Hydrodynamic Stability of Periodically Unsteady Axisymmetric and Swirling Jets

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(ABSTRACT)

Axisymmetric and swirling jets are generic flows that characterize many natural and man-made flows. These include cylindrical shear layer/mixing layer flows, aircraft jets and wakes, shedding of leading edge and wing tip vortices, tornadoes, astrophysical plasma flows and flows in mechanical devices such as supersonic combustion chambers and cyclone separators. These and other applications have resulted in a high level of interest in the stability of axisymmetric and swirling jets. To date, the majority of studies on stability of axisymmetric and swirling jets have been completed under the assumption of steady flow in both axial and azimuthal (swirl) directions. Yet, flows such as the ones mentioned above can have an inherent unsteadiness. Moreover, such unsteadiness can be used to control stability and thus flow characteristics in axisymmetric and swirling jets. In this work effects of periodic variations on the temporal stability of axisymmetric and swirling jets is examined. The unsteadiness is introduced in the former as a periodic variation of the axial velocity component of the flow, and in the latter as a periodic variation of the azimuthal (swirl) velocity component of the flow.

The temporal linear hydrodynamic stability of both steady inviscid axisymmetric and swirling jets is reviewed. An analytical dispersion relation is obtained in both cases and solved numerically. In the case of the steady axisymmetric jet, growth rate and celerity of unstable axisymmetric and helical modes are determined as functions of axial wavenumber. Results show that the inviscid axisymmetric jet is unstable to all values of axisymmetric and helical modes. In the case of the steady swirling jet, growth rate and celerity of axisymmetric modes are determined as functions of the axial wavenumber and swirl number. Results show that the inviscid swirling jet is unstable to all values of axial and azimuthal wavenumber, however, it is shown that increasing the swirl decreases the growth rate and increases the celerity of axisymmetric disturbances. The effects of periodic variations on the stability of a mixing layer is also reviewed. Results show that when the instability time scale is much smaller than the mean time scale a transformation of the time variable may be taken that, when the quasi-steady approach works, will reduce the unsteady field to that of the corresponding steady field in the new time scale. The price paid for this transformation, however, is a modulation of the amplitude and phase of the unsteady modes.

Extending the results from the unsteady mixing layer, the stability of a periodically unsteady inviscid axisymmetric jet is considered. An analytical dispersion relation is obtained and results show that for the unsteady inviscid axisymmetric jet, the quasi-steady approach works. Following this, the stability of a periodically unsteady swirling jet is considered and an analytical dispersion relation is obtained. It is shown that for the unsteady inviscid swirling jet, the quasi-steady approach does not work. Resulting modulations of unsteady modes are shown via a numerical solution to the unsteady dispersion relation. In both cases, using established results for unsteady mixing layers, these results are substantiated analytically by showing that the unsteady axisymmetric jet can be reduced the the exact equational form of the

steady axisymmetric jet in a new time scale, whereas the unsteady swirling jet cannot.

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nill illigitimi carborundum

Dedication

This thesis is dedicated to my beautiful wife, Michelle and our amazing gift form God, my perfect baby daughter Hannah.

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Chapter 1

Introduction

The theory of hydrodynamic stability is useful in predicting transitions between laminar and turbulent configurations for a given flow field. That is, hydrodynamic stability deals with predicting the most stable configuration for a given system and the transition from laminar flow to turbulence follows as a natural progression of the system marked by the initial instability. For example, it is intuitively obvious that the most stable configuration of a two-fluid interface is one in which the lighter of the two fluids exists on top of the heaviest fluid. In nature, this is what is observed and is not related to the fundamental physics of the problem, as theoretically both configurations are possible. The naturally observed configuration is a consequence of the stability of the physical system. In this case, an equilibrated system consisting of the opposite, heavy on top of light fluid, is unstable to all perturbations of the interface, thus, any infinitesimal perturbation grows without bound and the equilibrium is upset. We may say that the theory of hydrodynamic stability deals with predicting whether or not a given flow pattern is stable or unstable, and thus whether or not it can be observed in nature.

In practice, to determine the stability of a given flow field with respect to small deviations of the physical parameters of the system, a perturbation is added to the field variables that are then substituted back into the governing equations. Because the unperturbed field is necessarily admissible it already satisfies the governing equations, thus, the mathematical problem is reduced to determining the exact form of the perturbation eigenfunctions for the given problem. In the case of linear stability; that is, for a 'small' disturbance, the perturbations are taken such that nonlinear combinations of them may be disregarded, and the mathematical problem is simplified due to the subsequent linearization of the field equations. Furthermore, perturbations adhering to the L_2-norm may be expanded (for a linear system) in normal-modes and, following the theory of Liapounov, information may be extracted from the normal-mode solution that will help determine the stability of the flow [1] [2]. This is the idea behind the normal mode approach for linear stability analysis.

Axisymmetric and swirling jets/wakes are examples of fluid dynamical flow fields whose stability is of theoretical and practical interest. There are many physical problems in nature and technology involving jet flows, and, specifically swirling jets. Swirling jet flow is observed in aircraft jets and wakes, and, in the shedding of leading-edge and wing-tip vorticity. Furthermore, swirl is purposely imparted in mechanical devices such as supersonic combustion chambers and cyclone separators to manipulate flow dynamics for engineering purposes; in the former, swirl is introduced to aide in atomization of fuel jets, thereby increasing combustion efficiency [3]. The stability of certain natural flows of practical interest, such as tornadoes [4], and astrophysical flows of more theoretical interest, such as plasma jets in Active Galactic Nuclei (AGN), well-collimated flows in star formation regions, "black holes", and galaxy cores, have been analyzed using swirling plasma jet models [5] [6]. Swirl can also be generated in an initial swirl-free flow via the breaking of one or more symmetries, examples of such flows are, strong atmospheric vortical flows, rotational motions of astrophysical objects, whirlpools and flow in a draining bathtub [7]. In the case of the latter, it is the unbalanced Coriolis force that is responsible for the vortical motion of the flow, however, the initial instability that leads to this motion can be viewed as resulting from a broken symmetry.

Existing stability studies of the above-mentioned physical problems have been analyzed assuming steady mean flow; that is, all velocity components are considered independent of time. In general, stability studies of shear flows are completed assuming steady flow, and, this type of steady shear flow analysis receives more attention than the stability of unsteady shear flows. This is due to the fact that mathematically, the already complex nature of stability analyses is exacerbated by the introduction of time-dependent coefficients in the Navier-Stokes equation, and, under most circumstances analytical closed form solutions may be unattainable, and one must resort to a numerical or asymptotic analysis of the problem. Experimentally, generating a well-controlled unsteady mean flow is difficult and substantial complications may result from inadequate unsteady flow-field conditions. To avoid these difficulties, the effects of time-dependent mean flows on the stability of a fluid mechanical system have been analyzed via the quasi-steady approach; that is, applying steady state results to instantaneous velocity distributions.

The objective of this present study is to analyze the stability of periodically time-dependent jets directly form the unsteady linearized Navier-Stokes equation. Using results from the analysis of periodically unsteady mixing layers, conclusions are drawn regarding the applicability of the quasi-steady approach for unsteady axisymmetric and swirling jets. The paper is outlined as follows: Chapter 1 gives a brief motivation for the study of hydrodynamic stability, Chapter 2 gives, as a prerequisite for subsequent chapters, an overview of the temporal stability of steady axisymmetric and swirling jets and unsteady free shear flows. In chapter 3 the problem of the stability of an unsteady axisymmetric jet is formulated, solved and discussed. In chapter 4 the problem of the stability of an unsteady swirling jet is formulated, solved and discussed. A summary of the research and conclusions are given in Chapter 5.

Chapter 2

Stability of Steady Jets and Unsteady Shear Flows

In this chapter the background for the work performed herein is presented. First, the stability of a steady inviscid axisymmetric jet is detailed as given by Batchelor and Gill [8]. Second, the stability of a steady inviscid swirling jet following the work of Loiseleux *et.al* [9] is presented. And lastly, the effects of periodic variations on the stability of a mixing layer is considered as presented by Hajj [10].

2.1 Stability of Axisymmetric Jets

In this section, the stability of a steady axisymmetric jet is considered. An exhaustive treatment of the temporal stability of inviscid axisymmetric jets is given by Batchelor and Gill [8]. The spatial counterpart treatment is given by Michalke [11].

The velocity and pressure fields, $(\vec{v}, p)$, of inviscid Newtonian flows are governed by the Euler and continuity equations which are given by,

$$\frac{\partial \vec{v}}{\partial t} + \nabla \vec{v} \cdot \vec{v} + \frac{1}{\rho} \nabla p = \vec{0} \tag{2.1}$$

$$\nabla \cdot \vec{v} = 0 \tag{2.2}$$

The cylindrical coördinate components for these equations are written as,

$$\rho \Big(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \Big) = -\frac{\partial p}{\partial r} \tag{2.3}$$

$$\rho \Big(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \Big) = -\frac{1}{r} \frac{\partial p}{\partial \theta} \tag{2.4}$$

$$\rho \Big(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \Big) = -\frac{\partial p}{\partial z} \tag{2.5}$$

for equation (2.1) and,

$$\frac{1}{r} \frac{\partial}{\partial r} (\rho r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho v_\theta) + \frac{\partial}{\partial z} (\rho v_z) = 0 \tag{2.6}$$

for equation (2.2).

The inviscid axisymmetric jet, shown in figure 2.1, is modeled by a cylindrical vortex sheet of radius R such that,

$$\vec{V}(r,\theta,z) = \begin{cases} (0,0,v_\infty + \Delta v), & \text{for } r < R \\ (0,0,v_\infty), & \text{for } r > R \end{cases} \tag{2.7}$$

Where, v_∞ is the far-field free stream velocity and Δv is the axial velocity

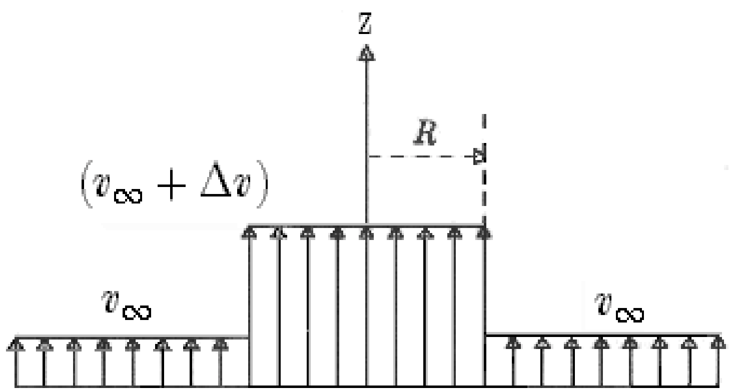

Figure 2.1: Sketch of Basic Flow for the Axisymmetric Jet

difference across the vortex sheet. For a generic field $\Big((0,0,V_z),P\Big)$, substitution into equations (2.3-5) shows that the mean flow is governed by,

$$\rho\Big(\frac{\partial V_z}{\partial t} + V_z\frac{\partial V_z}{\partial z}\Big) = -\frac{\partial P}{\partial z} \tag{2.8}$$

$$\frac{\partial P}{\partial r} = 0 \tag{2.9}$$

$$\frac{1}{r}\frac{\partial P}{\partial \theta} = 0 \tag{2.10}$$

$$\frac{\partial}{\partial z}\big(\rho V_z\big) = 0 \tag{2.11}$$

To examine the stability of the jet governed by the above equations, perturbations $\delta\vec{v}$ and δp are added to the mean flow $\vec{V}$ and P according to,

$$\vec{v} \mapsto \vec{V} + \delta\vec{v} = (\delta v_r, \delta v_\theta, V_z + \delta v_z) \tag{2.12}$$

$$p \mapsto P + \delta p \tag{2.13}$$

Substituting (2.12) and (2.13) into (2.3-5) yields the following non-linear equations for the total flow,

$$\rho\left(\frac{\partial}{\partial t}\delta v_r + \delta v_r\frac{\partial \delta v_r}{\partial r} + \frac{\delta v_\theta}{r}\frac{\partial \delta v_r}{\partial \theta} - \frac{\delta v_\theta^2}{r} + \big(V_z + \delta v_z\big)\frac{\partial \delta v_r}{\partial z}\right) = -\frac{\partial}{\partial r}\big(P + \delta p\big) \tag{2.14}$$

$$\rho\left(\frac{\partial}{\partial t}\delta v_\theta + \delta v_r\frac{\partial \delta v_\theta}{\partial r} + \frac{\delta v_\theta}{r}\frac{\partial \delta v_\theta}{\partial \theta} + \frac{\delta v_r \delta v_z}{r} + \big(V_z + \delta v_z\big)\frac{\partial \delta v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial}{\partial \theta}\big(P + \delta p\big) \tag{2.15}$$

and

$$\rho\left(\frac{\partial}{\partial t}\big(V_z + \delta v_z\big) + \delta v_r\frac{\partial}{\partial r}\big(V_z + \delta v_z\big) + \frac{\delta v_\theta}{r}\frac{\partial}{\partial \theta}\big(V_z + \delta v_z\big) + \big(V_z + \delta v_z\big)\frac{\partial}{\partial z}\big(V_z + \delta v_z\big)\right) = -\frac{\partial}{\partial \theta}\big(P + \delta p\big) \tag{2.16}$$

Substituting (2.12) into (2.6) yields the following conservation of mass equation for the total flow,

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(\rho r \delta v_r\Big) + \frac{1}{r}\frac{\partial}{\partial \theta}\Big(\rho \delta v_\theta\Big) + \frac{\partial}{\partial z}\Big(\rho V_z + \rho \delta v_z\Big) = 0 \tag{2.17}$$

Subtracting the mean equations (2.8-10) from the non-linear momentum transport equations yields the governing momentum transport equations for the perturbations,

$$\rho\left(\frac{\partial \delta v_r}{\partial t} + \delta v_r \frac{\partial \delta v_r}{\partial r} + \frac{\delta v_\theta}{r}\frac{\partial \delta v_r}{\partial \theta} - \frac{\delta v_\theta^2}{r} + v_z \frac{\partial \delta v_r}{\partial z} + \delta v_z \frac{\partial \delta v_r}{\partial z}\right) = -\frac{\partial \delta p}{\partial r} \tag{2.18}$$

$$\rho\left(\frac{\partial \delta v_\theta}{\partial t} + \delta v_r \frac{\partial \delta v_\theta}{\partial r} + \frac{\delta v_\theta}{r}\frac{\partial \delta v_\theta}{\partial \theta} + \frac{\delta v_r \delta v_z}{r} + v_z \frac{\partial \delta v_\theta}{\partial z} + \delta v_z \frac{\partial \delta v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial \delta p}{\partial \theta} \tag{2.19}$$

and

$$\rho\left(\frac{\partial \delta v_z}{\partial t} + \delta v_r \frac{\partial v_z}{\partial r} + \delta v_r \frac{\partial \delta v_z}{\partial r} + \frac{\delta v_\theta}{r}\frac{\partial v_z}{\partial \theta} + \frac{\delta v_\theta}{r}\frac{\partial \delta v_z}{\partial \theta} + \delta v_z \frac{\partial v_z}{\partial z} + \delta v_z \frac{\partial \delta v_z}{\partial z} + v_z \frac{\partial \delta v_z}{\partial z}\right) = -\frac{\partial \delta p}{\partial \theta} \tag{2.20}$$

Subtracting the mean continuity equation (2.11) from (2.17) yields

$$\frac{1}{r}\left(\delta v_r + r\frac{\partial \delta v_r}{\partial r}\right) + \frac{1}{r}\frac{\partial \delta v_\theta}{\partial \theta} + \frac{\partial \delta v_z}{\partial z} = 0 \tag{2.21}$$

At the onset of instability, perturbations are assumed to be small, *i.e.*, $\delta f = O(\varepsilon) \ni (\delta f) \gg (\delta f)^2$ and for sufficiently small disturbances, $(\delta f)^2 \simeq 0$. In essence, we linearize the system by taking perturbations of order ε where $\varepsilon \ll 1$ so that we may throw out quadratic combinations; that is, quantities of order (ε^2). Doing so, equations (2.18-20), representing the linearized momentum-transport equations for the fluctuation quantities δv and δp, become,

$$\rho\left(\frac{\partial \delta v_r}{\partial t} + v_z\frac{\partial \delta v_r}{\partial r}\right) = -\frac{\partial \delta p}{\partial r} \tag{2.22}$$

$$\rho\left(\frac{\partial \delta v_\theta}{\partial t} + v_z\frac{\partial \delta v_\theta}{\partial r}\right) = -\frac{1}{r}\frac{\partial \delta p}{\partial \theta} \tag{2.23}$$

$$\rho\left(\frac{\partial \delta v_z}{\partial t} + v_z\frac{\partial \delta v_z}{\partial z}\right) = -\frac{\partial \delta p}{\partial z} \tag{2.24}$$

We note that the continuity equation (2.21) is already linear. Therefore, equations (2.21-2.24) represent the linearized conservation of mass and transport of momentum for velocity and pressure perturbations δv and δp in cylindrical coördinates with a mean velocity profile $(0, 0, V_z)$.

