

RECEIVED: May 9, 2016

REVISED: June 27, 2016

ACCEPTED: August 8, 2016

PUBLISHED: August 16, 2016

# Toda-like (0,2) mirrors to products of projective spaces

**Zhuo Chen, Eric Sharpe and Ruoxu Wu**

*Physics Department, Robeson Hall 0435,  
Virginia Tech, Blacksburg, VA 24061, U.S.A.*

*E-mail:* [zhuo2012@vt.edu](mailto:zhuo2012@vt.edu), [ersharpe@vt.edu](mailto:ersharpe@vt.edu), [ronwu@vt.edu](mailto:ronwu@vt.edu)

**ABSTRACT:** One of the open problems in understanding (0,2) mirror symmetry concerns the construction of Toda-like Landau-Ginzburg mirrors to (0,2) theories on Fano spaces. In this paper, we begin to fill this gap by making an ansatz for (0,2) Toda-like theories mirror to (0,2) supersymmetric nonlinear sigma models on products of projective spaces, with deformations of the tangent bundle, generalizing a special case previously worked out for  $\mathbb{P}^1 \times \mathbb{P}^1$ . We check this ansatz by matching correlation functions of the B/2-twisted Toda-like theories to correlation functions of corresponding A/2-twisted nonlinear sigma models, computed primarily using localization techniques. These (0,2) Landau-Ginzburg models admit redundancies, which can lend themselves to multiple distinct-looking representatives of the same physics, which we discuss.

**KEYWORDS:** Topological Field Theories, Extended Supersymmetry

**ARXIV EPRINT:** [1603.09634](https://arxiv.org/abs/1603.09634)

---

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Review of Toda models in ordinary mirror symmetry</b>	<b>3</b>
<b>3</b>	<b>Toda-like duals to <math>\mathbb{P}^1 \times \mathbb{P}^1</math></b>	<b>5</b>
3.1	The (0,2) NLSM	5
3.2	The Toda-like mirror theory	6
3.3	Moduli	9
3.4	Redundancies and equivalent descriptions	9
<b>4</b>	<b>Generalization to <math>\mathbb{P}^n \times \mathbb{P}^m</math></b>	<b>12</b>
4.1	The A/2-twisted nonlinear sigma model	12
4.2	The Toda-like mirror theory	13
4.3	Example: $\mathbb{P}^1 \times \mathbb{P}^2$	16
<b>5</b>	<b>Conclusions</b>	<b>17</b>
<b>A</b>	<b>Correlation functions in some examples</b>	<b>18</b>
A.1	A/2 correlation functions on $\mathbb{P}^1 \times \mathbb{P}^1$	18
A.2	Toda-like dual to $\mathbb{P}^1 \times \mathbb{P}^1$	19
A.3	A/2 correlation functions on $\mathbb{P}^1 \times \mathbb{P}^2$	20
<b>B</b>	<b>Tangent bundle moduli</b>	<b>22</b>

---

## 1 Introduction

Historically, mirror symmetry has been one of the most productive arenas for mathematics to emerge from string theory. It has led to notions of curve counting, quantum cohomology, and Gromov-Witten theory, and has been generalized via e.g. homological mirror symmetry [1].

This paper concerns a different generalization of mirror symmetry, known as ‘(0,2) mirror symmetry,’ as it relates UV descriptions of theories with (0,2) supersymmetry, just as ordinary mirror symmetry relates UV descriptions of theories with (2,2) supersymmetry.

Although (0,2) mirror symmetry has not been developed to nearly the same extent as ordinary mirror symmetry, a number of crucial results do exist. One of the first accomplishments was a numerical scan through anomaly-free examples demonstrating the existence of pairs of (0,2) theories with matching spectrum computations [2], giving strong evidence for the existence of (0,2) mirrors. Other work includes a version [3] of the old Greene-Plesser orbifold construction [4], work on GLSM-based dualities [5], and most recently, a proposal

for a generalization of Batyrev’s construction involving reflexively plain polytopes [6]. In addition, there has been considerable work on quantum sheaf cohomology [7–27], the (0,2) analogue of ordinary quantum cohomology.

All that said, many basic gaps remain. For example, there is not yet a systematic description of (0,2) Landau-Ginzburg mirrors to (0,2) nonlinear sigma models on Fano spaces, aside from a special case discussed in [5]. This paper is a first pass at filling that gap.

Recall a (0,2) supersymmetric nonlinear sigma model is typically defined by a complex Kähler manifold  $X$  and holomorphic vector bundle  $\mathcal{E} \rightarrow X$  obeying

$$\mathrm{ch}_2(\mathcal{E}) = \mathrm{ch}_2(TX),$$

known as the Green-Schwarz or anomaly cancellation condition. In addition, to define the A/2-twist, we must also require that

$$\det \mathcal{E}^* \cong K_X.$$

For example, if  $\mathcal{E} = TX$ , both of these conditions are trivially satisfied. There is also a B/2-twist, which requires instead

$$\det \mathcal{E} \cong K_X.$$

If  $\mathcal{E} = TX$  and  $K_X^{\otimes 2}$  is trivial, these conditions are satisfied, which match the conditions for consistency of the closed-string B model [18]. The A/2 and B/2 twists are closely related: the A/2 twist of a nonlinear sigma model defined by  $(X, \mathcal{E})$  is equivalent to the B/2 twist of a nonlinear sigma model defined by  $(X, \mathcal{E}^*)$  [18].

For  $X$  a Calabi-Yau, the simplest version of (0,2) mirror symmetry asserts that the pair  $(X, \mathcal{E})$  define the same (0,2) SCFT as another pair  $(X', \mathcal{E}')$ , satisfying the same two conditions above, where  $X'$  is Calabi-Yau. This duality also exchanges the A/2 and B/2 twists, in the sense that the A/2 twist of the nonlinear sigma model defined by  $(X, \mathcal{E})$  is equivalent to the B/2 twist of the nonlinear sigma model defined by  $(X', \mathcal{E}')$ .

In this paper, we will be concerned with duals in cases where  $X$  is not Calabi-Yau. Specifically, we will consider duals to A/2 twists of nonlinear sigma models on Fano manifolds  $X$ , which will correspond to B/2 twists of certain (0,2) Landau-Ginzburg models.

For (2,2) theories, such dualities are well-known as Toda duals to Fano spaces. For (0,2) theories, one special case was worked out in [5], corresponding to particular deformations of the tangent bundle of  $\mathbb{P}^1 \times \mathbb{P}^1$ . The point of this paper is to construct (0,2) Landau-Ginzburg mirrors to more general tangent bundle deformations of arbitrary products of projective spaces, as deformations of (2,2) Landau-Ginzburg mirrors, and in so doing, pave the way for an understanding of such duals to arbitrary Fano manifolds.

We check our ansatz for (0,2) duals by comparing correlation functions of B/2 twists of the proposed (0,2) Landau-Ginzburg mirrors to correlation functions in A/2-twisted nonlinear sigma models, which can be computed as in [8, 9, 22, 25]. In particular, those nonlinear sigma models compute quantum sheaf cohomology, a generalization of ordinary quantum cohomology. Recall that in a (2,2) supersymmetric nonlinear sigma model, the

ordinary quantum cohomology is generated additively by

$$H^\bullet(X, \wedge^\bullet T^*X),$$

with  $T^*X$  the cotangent bundle of  $X$ . In the (0,2) case, the analogue (known as the quantum sheaf cohomology ring) is generated additively by

$$H^\bullet(X, \wedge^\bullet \mathcal{E}^*)$$

instead. Quantum sheaf cohomology was first introduced in [7], and the subject has been further developed in a number of works including [8–27].

We will see in sections 3, 4, that the (0,2) Toda-like duals have the property that their classical vacua are defined by the quantum sheaf cohomology ring relations of the dual A/2-twisted theories.

In section 2, we begin by reviewing old results from ordinary mirror symmetry. In section 3, we describe our ansatz for the (0,2) Toda-like dual to  $\mathbb{P}^1 \times \mathbb{P}^1$  with a general deformation of the tangent bundle, and check that (genus zero) correlation functions match those of the corresponding A/2 theory. We then compare the number of parameters in the theory to the number of expected infinitesimal moduli, and discuss some reparametrization symmetries that can be used to write physically-equivalent but different-looking forms of the Toda-like dual. In section 4, we generalize to products of projective spaces, discussing Toda-like duals, giving a general argument for matching of (genus zero) correlation functions to those of the corresponding A/2 theory, and also checking in detail in the special case of  $\mathbb{P}^1 \times \mathbb{P}^2$ . In appendices, we give detailed results for correlation functions in a number of examples, and also discuss the number of moduli appearing mathematically.

## 2 Review of Toda models in ordinary mirror symmetry

Let us quickly review ordinary Toda duals to A-twisted (2,2) supersymmetric nonlinear sigma models on projective spaces. First, recall that in the A-twisted<sup>1</sup> nonlinear sigma model on  $\mathbb{P}^n$ , all BRST-cohomology classes of local operators are generated by a single operator  $\psi$ , corresponding to a degree-two cohomology class on  $\mathbb{P}^n$ , with correlation functions of the form

$$\begin{aligned} \langle \psi^n \rangle &= 1, \\ \langle \psi^{2n+1} \rangle &= q, \\ \langle \psi^{n+d(n+1)} \rangle &= q^d, \end{aligned}$$

and OPE (quantum cohomology relation)  $\psi^{n+1} = q$ .

The mirror Toda theory is a B-twisted Landau-Ginzburg theory with superpotential of the form

$$W = \exp(Y_1) + \exp(Y_2) + \cdots + \exp(Y_n) + q \exp(-Y_1 - Y_2 - \cdots - Y_n).$$

---

<sup>1</sup>The reader should note that we do not couple this theory to worldsheet gravity — throughout this paper, we consider only topological field theories, not topological string theories.

(In effect, because of the exponentials, the superpotential is defined over  $(\mathbb{C}^\times)^n$ .) We define

$$X_i = e^{Y_i},$$

so that the superpotential can be written in the simpler form

$$W = X_1 + X_2 + \cdots + X_n + \frac{q}{X_1 \cdots X_n},$$

bearing in mind that the fundamental fields are  $Y_i$ .

As the superpotential is over a vector space, the correlation functions in this<sup>2</sup> theory are

$$\langle F_1 \cdots F_n \rangle = \sum_{dW=0} \frac{F_1 \cdots F_n}{H},$$

where  $H = \det(\partial_i \partial_j W)$  (with derivatives computed with respect to  $Y$ 's).

