2.5) and (8.3.1), one can write:

$$V(U_i) = V_1 - C_1 + (n-i+1)(C_1 - C_2) = \left[\frac{k-c_1}{a} + \frac{(n-i+1)(\lambda_1-\lambda_2)}{k\Delta}\right]\sigma^2 \ ,$$

hence:

$$\theta_i = \frac{k-c_1}{a} + \frac{(n-1)(\lambda_1-\lambda_2)}{k\Delta} \qquad \text{for } L_2 \text{ sub-type,}$$

and

$$\theta_i = \frac{k-c_1}{a} + \frac{(n-2)(\lambda_1-\lambda_2)}{k\Delta} \qquad \text{for } L_3 \text{ sub-type.}$$

Accordingly,

$$\hat{\tau}_i = U_i + \frac{[k\Delta(k-c_1)+(n-1)a(\lambda_1-\lambda_2)]Bs^2}{ak\Delta \sum_{\substack{j=1 \\ j\neq i}}^{v-1} (X_j - U_j)^2} (X_i - U_i) \quad , \tag{8.7.34}$$

$$i = n(n-1)+1, \ldots, n^2-1 \ ,$$

for L_2 sub-type, or LS1 through LS5 and LS10 through LS15 as listed in reference (1), and

$$\hat{\tau}_i = U_i + \frac{[k\Delta(k-c_1)+(n-2)a(\lambda_1-\lambda_2)]Bs^2}{ak\Delta \sum_{\substack{j=1 \\ j\neq i}}^{v-1} (X_j-U_j)^2} (X_i-U_i) \quad , \tag{8.7.35}$$

$$i = n(n-1)+1,\ldots,n^2-1 \; ,$$

for L_3 sub-type, or LS16 through LS20 as listed in reference (1).

8.8 Triangular P.B.I.B. Designs

8.8.1 Definition and Comment on the Singularity of the Inter-analysis Model

In triangular designs the number of treatments can be expressed as $v = \frac{n(n-1)}{2}$, and the association scheme is an array of n rows and n columns with the following properties:

(i) The positions in the principal diagonal are left blank.

(ii) The $n(n-1)/2$ positions above the principal diagonal are filled by the numbers $1,2,\ldots, n(n-1)/2$ corresponding to the treatments.

(iii) The $n(n-1)/2$ positions below the principal diagonal are filled so that the array is symmetrical about the principal diagonal.

(iv) For any treatment i the first associates are exactly those treatments which lie in the same row (or in the same column) as i.

The following relations hold

$$n_1 = 2n-4 \ , \quad n_2 = (n-2)(n-3)/2 \ , \tag{8.8.1}$$

$$P_1 = \begin{pmatrix} n-2 & n-3 \\ n-3 & (n-3)(n-4)/2 \end{pmatrix} ,$$

$$P_2 = \begin{pmatrix} 4 & 2n-8 \\ 2n-8 & (n-4)(n-5)/2 \end{pmatrix} . \tag{8.8.2}$$

The parameters v, r, k, b, λ_1, λ_2, and n, along with the constants c_1, c_2, Δ, and H, are given for 36 triangular designs in reference (1).

Reference (6) gives a set of three equations with three unknowns for each treatment resulting from Rao's combined analysis. These equations are:

$$\alpha T_j - \beta\lambda_1 G_{j1} - \beta\lambda_2 G_{j2} = P_j \quad , \tag{8.8.3}$$

$$-\beta\lambda_1 n_1 T_j + (\alpha - \beta\lambda_1 P^1_{11} - \beta\lambda_2 P^1_{12}) G_{j1}$$

$$- (\beta\lambda_1 P^2_{11} + \beta\lambda_2 P^2_{12}) G_{j2} = \sum_j P_{j1} \quad , \tag{8.8.4}$$

and

$$-\beta\lambda_2 n_2 T_j - (\beta\lambda_1 P_{21}^1 + \beta\lambda_2 P_{22}^1) G_{j1}$$
$$+ (\alpha - \beta\lambda_1 P_{21}^2 - \beta\lambda_2 P_{22}^2) G_{j2} = \sum_j P_{j2} \quad , \qquad (8.8.5)$$

where

$$\alpha = r[w\,\frac{k-1}{k} + \frac{w'}{k}] \quad , \qquad \beta = \frac{w-w'}{k} \quad , \qquad (8.8.6)$$

G_{j1} is the total of treatments which are first associates to the j-th treatment,

G_{j2} is the total of treatments which are second associates to the j-th treatment,

$$P_j = wQ_j + \frac{w'}{k}\,(T_j - \frac{rG}{b}) \quad , \qquad (8.8.7)$$

$\sum_j P_{j1}$ is the sum of P_j's for the treatments which are first associates to the j-th treatment,

and

$\sum_j P_{j2}$ is the sum of P_j's for the treatments which are second associates of treatment j.

From the definitions of G_{j1} and G_{j2}, one can write the further relation,

$$G_{j1} + G_{j2} + t_j = 0 \quad . \qquad (8.8.8)$$

If one lets w = 0 in (8.8.3) through (8.8.7), the corresponding normal equations for the inter-analysis above are obtained:

$$r\hat{t}'_j + \lambda_1 Gj_1 + \lambda_2 G_{j2} = Q'_j \quad , \tag{8.8.9}$$

$$n_1\lambda_1\hat{t}'_j + (r+\lambda_1 p^1_{11}+\lambda_2 p^1_{12})G_{j1} + (\lambda_1 p^2_{11}+\lambda_2 p^2_{12})G_{j2} = \Sigma Q'_{j1} \quad , \tag{8.8.10}$$

$$n_2\lambda_2\hat{t}'_j + (\lambda_1 p^1_{21}+\lambda_1 p^1_{22})G_{j1} + (r+\lambda_1 p^2_{21}+\lambda_2 p^2_{22})G_{j2} = \Sigma Q'_{j2} \quad , \tag{8.8.11}$$

and

$$\hat{t}'_j + G_{j1} + G_{j2} = 0 \quad . \tag{8.8.12}$$

Substituting the p^i_{jk}'s from (8.8.2) into (8.8.9) through (8.8.12), one obtains for the triangular designs:

$$r\hat{t}'_j + \lambda_1 G_{j1} + \lambda_2 G_{j2} = Q'_j \quad , \tag{8.8.13}$$

$$\begin{aligned}(2n-4)\lambda_1\hat{t}'_j &+ [r+\lambda_1(n-2)+\lambda_2(n-3)]G_{j1} \\ &+ [4\lambda_1+\lambda_2(2n-8)]G_{j2} = \sum_j Q'_{j1} \quad ,\end{aligned} \tag{8.8.14}$$

$$\begin{aligned}\tfrac{1}{2}(n-2)(n-3)\lambda_2\hat{t}'_j &+ [\lambda_1(n-3)+\tfrac{1}{2}\lambda_2(n-3)(n-4)]G_{j1} \\ &+ [r+\lambda_1(2n-8)+\tfrac{1}{2}\lambda_2(n-4)(n-5)]G_{j2} = \sum_j Q'_{j2} \quad ,\end{aligned} \tag{8.8.15}$$

and

$$\hat{t}'_j + G_{j1} + G_{j2} = 0 \quad . \tag{8.8.16}$$

Computing G_{j2} from (8.8.16) and substituting into (8.8.13) and (8.8.14), one can write:

$$(r-\lambda_2)\hat{t}'_j + (\lambda_1-\lambda_2)G_{j1} = Q'_j \quad ,$$

and

$$[2n-4)\lambda_1-4\lambda_1-(2n-8)\lambda_2]\hat{t}'_j$$

$$+ [r+\lambda_1(n-2)-4\lambda_1+\lambda_2(n-3)-\lambda_2(2n-8)]G_{j1} = \sum_j Q'_{j1} \quad .$$

Simplifying the above equations,

$$(r-\lambda_2)\hat{t}'_j + (\lambda_1-\lambda_2)G_{j1} = Q'_j \quad , \qquad (8.8.17)$$

and

$$(2n-8)(\lambda_1-\lambda_2)\hat{t}'_j + [(r-\lambda_1)+(n-5)(\lambda_1-\lambda_2)]G_{j1} = \sum_j Q'_{j1} \quad . \qquad (8.8.18)$$

This set of two equations has no unique solution when the coefficients matrix is singular, i.e.,

$$F = \begin{vmatrix} r-\lambda_2 & \lambda_1-\lambda_2 \\ (2n-8)(\lambda_1-\lambda_2) & (r-\lambda_1)+(n-5)(\lambda_1-\lambda_2) \end{vmatrix} \quad ,$$

$$= (r-\lambda_1)(r-\lambda_2) + (n-5)(r-\lambda_2)(\lambda_1-\lambda_2) - 2(n-4)(\lambda_1-\lambda_2)^2 = 0 \quad . \qquad (8.8.20)$$

8.8.2 <u>Application of Theorem 3 When $V(z_i)$ is of the Form $V-C+\rho(C-C')$</u>

The formula (8.4.7) in theorem 3 can be written here as:

$$\sum_{j\neq i}^{t} z^2 = \sum_{j\neq i}^{t} [V-C+\rho_j(C-C')]\chi_1^2 \quad . \qquad (8.8.21)$$

Suppose that $\rho = 0$ for ν_1 z_i's, and $\rho > 0$ for ν_2 z_i's, where $\nu_1 + \nu_2 = t-1$, then:

$$\sum_{j\neq i}^{t} z_j^2 = (V-C)\chi^2_{\nu_1} + \sum_{\substack{j=\nu_1+1 \\ j\neq i}}^{t} [(V-C)+\rho_j(C-C')]\chi^2_1 \quad .(8.8.22)$$

In the case $\lambda_1 > \lambda_2$ and in view of (8.3.8), (8.8.22) can be written as:

$$\sum_{j\neq i}^{t} z_j^2 < (V-C)\chi^2_{\nu_1} + [(V-C)+\rho_{min.}(C-C')]\chi^2_{\nu_2}$$

$$= \psi_1\chi^2_{\nu_1} + \psi_2\chi^2_{\nu_2} = X + Y_1 \quad , \qquad (8.8.23)$$

and

$$\sum_{j\neq i}^{t} z_j^2 > (V-C)\chi^2_{\nu_1} + [V-C+\rho_{max.}(C-C')]\chi^2_{\nu_2}$$

$$= \psi_1\chi^2_{\nu_1} + \psi_3\chi^2_{\nu_2} = X + Y_2 \quad , \qquad (8.8.24)$$

From (8.8.23) and (8.8.24) one can develop the formula (8.4.6) in theorem 3 as follows:

$$V(\hat{\tau}_i) = V_i + \frac{V_i^2B^2(f+2)\Phi_i}{f} E \frac{1}{(\sum_{j\neq i}^{t} z^2)^2} - 2V_i^2BE \frac{1}{\sum_{j\neq i}^{t} z_j^2}$$

$$< V_i + \frac{V_i^2B^2(f+2)\Phi_i}{f} E \frac{1}{(X+Y_2)^2} - 2V_i^2BE \frac{1}{X+Y_1} \quad . \qquad (8.8.25)$$

From (8.5.3) and (8.5.6), one can write:

$$E \frac{1}{(X+Y_2)^2} = \frac{\Gamma(\overline{\tfrac{1}{2}\nu_1+\nu_2}-2)}{4\Gamma(\tfrac{1}{2}\nu_1)\Gamma(\tfrac{1}{2}\nu_2)\psi_1^2} \int_0^1 \frac{u^{\frac{1}{2}\nu_1-1}(1-u)^{\frac{1}{2}\nu_2-1}}{\left[u + \frac{\psi_3}{\psi_1}(1-u)\right]^2} du$$

$$< \frac{1}{(v-4)(v-6)\beta(\tfrac{1}{2}\nu_1, \tfrac{1}{2}\nu_2)\psi_1^2} \int_0^1 \frac{u^{\frac{1}{2}\nu_1-1}(1-u)^{\frac{1}{2}\nu_2-1}}{[u + L(1-u)]^2} du$$

$$= \frac{G_2}{(v-4)(v-6)\beta(\tfrac{1}{2}\nu_1, \tfrac{1}{2}\nu_2)\psi_1^2} , \tag{8.8.26}$$

where L is a lower bound of ψ_3/ψ_1 .

