SMALL OSCILLATIONS OF A BEAM-COLUMN WITH FINITE ELECTRICAL CONDUCTIVITY IN A CONSTANT TRANSVERSE MAGNETIC FIELD

by

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Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

MASTER OF SCIENCE

in

ENGINEERING MECHANICS

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December 1966

Blacksburg, Virginia

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LIST OF SYMBOLS

$\vec{U}$	deflection of beam-column
N	normal force on beam-column
M	bending moment in beam-column
V	shear force in beam-column
$\vec{\vec{a}}$	acceleration of differential element of beam-column
$\vec{\vec{S}}$	stress tensor
$\vec{f}$	electromagnetic body force
$\vec{E}$	electric field
$\vec{D}$	electric displacement
$\vec{H}$	magnetic field
$\vec{B}$	magnetic induction
$\vec{J}$	current density
ρ_e	free charge density
$\vec{J}_s$	surface current density
ϵ	electric permeability
k	magnetic permeability
σ	electrical conductivity
χ	magnetic susceptibility
c	speed of light in vacuum

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I. INTRODUCTION

During the past decade increasing interest has been shown in the effects of electromagnetic fields on the mechanics of solids. Although investigation in this field has practical applications in the fields of optics and accoustics, the first attempts to study this phenemenon were made by geophysicists who wished to understand the effect of the earth's electromagnetic field on seismic waves. Since then many authors including Banos, Chadwick, Kaliski, Petykiewicz, Dunkin, and Eringen have solved problems concerned with infinite and semi-infinite spaces subjected to electric and magnetic fields. Contributions to the theory of this subject have been made by Dunkin, Eringen, Toupin, and others. For a complete list of references the interested reader should consult Suhubi [1].

Other shapes than the infinite and semi-infinite space have been studied, but not many problems have been solved in this area. Dunkin and Eringen considered the first problem of this type. They investigated the vibration of an infinite plate in a strong uniform magnetic field. Suhubi [1] studied small torsional oscillations of an infinite circular cylinder with finite conductivity in a constant axial magnetic field.

The following study is concerned with small oscillations of a beam-column with finite electrical conductivity in a constant transverse magnetic field. It is based on a linearized theory of electromagnetic-elastic interactions and simplified by using the elementary theory of bars rather than the more exact elasticity theory.

In what follows three problems are considered. First the small oscillations of a conducting column are examined. By equating the end load to zero the frequency equation for a conducting beam is obtained. The roots of this equation are examined by an approximate method. By retaining the end load the elastic stability of a conducting column is studied. Second a conducting beam-column is investigated. Both the frequencies of vibration and the elastic stability of this beam-column are examined. Finally a problem in the forced vibration of a conducting column is considered. Primary interest is focused on the elastic stability of this structure.

In all of the afore mentioned problems two questions are of primary importance. The first is the nature of the effect of the magnetic field. The second is the size of magnetic field required to make these effects noticeable.

II. DERIVATION OF EQUATIONS

It is desired to investigate various aspects of the behavior of an electrically conducting beam-column as it performs small oscillations in a transverse magnetic field $\vec{H}_0$ oriented as shown in figure 1. The bar has density ρ , modulus of elasticity E , moment of inertia I , and length L . It is subjected to mechanical loads $q(x)$ and P as shown. First the equations of motion will be derived for the beam-column. This will be done neglecting the effect of shear on the deflection of the bar. Also neglected will be the rotary inertia and axial acceleration of each element of the structure. Consider the free-body diagram of an element of the beam-column shown in figure 2.

$$\sum F_x = -N + f_x A\,dx + N + N_{,x}\,dx = 0$$

$$N_{,x} + A f_x = 0 \tag{2.1}$$

$$\sum F_y = -V + f_y A\,dx + V + V_{,x}\,dx + q\,dx = \rho A\,dx\,U_{,tt}$$

$$q + V_{,x} + A f_y = \rho A\,U_{,tt} \tag{2.2}$$

$$\sum \overset{\curvearrowright +}{M_p} = M + V\,dx - q\,dx\,\frac{dx}{2} - f_y A\,dx\,\frac{dx}{2}$$

$$-(M + M_{,x}\,dx) - (N + N_{,x}\,dx)U_{,x}\,dx = 0$$

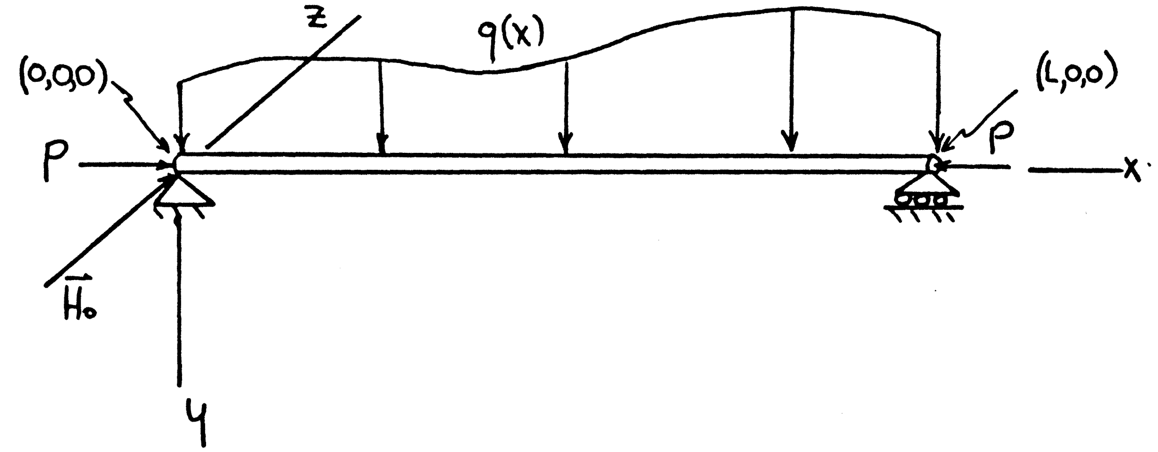

Figure 1.- Beam-column in magnetic field.

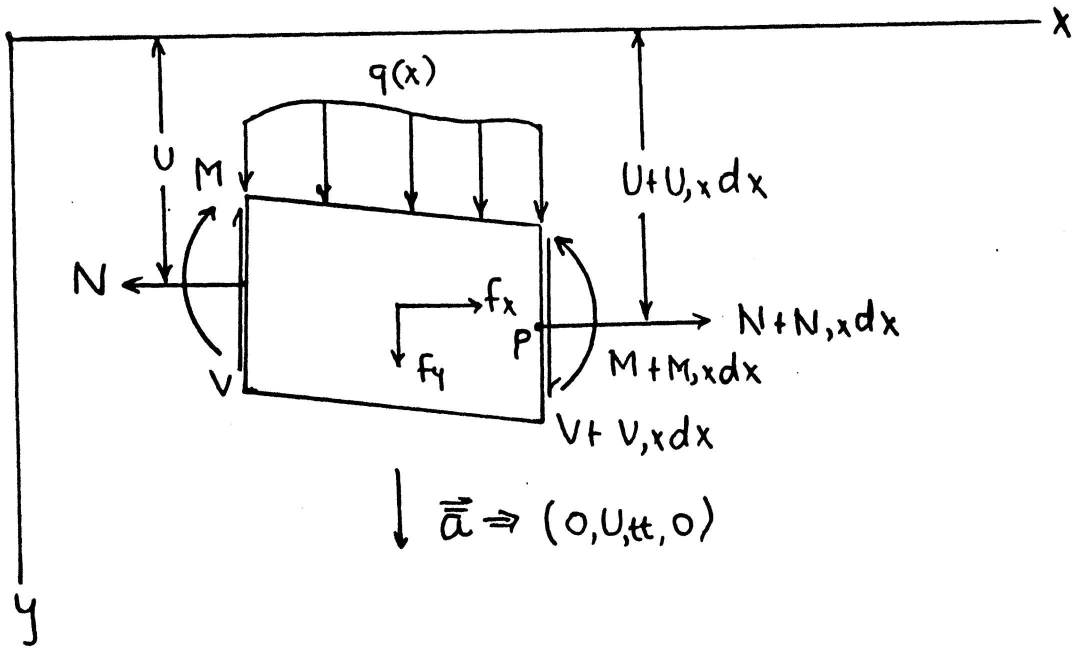

Figure 2.- Freebody diagram of differential element of beam-column.

Neglecting second order terms the following equation is obtained.

$$M_{,x} + N U_{,x} - V = 0 \tag{2.3}$$

From the elementary theory of beam deflections

$$EI U_{,xx} = -M \, . \tag{2.4}$$

Now equations (2.2) and (2.3) are substituted into equation (2.4) as follows.

$$EI U_{,xxx} = -M_{,x} = N U_{,x} - V$$

Thus

$$EI U_{,xxxx} - (N U_{,x})_{,x} + \rho A U_{,tt} = A f_y + q \, . \tag{2.5}$$

Equations (2.1) and (2.5) are the governing equations for the deflection of the bar and the normal force.

The electromagnetic force per unit volume is given by

$$\vec{f} = \vec{J} \times \vec{B} + \rho_e \vec{E} \, . \tag{2.6}$$

To determine the quantities in the above expression Maxwell's equations are necessary. They are

$$\begin{aligned} &\vec{\nabla} \times \vec{E} + \vec{B}_{,t} = 0 \qquad && \vec{\nabla} \cdot \vec{D} = \rho_e \\ &\vec{\nabla} \times \vec{H} - \vec{D}_{,t} = \vec{J} \qquad && \vec{\nabla} \cdot \vec{B} = 0 \, . \end{aligned} \tag{2.7}$$

Constituative equations are also required. For a homogeneous beam they are

$$\begin{aligned} \vec{D} &= \epsilon \vec{E} + \alpha U_{,t} \vec{I}_y \times \vec{H} \\ \vec{B} &= k \vec{H} - \alpha U_{,t} \vec{I}_y \times \vec{E} \end{aligned} \tag{2.8}$$

$$\vec{J} = \rho_e U_{,t} \vec{I}_y + \sigma(\vec{E} + U_{,t} \vec{I}_y \times \vec{B}).$$

In equations (2.8)

$$\alpha = k\epsilon - \frac{1}{c^2} \tag{2.9}$$

and $\vec{I}_y$ is a unit vector directed in the positive y direction.