Solving the constraint  $dW = 0$  (for derivatives with respect to the fundamental fields  $Y$ ), one finds that the classical vacua are given by

$$X_1 = X_2 = \cdots = X_n \equiv X, X = qX^{-n}.$$

In particular, the vacua are given by  $X$  such that  $X^{n+1} = q$ , which is the defining relation of the quantum cohomology ring of  $\mathbb{P}^n$ . (This is no accident, and in fact, is an important property we will apply later in working out duals to (0,2) theories.) Furthermore, after restriction to the classical vacua, the Hessian  $H$  is easily computed to be

$$H = (n+1)X^n.$$

Thus, the correlation functions of this model are

$$\langle X^m \rangle = \sum \frac{X^m}{(n+1)X^n},$$

where the sum runs over  $X$ 's solving  $X^{n+1} = q$ , i.e.  $(n+1)$ th roots of  $q$ . This expression can only be nonvanishing when  $m - n$  is divisible by  $n+1$ . We find that the nonzero correlation functions are<sup>3</sup>

$$\begin{aligned} \langle X^n \rangle &= 1, \\ \langle X^{2n+1} \rangle &= q, \\ \langle X^{n+d(n+1)} \rangle &= q^d, \end{aligned}$$

matching the A model correlation functions if we identify  $X$  with  $\psi$ .

In the rest of this paper, we shall describe an ansatz for Toda-like duals to (0,2) nonlinear sigma models on certain Fano spaces with deformations of the tangent bundle, generalizing the discussion above off the (2,2) locus, which we will check by comparing correlation functions (and quantum sheaf cohomology relations).

---

<sup>2</sup>Here we are considering Landau-Ginzburg models over vector spaces, for which this correlation function can be found in [28]. See [29] for a discussion of correlation functions in more general B-twisted Landau-Ginzburg models. The computation in this section, demonstrating how quantum cohomology appears in Toda duals, is also described in [30], as a prelude to the discussion of Toda duals to (2,2) theories on smooth Fano Deligne-Mumford stacks.

<sup>3</sup>In principle we should write the correlation functions in terms of  $X_1, \cdots, X_n$ ; however, since the  $X_i$  coincide on the set of vacua, and the correlation functions are computed by summing over vacua, it is an immediate result that

$$\langle f(X_1, \cdots, X_n) \rangle = \langle f(X, \cdots, X) \rangle,$$

so without loss of generality we merely write the correlation functions in terms of powers of  $X$ .

### 3 Toda-like duals to $\mathbb{P}^1 \times \mathbb{P}^1$

#### 3.1 The (0,2) NLSM

In the case of  $X = \mathbb{P}^1 \times \mathbb{P}^1$ , one can describe a general deformation  $\mathcal{E}$  of the tangent bundle as the cokernel of the following sequence:

$$0 \longrightarrow \mathcal{O} \otimes \mathcal{O} \xrightarrow{E} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \longrightarrow \mathcal{E} \longrightarrow 0,$$

where

$$E = \begin{bmatrix} Ax & Bx \\ C\tilde{x} & D\tilde{x} \end{bmatrix},$$

with  $A, B, C, D$   $2 \times 2$  matrices and

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \tilde{x} = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

are homogeneous coordinates on the two  $\mathbb{P}^1$  factors. The tangent bundle corresponds to  $A = D = I$ , and  $B = C = 0$ . For more general  $A, B, C, D$ , the vector bundle is (generically) a deformation of the tangent bundle. In this model, it has been argued in [8, 9, 22, 25] that the OPE ring relations in the A/2 twist (defining the quantum sheaf cohomology ring) are given by

$$\det(A\psi + B\tilde{\psi}) = q_1, \tag{3.1}$$

$$\det(C\psi + D\tilde{\psi}) = q_2. \tag{3.2}$$

Correlation functions in A/2 twisted theories on  $\mathbb{P}^1 \times \mathbb{P}^1$  with a deformation of the tangent bundle can be computed in several ways. One method is to use direct Cech techniques to compute sheaf cohomology products on  $\mathbb{P}^1 \times \mathbb{P}^1$ , as has been discussed in e.g. [7, 11, 26]. Another method is to use GLSM-based Coulomb branch results, as described in [25]. A third, more recent, method is to use residue formulas obtained via localization, as in [22]. In this last approach, correlation functions in the A/2 twisted theory on  $\mathbb{P}^1 \times \mathbb{P}^1$  are of the form<sup>4</sup>

$$\langle f(\psi, \tilde{\psi}) \rangle = \sum_{k_1, k_2} q_1^{k_1} q_2^{k_2} \text{JKG} - \text{Res} \left( \frac{1}{\det(A\psi + B\tilde{\psi})^{k_1+1}} \frac{1}{\det(C\psi + D\tilde{\psi})^{k_2+1}} f(\psi, \tilde{\psi}) \right).$$

However one computes the correlation functions, the results have the following form, in terms of the matrices  $A, B, C, D$  above. Let

$$\begin{aligned} a &= \det(A), & b &= \det(B), & c &= \det(C), & d &= \det(D), \\ e &= \det(A + B), & f &= \det(C + D). \end{aligned}$$

---

<sup>4</sup>As a matter of principle, there is a phase ambiguity in expressions of this form, due geometrically to possible phases of the isomorphism  $\det \mathcal{E}^* \xrightarrow{\sim} K_X$ , and physically to chiral left and right global U(1) actions on the worldsheet, that play a role closely analogous to that of the Bagger-Witten line bundle. The expression given here implicitly determines such phases.

Define

$$\begin{aligned}
\mu &= e - a - b, & \nu &= f - c - d, \\
\phi_1 &= \nu b - \mu d = ad + bf - de - bc, \\
\phi_2 &= ad - bc, \\
\phi_3 &= \mu c - \nu a = ad + ce - af - bc, \\
\Delta &= \phi_2^2 - \phi_1 \phi_3, \\
&= (c - d)(bc - ad)e + cde^2 + (a - b)(ad - bc)f - (bc + ad)ef + abf^2.
\end{aligned}$$

Then the two-point correlation functions, for example, can be expressed as:

$$\langle \psi \psi \rangle = \frac{\phi_1}{\Delta}, \quad \langle \psi \tilde{\psi} \rangle = \frac{\phi_2}{\Delta}, \quad \langle \tilde{\psi} \tilde{\psi} \rangle = \frac{\phi_3}{\Delta}. \quad (3.3)$$

Higher-point correlation functions have a similar form. We list four-point functions in this A/2-twisted theory in appendix A.1. More general correlation functions at genus zero are straightforward to compute with residue techniques, but the resulting expressions are rather unwieldy, so we do not include them in this paper.

### 3.2 The Toda-like mirror theory

We claim the mirror theory to the A/2 twisted theory just described, is a (0,2) Landau-Ginzburg model, defined by a (0,2) superpotential of the form

$$W = FJ + \tilde{F}\tilde{J}, \quad (3.4)$$

where  $F$  and  $\tilde{F}$  are Fermi superfields, and

$$\begin{aligned}
J &= X^{-1}(\det(AX + B\tilde{X}) - q_1) = aX + b\frac{\tilde{X}^2}{X} + \mu\tilde{X} - \frac{q_1}{X}, \\
\tilde{J} &= \tilde{X}^{-1}(\det(CX + D\tilde{X}) - q_2) = d\tilde{X} + c\frac{X^2}{\tilde{X}} + \nu X - \frac{q_2}{\tilde{X}},
\end{aligned}$$

for  $X = \exp(Y)$ ,  $\tilde{X} = \exp(\tilde{Y})$ , where  $Y$ ,  $\tilde{Y}$  are the fundamental fields, and

$$\begin{aligned}
a &= \det A, & b &= \det B, & c &= \det C, & d &= \det D, \\
\mu &= \det(A + B) - \det A - \det B, \\
\nu &= \det(C + D) - \det C - \det D,
\end{aligned}$$

for  $A$ ,  $B$ ,  $C$ ,  $D$  the matrices defining the tangent bundle deformation of the A/2 theory.

In passing, the form written here does not manifestly match the expression in [5] for the special case they considered. In section 3.4, we will study various field redefinitions yielding non-obviously-equivalent expressions, and discover the expression in [5] arising as a special case.

We will check the ansatz above by comparing correlation functions between the original A/2 theory and the B/2 twist of the Landau-Ginzburg theory above, but first, let us make a few quick observations.

As one consistency check, note that for

$$A = D = I, \quad B = C = 0,$$

then the vector bundle  $\mathcal{E}$  is the tangent bundle, and the theory has (2,2) supersymmetry. This also can be seen from the (0,2) superpotential

$$W = F\left(X - \frac{q_1}{X}\right) + \tilde{F}\left(\tilde{X} - \frac{q_2}{\tilde{X}}\right),$$

which matches the (2,2) superpotential in this case.

As another check, note that the space of classical vacua of this theory ( $J = \tilde{J} = 0$ ) matches the space of solutions to the quantum sheaf cohomology ring relations:

$$\det(AX + B\tilde{X}) = q_1, \tag{3.5}$$

$$\det(CX + D\tilde{X}) = q_2. \tag{3.6}$$

Now, let us compute and compare genus zero correlation functions. Given a B/2-twisted Landau-Ginzburg model with superpotential  $W$  over a vector space or a product of  $\mathbb{C}^\times$ 's, correlation functions at genus zero are given by<sup>5</sup> [24]

$$\langle \phi^{i_1}(x_1) \cdots \phi^{i_k}(x_k) \rangle = \sum_{J_i(\phi)=0} \phi^{i_1}(x_1) \cdots \phi^{i_k}(x_k) [\det_{i,j} J_{i,j}]^{-1} \tag{3.7}$$

where the sum is over classical vacua.

Using the formula above for B/2-twisted Landau-Ginzburg correlation functions, one finds that the two-point correlation functions in this model are given by

$$\begin{aligned} \langle XX \rangle &= \Delta^{-1}(b\nu - d\mu), \\ \langle X\tilde{X} \rangle &= \Delta^{-1}(ad - bc), \\ \langle \tilde{X}\tilde{X} \rangle &= \Delta^{-1}(c\mu - a\nu), \end{aligned}$$

where

$$\Delta = b^2c^2 - 2abcd + a^2d^2 + cd\mu^2 - (bc + ad)\mu\nu + ab\nu^2.$$

These match the A/2 correlation functions in equation (3.3), if we identify  $X$  with  $\psi$  and  $\tilde{X}$  with  $\tilde{\psi}$ .

We also checked that all four-point functions for general  $A, B, C, D$  (as listed in appendix A.2) match the results from the A/2 model. For the special case in which

---

<sup>5</sup>Correlation functions for more general B/2-twisted Landau-Ginzburg models are discussed in [21]. In passing, we should comment on the absence of worldsheet instanton corrections to the formulas above. On the (2,2) locus, the Toda duals to A model topological field theories are B-twisted, and correlation functions in the B model do not have worldsheet instanton corrections. In the present case, our Toda-like mirrors to A/2 model pseudo-topological field theories are B/2 twisted. Unlike the (2,2) case, however, in general B/2 twisted models can and will receive worldsheet instanton corrections.

However, our Toda-like theories are defined by superpotentials over algebraic tori, i.e.  $(\mathbb{C}^\times)^n$ , and there are no non-constant holomorphic maps from  $\mathbb{P}^1$  (or any projective variety) to an algebraic torus. All holomorphic maps are constant maps, hence there are no worldsheet instanton corrections in these theories [31]. Thus, we need only compute classically in the B/2 model, just as in ordinary Toda mirrors.