Again using (8.5.6),

$$E \frac{1}{X+Y_1} = \frac{\Gamma(\overline{\tfrac{1}{2}\nu_1+\nu_2}-1)}{2\Gamma(\tfrac{1}{2}\nu_1)\Gamma(\tfrac{1}{2}\nu_2)\psi_1} \int_0^1 \frac{u^{\frac{1}{2}\nu_1-1}(1-u)^{\frac{1}{2}\nu_2-1}}{u + \frac{\psi_2}{\psi_1}(1-u)} du$$

$$< \frac{1}{(v-3)\beta(\tfrac{1}{2}\nu_1, \tfrac{1}{2}\nu_2)\psi_1} \int_0^1 \frac{u^{\frac{1}{2}\nu_1-1}(1-u)^{\frac{1}{2}\nu_2-1}}{u + P(1-u)} du$$

$$= \frac{G_1}{(v-4)\beta(\tfrac{1}{2}\nu_1, \tfrac{1}{2}\nu_2)\psi_1} , \tag{8.8.27}$$

where P is an upper bound of ψ_2/ψ_1 .

Substituting (8.8.26) and (8.8.27) into (8.8.25),

$$V(\hat{\tau}_i) < V_i + \frac{V_i^2 B^2 (f+2)\Phi_i G_2}{f(v-4)(v-6)\beta(\tfrac{1}{2}\nu_1,\tfrac{1}{2}\nu_2)\psi_1^2} - \frac{2V_i^2 B G_1}{(v-4)\beta(\tfrac{1}{2}\nu_1,\tfrac{1}{2}\nu_2)\psi_1} . \tag{8.8.28}$$

It is required that

$$0 < B < \frac{2fG_1(v-6)\psi_1}{(f+2)G_2\Phi_i} \quad , \qquad (8.8.29)$$

and

$$B_{opt.} = \frac{fG_1(v-6)\psi_1}{(f+2)G_2\Phi_i} \quad . \qquad (8.8.30)$$

When $\Phi_i = \psi_1$, i.e., $\Phi_i = V-C$, (8.8.30) becomes:

$$B_{opt.} = \frac{fG_1(v-6)}{(f+2)G_2} \quad . \qquad (8.8.31)$$

Substituting (8.8.31) into (8.8.28),

$$V(\hat{\tau}) < V_i + \frac{V_i^2 B}{(v-4)\beta(\frac{1}{2}\nu_1,\frac{1}{2}\nu_2)\psi_1}\left[\frac{(f+2)\psi_1 G_2}{f(v-6)\psi_1} \cdot \frac{fG_1(v-6)\psi_1}{(f+2)\psi_1 G_2} - 2G_1\right] \quad ,$$

$$= V_i + \frac{V_i^2}{(v-4)\beta(\frac{1}{2}\nu_1,\frac{1}{2}\nu_2)\psi_1} \cdot \frac{fG_1(v-6)}{(f+2)G_2}(-G_1) \quad ,$$

$$= V_i - \frac{fG_1^2(v-6)}{(v-4)G_2(f+2)\beta(\frac{1}{2}\nu_1,\frac{1}{2}\nu_2)} \cdot \frac{V_i^2}{\psi_1} \quad , \qquad (8.8.32)$$

so that the recovery ratio is at least

$$D = \frac{fG_1^2(v-6)}{(f+2)G_2\beta(\frac{1}{2}\nu_1,\frac{1}{2}\nu_2)(v-4)} \quad . \qquad (8.8.33)$$

When $\Phi_i \neq \psi_1$, i.e., $V(z_i) = V-C+\rho(C-C')$ with $\rho > 0$, then

$$B_{opt.} > \frac{fG_1(v-6)\psi_1}{(f+2)\psi_2 G_2} > \frac{fG_1(v-6)}{(f+2)G_2 P} \quad , \qquad (8.8.34)$$

where P is an upper bound of $\frac{\psi_2}{\psi_1}$. Thus one can take

$$B = \frac{fG_1(v-6)}{(f+2)G_2P} \quad . \tag{8.8.35}$$

In view of (8.8.30), one can write (8.8.28) as:

$$V(\hat{\tau}_i) < V_i + \frac{V_i^2 B}{(v-4)\beta(\tfrac{1}{2}\nu_1,\tfrac{1}{2}\nu_2)\psi_1}\left[\frac{(f+2)\Phi_i G_2}{f(v-6)\psi_1} \cdot \frac{fG_1(v-6)\psi_1}{(f+2)G_2\Phi_i} - 2G_1\right] ,$$

$$< V_i - \frac{V_i^2}{(v-4)\beta(\tfrac{1}{2}\nu_1,\tfrac{1}{2}\nu_2)\psi_1} \cdot \frac{fG_1^2(v-6)}{(f+2)G_2P} ,$$

$$= V_i - \frac{fG_1^2(v-6)}{(v-4)(f+2)\beta(\tfrac{1}{2}\nu_1,\tfrac{1}{2}\nu_2)G_2P} \cdot \frac{\Phi_i}{\psi_1} \cdot \frac{V_i^2}{\Phi_i} ,$$

$$< V_i - \frac{fG_1^2(v-6)L}{(v-4)(f+2)\beta(\tfrac{1}{2}\nu_1,\tfrac{1}{2}\nu_2)G_2P} \cdot \frac{V_i^2}{\Phi_i} .$$

Hence the recovery ratio is at least:

$$D = \frac{fG_1^2(v-6)L}{(v-4)(f+2)\beta(\tfrac{1}{2}\nu_1,\tfrac{1}{2}\nu_2)G_2P} \quad . \tag{8.8.37}$$

If $\lambda_1 < \lambda_2$, then $\rho_{min.}$ in (8.8.23) and $\rho_{max.}$ in (8.8.24) exchange positions.

8.8.3 Recovery of Inter-block Information in Triangular Designs

The designs to be discussed here, are those in which the inter-analysis is not singular. Out of the 36 designs listed in (1), one finds, applying the formula (8.8.20), that the inter-analysis is not singular in the following designs:

T6, T7, T8, T12, T13, T14, T18, T19, in which $v = 10$, and in

T23, T24, T30, in which $v = 15$.

(a) For designs with $v = 10$, consider the matrix:

$$M_1 = \begin{array}{c} \begin{array}{cccccccccc} 1&2&3&4&5&6&7&8&9&10 \end{array} \\ \left[\begin{array}{cccc|ccc|ccc} 1&-1&&&&&&&& \\ 1&1&-2&&&&&&& \\ 1&1&1&-3&&&&&& \\ \hline &&&&1&-1&&&& \\ &&&&1&1&-2&&& \\ \hline &&&&&&&1&-1& \\ &&&&&&&1&1&-2 \\ \hline 3&3&3&3&-4&-4&-4&&& \\ 3&3&3&3&3&3&3&-7&-7&-7 \\ 1&1&1&1&1&1&1&1&1&1 \end{array}\right] \end{array} . \qquad (8.8.38)$$

Let M be the orthogonal matrix obtained by normalizing the rows of M_1.

Let

$$\underline{U} = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_{v-1} \\ 0 \end{bmatrix} = M \begin{bmatrix} \hat{t}_1 \\ \hat{t}_2 \\ \vdots \\ \hat{t}_v \end{bmatrix} = M\,\underline{t} \quad , \tag{8.8.39}$$

and

$$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{v-1} \\ 0 \end{bmatrix} = M \begin{bmatrix} \hat{t}'_1 \\ \hat{t}'_2 \\ \vdots \\ \hat{t}'_v \end{bmatrix} = M\,\underline{\hat{t}}' \quad , \tag{8.8.40}$$

where $\hat{t}_i$, $\hat{t}'_i$ are the intra- and inter-estimates of the i-th treatment effect t_i, respectively. The treatments are arranged in the vector $\underline{t}$ arithmetically from 1 to v. From the construction of M_1 in (8.8.38), it should be noted that the treatments have been divided into three groups; the first contains t_1, t_2, t_3, t_4; the second contains t_5, t_6, t_7; the treatments t_8, t_9, t_{10} form the third group. The first seven rows in M_1 are comparisons within groups; the next two comparisons are among groups. It is to be noticed that the treatments in the same group are mutually first associates. It is simple to verify that for the within comparisons:

$$V(U_i) = V_1 - C_1 \quad , \qquad i = 1,2,\ldots,7 \quad , \tag{8.8.41}$$

$$V(X_i) = V_1' - C_1' \quad , \qquad i = 1,2,\ldots,7 \quad . \tag{8.8.42}$$

For among comparisons:

$$V(U_8) = V_1 - C_1 + \frac{12}{7}(C_1 - C_2) \quad , \tag{8.8.43}$$

$$V(U_9) = V_1 - C_1 + \frac{9}{7}(C_1 - C_2) \quad , \tag{8.8.44}$$

$$V(X_8) = V_1' - C_1' + \frac{12}{7}(C_1' - C_2') \quad , \tag{8.8.45}$$

and

$$V(X_9) = V_1' - C_1' + \frac{9}{7}(C_1' - C_2') \quad . \tag{8.8.46}$$

Let

$$\underline{Z} = \underline{X} - \underline{U} \quad , \tag{8.8.47}$$

then

$$V(Z_i) = V(U_i) + V(X_i) \quad , \tag{8.8.48}$$

and using (8.8.41) through (8.8.46), one can write:

$$V(Z_i) = V_1 - C_1 + V_1' - C_1' = V - C \quad , \quad i = 1,2,\ldots,7 \ , \tag{8.8.51}$$

$$V(Z_8) = V - C + \frac{12}{7}(C - C') \quad , \tag{8.8.52}$$

and

$$V(Z_9) = V - C + \frac{9}{7}(C - C') \quad . \tag{8.8.53}$$

It should be noted that $V(Z_i)$ has been expressed in the general form $V-C+\rho(C-C')$, where $\rho = 0$ for Z_1 through Z_7, and $\rho = \frac{12}{7}, \frac{9}{7}$ for Z_8 and Z_9, respectively.

(b) Similarly for designs in which $v = 15$, consider the matrix:

$$
M_1 = \begin{array}{c}
\begin{array}{ccccc|cccc|ccc|ccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15
\end{array} \\
\left[\begin{array}{ccccc|cccc|ccc|ccc}
1 & -1 & & & & & & & & & & & & & \\
1 & 1 & -2 & & & & & & & & & & & & \\
1 & 1 & 1 & -3 & & & & & & & & & & & \\
1 & 1 & 1 & 1 & -4 & & & & & & & & & & \\
\hline
 & & & & & 1 & -1 & & & & & & & & \\
 & & & & & 1 & 1 & -2 & & & & & & & \\
 & & & & & 1 & 1 & 1 & -3 & & & & & & \\
\hline
 & & & & & & & & & 1 & -1 & & & & \\
 & & & & & & & & & 1 & 1 & -2 & & & \\
\hline
 & & & & & & & & & & & & 1 & -1 & \\
 & & & & & & & & & & & & 1 & 1 & -2 \\
\hline
4 & 4 & 4 & 4 & 4 & -5 & -5 & -5 & -5 & & & & & & \\
 & & & & & & & & & 1 & 1 & 1 & -1 & -1 & -1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & -3 & -3 & -3 & -3 & -3 & -3 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right]
\end{array} . \tag{8.8.54}
$$

Let M be the orthogonal matrix obtained from M_1 by normalizing its rows, and let $\underline{U}$ and $\underline{X}$ be as defined in (8.8.39) and (8.8.40), respectively. From the construction of M_1 in

(8.8.54), it should be noted that the 15 treatments have been divided into four groups. The first group contains t_1 through t_5, the second group contains t_6 through t_9, the treatments t_{10} through t_{12} form the third group, and the fourth group contains t_{13} through t_{15}. The first 11 rows of M_1 are comparisons within groups; the next three comparisons are among groups. The treatments within the same group are mutually first associates. For within comparisons:

$$V(U_i) = V_1 - C_1 \quad , \qquad i = 1,2,\ldots,11 \quad , \tag{8.8.55}$$

and

$$V(X_i) = V_1' - C_1' \quad , \qquad i = 1,2,\ldots,11 \quad . \tag{8.8.56}$$

For among comparisons:

$$V(U_{12}) = V_1 - C_1 + \frac{8}{3}(C_1 - C_2) \quad , \tag{8.8.57}$$

$$V(U_{13}) = V_1 - C_1 + (C_1 - C_2) \quad , \tag{8.8.58}$$

$$V(U_{14}) = V_1 - C_1 + \frac{7}{3}(C_1 - C_2) \quad ; \tag{8.8.59}$$

$$V(X_{12}) = V_1' - C_1' + \frac{8}{3}(C_1' - C_2') \quad , \tag{8.8.60}$$

$$V(X_{13}) = V_1' - C_1' + (C_1' - C_1') \quad , \tag{8.8.61}$$

and

$$V(X_{14}) = V_1' - C_1' + \frac{7}{3}(C_1' - C_2') \quad . \tag{8.8.62}$$

Taking into account (8.8.47) and (8.8.48), and using (8.8.55) through (8.8.62), one can write:

$$V(Z_i) = V-C \quad , \qquad i = 1,2,\ldots,11 \quad ; \tag{8.8.63}$$

$$V(Z_{12}) = V-C + \frac{8}{3}\,(C-C') \quad , \tag{8.8.64}$$

$$V(Z_{13}) = V-C + (C-C') \quad , \tag{8.8.65}$$

and

$$V(Z_{14}) = V-C + \frac{7}{3}\,(C-C') \quad . \tag{8.8.66}$$

It is to be noted also that $V(Z_i)$ has been expressed in the general form $V-C+\rho(C-C')$, where $\rho = 0$ for Z_1 through Z_{11}, and $\rho = \frac{8}{3}$, 1, $\frac{7}{3}$ for Z_{12}, Z_{13}, Z_{14}, respectively.