Consider a state of equilibrium of the beam-column characterized by the deflection U_o and the normal force N_o combined with a steady state electromagnetic field in turn characterized by the field vectors $\vec{E}_o$, $\vec{H}_o$, $\vec{D}_o$, $\vec{B}_o$, $\vec{J}_o$. The body force is denoted by $\vec{f}_o$. The governing equations are

$$EI\,U_{o,xxxx} - (N_o U_{o,x})_{,x} = A f_{oy} + q_o \tag{2.10}$$

$$N_o + A f_{ox} = 0 \tag{2.11}$$

$$\begin{aligned} \vec{\nabla} \times \vec{E}_o &= 0 \qquad & \vec{\nabla} \cdot \vec{D}_o &= 0 \\ \vec{\nabla} \times \vec{H}_o &= \vec{J}_o \qquad & \vec{\nabla} \cdot \vec{B}_o &= 0 \end{aligned} \tag{2.12}$$

$$\begin{aligned} \vec{D}_o &= \epsilon \vec{E}_o \\ \vec{B}_o &= k \vec{H}_o \\ \vec{J}_o &= \sigma \vec{E}_o \end{aligned} \tag{2.13}$$

$$\vec{f}_o = \vec{J}_o \times \vec{B}_o . \tag{2.14}$$

In the above equations $\rho_{oe} = 0$. Since the material has finite conductivity there can be no free charge except on the surface.

Now use equations (2.13) to put equations (2.12) and (2.14) in terms of $\vec{E}_o$ and $\vec{H}_o$. The results are

$$\vec{f}_o = \sigma K \vec{E}_o \times \vec{H}_o \tag{2.15}$$

$$\begin{aligned} &\vec{\nabla} \times \vec{E}_o = 0 \\ &\vec{\nabla} \times \vec{H}_o = \sigma \vec{E}_o \\ &\vec{\nabla} \cdot \vec{E}_o = 0 \\ &\vec{\nabla} \cdot \vec{H}_o = 0 \, . \end{aligned} \tag{2.16}$$

The governing equations are (2.10), (2.11), (2.15), and (2.16).

Now consider a small perturbation which is characterized by a small displacement of the beam $U(x,t)$. All variables will be perturbed as follows.

$$\begin{aligned} &U_1 = U_o + U && \vec{D}_1 = \vec{D}_o + \vec{d} \\ &N_1 = N_o + N && \vec{H}_1 = \vec{H}_o + \vec{h} \\ &\vec{f}_1 = \vec{f}_o + \vec{f} && \vec{J}_1 = \vec{J}_o + \vec{j} \\ &\vec{E}_1 = \vec{E}_o + \vec{e} && q_1 = q_o + q \\ &\vec{B}_1 = \vec{B}_o + \vec{b} \end{aligned} \tag{2.17}$$

Substitute these quantities into equations (2.1), (2.5), (2.6), (2.7), and (2.8). It is assumed that the products of perturbed quantities and their derivatives are of a higher order than the perturbed quantities themselves. After simplifying, using equations (2.10), (2.11), (2.15), and (2.16), and neglecting higher order terms the results are

$$EIU_{,xxxx} - (N_0 U_{,x})_{,x} - (NU_{0,x})_{,x} + \rho A U_{,tt} = A f_y + q \tag{2.18}$$

$$N_{,x} + A f_x = 0 \tag{2.19}$$

$$\vec{f} = \vec{J}_0 \times \vec{b} + \vec{J} \times \vec{B}_0 + \rho_e \vec{E}_0 \tag{2.20}$$

$$\begin{aligned} &\vec{\nabla} \times \vec{e} + \vec{b}_{,t} = 0 & &\vec{\nabla} \cdot \vec{d} = \rho_e \\ &\vec{\nabla} \times \vec{h} - \vec{d}_{,t} = \vec{J} & &\vec{\nabla} \cdot \vec{b} = 0 \end{aligned} \tag{2.21}$$

$$\begin{aligned} &\vec{d} = \epsilon \vec{e} + \alpha U_{,t} \vec{I}_y \times \vec{H}_0 = 0 \\ &\vec{b} = k \vec{h} - \alpha U_{,t} \vec{I}_y \times \vec{E}_0 = 0 \\ &\vec{J} = \sigma (\vec{e} + U_{,t} \vec{I}_y \times \vec{B}_0) \end{aligned} \tag{2.22}$$

or $\vec{J} = \sigma (\vec{e} + k U_{,t} \vec{I}_y \times \vec{H}_0)$.

Use equations (2.22) to put equations (2.20) and (2.21) in terms of $\vec{E}_0$, $\vec{H}_0$, $\vec{e}$ and $\vec{h}$. After simplification the following results are obtained.

$$\vec{f} = \sigma \vec{E}_0 \times (k\vec{h} - \alpha U_{,t} \vec{I}_y \times \vec{E}_0) + \sigma (\vec{e} + k U_{,t} \vec{I}_y \times \vec{H}_0) \times k \vec{H}_0 + \rho_e \vec{E}_0 \tag{2.23}$$

$$\begin{aligned} &\vec{\nabla} \times \vec{e} + (k\vec{h} - \alpha U_{,t} \vec{I}_y \times \vec{E}_0)_{,t} = 0 \\ &\vec{\nabla} \times \vec{h} - (\epsilon \vec{e} + \alpha U_{,t} \vec{I}_y \times \vec{H}_0)_{,t} = \sigma (\vec{e} + k U_{,t} \vec{I}_y \times \vec{H}_0) \\ &\vec{\nabla} \cdot (\epsilon \vec{e} + \alpha U_{,t} \vec{I}_y \times \vec{H}_0) = \rho_e \end{aligned} \tag{2.24}$$

$$\vec{\nabla}\cdot(k\vec{h} - \alpha U_{,t} I_y \times \vec{E}_o) = 0$$

The governing equations for the dynamic problem are (2.18), (2.19), (2.23), and (2.24). These equations hold for any magnetic field as long as the oscillations are confined to the x, y plane. For the specific problem under consideration

$$\vec{H}_o \Rightarrow (0, 0, H_o)$$

where H_o is constant. Since the one dimensional theory of bars was used to derive equations (2.18) and (2.19) it will be assumed that all perturbed electromagnetic quantities are functions of only the variables x and t. Since H_o is constant

$$\vec{\nabla} \times \vec{H}_o = 0.$$

By equations (2.16)

$$\vec{E}_o = 0$$

Thus breaking the vector equations (2.23) and (2.24) into components yields

$$f_x = \sigma k e_y H_o \tag{2.25}$$

$$f_y = -\sigma k e_x H_o - \sigma k^2 H_o^2 U_{,t} \tag{2.26}$$

$$f_z = 0 \tag{2.27}$$

$$k h_{x,t} = 0 \tag{2.28}$$

$$-e_{z,x} + K h_{y,t} = 0 \tag{2.29}$$

$$e_{y,x} + K h_{z,t} = 0 \tag{2.30}$$

$$-\epsilon e_{x,t} - \alpha H_o U_{,tt} = \sigma e_x + \sigma K H_o U_{,t} \tag{2.31}$$

$$-h_{z,x} - \epsilon e_{y,t} = \sigma e_y \tag{2.32}$$

$$h_{y,x} - \epsilon e_{z,t} = \sigma e_z \tag{2.33}$$

$$\epsilon e_{x,x} + \alpha H_o U_{,xt} = \rho_e \tag{2.34}$$

$$K h_{x,x} = 0 \; . \tag{2.35}$$

Substituting equations (2.25) and (2.26) into equations (2.18) and (2.19) yields

$$EI U_{,xxxx} - (N_o U_{,x})_{,x} - (N U_{o,x})_{,x} + \rho A U_{,tt} = -A\sigma K H_o e_x - A\sigma K^2 H_o^2 U_{,t} + q \tag{2.36}$$

$$N_{,x} + A\sigma K H_o e_y = 0. \tag{2.37}$$

From equations (2.36)

$$e_x = -\frac{1}{\sigma K A H_o}\left\{EI U_{,xxxx} - (N_o U_{,x})_{,x} - (N U_{o,x})_{,x} + \rho A U_{,tt} + \sigma K A H_o^2 U_{,t} - q\right\}.$$

Substituting the above result into equation (2.31) it is found that

$$\epsilon\left\{EI U_{,xxxx} - (N_o U_{,x})_{,x} - (N U_{o,x})_{,x} + \rho A U_{,tt} + \sigma K A H_o^2 U_{,t} - q\right\}_{,t}$$
$$+\sigma\left\{EI U_{,xxxx} - (N_o U_{,x})_{,x} - (N U_{o,x})_{,x} + \rho A U_{,tt} + \sigma K^2 A H_o^2 U_{,t} - q\right\}$$
$$-\sigma K \alpha A H_o^2 U_{,tt} - \sigma^2 K^2 A H_o^2 U_{,t} = 0.$$

Expanding out the above equation gives

$$\epsilon EIU_{,xxxxt} - \epsilon(N_oU_{,x})_{,xt} + \sigma EIU_{,xxxx} - \sigma(N_oU_{,x})_{,xt} + \epsilon\rho AU_{,ttt}$$

$$+\sigma A\left\{\rho + kH_o^2(\epsilon k - \alpha)\right\}U_{,tt} = \epsilon(NU_{o,x})_{,xt} + \sigma(NU_{o,x})_{,x}$$

$$+\ \epsilon q_{,t} + \sigma q. \tag{2.38}$$

From equation (2.9)

$$\epsilon k - \alpha = \frac{1}{c^2}.$$

Using this information equation (2.38) becomes

$$\epsilon EIU_{,xxxxt} - \epsilon(N_oU_{,x})_{,xt} + \sigma EIU_{,xxxx} - \sigma(N_oU_{,x})_{,x} + \epsilon\rho AU_{,ttt}$$

$$+\sigma A\left(\rho + \frac{kH_o^2}{c^2}\right)U_{,tt} = \epsilon(NU_{o,x})_{,xt} + \sigma(NU_{o,x})_{,x} + \epsilon q_{,t} + \sigma q. \tag{2.39}$$

Now an additional equation containing the variable N is required.

