Appendix E. Models

The variational method permits to solve for the displacement of the beam and its attached elements. The global displacements (described by u_b, w_b, ϕ_c, or w_m) is determined by solving a matrix equation (E.1). The matrices involved in these equations can be combined to model complex systems. The following items are presented in this appendix:

- Beam
- Springs (boundary conditions)
- Piezoelectric layer
- Constrained layer
- Point absorber
- Distributed active vibration absorber with uniform mass layer
- Distributed active vibration absorber with varying mass distribution

The global matrices (coupled system) can be subdivided into several sub-matrices which are indexed from 1 to 6. The matrix coefficients are indexed by p and q. Vector coefficients are indexed by p.

$$\left\{ -\omega^2 \begin{bmatrix} M^1 & M^2 & M^4 \\ M^{2t} & M^3 & M^5 \\ M^{4t} & M^{5t} & M^6 \end{bmatrix} + \begin{bmatrix} K^1 & K^2 & K^4 \\ K^{2t} & K^3 & K^5 \\ K^{4t} & K^{5t} & K^6 \end{bmatrix} \right\} \begin{Bmatrix} A \\ B \\ C \end{Bmatrix} = \begin{Bmatrix} F^1 \\ F^3 \\ F^6 \end{Bmatrix} \qquad \text{(E.1)}$$

The matrices or vector with subscript ^{1} are related to the axial displacement u_b, those with the subscript ^{3} are related to the transversal displacement w_b, and those with subscript ^{6}

are related to ϕ_c or w_m. The other subscripts indicate coupling between two types of displacement. The models use the functions described in Appendix D.

E.1 Beam

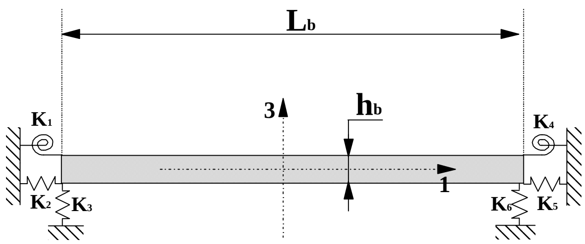

Figure E.1: Beam model with arbitrary boundary conditions

$$M^1_{pq} = \rho_b b h_b \frac{L_b}{8} F4_{pq}(L_b, 0) \tag{E.2}$$

$$M^3_{pq} = \rho_b b h_b^3 \frac{1}{24 L_b} F1_{pq}(L_b, 0) + \rho_b b h_b \frac{L_b}{8} F4_{pq}(L_b, 0) \tag{E.3}$$

$$K^1_{pq} = c^b_{11} b h_b \frac{1}{2 L_b} F1_{pq}(L_b, 0) \tag{E.4}$$

$$K^3_{pq} = c^b_{11} b h_b^3 \frac{1}{6 L_b^3} F2_{pq}(L_b, 0) \tag{E.5}$$

E.2 Springs (boundary conditions, cf. figure E.1)

$$K^1_{pq} = \frac{K_2}{4}\left[\cos(\alpha_p - \beta_p)\cos(\alpha_q - \beta_q) + \cos(\gamma_p - \delta_p)\cos(\gamma_q - \delta_q) - \cos(\alpha_p - \beta_p)\cos(\gamma_q - \delta_q) - \cos(\gamma_p - \delta_p)\cos(\alpha_q - \beta_q)\right]$$

$$+\frac{K_5}{4}\left[\cos(\alpha_p+\beta_p)\cos(\alpha_q+\beta_q)+\cos(\gamma_p+\delta_p)\cos(\gamma_q+\delta_q)\right.$$
$$\left.-\cos(\alpha_p+\beta_p)\cos(\gamma_q+\delta_q)-\cos(\gamma_p+\delta_p)\cos(\alpha_q+\beta_q)\right] \quad \text{(E.6)}$$