Because the new governing field equations are linear, the perturbations may be expanded in basis functions for the appropriate norm. In the space of L_2, the basis functions may be chosen as pure harmonics, and, the only requirement imposed on the perturbations is that they are of finite-energy;

that is, bounded in the mean square sense. Such an expansion yields the well known Fourier, or normal-mode representation, written as

$$\delta f = \widehat{f}(r)e^{(i\alpha(z-ct)+im\theta)}. \tag{2.25}$$

In cylindrical coördinates the constant harmonics α, c and m are the axial wavenumber, the wave speed and the azimuthal wavenumber, respectively.

What follows is a temporal stability analysis, thus, we take the wave speed c complex; *i.e.* $c = c_R + ic_I$. Based on equation (2.25), αc_I and c_R represent the growth rate and celerity of the disturbance. The perturbation fields $(\delta\vec{v}, \delta p)$ are given in terms of normal modes as,

$$\delta v_r = \widehat{v_r}(r)e^{(i\alpha(z-ct)+im\theta)} \tag{2.26}$$

$$\delta v_\theta = \widehat{v_\theta}(r)e^{(i\alpha(z-ct)+im\theta)} \tag{2.27}$$

$$\delta v_z = \widehat{v_z}(r)e^{(i\alpha(z-ct)+im\theta)} \tag{2.28}$$

$$\delta p = \widehat{p}(r)e^{(i\alpha(z-ct)+im\theta)} \tag{2.29}$$

Substituting (2.26-29) into (2.21-24) and dividing by the common (non-zero) exponential term $exp[(i\alpha(z-ct)+im\theta)]$, the following stability equations for the region inside the axisymmetric jet are obtained,

- Continuity-

$$\frac{d\widehat{v_r}(r)}{dr} + \frac{1}{r}\widehat{v_r}(r) + \frac{im}{r}\widehat{v_\theta}(r) + i\alpha\widehat{v_z}(r) = 0 \tag{2.30}$$

- Momentum-

$$\frac{d\widehat{p}(r)}{dr} - i\alpha c\rho\widehat{v_r}(r) + i\alpha V_z\rho\widehat{v_r}(r) = 0 \tag{2.31}$$

$$\frac{im}{\rho r}\widehat{p}(r) - i\alpha c\widehat{v}_\theta(r) + i\alpha V_z\widehat{v}_\theta(r) = 0 \tag{2.32}$$

$$\frac{i\alpha}{\rho}\widehat{p}(r) - i\alpha c\widehat{v}_z(r) + i\alpha V_z\widehat{v}_z(r) = 0 \tag{2.33}$$

where V_z is defined as $(v_\infty + \Delta v)$.

The above system may be reduced to a single equation by differentiating (2.30) with respect to r, then substituting velocity perturbation expressions from (2.31-33) into the differentiated form of equation (2.30). Doing this and rearranging results in a single second-order ordinary differential equation for the pressure eigenfunction which is given by

$$\frac{d^2\widehat{p}(r)}{dr^2} + \frac{1}{r}\frac{d\widehat{p}(r)}{dr} - \Big(\frac{m^2}{r^2} + \alpha^2\Big)\widehat{p}(r) = 0 \tag{2.34}$$

where equation (2.34) is readily recognized as a modified Bessel equation. Subsequently, (2.34) admits solutions of the form,

$$\widehat{p}(r) = A_1 I_m(\alpha r) + A_2 K_m(\alpha r) \tag{2.35}$$

where, I_m and K_m are modified Bessel functions and A_1 and A_2 are constants.

In the region outside the jet, the quiescent flow is irrotational and applying standard results from vector calculus, it follows that a velocity potential may be defined; that is,

$$\nabla \times \vec{v} = \vec{0} \;\; \Rightarrow \exists\Phi \ni \vec{v} = \nabla\Phi \tag{2.36}$$

furthermore, the field is incompressible, thus

$$div(\vec{v}) = div\big(\nabla\Phi\big) = \nabla^2\Phi = 0 \tag{2.37}$$

Hence, in the farfield the flow is potential and external disturbances are governed by the Laplacian for a perturbation velocity potential, $\delta\phi$. Expanding again in normal-modes, it follows that

$$\delta\phi = \widehat{\Phi}exp[(i\alpha(z-ct)+im\theta)] \tag{2.38}$$

In cylindrical coördinates the Laplacian operator is expressed as,

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} \tag{2.39}$$

Upon substitution of $\delta\phi$ into $\nabla^2\Phi = 0$ in cylindrical coördinates and dividing by the common exponential $exp[(i\alpha(z-ct)+im\theta)]$, one obtains,

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\widehat{\Phi}(r)}{dr}\right) + \frac{1}{r^2}(-m^2)\widehat{\Phi}(r) - \alpha^2\widehat{\Phi}(r) = 0 \tag{2.40}$$

Expanding the differential term and rearanging the equation yields

$$\frac{d^2\widehat{\Phi}(r)}{dr^2} + \frac{1}{r}\frac{d\widehat{\Phi}(r)}{dr} - \Big(\frac{m^2}{r^2} + \alpha^2\Big)\widehat{\Phi}(r) = 0 \tag{2.41}$$

Again, equation (2.41) is easily recognizable as a modified Bessel equation with a linear combination of the I_m and K_m Bessel functions as a general solution,

$$\widehat{\Phi}(r) = B_1 I_m(\alpha r) + B_2 K_m(\alpha r) \tag{2.42}$$

2.1.1 Dispersion Relation for the Axisymmetric Jet

The above analysis gives eigenfunction solutions in the core and the far field for the axisymmetric jet as,

$$\widehat{p}(r) = A_1 I_m(\alpha r) + A_2 K_m(\alpha r) \qquad \text{inside the jet, } (r < R) \tag{2.43a}$$

$$\widehat{\Phi}(r) = B_1 I_m(\alpha r) + B_2 K_m(\alpha r) \qquad \text{outside the jet, } (r > R) \tag{2.43b}$$

There are certain boundary conditions derived from physical constraints on the system that should be applied direclty to the above solutions: first, the pressure eigenfunction vanishes at $r = 0$, second, the velocity potential eigenfunction vanishes at $r = \infty$.

Applying these boundary condition to equations (2.43-a) and (2.43-b) yields,

$$K_m(0) \neq 0\ \forall m \ \Rightarrow \boxed{A_2 = 0} \tag{2.44a}$$

$$\lim_{b\to\infty} I_m(b) \neq 0\ \forall m \ \Rightarrow \boxed{B_1 = 0} \tag{2.44b}$$

respectively. Consequently, the solutions are written as

$$\widehat{p}(r) = A_1 I_m(\alpha r) \qquad \text{inside the jet, } (r < R) \tag{2.45a}$$

$$\widehat{\Phi}(r) = B_2 K_m(\alpha r) \qquad \text{outside the jet, } (r > R) \tag{2.45b}$$

Next, the kinematic and dynamic boundary conditions must be applied across the vortex sheet at $r = R$ to match the far-field and jet core solutions. First it is assumed that the radial displacement of the vortex sheet, $\eta(\theta, z, t)$, due the the disturbance may be expressed as a normal-mode; that is,

$$\eta(\theta, z, t) = Ce^{i(\alpha(z-c)+m\theta)} \tag{2.46}$$

where C is a constant.

Physically, the kinematic boundary condition is equivalent to the principle of conservation of mass at the interface; that is, $DG/Dt = 0$ on the material surface $G = 0$. Applying this inside and outside the vortex sheet at the displacement interface η yields

$$\frac{\partial \eta}{\partial t} + V_z \frac{\partial \eta}{\partial z} = \left(\frac{\partial \delta p}{\partial r} \right)_{r=R} \tag{2.47}$$

and

$$\frac{\partial \eta}{\partial t} = \left(\frac{\partial \delta \phi}{\partial r} \right)_{r=R} \tag{2.48}$$

respectively.

Applying (2.46) in equation (2.47) and eliminating the common exponential $exp[(i\alpha(z-ct)+im\theta)]$ yields

$$i\alpha\Big(V_z - c\Big)C = A_1 \frac{\partial I_m(x)}{\partial x}\bigg|_{x=\alpha R} \tag{2.49}$$

Similarly, from (2.48) we have

$$-i\alpha c C = B_2 \frac{\partial K_m(x)}{\partial x}\bigg|_{x=\alpha R} \tag{2.50}$$

Physically, the dynamic boundary condition is equivalent to the conservation of linear momentum at the interface. In this case both fluids are inviscid on either side of the vortex sheet, thus, the dynamic boundary condition reduces to a statement of pressure continuity across the interface, *i.e.*

$$\left(\frac{\partial \delta p}{\partial t} + V_z \frac{\partial \delta p}{\partial z} \right)_{r=R} = \frac{\partial \delta \phi}{\partial t}\bigg|_{r=R} \tag{2.51}$$

Applying this to the pressure and velocity potential eigenfunctions and again eliminating the common exponential and letting $v_\infty = 0$ without loss of generality yields

$$i\alpha\Big(\Delta v - c\Big)A_1 I_m(\alpha R) = -i\alpha c B_2 K_m(\alpha R) \tag{2.52}$$

Rejecting trivial solutions to the eigen-system (2.49-50) leaves,

$$A_1 = C\frac{i\alpha\Big(\Delta v - c\Big)}{I_m{}'(\alpha R)} \tag{2.53}$$

$$B_2 = C\frac{-i\alpha c}{K_m{}'(\alpha R)} \tag{2.54}$$

where,

$$I_m{}'(\alpha R) \equiv \frac{\partial I_m(x)}{\partial x}\bigg|_{x=\alpha R} \tag{2.55a}$$

$$K_m{}'(\alpha R) \equiv \frac{\partial K_m(x)}{\partial x}\bigg|_{x=\alpha R} \tag{2.55b}$$

Substituting these values for A_1 and B_2 into (2.52) produces the following dispersion relation for the steady axisymmetric jet:

$$\boxed{\frac{I_m{}'(\alpha R)}{I_m(\alpha R)} = \Big(\frac{\Delta v - c}{c}\Big)^2 \frac{K_m{}'(\alpha R)}{K_m(\alpha R)}} \tag{2.56}$$

The solution of equation (2.56) renders a relationship between growth rate, celerity and axial wavenumber at fixed m for the axisymmetric jet.

2.1.2 Numerical Solution Procedure: Axisymmetric Jet

The dispersion relation (2.56) is a non-trival complex valued function of c_R, c_I and α of the form $F(c_R, c_I, \alpha; m) = 0$. Here $F = 0$ was solved via a complex-valued Newton-Raphson method written in MATLAB. The complex-valued function $F(c, \alpha)$ was differentiated numerically by taking differential perturbations along both the real and complex axis in the Argand plane. The differentiation was carried out along both real and imaginary axes to ensure the validity of the derivative as the Cauchy-Riemann condition was built into the code guaranteeing the analyticity of F for all values of m, α and c_R, c_I. In this sense, given a complex initial guess c of the root, the numerical derivative is taken as $[F(c) - F(c + \Delta c_R)]/\Delta c_R$ -and- $[F(c) - F(c + i\Delta c_I)]/\Delta c_I$.

Following the standard procedure using a Taylor expansion to first order, an iterated guess of the root is based on the subsequent guess minus the function value at the initial guess divided by the complex-valued derivative. The dispersion relation $F = 0$ for the axisymmetric jet is well behaved inasmuch as there were no convergence problems using only the standard Newton-Raphson iteration. Thus, an initial guess of $c_R + ic_I$ that is equal to $\epsilon + i\epsilon$ where $\epsilon \ll 1$ was sufficient for appropriately small iteration steps to produce the solution.

2.1.3 Numerical Results for the Steady Axisymmetric Jet

Solving equation (2.56) using the numerical procedure outlined in section 2.1.2 produces the following results. Figure 2.2 shows the celerity as a function of the axial wavenumber for azimuthal wavenumbers $m = \{-1, 0, 1, 2, 3\}$. The results indicate that long waves, $(\alpha R) \to 0$, travel initially with celerity equal to the velocity of the center of the jet for the Kelvin-Helmholtz mode $(m = 0)$. Whereas, long waves travel initially with celerity equal to one-half of the jet-center velocity for helical modes ($m \neq 0$) where the axial symmetry is broken [7]. Figure 2.3 shows the growth rate as a function of axial wavenumber. Here, the numerical results indicate that the flow is unstable with respect to small disturbances for all values of azimuthal wave number

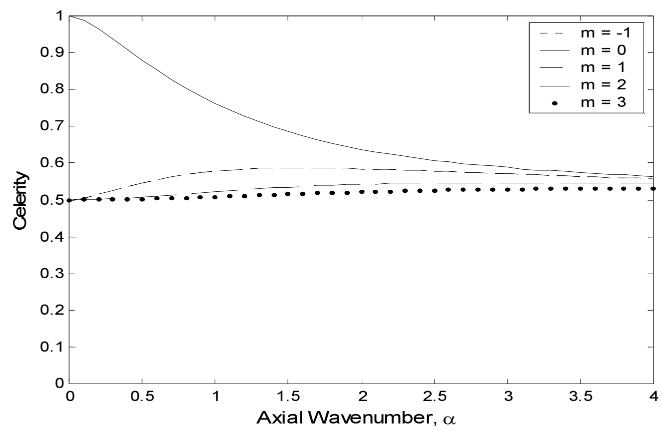

Figure 2.2: Celerity vs. Axial Wavenumber for the Steady Axisymmetric Jet

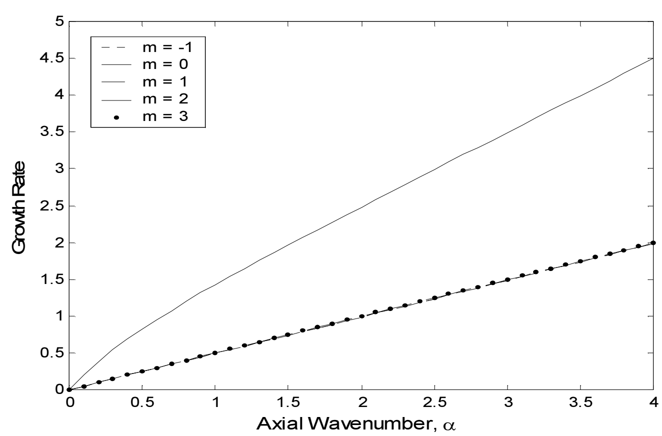

Figure 2.3: Growth Rate vs. Axial Wavenumber for the Steady Axisymmetric Jet

m and axial wave number α, as the growth rate of all disturbances increases monotonically. As $(\alpha R) \to \infty$, the results approach that of a disturbance of a plane vortex sheet, as expected [8]. That is, as $(\alpha R) \to \infty$ the vortex sheet effectively 'opens up' so that curvature is negligable and locally, the jet looks like a plane vortex sheet. The fact that the flow is unstable to all disturbances of the vortex sheet is due to the idealization of the model; specifically, the vortex sheet has zero thickness. In a realistic flow, the vortex sheet would have finite thickness and the flow would be stable to disturbances whose wavelength is shorter than that thickness. The result would be the existence of an extremum in the growth rate-axial wave number relation.

2.2 Stability of Swirling Jets

In this section, the stability of a steady swirling jet is considered. An exahustive treatment is given by, Loiseleux *et.al* [9] (spatio-temporal), Coleman [5] [6](spatial) and Billant *et.al* [12]. The swirling jet differs from the axisymmetric jet by the addition of a solid-body core rotation. Herein, the swirling jet is modeled as a Rankine vortex with axial flow; that is, rotation that increases linearly with the radial distance from the jet center inside the core and decreases proportional to $1/r$ outside the jet. This field is given by,

$$\vec{V}(r,\theta,z) = \begin{cases} (0, \Omega r, v_\infty + \Delta v), & \text{for } r < R \\ (0, \frac{\Omega R^2}{r}, v_\infty), & \text{for } r > R \end{cases} \tag{2.57}$$

where, v_∞ is the far-field free stream velocity, Δv is the axial velocity difference across the vortex sheet and Ω is the constant solid-body rotation of the

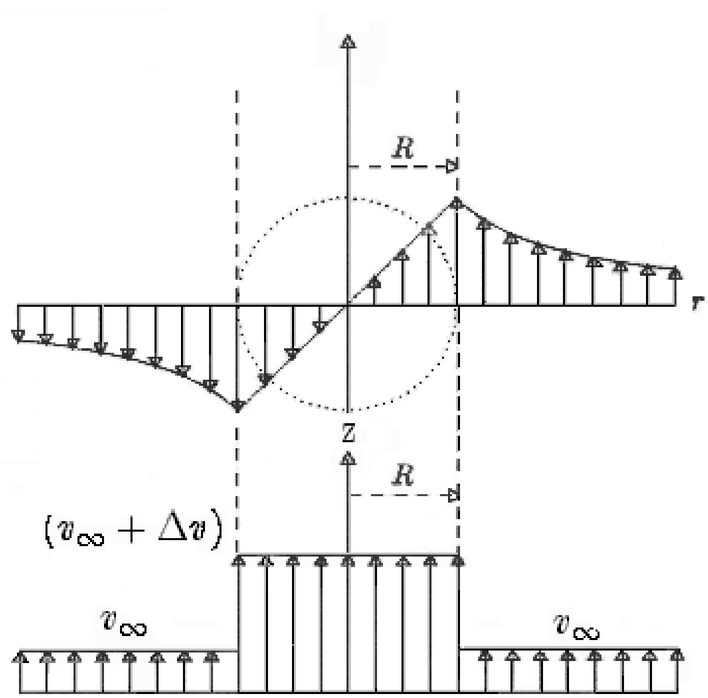

Figure 2.4: Sketch of Basic Flow for the Swirling Jet

core. In the following, the stability equations for the flow inside and outside the jet core are derived.