Differentiating equation (2.30) once with respect to x yields

$$h_{z,xt} = -\frac{1}{k}\,e_{y,xx}. \tag{2.40}$$

Differentiating equation (2.32) once with respect to t yields

$$h_{z,xt} = -\epsilon e_{y,tt} - \sigma e_{y,t}. \tag{2.41}$$

Equating the right sides of equations (2.40) and (2.41) one finds that

$$k\epsilon e_{y,tt} + k\sigma e_{y,t} - e_{y,xx} = 0. \tag{2.42}$$

From equation (2.37)

$$e_y = -\frac{N_{,x}}{\sigma kAH_o}. \tag{2.43}$$

Substituting equation (2.43) into equation (2.42) it is found that

$$N_{,xxx} - \epsilon kN_{,xtt} - \sigma kN_{,xt} = 0. \tag{2.44}$$

Equations (2.39) and (2.44) are two equations containing the four variables U, N, U_o and N_o. Thus two more equations are required. They are found by substituting $\vec{E}_o = 0$ into equation (2.15) and using this result in equations (2.10) and (2.11). This process yields

$$EI\,U_{o,xxxx} - (N_o U_{o,x})_{,x} = q_o \tag{2.45}$$

$$N_{o,x} = 0 \; . \tag{2.46}$$

Equations (2.39), (2.44), (2.45), and (2.46) are the complete set of governing equations necessary to find the deflection of the beam-column.

III. BOUNDARY CONDITIONS

To derive the boundary conditions necessary to solve equations (2.39), (2.44), (2.45), and (2.46) it will be assumed that the beam-column is simply supported and free from externally applied end moments. The mechanical boundary conditions for the static problem are

$$U_0(0,t) = U_0(L,t) = U_{0,xx}(0,t) = U_{0,xx}(L,t) = 0 . \qquad (3.1)$$

It will be assumed that the electromagnetic boundary conditions at the left end of the beam can be adjusted at will. The right end of the beam will be thought of as the boundary between the interior region (beam) and the exterior region (surrounding environment). No attempt is made to satisfy the electromagnetic boundary conditions on the sides of the bar. This would be inconsistent with the elementary theory of beams which does not attempt to satisfy mechanical boundary conditions on the sides of the bar. The cross section of the bar at $x = L$ is now the boundary between two regions with different electromagnetic properties. The fields on either side of this boundary are related by the jump conditions of electromagnetic theory. The following notation will be used to denote a jump condition.

$$[\vec{A}] = \vec{A}^E - \vec{A}^I$$

where $\vec{A}^E$ is the value of $\vec{A}$ on the exterior side of the boundary surface and $\vec{A}^I$ is the value of $\vec{A}$ on the interior side of the boundary surface. Two jump conditions, as given by Stratton [2], are appropriate to this problem. One is

$$\left[\overline{\overline{S}} + \vec{E}\vec{D} + \vec{H}\vec{B} - \tfrac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}\right)\overline{\overline{\delta}} + \left(\vec{D}\times\vec{B}\right)U_{,t}\,\vec{1}_y\right]\cdot\vec{n} = 0 . \qquad (3.2)$$

where $\vec{n}$ is a normal unit vector pointing outward from the discontinuity surface. The other is

$$[\vec{H}]_t = \vec{J}_s \tag{3.3}$$

where the subscript on the left indicates that only the component of $\vec{H}$ tangent to the discontinuity surface is considered in the above equation.

Equation (3.2) will now be applied to the static problem. For the surface in question

$$\vec{n} \Rightarrow (1,0,0)$$

and

$$\vec{H}_o \Rightarrow (0,0,H_o).$$

The desired component of equation (3.2) is

$$S_{oxx} - \frac{1}{2} K H_o^2 = -\frac{1}{2} K_o H_o^2 - P/A$$

$$S_{oxx} = \frac{1}{2}(K-K_o) H_o^2 - P/A.$$

The first term on the right hand side is the Maxwell stress on the boundary. Introducing $\chi = K/K_o - 1$ the above equation becomes

$$S_{oxx} = \frac{1}{2} K_o \chi H_o^2 - P/A. \tag{3.4}$$

To convert the Maxwell stress to force, equation (3.4) is multiplied by A since S_{oxx} is considered to be constant across the cross section. The final boundary condition is thus

$$N_o(L,t) = -P + \frac{1}{2} K_o \chi A H_o^2. \tag{3.5}$$

The boundary conditions for the dynamic problem must also be found. The fact that the beam-column is simply supported combined with equations (3.1) yields

$$U(0,t) = U_{,xx}(0,t) = U(L,t) = U_{,xx}(L,t) = 0. \tag{3.6}$$

Now equation (3.2) will be applied to the dynamic problem.

$$\vec{h} \Rightarrow (h_x, h_y, h_z)$$

Using the above information together with equation (3.3) the required component of equation (3.2) becomes

$$S_{xx} - KH_o h_z^I = -K_o H_o h_z^E. \tag{3.7}$$

With the assumption of zero surface currents equation (3.3) becomes

$$[\vec{H}]_t = 0.$$

Thus

$$[\vec{H}_o + \vec{h}]_t = 0. \tag{3.8}$$

In the z direction the component of equation (3.8) is

$$H_o + h_z^I = H_o + h_z^E$$

or

$$h_z^I = h_z^E. \tag{3.9}$$

Substituting equation (3.9) into equation (3.7) yields

$$S_{xx} = (K - K_o) H_o h_z$$

or

$$N = (K - K_o) A H_o h_z. \tag{3.10}$$

Differentiating equation (3.10) once with respect to t yields

$$N_{,t} = (K - K_o) A H_o h_{z,t}. \tag{3.11}$$

From equations (2.30) and (2.43)

$$h_{z,t} = -\frac{1}{K} e_{y,x} = \frac{N_{,xx}}{\sigma K^2 A H_o}. \tag{3.11}$$

Substituting equation (3.12) into equation (3.11) gives

$$N_{,t} = (K - K_0)\frac{AH_0}{\sigma K^2 AH_0} N_{,xx} . \tag{3.13}$$

Recalling the definition of χ the boundary condition finally becomes

$$\sigma K^2 N_{,t}(L,t) - K_0 \chi N_{,xx}(L,t) = 0 . \tag{3.14}$$

The boundary conditions are now contained in equations (3.1), (3.5), (3.6), and (3.14).

IV. OSCILLATION OF A COLUMN

For a column one takes

$$q_0 = q = 0$$

in the governing equations.

Thus equations (2.39), (2.44), (2.45), and (2.46) become

$$\epsilon EI U_{,xxxxt} - \epsilon (N_0 U_{,x})_{,xt} + \sigma EI U_{,xxxx} - \sigma (N_0 U_{,x})_{,x} + \epsilon \rho A U_{,ttt}$$
$$+ \sigma A \left(\rho + \frac{kH_0^2}{C^2}\right) U_{,tt} = \epsilon (N U_{0,x})_{,xt} + \sigma (N U_{0,x})_{,x} \tag{4.1}$$

$$N_{,xxx} - \epsilon K N_{,xtt} - \sigma K N_{,xt} = 0 \tag{4.2}$$

$$EI U_{0,xxxx} - (N_0 U_{0,x})_{,x} = 0 \tag{4.3}$$

$$N_{0,x} = 0 . \tag{4.4}$$

The boundary conditions are given by equations (3.1), (3.5), (3.6), and (3.14).

The solution to equation (4.4) combined with boundary condition (3.5) is

$$N_0 = -P + \frac{1}{2} K_0 \chi A H_0^2 . \tag{4.5}$$

From the elementary theory of elastic stability it is known that if N_0 is less than Euler's load the column can be in equilibrium only in the undeflected position. Thus the solution to equation (4.3) combined with boundary conditions (3.1) is

$$U_0 = 0. \tag{4.6}$$

Combining equations (4.1) and (4.6) yields

$$\epsilon EIU_{,xxxxt} - \epsilon N_0 U_{,xxt} + \sigma EIU_{,xxxx} - \sigma N_0 U_{,xx} + \epsilon\rho AU_{,ttt}$$

$$+\sigma A\left(\rho + \frac{KH_0^2}{C^2}\right)U_{,tt} = 0. \tag{4.7}$$

To solve this equation the following series solution is assumed.

$$U = \sum_{n=1}^{\infty} \bar{U}_n \sin\frac{n\pi x}{L} e^{\omega t} \tag{4.8}$$

Substituting solution (4.8) into governing equation (4.7) gives

$$\frac{\epsilon EI\omega n^4\pi^4}{L^4} + \frac{\epsilon N_0\omega n^2\pi^2}{L^2} + \frac{\sigma EIn^4\pi^4}{L^4} + \frac{\sigma N_0 n^2\pi^2}{L^2}$$

$$+\omega^3\epsilon\rho A + \omega^2\sigma A\left(\rho + \frac{KH_0^2}{C^2}\right) = 0.$$

Rearranging terms this becomes

$$\omega^3 + \left(\rho + \frac{KH_0^2}{C^2}\right)\frac{\sigma}{\epsilon\rho}\omega^2 + \frac{n^2\pi^2}{\rho AL^2}\left(\frac{EIn^2\pi^2}{L^2} + N_0\right)\omega$$

$$+\frac{\sigma n^2\pi^2}{\epsilon\rho AL^2}\left(\frac{EIn^2\pi^2}{L^2} + N_0\right) = 0. \tag{4.9}$$

Equation (4.9) is the frequency equation for this vibration problem.