$\det B = \det C = 0$ , we have checked that all correlation functions up to ten-point correlation functions and one twelve-point correlation function  $\langle X^6 \tilde{X}^6 \rangle$  match the results from the A/2 model.

Beyond special cases, there is also a general argument that all correlation functions must match. We will utilize a formula for the A/2 model correlation functions given in [25, Section 3.4], which is similar in form to the formula above for B/2 Landau-Ginzburg model correlation functions, and argue that after some algebra, the formula for A/2 correlation functions in [25] matches the formula for B/2 correlation functions above. As a result, all correlation functions in our B/2-twisted Landau-Ginzburg model will necessarily match those of the A/2 nonlinear sigma model.

Let us describe this argument for general matching correlation functions. From [25, Section 3.4], all correlation functions in an A/2-twisted (0,2) nonlinear sigma model on  $\mathbb{P}^1 \times \mathbb{P}^1$ , at genus zero, take the form

$$\begin{aligned} \langle f(\psi, \tilde{\psi}) \rangle &= \sum_{\psi, \tilde{\psi} | \mathcal{J}_a=0} f(\psi, \tilde{\psi}) \left[ \left( \det_{a,b} \mathcal{J}_{a,b} \right) \det(A\psi + B\tilde{\psi}) \det(C\psi + D\tilde{\psi}) \right]^{-1}, \\ &= \sum_{\psi, \tilde{\psi} | \mathcal{J}_a=0} f(\psi, \tilde{\psi}) \det \begin{bmatrix} \partial_\psi \det(A\psi + B\tilde{\psi}) & \partial_{\tilde{\psi}} \det(A\psi + B\tilde{\psi}) \\ \partial_\psi \det(C\psi + D\tilde{\psi}) & \partial_{\tilde{\psi}} \det(C\psi + D\tilde{\psi}) \end{bmatrix}^{-1} \end{aligned}$$

where the  $\mathcal{J}_a$  (not to be confused with the  $J, \tilde{J}$  we used in our dual theory earlier) are defined by

$$\begin{aligned} \mathcal{J}_1 &= \ln \left( q_1^{-1} \det(A\psi + B\tilde{\psi}) \right), \\ \mathcal{J}_2 &= \ln \left( q_2^{-1} \det(C\psi + D\tilde{\psi}) \right). \end{aligned}$$

To compare the correlation functions above with the B/2 correlation functions in our dual theory, which take a similar form, first note that the constraint  $\tilde{J}_a = 0$  implies

$$\det(A\psi + B\tilde{\psi}) = q_1, \quad \det(C\psi + D\tilde{\psi}) = q_2,$$

the quantum sheaf cohomology relations and also the relations defining the vacua of the B/2 Landau-Ginzburg model. Then, matching follows as a consequence of

$$\det_{i,j} J_{i,j} = \left( \det_{a,b} \mathcal{J}_{a,b} \right) \det(A\psi + B\tilde{\psi}) \det(C\psi + D\tilde{\psi}),$$

or more explicitly

$$\begin{aligned} \det \begin{bmatrix} \partial_Y \left( aX + b\tilde{X}^2/X + \mu\tilde{X} - q_1/X \right) & \partial_{\tilde{Y}} \left( aX + b\tilde{X}^2/X + \mu\tilde{X} - q_1/X \right) \\ \partial_Y \left( d\tilde{X} + cX^2/\tilde{X} + \nu X - q_2/\tilde{X} \right) & \partial_{\tilde{Y}} \left( d\tilde{X} + cX^2/\tilde{X} + \nu X - q_2/\tilde{X} \right) \end{bmatrix} \\ = \det \begin{bmatrix} \partial_\psi \det(A\psi + B\tilde{\psi}) & \partial_{\tilde{\psi}} \det(A\psi + B\tilde{\psi}) \\ \partial_\psi \det(C\psi + D\tilde{\psi}) & \partial_{\tilde{\psi}} \det(C\psi + D\tilde{\psi}) \end{bmatrix}, \end{aligned}$$

where  $X = \exp(Y)$ ,  $\tilde{X} = \exp(\tilde{Y})$ , after identifying  $X$  with  $\psi$  and  $\tilde{X}$  with  $\tilde{\psi}$ , which is straightforward to verify. Thus, all genus zero correlation functions of our B/2 Landau-Ginzburg model, the proposed dual to  $\mathbb{P}^1 \times \mathbb{P}^1$ , do indeed match the correlation functions of the (0,2) theory on  $\mathbb{P}^1 \times \mathbb{P}^1$ .

We will apply a more general version of this argument when checking genus zero correlation functions of the proposed B/2 Landau-Ginzburg dual to A/2 theories on  $\mathbb{P}^n \times \mathbb{P}^n$  in section 4.2.

### 3.3 Moduli

On the face of it, the correlation functions above are determined by six numbers:

$$\det A, \quad \det B, \quad \det C, \quad \det D, \quad \det(A + B), \quad \det(C + D),$$

(in addition, of course, to  $q_1, q_2$ ). Not all of the individual elements of each of the four matrices  $A, B, C, D$  are pertinent, essentially because this theory admits global  $GL(2)$  actions rotating those matrices. In addition, in principle field redefinitions could be used to also eliminate some of the parameters above.

Mathematically, the tangent bundle of  $\mathbb{P}^1 \times \mathbb{P}^1$  also has six moduli (as counted in appendix B), matching the count above. However, if one deforms to a finite distance away from the tangent bundle, the number of mathematical bundle moduli (counted by  $H^1(\mathbb{P}^1 \times \mathbb{P}^1, \text{End } \mathcal{E})$  for bundle  $\mathcal{E}$ ) may drop, as we discuss in appendix B. Furthermore, not all of those moduli need necessarily be expressible monadically, as polynomial deformations of the GLSM, so the true number of parameters that the GLSM can access may be significantly smaller (reflecting e.g. the symmetries and field redefinitions mentioned above). We will see this in an example in section 3.4, where we will take models with matrices  $B$  such that  $\det B \neq 0$ , and construct equivalent theories with  $\det B = 0$ .

### 3.4 Redundancies and equivalent descriptions

As the moduli counts in the last section suggest, our description of the theories in terms of four matrices  $A, B, C, D$  has a great deal of redundancy. This can be expressed in the fact that there are three  $GL(2)$  actions<sup>6</sup> on these matrices. Specifically, three matrices  $P, Q, R$  each in  $GL(2)$  act on the matrices  $A, B, C, D$  as follows:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \mapsto \begin{bmatrix} PA & PB \\ QC & QD \end{bmatrix} R$$

at the same time that

$$\begin{bmatrix} \psi \\ \tilde{\psi} \end{bmatrix} \mapsto R \begin{bmatrix} \psi \\ \tilde{\psi} \end{bmatrix},$$

and

$$q_1 \mapsto (\det P)q_1, \quad q_2 \mapsto (\det Q)q_2.$$

---

<sup>6</sup>We would like to thank R. Donagi for making this observation originally. See also a related discussion in [27].

Of course, these three  $GL(2)$  actions are not completely independent, but in broad brushstrokes, they are the reason that there are no more than six independent moduli yet sixteen naive parameters (the elements of the four  $2 \times 2$  matrices).

To understand how correlation functions behave, let us consider a residue expression for correlation functions from [22]:

$$\begin{aligned} & \langle f(\psi, \tilde{\psi}) \rangle \\ &= \sum_{k_1, k_2} q_1^{k_1} q_2^{k_2} \text{JKG} - \text{Res} \frac{f(\psi, \tilde{\psi})}{(\det(A\psi + B\tilde{\psi}))^{k_1+1} \det(C\psi + D\tilde{\psi})^{k_2+1}}. \end{aligned}$$

Formally, if we rotate  $\psi, \tilde{\psi}$  by the matrix  $R$  at the same time that  $A, B, C, D$  are also rotated by  $R$ , the new resulting expression is equivalent to the original one, after a linear field redefinition. In other words,

$$\langle f(R(\psi, \tilde{\psi})) \rangle_{R(A,B,C,D)} = \frac{1}{|\det R|} \langle f(\psi, \tilde{\psi}) \rangle_{A,B,C,D}.$$

That said, the expressions for correlation functions we utilize in this paper assume that  $A$  and  $D$  are both invertible, and a general  $R$ -rotation could change that. In such cases, the pole prescription implicit in the definition of the JKG residue in [22] would yield different results, so one should be careful in applying the formal statement above.

An example of such equivalences is as follows. Define  $\beta$  to be a solution of

$$(\det A)\beta^2 + \mu\beta + (\det B) = 0,$$

i.e.

$$\beta = \frac{1}{2a} \left( -\mu \pm \sqrt{\mu^2 - 4ab} \right),$$

(where  $A$  is assumed invertible,) and let us assume  $C = 0$ . Take

$$R = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}.$$

This matrix  $R$  rotates  $B$  to a noninvertible matrix. Specifically, under the action of  $R$ ,

$$B \mapsto B' = B + \beta A,$$

and the other matrices are invariant. It is straightforward to check that  $\det B' = 0$ . In principle, correlation functions in the original theory should match correlation functions with these parameters so long as  $\psi, \tilde{\psi}$  are suitably rotated:

$$\langle f(\psi, \tilde{\psi}) \rangle_{\text{original}} = \langle f(\psi + \beta\tilde{\psi}, \tilde{\psi} + \gamma\psi) \rangle_{\text{new}}.$$

Now, having constructed an equivalent model for which  $\det B' = 0$ , we can construct the dual (0,2) Landau-Ginzburg theory. This is defined by the (0,2) superpotential with

$$\begin{aligned} J' &= aX + \mu' \tilde{X} - \frac{q_1}{X}, \\ \tilde{J}' &= d\tilde{X} + \nu' X - \frac{q_2}{\tilde{X}}, \end{aligned}$$

where

$$\mu' = \det(A + B') - \det A - \det B', \quad \nu' = \det(C + D) - \det C - \det D = \nu.$$

This is just the specialization of our previous proposed dual to case with  $B'$  instead of  $B$  and with  $C = 0$ , so that the  $\tilde{X}^2/X$  and  $X^2/\tilde{X}$  terms vanish.

Given the rotation on the original  $\psi, \tilde{\psi}$ , we see that in principle the original correlation functions should match the correlation functions in the final Landau-Ginzburg model above as

$$\langle f(\psi, \tilde{\psi}) \rangle_{\text{original}} = \langle f(X + \beta\tilde{X}, \tilde{X}) \rangle_{\text{final}}.$$

Now, let us turn to a particular special case, appearing in [5]. This special case is the sole previous example of a (0,2) Landau-Ginzburg mirror to a A/2-twisted theory that had previously appeared in the literature. More to the point, this sole example in the literature does not fit the pattern we have discussed in previous sections, and instead is related to them via a field redefinition of the form discussed in this section.