1. *Inside the jet* ($r < R$) Inside the jet core, we start again from the governing equations for the given fields $(\vec{v}, p)$ *i.e.* Euler's and the continuity equations written as,

$$\frac{\partial \vec{v}}{\partial t} + \nabla \vec{v} \cdot \vec{v} + \frac{1}{\rho} \nabla p = \vec{0} \tag{2.58}$$

$$\nabla \cdot \vec{v} = 0 \tag{2.59}$$

Substituting (2.57) into the cylindrical coördinate component form of (2.58-59) given by (2.3-6) it follows that for the mean flow components, $\big((0, V_\theta, V_z), P\big)$ satisfy the following equations.

$$\frac{V_\theta^2}{r} = \frac{1}{\rho}\frac{\partial P}{\partial r} \tag{2.60}$$

$$\rho\Big(\frac{\partial V_\theta}{\partial t} + \frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + V_z\frac{\partial V_\theta}{\partial z}\Big) = -\frac{1}{r}\frac{\partial P}{\partial \theta} \tag{2.61}$$

$$\rho\Big(\frac{\partial V_z}{\partial t} + \frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z\frac{\partial V_z}{\partial z}\Big) = -\frac{\partial P}{\partial z} \tag{2.62}$$

and

$$\frac{1}{r}\frac{\partial}{\partial \theta}\big(\rho V_z\big) + \frac{\partial}{\partial z}\big(\rho V_z\big) = 0 \tag{2.63}$$

Again, adding perturbations $\delta\vec{v}$, and δp in the following way,

$$\vec{v} \mapsto \vec{V} + \delta\vec{v} = (\delta v_r, V_\theta + \delta v_\theta, V_z + \delta v_z) \tag{2.64}$$

$$p \mapsto P + \delta p \tag{2.65}$$

and, substituting (2.64) and (2.65) into (2.3-6) yields the following non-linear equations that govern the total flow,

$$\rho\Bigg(\frac{\partial \delta v_r}{\partial t} + \delta v_r\frac{\partial \delta v_r}{\partial r} + \frac{V_\theta}{r}\frac{\partial \delta v_r}{\partial \theta} + \frac{\delta v_\theta}{r}\frac{\partial \delta v_r}{\partial \theta} - \frac{V_\theta^2}{r} - 2\frac{V_\theta \delta v_\theta}{r} - \frac{\delta v_\theta^2}{r} + V_z\frac{\partial \delta v_z}{\partial z} + \delta v_z\frac{\partial \delta v_r}{\partial z}\Bigg) = -\frac{\partial P}{\partial r} - \frac{\partial \delta p}{\partial r} \tag{2.66}$$

$$\rho\Bigg(\frac{\partial V_\theta}{\partial t}+\frac{\partial \delta v_\theta}{\partial t}+\delta v_r\frac{\partial V_\theta}{\partial r}+\delta v_r\frac{\partial \delta v_\theta}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{V_\theta}{r}\frac{\partial \delta v_\theta}{\partial \theta}$$
$$+\frac{\delta v_\theta}{r}\frac{\partial V_\theta}{\partial \theta}+\frac{\delta v_\theta}{r}\frac{\partial \delta v_\theta}{\partial \theta}+\frac{\delta v_r V_\theta}{r}+\frac{\delta v_\theta \delta v_r}{r}+V_z\frac{\partial V_\theta}{\partial z}$$
$$+V_z\frac{\partial \delta v_\theta}{\partial z}+\delta v_z\frac{\partial V_\theta}{\partial z}+\delta v_z\frac{\partial \delta v_\theta}{\partial z}\Bigg)=-\frac{1}{r}\Bigg(\frac{\partial P}{\partial \theta}+\frac{\partial \delta p}{\partial \theta}\Bigg) \quad (2.67)$$

$$\rho\Bigg(\frac{\partial V_z}{\partial t}+\frac{\partial \delta v_z}{\partial t}+\delta v_r\frac{\partial V_z}{\partial r}+\delta v_r\frac{\partial \delta v_z}{\partial r}+\frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta}+\frac{V_\theta}{r}\frac{\partial \delta v_z}{\partial \theta}$$
$$+\frac{\delta v_\theta}{r}\frac{\partial V_z}{\partial \theta}+\frac{\delta v_\theta}{r}\frac{\partial \delta v_z}{\partial \theta}+V_z\frac{\partial V_z}{\partial z}+V_z\frac{\partial \delta v_z}{\partial z}$$
$$+\delta v_z\frac{\partial V_z}{\partial z}+\delta v_z\frac{\partial \delta v_z}{\partial z}\Bigg)=-\frac{\partial P}{\partial z}-\frac{\partial \delta p}{\partial z} \quad (2.68)$$

and,

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(\rho r\delta v_r\Big)+\frac{1}{r}\frac{\partial}{\partial \theta}\Big(\rho\delta v_\theta\Big)+\frac{\partial}{\partial z}\Big(\rho\delta v_z)+\frac{\partial}{\partial z}\Big(\rho V_z\Big)+\frac{\partial}{\partial \theta}\Big(\rho V_\theta\Big)=0 \quad (2.69)$$

Subtracting the mean flow equations (2.60-63) from the total flow equations leaves the equations governing the perturbations as,

$$\rho\Bigg(\frac{\partial \delta v_r}{\partial t}+\delta v_r\frac{\partial \delta v_r}{\partial r}+\frac{V_\theta}{r}\frac{\partial \delta v_r}{\partial \theta}+\frac{\delta v_\theta}{r}\frac{\partial \delta v_r}{\partial \theta}-2\frac{V_\theta \delta_\theta}{r}-\frac{\delta v_\theta^2}{r}$$
$$+V_z\frac{\partial \delta v_z}{\partial z}+\delta v_z\frac{\partial \delta v_r}{\partial z}\Bigg)=-\frac{\partial \delta p}{\partial r} \quad (2.70)$$

$$\rho\Bigg(\frac{\partial\delta v_\theta}{\partial t}+\delta v_r\frac{\partial V_\theta}{\partial r}+\delta v_r\frac{\partial\delta v_\theta}{\partial r}+\frac{V_\theta}{r}\frac{\partial\delta v_\theta}{\partial\theta}+\frac{\delta v_\theta}{r}\frac{\partial V_\theta}{\partial\theta}+\frac{\delta v_\theta}{r}\frac{\partial\delta v_\theta}{\partial\theta}$$
$$+\frac{\delta v_r V_\theta}{r}+\frac{\delta v_\theta\delta v_r}{r}+V_z\frac{\partial\delta v_\theta}{\partial z}+\delta v_z\frac{\partial V_\theta}{\partial z}+\delta v_z\frac{\partial\delta v_\theta}{\partial z}\Bigg)=-\frac{1}{r}\frac{\partial\delta p}{\partial\theta}\quad(2.71)$$

$$\rho\Bigg(\frac{\partial\delta v_z}{\partial t}+\delta v_r\frac{\partial V_z}{\partial r}+\delta v_r\frac{\partial\delta v_z}{\partial r}+\frac{V_\theta}{r}\frac{\partial\delta v_z}{\partial\theta}+\frac{\delta v_\theta}{r}\frac{\partial V_z}{\partial\theta}+\frac{\delta v_\theta}{r}\frac{\partial\delta v_z}{\partial\theta}$$
$$+V_z\frac{\partial\delta v_z}{\partial z}+\delta v_z\frac{\partial V_z}{\partial z}+\delta v_z\frac{\partial\delta v_z}{\partial z}\Bigg)=-\frac{\partial\delta p}{\partial z}\quad(2.72)$$

and,

$$\frac{1}{r}\frac{\partial}{\partial r}\Big(\rho r\delta v_r\Big)+\frac{1}{r}\frac{\partial}{\partial\theta}\Big(\rho\delta v_\theta\Big)+\frac{\partial}{\partial z}\Big(\rho\delta v_z\Big)=0\quad(2.73)$$

Because a linear stability analysis is of interest, the perturbations are taken as $\delta f = O(\varepsilon)$ so that combinations $O(\varepsilon^2)$ may be neglected. Doing this, equations (2.70-72) become,

$$\rho\Bigg(\frac{\partial\delta v_r}{\partial t}+\frac{V_\theta}{r}\frac{\partial\delta v_r}{\partial\theta}-2\frac{V_\theta\delta v_\theta}{r}+V_z\frac{\partial\delta v_z}{\partial z}\Bigg)=-\frac{\partial\delta p}{\partial r}\quad(2.74)$$

$$\rho\left(\frac{\partial \delta v_\theta}{\partial t} + \delta v_r \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial \delta v_\theta}{\partial \theta} + \frac{\delta v_\theta}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\delta v_r V_\theta}{r} + V_z \frac{\partial \delta v_\theta}{\partial z} + \delta v_z \frac{\partial V_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial \delta p}{\partial \theta} \quad (2.75)$$

$$\rho\left(\frac{\partial \delta v_z}{\partial t} + \delta v_r \frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial \delta v_z}{\partial \theta} + \frac{\delta v_\theta}{r}\frac{\partial V_z}{\partial \theta} + V_z \frac{\partial \delta v_z}{\partial z} + \delta v_z \frac{\partial V_z}{\partial z}\right) = -\frac{\partial \delta p}{\partial z} \quad (2.76)$$

Requiring perturbations to be bounded in the sense of the L_2-norm, we again may expand the perturbations in harmonics,

$$\delta f = \widehat{f}(r) e^{(i\alpha(z-ct)+im\theta)}. \quad (2.77)$$

After substituting (2.57) for $(r < R)$ for the generic velocity field $(0, V_\theta, V_z)$ and using the harmonic form of the perturbations (2.77), the following system of normal-mode stability equations for the steady swirling jet is obtained,

$$\frac{1}{\rho}\frac{d\widehat{p}(r)}{dr} = i\left(\alpha c - m\Omega - \alpha V_z\right)\widehat{v}_r(r) + 2\Omega\widehat{v}_\theta(r) \quad (2.78)$$

$$\frac{im}{\rho r}\widehat{p}(r) = i\left(\alpha c - m\Omega - \alpha V_z\right)\widehat{v}_\theta(r) - 2\Omega\widehat{v}_r(r) \quad (2.79)$$

$$\frac{i\alpha}{\rho}\widehat{p}(r) = i\left(\alpha c - m\Omega - \alpha V_z\right)\widehat{v}_z(r) \quad (2.80)$$

$$\frac{d\widehat{v}_r(r)}{dr} = -\frac{1}{r}\widehat{v}_r(r) - \frac{im}{r}\widehat{v}_\theta(r) - i\alpha\widehat{v}_z(r) \tag{2.81}$$

where V_z is defined as $(v_\infty + \Delta v)$

Using the definition of the celerity, $c = \omega/\alpha$ and following Loiseleux *et al.* [9], the above system may be simplified by recognizing that

$$\widetilde{\omega} \equiv \Big(\omega - m\Omega - \alpha V_z\Big) \tag{2.82}$$

represents the Doppler shifted frequency in an inertial frame moving with the jet core. Substituting the slightly simplified system of coupled differential equations results in,

$$\frac{1}{\rho}\frac{d\widehat{p}(r)}{dr} = i\widetilde{\omega}\widehat{v}_r(r) + 2\Omega\widehat{v}_\theta(r) \tag{2.83}$$

$$\frac{im}{\rho r}\widehat{p}(r) = i\widetilde{\omega}\widehat{v}_\theta(r) - 2\Omega\widehat{v}_r(r) \tag{2.84}$$

$$\frac{i\alpha}{\rho}\widehat{p}(r) = i\widetilde{\omega}\widehat{v}_z(r) \tag{2.85}$$

$$\frac{d\widehat{v}_r(r)}{dr} = -\frac{1}{r}\widehat{v}_r(r) - \frac{im}{r}\widehat{v}_\theta(r) - i\alpha\widehat{v}_z(r) \tag{2.86}$$

Substituting (2.85) into (2.86) directly produces the equation,

$$\frac{d\widehat{v}_r(r)}{dr} = -\frac{1}{r}\widehat{v}_r(r) - \frac{im}{r}\widehat{v}_\theta(r) - \frac{i\alpha^2}{\widetilde{\omega}\rho}\widehat{p}(r) \tag{2.87}$$

Solving for $\widehat{v}_\theta$ in (2.83) and (2.84) and equating leaves an equation in terms of $\widehat{v}_r$, and $\widehat{p}$ which may be solved for $\widehat{v}_r$.

$$\widehat{v}_r(r) = -\frac{2i\Omega\widetilde{\omega}}{\Xi}\left(\frac{1}{2\Omega\rho}\frac{d\widehat{p}(r)}{dr} - \frac{m}{\widetilde{\omega}\rho r}\widehat{p}(r)\right) \tag{2.88}$$

where,

$$\Xi \equiv \widetilde{\omega}^2 - 4\Omega^2$$

Now, substituting (2.88) into (2.83) and (2.84) then solving in both equations for $\widehat{v}_\theta$ leaves two equations. Equating these leaves a single equation for $\widehat{v}_\theta$ in terms of $\widehat{p}$ and $\widehat{p}'$,

$$\widehat{v}_\theta(r) = \left(\frac{2\Omega}{\rho\Xi}\right)\frac{d\widehat{p}(r)}{dr} - \left(\frac{m}{\widetilde{\omega}\rho r\Xi}\right)\widehat{p}(r) \tag{2.89}$$

Differentiating (2.83) with respect to r yields

$$\frac{1}{\rho}\frac{d^2\widehat{p}(r)}{dr^2} = i\widetilde{\omega}\frac{d\widehat{v}_r(r)}{dr} + 2\Omega\frac{d\widehat{v}_\theta(r)}{dr}. \tag{2.90}$$

Differentiating (2.84) with respect to r leaves

$$\frac{d\widehat{v}_\theta(r)}{dr} = \frac{m}{\rho\widetilde{\omega}r}\frac{d\widehat{p}(r)}{dr} - \frac{m}{\rho\widetilde{\omega}r^2}\widehat{p}(r) - \frac{2i\Omega}{\widetilde{\omega}}\frac{d\widehat{v}_r(r)}{dr}. \tag{2.91}$$

Substituting (2.87) into (2.90) and using (2.88), (2.89) and (2.91) leaves, after some manipulation,

$$\frac{d^2\widehat{p}(r)}{dr^2} + \frac{1}{r}\frac{d\widehat{p}(r)}{dr} - \left(\frac{\alpha^2\Xi}{\widetilde{\omega}^2} - \frac{m^2}{r^2}\right)\widehat{p}(r) = 0 \tag{2.92}$$

Substituting Ξ into (2.92) and rearranging terms produces the following equation governing pressure perturbations in the swirling jet core, $(r < R)$,

$$\frac{d^2\widehat{p}(r)}{dr^2} + \frac{1}{r}\frac{d\widehat{p}(r)}{dr} - \left(\alpha^2 r^2\left(\frac{4\Omega^2}{\widetilde{\omega}^2} - 1\right) - m^2\right)\widehat{p}(r) = 0 \tag{2.93}$$

Equation (2.80) is readily recognized as a modified Bessel equation and admits solutions of the form,

$$\widehat{p}(r) = A_1 J_m\left(r\sqrt{\alpha^2\left(\frac{4\Omega^2}{\widetilde{\omega}^2} - 1\right)}\right) + A_2 Y_m\left(r\sqrt{\alpha^2\left(\frac{4\Omega^2}{\widetilde{\omega}^2} - 1\right)}\right) \tag{2.94}$$

where, J_m and Y_m are modified Bessel functions and A_1 and A_2 are constants. Again, considering physical constraints on the pressure eigenfunction, we note that

$$\lim_{b\to\infty} Y_m(b) \to \infty \;\forall m \;\Rightarrow \boxed{A_2 = 0} \tag{2.95a}$$