For simplicity let

$$\begin{aligned}
a_1 &= \left(\rho + \frac{KH_0^2}{C^2}\right)\frac{\sigma}{\epsilon\rho} \\
a_2 &= \frac{n^2\pi^2}{\rho AL^2}\left(\frac{EI\pi^2 n^2}{L^2} + N_0\right) \\
a_3 &= \frac{\sigma n^2\pi^2}{\epsilon\rho AL^2}\left(\frac{EIn^2\pi^2}{L^2} + N_0\right).
\end{aligned} \tag{4.10}$$

Now equation (4.9) becomes

$$\omega^3 + a_1\omega^2 + a_2\omega + a_3 = 0. \tag{4.11}$$

In order for the solution of equation (4.9) to be stable it is necessary that the roots of equation (4.11) have no positive real parts since these lead to solutions which are ever increasing in time. A method of determining whether or not the roots of a polynomial equation have positive real parts is due to Routh. To apply this test it is necessary to set up the Routh tabulation as described, for instance, by Kuo [3]. The tabulation has the following form when applied to equation (4-11).

$$\begin{array}{c|ccc|}
\omega^3 & 1 & a_2 & 0 \\
\omega^2 & a_1 & a_3 & 0 \\
\omega^1 & \dfrac{a_1 a_2 - a_3}{a_1} & 0 & 0 \\
\omega^0 & a_3 & 0 & 0
\end{array}$$

The Routh criterion for stability states that all terms in the first column of the tabulation must be positive. Thus for stability

$$a_1 = \left(\rho + \frac{kH_0^2}{C^2}\right)\frac{\sigma}{\epsilon\rho} > 0 \tag{4.12}$$

$$a_1 a_2 - a_3 = \left(\rho + \frac{kH_0^2}{C^2}\right)\frac{\sigma}{\epsilon\rho}\,\frac{n^2\pi^2}{\rho A L^2}\left(\frac{EIn^2\pi^2}{L^2} + N_0\right) - \frac{\sigma n^2\pi^2}{\epsilon\rho A L^2}\left(\frac{EIn^2\pi^2}{L^2} + N_0\right) > 0 \tag{4.13}$$

$$a_3 = \frac{\sigma h^2\pi^2}{\epsilon\rho A L^2}\left(\frac{EIh^2\pi^2}{L^2} + N_0\right) > 0 . \tag{4.14}$$

Clearly inequality (4.12) is always satisfied. Inequality (4.14) is satisfied if

$$\frac{EIh^2\pi^2}{L^2} + N_0 > 0 . \tag{4.15}$$

Inequality (4.13) can be simplified as follows.

$$\left(\frac{EIh^2\pi^2}{L^2} + N_0\right)\frac{\sigma h^2\pi^2}{\epsilon\rho A L^2}\left(1 + \frac{KH_0^2}{\rho C^2} - 1\right) > 0$$

or

$$\left(\frac{EIh^2\pi^2}{L^2} + N_0\right)\frac{\sigma h^2\pi^2}{\epsilon\rho A L^2}\;\frac{KH_0^2}{\rho C^2} > 0 . \tag{4.16}$$

Inequality (4.16) is satisfied only if inequality (4.15) is satisfied. The condition for instability is seen from inequality (4.15). It is

$$N_0 + \frac{EIh^2\pi^2}{L^2} < 0$$

or

$$-P + \frac{1}{2}KH_0^2\chi A + \frac{EIh^2\pi^2}{L^2} < 0$$

or

$$P = \frac{EIh^2\pi^2}{L^2} + \frac{1}{2}K_0\chi A H_0^2 . \tag{4.17}$$

The buckling load for this column is thus changed over the classical Euler load by the addition of the Maxwell force. χ can be positive or negative depending on the material. Therefore the Maxwell contribution can be either positive or negative.

Now it is desired to investigate the roots of the frequency equation. For this discussion it will be assumed that

$$P = 0 .$$

Thus the frequency equation now corresponds to the problem of free vibrations of a conducting beam. The frequency equation is

$$\omega^3 + \left(1 + \frac{KH_0^2}{\rho c^2}\right)\frac{\sigma}{\epsilon}\omega^2 + \frac{n^2\pi^2}{\rho A L^2}\left(\frac{EI\pi^2 n^2}{L^2} + N_0\right)\omega + \frac{\sigma}{\epsilon}\frac{n^2\pi^2}{\rho A L^2}\left(\frac{EI\pi^2 n^2}{L^2} + N_0\right) = 0. \qquad (4.18)$$

For given numerical values of the parameters equation (4.18) can be solved by the use of the cubic formula. Without resorting to such a procedure it is possible, however, to discover the general effect of the magnetic field on the three roots. Toward this end substitute into equation (4.18)

$$\omega = m\frac{\sigma}{\epsilon}$$

where m is any number.

The result is

$$\frac{\sigma^3}{\epsilon^3} m^3 + \left(1 + \frac{KH_0^2}{\rho c^2}\right)\frac{\sigma^3}{\epsilon^3} m^2 + \frac{n^2\pi^2}{\rho A L^2}\left(\frac{EIn^2\pi^2}{L^2} + N_0\right)\frac{\sigma}{\epsilon} m + \frac{n^2\pi^2}{\rho A L^2}\left(\frac{EIn^2\pi^2}{L^2} + N_0\right)\frac{\sigma}{\epsilon} = 0$$

or

$$m^2\left(m + 1 + \frac{KH_0^2}{\rho c^2}\right) + \frac{\epsilon^2}{\sigma^2}\frac{n^2\pi^2}{\rho A L^2}\left(\frac{EIn^2\pi^2}{L^2} + \frac{K_0 \chi A H_0^2}{2}\right)(m+1) = 0. \qquad (4.19)$$

Now find frequencies of the form

$$\omega = \frac{\sigma}{\epsilon}\left(m^* + \Delta m\right)$$

where m^* is the solution of equation (4.19) with $H_0^2 = 0$ and Δm is a linear correction factor based on a small deviation of H_0^2 from zero. First m^* must be determined. With $H_0^2 = 0$ equation (4.19) becomes

$$m^{*2}(m^* + 1) + \frac{\epsilon^2}{\sigma^2}\frac{EIn^4\pi^4}{\rho A L^4}(m^* + 1) = 0$$

or

$$(m^* + 1)\left(m^* + \frac{\epsilon^2}{\sigma^2}\frac{EIn^4\pi^4}{\rho A L^4}\right) = 0. \qquad (4.20)$$

The roots of equation (4.20) are

$$m^* = -1 \qquad \text{and} \quad m^* = \pm i\frac{\epsilon}{\sigma}\left(\frac{EIn^4\pi^4}{\rho AL^4}\right)^{1/2}. \tag{4.21}$$

To find Δm equation (4.19) is thought of as a function of the variables m and H_o^2. Thus

$$f(m,H_o^2) = m^2\left(m+1+\frac{KH_o^2}{\rho c^2}\right)+\frac{\epsilon^2}{\sigma^2}\frac{n^2\pi^2}{\rho AL^2}\left(\frac{EIn^2\pi^2}{L^2}+\frac{K_o\chi AH_o^2}{2}\right)(m+1)=0. \tag{4.22}$$

From equation (4.22) it is seen that

$$df = \frac{\partial f}{\partial m}dm + \frac{\partial f}{\partial H_o^2}dH_o^2 = 0. \tag{4.23}$$

If H_o^2 differs from zero by only a small amount dm can, as a first approximation, be replaced by Δm. Also dH_o^2 becomes H_o^2. Now equation (4.23) can be written as

$$\left(\frac{\partial f}{\partial m}\right)_{H_o^2=0}\Delta m + \left(\frac{\partial f}{\partial H_o^2}\right)_{H_o^2=0} H_o^2 = 0$$

or

$$\Delta m = -\frac{(\partial f/\partial H_o^2)_{H_o^2=0}}{(\partial f/\partial m)_{H_o^2=0}}H_o^2. \tag{4.24}$$

Now evaluate the required partial derivatives.

$$\left(\frac{\partial f}{\partial H_o^2}\right)_{H_o^2=0} = m^{*2}\frac{K}{\rho c^2}+\frac{\epsilon^2}{\sigma^2}\frac{n^2\pi^2}{\rho AL^2}\frac{K_o\chi A}{2}(m^*+1)=0$$

$$\left(\frac{\partial f}{\partial m}\right)_{H_o^2=0} = 2m^*(m^*+1)+m^{*2}+\frac{\epsilon^2}{\sigma^2}\frac{n^4\pi^4EI}{\rho AL^4} \tag{4.25}$$

Substituting equations (4.25) into equation (4.24) gives

$$\Delta m = \frac{m^{*2}\frac{K}{\rho c^2}+\frac{\epsilon^2}{\sigma^2}\frac{n^2\pi^2}{\rho AL^2}\frac{K_o\chi A}{2}(m^*+1)}{2m^*(m^*+1)+m^{*2}+\frac{\epsilon^2}{\sigma^2}\frac{n^4\pi^4EI}{\rho AL^4}}H_o^2. \tag{4.26}$$

When $m^* = -1$

$$\Delta m = -\frac{\frac{k}{\rho c^2}}{1+\frac{\epsilon^2}{\sigma^2}\frac{n^4\pi^4 EI}{\rho A L^4}} H_0^2$$

and

$$\Delta \omega = -\frac{\sigma}{\epsilon}\frac{\frac{k}{\rho c^2}}{1+\frac{\epsilon^2}{\sigma^2}\frac{n^4\pi^4 EI}{\rho A L^4}} H_0^2 \quad . \tag{4.27}$$