8.8.3.1 <u>Study of the Ratio $[V-C+\rho(C-C')]/(V-C)$</u>

Now

$$\frac{V-C+\rho(C-C')}{V-C} = 1 + \rho\,\frac{C-C'}{V-C} \quad . \tag{8.8.67}$$

By (8.2.16) and (8.2.17),

$$C-C' = C_1+C_1' - C_2-C_2' = (C_1-C_2) + (C_1'-C_2') \quad . \tag{8.8.68}$$

Substituting (8.2.7) and (8.3.2) into (8.8.68), one obtains:

$$C-C' = \frac{c_1-c_2}{a}\,\sigma^2 + \frac{c_1'-c_2'}{r}\,\sigma'^2 \quad . \tag{8.8.69}$$

Substituting (8.8.69) and (8.6.16) into (8.8.67), one obtains:

$$\frac{V-C+\rho(C-C')}{V-C} = 1 + \rho \frac{\frac{c_1-c_2}{a}\sigma^2 + \frac{c_1'-c_2'}{r}\sigma'^2}{\frac{k-c_1}{a}\sigma^2 + \frac{k-c_1'}{r}\sigma'^2} ,$$

and using (8.1.13),

$$= 1 + \rho \frac{\left(\frac{c_1-c_2}{a} + \frac{c_1'-c_2'}{r}\right)\sigma^2 + \frac{k(c_1'-c_2')}{r}\sigma_b^2}{\left(\frac{k-c_1}{a} + \frac{k-c_1'}{r}\right)\sigma^2 + \frac{k(k-c_1')}{r}\sigma_b^2} . \quad (8.8.70)$$

Let $\frac{\sigma_b^2}{\sigma^2} = R > 1$, then (8.8.70) becomes:

$$\frac{V-C+\rho(C-C')}{V-C} = 1 + \rho \frac{\alpha_1+\beta_1 R}{\alpha_2+\beta_2 R} = F(R) , \quad (8.8.71)$$

where:

$$\alpha_1 = \frac{c_1-c_2}{a} + \frac{c_1'-c_2'}{r} , \qquad \beta_1 = \frac{k(c_1'-c_2')}{r} ,$$

$$\alpha_2 = \frac{k-c_1}{a} + \frac{k-c_1'}{r} , \qquad \beta_2 = \frac{k(k-c_1')}{r} , \quad (8.8.72)$$

and c_1', c_2' were defined in (8.1.11).

F(R) is always between the limits:

$$F(1) = 1 + \rho \frac{\alpha_1+\beta_1}{\alpha_2+\beta_2} , \quad (8.8.73)$$

and

$$F(\infty) = 1 + \rho \frac{\beta_1}{\beta_2} . \quad (8.8.74)$$

8.8.3.2 Combined Estimates for Triangular Designs

The results of Section (8.8.2) are applicable in the triangular subclass for combining the comparisons U_i's and X_i's.

(a) For within comparisons, the combined estimate given in (8.4.1) is:

$$\hat{\tau}_i = U_i + \frac{\theta_i B s^2}{\sum\limits_{\substack{j=1 \\ j\neq i}}^{t} (X_j - U_j)^2} (X_i - U_i) \quad , \qquad (8.8.75)$$

where θ_i is the coefficient of σ^2 in $V(U_i)$. Using (8.2.5),

$$V(U_i) = V_1 - C_1 = \frac{k-c_1}{a}\sigma^2 \quad . \qquad (8.8.76)$$

Thus $\theta_i = \frac{k-c_1}{a}$ for every within comparison, and (8.8.75) becomes finally:

$$\hat{\tau}_i = U_i + \frac{(k-c_1) B s^2}{a \sum\limits_{j\neq i}^{t} (X_j - U_j)^2} (X_i - U_i) \quad , \qquad (8.8.77)$$

where k, c_1, a=rk-r, are known parameters, and B is a constant to be taken for a specific design from Table VII. A conservative lower bound D of the ratio of recovery achieved is given for each design in the same table.

(b) For among comparisons, the variance of U_i is of the general form:

$$V(U_i) = V_1 - C_1 + \rho_i(C_1 - C_2) \quad . \tag{8.8.78}$$

By (8.2.5) and (8.2.7), one can write (8.8.78) as:

$$V(U_i) = \left[\frac{k-c_1}{a} + \rho_i \frac{c_1-c_2}{a}\right]\sigma^2 = \frac{(k-\rho_i c_2)+(\rho_i-1)c_1}{a}\sigma^2 \quad . \tag{8.8.79}$$

Thus

$$\theta_i = \frac{(k-\rho_i c_2)+(\rho_i-1)c_1}{a} \quad , \tag{8.8.80}$$

and (8.8.75) becomes:

$$\hat{\tau}_i = U_i + \frac{[(k-\rho_i c_2)+(\rho_i-1)c_1]Bs^2}{a\sum\limits_{j\neq i}^{t}(X_j-U_j)^2}(X_i-U_i) \quad . \tag{8.8.81}$$

B is given for each design in Table VIII. Also in the same table, a lower bound D of the recovery ratio is given. The ρ_i's are given in Table XI.

8.9 Cyclic P.B.I.B. Designs with Two Associate Classes

8.9.1 Definition

A non-group divisible partially balanced incomplete block design is called cyclic if the set of first associates of the i-th treatment is obtained by adding i-1 to the numbers in the set of first associates of the first treatment and subtracting v, whenever the sum exceeds v, where v is the number of treatments in the design. In giving the association scheme of such a design, it is, therefore, sufficient to give the first associates of the first treatment. The parameters and plans of these designs are given in reference (1). The inter-analysis in all ten designs listed is not singular.

8.9.2 Recovery of Inter-block Information in Cyclic Designs

Following a similar approach to that in traingular designs, the treatments in each design will be grouped into a number of groups, with the treatments falling in one group being mutually first associates. An orthogonal matrix M will be defined in each case. The vectors $\underline{U}$ and $\underline{X}$ will be defined as:

$$\underline{U} = M \, \hat{\underline{t}} \quad , \tag{8.9.1}$$

and

$$\underline{X} = M \, \hat{\underline{t}}' \quad . \tag{8.9.2}$$

The treatments $t_1, t_2, \ldots, t_v$ are arranged in the vector $\hat{\underline{t}}$ or $\hat{\underline{t}}'$ according to the grouping plan; that is, the first number of treatments are those of the first group, followed by treatments of the second group, etc. A vector,

$$\underline{Z} = \underline{X} - \underline{U} \quad , \tag{8.9.3}$$

will be considered, and the variances of Z_i's will be computed. Finally, the two linear functions U_i and X_i will be combined, $i = 1,2,\ldots,v-1$. The U_v and X_v are both zero's and their combined estimate $\hat{\tau}_v$ will be assumed to be zero also.

(a) For cyclic designs c1 through c4, where v = 13, consider the 13 x 13 matrix:

$$M_1 = \begin{array}{c}
\begin{array}{ccccccccccccc} 1 & 3 & 8 & 2 & 4 & 9 & 5 & 7 & 12 & 6 & 11 & 13 & 10 \end{array} \\
\left[\begin{array}{ccc|ccc|ccc|ccc|c}
1 & -1 & & & & & & & & & & & \\
1 & 1 & -2 & & & & & & & & & & \\
\hline
 & & & 1 & -1 & & & & & & & & \\
 & & & 1 & 1 & -2 & & & & & & & \\
\hline
 & & & & & & 1 & -1 & & & & & \\
 & & & & & & 1 & 1 & -2 & & & & \\
\hline
 & & & & & & & & & 1 & -1 & & \\
 & & & & & & & & & 1 & 1 & -2 & \\
\hline
1 & 1 & 1 & -1 & -1 & -1 & & & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & -2 & -2 & -2 & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 & -3 & -3 & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -12 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right]
\end{array} . \tag{8.9.4}$$

The grouping plan appears on the top of the matrix, i.e., the first group includes t_1, t_3, t_8; the second group includes t_2, t_4, t_9; the third group includes t_5, t_7, t_{12}; the fourth group includes t_6, t_{11}, t_{13}; and the treatment t_{10} alone forms the fifth group. Normalizing the rows of M_1, the orthogonal matrices M that appear in (8.9.1) and (8.9.2) are obtained. The column vectors, $\hat{\underline{t}}$ in (8.9.1) and $\hat{\underline{t}}'$ in (8.9.2), are in this case,

$$\hat{\underline{t}} = [\hat{t}_1,\hat{t}_3,\hat{t}_8,\hat{t}_2,\hat{t}_4,\hat{t}_9,\hat{t}_5,\hat{t}_7,\hat{t}_{12},\hat{t}_6,\hat{t}_{11},\hat{t}_{13},\hat{t}_{10}] \quad , \tag{8.9.5}$$

and

$$\hat{\underline{t}}' = [\hat{t}'_1,\hat{t}'_3,\hat{t}'_8,\hat{t}'_2,\hat{t}'_4,\hat{t}'_9,\hat{t}'_5,\hat{t}'_7,\hat{t}'_{12},\hat{t}'_6,\hat{t}'_{11},\hat{t}'_{13},\hat{t}'_{10}] \quad . \tag{8.9.6}$$

It should be noted that the number of within comparisons is eight and that of among comparisons is four.

The variances of Z_i's are as follows:

$$V(Z_i) = V(U_i) + V(X_i) = V_1 - C_1 + V'_1 - C'_1 = V-C \quad , \quad i = 1,2,\ldots,8 \quad ; \tag{8.9.7}$$

$$V(Z_9) = V(U_9) + V(X_9) = V-C + 2(C-C') \quad , \tag{8.9.8}$$

$$V(Z_{10}) = V(U_{10}) + V(X_{10}) = V-C + 2(C-C') \quad , \tag{8.9.9}$$

$$V(Z_{11}) = V(U_{11}) + V(X_{11}) = V-C + \frac{3}{2}(C-C') \quad , \tag{8.9.10}$$

and

$$V(Z_{12}) = V(U_{12}) + V(X_{12}) = V-C + \frac{1}{2}(C-C') \quad . \tag{8.9.11}$$

(b) For cyclic designs c5 through c7, where v = 17, consider the 17 x 17 matrix:

$$
M_1 = \begin{array}{c}
\begin{array}{ccc|ccc|ccc|ccc|ccc|cc}
1 & 4 & 7 & 2 & 5 & 8 & 3 & 6 & 9 & 10 & 13 & 16 & 11 & 14 & 17 & 12 & 15
\end{array} \\
\left[\begin{array}{ccc|ccc|ccc|ccc|ccc|cc}
1 & -1 & & & & & & & & & & & & & & & \\
1 & 1 & -2 & & & & & & & & & & & & & & \\
\hdashline
 & & & 1 & -1 & & & & & & & & & & & & \\
 & & & 1 & 1 & -2 & & & & & & & & & & & \\
\hdashline
 & & & & & & 1 & -1 & & & & & & & & & \\
 & & & & & & 1 & 1 & -2 & & & & & & & & \\
\hdashline
 & & & & & & & & & 1 & -1 & & & & & & \\
 & & & & & & & & & 1 & 1 & -2 & & & & & \\
\hdashline
 & & & & & & & & & & & & 1 & -1 & & & \\
 & & & & & & & & & & & & 1 & 1 & -2 & & \\
\hdashline
 & & & & & & & & & & & & & & & 1 & -1 \\
\hdashline
1 & 1 & 1 & -1 & -1 & -1 & & & & & & & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & -2 & -2 & -2 & & & & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 & -3 & -3 & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -4 & -4 & -4 & & \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & -15 & -15 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right]
\end{array} . \tag{8.9.12}
$$