$$K^3_{pq}=\frac{K_3}{4}\left[\cos(\alpha_p-\beta_p)\cos(\alpha_q-\beta_q)+\cos(\gamma_p-\delta_p)\cos(\gamma_q-\delta_q)\right.$$
$$\left.-\cos(\alpha_p-\beta_p)\cos(\gamma_q-\delta_q)-\cos(\gamma_p-\delta_p)\cos(\alpha_q-\beta_q)\right]$$
$$+\frac{K_6}{4}\left[\cos(\alpha_p+\beta_p)\cos(\alpha_q+\beta_q)+\cos(\gamma_p+\delta_p)\cos(\gamma_q+\delta_q)\right.$$
$$\left.-\cos(\alpha_p+\beta_p)\cos(\gamma_q+\delta_q)-\cos(\gamma_p+\delta_p)\cos(\alpha_q+\beta_q)\right]$$
$$+\frac{K_1}{L_b^2}\left[\alpha_p\alpha_q\sin(\alpha_p-\beta_p)\sin(\alpha_q-\beta_q)+\gamma_p\gamma_q\sin(\gamma_p-\delta_p)\sin(\gamma_q-\delta_q)\right.$$
$$\left.-\alpha_p\gamma_q\sin(\alpha_p-\beta_p)\sin(\gamma_q-\delta_q)-\gamma_p\alpha_q\sin(\gamma_p-\delta_p)\sin(\alpha_q-\beta_q)\right]$$
$$+\frac{K_4}{L_b^2}\left[\alpha_p\alpha_q\sin(\alpha_p+\beta_p)\sin(\alpha_q+\beta_q)+\gamma_p\gamma_q\sin(\gamma_p+\delta_p)\sin(\gamma_q+\delta_q)\right.$$
$$\left.-\alpha_p\gamma_q\sin(\alpha_p+\beta_p)\sin(\gamma_q+\delta_q)-\gamma_p\alpha_q\sin(\gamma_p+\delta_p)\sin(\alpha_q+\beta_q)\right] \quad \text{(E.7)}$$

E.3 Piezoelectric layer

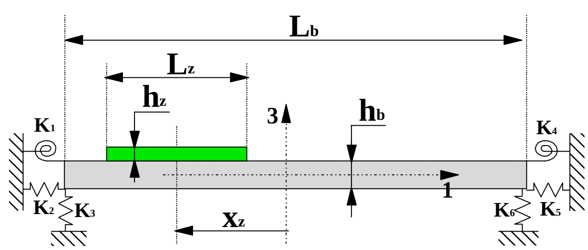

Figure E.2: Piezoelectric layer on top of a beam

$$M^1_{pq}=\rho_z b h_z\frac{L_b}{8}F4_{pq}(L_z,x_z) \quad \text{(E.8)}$$

$$M^3_{pq}=\rho_z b\left[\frac{h_z^3}{12}+\left(\frac{h_b+h_z}{2}\right)^2 h_z\right]\frac{1}{2L_b}F1_{pq}(L_z,x_z)$$
$$+\rho_x b h_x\frac{L_b}{8}F4_{pq}(L_x,x_z) \quad \text{(E.9)}$$

$$M_{pq}^{2} = -\rho_z b(h_b + h_z)h_z \frac{1}{8} F6_{pq}(L_z, x_z) \tag{E.10}$$

$$K_{pq}^{1} = c_{11}^{z} b h_z \frac{1}{2L_b} F1_{pq}(L_z, x_z) \tag{E.11}$$

$$K_{pq}^{3} = c_{11}^{z} b \left[\frac{h_z^3}{12} + \left(\frac{h_b + h_z}{2} \right)^2 h_z \right] \frac{2}{L_b^3} F2_{pq}(L_z, x_z) \tag{E.12}$$

$$K_{pq}^{2} = -c_{11}^{z} b(h_b + h_z)h_z \frac{1}{2L_b^2} F3_{pq}(L_z, x_z) \tag{E.13}$$

$$F_{p}^{1} = e_{31}^{z} b \Delta\varphi_z G2_p(L_z, x_z) \tag{E.14}$$

$$F_{p}^{3} = -e_{31}^{z} b(h_b + h_z)\Delta\varphi_z \frac{1}{L_b} G3_p(L_z, x_z) \tag{E.15}$$