When $m^* = \pm i\frac{\epsilon}{\sigma}\left(\frac{n^4\pi^4 EI}{\rho A L^4}\right)^{1/2}$

$$\Delta m = -\left\{\frac{k}{2\rho c^2}\frac{m^*}{m^*+1} + \frac{\epsilon^2}{\sigma^2}\frac{n^2\pi^2}{\rho A L^2}\frac{K_0 \chi A}{4}\frac{1}{m^*}\right\} H_0^2 \tag{4.28}$$

$$\frac{m^*}{m^*+1} = \frac{\pm i\frac{\epsilon}{\sigma}\left(\frac{n^4\pi^4 EI}{\rho A L^4}\right)^{1/2}}{1 \pm i\frac{\epsilon}{\sigma}\left(\frac{n^4\pi^4 EI}{\rho A L^4}\right)^{1/2}}$$

$$= \frac{\pm i\frac{\epsilon}{\sigma}\left(\frac{n^4\pi^4 EI}{\rho A L^4}\right)^{1/2} + \frac{\epsilon^2}{\sigma^2}\frac{n^4\pi^4 EI}{\rho A L^4}}{1+\frac{\epsilon^2}{\sigma^2}\frac{n^4\pi^4 EI}{\rho A L^4}} \quad .$$

$$\frac{m^*}{m^*+1} = \frac{\frac{\epsilon^2}{\sigma^2}\frac{n^4\pi^4 EI}{\rho A L^4}}{1+\frac{\epsilon^2}{\sigma^2}\frac{n^4\pi^4 EI}{\rho A L^4}} \pm i\frac{\frac{\epsilon}{\sigma}\left(\frac{n^4\pi^4 EI}{\rho A L^4}\right)^{1/2}}{1+\frac{\epsilon^2}{\sigma^2}\frac{n^4\pi^4 EI}{\rho A L^4}} \tag{4.29}$$

$$\frac{1}{m^{+}} = \mp i \frac{\sigma}{\epsilon}\left(\frac{\rho A L^4}{n^4\pi^4 EI}\right)^{1/2} \tag{4.30}$$

Combining equations (4.28), (4.29), and (4.30) yields

$$\Delta m = -\frac{k}{2\rho c^2}\frac{\frac{\epsilon^2}{\sigma^2}\frac{n^4\pi^4 EI}{\rho A L^4}}{1+\frac{\epsilon^2}{\sigma^2}\frac{n^4\pi^4 EI}{\rho A L^4}} H_0^2 \pm i\left\{\frac{\epsilon}{\sigma}\frac{K_0\chi}{4\rho}\left(\frac{\rho A}{EI}\right)^{1/2} - \frac{k}{2\rho c^2}\frac{\frac{\epsilon}{\sigma}\left(\frac{n^4\pi^4 EI}{\rho A L^4}\right)^{1/2}}{1+\frac{\epsilon^2}{\sigma^2}\frac{n^4\pi^4 EI}{\rho A L^4}}\right\} H_0^2$$

and

$$\Delta\omega = -\frac{k}{2\rho c^2}\frac{\frac{\epsilon}{\sigma}\frac{n^4\pi^4 EI}{\rho A L^4}}{1+\frac{\epsilon^2}{\sigma^2}\frac{n^4\pi^4 EI}{\rho A L^4}} H_0^2 \pm i\left\{\frac{K_0\chi}{4\rho}\left(\frac{\rho A}{EI}\right)^{1/2} - \frac{k}{2\rho c^2}\frac{\left(\frac{n^4\pi^4 EI}{\rho A L^4}\right)^{1/2}}{1+\frac{\epsilon^2}{\sigma^2}\frac{n^4\pi^4 EI}{\rho A L^4}}\right\} H_0^2 . \tag{4.31}$$

When $H_0^2=0$ the three roots consist of one which leads to pure damping and a conjugate pair which leads to simple harmonic vibrations. Equation (4.27) indicates that the previously purely damped mode remains purely damped but its relaxation time is decreased. Equation (4.31) shows that the previously simple harmonic modes are now damped harmonic modes. The frequencies of these modes can be increased or decreased depending on the relative magnitudes of the two terms inside the brackets.

V. OSCILLATIONS OF A BEAM-COLUMN

Consider a beam-column loaded by a distributed load in the form of a half sine wave. In this case

$$q_0 = \bar{q} \sin \frac{\pi x}{L} \,, \quad q = 0 \,.$$

The governing equations (2.39), (2.44), (2.45), and (2.46) now become

$$\epsilon EI U_{,xxxxt} - \epsilon (N_0 U_{,x})_{,xt} + \sigma EI U_{,xxxx} - \sigma (N_0 U_{,x})_{,x} + \epsilon \rho A U_{,ttt}$$

$$+ \sigma A \left(\rho + \frac{KH_0^2}{c^2}\right) U_{,tt} = \epsilon (N U_{0,x})_{,xt} + \sigma (N U_{0,x})_{,x} \tag{5.1}$$

$$N_{,xxx} - \epsilon K N_{,xtt} - \sigma K N_{,xt} = 0 \tag{5.2}$$

$$EI U_{0,xxxx} - (N_0 U_{0,x})_{,x} = \bar{q} \sin \frac{\pi x}{L} \tag{5.3}$$

$$N_{0,x} = 0 \,. \tag{5.4}$$

The boundary conditions are given by equations (3.1), (3.5), (3.6), and (3.14).

As in the previous section, when equation (5.4) is solved subject to boundary condition (3.5) the result is

$$N_0 = -P + \frac{1}{2} K_0 \chi A H_0^2 \,. \tag{5.5}$$

To solve equation (5.3) together with boundary conditions (3.1) assume

$$U_0 = \bar{U}_0 \sin \frac{\pi x}{L} \,. \tag{5.6}$$

Substituting solution (5.6) into equation (5.3) yields

$$\bar{U}_0 \left(\frac{EI\pi^4}{L^4} + \frac{N_0 \pi^2}{L^2} \right) = \bar{q}$$

or

$$\bar{U}_0 = \frac{\bar{q}}{\frac{\pi^2}{L^2}\left(\frac{EI\pi^2}{L^2}+N_0\right)} \; .$$

Thus the solution to equation (5.3) is

$$U_0 = \frac{L^2\bar{q}}{\pi^2\left(\frac{EI\pi^2}{L^2}+N_0\right)} \sin\frac{\pi x}{L} \; . \tag{5.7}$$

A solution to equation (5.2) which satisfies boundary condition (3.14) is

$$N = \bar{N} \sin\frac{\pi x}{L} e^{\omega t} \; . \tag{5.8}$$

When equation (5.8) is substituted into equation (5.2) the result is

$$\frac{\pi^2}{L^2} + \epsilon k\omega^2 + \sigma k\omega = 0$$

or

$$\omega^2 + \frac{\sigma}{\epsilon}\omega + \frac{\pi^2}{\epsilon k L^2} = 0 \; . \tag{5.9}$$

The roots of equation (5.9) are

$$\omega = -\frac{\sigma}{2\epsilon} \pm \left(\frac{\sigma^2}{4\epsilon^2} - \frac{\pi^2}{\epsilon k L^2}\right)^{1/2} . \tag{5.10}$$

Using the results obtained so far equation (5.1) can be written as

$$\begin{aligned} &\epsilon EIU_{,xxxxt} - \epsilon N_0 U_{,xxt} + \sigma EIU_{,xxxx} - \sigma N_0 U_{,xxt} + \epsilon\rho A U_{,ttt} + \sigma A\left(\rho + \frac{kH_0^2}{C^2}\right)U_{,tt} = \\ &\epsilon \bar{N}\bar{U}_0 \frac{\omega\pi^2}{L^2} \cos^2\frac{\pi x}{L} e^{\omega t} - \epsilon\bar{N}\bar{U}_0 \frac{\omega\pi^2}{L^2} \sin^2\frac{\pi x}{L} e^{\omega t} \\ &+ \sigma\bar{N}\bar{U}_0 \frac{\pi^2}{L^2} \cos^2\frac{\pi x}{L} e^{\omega t} - \sigma\bar{N}\bar{U}_0 \sin^2\frac{\pi x}{L} e^{\omega t} \; . \end{aligned} \tag{5.11}$$

After a slight simplification equation (5.11) becomes

$$\begin{aligned} &\epsilon EIU_{,xxxxt} - \epsilon N_0 U_{,xxt} + \sigma EIU_{,xxxx} - \sigma N_0 U_{,xx} + \epsilon\rho A U_{,ttt} + \sigma A\left(\rho + \frac{kH_0^2}{C^2}\right) = \\ &\frac{\pi^2\bar{N}\bar{U}_0}{L^2}(\sigma + \epsilon\omega) \cos\frac{2\pi x}{L} e^{\omega t} . \end{aligned} \tag{5.12}$$

For ease of solution it is desirable to expand the right side of equation (5.12) in a Fourier sine series in the interval $0 \leq x \leq L$. The function to be expanded is

$$f(x) = \cos \frac{2\pi x}{L} . \tag{5.13}$$

Its sine series expansion is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \tag{5.14}$$

where the Fourier coefficients are fiven by

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx . \tag{5.15}$$

Substituting equation (5.13) into equation (5.15) yields

$$b_n = \frac{2}{L} \int_0^L \cos \frac{2\pi x}{L} \sin \frac{n\pi x}{L} dx \qquad n \neq 2$$

$$= \frac{2}{L} \left\{ \frac{\cos \frac{\pi}{L}(2-n)x}{\frac{2\pi}{L}(2-n)} - \frac{\cos \frac{\pi}{L}(2+n)x}{\frac{2\pi}{L}(2+n)} \right\}_0^L \qquad n \neq 2 .$$

$$b_n = \frac{1}{\pi(2-n)} \left\{ \cos \pi(2-n) - 1 \right\} - \frac{1}{\pi(2+n)} \left\{ \cos \pi(2+n) - 1 \right\} \tag{5.16}$$

$$n \neq 2$$

$$b_2 = \frac{2}{L} \int_0^L \cos \frac{2\pi x}{L} \sin \frac{2\pi x}{L} dx$$

$$= \frac{1}{\pi} \left\{ \sin \frac{2\pi x}{L} \right\}_0^L = 0 \tag{5.17}$$

From equations (5.16) and (5.17) one deduces that

$$b_n = \begin{cases} 0 & n \text{ even} \\ \frac{4}{\pi} \frac{n}{n^2-4} & n \text{ odd.} \end{cases} \tag{5.18}$$