Therefore, the perturbation pressure eigenfunction in the jet core is then given by,

$$\widehat{p}(r) = A_1 J_m\left(r\sqrt{\alpha^2\left(\frac{4\Omega^2}{\widetilde{\omega}^2} - 1\right)}\right) \tag{2.96}$$

For convenience we will let

$$\sqrt{\alpha^2\left(\frac{4\Omega^2}{\widetilde{\omega}^2}-1\right)} \equiv \Psi \tag{2.97}$$

so that

$$\widehat{p}(r) = A_1 J_m\big(r\Psi\big) \tag{2.98}$$

Now that $\widehat{p}$ is determined, the stability eigen-system (2.83), (2.84) and (2.87) is given by,

$$\frac{1}{\rho}\Psi A_1 J_m{}'\big(r\Psi\big) = i(\widetilde{\omega})\widehat{v}_r + 2\Omega\widehat{v}_\theta \tag{2.99}$$

$$\frac{im}{\rho r}A_1 J_m\big(r\Psi\big) = i(\widetilde{\omega})\widehat{v}_\theta - 2\Omega\widehat{v}_r \tag{2.100}$$

$$\frac{d\widehat{v}_r(r)}{dr} = -\frac{1}{r}\widehat{v}_r(r) - \frac{im}{r}\widehat{v}_\theta(r) - \frac{i\alpha^2}{\widetilde{\omega}\rho}A_1 J_m\big(r\Psi\big) \tag{2.101}$$

These equations may be formally manipulated to yield an expression for the radial perturbation velocity eigenfunction in terms of the pressure eigenfunction $\widehat{p}(r)$. Equations (2.99) and (2.100) may be solved for $\widehat{v}_\theta$ in terms of $\widehat{v}_r$, $J_m\big(r\Psi\big)$ and $J_m{}'\big(r\Psi\big)$, then, equating the resulting expressions for $\widehat{v}_r$ yields, after some algebra,

$$\widehat{v}_r = \frac{i\alpha^2 A_1}{\rho\widetilde{\omega}\Psi^2}\left[-\frac{2\Omega m}{r\widetilde{\omega}}J_m\big(r\Psi\big) + \Psi J_m{}'\big(r\Psi\big)\right] \tag{2.102}$$

2. *Outside the jet* $(r > R)$

It is easilly verified that $\nabla \times \vec{v} = \vec{0}$ in the external region of the swirling jet $(r > R)$ where $\vec{v} = (0, \frac{\Omega R^2}{r}, v_\infty + \Delta v)$. Thus, in the far field the flow is irrotational and again it follows that there exists a velocity potential,

$$\vec{v} = \nabla\Phi \text{ for some } \Phi$$

Moreover, the field is incompressible, thus

$$div(\vec{v}) = div\big(\nabla\Phi\big) = \nabla^2\Phi = 0$$

Hence, in the farfield the flow is potential and external disturbances are governed by the Laplacian for a perturbation velocity potential, $\delta\phi$. Expanding again in normal-modes, $\delta\phi = \widehat{\Phi} exp[(i\alpha(z - ct) + im\theta)]$ is substituted this into the Laplacian, $\nabla^2\delta\phi = 0$. Once this is done utilizing the the cylindrical coördinate form of the Laplacian operator (2.93), and the common terms $exp[(i\alpha(z - ct) + im\theta)]$ are eliminated yields,

$$\frac{d^2\widehat{\Phi}(r)}{dr^2} + \frac{1}{r}\frac{d\widehat{\Phi}(r)}{dr} - \Big(\frac{m^2}{r^2} + \alpha^2\Big)\widehat{\Phi}(r) = 0 \tag{2.103}$$

Equation (2.103) is easily recognizable as a modified Bessel equation with a linear combination of the I_m and K_m Bessel functions of order $|m|$ as a general solution,

$$\widehat{\Phi}(r) = B_1 I_m(\alpha r) + B_2 K_m(\alpha r) \tag{2.104}$$

Finite velocity is required as $r \to \infty$, thus,

$$\lim_{b \to \infty} I_m(b) \to \infty \; \forall m \;\Rightarrow \boxed{B_1 = 0} \tag{2.105a}$$

Therefore the pressure eigenfunction outside the jet takes the form,

$$\widehat{\Phi}(r) = B_2 K_m(\alpha r) \tag{2.106}$$

There is another condition to impose on the solution, in light of condition (2.105a) where the asymptotic behavior of the K_m Bessel function as $r \to \infty$ is given by,

$$K_m(s\alpha r) \sim \frac{e^{-s\alpha r}}{\sqrt{2s\alpha r}} \tag{2.107}$$

where, $s \equiv sgn(\alpha)$ to ensure the decay of K_m as $r \to \infty$, then the sign function s must be introduced in the argument of K_m. Hence, the correct form of the solution is written as,

$$\widehat{\Phi}(r) = B_2 K_m(s\alpha r) \tag{2.108}$$

To determine the pressure eigenfunction in the far-field, we recall that the velocity vector is related to the velocity potential via the gradient operator and the r-component is given as follows

$$\vec{v} = \nabla \Phi \;\Rightarrow \langle \vec{v} | \underline{e}_r \rangle = \frac{\partial \Phi}{\partial r} \tag{2.109}$$

It follows, then, that

$$\widehat{v}_r(r) = B_2 \frac{\partial K_m(x)}{\partial x}\bigg|_{x=\alpha r} \tag{2.110}$$

Because we have a velocity potential in the freestream, Bernoulli's equation yields the pressure. That is, the mean pressure is given in terms of the mean velocity potential by the relation,

$$\frac{P}{\rho} + \frac{\partial \Phi}{\partial t} + \frac{1}{2}\langle \nabla\Phi \big| \nabla\Phi \rangle = 0 \tag{2.111}$$

Perturbations of the mean pressure and velocity potential are taken, as usual, in the following way,

$$p \mapsto P + \delta p \tag{2.112}$$

$$\phi \mapsto \Phi + \delta\phi \tag{2.113}$$

with,

$$\delta f = \widehat{f}(r) e^{(i\alpha(z-ct)+im\theta)} \tag{2.114}$$

where Φ and P are mean values and $\delta\phi$ and δp are perturbations of the field. After substituting (2.112-113) into (2.111) it follows directly that,

$$\frac{P}{\rho} + \frac{\delta p}{\rho} + \frac{\partial \Phi}{\partial t} + \frac{\partial \delta\phi}{\partial t} + \frac{1}{2}\Big[\langle \nabla\Phi \big| \nabla\Phi \rangle + 2\langle \nabla\delta\phi \big| \nabla\Phi \rangle + \langle \nabla\delta\phi \big| \nabla\delta\phi \rangle\Big] = 0 \tag{2.115}$$

Subtracting the mean (2.111) from (2.115) and throwing out $O(\varepsilon^2)$ combinations of the $O(\varepsilon)$ perturbations $\delta\phi$ it follows that the linearized pressure and velocity potential perturbations are governed in the farfield by the following,

$$\frac{\delta p}{\rho} + \frac{\partial \delta\phi}{\partial t} + \langle \nabla\delta\phi \big| \nabla\Phi \rangle = 0 \tag{2.116}$$

where,

$$\delta p = \widehat{p}(r) exp[(i\alpha(z - ct) + im\theta)] \tag{2.117a}$$

$$\delta\phi = \widehat{\phi}(r) exp[(i\alpha(z - ct) + im\theta)] \tag{2.117b}$$

$$= B_2 K_m(\alpha r) exp[(i\alpha(z - ct) + im\theta)] \tag{2.117c}$$

$$\nabla\Phi = \big(0, \Omega R^2/r, v_\infty + \Delta v\big) \; ; \; (r > R) \tag{2.117d}$$

Using (2.117a-d) with (2.116) and eliminating the exponential $exp[(i\alpha(z - ct) + im\theta)]$ leaves an expression for the perturbation pressure eigenfunction in the farfield of the swirling jet,

$$\widehat{p} = -i\rho\Big(- \alpha c + \frac{m\Omega R^2}{r^2} + \alpha v_\infty\Big) B_2 K_m(\alpha r) \tag{2.118}$$

2.2.1 Dispersion Relation for the Steady Swirling Jet

General solutions for pressure and radial velocity perturbations in the jet core and outside the jet have been obtained. So, to summarize the results thus far, we have the following stability eigenfunctions for the swirling jet.

♢ Inside the jet, $(r < R)$

$$\begin{cases} \widehat{p}(r) = A_1 J_m\big(r\Psi\big) \\ \widehat{v}_r(r) = [i\alpha^2 A_1]/[\rho\widetilde{\omega}\Psi^2]\Big([-2\Omega m]/[r\widetilde{\omega}]J_m\big(r\Psi\big) + \Psi J_m{}'\big(r\Psi\big)\Big) \end{cases} \tag{2.119}$$

♢ Outside the jet, $(r > R)$

$$\begin{cases} \widehat{v}_r(r) = B_2 \frac{\partial K_m(x)}{\partial x}\Big|_{x=\alpha r} \\ \widehat{p}(r) = -i\rho\Big(-\alpha c + \frac{m\Omega R^2}{r^2} + \alpha v_\infty\Big) B_2 K_m(\alpha r) \end{cases} \tag{2.120}$$

Across the vortex sheet $r = R$, pressure and velocity must be continuous; again following Loiseleux *et.al* [9] we obtain,

$$\frac{\widehat{v}_r(R^-)}{\widetilde{\omega}} = \frac{\widehat{v}_r(R^+)}{\widetilde{\omega} + \alpha\Delta v} \tag{2.121}$$

$$\widehat{p}(R^-) = \widehat{p}(R^+) \tag{2.122}$$

In light of the matching conditions (2.121) and (2.122), the eigenfunction equations in the regions $r < R$ given by (2.119) and $r > R$ given by (2.120) reduce to two equations in two unknowns A_1 and B_2. Disregarding trivial solutions to the system of equations requires that the determinant of the system equal zero. Doing this and manipulating the result algebraically leads to the following dispersion relation for the steady swirling jet:

$$(\widetilde{\omega}+\alpha\Delta v)^2\left[-\frac{2\Omega m}{\widetilde{\omega}}+\Psi R\frac{J_m{}'(R\Psi)}{J_m(R\Psi)}\right]=-\frac{\widetilde{\omega}^2\Psi^2}{s\alpha R}\frac{K_m{}'(s\alpha R)}{K_m(s\alpha R)} \tag{2.123}$$

It is straight forward to verify that the above dispersion relation (2.123) reduces to exactly that of the axisymmetric jet (2.56) when the rotation rate Ω is set equal to zero. This is, of course, a necessary requirment for validity of the analysis.

2.2.2 Nondimensionalization of the Dispersion Relation

The dispersion relation for the swirling jet in the form (2.123) is dimensional inasmuch as,

$$\left.\begin{aligned}\alpha &\sim [1]\\ m &\sim [1]\\ \Omega &\sim [1/T]\\ c &\sim [1/T]\end{aligned}\right\}\Rightarrow \widetilde{\omega}\sim[1/T] \tag{2.124}$$

Therefore, we may non-dimensionalize via $\Big(R/\Delta v\Big)\sim[T]$. Proceding this way equation (2.123) is multiplied by the nondimensionalization factor, and, the following non-dimensional variables are defined:

$$c^* \equiv \frac{cR}{\Delta v}$$

$$\alpha^* \equiv \alpha R$$

$$\Psi^* \equiv \Psi R$$

the new control parameter S defined as the swirl number is given as follows,

$$S \equiv \frac{\Omega R}{\Delta v}$$

Considering these new parameters, algebraic manipulation yields the following non-dimensional dispersion relation for the steady swirling jet where the $*$ superscripts have been suppressed for convenience. (*e.g.*, we let $c^* = c$ where it is understood that c is now non-dimensional)

$$\boxed{\left(\widetilde{\omega} + \alpha\right)^2 \left(- 2mS + \widetilde{\omega}\Psi \frac{J_m{}'(\Psi)}{J_m(\Psi)} \right) + \frac{\widetilde{\omega}^3 \Psi^2}{s\alpha} \frac{K_m{}'(s\alpha)}{K_m(s\alpha)} = 0} \tag{2.125}$$

2.2.3 Numerical Solution Procedure: Swirling Jet

Letting $v_\infty = 0$ without loss of generality, the dispersion relation (2.125) may be solved numerically for c_R and c_I vs axial wavenumber, α at a fixed azimuthal wavenumber m and a fixed swirl S using a numerical method slightly modified from that outlined in chapter 2.1.2. Equation (2.125) differs from (2.56) in that the axial wavenumber, azimuthal wavenumber and complex celerity are coupled in the argument of the J_m Bessel functions. The result is that the standard complex-valued Newton-Raphson method fails to converge to roots close to $\alpha = 0$ for helical modes and for the Kelvin-Helmholtz mode for large values of swirl. The function (2.125) is still well behaved as extending the method to utilize a modified Newton-Raphson algorithm and adding a relaxation paramater to 'lessen' the slope of the derivative for the next guess of the root allievated the problem resulting in convergence for all values of α, m and S.

2.2.4 Numerical Results for the Steady Swirling Jet

Solving equation (2.125) using the procedure of section 2.2.3 at various values of the swirl results in figures 2.5 and 2.6 for the celerity and growth rate of axisymmetric ($m = 0$) disturbances respectively. Figure 2.5 is a plot of the celerity vs. axial wavenumber for swirl varying according to ($0 \leq S \leq 2.5$) for the steady swirling jet. Figure 2.5 shows that the celerity of axisymmetric disturbances increase as swirl increases. Consequently, fast waves become less sensitive to changes in the swirl as the swirl increases.

Figure 2.5 shows that for a finite amount of swirl the growth rate of axisymmetric disturbances decreases. That is, increasing swirl decreases the growth rate of axisymmetric disturbances. For non-axisymmetric, or helical modes ($m \neq 0$), it has been shown that positive helicity modes are stabilized by a small amount of swirl, whereas negative helicity modes exhibit decreased growth rate [5]. That is to say, for a given value of swirl, negative helical modes ($m < 0$) are more unstable than positive ones ($m > 0$) [9]. Notice that in both cases when $S = 0$ the numerical results matching the axisymmetric jet in figures 2.2 and 2.3 are recovered.

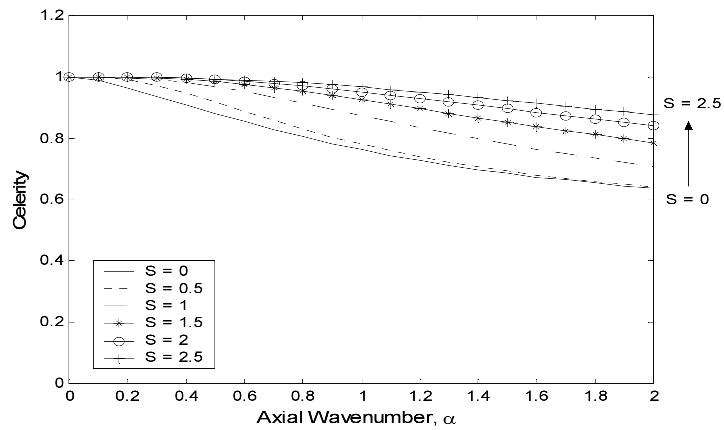

Figure 2.5: Celerity vs. Axial Wavenumber for the Steady Swirling Jet

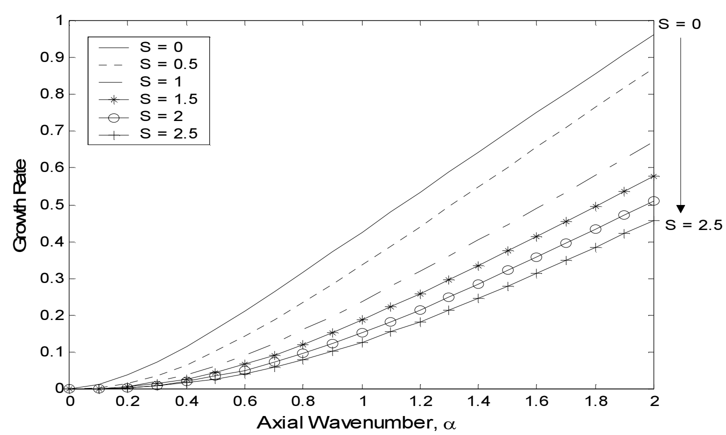

Figure 2.6: Growth Rate vs. Axial Wavenumber for the Steady Swirling Jet

2.3 Stability of Time Dependent Shear Flows

In the majority of shear layer stability studies, the mean flow, as in the previous examples of axisymmetric and swirling jets, is taken to be time independent. In general, analyses of unsteady shear flows have been carried out by applying steady state results to instantaneous velocity distributions; that is, using the quasi-steady approach. Hajj [10] extended the idea of the applicability of the quasi-steady approach for flows wherein the unsteady mean time scale is much larger than the instability time scale. Hajj [10] showed that when a steady velocity profile is modified by a periodic temporal function or the form,

$$T(t) = \left(1 + \epsilon e^{i\omega_m t}\right) \tag{2.126}$$

where, ϵ is the perturbation size and ω_m is the perturbation frequency in the time scale t, the solution of the unsteady inviscid problem can be reduced to that of the steady flow by a transformation of the time variable. Specifically, the instability time scale is 'stretched', and a new time scale τ is defined,

$$\tau \equiv \int_0^t dt'(1 + \epsilon exp(i\omega_m t')) = t + \frac{\epsilon}{i\omega_m} e^{i\omega_m t} - \frac{\epsilon}{i\omega_m} \tag{2.127}$$

Based on this τ, the temporal derivative operator in the instability time scale may be replaced by,

$$\frac{\partial}{\partial t} \mapsto T(t)\frac{\partial}{\partial \tau} \tag{2.128}$$

Because this transformation in time results in similar linearized stability equations for both steady and unsteady mean flows, the solution for the unsteady equations admit normal-mode perturbations given by

$$\delta f(x,y,\tau) = \widehat{f}(y)e^{(i\alpha(x-c\tau)} \tag{2.129}$$

Transformation from the time scale τ into the instability time scale t within the perturbation is obtained via substitution of (2.127) into (2.129)

$$\delta f(x,y,t) = \widehat{f}(y)e^{\alpha c_I + Acos(\omega_m t+\beta)}e^{i(\alpha(x-c_R t)+Asin(\omega_m t+\beta))}e^{-Acos(\beta)}e^{-iAsin(\beta)} \tag{2.130}$$

where,

$$A \equiv \frac{\alpha\epsilon}{\omega_m}\sqrt{c_I^2 + c_R^2} \tag{2.131}$$

and

$$\beta \equiv arctan\left(\frac{c_I}{c_R}\right) \tag{2.132}$$

The term $Acos(\omega_m t + \beta)$ in the first exponential produces an amplitude modulation of the unstable mode; the term $Asin(\omega_m t + \beta)$ in the second exponential produces a phase modulation of the unstable mode. The last two temporally constant terms may be taken *totus in re* into the phase of $\widehat{f}(y)$. Hence, for a periodically unsteady transiently separable shear layer, the unsteady basic state can be 're-sorted' to the steady state by way of a transformation of the time variable, at a cost of modulating the amplitude and phase of the unstable modes.