E.4 Constrained layer damping

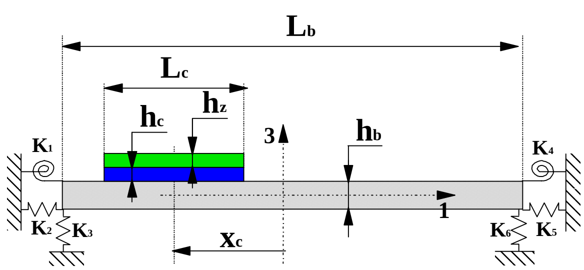

Figure E.3: Constrained layer damping on top of a beam

$$M_{pq}^{1} = \rho_c b h_c \frac{L_b}{8} F4_{pq}(L_c, x_c) \tag{E.16}$$

$$M_{pq}^{3} = \rho_c b \left[\frac{h_c^3}{12} + \left(\frac{h_b + h_c}{2} \right)^2 h_c \right] \frac{1}{2L_b} F1_{pq}(L_c, x_c)$$

$$+ \rho_c b h_c \frac{L_b}{8} F4_{pq}(L_c, x_c) \tag{E.17}$$

$$M^{6}_{pq} = \rho_c b h_c^3 \frac{L_b}{24} F4_{pq}\left(L_c, x_c\right) \tag{E.18}$$

$$M^{2}_{pq} = -\rho_c b\left(h_b + h_c\right) h_c \frac{1}{8} F6_{pq}\left(L_c, x_c\right) \tag{E.19}$$

$$M^{4}_{pq} = -\rho_c b h_c^2 \frac{L_b}{16} F4_{pq}\left(L_c, x_c\right) \tag{E.20}$$

$$M^{5}_{pq} = \rho_c b h_c^2 \left(\frac{h_b}{4} + \frac{h_c}{3}\right)\frac{1}{4} F6_{qp}\left(L_c, x_c\right) \tag{E.21}$$

$$K^{1}_{pq} = c^c_{11} b h_c \frac{1}{2L_b} F1_{pq}\left(L_c, x_c\right) \tag{E.22}$$

$$K^{3}_{pq} = c^c_{11} b\left[\frac{h_c^3}{12} + \left(\frac{h_b + h_c}{2}\right)^2 h_c\right]\frac{2}{L_b^3} F2_{pq}\left(L_c, x_c\right) \tag{E.23}$$

$$K^{6}_{pq} = c^c_{11} b h_c^3 \frac{1}{6L_b} F1_{pq}\left(L_c, x_c\right)$$

$$+ c^c_{55} b h_c \frac{L_b}{8} F4_{pq}\left(L_c, x_c\right) \tag{E.24}$$

$$K^{2}_{pq} = -c^c_{11} b\left(h_b + h_c\right) h_c \frac{1}{2L_b^2} F3_{pq}\left(L_c, x_c\right) \tag{E.25}$$

$$K^{4}_{pq} = -c^c_{11} b h_c^2 \frac{1}{4L_b} F1_{pq}\left(L_c, x_c\right) \tag{E.26}$$

$$K^{5}_{pq} = c^c_{11} b h_c^2 \left(\frac{h_b}{4} + \frac{h_c}{3}\right)\frac{1}{L_b^2} F3_{pq}\left(L_c, x_c\right) \tag{E.27}$$

E.5 Point vibration absorber (TVA)