Specifically, let us consider the case

$$A = D = I, \quad C = 0, \quad B = \begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{bmatrix}.$$

Following the methods we have discussed prior to this section, the dual Landau-Ginzburg theory has the parameters

$$a = 1, \quad b = \epsilon_1\epsilon_2, \quad c = 0, \quad d = 1, \quad \mu = \epsilon_1 + \epsilon_2, \quad \nu = 0,$$

and superpotential

$$W = F\left(X + \epsilon_1\epsilon_2 \frac{\tilde{X}^2}{X} + (\epsilon_1 + \epsilon_2)\tilde{X} - \frac{q_1}{X}\right) + \tilde{F}\left(\tilde{X} - \frac{q_2}{\tilde{X}}\right).$$

The two-point correlation functions in this theory, for example, are

$$\begin{aligned} \langle XX \rangle &= -(\epsilon_1 + \epsilon_2), \\ \langle X\tilde{X} \rangle &= 1, \\ \langle \tilde{X}\tilde{X} \rangle &= 0. \end{aligned}$$

Unfortunately, although this does correctly capture the A/2 correlation functions, neither the superpotential nor the correlation functions above match those given in [5] as the dual.

To find the presentation of the dual given in [5], one must instead perform a  $R$ -rotation of the sort described above, rotating  $B$  and  $C$  to noninvertible matrices. One then computes

$$\begin{aligned} \beta &= \frac{1}{2}(-\epsilon_1 - \epsilon_2 + (\epsilon_1 - \epsilon_2)) = -\epsilon_2, \\ \gamma &= 0, \end{aligned}$$

(taking the positive square root in  $\beta$ ). After transforming by

$$R = \begin{bmatrix} 1 & \beta \\ \gamma & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\epsilon_2 \\ 0 & 1 \end{bmatrix},$$

one has the new dual defined by parameters

$$\begin{aligned} a' &= 1, \quad b' = \det \begin{bmatrix} \epsilon_1 - \epsilon_2 & 0 \\ 0 & 0 \end{bmatrix} = 0, \quad c' = 0, \quad d' = 1, \\ \mu' &= \det(A' + B') - \det A' - \det B' = \epsilon_1 - \epsilon_2, \\ \nu' &= \det(C' + D') - \det C' - \det D' = 0, \end{aligned}$$

hence the superpotential

$$W = F\left(X + (\epsilon_1 - \epsilon_2)\tilde{X} - \frac{q_1}{X}\right) + \tilde{F}\left(\tilde{X} - \frac{q_2}{\tilde{X}}\right).$$

From the results in appendix A.2, the two-point functions in this Landau-Ginzburg model are given by

$$\langle XX \rangle = \epsilon_2 - \epsilon_1, \quad \langle X\tilde{X} \rangle = 1, \quad \langle \tilde{X}\tilde{X} \rangle = 0,$$

matching the results of [5].

Also note that, in this same theory,

$$\begin{aligned} \langle (X - \epsilon_2\tilde{X})^2 \rangle &= \langle X^2 \rangle - 2\epsilon_2\langle X\tilde{X} \rangle + \epsilon_2^2\langle \tilde{X}^2 \rangle = -\epsilon_1 - \epsilon_2, \\ \langle (X - \epsilon_2\tilde{X})\tilde{X} \rangle &= \langle X\tilde{X} \rangle - \epsilon_2\langle \tilde{X}^2 \rangle = 1, \\ \langle \tilde{X}\tilde{X} \rangle &= 0, \end{aligned}$$

matching the correlation functions of the original A/2-twisted theory, as expected. In [11], the change of variables above was given to correlate A/2 correlation functions with those of the proposed dual theory, and here we see that this is a special case of a much more general redundancy in the description.

## 4 Generalization to $\mathbb{P}^n \times \mathbb{P}^m$

### 4.1 The A/2-twisted nonlinear sigma model

Let us begin by briefly reviewing pertinent properties of the (0,2) nonlinear sigma model on  $\mathbb{P}^n \times \mathbb{P}^m$ , whose dual we shall describe. First, the gauge bundle in this (0,2) theory is a deformation  $\mathcal{E}$  of the tangent bundle of  $\mathbb{P}^n \times \mathbb{P}^m$ , which can be described as a cokernel

$$0 \longrightarrow \mathcal{O}^2 \xrightarrow{E} \mathcal{O}(1,0)^{n+1} \oplus \mathcal{O}(0,1)^{m+1} \longrightarrow \mathcal{E} \longrightarrow 0,$$

where

$$E = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

in which  $A, B$  are  $(n+1) \times (n+1)$  matrices and  $C, D$  are  $(m+1) \times (m+1)$  matrices. The quantum sheaf cohomology ring of an A/2-twisted nonlinear sigma model on  $\mathbb{P}^n \times \mathbb{P}^m$  with the bundle above takes the form [8, 9, 22, 25]

$$\det(A\psi + B\tilde{\psi}) = q_1, \quad \det(C\psi + D\tilde{\psi}) = q_2,$$

and for later use, we expand the determinants as follows:

$$\det(A\psi + B\tilde{\psi}) = a\psi^{n+1} + b\tilde{\psi}^{n+1} + \sum_{i=1}^n \mu_i \psi^i \tilde{\psi}^{n+1-i}, \quad (4.1)$$

$$\det(C\psi + D\tilde{\psi}) = c\psi^{m+1} + d\tilde{\psi}^{m+1} + \sum_{k=1}^m \nu_k \psi^k \tilde{\psi}^{m+1-k}, \quad (4.2)$$

where

$$a = \det A, \quad b = \det B, \quad c = \det C, \quad d = \det D,$$

$\mu_i$  is a sum of determinants of matrices, each of which is formed by taking  $i$  rows of  $A$  and  $n+1-i$  rows of  $B$ , and  $\nu_i$  is formed similarly from  $C, D$ .

## 4.2 The Toda-like mirror theory

We claim the (0,2) superpotential of the (0,2) Landau-Ginzburg Toda-like mirror to  $\mathbb{P}^n \times \mathbb{P}^m$  is

$$W = \sum_{i=1}^n F_i J_i + \sum_{k=1}^m \tilde{F}_k \tilde{J}_k, \quad (4.3)$$

where

$$J_i = a^{(1-n)/n} \left( aX_i - \frac{q_1}{X_1 \cdots X_n} + b \frac{\tilde{X}_1^{n+1}}{X_1^n} + \sum_{i=1}^n \mu_{n+1-i} \frac{\tilde{X}_1^i}{X_1^{i-1}} \right), \quad (4.4)$$

$$\tilde{J}_k = d^{(1-m)/m} \left( d\tilde{X}_k - \frac{q_2}{\tilde{X}_1 \cdots \tilde{X}_m} + c \frac{X_1^{m+1}}{\tilde{X}_1^m} + \sum_{k=1}^m \nu_k \frac{X_1^k}{\tilde{X}_1^{k-1}} \right), \quad (4.5)$$

which clearly generalizes the dual to  $\mathbb{P}^1 \times \mathbb{P}^1$  discussed in section 3.2.

First, note that if the parameters  $a, b, c, d$ , and the  $\mu_i, \nu_k$  are related to the matrices  $A, B, C, D$  of the A/2 model as above, then the vacua of this theory, defined by  $J_i = 0 = \tilde{J}_k$ , are the solutions of

$$\begin{aligned} X_1 = X_2 = \cdots = X_n &\equiv X, & \tilde{X}_1 = \tilde{X}_2 = \cdots = \tilde{X}_m &\equiv \tilde{X}, \\ \det(AX + B\tilde{X}) &= q_1, & \det(CX + D\tilde{X}) &= q_2, \end{aligned}$$

identical to the solutions of the quantum sheaf cohomology relations, as one would expect for a sensible Toda-like dual.

One can show that the correlation functions of this B/2-twisted Landau-Ginzburg model computed by equation (3.7) equal the correlation functions of A/2-twisted model on  $\mathbb{P}^n \times \mathbb{P}^m$  [25]:

$$\langle \sigma_{a_1} \cdots \sigma_{a_l} \rangle = \sum_{\sigma | \mathcal{J}=0} \sigma_{a_1} \cdots \sigma_{a_l} \left[ \det_{a,b} \mathcal{J}_{a,b} \prod_{\alpha} \det M_{(\alpha)} \right]^{-1} \quad (4.6)$$

with

$$\mathcal{J}_a = \ln \left[ q_a^{-1} \prod_{\alpha} \det M_{(\alpha)}^{Q_a^a} \right]. \quad (4.7)$$

In the present case, for  $\mathbb{P}^n \times \mathbb{P}^m$ , there are only two  $\sigma$ 's, which we label  $\sigma_1$ ,  $\sigma_2$ , and

$$\begin{aligned} \mathcal{J}_1 &= \ln [q_1^{-1} \det(A\sigma_1 + B\sigma_2)], \\ \mathcal{J}_2 &= \ln [q_2^{-1} \det(C\sigma_1 + D\sigma_2)]. \end{aligned}$$

To show that the two expressions for correlation functions match, it suffices to show that

$$\det |J_{i,j}| = \det_{a,b} |\mathcal{J}_{a,b}| \prod_{\alpha} \det M_{(\alpha)}, \quad (4.8)$$

by identifying  $X_i$  with  $\sigma_1$  and  $\tilde{X}_k$  with  $\sigma_2$  on the space of vacua, since

$$\det_{a,b} |\mathcal{J}_{a,b}| \prod_{\alpha} \det M_{(\alpha)} = \det \begin{bmatrix} \partial_{\sigma_1} \det(A\sigma_1 + B\sigma_2) & \partial_{\sigma_2} \det(A\sigma_1 + B\sigma_2) \\ \partial_{\sigma_1} \det(C\sigma_1 + D\sigma_2) & \partial_{\sigma_2} \det(C\sigma_1 + D\sigma_2) \end{bmatrix} \quad (4.9)$$

on the classical vacua  $\mathcal{J}_a(\sigma) = 0$ .