Notice, however, if the actual value of the modulation is compared at each wavenumber; that is, each point is normalized with respect to its respective wavenumber, the modulation is given by $(\epsilon\|c\|/\omega_m)$. This modulation is inversely proportional to the perturbation frequency ω_m and directly proportional to the size of the perturbation ϵ. We cannot choose ϵ arbitrarily large in an effort to drastically increase the perturbation size as the disturbance will no longer be small and the linearization breaks down. Because ω_m and ϵ are fixed, the real behavior of the amplitude modulation is related to the complex phase speed, $\|c\|$.

When, for a given problem, the unsteady velocity profile yields an analytical solution that can be recast into the steady state, the quasi-static approach is guaranteed to work. What is more, the subsequent modulations of phase and amplitude of the unsteady modes can be seen directly by transforming from the stretched τ time scale back into the t time scale.

Chapter 3

Stability of a Periodically Unsteady Axisymmetric Jet

In this chapter, the stability of a periodically unsteady inviscid axisymmetric jet is considered. The jet is modeled as a cylindrical vortex sheet of finite radius with the flow varied periodically, governing equations are derived, perturbations are introduced and the effects of the time variations of the jet stability are examined.

3.1 Formulation of the Unsteady Swirling Jet

The jet is modeled by a cylindrical vortex sheet of radius R (plug flow) such that the core flow varies sinusoidially, thus, in essence, the jet is pulsed axially. Mathematically, this model is written as

$$\vec{V}(r,\theta,z) = \begin{cases} (0,0,v_\infty + T(t)\Delta v), & \text{for } r < R \\ (0,0,v_\infty), & \text{for } r > R \end{cases} \tag{3.1}$$

Where, v_∞ is the far-field free stream velocity, Δv is the axial velocity dif-

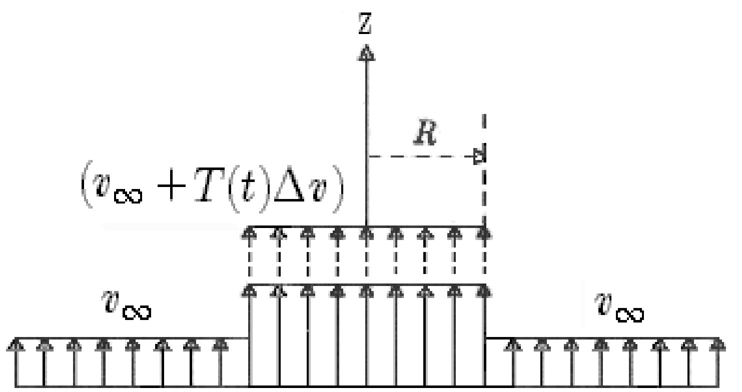

Figure 3.1: Sketch of Basic Flow for the Pulsed Axisymmetric Jet

ference across the vortex sheet and the periodic axial perturbation is given by,

$$T(t) = 1 + \epsilon \ cos(\omega_m t) \tag{3.2}$$

where, ϵ is the perturbation size, and ω_m is the perturbation frequency in the time scale t.

3.1.1 Field equations for the Axisymmetric Jet: Consequences of Unsteadiness

Following Hajj [10], a new time scale τ is defined such that,

$$\tau \equiv \int_0^t dt'(1+\epsilon cos(\omega_m t')) = t + \frac{\epsilon}{\omega_m} sin(\omega_m t) \tag{3.3}$$

Where τ is valid in the set $0 < \omega_m t < 2\pi$. Based on this τ, the temporal derivative operator in the instability time scale may be replaced by,

$$\frac{\partial}{\partial t} \mapsto T(t)\frac{\partial}{\partial \tau} \tag{3.4}$$

Considering the transformation given by (3.4), the field equations for the unsteady inviscid flow are governed for fields $(\vec{v}, p)$ by

$$T(t)\frac{\partial \vec{v}}{\partial \tau} + \nabla\vec{v}\cdot\vec{v} + \frac{1}{\rho}\nabla p = \vec{0} \tag{3.5}$$

$$\nabla\cdot\vec{v} = 0 \tag{3.6}$$

In cylindrical coördinate component form, equations (3.5) and (3.6) are written as

$$\rho\Big(T\frac{\partial v_r}{\partial \tau} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\Big) = -\frac{\partial p}{\partial r} \tag{3.7}$$

$$\rho\Big(T\frac{\partial v_\theta}{\partial \tau} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\Big) = -\frac{1}{r}\frac{\partial p}{\partial \theta} \tag{3.8}$$

$$\rho\Big(T\frac{\partial v_z}{\partial \tau} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\Big) = -\frac{\partial p}{\partial z} \tag{3.9}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \tag{3.10}$$

Following the procedure for the steady axisymmetric jet, the total flow is divided into a mean flow given by the fields $\Big((0, 0, V_z), P\Big)$ plus perturbations such that

$$\vec{v} \mapsto \vec{V} + \delta\vec{v} = (\delta v_r, \delta v_\theta, V_z + \delta v_z) \tag{3.11}$$

$$p \mapsto P + \delta p \tag{3.12}$$

where without loss of generality, $V_z = T(t)\Delta v$ so that the far-field in figure 3.1 is quisecent. Substituting (3.11-12) into the governing equations (3.7-10) produces a set of non-linear stability differential equations. Subtracting the mean and disregarding $O(\epsilon^2)$ combinations of the $O(\epsilon)$ perturbations as outlined in chapter 2.1 yields the following set of linearized stability equations for the periodically unsteady (pulsed) axisymmetric jet.

$$\rho\left(T\frac{\partial \delta v_r}{\partial \tau} + T\Delta v\frac{\partial \delta v_r}{\partial r}\right) = -\frac{\partial \delta p}{\partial r} \tag{3.13}$$

$$\rho\left(T\frac{\partial\delta v_\theta}{\partial\tau}+T\Delta v\frac{\partial\delta v_\theta}{\partial r}\right)=-\frac{1}{r}\frac{\partial\delta p}{\partial\theta} \tag{3.14}$$

$$\rho\left(T\frac{\partial\delta v_z}{\partial\tau}+T\Delta v\frac{\partial\delta v_z}{\partial z}\right)=-\frac{\partial\delta p}{\partial z} \tag{3.15}$$

and

$$\frac{1}{r}\left(\delta v_r+r\frac{\partial\delta v_r}{\partial r}\right)+\frac{1}{r}\frac{\partial\delta v_\theta}{\partial\theta}+\frac{\partial\delta v_z}{\partial z}=0 \tag{3.16}$$

Equations (3.13-16) represent the linearized governing equations for velocity and pressure perturbations δv and δp in cylindrical coördinates corresponding to a mean velocity profile $(0, 0, T\Delta v)$. These equations can be re-written as

$$\rho T\left(\frac{\partial\delta v_r}{\partial\tau}+\Delta v\frac{\partial\delta v_r}{\partial r}\right)=-\frac{\partial\delta p}{\partial r} \tag{3.17}$$

$$\rho T\left(\frac{\partial\delta v_\theta}{\partial\tau}+\Delta v\frac{\partial\delta v_\theta}{\partial r}\right)=-\frac{1}{r}\frac{\partial\delta p}{\partial\theta} \tag{3.18}$$

$$\rho T\left(\frac{\partial\delta v_z}{\partial\tau}+\Delta v\frac{\partial\delta v_z}{\partial z}\right)=-\frac{\partial\delta p}{\partial z} \tag{3.19}$$

By defining,

$$\delta p_* \equiv \frac{\delta p}{T} \tag{3.20}$$

equations (3.17-19) can be re-written as

$$\rho\left(\frac{\partial \delta v_r}{\partial \tau} + \Delta v \frac{\partial \delta v_r}{\partial r}\right) = -\frac{\partial \delta p_*}{\partial r} \tag{3.21}$$

$$\rho\left(\frac{\partial \delta v_\theta}{\partial \tau} + \Delta v \frac{\partial \delta v_\theta}{\partial r}\right) = -\frac{1}{r}\frac{\partial \delta p_*}{\partial \theta} \tag{3.22}$$

$$\rho\left(\frac{\partial \delta v_z}{\partial \tau} + \Delta v \frac{\partial \delta v_z}{\partial z}\right) = -\frac{\partial \delta p_*}{\partial z} \tag{3.23}$$

Because these equations are similar to equations governing the stability of the steady axisymmetric jet, one can apply the same analysis outlined for the steady axisymmetric jet to produce stability eigenfunctions in the jet core and in the far-field region. There are, however, two major differences. First, the time scale for equations (3.21-23) is τ. Thus, the time t in the normal-modes is replaced by τ, *i.e.*,

$$\delta f(r, \theta, z, t) \mapsto \delta f(r, \theta, z, \tau)$$

so that

$$\delta f(r, \theta, z, \tau) = \widehat{f}(r) e^{i\alpha(z - c\tau) + im\theta} \tag{3.24}$$

Second, for the unsteady axisymmetric jet the pressure is δp_* where

$$\widehat{p}_*(r) \equiv \frac{\widehat{p}(r)}{T} \tag{3.25}$$

The solution is then written as

$$\begin{cases} \widehat{p}_*(r) = A_1 I_m(\alpha r) & \text{inside the jet, } (r < R) \\ \widehat{\Phi}(r) = B_2 K_m(\alpha r) & \text{outside the jet, } (r > R) \end{cases} \tag{3.26}$$

3.1.2 Dispersion Relation for the Unsteady Axisymmetric Jet

As before, the stability eigenfunctions are matched across the vortex sheet by applying the kinematic and dynamic boundary conditions for a radial interface displacement of the vortex sheet $\eta(\theta, z, \tau)$ due to the disturbance. By expressing η as a normal-mode such that,

$$\eta = Ae^{i\alpha(z-c\tau)+im\theta} \tag{3.27}$$

where, A is a constant and following the procedure given in chapter 2.1.3, except in the new time scale τ, the kinematic boundary condition gives the conditions,

$$T\left(\frac{\partial \eta}{\partial \tau} + \Delta v \frac{\partial \eta}{\partial z}\right) = \left(\frac{\partial \delta p_*}{\partial r}\right)_{r=R} \tag{3.28}$$

$$T\frac{\partial \eta}{\partial \tau} = \left(\frac{\partial \delta \phi}{\partial r}\right)_{r=R} \tag{3.29}$$

Using (3.28) yields,

$$i\alpha(\Delta v - c)TA = \left.\frac{d\widehat{p}_*(r)}{dr}\right|_{r=R} \tag{3.30}$$

Using (3.26) for ($r < R$),

$$\left.\frac{d\widehat{p}_*(r)}{dr}\right|_{r=R} = A_1 \left.\frac{\partial I_m(x)}{\partial x}\right|_{x=\alpha R} \tag{3.31}$$

It follows from (3.30-31) that,

$$i\alpha(\Delta v - c)TA = A_1 {I_m}'(\alpha R) \tag{3.32}$$

Similarly, applying (3.29) and using the eigenfunction solution for the velocity potential perturbation; that is, (3.26) for ($r > R$) one obtains,

$$-i\alpha cTA = B_2 {K_m}'(\alpha R) \tag{3.33}$$

The dynamic boundary condition in the time scale τ is given by,

$$T\left(\frac{\partial \delta p_*}{\partial \tau} + \Delta v \frac{\partial \delta p_*}{\partial z}\right)_{r=R} = T\left(\frac{\partial \delta \phi}{\partial \tau}\right)_{r=R} \tag{3.34}$$

Thus,

$$i\alpha(\Delta v - c)A_1 I_m(\alpha R) = -i\alpha c B_2 K_m(\alpha R) \tag{3.35}$$

Disregarding trival solutions among equations (3.32), (3.33) and (3.35), one can solve for A_1 and B_2 in terms of A as

$$A_1 = \frac{i\alpha T(\Delta v - c)}{I_m{'}(\alpha R)} \tag{3.36}$$

and

$$B_2 = -\frac{i\alpha c T}{K_m{'}(\alpha R)} \tag{3.37}$$

Substituting these values into equation (3.35) yields,

$$i\alpha(\Delta v - c)\left\{\frac{i\alpha T(\Delta v - c)}{I_m{'}(\alpha R)}\right\} A I_m(\alpha R) = -i\alpha c\left\{-\frac{i\alpha c T}{K_m{'}(\alpha R)}\right\} A K_m(\alpha R) \tag{3.38}$$

Algebraic manipulation of (3.38) yields the following form of the dispersion relation for the periodically unsteady inviscid axisymmetric jet,

$$\boxed{\frac{I_m{'}(\alpha R)}{I_m(\alpha R)} = \left(\frac{\Delta v - c}{c}\right)^2 \frac{K_m{'}(\alpha R)}{K_m(\alpha R)}} \tag{3.39}$$

It should be noted here that this equation is the same as the one found for the steady jet, however, the solution is in the time scale τ and not t. Consequently, the stability of the sinusoidally pulsed axisymmetric jet can be determined by simply applying steady state results to instantaneous velocity profiles; that is, the quasi-steady approach can be used to predict the stability of the periodically varying axisymmetric jet.

3.1.3 Results in the Quasi-steady Case

Because the quasi-steady approach works, one can re-write for the perturbations,

$$\delta f(r,\theta,z,\tau) = \widehat{f}(r)e^{(i\alpha(z-c\tau)+im\theta)} \tag{3.40}$$

and, in the temporal analysis c is complex, thus,

$$\begin{aligned}\delta f &= \widehat{f}(r)e^{(i\alpha\left(z-(c_R+ic_I)\tau\right)+im\theta)} \\ &= \widehat{f}(r)e^{i\alpha(z-c_R\tau)+im\theta}e^{\alpha c_I\tau}\end{aligned} \tag{3.41}$$

Substituting equation (3.3) into (3.41) transforms the perturbation from the instability to the mean time scale.

$$\begin{aligned}\delta f &= \widehat{f}(r)e^{i\alpha(z-c_R\tau)+im\theta}e^{\alpha c_I\tau} \\ &= \widehat{f}(r)e^{i\alpha(z-c_R\left[t+\epsilon/\omega_m sin(\omega_m t)\right])+im\theta}e^{\alpha c_I\left[t+\epsilon/\omega_m sin(\omega_m t)\right]}\end{aligned} \tag{3.42}$$

The first exponential term in equation (3.42) governs the wavelike nature of the instability mode and the portion $\epsilon/\omega_m sin(\omega_m t)$ represents a modulation of the phase of the instability mode. The second exponential term in equation (3.42) governs the amplitude behavior of the instability mode, and, the sinusoidal term $\epsilon/\omega_m sin(\omega_m t)$ in this context represents a modulation of the amplitude of the instability mode.