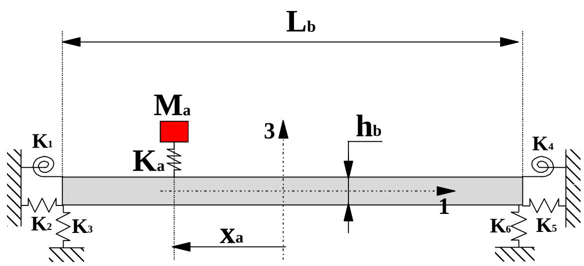

Figure E.4: Point vibration absorber on top of a beam

The equation of motion for a beam with a point TVA is presented equation (E.28).

$$-\omega^2[M]+[K]+D(\omega)[G]=\{F\}$$

$$with \quad D(\omega)=\frac{\alpha^2}{1-\alpha^2+j\frac{\alpha}{Q}}, \quad \alpha=\frac{\omega}{\omega_r}, and \quad Q=\frac{M\omega_r}{C} \qquad \text{(E.28)}$$

$$G_{pq}^3=-k\left[cos\left(\frac{2x_a}{L_b}\alpha_p-\beta_p\right)cos\left(\frac{2x_a}{L_b}\alpha_q-\beta_q\right)+cos\left(\frac{2x_a}{L_b}\gamma_p-\delta_p\right)cos\left(\frac{2x_a}{L_b}\gamma_q-\delta_q\right)\right.$$
$$\left.-cos\left(\frac{2x_a}{L_b}\alpha_p-\beta_p\right)cos\left(\frac{2x_a}{L_b}\gamma_q-\delta_q\right)-cos\left(\frac{2x_a}{L_b}\gamma_p-\delta_p\right)cos\left(\frac{2x_a}{L_b}\alpha_q-\beta_q\right)\right]$$

(E.29)

E.6 Distributed active vibration absorber (DAVA) with uniform mass layer

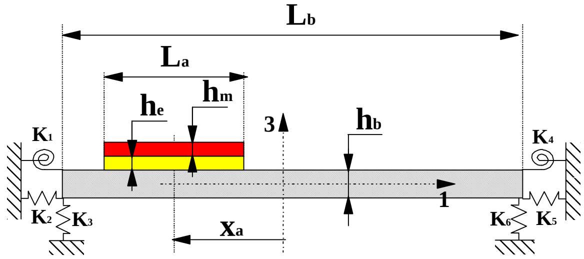

Figure E.5: DAVA with uniform mass layer on top of a beam

$$M_{pq}^{1} = \rho_m b h_m \frac{L_b}{8} F4_{pq}(L_a, x_a) + \rho_e b h_e \frac{L_b}{8} F4_{pq}(L_a, x_a) \tag{E.30}$$

$$M_{pq}^{3} = \rho_m b \left[\frac{h_m^3}{12} + \left(\frac{h_b + 2h_e + h_m}{2} \right)^2 h_m \right] \frac{1}{2L_b} F1_{pq}(L_a, x_a) + \rho_e b \left[\frac{h_e^3}{12} + \left(\frac{h_b + h_e}{2} \right)^2 h_e \right] \frac{1}{2L_b} F1_{pq}(L_a, x_a) + \frac{5}{6} \rho_e b h_e \frac{L_b}{8} F4_{pq}(L_a, x_a) \tag{E.31}$$

$$M_{pq}^{6} = \rho_m b h_m \frac{L_b}{8} F4_{pq}(L_a, x_a) + \frac{1}{3} \rho_e b h_e \frac{L_b}{8} F4_{pq}(L_a, x_a) \tag{E.32}$$

$$M_{pq}^{2} = -\rho_m b (h_b + 2h_e + h_m) h_m \frac{1}{8} F6_{pq}(L_a, x_a) - \rho_e b (h_b + h_e) h_e \frac{1}{8} F6_{pq}(L_a, x_a) \tag{E.33}$$

$$M_{pq}^{5} = -\frac{1}{12} \rho_e b h_e \frac{L_b}{8} F4_{pq}(L_a, x_a) \tag{E.34}$$