The grouping plan put in a rectangle where each row represents a group is:

$$
\begin{bmatrix}
1 & 4 & 7 \\
2 & 5 & 8 \\
3 & 6 & 9 \\
10 & 13 & 16 \\
11 & 14 & 17 \\
12 & 15 &
\end{bmatrix} . \tag{8.9.13}
$$

The variances of Z_i's are:

$$V(Z_i) = V-C \quad , \qquad i = 1,2,\ldots,11; \tag{8.9.14}$$

$$V(Z_{12}) = V-C + \frac{7}{3}(C-C') \quad , \tag{8.9.15}$$

$$V(Z_{13}) = V-C + \frac{7}{3}(C-C') \quad , \tag{8.9.16}$$

$$V(Z_{14}) = V-C + \frac{5}{6}(C-C') \quad , \tag{8.9.17}$$

$$V(Z_{15}) = V-C + \frac{43}{30}(C-C') \quad , \tag{8.9.18}$$

and

$$V(Z_{16}) = V-C + \frac{16}{15}(C-C') \quad . \tag{8.9.19}$$

(c) For cyclic designs c8 and c9, where $v = 29$, consider the 29 x 29 matrix:

$$
M_1 = \begin{array}{c}
\begin{array}{cccccccccccccccccccccccccccccc}
1 & 2 & 6 & 7 & 3 & 4 & 8 & 9 & 10 & 11 & 15 & 16 & 17 & 18 & 22 & 23 & 19 & 20 & 24 & 25 & 5 & 12 & 21 & 28 & 13 & 14 & 26 & 27 & 29
\end{array} \\
\left[\begin{array}{cccc|cccc|cccc|cccc|cccc|cccc|cc|cc|c}
1 & -1 & & & & & & & & & & & & & & & & & & & & & & & & & & & \\
1 & 1 & -2 & & & & & & & & & & & & & & & & & & & & & & & & & & \\
1 & 1 & 1 & -3 & & & & & & & & & & & & & & & & & & & & & & & & & \\
\hline
& & & & 1 & -1 & & & & & & & & & & & & & & & & & & & & & & & \\
& & & & 1 & 1 & -2 & & & & & & & & & & & & & & & & & & & & & & \\
& & & & 1 & 1 & 1 & -3 & & & & & & & & & & & & & & & & & & & & & \\
\hline
& & & & & & & & 1 & -1 & & & & & & & & & & & & & & & & & & & \\
& & & & & & & & 1 & 1 & -2 & & & & & & & & & & & & & & & & & & \\
& & & & & & & & 1 & 1 & 1 & -3 & & & & & & & & & & & & & & & & & \\
\hline
& & & & & & & & & & & & 1 & -1 & & & & & & & & & & & & & & & \\
& & & & & & & & & & & & 1 & 1 & -2 & & & & & & & & & & & & & & \\
& & & & & & & & & & & & 1 & 1 & 1 & -3 & & & & & & & & & & & & & \\
\hline
& & & & & & & & & & & & & & & & 1 & -1 & & & & & & & & & & & \\
& & & & & & & & & & & & & & & & 1 & 1 & -2 & & & & & & & & & & \\
& & & & & & & & & & & & & & & & 1 & 1 & 1 & -3 & & & & & & & & & \\
\hline
& & & & & & & & & & & & & & & & & & & & 1 & -1 & & & & & & & \\
& & & & & & & & & & & & & & & & & & & & 1 & 1 & -2 & & & & & & \\
& & & & & & & & & & & & & & & & & & & & 1 & 1 & 1 & -3 & & & & & \\
\hline
& & & & & & & & & & & & & & & & & & & & & & & & 1 & -1 & & & \\
\hline
& & & & & & & & & & & & & & & & & & & & & & & & & & 1 & -1 & \\
\hline
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & & & & & & & & & & & & & & & & & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -2 & -2 & -2 & -2 & & & & & & & & & & & & & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 & -3 & -3 & -3 & & & & & & & & & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -4 & -4 & -4 & -4 & & & & & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -5 & -5 & -5 & -5 & & & & & \\
& & & & & & & & & & & & & & & & & & & & & & & & 1 & 1 & -1 & -1 & \\
& & & & & & & & & & & & & & & & & & & & & & & & 1 & 1 & 1 & 1 & -4 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & -24 & -24 & -24 & -24 & -24 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right]
\end{array}
$$

(8.9.20)

The grouping plan put in a rectangle where each row represents a group is:

$$\begin{bmatrix} 1 & 2 & 6 & 7 \\ 3 & 4 & 8 & 9 \\ 10 & 11 & 15 & 16 \\ 17 & 18 & 22 & 23 \\ 19 & 20 & 24 & 25 \\ 5 & 12 & 21 & 28 \\ 13 & 14 & & \\ 26 & 27 & & \\ 29 & & & \end{bmatrix} . \tag{8.9.21}$$

The variances of the Z_i's are:

$$V(Z_i) = V-C \quad , \quad i = 1,2,\ldots,20 \; ; \tag{8.9.22}$$

$$V(Z_{21}) = V-C + \frac{5}{2}\,(C-C') \quad , \tag{8.9.23}$$

$$V(Z_{22}) = V-C + \frac{11}{6}\,(C-C') \quad , \tag{8.9.24}$$

$$V(Z_{23}) = V-C + \frac{61}{24}\,(C-C') \quad , \tag{8.9.25}$$

$$V(Z_{24}) = V-C + \frac{97}{40}\,(C-C') \quad , \tag{8.9.26}$$

$$V(Z_{25}) = V-C + \frac{11}{5}\,(C-C') \quad , \tag{8.9.27}$$

$$V(Z_{26}) = V-C + (C-C') \quad , \tag{8.9.28}$$

$$V(Z_{27}) = V-C + (C-C') \quad , \tag{8.9.29}$$

and

$$V(Z_{28}) = V - C + \frac{1}{2}(C - C') \quad . \tag{8.9.30}$$

(d) For cyclic design c10 in which v = 37, consider the 37 x 37 matrix:

$$
M_1 = \begin{array}{c}
\begin{array}{cccccccccccccccccccccccccccccccccccccc}
1 & 2 & 5 & 12 & 3 & 4 & 7 & 14 & 6 & 9 & 10 & 13 & 8 & 11 & 15 & 18 & 16 & 17 & 20 & 27 & 19 & 23 & 26 & 30 & 21 & 24 & 25 & 28 & 22 & 29 & 32 & 33 & 31 & 34 & 35 & 36 & 37
\end{array} \\
\left[\begin{array}{cccc|cccc|cccc|cccc|cccc|cccc|cccc|cccc|ccc|cc}
1 & -1 & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & \\
1 & 1 & -2 & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & \\
1 & 1 & 1 & -3 & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & \\
\hline
 & & & & 1 & -1 & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & \\
 & & & & 1 & 1 & -2 & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & \\
 & & & & 1 & 1 & 1 & -3 & & & & & & & & & & & & & & & & & & & & & & & & & & & & & \\
\hline
- & - \\
\hline
 & & & & & & & & & & & & & & & & & & & & & & & & 1 & -1 & & & & & & & & & & & \\
 & & & & & & & & & & & & & & & & & & & & & & & & 1 & 1 & -2 & & & & & & & & & & \\
 & & & & & & & & & & & & & & & & & & & & & & & & 1 & 1 & 1 & -3 & & & & & & & & & \\
\hline
 & & & & & & & & & & & & & & & & & & & & & & & & & & & & 1 & -1 & & & & & & & \\
 & & & & & & & & & & & & & & & & & & & & & & & & & & & & 1 & 1 & -2 & & & & & & \\
 & & & & & & & & & & & & & & & & & & & & & & & & & & & & 1 & 1 & 1 & -3 & & & & & \\
\hline
 & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & 1 & -1 & & & \\
 & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & 1 & 1 & -2 & & \\
\hline
 & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & 1 & -1 \\
\hline
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & & & & & & & & & & & & & & & & & & & & & & & & & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -2 & -2 & -2 & -2 & & & & & & & & & & & & & & & & & & & & & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -3 & -3 & -3 & -3 & & & & & & & & & & & & & & & & & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -4 & -4 & -4 & -4 & & & & & & & & & & & & & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -5 & -5 & -5 & -5 & & & & & & & & & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -6 & -6 & -6 & -6 & & & & & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -7 & -7 & -7 & -7 & & & & & \\
 & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & 2 & 2 & 2 & -3 & -3 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & -32 & -32 & -32 & -32 & -32 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right]
\end{array}
\qquad (8.9.31)
$$

The grouping plan put in a rectangle, where each row represents a group, is:

$$\begin{bmatrix} 1 & 2 & 5 & 12 \\ 3 & 4 & 7 & 14 \\ 6 & 9 & 10 & 13 \\ 8 & 11 & 15 & 18 \\ 16 & 17 & 20 & 27 \\ 19 & 23 & 26 & 30 \\ 21 & 24 & 25 & 28 \\ 22 & 29 & 32 & 33 \\ 31 & 34 & 35 & \\ 36 & 37 & & \end{bmatrix} . \qquad (8.9.32)$$

The variances of Z_i's are:

$$V(Z_i) = V-C \quad , \qquad i = 1,2,\ldots,27; \qquad (8.9.33)$$

$$V(Z_{28}) = V-C + \frac{5}{2}(C-C') \quad , \qquad (8.9.34)$$

$$V(Z_{29}) = V-C + \frac{5}{3}(C-C') \quad , \qquad (8.9.35)$$

$$V(Z_{30}) = V-C + \frac{47}{23}(C-C') \quad , \qquad (8.9.36)$$

$$V(Z_{31}) = V-C + \frac{91}{40}(C-C') \quad , \qquad (8.9.37)$$

$$V(Z_{32}) = V-C + \frac{11}{5}(C-C') \quad , \qquad (8.9.38)$$

$$V(Z_{33}) = V-C + \frac{37}{14}(C-C') \quad , \qquad (8.9.39)$$

$$V(Z_{34}) = V-C + \frac{243}{112}(C-C') \quad , \qquad (8.9.40)$$

$$V(Z_{35}) = V\text{-}C + \frac{8}{5}(C\text{-}C') \quad , \tag{8.9.41}$$

and

$$V(Z_{36}) = V\text{-}C + \frac{3059}{2960}(C\text{-}C') \ . \tag{8.9.42}$$

8.9.2.1 <u>Combined Estimates for Cyclic Designs</u>

It should be noted that all the variances of Z_i's have been expressed in the general form $V\text{-}C+\rho_i(C\text{-}C')$, where $\rho_i \geq 0$. Hence the results of Section (8.8.2) and Section (8.8.3.1) are applicable for combining U_i's and X_i's. Thus the combined estimates in Section (8.8.3.2) are suitable for the cyclics under study.

(a) For within comparisons, the combined estimate is as in (8.8.77):

$$\hat{\tau}_i = U_i + \frac{(k-c_1)Bs^2}{a\sum\limits_{j\neq i}^{t}(X_j-U_j)^2}(X_i-U_i) \quad , \tag{8.9.43}$$

where k, c_1, a=rk-r, are known parameters, and B is a constant to be taken from Table IX. A conservative lower bound D of the ratio of recovery achieved is given for each design in the same table.

(b) For among comparisons, the combined estimate is, as in (8.8.81):

$$\hat{\tau}_i = U_i + \frac{[(k-\rho_i c_2)+(\rho_i-1)c_1]Bs^2}{a\sum_{j\neq i}^{t}(X_i-U_j)^2}(X_i-U_i) \quad . \qquad (8.9.44)$$

B is given for each design in Table X. Also in the same table, a very conservative lower bound of the recovery ratio D is given. The ρ_i's are given in Table XII.

8.10 General Procedure for Recovery of Inter-block Information in P.B.I.B. with Two Associate Classes

Compute:

1. $Y_{i.} = \sum_{j=1}^{b}\delta_{ij}y_{ij}$, $i = 1,2,\ldots,v$; (i-th treatment total).

2. $Y_{.j} = \sum_{i=1}^{v}\delta_{ij}y_{ij}$, $j = 1,2,\ldots,b$; (j-th block total).

3. $Y_{..} = \sum_{ij}\delta_{ij}y_{ij}$; (grand total).

4. $Q_i = Y_{i.} - \frac{1}{k}$ (sum of block totals in which treatment i occurs), $i = 1,2,\ldots,v$; (adjusted treatment totals).