Substituting coefficients (5.18) into series (5.14) and this into equation (5.12) gives the following form to the governing equation.

$$\epsilon EI U_{,xxxxt} - \epsilon N_o U_{,xxt} + \sigma EI U_{,xxxx} - \sigma N_o U_{,xx} + \epsilon \rho A U_{,ttt} + \sigma A\left(\rho + \frac{KH_o^2}{c^2}\right) = \frac{4\pi \bar{N} \bar{U}_o}{L^2}(\sigma + \epsilon\omega) e^{\omega t} \sum_{n=1,3,\ldots} \frac{n}{n^2-4} \sin\frac{n\pi x}{L} \tag{5.19}$$

To solve equation (5.19) one seeks a solution of the form

$$U = U_H + U_p$$

where U_H is a solution to the homogeneous equation and U_p is a particular solution to the complete equation. The particular solution will be found first. Assume

$$U_p = \sum_{n=1,3,\ldots} \bar{U}_n \sin\frac{n\pi x}{L} e^{\omega t}. \tag{5.20}$$

When this is substituted into the governing equation (5.19) the result is

$$\left\{ (\sigma + \epsilon\omega)\frac{n^2\pi^2}{L^2}\left(\frac{n^2\pi^2 EI}{L^2} + N_o\right) + \omega^2 A\left(\epsilon\rho\omega + \sigma\rho + \frac{\sigma K H_o^2}{c^2}\right)\right\}\bar{U}_n$$

$$= \frac{4\pi \bar{N} \bar{U}_o}{L^2}(\sigma + \epsilon\omega)\frac{n}{n^2-4}$$

or

$$\left\{(\sigma + \epsilon\omega)\left[\frac{n^2\pi^2}{L^2}\left(\frac{n^2\pi^2 EI}{L^2} + N_o\right) + \omega^2\rho A\right] + \frac{\omega^2 \sigma K A H_o^2}{c^2}\right\}\bar{U}_n =$$

$$\frac{4\pi \bar{N} \bar{U}_o}{L^2}(\sigma + \epsilon\omega)\frac{n}{n^2-4}.$$

$$\therefore \quad \bar{U}_n = \frac{\frac{4\pi \bar{N} \bar{U}_o}{L^2} \frac{n}{n^2-4}}{\frac{n^2\pi^2}{L^2}\left(\frac{n^2\pi^2 EI}{L^2} + N_o\right) + \omega^2 A \left\{\rho + \frac{\sigma K H_o^2}{(\sigma+\epsilon\omega)C^2}\right\}} \tag{5.21}$$

Substituting coefficients (5.21) into solution (5.20) gives the following particular solution.

$$U_p = \frac{4\pi^2 \bar{N} \bar{U}_o}{L^2} e^{\omega t} \sum_{n=1,3,\ldots} \frac{\frac{n}{n^2-4} \sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}\left(\frac{n^2\pi^2 EI}{L^2} + N_o\right) + \omega^2 A \left\{\rho + \frac{\sigma K H_o^2}{(\sigma+\epsilon\omega)C^2}\right\}} \tag{5.22}$$

To find U_H equation (5.19) must be solved with the right side set equal to zero. The governing equation then becomes identical to equation (4.7) which was treated in the previous section. Therefore attention will be focused on particular solution (5.22). Clearly an instability will occur in this solution when its denominator is set equal to zero. The equation to be studied is

$$\frac{n^2\pi^2 EI}{L^2} + N_o + \frac{\omega^2 A L^2}{n^2\pi^2} \left\{\rho + \frac{\sigma K H_o^2}{(\sigma+\epsilon\omega)C^2}\right\} = 0 . \tag{5.23}$$

From equation (5.10) it can be seen that ω can be either real or complex. The case of a real ω will be considered first.

Reference to equations (5.7) and (5.22) shows that U_p has one instability when

$$\frac{EI n^2\pi^2}{L^2} + N_o = 0$$

or

$$P = \frac{EI n^2 \pi^2}{L^2} + \frac{1}{2} K_o \chi A H_o^2 . \tag{5.24}$$

Equation (5.23) indicates that U_P has another instability when

$$P = \frac{EI n^2 \pi^2}{L^2} + \frac{1}{2} K_o \chi A H_o^2 + \frac{\omega^2 A L^2}{n^2 \pi^2} \left\{ \rho + \frac{\sigma K H_o^2}{(\sigma + \epsilon\omega) c^2} \right\} . \tag{5.25}$$

To find the buckling load one searches for the lowest value of the end load which causes an instability. If ω is real the last term of equation (5.25) is real. If this term is positive the critical value of P is given by equation (5.24). If this term is negative the critical value is given by equation (5.25). Clearly the sign of the term in question is the same as the sign of $(\sigma + \epsilon\omega)$. From equation (5.10) it is seen that two cases must be considered for which ω is real. The first is when

$$\frac{\sigma^2}{4\epsilon^2} - \frac{\pi^2}{\epsilon K L^2} = 0 .$$

In this case

$$\omega = -\frac{\sigma}{2\epsilon}$$

and

$$\sigma + \epsilon\omega = \sigma - \frac{\sigma}{2} = \frac{\sigma}{2} > 0 .$$

The second case to consider is when

$$\frac{\sigma^2}{4\epsilon^2} - \frac{\pi^2}{\epsilon K L^2} > 0 ,$$

Then

$$\omega = -\frac{\sigma}{2\epsilon} \pm \left(\frac{\sigma^2}{4\epsilon^2} - \frac{\pi^2}{\epsilon K L^2} \right)^{1/2}$$

and

$$\sigma + \epsilon\omega = \sigma - \frac{\sigma}{2} \pm \epsilon \left(\frac{\sigma^2}{4\epsilon^2} - \frac{\pi^2}{\epsilon K L^2} \right)^{1/2}$$

or

$$\sigma + \epsilon\omega = \frac{\sigma}{2} \pm \left(\frac{\sigma^2}{4} - \frac{\pi^2\epsilon}{kL^2}\right)^{1/2}.$$

In order for the above expression to be negative the second term would have to be larger than the first. This is obviously impossible. Thus

$$\sigma + \epsilon\omega > 0.$$

From the above expressions it can be concluded that the electromagnetic forces do not effect the buckling load. The beam-column buckles at the same load that would be required if it were static and no electromagnetic forces (other than the static Maxwell force) were present.

It is also possible for ω to be complex. In this case the demominator of U_p consists of a real and an imaginary part. Both of these must equal zero simultaneously for an instability to occur. From equation (5.10) it is noted that ω is complex when

$$\frac{\sigma^2}{4\epsilon^2} - \frac{\pi^2}{\epsilon K L^2} < 0.$$

Inspection of equation (5.23) reveals that the real and imaginary parts of the equation are not identical. Thus one value of the end load would not make both the real and imaginary parts zero simultaneously as required. Consequently in this case also it is seen that the electromagnetic forces do not lower the buckling load which remains

$$P = \frac{n^2\pi^2 EI}{L^2} + \frac{1}{2} K_o \chi A H_o^2.$$

VI. FORCED OSCILLATIONS OF A COLUMN

Consider the following case of forced vibration.

$$q_o = 0 \qquad q = \bar{q} \sin \frac{\pi x}{L} e^{-\zeta t} \tag{6.1}$$

This column has a load applied to it at $t = 0$. It is noted that the solutions to governing equations (2.45) and (2.46) are the same as those presented in section IV, namely

$$U_o = 0 \tag{6.2}$$

$$N_o = -P + \frac{1}{2} K_o \chi A H_o^2 \; . \tag{6.3}$$

Substituting equations (6.1) and (6.2) into equation (2.39) yields the following governing equation.

$$\epsilon EI U_{,xxxxt} - \epsilon N_o U_{,xxt} + \sigma EI U_{,xxxx} - \sigma N_o U_{,xx} + \epsilon \rho A U_{,ttt} + \sigma A\left(\rho + \frac{K H_o^2}{C^2}\right) U_{,tt} = (\sigma - \epsilon \zeta) \bar{q} \sin \frac{\pi x}{L} e^{-\zeta t} \tag{6.4}$$

Again a solution is sought in the form

$$U = U_P + U_H$$

where U_P is a particular solution to the complete equation and U_H is a solution to the homogeneous equation. It can be seen that the homogeneous equation is identical to equation (4.8). Thus all the results of section IV apply equally well to the problem now under discussion.

Now a particular solution to equation (6.4) will be found.