In order to show (4.8), we will need a minor linear algebra result. For an  $(n+m) \times (n+m)$  matrix of the form

$$\begin{array}{c} n \\ \left\{ \begin{array}{l} a_{11} \ a_{12} \ a_{13} \ \cdots \ a_{1n} \ \beta \ 0 \ \cdots \ 0 \\ -\alpha \ \alpha \ 0 \ \cdots \ 0 \ 0 \ 0 \ \cdots \ 0 \\ -\alpha \ 0 \ \alpha \ \cdots \ 0 \ 0 \ \cdots \ 0 \\ \vdots \ \quad \quad \quad \ddots \ \vdots \ \vdots \ \quad \quad \quad \ddots \ \vdots \\ -\alpha \ 0 \ \cdots \ 0 \ \alpha \ 0 \ \cdots \ 0 \end{array} \right. \\ m \\ \left\{ \begin{array}{l} \rho \ 0 \ \cdots \ 0 \ d_{11} \ d_{12} \ d_{13} \ \cdots \ d_{1m} \\ 0 \ 0 \ \cdots \ 0 \ -\delta \ \delta \ 0 \ \cdots \ 0 \\ 0 \ \quad \quad \quad 0 \ -\delta \ 0 \ \delta \ \cdots \ 0 \\ \vdots \ \quad \quad \quad \ddots \ \vdots \ \vdots \ \quad \quad \quad \ddots \ \vdots \\ 0 \ \quad \quad \quad \cdots \ 0 \ -\delta \ 0 \ \cdots \ 0 \ \delta \end{array} \right. \end{array}, \quad (4.10)$$

its determinant has the form

$$(\det \zeta)(\det \eta) - \beta \rho \alpha^{n-1} \delta^{m-1}, \quad (4.11)$$

where  $\det \zeta$  is the determinant of the upper-left  $n \times n$  submatrix and  $\det \eta$  is the determinant of the lower-right  $m \times m$  submatrix, given by

$$\det \zeta = \alpha^{n-1} \sum_{i=1}^n a_{1i}, \quad (4.12)$$

$$\det \eta = \delta^{m-1} \sum_{k=1}^m d_{1k}. \quad (4.13)$$

Next, we need to compute

$$\det |J_{i,j}| = \det \begin{bmatrix} \partial_{Y_1} J_1 & \cdots & \partial_{Y_n} J_1 & \partial_{\tilde{Y}_1} J_1 & \cdots & \partial_{\tilde{Y}_m} J_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \partial_{Y_1} J_n & \cdots & \partial_{Y_n} J_n & \partial_{\tilde{Y}_1} J_n & \cdots & \partial_{\tilde{Y}_m} J_n \\ \partial_{Y_1} \tilde{J}_1 & \cdots & \partial_{Y_n} \tilde{J}_1 & \partial_{\tilde{Y}_1} \tilde{J}_1 & \cdots & \partial_{\tilde{Y}_m} \tilde{J}_1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \partial_{Y_1} \tilde{J}_m & \cdots & \partial_{Y_n} \tilde{J}_m & \partial_{\tilde{Y}_1} \tilde{J}_m & \cdots & \partial_{\tilde{Y}_m} \tilde{J}_m \end{bmatrix},$$

where  $X_i = \exp(Y_i)$  and  $\tilde{X}_i = \exp(\tilde{Y}_i)$ . By taking suitable linear combinations, one can rewrite the matrix above in the form of the matrix (4.10), with the following identifications:

$$\begin{aligned} a_{11} &= a^{(1-n)/n} \left( aX_1 + \frac{q_1}{X_1 \cdots X_n} - nb \frac{\tilde{X}_1^{n+1}}{X_1^n} + \sum_{i=1}^n (1-i) \mu_{n+1-i} \frac{\tilde{X}_1^i}{X_1^{i-1}} \right), \\ &= a^{(1-n)/n} \left( 2aX + (1-n)b \frac{\tilde{X}^{n+1}}{X^n} + \sum_{i=1}^n (2-i) \mu_{n+1-i} \frac{\tilde{X}^i}{X^{i-1}} \right), \\ a_{12} &= a_{13} = \cdots = a_{1n} = a^{(1-n)/n} \left( + \frac{q_1}{X_1 \cdots X_n} \right), \\ &= a^{(1-n)/n} \left( aX + b \frac{\tilde{X}^{n+1}}{X^n} + \sum_{i=1}^n \mu_{n+1-i} \frac{\tilde{X}^i}{X^{i-1}} \right), \\ \alpha &= a^{(1-n)/n} (aX_1), \\ &= a^{1/n} X, \\ \beta &= a^{(1-n)/n} \left( (n+1)b \frac{\tilde{X}_1^{n+1}}{X_1^n} + \sum_{i=1}^n i \mu_{n+1-i} \frac{\tilde{X}_1^i}{X_1^{i-1}} \right), \\ &= a^{(1-n)/n} \left( (n+1)b \frac{\tilde{X}^{n+1}}{X^n} + \sum_{i=1}^n i \mu_{n+1-i} \frac{\tilde{X}^i}{X^{i-1}} \right), \\ d_{11} &= d^{(1-m)/m} \left( d\tilde{X}_1 + \frac{q_2}{\tilde{X}_1 \cdots \tilde{X}_m} - cm \frac{X_1^{m+1}}{\tilde{X}_1^m} + \sum_{k=1}^m (1-k) \nu_k \frac{X_1^k}{\tilde{X}_1^{k-1}} \right), \\ &= d^{(1-m)/m} \left( 2d\tilde{X} + (1-m)c \frac{X^{m+1}}{\tilde{X}^m} + \sum_{k=1}^m (2-k) \nu_k \frac{X^k}{\tilde{X}^{k-1}} \right), \\ d_{12} &= d_{13} = \cdots = d_{1n} = d^{(1-m)/m} \left( + \frac{q_2}{\tilde{X}_1 \cdots \tilde{X}_m} \right), \\ &= d^{(1-m)/m} \left( d\tilde{X} + c \frac{X^{m+1}}{\tilde{X}^m} + \sum_{k=1}^m \nu_k \frac{X^k}{\tilde{X}^{k-1}} \right), \\ \delta &= d^{(1-m)/m} (d\tilde{X}_1), \\ &= d^{1/m} \tilde{X}, \end{aligned}$$



$$\begin{aligned}\rho &= d^{(1-m)/m} \left( (m+1)c \frac{X_1^{m+1}}{\tilde{X}_1^m} + \sum_{k=1}^m k\nu_k \frac{X_1^k}{\tilde{X}_1^{k-1}} \right), \\ &= d^{(1-m)/m} \left( (m+1) \frac{X^{m+1}}{\tilde{X}^m} + \sum_{k=1}^m k\nu_k \frac{X^k}{\tilde{X}^{k-1}} \right).\end{aligned}$$

(In the expressions above, the second line is obtained by evaluation on vacua.)

Putting this together, we can write

$$\begin{aligned}\det |J_{i,j}| &= \det \begin{bmatrix} \det \zeta & \beta \alpha^{n-1} \\ \rho \delta^{m-1} & \det \eta \end{bmatrix} \\ &= \det \begin{bmatrix} \alpha^{n-1}(a_{11} + (n-1)a_{12}) & \alpha^{n-1}\beta \\ \delta^{m-1}\rho & \delta^{m-1}(d_{11} + (n-1)d_{12}) \end{bmatrix}\end{aligned}$$

which is easily checked to be the determinant of

$$\begin{bmatrix} (n+1)aX^n + \sum_{i=1}^n (n+1-i)\mu_{n+1-i}\tilde{X}^i X^{n-i} & (n+1)b\tilde{X}^{n+1}X^{-1} + \sum_{i=1}^n i\mu_{n+1-i}\tilde{X}^i X^{n-i} \\ (m-1)cX^{m+1}\tilde{X}^{-1} + \sum_{k=1}^m k\nu_k X^k \tilde{X}^{m-k} & (m-1)d\tilde{X}^m + \sum_{k=1}^m (m+1-k)\nu_k X^k \tilde{X}^{m-k} \end{bmatrix}.$$

By identifying  $X_i$  with  $\sigma_1$  and  $\tilde{X}_k$  with  $\sigma_2$ , we see that the determinant above matches (4.9).

Thus, all genus-zero correlation functions in our proposed Toda dual match those of the (0,2) theory on  $\mathbb{P}^n \times \mathbb{P}^m$  with a deformation of the tangent bundle. In addition to constructing a general argument that correlation functions should match, we have also compared correlation functions in special cases, as we shall outline next.

### 4.3 Example: $\mathbb{P}^1 \times \mathbb{P}^2$

As a consistency check, as we have already studied the dual to  $\mathbb{P}^1 \times \mathbb{P}^1$ , we next consider the special case  $\mathbb{P}^1 \times \mathbb{P}^2$ . Specializing the results for  $\mathbb{P}^n \times \mathbb{P}^m$ , the mirror (0,2) Landau-Ginzburg model is defined by the superpotential

$$W = FJ + \widetilde{F}_1 \widetilde{J}_1 + \widetilde{F}_2 \widetilde{J}_2$$

with

$$J = aX - \frac{q_1}{X} + b\frac{\tilde{X}_1^2}{X} + \mu\tilde{X}_1, \quad (4.14)$$

$$\tilde{J}_1 = d^{-\frac{1}{2}} \left( d\tilde{X}_1 - \frac{q_2}{\tilde{X}_1\tilde{X}_2} + c\frac{X^3}{\tilde{X}_1^2} + fX + g\frac{X^2}{\tilde{X}_1} \right), \quad (4.15)$$

$$\tilde{J}_2 = d^{-\frac{1}{2}} \left( d\tilde{X}_2 - \frac{q_2}{\tilde{X}_1\tilde{X}_2} + c\frac{X^3}{\tilde{X}_1^2} + fX + g\frac{X^2}{\tilde{X}_1} \right). \quad (4.16)$$

In the expression above,

$$\begin{aligned}a &= \det A, \quad b = \det B, \quad c = \det C, \quad d = \det D, \\ \mu &= \det(A+B) - \det A - \det B,\end{aligned}$$

for the matrices  $A, B, C, D$  defining the gauge bundle deformation in the  $A/2$ -twisted nonlinear sigma model, and where  $g$  is a sum of determinants of three matrices, each of which is formed from taking two rows of  $C$  and one row of  $D$ , and  $f$  is similarly a sum of three determinants, involving matrices formed as two rows of  $D$  and one row of  $C$ .

We have directly computed correlation functions in the proposed dual Landau-Ginzburg theory above, in the special case  $c = f = g = 0$ . On the vacua,  $\tilde{X}_1 = \tilde{X}_2$ , so in computing correlation functions, we will use  $\tilde{X}$  to denote either  $\tilde{X}_1$  or  $\tilde{X}_2$ . In any event, the three-point correlation functions in this case are given by

$$\begin{aligned}\langle XXX \rangle &= -(ab - \mu^2)(a^3d)^{-1}, \\ \langle XX\tilde{X} \rangle &= -\mu(a^2d)^{-1}, \\ \langle X\tilde{X}\tilde{X} \rangle &= (ad)^{-1}, \\ \langle \tilde{X}\tilde{X}\tilde{X} \rangle &= 0.\end{aligned}$$

The five-point correlation functions are given by

$$\begin{aligned}\langle X^5 \rangle &= -(a^4d)^{-1}(2ab - 3\mu^2)q_1, \\ \langle X^4\tilde{X} \rangle &= -2(a^3d)^{-1}\mu q_1, \\ \langle X^3\tilde{X}^2 \rangle &= (a^2d)^{-1}q_1, \\ \langle X^2\tilde{X}^3 \rangle &= 0, \\ \langle X\tilde{X}^4 \rangle &= 0, \\ \langle \tilde{X}^5 \rangle &= 0.\end{aligned}$$

The six-point correlation functions are given by

$$\begin{aligned}\langle X^6 \rangle &= -(a^6d^2)^{-1}\mu(3a^2b^2 - 4ab\mu^2 + \mu^4)q_2, \\ \langle X^5\tilde{X} \rangle &= (a^5d^2)^{-1}(a^2b^2 - 3ab\mu^2 + \mu^4)q_2, \\ \langle X^4\tilde{X}^2 \rangle &= -(a^4d^2)^{-1}\mu(-2ab + \mu^2)q_2, \\ \langle X^3\tilde{X}^3 \rangle &= -(a^3d^2)^{-1}(ab - \mu^2)q_2, \\ \langle X^2\tilde{X}^4 \rangle &= -(a^2d^2)^{-1}\mu q_2, \\ \langle X\tilde{X}^5 \rangle &= (ad^2)^{-1}q_2, \\ \langle \tilde{X}^6 \rangle &= 0.\end{aligned}$$

If we identify  $X$  with  $\psi$  and  $\tilde{X}$  with  $\tilde{\psi}$ , then these correlation functions match those of the corresponding  $A/2$  model, for this case ( $c = f = g = 0$ ). We have listed the  $A/2$  model correlation functions (for the general case) in appendix [A.3](#).