5. $S_1(Q_i)$, $i = 1,2,\ldots,v$; (the sum of Q's for all treatments which are first associates of treatment i).

6. $\hat{t}_i = \frac{k-c_2}{a} Q_i + \frac{c_1-c_2}{a} S_1(Q_i)$, $i = 1,2,\ldots,v$;

(the intra-treatment estimate).

7. $Q_i' = Y_{i.} - Q_i - \frac{Y_{..}}{v}$, $i = 1,2,\ldots,v$.

8. $S_1(Q_i')$, $i = 1,2,\ldots,v$.

9. $\hat{t}_i' = \frac{k-c_2'}{r} Q_i' + \frac{c_1'-c_2'}{a} S_1(Q_i')$, $i = 1,2,\ldots,v$;

(the inter-treatment estimate).

10. $\hat{t}_i' - \hat{t}_i$, $i = 1,2,\ldots,v$.

11. $$\underline{X} - \underline{U} = \begin{pmatrix} X_1-U_1 \\ X_2-U_2 \\ \vdots \\ X_{v-1}-U_{v-1} \\ 0 \end{pmatrix} = M \begin{pmatrix} \hat{t}_1'-\hat{t}_1 \\ \hat{t}_2'-\hat{t}_2 \\ \vdots \\ \hat{t}_{v-1}'-\hat{t}_{v-1} \\ \hat{t}_v \quad -\hat{t}_v \end{pmatrix} = M(\hat{\underline{t}}'-\hat{\underline{t}}) \ ;$$

where M is the orthogonal matrix defined in each case previously.

12. $s^2 = \frac{1}{f}(\sum_{ij} \delta_{ij} Y_{ij}^2 - SST - \frac{1}{k} \sum_{j=1}^{b} Y_{.j}^2)$; (error mean square, $f = bk-b-v+1$).

To combine X_i and U_i, compute:

13. $\sum_{j=1}^{v}(X_j-U_j)^2$.

14. $\sum_{j\neq i}^{v}(X_j-U_j)^2 = \sum_{j=1}^{v}(X_j-U_j)^2 - (X_i-U_i)^2$.

15. $J = \dfrac{\theta_i Bs^2}{\sum_{j\neq i}(X_j-U_j)^2}$, where θ_i was defined previously in each case; B is a constant to be taken from the corresponding table.

16. The combined estimate of U_i and X_i,

$$\hat{\tau}_i = U_i + J(X_i-U_i) \ , \quad i = 1,2,\ldots,v \ .$$

If the combined estimates of the treatments t_i's themselves are desired, one should compute:

$$\underline{T} = \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_v \end{pmatrix} = M' \begin{pmatrix} \hat{\tau}_1 \\ \hat{\tau}_2 \\ \vdots \\ \hat{\tau}_v \end{pmatrix} = M'\hat{\underline{\tau}} \quad ,$$

where M' is the transpose of M.

8.11 Comment on the Numerical Methods Used in Table VII Through Table X

In Section 7.1.1, a method of utilizing the Hypergeometric series and Gauss' continued fraction was discussed. In Table VII through X, the same idea was applied for computing E_1 and E_2, where:

$$E_1 = \frac{1}{\beta(\frac{1}{2}\nu_1, \frac{1}{2}\nu_2)} \int_0^1 \frac{u^{\frac{1}{2}\nu_1 - 1} (1-u)^{\frac{1}{2}\nu_2 - 1}}{u + P(1-u)} du \quad , \qquad (8.11.1)$$

and

$$E_2 = \frac{1}{\beta(\frac{1}{2}\nu_1, \frac{1}{2}\nu_2)} \int_0^1 \frac{u^{\frac{1}{2}\nu_1} (1-u)^{\frac{1}{2}\nu_2 - 1}}{[u + L(1-u)]^2} du \quad . \qquad (8.11.2)$$

In view of (7.1.8), (7.1.9), and (7.1.11), (8.11.1) may be written as:

$$E_1 = \frac{1}{P\beta(\frac{1}{2}\nu_1, \frac{1}{2}\nu_2)} \int_0^1 \frac{u^{\frac{1}{2}\nu_1 - 1} (1-u)^{\frac{1}{2}\nu_2 - 1}}{1 - \frac{P-1}{P} u} du \quad ,$$

$$= \frac{F(\frac{1}{2}\nu_1, 1, \frac{v-2}{2}, \frac{P-1}{P})}{P} \quad . \qquad (8.11.3)$$

Now,

$F(\frac{1}{2}\nu_1, 1, \frac{v-2}{2}, \frac{P-1}{P})$ could be put in the form of (7.1.12) and evaluated.

Similarly, for E_2, one can write:

$$E_2 = \frac{1}{L^2\beta(\frac{1}{2}\nu_1, \frac{1}{2}\nu_2)} \int_0^1 \frac{u^{\frac{1}{2}\nu_1 - 1} (1-u)^{\frac{1}{2}\nu_2 - 1}}{[1 - \frac{L-1}{L} u]^2} du \quad ,$$

$$= \frac{F(2, \frac{1}{2}\nu_1, \frac{\nu-2}{2}, \frac{L-1}{L})}{L^2} \quad . \tag{8.11.4}$$

To evaluate $F(2, \frac{1}{2}\nu_1, \frac{\nu-2}{2}, \frac{L-1}{L})$, the following recursion formula (see reference 13, page 363) is needed:

$$F(a,b,c;z) = F(a,b+1,c+1;z) - \frac{a(c-b)z}{c(c+1)} F(a+1,b+1,c+2;z) \quad , \tag{8.11.5}$$

or

$$F(a+1,b+1,c+2;z) = \frac{c(c+1)}{a(c-b)z}[F(a,b+1,c+1;z)-F(a,b,c;z)] \quad ; \tag{8.11.6}$$

putting a = 1, one gets:

$$F(2,b+1,c+2;z) = \frac{c(c+1)}{(c-b)z}[F(1,b+1,c+1;z)-F(1,b,c;z)] \quad . \tag{8.11.7}$$

Comparing (8.11.4) and (8.11.7), one obtains:

$$b+1 = \tfrac{1}{2}\nu_1 \quad , \qquad b = \tfrac{1}{2}\nu_1 - 1 \quad ,$$

$$c+2 = \frac{\nu-2}{2} \quad , \qquad c = \frac{\nu-6}{2} \tag{8.11.8}$$

and

$$z = \frac{L-1}{L} \quad .$$

Thus, substituting in (8.11.7),

$$F(2,\ \tfrac{1}{2}\nu_1,\ \frac{v-2}{2};\ \frac{L-1}{L}) = \frac{\frac{v-6}{2}\cdot\frac{v-4}{2}}{(\frac{v-6}{2}-\tfrac{1}{2}\nu_1+1)\frac{L-1}{L}}[F(1,\ \tfrac{1}{2}\nu_1,\ \frac{v-4}{2};\ \frac{L-1}{L})$$

$$-\ F(1,\ \tfrac{1}{2}\nu_1-1,\ \frac{v-6}{2};\ \frac{L-1}{L})] \quad .(8.11.9)$$

In view of (7.1.11) and the fact that $\nu_1+\nu_2 = v-2$, (8.11.9) may be written as:

$$F(\tfrac{1}{2}\nu_2,\ 2,\ \frac{v-2}{2};\ \frac{L-1}{L}) = \frac{(v-6)(v-4)\cdot L}{4(\tfrac{1}{2}\nu_2-1)(L-1)}[F(\tfrac{1}{2}\nu_1,\ 1,\ \frac{v-4}{2};\ \frac{L-1}{L})$$

$$-\ F(\tfrac{1}{2}\nu_1-1,\ 1,\ \frac{v-6}{2};\ \frac{L-1}{L})] \ .\ (8.11.10)$$

Now, $F(\tfrac{1}{2}\nu_1,\ 1,\ \frac{v-4}{2};\ \frac{L-1}{L})$ and $F(\tfrac{1}{2}\nu_1-1,\ 1,\ \frac{v-6}{2};\ \frac{L-1}{L})$ can be put in the form of Gauss' continued fraction as in (7.1.12) and evaluated.

The above was programmed for an IBM 1620 and used for cyclic designs (Tables IX and X); however, as we notice from (8.11.10), the above method is not applicable when $\tfrac{1}{2}\nu_2-1 = 0$ or $\nu_2 = 2$, the case that was encountered in triangular designs (Tables VII and VIII). Whenever this happened the integrals,

$$G_1 = \int_0^1 \frac{u^{\frac{1}{2}\nu_1}(1-u)^{\frac{1}{2}\nu_2-1}}{u + P(1-u)}\, du \quad , \tag{8.11.11}$$

and

$$G_2 = \int_0^1 \frac{u^{\frac{1}{2}\nu_1-1}(1-u)^{\frac{1}{2}\nu_2-1}}{[u + L(1-u)]^2}\, du \quad , \tag{8.11.12}$$

were evaluated by the Simpson rule. The accuracy of the Simpson rule was listed in the tables; the accuracy of E_1 or E_2 was set to the sixth decimal place constantly.

9. SUMMARY AND DISCUSSION

In summary, this dissertation has brought up the following points:

1. Under certain conditions, a new method of combining two independent estimates has been given in theorem 1. This new method has its immediate application in incomplete block designs, in similar experiments, and in randomized block designs with heterogeneous variances. The amount of improvement obtained by this new method is very satisfactory, compared with the utmost possible theoretical improvement.

2. A procedure for recovering the inter-block information in balanced incomplete block designs was given which is applicable in experiments as small as $t = 4$.

3. A generalization of theorem 1 was given which shows that the idea of recovering the inter-block information can be practically utilized in any incomplete block design with seven treatments or more.

4. For a partially balanced incomplete block design with two associate classes, a separate development was given to each of its four subclasses, namely the group divisible (G.D.), Latin square type (L_2S type and L_3S type), the

triangular, and the cyclic designs. A combined estimate was given for each case, and a general procedure for recovery of inter-block information in these subclasses was developed. Because of the special nature of singular and semi-regular G.D. designs, an inter-analysis was discussed for G.D.'s in general and for these two cases in special, and a partial utilization of the inter-information was made possible.

The essence of this work arises from the fact that so far a method for recovering the inter-block information was based on the common sense of the consistency property of an estimate. Consequently, the applicability of the resulting method would make sense only in big size experiments. The resulting combined estimates are of so complicated expression that a mathematical study of their merits or demerits is not feasible, even in their domain of applicability. On the contrary, this dissertation provides a general method by a mathematical treatment. This work can be said to have the following two merits:

1. It makes possible the utilization of the inter-block information in small size experiments, as small as four treatments in B.I.B. designs and as small as seven treatments in P.B.I.B. designs. This is of special importance because of

the fact that the intra-block efficiency is in general lower in small and moderate size designs. It follows that the recovery is needed more in this sort of designs.

2. As a ratio of the utmost possible theoretical recovery, either exactly or a lower bound of the amount of the ratio of recovery is always computable; this ratio was tabled for all B.I.B. designs listed in reference (2), and P.B.I.B. designs listed in reference (1). The ratio of recovery depends on the structure of the design; it always increases with v, the number of treatments.