To reiterate, the growth rate is now given by,

$$\omega_I = \omega_I \Big[1 + \frac{\epsilon}{\omega_m t} sin(\omega_m t) \Big] \tag{3.43}$$

where the second term in the brackets represents a frequency modulation, and, the celerity is now given by,

$$c_R = c_R \Big[1 + \frac{\epsilon}{\omega_m t} sin(\omega_m t) \Big] \tag{3.44}$$

where here, the second term in the brackes represents a phase modulation with magnitude $(\epsilon c_R)/\omega_m$.

Chapter 4

Stability of a Periodically Unsteady Swirling Jet

In this chapter, the stability of a periodically unsteady inviscid swirling jet is considered. The problem is formulated mathematically to obtain an analytical dispersion relation for the unsteady field. This relation is solved numerically in the time variable at fixed control parameters namely, axial wavebumber α, azimuthal wavenumber m and swirl number S. The effects of unsteadiness on celerity and growth rates are determined and discussed.

4.1 Formulation of the Unsteady Swirling Jet

In this section, the problem of the periodically unsteady swirling jet is formulated mathematically. The jet is modeled as a Rankine vortex with axial flow as given in chapter 2.2 for the steady inviscid swirling jet, however, now the azimuthal velocity, and therefore the swirl, is varied periodically both in the jet core and in the far-field. In figure 4.1 a schematic of the swirling jet is shown. Periodic unsteadiness is introduced in the swirl such that

$$\Omega(t) = T(t)\Omega. \tag{4.1}$$

and, v_∞ is the far-field free stream velocity, Δv is the axial velocity difference across the vortex sheet.

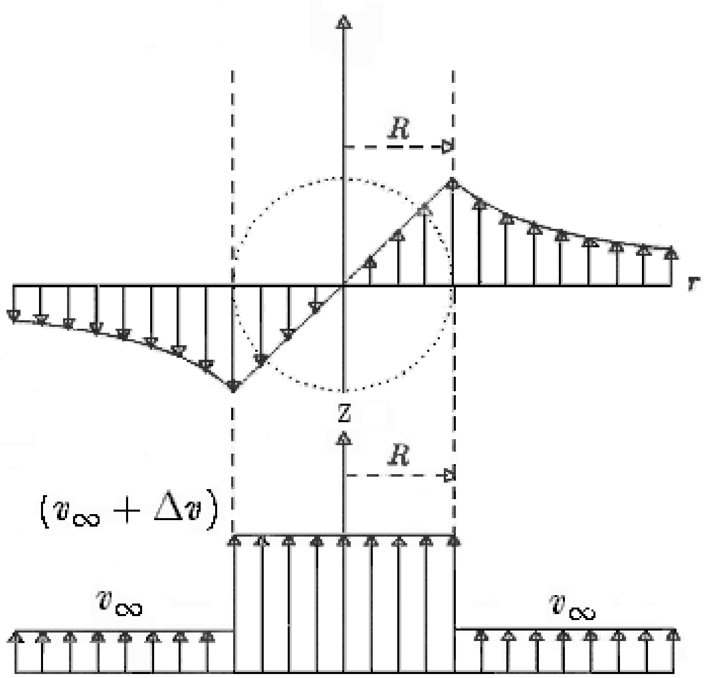

Figure 4.1: Sketch of Basic Flow for the Unsteady Swirling Jet

The velocity field for the periodically unsteady axisymmetric jet is written as,

$$\vec{V}(r,\theta,z) = \begin{cases} (0, T(t)\Omega r, v_\infty + \Delta v), & \text{for } r < R \\ (0, \frac{T(t)\Omega R^2}{r}, v_\infty), & \text{for } r > R \end{cases} \tag{4.2}$$

from figure 4.1, v_∞ is the far-field free stream velocity, Δv is the axial velocity difference across the vortex sheet. In this work, the periodic unsteadiness is given by,

$$T(t) = 1 + \epsilon \ cos(\omega_m t) \tag{4.3}$$

where, ϵ represents the amplitude of the unsteady perturbation, and ω_m is the corresponding frequency.

4.1.1 Field equations for the Swirling Jet: Consequences of Unsteadiness

Following the analysis by Hajj [10] for the unsteady mixing layer, a new time scale τ may be defined as,

$$\tau \equiv \int_0^t dt' T(t') = t + \frac{\epsilon}{\omega_m} sin(\omega_m t) \tag{4.4}$$

Based of this relation, it is noted that the time scales t and τ are related by,

$$\frac{\partial}{\partial t} \mapsto T(t) \frac{\partial}{\partial \tau} \tag{4.5}$$

4.1.2 Analysis in the Jet Core and Far Field: Unsteady Stability Eigenfunctions for the Swirling Jet

The governing equations for the unsteady inviscid swirling jet are the same as the ones given by equations (3.7) and (3.10) with the time derivative operator $\frac{\partial}{\partial t}$ replaced by $T(t)\frac{\partial}{\partial \tau}$, *viz.*

$$\rho\Big(T\frac{\partial v_r}{\partial \tau} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\Big) = -\frac{\partial p}{\partial r} \tag{4.6}$$

$$\rho\Big(T\frac{\partial v_\theta}{\partial \tau} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\Big) = -\frac{1}{r}\frac{\partial p}{\partial \theta} \tag{4.7}$$

$$\rho\Big(T\frac{\partial v_z}{\partial \tau} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\Big) = -\frac{\partial p}{\partial z} \tag{4.8}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \tag{4.9}$$

Following the same procedure as for the steady swirling jet, the total flow is considered as the sum of the generic field $\Big((0, V_\theta, V_z), P\Big)$ and perturbations δp, $\delta\vec{v}$; that is,

$$\vec{v} \mapsto \vec{V} + \delta\vec{v} = (\delta v_r, V_\theta + \delta v_\theta, V_z + \delta v_z) \tag{4.10}$$

$$p \mapsto P + \delta p \tag{4.11}$$

Applying (4.10) and (4.11) with equations (4.6-9), subtracting the equations governing the mean flow and eliminating the $O(\epsilon)$ terms, the linearized stability differential equations for the periodically unsteady swirling jet in the new time scale τ are obtained as,

$$\rho\left(T(t)\frac{\partial\delta v_r}{\partial\tau} + \frac{V_\theta}{r}\frac{\partial\delta v_r}{\partial\theta} - 2\frac{V_\theta\delta v_\theta}{r} + V_z\frac{\partial\delta v_z}{\partial z}\right) = -\frac{\partial\delta p}{\partial r} \tag{4.12}$$

$$\rho\left(T(t)\frac{\partial\delta v_\theta}{\partial\tau} + \delta v_r\frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r}\frac{\partial\delta v_\theta}{\partial\theta} + \frac{\delta v_\theta}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{\delta v_r V_\theta}{r} + V_z\frac{\partial\delta v_\theta}{\partial z} + \delta v_z\frac{\partial V_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial\delta p}{\partial\theta} \tag{4.13}$$

$$\rho\left(T(t)\frac{\partial\delta v_z}{\partial\tau} + \delta v_r\frac{\partial V_z}{\partial r} + \frac{V_\theta}{r}\frac{\partial\delta v_z}{\partial\theta} + \frac{\delta v_\theta}{r}\frac{\partial V_z}{\partial\theta} + V_z\frac{\partial\delta v_z}{\partial z} + \delta v_z\frac{\partial V_z}{\partial z}\right) = -\frac{\partial\delta p}{\partial z} \tag{4.14}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\rho r\delta v_r\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\rho\delta v_\theta\right) + \frac{\partial}{\partial z}\left(\rho\delta v_z\right) = 0 \tag{4.15}$$

The above system of equations govern the stability of a periodically unsteady swirling jet. Here, the azimuthal V_θ component of the field is given by $V_\theta = T(t)\Omega r$ in the jet core ($r < R$). Substituting this value into equations (4.12-15) results in,

$$T(t)\left[\frac{\partial \delta v_r}{\partial \tau} + \Omega \frac{\partial \delta v_r}{\partial \theta} - 2\Omega \delta v_\theta\right] + V_z \frac{\partial \delta v_r}{\partial z} = -\frac{1}{\rho}\frac{\partial \delta p}{\partial r} \tag{4.16}$$

$$T(t)\left[\frac{\partial \delta v_\theta}{\partial \tau} + \Omega \frac{\partial \delta v_\theta}{\partial \theta} + 2\Omega \delta v_r\right] + V_z \frac{\partial \delta v_\theta}{\partial z} = -\frac{1}{r\rho}\frac{\partial \delta p}{\partial \theta} \tag{4.17}$$

$$T(t)\left[\frac{\partial \delta v_z}{\partial \tau} + \Omega \frac{\partial \delta v_z}{\partial \theta}\right] + V_z \frac{\partial \delta v_z}{\partial z} = -\frac{1}{\rho}\frac{\partial \delta p}{\partial z} \tag{4.18}$$

and

$$\frac{\delta v_r}{r} + \frac{\partial \delta v_r}{\partial r} + \frac{1}{r}\frac{\partial \delta v_\theta}{\partial \theta} + \frac{\partial \delta v_z}{\partial z} = 0 \tag{4.19}$$

Equations (4.16-4.19) represent the linearized equations governing the velocity and pressure perturbations δv and δp in cylindrical coördinates corresponding to a mean velocity profile $(0, T(t)\Omega r, V_z)$ in the time scale τ. Notice, after the periodic perturbation $T(t)$ is factored out, the system is such that $T(t)$ cannot be absorbed in a redefinition of only one term as in the case of the unsteady axisymmetric jet considered in chapter 3. Hence, equations (4.16-18) do not reduce to same form as the steady state equations once the transformation from t to τ is made. Therefore, it is already apparent that for the periodically unsteady swirling jet, the quasi-steady approach does not work. That is, the unsteady stability cannot be determined by applying steady-state results to instantaneous velocity profiles. By examining equations (4.16-18), it is apparent that the problem lies in the combination of the axial velocity component and the spatial derivative, $V_z \frac{\partial}{\partial z}$. If the field were unsteady in the axial and azimuthal directions, then the quasi-steady approach would work, however, this would amount to simply adjusting the field so that the motion in one direction accommodates the motion in another allowing for the total unsteady effects to be seperable, that is to be an overall addition to the steady flow.

To determine the analytical dispersion relation, the perturbations are taken as normal-modes of the form,

$$\delta f(r, \theta, z, \tau) = \widehat{f}(r) e^{i(z - c\tau) + im\theta} \tag{4.20}$$

Substituting this form for the perturbations into equations (4.16-19) and dividing by the common exponential $exp[i\alpha(z - c\tau) + im\theta]$, yields the following system,

$$\frac{1}{\rho}\frac{d\widehat{p}(r)}{dr} = i\Big(\alpha c T(t) - m\Omega T(t) - \alpha V_z\Big)\widehat{v}_r(r) + 2\Omega T(t)\widehat{v}_\theta(r) \tag{4.21}$$

$$\frac{im}{\rho r}\widehat{p}(r) = i\Big(\alpha c T(t) - m\Omega T(t) - \alpha V_z\Big)\widehat{v}_\theta(r) - 2\Omega T(t)\widehat{v}_r(r) \tag{4.22}$$

$$\frac{i\alpha}{\rho}\widehat{p}(r) = i\Big(\alpha c T(t) - m\Omega T(t) - \alpha V_z\Big)\widehat{v}_z(r) \tag{4.23}$$

$$\frac{d\widehat{v}_r(r)}{dr} = -\frac{1}{r}\widehat{v}_r(r) - \frac{im}{r}\widehat{v}_\theta(r) - i\alpha\widehat{v}_z(r) \tag{4.24}$$

where V_z is defined as $(v_\infty + \Delta v)$

Using the definition of the celerity, $c = \omega/\alpha$, the above system may be simplified by recognizing that when $T(t) = 1$

$$\widetilde{\omega} \equiv \Big(\omega - m\Omega - \alpha V_z\Big) \tag{4.25}$$

represents the Doppler shifted frequency in an inertial frame moving with the jet core [9]. Therefore, by defining

$$\widetilde{\omega}' \equiv \Big(\omega T(t) - m\Omega T(t) - \alpha V_z\Big) \tag{4.26}$$

and substituting into the system (4.21-24) yields a slightly simplified system of coupled differential equations given by,

$$\frac{1}{\rho}\frac{d\widehat{p}(r)}{dr} = i\widetilde{\omega}'\widehat{v}_r(r) + 2\Omega T(t)\widehat{v}_\theta(r) \tag{4.27}$$

$$\frac{im}{\rho r}\widehat{p}(r) = i\widetilde{\omega}'\widehat{v}_\theta(r) - 2\Omega T(t)\widehat{v}_r(r) \tag{4.28}$$

$$\frac{i\alpha}{\rho}\widehat{p}(r) = i\widetilde{\omega}'\widehat{v}_z(r) \tag{4.29}$$

$$\frac{d\widehat{v}_r(r)}{dr} = -\frac{1}{r}\widehat{v}_r(r) - \frac{im}{r}\widehat{v}_\theta(r) - i\alpha\widehat{v}_z(r) \tag{4.30}$$

The above system may be reduced to a single equation for the unknown pressure perturbation eigenfunction by first substituting (4.29) into (4.30) which yields,

$$\frac{d\widehat{v}_r(r)}{dr} = -\frac{1}{r}\widehat{v}_r(r) - \frac{im}{r}\widehat{v}_\theta(r) - \frac{i\alpha^2}{\widetilde{\omega}'\rho}\widehat{p}(r) \tag{4.31}$$

Then, equations (4.27) and (4.28) are solved for $\widehat{v}_\theta$ and equated leaving an equation in terms of $\widehat{v}_r$, and $\widehat{p}$ which may be solved for $\widehat{v}_r$.

$$\widehat{v}_r(r) = -\frac{2i\Omega T(t)\widetilde{\omega}'}{\Xi'}\left(\frac{1}{2\Omega T(t)\rho}\frac{d\widehat{p}(r)}{dr} - \frac{m}{\widetilde{\omega}'\rho r}\widehat{p}(r)\right) \tag{4.32}$$

where

$$\Xi' \equiv \widetilde{\omega}'^2 - 4\Omega^2 T^2(t)$$

Next, substituting (4.32) into (4.27) and (4.28) then solving in terms of $\widehat{v}_r$ leaves two equations in terms of $\widehat{v}_\theta$. Equating these leaves a single equation for $\widehat{v}_\theta$ in terms of $\widehat{p}$ and $\widehat{p}'$,

$$\widehat{v}_\theta(r) = \left(\frac{2\Omega T(t)}{\rho\Xi'}\right)\frac{d\widehat{p}(r)}{dr} - \left(\frac{m}{\widetilde{\omega}'\rho r\Xi'}\right)\widehat{p}(r) \tag{4.33}$$

Differentiating (4.27) with respect to r yields,

$$\frac{1}{\rho}\frac{d^2\widehat{p}(r)}{dr^2} = i\widetilde{\omega}'\frac{d\widehat{v}_r(r)}{dr} + 2\Omega T(t)\frac{d\widehat{v}_\theta(r)}{dr}. \tag{4.34}$$

Differentiating (4.28) with respect to r leaves,

$$\frac{d\widehat{v}_\theta(r)}{dr} = \frac{m}{\rho\widetilde{\omega}' r}\frac{d\widehat{p}(r)}{dr} - \frac{m}{\rho\widetilde{\omega}' r^2}\widehat{p}(r) - \frac{2i\Omega T(t)}{\widetilde{\omega}'}\frac{d\widehat{v}_r(r)}{dr}. \tag{4.35}$$

Substituting (4.31) into (4.34) and using (4.33), (4.35) and (4.37) leaves, after some manipulation

$$\frac{d^2\widehat{p}(r)}{dr^2} + \frac{1}{r}\frac{d\widehat{p}(r)}{dr} - \left(\frac{\alpha^2\Xi'}{\widetilde{\omega}'^2} - \frac{m^2}{r^2}\right)\widehat{p}(r) = 0 \tag{4.36}$$

Substituting Ξ' into (4.36) and rearranging terms produces the following equation governing pressure perturbations in the unsteady swirling jet core, $(r < R)$,

$$\frac{d^2\widehat{p}(r)}{dr^2} + \frac{1}{r}\frac{d\widehat{p}(r)}{dr} - \left(\alpha^2 r^2\Big(\frac{4\Omega^2 T^2(t)}{\widetilde{\omega}'^2} - 1\Big) - m^2\right)\widehat{p}(r) = 0 \tag{4.37}$$

Equation (4.37) is again readily recognized as a modified Bessel equation and admits solutions of the form,

$$\widehat{p}(r) = A_1 J_m\left(r\sqrt{\alpha^2\Big(\frac{4\Omega^2 T^2(t)}{\widetilde{\omega}'^2} - 1\Big)}\right) + A_2 Y_m\left(r\sqrt{\alpha^2\Big(\frac{4\Omega^2 T^2(t)}{\widetilde{\omega}'^2} - 1\Big)}\right) \tag{4.38}$$

where, J_m and Y_m are modified Bessel functions and A_1 and A_2 are constants. Physical constraints on the pressure eigenfunction require that,

$$\lim_{b \to \infty} Y_m(b) \to \infty \ \forall m \ \Rightarrow \boxed{A_2 = 0} \tag{4.39a}$$