## 5 Conclusions

In this paper we establish the  $(0,2)$  Toda-like dual models to  $(0,2)$  nonlinear sigma models on  $\mathbb{P}^n \times \mathbb{P}^m$  with a deformation of the tangent bundle, solving an old problem on the road to understanding  $(0,2)$  mirror symmetry. We checked our ansatz via a general argument demonstrating that all genus zero correlation functions match, and also checked matching of low-order correlation functions explicitly.

We have only checked our ansatz for duals at genus zero. It would be useful to check at higher genera, but unfortunately at this time it is not known how to compute higher genus correlation functions in A/2 twisted theories, so such checks are left for the future.

The methods used here, such as our use of quantum sheaf cohomology to determine the vacua of the correct dual theory, reminiscent of methods in e.g. [32], should be straightforward to extend to more general Fano toric varieties.

## Acknowledgments

We would like to thank L. Anderson, R. Donagi, J. Gray, J. Guffin, S. Katz, Z. Lu, and I. Melnikov for useful conversations. E.S. was partially supported by NSF grant PHY-1417410.

## A Correlation functions in some examples

### A.1 A/2 correlation functions on $\mathbb{P}^1 \times \mathbb{P}^1$

In this appendix we list the two- and four-point correlation functions for A/2 twisted nonlinear sigma models on  $\mathbb{P}^1 \times \mathbb{P}^1$  with a deformation  $\mathcal{E}$  of the tangent bundle, defined by

$$0 \longrightarrow \mathcal{O} \otimes \mathcal{O} \xrightarrow{E} \mathcal{O}(1, 0)^2 \oplus \mathcal{O}(0, 1)^2 \longrightarrow \mathcal{E} \longrightarrow 0,$$

defined as in section 3.1 by four matrices  $A, B, C, D$ .

In writing the correlation functions, we use the following notation:

$$\begin{aligned} a &= \det(A), & b &= \det(B), & c &= \det(C), & d &= \det(D), \\ e &= \det(A + B), & f &= \det(C + D), \\ \mu &= e - a - b, & \nu &= f - c - d, \\ \phi_1 &= \nu b - \mu d = ad + bf - de - bc, \\ \phi_2 &= ad - bc, \\ \phi_3 &= \mu c - \nu a = ad + ce - af - bc, \\ \Delta &= \phi_2^2 - \phi_1 \phi_3, \\ &= (c - d)(bc - ad)e + cde^2 + (a - b)(ad - bc)f - (bc + ad)ef + abf^2. \end{aligned}$$

The two-point correlation functions are given by

$$\langle \psi \psi \rangle = \frac{\phi_1}{\Delta}, \quad \langle \psi \tilde{\psi} \rangle = \frac{\phi_2}{\Delta}, \quad \langle \tilde{\psi} \tilde{\psi} \rangle = \frac{\phi_3}{\Delta}. \quad (\text{A.1})$$

The four-point correlation functions are given by

$$\begin{aligned} \langle \psi \psi \psi \psi \rangle_{10} &= \frac{\phi_1}{\Delta^2} (\nu \phi_1 + 2\phi_2 d) = \frac{1}{\Delta^2} (\phi_1 ((f - c)\phi_1 + ad^2 + d^2 e - bcd - bdf)), \\ \langle \psi \psi \psi \tilde{\psi} \rangle_{10} &= \frac{1}{\Delta^2} (-\phi_1^2 c + \phi_2^2 d), \\ \langle \psi \psi \tilde{\psi} \tilde{\psi} \rangle_{10} &= \frac{\phi_2}{\Delta^2} (\phi_3 d - \phi_1 c), \end{aligned}$$

$$\begin{aligned}
 \langle \psi \tilde{\psi} \tilde{\psi} \tilde{\psi} \rangle_{10} &= \frac{1}{\Delta^2} (\phi_3^2 d - \phi_2^2 c), \\
 \langle \tilde{\psi} \tilde{\psi} \tilde{\psi} \tilde{\psi} \rangle_{10} &= \frac{\phi_3}{\Delta^2} (\nu \phi_3 + 2\phi_2 c), \\
 &= \frac{1}{\Delta^2} (\phi_3 (ce(c+d-f) + bc(c-d+f) + a((d-f)^2 - c(d+f)))), \\
 \langle \psi \psi \psi \psi \rangle_{01} &= -\frac{\phi_1}{\Delta^2} (\mu \phi_1 + 2\phi_2 b), \\
 &= \frac{1}{\Delta^2} (-\phi_1 (2b(ad-bc) - d(a+b-e)^2 + b(a+b-e)(c+d-f))), \\
 \langle \psi \psi \psi \tilde{\psi} \rangle_{01} &= \frac{1}{\Delta^2} (\phi_1^2 a - \phi_2^2 b), \\
 \langle \psi \psi \tilde{\psi} \tilde{\psi} \rangle_{01} &= \frac{\phi_2}{\Delta^2} (-\phi_3 b + \phi_1 a), \\
 \langle \tilde{\psi} \tilde{\psi} \tilde{\psi} \tilde{\psi} \rangle_{01} &= \frac{1}{\Delta^2} (-\phi_3^2 b + \phi_2^2 a), \\
 \langle \tilde{\psi} \tilde{\psi} \tilde{\psi} \tilde{\psi} \rangle_{01} &= -\frac{\phi_3}{\Delta^2} (\mu \phi_3 - 2\phi_2 a) = \frac{1}{\Delta^2} (\phi_3 ((e-b)\phi_3 + a^2 d + a^2 f - abc - acd)),
 \end{aligned}$$

where the subscripts 10 and 01 denote contributions from the degree one sector on either  $\mathbb{P}^1$  factor:

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle = q_1 \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle_{10} + q_2 \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \mathcal{O}_4 \rangle_{01}.$$

## A.2 Toda-like dual to $\mathbb{P}^1 \times \mathbb{P}^1$

In this appendix we list the two-point and four-point correlation functions for our proposed Toda-like dual (0,2) Landau-Ginzburg model, with superpotential of the form

$$\begin{aligned}
 J &= aX + b\frac{\tilde{X}^2}{X} + \mu\tilde{X} - \frac{q_1}{X}, \\
 \tilde{J} &= d\tilde{X} + c\frac{X^2}{\tilde{X}} + \nu X - \frac{q_2}{\tilde{X}}.
 \end{aligned}$$

The two-point correlation functions in this (0,2) Landau-Ginzburg model can be shown to be

$$\begin{aligned}
 \langle XX \rangle &= \gamma^{-1}(b\nu - d\mu), \\
 \langle X\tilde{X} \rangle &= \gamma^{-1}(ad - bc), \\
 \langle \tilde{X}\tilde{X} \rangle &= \gamma^{-1}(c\mu - a\nu),
 \end{aligned}$$

where  $\gamma = b^2 c^2 - 2abcd + a^2 d^2 + cd\mu^2 - (bc + ad)\mu\nu + ab\nu^2$ .

The four-point correlation functions in this (0,2) Landau-Ginzburg model can be shown to be

$$\begin{aligned}
 \langle XXXX \rangle_{10} &= \gamma^{-2}(-(d\mu - b\nu)(d(2ad - \mu\nu) + b(-2cd + \nu^2))), \\
 \langle XXXX \rangle_{01} &= \gamma^{-2}(-d\mu + b\nu)(2b^2 c + d\mu^2 - b(2ad + \mu\nu)), \\
 \langle XXX\tilde{X} \rangle_{10} &= \gamma^{-2}(d((bc - ad)^2 - cd\mu^2) + 2bcd\mu\nu - b^2 c\nu^2), \\
 \langle XXX\tilde{X} \rangle_{01} &= \gamma^{-2}(-b^3 c^2 + ad^2 \mu^2 - adb(ad + 2\mu\nu) + ab^2(2cd + \nu^2)), \\
 \langle XX\tilde{X}\tilde{X} \rangle_{10} &= \gamma^{-2}(bc - ad)(-2cd\mu + bc\nu + adv),
 \end{aligned}$$

$$\begin{aligned}
 \langle XX\tilde{X}\tilde{X} \rangle_{01} &= \gamma^{-2}(bc - ad)(bc\mu + ad\mu - 2ab\nu), \\
 \langle X\tilde{X}\tilde{X}\tilde{X} \rangle_{10} &= \gamma^{-2}(c(-(bc - ad)^2 + cd\mu^2) - 2acd\mu\nu + a^2d\nu^2), \\
 \langle X\tilde{X}\tilde{X}\tilde{X} \rangle_{01} &= \gamma^{-2}(a^3d^2 - bc^2\mu^2 + abc(bc + 2\mu\nu) - a^2b(2cd + \nu^2)), \\
 \langle \tilde{X}\tilde{X}\tilde{X}\tilde{X} \rangle_{10} &= \gamma^{-2}(c\mu - a\nu)(2bc^2 - c\mu\nu + a(-2cd + \nu^2)), \\
 \langle \tilde{X}\tilde{X}\tilde{X}\tilde{X} \rangle_{01} &= \gamma^{-2}(c\mu - a\nu)(2a^2d + c\mu^2 - a(2bc + \mu\nu)),
 \end{aligned}$$

where the 10 and 01 subscripts indicate the coefficients of  $q_1$ ,  $q_2$ , as in the previous subsection.

As remarked in section 3.2, if we identify the parameters above with matrix determinants as

$$\begin{aligned}
 a &= \det A, \quad b = \det B, \quad c = \det C, \quad d = \det D, \\
 \mu &= \det(A + B) - \det A - \det B, \\
 \nu &= \det(C + D) - \det C - \det D,
 \end{aligned}$$

for  $A$ ,  $B$ ,  $C$ ,  $D$  the matrices appearing in the A/2-twisted (0,2) model on  $\mathbb{P}^1 \times \mathbb{P}^1$ , the correlation functions in the Landau-Ginzburg model above match those of the A/2 model.