A glance at the tables, where these ratios are listed, shows that the new methods, for B.I.B. and P.B.I.B. designs, give good results where the old method is not applicable; where the old method starts, hopefully, to be valid, the ratio of recovery achieved by the new methods begins to approach the theoretical value that can be achieved, assuming the intra- and inter-variance are known

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T A B L E S

Table I: $\frac{1}{\sigma^2}$ [percentage gain due to the Rao Method]

No. of Design \ u	0	$\frac{1}{10(1+k)}$	$\frac{1}{5(1+k)}$	$\frac{3}{10(1+k)}$	$\frac{4}{10(1+k)}$	$\frac{5}{10(1+k)}$	$\frac{6}{10(1+k)}$	$\frac{7}{10(1+k)}$	$\frac{8}{10(1+k)}$	$\frac{9}{10(1+k)}$	$\frac{1}{1+k}$
R1	0	.000	.001	.002	.003	.004	.005	.006	.008	.010	.012
R2	0	.000	.001	.001	.002	.002	.003	.004	.005	.006	.007
R3	0	.000	.001	.001	.002	.003	.004	.005	.006	.007	.009
R4	0	.000	.000	.001	.001	.002	.003	.003	.004	.005	.007
R5	0	.000	.001	.002	.003	.003	.004	.006	.007	.008	.009
R6	0	.000	.001	.002	.003	.003	.004	.006	.007	.008	.009
R7	0	.000	.001	.001	.002	.002	.003	.004	.005	.006	.007
R8	0	.001	.001	.002	.002	.003	.004	.005	.006	.007	.008
R9	0	.000	.001	.001	.002	.002	.003	.004	.004	.005	.006
R10	0	.001	.001	.002	.003	.004	.005	.005	.006	.007	.008
R11	0	.000	.000	.001	.001	.002	.003	.004	.004	.005	.006
R12	0	.000	.000	.001	.001	.002	.002	.003	.004	.004	.005
R13	0	.000	.001	.001	.002	.003	.004	.004	.005	.007	.008
R14	0	.000	.001	.001	.001	.002	.002	.003	.004	.004	.005
R15	0	.000	.000	.000	.001	.001	.001	.002	.002	.003	.003
R16	0	.000	.000	.001	.001	.001	.002	.002	.003	.004	.004
R17	0	.000	.000	.001	.001	.001	.002	.002	.003	.003	.004
R18	0	.000	.000	.001	.001	.001	.002	.002	.003	.003	.004
R19	0	.000	.000	.001	.001	.001	.002	.002	.002	.003	.003
R20	0	.000	.001	.001	.002	.002	.003	.004	.004	.005	.006
R21	0	.000	.000	.001	.001	.001	.002	.002	.003	.004	.004
R22	0	.000	.000	.001	.001	.001	.002	.002	.003	.003	.004
R23	0	.000	.000	.000	.001	.001	.001	.002	.002	.003	.003
R24	0	.000	.000	.000	.001	.001	.001	.002	.002	.002	.003
R25	0	.000	.000	.000	.001	.001	.001	.002	.002	.003	.003

Table I: (continued)

u / No. of Design	0	$\frac{1}{10(1+k)}$	$\frac{1}{5(1+k)}$	$\frac{3}{10(1+k)}$	$\frac{4}{10(1+k)}$	$\frac{5}{10(1+k)}$	$\frac{6}{10(1+k)}$	$\frac{7}{10(1+k)}$	$\frac{8}{10(1+k)}$	$\frac{9}{10(1+k)}$	$\frac{1}{1+k}$
R26	0	.000	.000	.000	.001	.001	.001	.002	.002	.002	.003
R27	0	.000	.000	.001	.001	.001	.002	.002	.003	.003	.004
R28	0	.000	.000	.001	.001	.001	.002	.002	.003	.003	.004
R29	0	.000	.000	.000	.001	.001	.001	.002	.002	.003	.003
R30	0	.000	.000	.001	.001	.001	.002	.002	.003	.003	.004
R31	0	.000	.000	.000	.001	.001	.001	.002	.002	.003	.003
R32	0	.000	.000	.000	.001	.001	.001	.002	.002	.002	.003
R33	0	.000	.000	.001	.001	.001	.002	.002	.003	.003	.004
R34	0	.000	.000	.001	.001	.001	.001	.002	.002	.002	.003
R35	0	.000	.000	.001	.001	.002	.002	.003	.003	.004	.004
R36	0	.000	.000	.000	.001	.001	.001	.001	.002	.002	.002
R37	0	.000	.000	.001	.001	.001	.001	.002	.002	.003	.003
R38	0	.000	.000	.000	.001	.001	.001	.002	.002	.002	.003
R39	0	.000	.000	.000	.000	.001	.001	.001	.002	.002	.002
R40	0	.000	.000	.001	.001	.001	.002	.002	.003	.003	.004
R41	0	.000	.000	.000	.000	.001	.001	.001	.002	.002	.002
R42	0	.000	.000	.000	.000	.001	.001	.001	.001	.002	.002
R43	0	.000	.000	.000	.001	.001	.001	.001	.002	.002	.002
R44	0	.000	.000	.000	.000	.001	.001	.001	.002	.002	.002
R45	0	.000	.000	.000	.000	.001	.001	.001	.001	.001	.002
R46	0	.000	.000	.000	.000	.000	.001	.001	.001	.001	.002
R47	0	.000	.000	.000	.001	.001	.001	.002	.002	.002	.003
R48	0	.000	.000	.000	.000	.001	.001	.001	.001	.002	.002
R49	0	.000	.000	.000	.000	.000	.001	.001	.001	.001	.001
R50	0	.000	.000	.000	.000	.001	.001	.001	.001	.001	.002

Table I: (continued)

No. of Design \ u	0	$\frac{1}{10(1+k)}$	$\frac{1}{5(1+k)}$	$\frac{3}{10(1+k)}$	$\frac{4}{10(1+k)}$	$\frac{5}{10(1+k)}$	$\frac{6}{10(1+k)}$	$\frac{7}{10(1+k)}$	$\frac{8}{10(1+k)}$	$\frac{9}{10(1+k)}$	$\frac{1}{1+k}$
R51	0	.000	.000	.000	.000	.000	.000	.001	.001	.001	.001
R52	0	.000	.000	.000	.000	.000	.001	.001	.001	.001	.001
R53	0	.000	.000	.000	.000	.000	.001	.001	.001	.001	.001
R54	0	.000	.000	.000	.000	.000	.001	.001	.001	.001	.001
R55	0	.000	.000	.000	.000	.000	.001	.001	.001	.001	.001
R56	0	.000	.000	.000	.000	.000	.000	.001	.001	.001	.001
R57	0	.000	.000	.000	.000	.000	.000	.001	.001	.001	.001
R58	0	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
R59	0	.000	.000	.000	.000	.000	.000	.000	.000	.001	.001
R60	0	.000	.000	.000	.000	.000	.000	.000	.000	.001	.001
R61	0	.000	.000	.000	.000	.000	.000	.000	.000	.001	.001
R62	0	.000	.000	.000	.000	.000	.000	.000	.000	.001	.001
R63	0	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
R64	0	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
R65	0	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
R66	0	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
R67	0	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000
R68	0	.000	.000	.000	.000	.000	.000	.000	.000	.000	.000

Table II: Balanced Designs

s	v	r	k	b	λ	f	E	F*	B	D_2	D_3	D_1
1	4	3	2	6	1	3	.67	.38017290	.526	.18	.20	0.00
2	4	3	3	4	2	5	.89	"	.626	.20	.24	0.00
3	5	4	2	10	1	6	.62	.27639332	1.357	.34	.38	.23
4	5	6	3	10	3	16	.83	"	1.608	.40	.44	.28
5	5	4	4	5	3	11	.94	"	1.531	.38	.42	.26
6	6	5	2	15	1	10	.60	.21540524	2.321	.46	.50	.40
7	6	5	3	10	2	15	.80	"	2.458	.49	.53	.42
8	6	10	3	20	4	35	.80	"	2.635	.53	.57	.45
9	6	10	4	15	6	40	.90	"	2.653	.53	.57	.46
10	6	5	5	6	4	19	.96	"	2.520	.50	.54	.43
11	7	6	2	21	1	15	.58	.17614569	3.339	.56	.59	.51
12	7	3	3	7	1	8	.78	"	3.028	.50	.53	.47
13	7	4	4	7	2	15	.88	"	3.339	.56	.59	.51
14	7	6	6	7	5	29	.97	"	3.541	.59	.62	.55
15	8	7	2	28	1	21	.57	.14899933	4.377	.63	.65	.60
16	8	7	4	14	3	35	.86	"	4.535	.65	.68	.61
17	8	7	7	8	6	41	.98	"	4.571	.65	.68	.62
18	9	8	2	36	1	28	.56	.12916656	5.419	.68	.70	.66
19	9	8	4	18	3	46	.84	"	5.565	.70	.72	.63
20	9	10	5	18	5	64	.90	"	5.631	.70	.73	.68
21	9	8	6	12	5	52	.94	"	5.591	.70	.72	.68
22	9	8	8	9	7	55	.98	"	5.603	.70	.72	.68
23	10	9	2	45	1	36	.56	.11405141	6.461	.72	.74	.70
24	10	9	3	30	2	51	.74	"	6.562	.73	.75	.71
25	10	6	4	15	2	36	.83	"	6.461	.72	.74	.70
26	10	9	5	18	4	63	.89	"	6.610	.73	.75	.72
27	10	9	6	15	5	66	.93	"	6.619	.74	.75	.72
28	10	9	9	10	8	71	.99	"	6.633	.74	.76	.72

Table II: (continued)

s	v	r	k	b	λ	f	E	F*	B	D_2	D_3	D_1
29	11	10	2	55	1	45	.55	.10214480	7.499	.75	.77	.74
30	11	5	5	11	2	34	.88	"	7.397	.74	.76	.73
31	11	6	6	11	3	45	.92	"	7.499	.75	.77	.74
32	11	10	10	11	9	89	.99	"	7.660	.77	.78	.75
33	13	6	3	26	1	40	.72	.08457048	9.384	.78	.79	.77
34	13	4	4	13	1	27	.81	"	9.174	.76	.78	.76
35	13	9	9	13	6	92	.96	"	9.644	.80	.82	.80
36	15	7	3	35	1	56	.71	.07220328	11.462	.82	.84	.81
37	15	7	7	15	3	76	.92	"	11.567	.83	.84	.82
38	15	8	8	15	4	91	.94	"	11.616	.83	.84	.82
39	16	6	6	16	2	65	.89	.06729477	12.494	.83	.84	.83
40	16	9	6	24	3	105	.89	"	12.638	.84	.85	.84
41	16	10	10	16	6	129	.96	"	12.682	.85	.85	.84
42	19	9	3	57	1	97	.70	.05591668	15.576	.87	.87	.86
43	19	9	9	19	4	134	.94	.05591668	15.663	.87	.88	.87
44	19	10	10	19	5	153	.95	"	15.692	.87	.88	.87
45	21	10	3	70	1	120	.70	.05026207	17.613	.88	.89	.88
46	21	5	5	21	1	64	.84	"	17.364	.87	.87	.87
47	21	10	7	30	3	160	.90	"	17.685	.88	.89	.88
48	25	8	4	50	1	126	.78	.04181724	21.578	.90	.90	.90
49	25	9	9	25	3	176	.93	"	21.674	.90	.91	.90
50	28	9	4	63	1	162	.78	.03714312	24.625	.91	.91	.91
51	28	9	7	36	2	189	.89	"	24.668	.91	.92	.91
52	31	6	6	31	1	125	.86	.03341029	27.496	.92	.92	.92
53	31	10	10	31	3	249	.93	"	27.713	.92	.93	.92
54	37	9	9	37	2	260	.91	.02782200	33.687	.94	.94	.93
55	41	10	5	82	1	288	.82	.02503212	37.690	.94	.94	.94
56	57	8	8	57	1	343	.89	.01786875	53.652	.96	.96	.96
57	73	9	9	73	1	512	.90	.01389509	69.697	.97	.97	.97
58	91	10	10	91	1	729	.91	.01111421	87.735	.97	.98	.97

Table* III: Regular G.D., Within Comparisons

s	v	Lower bound of Φ_2/Φ_1	Upper bound of Φ_2/Φ_1	G_1	Error of G_1	G_2	Error of G_2	B	D
R5	8	2.75000000	3.00000000	.21033814	-.00018116	.13394966	-.00015593	2.57	.34
R6	8	2.75000000	3.00000000	.21033814	-.00018116	.13394966	-.00015593	2.99	.40
R7									
R8	9	.14285714	.16290727	.56841986	.00000717	.94426396	.00006088	1.63	.46
R9	9	.50000000	.54901961	.46543160	.00000205	.57522229	.00000454	2.23	.52
R10	9	.33333333	.39682540	.49736833	.00000285	.69030269	.00001040	2.02	.50
R11	9	1.89393940	2.00000000	.31913521	.00000056	.27019144	.00000031	3.35	.53
R12	9	.66666667	.70000000	.43977843	.00000160	.49866676	.00000253	2.54	.56
R13	9	2.78494620	3.00000000	.27239956	.00000037	.20781181	.00000013	3.79	.52
R14	10	3.80000000	4.00000000	.07240079	.00000000	.03680778	.00000000	7.57	.55
R15	12	.50000000	.51851852	.09866736	.00000332	.13872044	.00000572	3.95	.66
R16	12	1.57407410	1.66666670	.05581265	.00000181	.04559198	.00000160	6.87	.65
R17	12	.66666667	.70000000	.08708004	.00000275	.10820428	.00000369	4.58	.68
R18	12	.55555556	.61191626	.27243560	.00000000	.30782790	.00000000	5.08	.69
R19	12								
R20	12	2.75000000	3.00000000	.08072434	-.00013892	.05994471	-.00014068	7.82	.64
R21	12	1.57407410	1.66666670	.05581265	.00000181	.04559198	.00000160	7.14	.68
R22	12	1.43951610	1.50000000	.10731251	-.00013718	.09716970	-.00013761	6.44	.70
R23	12	1.94708990	2.00000000	.21184402	-.00000000	.18474452	-.00000000	6.71	.71
R24	14	1.92857140	2.00000000	.02258870	-.00000000	.01625612	-.00000000	10.40	.70
R25	14	1.42857140	1.50000000	.02682109	-.00000000	.02291372	-.00000000	8.95	.72
R26	14	1.92857140	2.00000000	.02258870	-.00000000	.01625612	-.00000000	10.81	.73

*G_1 and G_2 are computed by the Simpson rule in Table III through Table VI.