Therefore, the perturbation pressure eigenfunction in the unsteady swirling jet core is then given by,

$$\widehat{p}(r) = A_1 J_m\left(r\sqrt{\alpha^2\left(\frac{4\Omega^2 T^2(t)}{\widetilde{\omega}'^2} - 1\right)}\right) \tag{4.40}$$

For convenience we will let

$$\sqrt{\alpha^2\left(\frac{4\Omega^2 T^2(t)}{\widetilde{\omega}'^2} - 1\right)} \equiv \Psi' \tag{4.41}$$

so that

$$\widehat{p}(r) = A_1 J_m\big(r\Psi'\big) \tag{4.42}$$

Notice the similarity of this result to that of the steady swirling jet given in equation (2.96). As in section 2.2, $\widehat{p}(r)$ is substituted into the stability eigen-system (4.27), (4.28) and (4.31) resulting in the following system of equations wherein the radial velocity perturbation eigenfunction may be obtained.

$$\frac{1}{\rho}\Psi' A_1 {J_m}'\big(r\Psi'\big) = i(\widetilde{\omega}')\widehat{v}_r + 2\Omega T(t)\widehat{v}_\theta \tag{4.43}$$

$$\frac{im}{\rho r} A_1 J_m\big(r\Psi'\big) = i(\widetilde{\omega}')\widehat{v}_\theta - 2\Omega T(t)\widehat{v}_r \tag{4.44}$$

$$\frac{d\widehat{v}_r(r)}{dr} = -\frac{1}{r}\widehat{v}_r(r) - \frac{im}{r}\widehat{v}_\theta(r) - \frac{i\alpha^2}{\widetilde{\omega}'\rho} A_1 J_m\big(r\Psi'\big) \tag{4.45}$$

The prime on all Bessel functions denotes differentiation with respect to their argument and should not be confused with primes denoting time-varying coefficients, Ψ' and $\widetilde{\omega}'$. Throughout this analysis, primed operators denote differentiation with respect to their argument, primed functions or coefficients denote time dependence. Following the procedure in chapter 2, with the newly defined variables Ψ' and $\widetilde{\omega}'$ with $\Omega = \Omega T(t)$, the radial velocity perturbation eigenfunction for the unsteady swirling jet in the region $(r < R)$ is obtained algebraically from equations (4.43-45) and is given by,

$$\widehat{v}_r = \frac{i\alpha^2 A_1}{\rho\widetilde{\omega}'\Psi'^2}\left[-\frac{2\Omega T(t)m}{r\widetilde{\omega}'} J_m\big(r\Psi'\big) + \Psi {J_m}'\big(r\Psi'\big)\right] \tag{4.46}$$

Because the vorticity (the curl operator) is time independent, the condition $\nabla \times \vec{v} = \vec{0}$ still holds in the external region of the unsteady swirling jet $(r > R)$ where $\vec{v} = (0, \frac{\Omega T(t)R^2}{r}, v_\infty + \Delta v)$. Thus, in the far field, the flow is irrotational and again it follows that there exists a velocity potential Φ such that,

$$\vec{v} = \nabla\Phi \text{ for some } \Phi$$

Also, because the field is incompressible, it follows that

$$div(\vec{v}) = div\big(\nabla\Phi\big) = \nabla^2\Phi = 0$$

In the farfield the flow is potential and external disturbances are governed by the Laplacian for a perturbation velocity potential, $\delta\phi$. Following chapters 2.1 and 2.2 an expansion in normal-modes and elimination of the common term $exp[(i\alpha(z - ct) + im\theta)]$ yields the governing equation for the velocity potential perturbations.

$$\frac{d^2\widehat{\Phi}(r)}{dr^2} + \frac{1}{r}\frac{d\widehat{\Phi}(r)}{dr} - \left(\frac{m^2}{r^2} + \alpha^2\right)\widehat{\Phi}(r) = 0 \tag{4.47}$$

With general solution,

$$\widehat{\Phi}(r) = B_1 I_m(\alpha r) + B_2 K_m(\alpha r) \tag{4.48}$$

As before finite velocity is required as $r \to \infty$, thus,

$$\lim_{b\to\infty} I_m(b) \to \infty \ \forall m \ \Rightarrow \boxed{B_1 = 0} \tag{4.49a}$$

Therefore the pressure eigenfunction outside the unsteady jet is given by,

$$\widehat{\Phi}(r) = B_2 K_m(\alpha r) \tag{4.50}$$

The asymptotic behavior of the K_m Bessel function must again be considered. As in chapter 2.2 $s \equiv sgn(\alpha)$ is introduced into the argument of K_m to ensure decay of $\widehat{v}_r$ as $r \to \infty$. Consequently, $\widehat{\Phi}(r)$ must be written as,

$$\widehat{\Phi}(r) = B_2 K_m(s\alpha r) \tag{4.51}$$

The velocity vector is related to the velocity potential via the gradient operator, and the r-component is given as follows,

$$\vec{v} = \nabla\Phi \;\Rightarrow \langle \vec{v} \big| \underline{e}_r \rangle \tag{4.52}$$

it follows, then, that

$$\widehat{v}_r(r) = B_2 \frac{\partial K_m(x)}{\partial x}\bigg|_{x=\alpha r} \tag{4.53}$$

Using Bernoulli's equation, the pressure can then be determined. After perturbations are taken and the equation is linearized, the pressure perturbations are related to the velocity potential in the time scale τ by,

$$\frac{\delta p}{\rho} + T(t)\frac{\partial \delta\phi}{\partial \tau} + \langle \nabla\delta\phi \big| \nabla\Phi \rangle = 0 \tag{4.54}$$

Taking perturbations of the mean pressure and velocity potential in usual normal-mode expansion in the time scale τ it follows that the pressure perturbation eigenfunction in the far-field is given by,

$$\widehat{p}(r) = -i\rho\Big(- \alpha c T(t) + \frac{m\Omega T(t) R^2}{r^2} + \alpha(v_\infty + \Delta v)\Big) B_2 K_m(\alpha r) \tag{4.55}$$

4.1.3 Dispersion Relation for the Unsteady Swirling Jet

In this section, general solutions for pressure and radial velocity perturbations in the jet core and far-field for the unsteady swirling jet are obtained. To summarize the results thus far, the following stability eigenfunctions for the swirling jet hold.

♢ Inside the jet, $(r < R)$

$$\begin{cases} \widehat{p}(r) = A_1 J_m\big(r\Psi'\big) \\ \widehat{v}_r(r) = [i\alpha^2 A_1]/[\rho\widetilde{\omega}'\Psi'^2]\Big([-2\Omega T(t)m]/[r\widetilde{\omega}]J_m\big(r\Psi'\big) + \Psi' {J_m}'\big(r\Psi'\big)\Big) \end{cases} \tag{4.56}$$

♢ Outside the jet, $(r > R)$

$$\begin{cases} \widehat{v}_r(r) = B_2 \frac{\partial K_m(x)}{\partial x}\Big|_{x=\alpha r} \\ \widehat{p}(r) = -i\rho\Big(-\alpha c T(t) + \frac{m\Omega T(t) R^2}{r^2} + \alpha(v_\infty + \Delta v)\Big) B_2 K_m(\alpha r) \end{cases} \tag{4.57}$$

Across the vortex sheet at $r = R$, both pressure and velocity must be continuous, *i.e.*,

$$\frac{\widehat{v}_r(R^-)}{\widetilde{\omega}'} = \frac{\widehat{v}_r(R^+)}{\widetilde{\omega}' + \alpha\Delta v} \tag{4.58}$$

$$\widehat{p}(R^-) = \widehat{p}(R^+) \tag{4.59}$$

In light of the matching conditions (4.58) and (4.59), the eigenfunction equations in the regions ($r < R$) given by (4.56) and ($r > R$) given by (4.57) reduce to two equations in two unknowns A_1 and B_2. Disregarding trivial solutions to the system of equations requires that the determinant of the system be set equal to zero. Doing this and manipulating the result algebraically leads to the following dispersion relation for the periodically unsteady swirling jet:

$$(\widetilde{\omega}' + \alpha\Delta v)^2 \left[-\frac{2\Omega T(t) m}{\widetilde{\omega}'} + \Psi' R \frac{J_m{}'(R\Psi')}{J_m(R\Psi')} \right] = -\frac{\widetilde{\omega}'^2 \Psi'^2}{s\alpha R} \frac{K_m{}'(s\alpha R)}{K_m(s\alpha R)} \quad (4.60)$$

where,

$$\widetilde{\omega}' \equiv \alpha c T(t) - m\Omega T(t) - \alpha(v_\infty + \Delta v) \quad (4.61)$$

$$\Psi' \equiv \sqrt{\alpha^2 \Big(\frac{4\Omega^2 T(t)^2}{\widetilde{\omega}'^2} - 1 \Big)} \quad (4.62)$$

It is straight forward to verify that the above dispersion relation (4.60) reduces to exactly that of the steady swirling jet if $T(t) = 1$. And exactly to that of the axisymmetric jet if $T(t) = 1$ and the rotation rate Ω is set equal to zero. This is, of course, a necessary requirment for validity of the analysis.

4.1.4 Nondimensionalization of the Dispersion Relation

The dispersion relation for the unsteady swirling jet in the form (4.60) is dimensional and because the transient perturbation $T(t) = 1 + \epsilon\ cos(\omega_m t)$

is taken dimensionless from the start, non-dimensionalization is carried out exactly as was done for the steady swirling jet; by non-dimensionalizing with the factor $\left(R/\Delta v\right) \sim [T]$, where again R is the radius of the jet and Δv is the velocity difference across the vortex sheet. After doing this, the following dimensionless variables are defined,

$$c^* \equiv \frac{cR}{\Delta v} \tag{4.63}$$

$$\alpha^* \equiv \alpha R \tag{4.64}$$

$$\Psi'^* \equiv \Psi' R \tag{4.65}$$

the control parameter S defined as the unsteady swirl number is given as follows,

$$S \equiv \frac{\Omega T(t) R}{\Delta v} \tag{4.66}$$

Using these dimensionless variables results in the following dimensionless dispersion relation for the periodically unsteady swirling jet.

$$\boxed{\left(\widetilde{\omega} + \alpha\right)^2 \left(- 2mS + \widetilde{\omega}\Psi \frac{J_m{}'(\Psi)}{J_m(\Psi)} \right) + \frac{\widetilde{\omega}^3 \Psi^2}{s\alpha} \frac{K_m{}'(s\alpha)}{K_m(s\alpha)} = 0} \tag{4.67}$$

Again, the $*$ superscripts are suppressed for convenience. ($e.g.$, we let $c^* = c$ where it is understood that c is now non-dimensional)

Notice that because the quasi-steady approach does not work, the above equation is not equivalent to the one obtained for the steady swirling jet.

In fact, it is a completely different function. It is not as straight forward, then, to determine the effect of the unsteadiness on the stability of the unsteady swirling jet as was done in the case of the unsteady axisymmetric jet. Recall, for the unsteady axisymmetric jet, the modulations of phase (or frequency) and amplitude were completely determined from the time-variable-transformed celerity and growth rate. Here, those relationships do not hold and because the time-dependent perturbation is coupled in the stability equations. The effect of the time-dependent perturbation on the growth rate and celerity must be determined from a numerical solution of the dispersion relation in time.

4.1.5 Numerical Solution Procedure: Swirling Jet

Equation (4.67) is a complex-valued non-trivial function of three variables (c_R, c_I, t) and three control parameters (α, m, S). Equation (4.67) has the same functional characteristics as the dispersion relation for the steady swirling jet inasmuch as a standard Newton-Raphson numerical method is insufficient to produce the solution for all values of α, and t. Therefore, a modified Newton's method is employed and now the function is solved for c_R vs. t and c_I vs. t at fixed azimuthal and axial wave numbers m, α and at a fixed swirl S. In essence, then, the unsteady stability characteristics are investigated at a single point with respect to m and α over one complete period of time steps t.

4.1.6 Numerical Results for the Unsteady Swirling Jet

In this section, results are presented for the unsteady swirling jet. The case considered is that of a jet with swirl parameter $S = 0.7$ undergoing periodic unsteady variations with $\epsilon = 0.01$ and $\omega_m = 0.1$. The unsteady perturbation is shown in figure 4.2.

Modulation of the growth rate for the Kelvin-Helmholtz mode ($m = 0$)

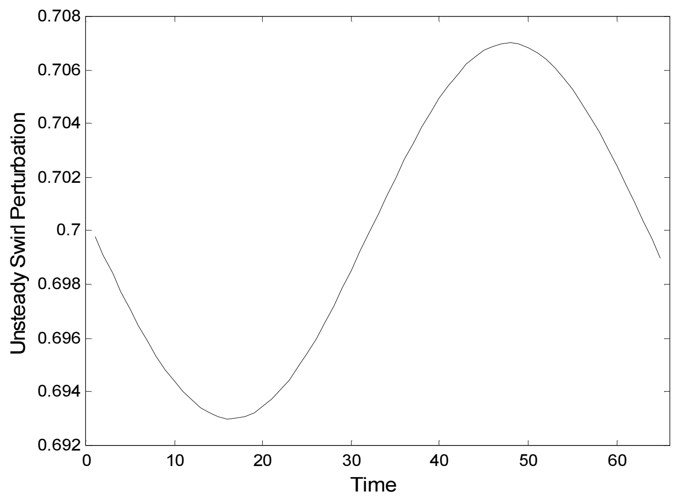

Figure 4.2: Swirl Perturbation vs. Time

at fixed values of the axial wavenumber, specifically, $\alpha = 0.1$, $\alpha = 0.5$, $\alpha = 1.0$, $\alpha = 1.5$ and $\alpha = 2.0$ over one period of the periodic perturbation is given in figures 4.3-7. The growth rates shown are for the unsteady swirling jet obtained directly fom equation (4.67) and the quasi-steady results are obtained from instantaneous velocity profiles using the steady state equation.

Examining figure 4.3, we note that in both the unsteady and quasi-steady

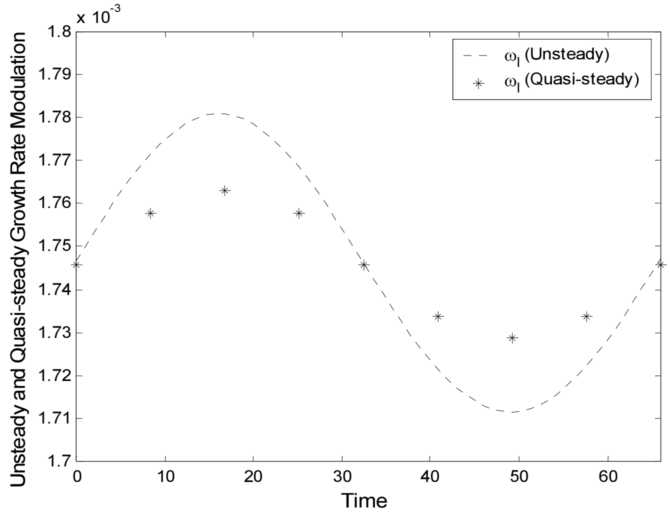

Figure 4.3: Unsteady and Quasi-steady Growth Rate Modulation vs. Time at $\alpha = 0.1$

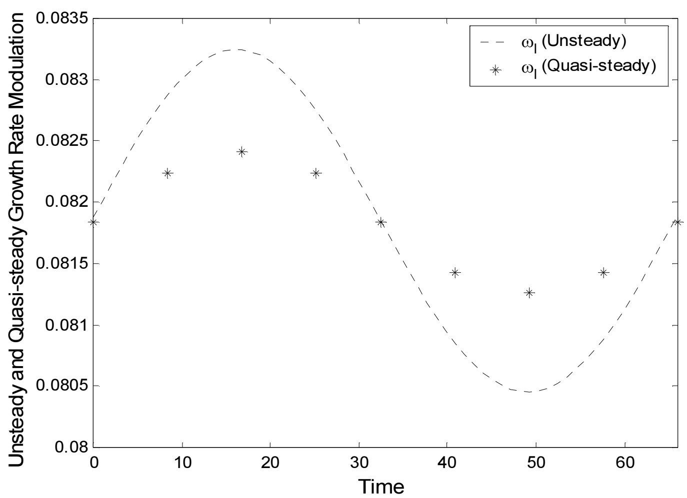

Figure 4.4: Unsteady and Quasi-steady Growth Rate Modulation vs. Time at $\alpha = 0.5$