Assume

$$U_P = \bar{U} \sin \frac{\pi x}{L} e^{-\zeta t} \; . \tag{6.5}$$

Substitution of solution (6.5) into equation (6.4) yields

$$\left\{(\sigma-\epsilon\varsigma)\frac{\pi^2}{L^2}\left(\frac{\pi^2 EI}{L^2}+N_o\right)+\varsigma^2 A\left(\sigma\rho+\frac{\sigma K H_o^2}{C^2}-\epsilon\rho\varsigma\right)\right\}\bar{U}=(\sigma-\epsilon\varsigma)\bar{q}$$

or

$$\left\{(\sigma-\epsilon\varsigma)\left[\frac{\pi^2}{L^2}\left(\frac{\pi^2 EI}{L^2}+N_o\right)+\varsigma^2 A\rho\right]+\frac{\varsigma^2 A\sigma K H_o^2}{C^2}\right\}\bar{U}=(\sigma-\epsilon\varsigma)\bar{q}$$

or

$$\bar{U}=\frac{\bar{q}}{\frac{\pi^2 EI}{L^2}+N_o+\frac{\varsigma^2 A L^2}{\pi^2}\left\{\rho+\frac{\sigma K H_o^2}{(\sigma-\epsilon\varsigma)C^2}\right\}}\,. \tag{6.6}$$

Combining equations (6.5) and (6.6) gives the following result.

$$U_p=\frac{\bar{q}}{\frac{\pi^2 EI}{L^2}+N_o+\frac{\varsigma^2 A L^2}{\pi^2}\left\{\rho+\frac{\sigma K H_o^2}{(\sigma-\epsilon\varsigma)C^2}\right\}}\sin\frac{\pi x}{L}e^{-\varsigma t}\,. \tag{6.7}$$

For an instability to occur in U_p the following condition must be satisfied.

$$-P+\frac{\pi^2 EI}{L^2}+\frac{1}{2}K_o\chi A H_o^2+\frac{\varsigma^2 A L^2}{\pi^2}\left\{\rho+\frac{\sigma K H_o^2}{(\sigma-\epsilon\varsigma)C^2}\right\}=0\,. \tag{6.8}$$

Using solution (6.3) equation (6.8) becomes

$$P=\frac{\pi^2 EI}{L^2}+\frac{1}{2}K_o\chi A H_o^2+\frac{\varsigma^2 A L^2}{\pi^2}\left\{\rho+\frac{\sigma K H_o^2}{(\sigma-\epsilon\varsigma)C^2}\right\}. \tag{6.9}$$

From equation (4.17) it is seen that U_H exhibits its first instability when

$$P=\frac{EI\pi^2}{L^2}+\frac{1}{2}K_o\chi A H_o^2\,. \tag{6.10}$$

Equation (6.10) will determine the buckling load for the column unless the third term of equation (6.9) is negative. For the electromagnetic forces to lower the buckling load it is necessary that

$$\rho+\frac{\sigma K H_o^2}{(\sigma-\epsilon\varsigma)C^2}<0\,. \tag{6.11}$$

Let

$$\zeta = m\frac{\sigma}{\epsilon} \quad , \quad m > 1 \ . \tag{6.12}$$

Equation (6.11) now reads

$$\rho + \frac{kH_o^2}{(1-m)C^2} < 0$$

or

$$H_o^2 > (m-1)\,\frac{\rho c^2}{k} \ . \tag{6.13}$$

From this expression one can determine the magnetic field which must be present in order for an instability to occur in the particular solution at a lower value of ρ than that predicted by the homogeneous solution.

VII. NUMERICAL RESULTS

All the results presented in this section are for aluminium for which

$$K \doteq K_0 , \quad \epsilon \doteq \epsilon_0 .$$

From equation (4.17) it is seen that the additional buckling stress due to the magnetic field is

$$\Delta S_{xx} = \frac{K_0 \chi}{2} H_0^2 .$$

Combining this with equation (2.13) yields

$$\Delta S_{xx} = \frac{\chi}{2K_0} B_0^2 . \tag{7.1}$$

Equation (7.1) is plotted in figure 3.

Equation (4.31) shows that the damping coefficient added to the previously undamped modes is

$$\Delta \omega_D = \frac{K H_0^2}{2\rho c^2} \frac{\frac{\epsilon}{\sigma} \omega^{*2}}{1 + \frac{\epsilon^2}{\sigma^2} \omega^{*2}} , \tag{7.2}$$

where

$$\omega^* = \left(\frac{n^4 \pi^4 EI}{\rho A L^4} \right)^{1/2} = \frac{n^2 \pi^2}{L^2} \left(\frac{EI}{\rho A} \right)^{1/2} .$$

Equation (7.2) can be written as

$$\Delta \omega_D = \frac{K H_0^2}{2\rho c^2} g(\omega^*) \tag{7.3}$$

where

$$g(\omega^*) = \frac{\frac{\epsilon}{\sigma} \omega^{*2}}{1 + \frac{\epsilon^2}{\sigma^2} \omega^{*2}}$$

or

$$g(\omega^*) = \frac{\sigma}{\epsilon} \frac{\omega^{*2}}{\frac{\sigma^2}{\epsilon^2} + \omega^{*2}} . \tag{7.4}$$

The above expression shows that

$$0 \leq g(\omega^*) \leq \frac{\sigma}{\epsilon} \; . \tag{7.5}$$

Combining inequality (7.5) with equation (7.3) yields

$$(\Delta\omega_0)_{max} = \frac{\sigma K}{2\epsilon\rho C^2} H_0^2 \tag{7.6}$$

Using equation (2.13), equation (7.6) becomes

$$(\Delta\omega_0)_{max} = \frac{\sigma}{2K_0\epsilon_0\rho C^2} B_0^2$$

or

$$(\Delta\omega_0)_{max} = \frac{\sigma}{2\rho} B_0^2 \; . \tag{7.7}$$

Thus if the relaxation time is denoted by τ,

$$\tau_{min} = \frac{2\rho}{\sigma B_0^2} \; . \tag{7.8}$$

Equation (7.8) is plotted in figure 4.

A 16WF50 aluminum beam has the following specifications.

$$E = 10^7 \; lb/in^2$$
$$\rho = 2.96 \times 10^{-3} \; slugs/in^3$$
$$I = 6.55 \times 10^2 \; in^4$$
$$A = 1.47 \times 10 \; in^2 .$$

If L is picked as follows

$$L = 1.6 \times 10^2 \; in$$

then

$$\omega^* = 241 n^2 \; cps \; .$$

For aluminum $\sigma/\epsilon = 1.27 \times 10^{19}$ cps.

Clearly

$$\frac{\sigma^2}{\epsilon^2} >> \omega^{*2} . \tag{7.9}$$

except at extremely high modes of vibration. This result is typical of those obtained for beams of common shapes. Substituting inequality (7.9) into equation (7.4) yields

$$g(\omega^*) = \frac{\epsilon}{\sigma}\,\omega^{*2} \tag{7.10}$$

Substituting equation (7.10) into equation (7.3) gives

$$\Delta\omega_D = \frac{\epsilon K H_o^2}{2\rho\sigma c^2}\,\omega^{*2}$$

or

$$\Delta\omega_D = \frac{\omega^{*2} B_o^2}{2\sigma K_o^2 \rho c^4} \,. \tag{7.11}$$

The relaxation time is thus

$$\tau = \frac{2\sigma K_o^2 \rho c^4}{\omega^{*2} B_o^2} \,. \tag{7.12}$$

Equation (7.12) is plotted in figure 5.

From equation (6.13) it is evident that the lowest value of which will effect the buckling load is

$$B_o = (\rho K_o)^{1/2} c\,(m-1)^{1/2}. \tag{7.13}$$

Equation (7.13) is plotted in figure 6.

VIII. CONCLUSION

Before discussing the results of the previous work it is useful to review the assumptions that were made to obtain those results. First it was assumed that all variables were perturbed only a small amount. Second the assumptions of elementary beam theory were used. In addition the effect of shear, rotary inertia, and the acceleration of a differential element of the bar in the axial direction were neglected. Finally, to be consistent with the elementary theory of beams, it was assumed that all electromagnetic variables were functions only of x and t. While none of the above assumptions are unreasonable, they are certainly not the only assumptions that could be made. It is to be remembered that the results discussed in what follows are valid only to the extent that these assumptions are satisfied.

The work presented concerned itself with two areas. They were the vibration and elastic stability of the structures studied. Conclusions with respect to vibration will be summarized first.

In sections IV and V the oscillations of a column and a beam-column respectively were studied. It was evident that, at least from a theoretical point of view, the addition of a magnetic field had marked effects. In the column problem three roots to the frequency equation were obtained as compared to two roots in the absence of the magnetic field. The magnetic field introduced damping into the previously undamped modes, changed the frequencies of these modes, and introduced an additional totally damped mode. In the beam-column more additional frequencies were introduced due to the appearance of the perturbed normal force in the governing equation for deflection.

For the two problems discussed above the effect of the magnetic field on the elastic stability of the structures was not so great. It is true that the buckling load was changed. This change was not, however, caused by the dynamic electromagnetic forces. It was caused by the addition of the static Maxwell force to Euler's load. It must be noted that the small perturbation assumption precluded the inclusion of the term NU in the governing equation for deflection. It is well known that this term has a prominent role to play in any discussion of elastic stability. A more inclusive theory would be required to investigate the effect of this term. In section VI it was discovered that for forced vibration of a column the dynamic electromagnetic forces can indeed lower the buckling load.

The numerical results presented in the following graphs indicate that the effect of the magnetic field is of slight practical importance. Figure 4. shows that the relaxation times can be small but only for extremely high frequencies. Figures 3., 5., and 6. indicate that no appreciable effect can be noticed for magnetic fields of reasonable size. One cannot, however, rule out completely the effect of the magnetic field. It is certainly possible, for instance, to find some column whose Euler load is so small that the electromagnetic contribution is significant.

It is to be recalled that numerical results are presented for only one material. It would be incorrect, therefore, to conclude that the contribution of the magnetic field is without significance for all materials. Since aluminum is fairly typical of the usual structural metals it can be said that for ordinary beams and beam-columns made of ordinary structural materials the effect of a magnetic field on their vibrations and elastic stability is very small.

IX. ACKNOWLEDGEMENT

The author would like to gratefully acknowledge the contributions to the aforegoing work made by his two advisors; Dr. M. A. Garcia and Dr. R. P. McNitt. Although Dr. Garcia left V. P. I. before work on this thesis was begun, he helped the author plan his graduate program and displayed a spirit of inquiry that could not help but rub off on those around him. Dr. McNitt made valuable suggestions at every stage of the development of this work and helped to crystalize the author's ideas through many hours of patient discussion.

Thanks are also due to Professor D. H. Pletta, Head of the Department of Engineering Mechanics at V. P. I.. He offered helpful suggestions in regard to the author's graduate program and allowed the author to support himself over the past two years by means of a teaching assistantship.