### A.3 A/2 correlation functions on $\mathbb{P}^1 \times \mathbb{P}^2$

In this appendix we list the three-point, five-point and six-point correlation functions for A/2 twisted nonlinear sigma models on  $\mathbb{P}^1 \times \mathbb{P}^2$  with a deformation  $\mathcal{E}$  of the tangent bundle, defined by

$$0 \longrightarrow \mathcal{O}^2 \xrightarrow{E} \mathcal{O}(1, 0)^2 \oplus \mathcal{O}(0, 1)^3 \longrightarrow \mathcal{E} \longrightarrow 0,$$

with  $A$ ,  $B$   $2 \times 2$  matrices and  $C$ ,  $D$   $3 \times 3$  matrices.

Correlation functions in this theory can be computed in a variety of methods, such as e.g. residues [22]. In writing the correlation functions, we use the following notation:

$$\begin{aligned}
 a &= \det A, \quad b = \det B, \quad c = \det C, \quad d = \det D, \\
 \mu &= \det(A + B) - \det A - \det B,
 \end{aligned}$$

$g$  is a sum of determinants of three matrices, each formed from two rows of  $C$  and one row of  $D$ , and  $f$  is similarly a sum of three determinants, each having two rows of  $D$  and one row of  $C$ .

Three-point functions in the A/2 theory are given by

$$\begin{aligned}
 \langle \psi^3 \rangle &= \Delta^{-1}(-abd + b^2g - bf\mu + d\mu^2), \\
 \langle \psi^2\tilde{\psi} \rangle &= \Delta^{-1}(-b^2c + abf - ad\mu), \\
 \langle \psi\tilde{\psi}^2 \rangle &= \Delta^{-1}(a^2d - abg + bc\mu), \\
 \langle \tilde{\psi}^3 \rangle &= \Delta^{-1}(abc - a^2f + ag\mu - c\mu^2),
 \end{aligned}$$

where

$$\begin{aligned}
 \Delta &= a^3d^2 + b((bc - af)^2 - 2a^2dg + abg^2) + (bcf + adg)\mu^2 - cd\mu^3 \\
 &\quad - (ad(-3bc + af) + b(bc + af)g)\mu.
 \end{aligned}$$

Five-point correlation functions in the A/2 theory are given by

$$\begin{aligned}
 \langle \psi^5 \rangle &= q_1 \Delta^{-2} (b^4(c^2d - 2cfg + g^3) + d^2\mu^2(3a^2d - 2af\mu + g\mu^2) \\
 &\quad + 2b^3(ag(f^2 - 2dg) + (cf^2 + cdg - fg^2)\mu) \\
 &\quad + b^2(a^2d(-f^2 + 5dg) - 2af(f^2 - dg)\mu + (-4cdf + g(f^2 + 2dg))\mu^2) \\
 &\quad - 2bd(a^3d^2 + a^2df\mu - 2a(f^2 - dg)\mu^2 + (-cd + fg)\mu^3)), \\
 \langle \psi^4 \tilde{\psi} \rangle &= q_1 \Delta^{-2} (2a^3d^2(bf - d\mu) + 2ab^2c(-bf^2 + 2bdg + df\mu) \\
 &\quad + a^2(b^2(-3cd^2 + f^3 - 2dfg) + 2bd(-f^2 + dg)\mu + d^2f\mu^2) \\
 &\quad + c(b^4(cf - g^2) - b^2(f^2 + 2dg)\mu^2 + 2bdf\mu^3 - d^2\mu^4) \\
 &\quad + b^3(-2cd\mu + 2fg\mu)), \\
 \langle \psi^3 \tilde{\psi}^2 \rangle &= q_1 \Delta^{-2} (a^4d^3 - 2a^3bd^2g + b^2c^2(b^2g - 2bf\mu + 3d\mu^2) \\
 &\quad + a^2(b^2(2cdf - f^2g + dg^2) + 2bdfg\mu - d^2g\mu^2) \\
 &\quad - 2ac(b^3cd + 2bdf\mu^2 - d^2\mu^3 + b^2(-f^2\mu + dg\mu))), \\
 \langle \psi^2 \tilde{\psi}^3 \rangle &= q_1 \Delta^{-2} (-b^4c^3 + 2ab^3c^2f - a^2d^2(a^2f - 2ag\mu + 3c\mu^2) \\
 &\quad + b^2(a^2(fg^2 - c(f^2 + 2dg)) - 2acfg\mu + c^2f\mu^2) \\
 &\quad + 2bd(a^3cd + a^2(cf - g^2)\mu + 2acg\mu^2 - c^2\mu^3)), \\
 \langle \psi \tilde{\psi}^4 \rangle &= q_1 \Delta^{-2} (a^4d(f^2 - dg) + 2a^3d(-2bcf + bg^2 + cd\mu - fg\mu) \\
 &\quad + a^2(b^2(3c^2d + 2cfg - g^3) - 2bcdg\mu + d(2cf + g^2)\mu^2) \\
 &\quad + c^2\mu(2b^3c - b^2g\mu + d\mu^3) - 2ac(b^3cg + b^2(cf - g^2)\mu + dg\mu^3)), \\
 \langle \tilde{\psi}^5 \rangle &= q_1 \Delta^{-2} (-a^4(cd^2 + f^3 - 2dfg) - c^2\mu^2(3b^2c - 2bg\mu + f\mu^2) \\
 &\quad + 2a^3(bf(2cf - g^2) - (cdf - f^2g + dg^2)\mu) \\
 &\quad + a^2(b^2c(-5cf + g^2) + 2bg(-cf + g^2)\mu - (2cf^2 - 4cdg + fg^2)\mu^2) \\
 &\quad + 2ac(b^3c^2 + b^2cg\mu + 2b(cf - g^2)\mu^2 + (-cd + fg)\mu^3)).
 \end{aligned}$$

Six-point correlation functions in the A/2 theory are given by

$$\begin{aligned}
 \langle \psi^6 \rangle &= q_2 \Delta^{-2} ((abd - b^2g + bf\mu - d\mu^2)(-2b^3c + d\mu^3 - b\mu(3ad + f\mu) \\
 &\quad + b^2(2af + g\mu))), \\
 \langle \psi^5 \tilde{\psi} \rangle &= q_2 \Delta^{-2} (-b^5c^2 + ab^4(2cf + g^2) + ad^2\mu^4 - abd\mu^2(3ad + 2f\mu) \\
 &\quad - ab^3(a(f^2 + 2dg) + 2(cd + fg)\mu) + ab^2(a^2d^2 + 4adf\mu \\
 &\quad + (f^2 + 2dg)\mu^2)), \\
 \langle \psi^4 \tilde{\psi}^2 \rangle &= q_2 \Delta^{-2} ((b^2c - abf + ad\mu)(2a^2bd + b^2c\mu + a(-2b^2g + bf\mu - d\mu^2))), \\
 \langle \psi^3 \tilde{\psi}^3 \rangle &= q_2 \Delta^{-2} (-a^4bd^2 - a^2b^3(2cf + g^2) - b^3c^2\mu^2 + ab^3c(bc + 2g\mu) \\
 &\quad + a^3(b^2(f^2 + 2dg) - 2bdf\mu + d^2\mu^2)), \\
 \langle \psi^2 \tilde{\psi}^4 \rangle &= q_2 \Delta^{-2} (-(a^2d - abg + bc\mu)(-bc\mu^2 + a^2(-2bf + d\mu) \\
 &\quad + ab(2bc + g\mu))), \\
 \langle \psi \tilde{\psi}^5 \rangle &= q_2 \Delta^{-2} (a^5d^2 - a^4b(f^2 + 2dg) - bc^2\mu^4 + abc\mu^2(3bc + 2g\mu) \\
 &\quad + a^3b(b(2cf + g^2) + 2(cd + fg)\mu) \\
 &\quad - a^2b(b^2c^2 + 4bcg\mu + (2cf + g^2)\mu^2)),
 \end{aligned}$$

$$\langle \tilde{\psi}^6 \rangle = q_2 \Delta^{-2} \left( -(a^2 f + c\mu^2 - a(bc + g\mu))(2a^3 d - c\mu^3 - a^2(2bg + f\mu) + a\mu(3bc + g\mu)) \right).$$

## B Tangent bundle moduli

In this appendix we compute<sup>7</sup> the dimension of the tangent space to the moduli space of tangent bundle deformations, at the tangent bundle and ‘near’ the tangent bundle. We will see that the rank of the tangent space to the moduli space of tangent bundle deformations can change as one moves away from the (2,2) locus.

Define

$$W = V \otimes \mathcal{O}(1, 0) + \tilde{V} \otimes \mathcal{O}(0, 1),$$

where

$$V \cong \mathbb{C}^{n+1}, \quad \tilde{V} \cong \mathbb{C}^{m+1},$$

so that we can write the definition of the tangent bundle deformation  $\mathcal{E}$  as

$$0 \longrightarrow \mathcal{O}^2 \longrightarrow W \longrightarrow \mathcal{E} \longrightarrow 0.$$

First, if we dualize the definition above and take the associated long exact sequence, then from the fact that

$$H^q(W^*) = 0 \quad \text{for all } q,$$

(from the Bott formula, [33, Section 1.1]), we have that

$$H^q(\mathcal{E}^*) = H^{q-1}(\mathcal{O}^2)$$

and so vanishes unless  $q = 1$ .