$$K_{pq}^{1} = c_{11}^{e} b h_e \frac{1}{2L_b} F1_{pq}(L_a, x_a) \tag{E.35}$$

$$K_{pq}^{3} = c_{11}^{e} b \left[\frac{h_e^3}{12} + \left(\frac{h_b + h_e}{2} \right)^2 h_e \right] \frac{2}{L_b^3} F2_{pq}(L_a, x_a) + c_{33}^{e} b \frac{1}{h_e} \frac{L_b}{8} F4_{pq}(L_a, x_a) + c_{13}^{e} b (h_b + h_e) \frac{1}{4L_b} F5_{qp}(L_a, x_a) \tag{E.36}$$

$$K_{pq}^{6} = c_{33}^{e} b \frac{1}{h_e} \frac{L_b}{8} F4_{pq}(L_a, x_a) \tag{E.37}$$

$$K_{pq}^{2} = -c_{11}^{e} b (h_b + h_e) h_e \frac{1}{2L_b^2} F3_{pq}(L_a, x_a) - c_{13}^{e} b \frac{1}{8} F6_{qp}(L_a, x_a) \tag{E.38}$$

$$K_{pq}^{4} = -c_{13}^{e} b \frac{1}{8} F6_{qp}(L_a, x_a) \tag{E.39}$$

$$K_{pq}^{5} = -c_{33}^{e} b \frac{1}{h_e} \frac{L_b}{8} F4_{pq}(L_a, x_a) - c_{13}^{e} b (h_b + h_e) \frac{1}{8L_b} F5_{qp}(L_a, x_a) \tag{E.40}$$

$$F_{p}^{1} = e_{31}^{e} b \Delta\varphi_e G2_p(L_a, x_a) \tag{E.41}$$

$$F_{p}^{3} = -e_{31}^{e} b (h_b + h_e) \Delta\varphi_e G3_p(L_a, x_a) - e_{33}^{e} b \frac{1}{h_e} \frac{L_b}{2} \Delta\varphi_e G1_p(L_a, x_a) \tag{E.42}$$

$$F_{p}^{6} = e_{33}^{e} b \frac{1}{h_e} \frac{L_b}{2} \Delta\varphi_e G1_p(L_a, x_a) \tag{E.43}$$

E.7 Distributed active vibration absorber (DAVA) with varying mass distribution

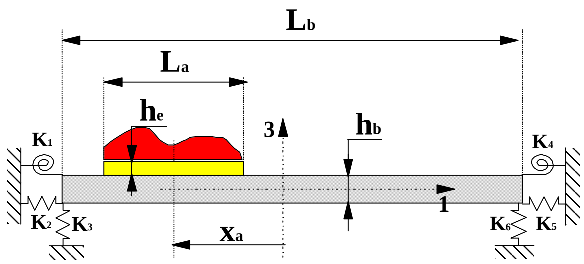

Figure E.6: DAVA with varying mass distribution on top of a beam

Only the mass terms of the mass layer are changed. A third dimension as been introduced by the mass distribution. Each mass element is in fact the result of the sum according to this third dimension.

$$M_{pq}^{1} = \sum_{s=1}^{S} d_s M_{pqs}^{1}$$

Each of the coefficients d_s is known and is provided by the design of the absorber (cf. equation (3.33)

$$M_{pqs}^{1} = \rho_m b h_m \frac{L_b}{32} E4_{pqs}(L_a, x_a) \tag{E.44}$$

$$M_{pqs}^{3} = \rho_m b h_m \left[\frac{h_m{}^{3}}{12} + \left(\frac{h_b + 2h_e + h_m}{2} \right)^{2} h_m \right] \frac{1}{8L_b} E1_{pqs}(L_a, x_a) \tag{E.45}$$

$$M_{pqs}^{6} = \rho_m b h_m \frac{L_b}{32} E4_{pqs}(L_a, x_a) \tag{E.46}$$

$$M_{pq}^{2} = -\frac{1}{2} \rho_m b (h_b + 2h_e + h_m) h_m \frac{1}{16} E6_{pqs}(L_a, x_a) \tag{E.47}$$