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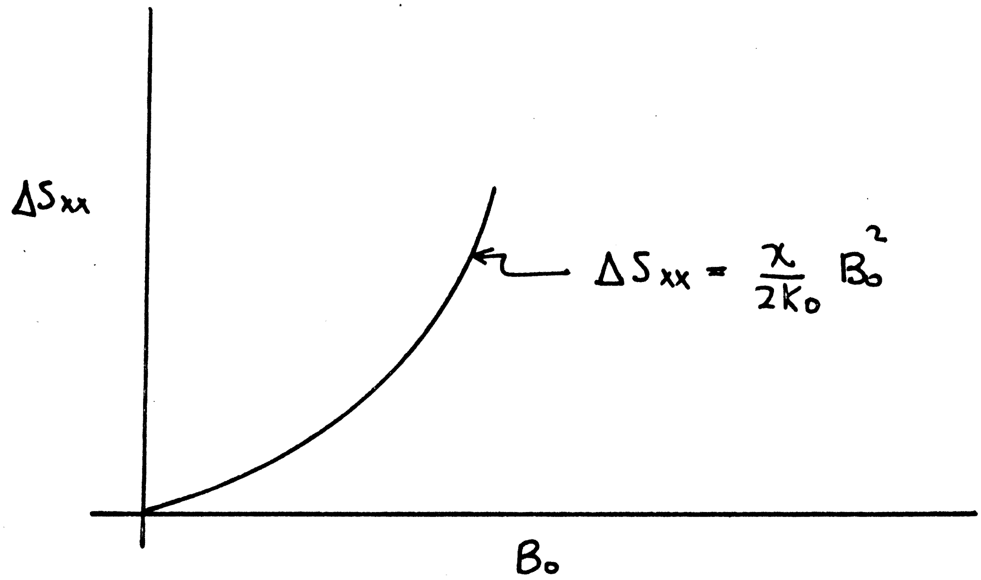
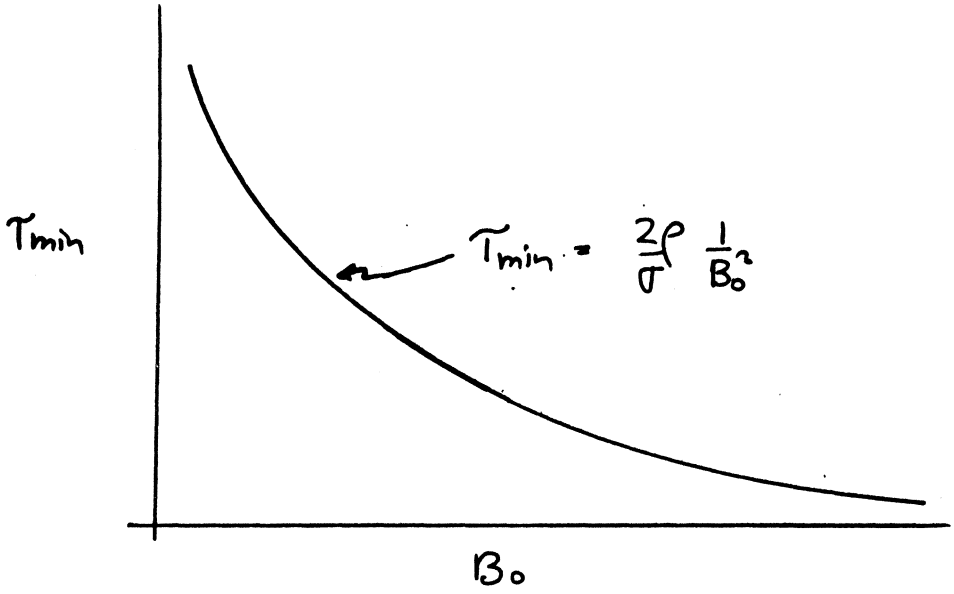


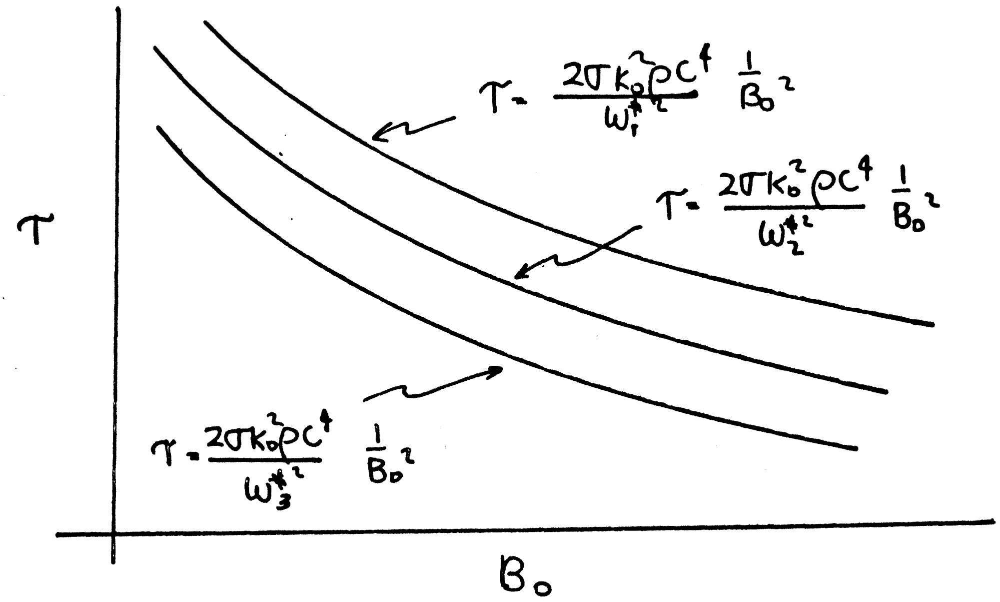
ΔS_{xx} (psi)	B_0 (gauss)
0	0
1.33×10^{-11}	1
1.33×10^{-9}	10
1.33×10^{-7}	10^2
1.33×10^{-3}	10^4
1.33×10	10^6

Figure 3.- Additional buckling stress vs. magnetic induction.



T_{min} (sec.)	B_o (gauss)
∞	0
1.54×10^4	1
1.54×10^2	10
1.54	10^2
1.54×10^{-4}	10^4
1.54×10^{-8}	10^6

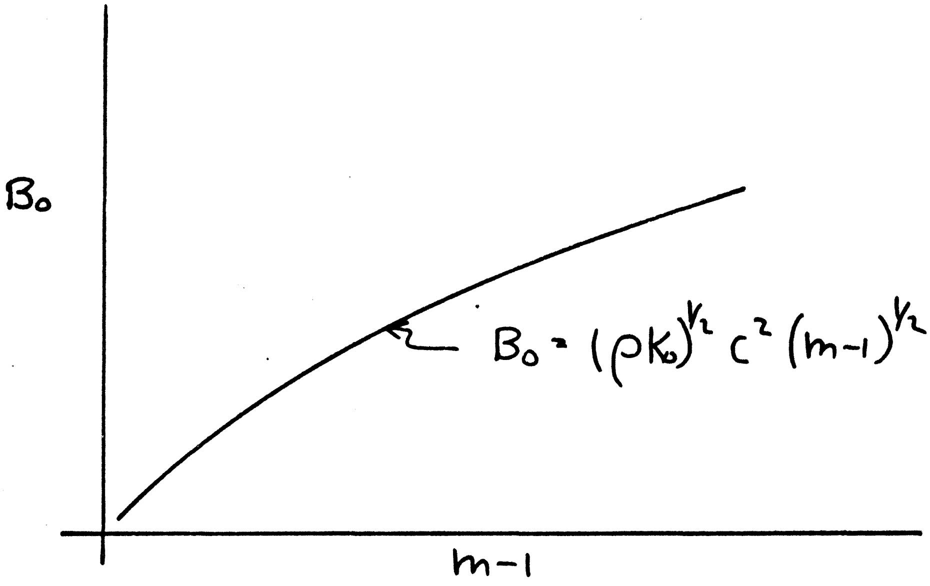
Figure 4.- Minimum relaxation time vs. magnetic induction.



$$\omega_1^* < \omega_2^* < \omega_3^*$$

τ (sec)			B_o (gauss)
$\omega^* = 10^2$ cps	$\omega^* = 25\times10^2$ cps	$\omega^* = 10^4$ cps	
∞	∞	∞	0
9.04×10^{35}	1.45×10^{33}	9.04×10^{31}	1
9.04×10^{33}	1.45×10^{31}	9.04×10^{29}	10
9.04×10^{31}	1.45×10^{29}	9.04×10^{27}	10^2
9.04×10^{27}	1.45×10^{25}	9.04×10^{23}	10^4
9.04×10^{23}	1.45×10^{21}	9.04×10^{19}	10^6

Figure 5.- Relaxation time vs. magnetic induction.



B_o (gauss)	$m-1$
1.75×10^{4}	10^{-14}
1.75×10^{5}	10^{-12}
1.75×10^{6}	10^{-10}
1.75×10^{9}	10^{-4}
1.75×10^{10}	10^{-2}

Figure 6.- Magnetic induction required to influence buckling load vs. load duration factor.

SMALL OSCILLATIONS OF A BEAM-COLUMN WITH FINITE ELECTRICAL CONDUCTIVITY IN A CONSTANT TRANSVERSE MAGNETIC FIELD

By

John Peddieson, Jr.

ABSTRACT

Small oscillations of a beam-column with finite electrical conductivity in a transverse magnetic field are examined under the assumption that the vibration of the bar causes only a weak perturbation in the electromagnetic field. The frequency equation is derived for a column and reduced to that for free vibration of a beam by equating the end load to zero. The roots of this equation are obtained approximately and the effect of the magnetic field on the frequencies is noted. In addition, the elastic stability of a conducting column and beam-column are investigated. The effect of the magnetic field on the buckling load is determined. Numerical results are presented which indicate that the effect of the dynamic electromagnetic forces is negligible except at extremely high frequencies of vibration.