Table III: (continued)

s	v	Lower bound of Φ_2/Φ_1	Upper bound of Φ_2/Φ_1	G_1	Error of G_1	G_2	Error of G_2	B	D
R27	15	3.80000000	4.00000000	.02249261	.00000000	.01406606	.00000002	13.52	.68
R28	15	1.86666670	2.00000000	.03139546	.00000000	.02615760	.00000000	10.35	.73
R29	15	.66666667	.70769231	.04453214	.00000000	.05079515	.00000000	7.66	.77
R30	15	3.80000000	4.00000000	.02249261	.00000000	.01406606	.00000002	14.02	.71
R31	15	3.35858590	3.50000000	.13773795	-.00000002	.11094994	-.00000006	10.89	.75
R32	15	.58333333	.64367816	.19287289	.00000000	.20999550	.00000000	8.05	.78
R33	15	.50000000	.56000000	.04709829	.00000000	.05832701	.00000000	7.08	.75
R34	15	1.57948720	1.60000000	.03434892	.00000000	.02968222	.00000000	10.22	.79
R35	16	2.75000000	3.00000000	.04674152	-.00014124	.03722217	-.00014304	12.06	.73
R36	16	.50000000	.52884615	.07203230	-.00013834	.08256627	-.00013773	8.43	.79
R37	16	.33333333	.37096774	.07533001	-.00013816	.09176736	-.00013735	7.98	.78
R38	16	.63636364	.68573668	.06917094	-.00013852	.07650495	-.00013805	8.82	.79
R39	16	1.93918920	2.00000000	.05368398	-.00014006	.04616141	-.00014109	11.38	.79
R40	18	1.28395060	1.33333333	.00613640	.00000000	.00550561	.00000000	13.04	.80
R41	18	.77777778	.80459770	.00792662	.00000000	.00908759	.00000000	10.24	.81
R42	20	1.24285710	1.28571430	.00294074	.00000000	.00267767	.00000000	15.08	.83
R43	20	.37500000	.41826923	.04575421	-.00014053	.05238933	-.00013975	12.05	.84
R44	21	1.42857140	1.50000000	.00418409	-.00000000	.00377673	-.00000000	16.31	.83
R45	24	4.83333333	5.00000000	.00286382	.00000123	.00174614	.00000127	28.73	.80
R46	24	1.69531250	1.75000000	.00139259	-.00000001	.00116892	-.00000001	21.04	.85
R47	24	2.75000000	3.00000000	.02330743	-.00014602	.01989910	-.00014788	20.74	.83
R48	24	1.66666670	1.57407410	.00448605	.00000120	.00406692	.00000120	19.57	.85
R49	24	.50000000	.53968254	.10454512	-.00000002	.11044128	-.00000001	16.82	.88
R50	24	4.83333330	5.00000000	.00286382	.00000123	.00174614	.00000127	29.18	.81

Table III: (continued)

s	v	Lower bound of Φ_2/Φ_1	Upper bound of Φ_2/Φ_1	G_1	Error of G_1	G_2	Error of G_2	B	D
R51	25	.50000000	.51891892	.01097727	.00000000	.01216185	.00000000	16.86	.88
R52	25	.33333333	.35813953	.01135905	.00000000	.01311656	.00000000	16.22	.87
R53	26	1.30769230	1.33333333	.00030970	-.00000000	.00027208	.00000000	22.42	.87
R54	27	1.55555556	1.60000000	.00051831	.00000000	.00044830	.00000000	23.93	.88
R55	28	1.92857140	2.00000000	.00124101	-.00000000	.00104955	-.00000000	25.65	.88
R56	28	.66666667	.68934240	.00163455	-.00000000	.00178741	-.00000000	19.90	.89
R57	30	.80000000	.81269841	.00009192	-.00000000	.00010303	-.00000000	21.20	.90
R58	33	.31250000	.32679739	.07223852	-.00000009	.07599691	-.00000007	25.36	.92
R59	35	.53333333	.55384615	.00471039	.00000003	.00502271	.00000002	26.98	.92
R60	39	.72727273	.73803120	.00001199	-.00000000	.00001323	-.00000000	29.69	.92
R61	40	1.80000000	1.76400000	.00003541	.00000000	.00003041	.00000000	39.32	.91
R62	45	1.95555560	2.00000000	.00003927	.00000000	.00003385	.00000000	44.96	.93
R63	48	6.87500000	7.00000000	.00004504	-.00000001	.00002640	-.00000001	71.06	.88
R64	49	.50000000	.50989011	.00021542	-.00000000	.00023106	-.00000000	39.85	.95
R65	49	.33333333	.34637965	.00022058	-.00000000	.00024282	-.00000000	38.85	.96
R66	63	7.88888889	8.00000000	.00000534	.00000000	.00000309	.00000000	97.99	.91
R67	64	.50000000	.50762195	.00002770	-.00000001	.00002943	-.00000001	54.37	.96
R68	80	8.90000000	9.00000000	.00000061	.00000000	.00000035	.00000000	128.51	1.00

Table** IV: Regular G.D., Among Comparisons

s	v	G_1	Error of G_1	G_2	Error of G_2	B	D	$\frac{1}{2}\nu_1$	$\frac{1}{2}\nu_2$
R5	8	.92660598	-.00000000	2.02812200	-.00000001	.75	.35	2	1
R6	8	.92660598	-.00000000	2.02812200	-.00000001	.87	.40	2	1
R7									
R8*	9	.18084145		.04166931		11.78	.40	3	1/2
R9*	9	.58054100		.37645660		4.24	.46	3	1/2
R10*	9	.40114170		.21142836		5.31	.40	"	"
R11*	9	1.82060710		3.43970230		1.50	.51	"	"
R12*	9	.75044440		.57646628		3.74	.53	"	"
R13*	9	2.47599780		6.70399280		1.07	.50	"	"
R14	10	.39800645	-.00013841	.92952501	-.00014245	1.65	.56	5/2	3/2
R15	12	.05296100	-.00000000	.03599987	-.00000000	8.17	.65	3	2
R16	12	.10760015	.00000000	.14907694	.00000000	4.05	.65	"	"
R17	12	.06449589	-.00000000	.05349805	-.00000000	6.86	.66	"	"
R18*	12	.50211678		.35168153		8.20	.60	9/2	1/2
R19									
R20	12	.52964673	-.00000000	1.31482570	-.00000000	2.34	.62	4	1
R21	12	.10760015	.00000000	.14907694	.00000000	4.21	.68	3	2
R22	12	.33216065	-.00000000	.47197854	-.00000000	4.10	.68	4	1
R23*	12	1.54557910		2.93234110		3.08	.70	9/2	1/2
R24	14	.05199792	.00000114	.07729396	.00000116	5.03	.71	7/2	5/2
R25	14	.04482904	.00000114	.05774582	.00000115	5.93	.72	"	"
R26	14	.05199792	.00000114	.07729396	.00000116	5.23	.74	"	"

*Starred experiments have G_2 and G_1 as improper integrals; these integrals were evaluated to the fourth decimal place.

**Lower and upper bounds of Φ_2/Φ_1 are given in Table III.

Table IV: (continued)

s	v	G_1	Error of G_1	G_2	Error of G_2	B	D	$\frac{1}{2}\nu_1$	$\frac{1}{2}\nu_2$
R27	15	.18129859	-.00014162	.50217754	-.00014551	3.05	.68	5	3/2
R28	15	.11611093	-.00013932	.20327436	-.00014065	4.93	.70	"	"
R29	15	.05347859	-.00013793	.04282467	-.00013764	10.91	.72	"	"
R30	15	.18129859	-.00014162	.50217754	-.00014551	3.17	.71	"	"
R31*	15	2.16696860		6.96629570		2.73	.73	6	1/2
R32*	15	.44699438		.32561219		12.04	.66	"	"
R33	15	.04206644	-.00013774	.02922470	-.00013730	12.62	.65	5	3/2
R34	15	.10340764	-.00013898	.14842798	-.00013971	6.15	.78	"	"
R35	16	.37627084	-.00000001	.99309690	-.00000000	3.64	.68	6	1
R36	16	.09018610	-.00000000	.05430287	.00000000	16.04	.72	"	"
R37	16	.06200081	-.00000000	.02843526	.00000000	21.19	.66	"	"
R38	16	.11214239	-.00000000	.08637660	-.00000000	12.67	.71	"	"
R39	16	.28767554	-.00000000	.52623532	-.00000001	5.35	.77	"	"
R40	18	.00877483	-.00000000	.01045750	-.00000000	9.82	.80	9/2	7/2
R41	18	.00661897	-.00000000	.00595953	-.00000000	13.04	.80	"	"
R42	20	.00401108	.00000000	.00467043	.00000000	11.79	.83	5	4
R43	20	.05062569		.02532839	-.00000000	27.58	.70	8	1
R44	21	.01032799	.00000118	.01424629	.00000118	10.67	.81	7	5/2
R45	24	.03353262	.00000002	.11118008	.00000005	5.28	.80	9	2
R46	24	.00398032	-.00000000	.00597248	-.00000000	11.77	.84	8	3
R47	24	.24051216	-.00000006	.67638281	.00000012	6.30	.76	10	1
R48	24	.01588761	.00000001	.02485663	.00000001	11.34	.81	9	2
R49*	24	.28425893		.16907593		29.88	.77	21/2	1/2
R50	24	.03353262	.00000002	.11118008	.00000005	5.37	.81	9	2

Table IV: (continued)

s	v	G_1	Error of G_1	G_2	Error of G_2	B	D	$\frac{1}{2}\nu_1$	$\frac{1}{2}\nu_2$
R51	25	.01445477	-.00014358	.00832752	-.00014303	32.42	.82	10	3/2
R52	25	.00989200	-.00014338	.00418149	-.00014265	44.30	.77	"	"
R53	26	.00043324	-.00000000	.00050744	-.00000000	16.82	.88	6.5	5.5
R54	27	.00132261	-.00000001	.00185870	-.00000001	14.73	.86	9	7/2
R55	28	.00518073	.00000122	.00904618	.00000123	12.42	.85	10.5	2.5
R56	28	.00224372	.00000121	.00169697	.00000120	28.77	.86	"	"
R57	30	.00007635	-.00000000	.00006869	-.00000000	26.41	.90	7.5	6.5
R58*	33	.14811017		.05181450		76.26	.84	15	1/2
R59	35	.00825644	-.00014972	.00495119	-.00014919	47.97	.86	15	1.5
R60	39	.00001325	-.00000000	.00001075	-.00000000	40.38	.91	13	5.5
R61	40	.00012458	.00000000	.00019609	.00000000	21.45	.91	15	4
R62	45	.00018122	-.00000001	.00031959	-.00000001	21.98	.91	18	3.5
R63	48	.00088430	-.00000000	.00386011	-.00000001	9.54	.89	20	3
R64	49	.00031901	.00000132	.00017518	.00000132	77.83	.91	21	2.5
R65	49	.00021694	.00000132	.00008413	.00000131	110.29	.88	"	"
R66	63	.00012800	-.00000001	.00062472	-.00000002	11.62	.91	27	3.5
R67	64	.00004318	-.00000000	.00002338	-.00000000	106.69	.94	28	3
R68	80	.00001735	.00000000	.00009354	.00000000	13.68	.92	35	4