Then, applying  $\text{Hom}(\mathcal{E}, -)$  to the definition of  $\mathcal{E}$  and taking the associated long exact sequence, one finds

$$0 \rightarrow H^0(\mathcal{E}^* \otimes W) \rightarrow H^0(\mathcal{E}^* \otimes \mathcal{E}) \rightarrow \mathbb{C}^4 \rightarrow H^1(\mathcal{E}^* \otimes W) \rightarrow H^1(\mathcal{E}^* \otimes \mathcal{E}) \rightarrow 0.$$

From this expression we find

$$h^1(\mathcal{E}^* \otimes \mathcal{E}) = h^0(\mathcal{E}^* \otimes \mathcal{E}) - (h^0(\mathcal{E}^* \otimes W) - h^1(\mathcal{E}^* \otimes W)) - 4. \quad (\text{B.1})$$

Next, we will derive a relation between  $h^0(\mathcal{E}^* \otimes W)$  and  $h^1(\mathcal{E}^* \otimes W)$ . Apply  $\text{Hom}(-, W)$  to the definition of  $\mathcal{E}$  to get, from the associated long exact sequence,

$$0 \rightarrow H^0(\mathcal{E}^* \otimes W) \rightarrow H^0(W^* \otimes W) \rightarrow H^0(\mathcal{O}^2 \otimes W) \rightarrow H^1(\mathcal{E}^* \otimes W) \rightarrow 0,$$

where we have used the fact that

$$H^q(W^* \otimes W) = 0 \quad \text{for } q > 0,$$

---

<sup>7</sup>These computations were originally worked out in collaboration with R. Donagi and J. Guffin for another project.

as none of  $\mathcal{O}$ ,  $\mathcal{O}(1, -1)$ ,  $\mathcal{O}(-1, 1)$  have any cohomology in degree greater than zero. From the sequence above, we have that

$$h^0(\mathcal{E}^* \otimes W) - h^1(\mathcal{E}^* \otimes W) = h^0(W^* \otimes W) - h^0(\mathcal{O}^2 \otimes W).$$

To simplify further, we use the fact that

$$H^0(W^* \otimes W) = V \otimes V^* + \tilde{V} \otimes \tilde{V}^*,$$

and so has dimension

$$(n+1)^2 + (m+1)^2$$

Similarly, from Bott-Borel-Weil,

$$H^0(W) = V \otimes V^* + \tilde{V} \otimes \tilde{V}^*,$$

and so has the same dimension. Thus,

$$h^0(\mathcal{E}^* \otimes W) - h^1(\mathcal{E}^* \otimes W) = -(n+1)^2 - (m+1)^2.$$

Plugging into equation (B.1), we find

$$h^1(\text{End } \mathcal{E}) = h^0(\text{End } \mathcal{E}) + (n+1)^2 + (m+1)^2 - 4. \quad (\text{B.2})$$

From the relation above, we immediately see that

$$h^1(\text{End } \mathcal{E}) \geq (n+1)^2 + (m+1)^2 - 4 = n(n+2) + m(m+2) - 2.$$

Let us compute  $h^0(\text{End } \mathcal{E})$  on the (2,2) locus, where  $\mathcal{E}$  is the tangent bundle of  $\mathbb{P}^n \times \mathbb{P}^m$ . From the Bott formula [33, Section 1.1], one has

$$H^0(\mathbb{P}^n, \Omega^1) = 0$$

and from applying  $\text{Hom}(T\mathbb{P}^n, -)$  to the Euler sequence, one can similarly derive

$$h^0(\text{End } T\mathbb{P}^n) = h^1(\Omega^1) = 1,$$

from which one quickly derives that

$$h^0(\text{End } \mathcal{E}) = 2.$$

Thus, on the (2,2) locus, we find

$$h^1(\text{End } \mathcal{E}) = n(n+2) + m(m+2).$$

For  $\mathbb{P}^1 \times \mathbb{P}^1$ , the predicted number of infinitesimal deformations of the tangent bundle is  $3 + 3 = 6$ , matching the number of parameters on which (0,2) computations depend, namely

$$a, b, c, d, \mu, \nu,$$

as described in section 3.3.



Away from the tangent bundle itself, the computations above suggest that the correct number of moduli is smaller, which can be confirmed from other computations. For example, if we twist the tangent bundle by  $\mathcal{O}(0, -1)$ , we get a rank two vector bundle of  $c_2 = 2$ , and from [34, Chapter 6, theorem 20], the moduli space of such vector bundles has dimension<sup>8</sup>  $4c_2 - 3 = 5$ . As twisting by line bundles does not affect Mumford stability, the space of tangent bundle deformations should have the same dimension, so we see that the tangent bundle of  $\mathbb{P}^1 \times \mathbb{P}^1$  represents an unstable point on the moduli space.

For higher-dimensional products, not all of the deformations can be realized in the Euler sequence, or as  $E$  moduli in the GLSM [27]. For example, the predicted number of infinitesimal moduli of the tangent bundle of  $\mathbb{P}^1 \times \mathbb{P}^2$  is  $3 + 2(4) = 11$ . However, only seven parameters appear in the (0,2) GLSMs:

$$a, b, c, d, \mu, \nu_1, \nu_2.$$

**Open Access.** This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

## References

- [1] M. Kontsevich, *Homological algebra of mirror symmetry*, [alg-geom/9411018](#) [[INSPIRE](#)].
- [2] R. Blumenhagen, R. Schimmrigk and A. Wisskirchen, (0,2) *mirror symmetry*, *Nucl. Phys. B* **486** (1997) 598 [[hep-th/9609167](#)] [[INSPIRE](#)].
- [3] R. Blumenhagen and S. Sethi, *On orbifolds of (0,2) models*, *Nucl. Phys. B* **491** (1997) 263 [[hep-th/9611172](#)] [[INSPIRE](#)].
- [4] B.R. Greene and M.R. Plesser, *Duality in Calabi-Yau moduli space*, *Nucl. Phys. B* **338** (1990) 15 [[INSPIRE](#)].
- [5] A. Adams, A. Basu and S. Sethi, (0,2) *duality*, *Adv. Theor. Math. Phys.* **7** (2003) 865 [[hep-th/0309226](#)] [[INSPIRE](#)].
- [6] I.V. Melnikov and M.R. Plesser, *A (0,2) mirror map*, *JHEP* **02** (2011) 001 [[arXiv:1003.1303](#)] [[INSPIRE](#)].
- [7] S.H. Katz and E. Sharpe, *Notes on certain (0,2) correlation functions*, *Commun. Math. Phys.* **262** (2006) 611 [[hep-th/0406226](#)] [[INSPIRE](#)].
- [8] R. Donagi, J. Guffin, S. Katz and E. Sharpe, *A mathematical theory of quantum sheaf cohomology*, *Asian J. Math.* **18** (2014) 387 [[arXiv:1110.3751](#)] [[INSPIRE](#)].
- [9] R. Donagi, J. Guffin, S. Katz and E. Sharpe, *Physical aspects of quantum sheaf cohomology for deformations of tangent bundles of toric varieties*, *Adv. Theor. Math. Phys.* **17** (2013) 1255 [[arXiv:1110.3752](#)] [[INSPIRE](#)].
- [10] J. Guffin, *Quantum sheaf cohomology, a precis*, *Mat. Contemp.* **41** (2012) 17 [[arXiv:1101.1305](#)] [[INSPIRE](#)].
- [11] J. Guffin and S. Katz, *Deformed quantum cohomology and (0,2) mirror symmetry*, *JHEP* **08** (2010) 109 [[arXiv:0710.2354](#)] [[INSPIRE](#)].

---

<sup>8</sup>We would like to thank Z. Lu for pointing out this computation to us.

- [12] M. Kreuzer, J. McOrist, I.V. Melnikov and M.R. Plesser,  $(0, 2)$  deformations of linear  $\sigma$ -models, *JHEP* **07** (2011) 044 [[arXiv:1001.2104](#)] [[INSPIRE](#)].
- [13] J. McOrist, *The revival of  $(0, 2)$  linear  $\sigma$ -models*, *Int. J. Mod. Phys. A* **26** (2011) 1 [[arXiv:1010.4667](#)] [[INSPIRE](#)].
- [14] J. McOrist and I.V. Melnikov, *Half-twisted correlators from the Coulomb branch*, *JHEP* **04** (2008) 071 [[arXiv:0712.3272](#)] [[INSPIRE](#)].
- [15] J. McOrist and I.V. Melnikov, *Old issues and linear  $\sigma$ -models*, *Adv. Theor. Math. Phys.* **16** (2012) 251 [[arXiv:1103.1322](#)] [[INSPIRE](#)].
- [16] I.V. Melnikov,  $(0, 2)$  Landau-Ginzburg models and residues, *JHEP* **09** (2009) 118 [[arXiv:0902.3908](#)] [[INSPIRE](#)].
- [17] E. Sharpe, *Notes on correlation functions in  $(0, 2)$  theories*, [hep-th/0502064](#) [[INSPIRE](#)].
- [18] E. Sharpe, *Notes on certain other  $(0, 2)$  correlation functions*, *Adv. Theor. Math. Phys.* **13** (2009) 33 [[hep-th/0605005](#)] [[INSPIRE](#)].
- [19] M.-C. Tan, *Two-dimensional twisted  $\sigma$ -models and the theory of chiral differential operators*, *Adv. Theor. Math. Phys.* **10** (2006) 759 [[hep-th/0604179](#)] [[INSPIRE](#)].
- [20] M.-C. Tan, *Two-dimensional twisted  $\sigma$ -models, the mirror chiral de Rham complex and twisted generalised mirror symmetry*, *JHEP* **07** (2007) 013 [[arXiv:0705.0790](#)] [[INSPIRE](#)].
- [21] J. Guffin and E. Sharpe, *A-twisted heterotic Landau-Ginzburg models*, *J. Geom. Phys.* **59** (2009) 1581 [[arXiv:0801.3955](#)] [[INSPIRE](#)].
- [22] C. Closset, W. Gu, B. Jia and E. Sharpe, *Localization of twisted  $\mathcal{N} = (0, 2)$  gauged linear  $\sigma$ -models in two dimensions*, *JHEP* **03** (2016) 070 [[arXiv:1512.08058](#)] [[INSPIRE](#)].
- [23] J. Guo, Z. Lu and E. Sharpe, *Quantum sheaf cohomology on Grassmannians*, [arXiv:1512.08586](#) [[INSPIRE](#)].
- [24] I.V. Melnikov and S. Sethi, *Half-twisted  $(0, 2)$  Landau-Ginzburg models*, *JHEP* **03** (2008) 040 [[arXiv:0712.1058](#)] [[INSPIRE](#)].
- [25] J. McOrist and I.V. Melnikov, *Summing the instantons in half-twisted linear  $\sigma$ -models*, *JHEP* **02** (2009) 026 [[arXiv:0810.0012](#)] [[INSPIRE](#)].
- [26] E. Sharpe, *An introduction to quantum sheaf cohomology*, [PoS\(ICMP 2012\)026](#).
- [27] R. Donagi, Z. Lu and I.V. Melnikov, *Global aspects of  $(0, 2)$  moduli space: toric varieties and tangent bundles*, *Commun. Math. Phys.* **338** (2015) 1197 [[arXiv:1409.4353](#)] [[INSPIRE](#)].
- [28] C. Vafa, *Topological Landau-Ginzburg models*, *Mod. Phys. Lett. A* **6** (1991) 337 [[INSPIRE](#)].
- [29] J. Guffin and E. Sharpe, *A-twisted Landau-Ginzburg models*, *J. Geom. Phys.* **59** (2009) 1547 [[arXiv:0801.3836](#)] [[INSPIRE](#)].
- [30] T. Pantev and E. Sharpe, *GLSM's for gerbes (and other toric stacks)*, *Adv. Theor. Math. Phys.* **10** (2006) 77 [[hep-th/0502053](#)] [[INSPIRE](#)].
- [31] S. Katz, private communication.
- [32] H. Iritani, *A mirror construction for the big equivariant quantum cohomology of toric manifolds*, [arXiv:1503.02919](#) [[INSPIRE](#)].
- [33] C. Okonek, M. Schneider and H. Spindler, *Vector bundles on complex projective spaces*, Birkhäuser, Boston, U.S.A. (1980).
- [34] R. Friedman, *Algebraic surfaces and holomorphic vector bundles*, Springer-Verlag, New York U.S.A. (1998).