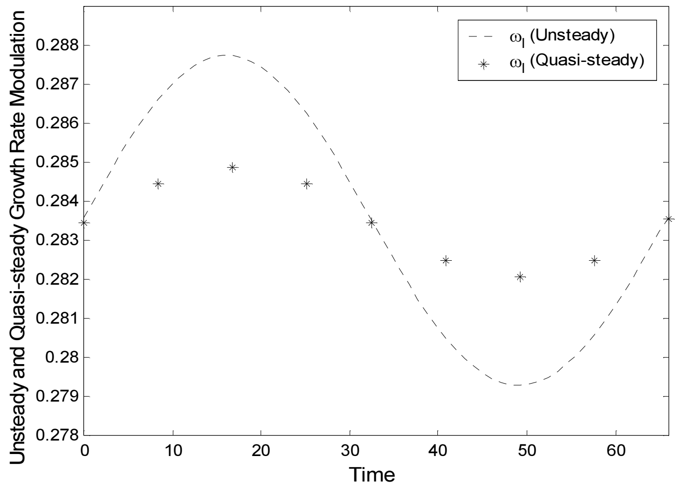

Figure 4.5: Unsteady and Quasi-steady Growth Rate Modulation vs. Time at $\alpha = 1.0$

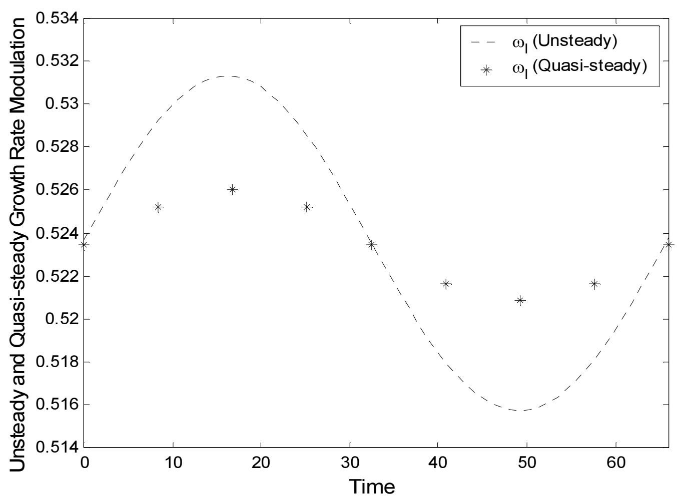

Figure 4.6: Unsteady and Quasi-steady Growth Rate Modulation vs. Time at $\alpha = 1.5$

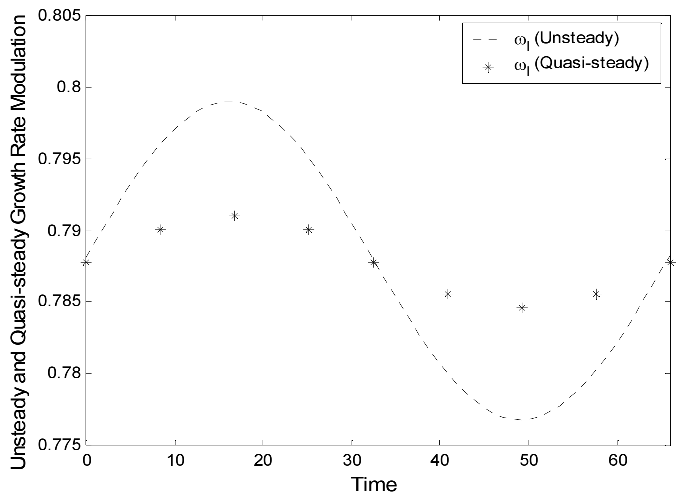

Figure 4.7: Unsteady and Quasi-steady Growth Rate Modulation vs. Time at $\alpha = 2.0$

approaches the amplitude of the modulated growth rate decreases as the swirl increases (refer to figure 4.2) for all values of α given. This is in agreement with the characteristics of the steady swirling jet wherein figure 2.6 shows that as swirl increases the growth rate of axisymmetric disturbances decrease. However, it is obvious that the quasi-steady approach underpredicts the level of the modulation.

Modulation of the celerity as obtained from the unsteady formulation and quasi-steady approach for the Kelvin-Helmholtz mode ($m = 0$) at the same fixed values of the axial wavenumber, $\alpha = 0.1$, $\alpha = 0.5$, $\alpha = 1.0$, $\alpha = 1.5$ and $\alpha = 2.0$ over one period of the periodic perturbation is given in figures 4.9-13. The unsteady swirl distribution associated with these modulations wherein $\Omega = 0.7$ is shown in figure 4.2

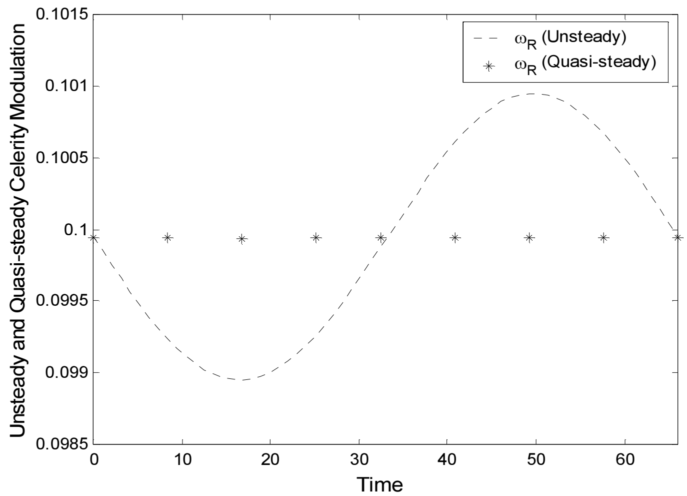

Figure 4.8: Unsteady and Quasi-steady Celerity Modulation vs. Time at $\alpha = 0.1$

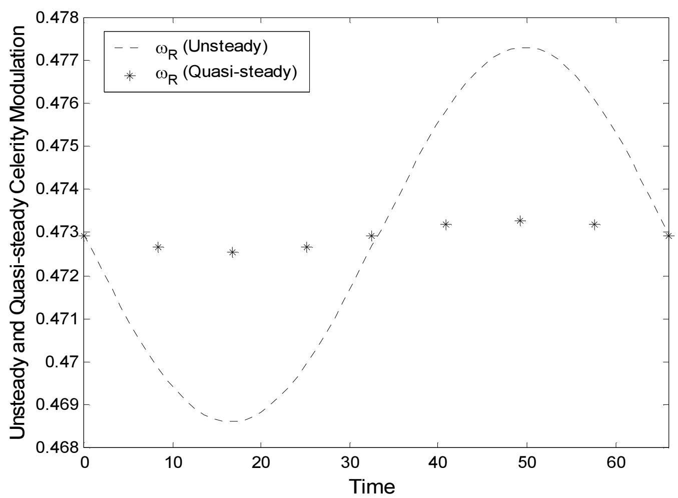

Figure 4.9: Unsteady and Quasi-steady Celerity Modulation vs. Time at $\alpha = 0.5$

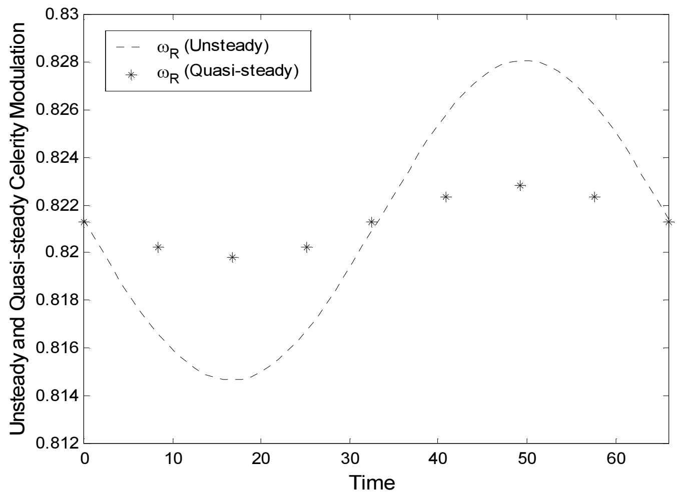

Figure 4.10: Unsteady and Quasi-steady Celerity Modulation vs. Time at $\alpha = 1.0$

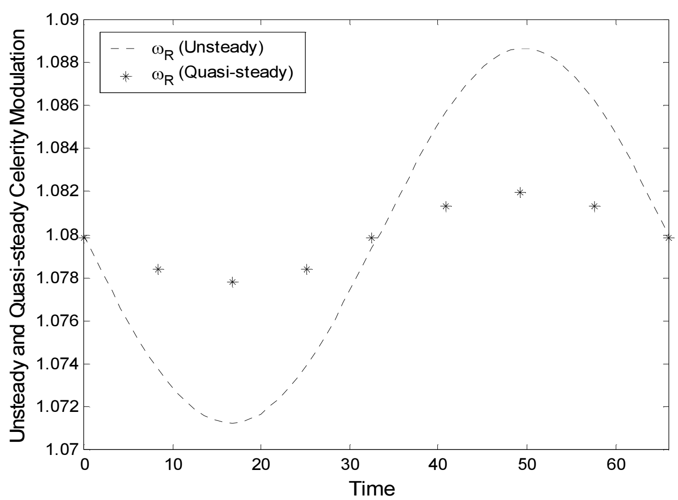

Figure 4.11: Unsteady and Quasi-steady Celerity Modulation vs. Time at $\alpha = 1.5$

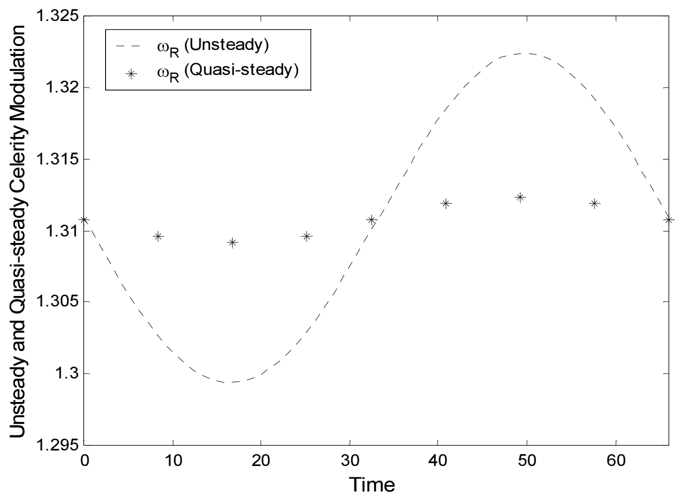

Figure 4.12: Unsteady and Quasi-steady Celerity Modulation vs. Time at $\alpha = 2.0$

It is apparent that the amplitude of the celerity modulation decreases with decreasing swirl, again consistent with the steady case given in figure 2.5 wherein it is shown that the celerity of axisymmetric disturbances increase with increasing swirl. Figures 4.9-13 show that the quasi-steady and unsteady growth rate values do not match. The quasi-steady approach underpredicts the level of modulation caused by the unstediness in the mean flow.

Notice that the celerity modulation seems to be less sensitive to the periodic perturbation at low wavenumbers than the growth rate modulation. This is partly due to the fact that initially the celerity travels at the same velocity as the jet, and, figure 2.5 shows that the celerity for the steady swirling jet is relatively insensitive to changes in the swirl for small values of the axial wavenumber α. Thus, we expect the difference between the maximum unsteady and quasi-steady modulation value over one period to increase as α

increases.

To see the effect of the unsteady modulation on celerity and growth rate with respect to the steady state celerity and growth rate, it is possible to extract the maximum unsteady modulation values over one period for a series of axial wavenumbers α. That is, the maximum value of the unsteady modulation for celerity and growth rate may be extracted form figures 4.3-7 and figures 4.9-13 at specific values of the axial wavenumber. Even though the unsteady modulation is a function of time, the steady state and unsteady maximum modulation values may be compared by plotting the maximum modulation of celerity and growth rate at the appropriate α thus producing a maximum modulation vs. α curve. Figure 4.14 shows the maximum modulation value of the unsteady growth rate for $0 \leq \alpha \leq 0.6$ for the Kelvin-Helmholtz mode (m=0) at $S = 0.7$ plotted with the steady state value of growth rate at the same value of m and S.

Figure 4.14 shows that the effect of the unsteady azimuthal perturbation results in a decrease in growth rate of the unsteady modes as expected, however, in light of figures 4.3-7 this decrease is much larger than what would have been guessed using the inappropriate quasi-steady approach.

Figure 4.15 shows the maximum modulation of the celerity for axisymmetric disturbances when $S = 0.7$. It is apparent that the wave speed increases with increasing swirl. Again, from figure 2.5 this behavior is expected, however, the actual amount by which the celerity increases with S is much larger than what would have been guessed using the inappropriate quasi-steady approach.

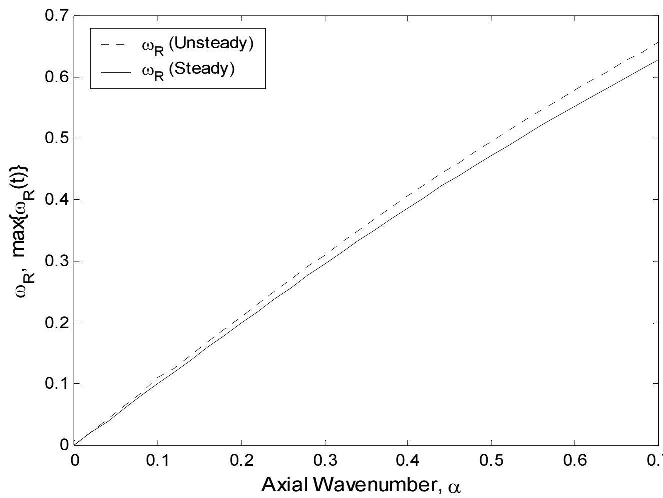

Figure 4.13: Growth Rate for Steady and Unsteady Swirling Jets at $S = 0.7$

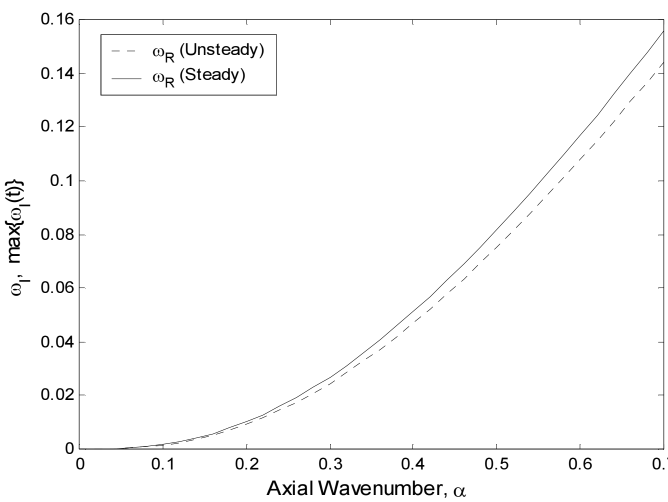

Figure 4.14: Celerity for Steady and Unsteady Swirling Jets at $S = 0.7$

Chapter 5

Conclusions

In this work the effects of periodic variations on the temporal stability of axisymmetric and swirling jets are examined. The unsteadiness is introduced in the axisymmetric jet as a periodic variation of the axial velocity component of the flow so that the jet is pulsed. Whereas, in the case of the unsteady swirling jet, the unsteadiness is introduced as a periodic variation of the azimuthal (swirl) velocity component of the flow. An analytical dispersion relation is obtained in both cases and solved numerically in the case of the swirling jet. Results for the unsteady inviscid axisymmetric jet show that the quasi-steady approach works; that is, it is possible to formulate the unsteady dispersion relation in the exact form as the dispersion relation for the steady axisymmetric jet. However, the unsteady jet is forced into a new time scale that is a 'stretched' version of the initial scale. In the case of a periodically unsteady swirling jet, an analytical dispersion relation is also obtained; however, in this case the unsteady dispersion relation does not reduce to the steady state in the stretched time scale. Analytically, this shows that for the periodically unsteady swirling jet, the quasi-steady approach does not work. In solving the unsteady dispersion relation fo the unsteady swirling jet, it is found that the quasi-steady approach substantially underpredicts the values of the modulations of celerity and growth rate resulting from the unsteadiness of the flow.

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Vita

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Mark Carrara was born on April 15, 1974 in Roswell, New Mexico and since childhood lived in Santa Fe, New Mexico until he left to enter college at the *University of New Mexico* in 1995. He was awarded a Bachelor of Science degree in Applied Mathematics/Physics from the *University of New Mexico* in May of 1999. At the same university he carried out research work at the *New Mexico Engineering Research Institute* associated with the school of engineering under the supervision of Adjunct Prof. R. Gordon McKeen until May of 1998. After this, he carried out research in the field of high energy physics in the department of Physics and Astronomy at UNM under the supervision of Dr. Sally Seidel. Wanting to further his education in the area of theoretical classical physics, after graduating from UNM he chose to study theoretical mechanics at the *Virginia Polytechnic Institute and State University* by enrolling as a graduate student in the department of Engineering Science and Mechanics in August 1999. He was awarded the degree of Master of Science in Engineering Mechanics conducting research in the area of mathematical fluid dynamics in May 2001.