Table V: LS Type, Within Comparisons

s	v	Lower bound of Φ_2/Φ_1=L	Upper bound of Φ_2/Φ_1=P	G_1	Error of G_1	G_2	Error of G_2	B	D
1	9	1.96551720	2.00000000	.31913521	.00000056	.26366817	.00000028	3.29	.52
2	9	1.96551720	2.00000000	.31913521	.00000056	.26366817	.00000028	3.48	.55
3	9	1.44827580	1.49999990	.35289453	.00000074	.32071417	.00000053	3.08	.54
4	9	1.72391020	1.79999990	.33148280	.00000062	.28723591	.00000037	3.33	.55
5	9	1.25556860	1.28571420	.37094270	.00000087	.34998933	.00000071	3.06	.57
6	9								
7	9								
8	9								
9	9								
10	9	1.98371340	2.00000000	.31913521	.00000056	.26206513	.00000028	3.41	.54
11	9	1.98371330	2.00000000	.31913521	.00000056	.26206513	.00000028	3.54	.57
12	16	.57142860	.59340660	.07081078	-.00013842	.07925708	-.00013790	8.68	.80
13	16	.40000000	.42615020	.07412205	-.00013822	.08779910	-.00013751	8.26	.79
14	16	1.89130490	2.00000000	.05368397	-.00014006	.04682753	-.00014098	10.26	.71
15	16	.50000000	.52439030	.07211909	-.00013834	.08256627	-.00013773	8.52	.80
16	25	.50000010	.55555560	.01089548	.00000000	.01216185	.00000000	16.59	.86
17	36	2.30390930	2.33333333	.00123686	.00000127	.00105736	.00000128	34.46	.91
18	49	2.64363930	2.66666740	.00016765	-.00000000	.00014105	-.00000000	50.60	.93
19	64	2.98104600	3.00001610	.00002149	-.00000001	.00001786	-.00000001	69.36	.95
20	100	3.65261530	3.66668620	.00000030	.00000000	.00000025	.00000000	112.48	.97

Table* VI: LS Type, Among Comparisons

s	v	G_1	Error of G_1	G_2	Error of G_2	B	D
1	9	1.87617270	.00028460	3.43970230	.00028460	1.48	.52
2	9	1.87617270	.00028460	3.43970230	.00028460	1.57	.55
3	9	1.46043870	.00028460	2.12674590	.00028460	1.92	.52
4	9	1.68620910	.00028460	2.88645690	.00028460	1.67	.53
5	9	1.29601880	.00028460	1.63940220	.00028460	2.28	.55
6	9						
7	9						
8	9						
9	9						
10	9	1.89020710	.00028460	3.43970230	.00028460	1.54	.54
11	9	1.89020700	.00028460	3.43970230	.00028460	1.60	.57
12	16	.10180204	-.00000000	.06678308	.00000000	14.81	.75
13	16	.07346688	-.00000000	.03668556	.00000000	19.60	.72
14	16	.28204712	-.00000000	.52623560	-.00000001	4.80	.68
15	16	.09018610	-.00000000	.05348033	.00000000	16.46	.74
16	25	.01445477	-.00014358	.00943603	-.00014312	28.36	.72
17	36	.00838014	.00000004	.01730837	.00000004	14.26	.90
18	49	.00136756	.00000135	.00316312	-.00000137	18.40	.93
19	64	.00020687	-.00000000	.00052993	-.00000001	22.50	.94
20	100	.00000388	.00000000	.00001188	.00000000	30.61	.96

*The lower and upper bounds of Φ_2/Φ_1 are given in Table V.

Table* VII: Triangular Designs, Within Comparisons

No. of design	v	Lower bound of ψ_3/ψ_1	Upper bound of ψ_3/ψ_1	Lower bound of ψ_2/ψ_1	Upper bound of ψ_2/ψ_1	G_2 or E_2	Error of G_2	G_1 or E_1	Error of G_1	B	D
T6	10	.42857120	.46938750	.57142840	.60204070	.48272526	-.00000000	.37314616	.00011290	2.62	.49
T7	10	1.00137230	1.02147270	1.00102920	1.01610450	.33310462	.00000000	.33199972	.00011336	3.75	.62
T8	10	.42857150	.46938780	.57142860	.60204090	.48272516	-.00000000	.37314613	.00011290	2.98	.56
T12	10	1.62068960	1.64285710	1.82758620	1.85714280	.25731060	-.00000000	.27927481	.00011432	3.96	.55
T13	10	1.62068960	1.64285710	1.82758610	1.85714280	.25731060	-.00000000	.27927481	.00011432	4.18	.58
T14	10	1.28817730	1.32142850	1.38423640	1.42857140	.29228316	.00000000	.30268360	.00011383	3.89	.59
T18	10	1.63700830	1.64286130	1.84934440	1.85714840	.25583841	.00000000	.27927454	.00011432	4.16	.58
T19	10	.42857450	.43072750	.57143090	.57304560	.48272390	-.00000000	.37672160	.00011286	3.00	.57
T23	15	.46666660	.52647970	.80000000	.82242990	1.34026490		1.04359710		6.51	.62
T24	15	.46666660	.52647970	.80000000	.82242990	1.34026490		1.04359710		6.80	.65
T30	15	1.12550610	1.14285720	1.33468310	1.38095270	.94547610		.92176423		8.57	.72

*The accuracy of E_1 and E_2 is to the sixth decimal place (see Section 8.11).

Table* VIII: Traingular Designs, Among Comparisons

No. of design	v	E_2 or G_2	Error of G_2	E_1 or G_1	Error of G_1	B	D
T6	10	1.19818240		1.05692300		4.96	.37
T7	10	.99986668		.99799656		3.69	.61
T8	10	1.19818080		1.05692300		5.64	.43
T12	10	.87732533		.91344069		2.05	.51
T13	10	.87732533		.91344069		2.16	.53
T14	10	.93602815		.95229941		2.68	.55
T18	10	.87476619		.91344014		2.14	.53
T19	10	1.19817970		1.06176610		5.95	.45
T23	15	.22027083	.00000000	.18703621	.00011599	8.63	.38
T24	15	.22027083	.00000000	.18703621	.00011599	9.02	.39
T30	15	.17512971	-.00000000	.17211884	.00011664	6.26	.61

*The accuracy of E_1 or E_2 is to the sixth decimal place (see Section 8.11).

Table* IX: Cyclic Designs, Within Comparisons

No. of design	v	Lower bound of ψ_3/ψ_1	Upper bound of ψ_3/ψ_1	Lower bound of ψ_2/ψ_1	Upper bound of ψ_2/ψ_1	E_1	E_2	B	D
c1	13	.33333333	.37037010	.83333333	.84259260	1.06178250	1.92264240	3.38	.40
c2	13	1.19143980	1.20000000	1.76575940	1.80000000	.78509506	.87702937	6.08	.53
c3	13	1.06262920	1.07142860	1.25051680	1.28571460	.90801427	.95631297	6.45	.65
c4	13	.33333340	.33397050	.83333340	.83349270	1.06565970	1.92264370	3.81	.45
c5	17	1.13059700	1.13888880	1.36567160	1.38888880	.88794416	.91941955	10.38	.71
c6	17	.53333340	.53631880	.83333340	.83439960	1.05928760	1.43911020	7.94	.65
c7	17	.53333270	.53539250	.83333310	.83406880	1.05941500	1.43910700	7.97	.65
c8	29	.57639040	.58160980	.91666700	.91769380	1.02510200	1.32158740	17.60	.72
c9	29	.27380540	.27535080	.85714210	.85744610	1.04443640	1.69198110	14.04	.59
c10	37	.70634920	.74441510	.88517270	.90005770	1.02648960	1.17377540	26.82	.83

*The accuracy of E_1 or E_2 is to the sixth decimal place (see Section 8.11).

Table* X: Cyclic Designs, Among Comparisons

No. of design	v	E_1	E_2	B	D
c1	13	1.04573670	1.61058820	4.72	.18
c2	13	.83119494	.90590690	3.46	.38
c3	13	.92964747	.96708089	5.08	.56
c4	13	1.04856730	1.61059310	5.37	.21
c5	17	.90855906	.93483468	7.52	.59
c6	17	1.04693670	1.33285940	10.16	.44
c7	17	1.04703650	1.33286350	10.19	.44
c8	29	1.02190010	1.27396560	19.83	.47
c9	29	1.03868120	1.57294470	17.51	.20
c10	37	1.02348070	1.15243270	30.26	.66

*The accuracy of E_1 or E_2 is to the sixth decimal place (see Section 8.11).

Table XI: The ρ_i's for Triangular Designs

No. of design					
T6	$\rho_i = 0$,	$i = 1,2,\ldots,7$;	$\rho_8 = \frac{12}{7}$,	$\rho_9 = \frac{9}{7}$	
T7	"	"	"	"	
T8	"	"	"	"	
T12	"	"	"	"	
T13	"	"	"	"	
T14	"	"	"	"	
T18	"	"	"	"	
T19	"	"	"	"	
T23	$\rho_i = 0$,	$i = 1,2,\ldots,11$;	$\rho_{12} = \frac{8}{3}$,	$\rho_{13} = 1$,	$\rho_{14} = \frac{7}{3}$
T24	"	"	"	"	"
T30	"	"	"	"	"

Table XII: The ρ_i's for Cyclic Designs

No. of design											
c1	$\rho_i=0$,	$i=1,2,\ldots,8$;	$\rho_9=2$,	$\rho_{10}=2$,	$\rho_{11}=\frac{3}{2}$,	$\rho_{12}=\frac{1}{2}$					
c2	"	"	"	"	"	"					
c3	"	"	"	"	"	"					
c4	"	"	"	"	"	"					
c5	$\rho_i=0$,	$i=1,2,\ldots,11$;	$\rho_{12}=\frac{7}{3}$,	$\rho_{13}=\frac{7}{3}$,	$\rho_{14}=\frac{5}{6}$,	$\rho_{15}=\frac{43}{30}$,	$\rho_{16}=\frac{16}{15}$				
c6	"	"	"	"	"	"	"				
c7	"	"	"	"	"	"	"				
c8	$\rho_i=0$,	$i=1,2,\ldots,20$;	$\rho_{21}=\frac{5}{2}$,	$\rho_{22}=\frac{11}{6}$,	$\rho_{23}=\frac{61}{24}$,	$\rho_{24}=\frac{97}{40}$,	$\rho_{25}=\frac{11}{5}$,	$\rho_{26}=1$,	$\rho_{27}=1$,	$\rho_{28}=\frac{1}{2}$	
c9	"	"	"	"	"	"	"	"	"	"	
c10	$\rho_i=0$,	$i=1,2,\ldots,27$;	$\rho_{28}=\frac{5}{2}$,	$\rho_{29}=\frac{5}{3}$,	$\rho_{30}=\frac{47}{23}$,	$\rho_{31}=\frac{91}{40}$,	$\rho_{32}=\frac{11}{5}$,	$\rho_{33}=\frac{37}{14}$,	$\rho_{34}=\frac{243}{112}$,	$\rho_{35}=\frac{8}{5}$,	$\rho_{36}=\frac{3059}{2960}$

ABSTRACT

We know that the best linear combination of the intra- and inter-block estimates is

$$\frac{\text{Intra-estimate}\times\text{Inter-variance}+\text{Inter-estimate}\times\text{Intra-variance}}{\text{Intra-variance} + \text{Inter-variance}};$$

however, this combined estimate is merely theoretical, since we do not know in practice the exact inter- and intra-variances. A reasonable solution is to use a random weight which can be computed from the data of our experiment, but so far there has been no practical solution without severe restrictions on the size of the experiment, and no solution at all for a clear answer to the question of how much we recovered. In fact, the experimenter applying the methodology available to him now, cannot be sure that he is really improving the accuracy of his estimation.

This research has achieved the following:

1. A new method of combining two independent estimates has been developed. This method has its use in incomplete block designs, in similar experiments, and in randomized block designs with heterogeneous variances. The improvement introduced by this method is very satisfactory, compared with the utmost possible theoretical improvement.

2. A procedure for recovering the inter-block information in B.I.B. designs was given, which is applicable in experiments of as small as $t = 4$.

3. It has been proven that the practical utilization of inter-block information is possible in any P.B.I.B. with seven treatments or more.

4. A general procedure for recovering the inter-block information in P.B.I.B.'s with two associate classes was given.

5. An inter-block analysis of singular and semi-regular group divisible designs was discussed, which makes a partial utilization of the inter-information possible.

In general, this work has two merits:

1. It makes possible the utilization of the inter-block information in small and moderate size experiments.

2. As a ratio of the utmost possible theoretical recovery (by combining linearly), either exactly or a lower bound of the ratio of recovery is always computable.

Tables which enable the experimenter to use the procedures described in this dissertation were given. The ratios of recovery listed in these tables show that the new method gives good results where the old method is not applicable, and when the old method starts, hopefully, to be valid, the ratio of recovery achieved by the new method starts to approach the theoretical value that can be achieved, assuming the intra- and inter-variance are known.