Continuum Analytical Shape Sensitivity Analysis of 1-D Elastic Bar

Soumya Sambit Nayak

Thesis submitted to the faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

Master of Science

In

Mechanical Engineering

Mayuresh Patil, Chair

Pinar Acar, Co-Chair

Danesh Tafti

December 2, 2020

Blacksburg, Virginia

Keywords: Continuum Sensitivity Analysis, CASSA, material sensitivity, local sensitivity, design velocity, variational form, convergence

Continuum Analytical Shape Sensitivity Analysis of 1-D Elastic Bar

Soumya Sambit Nayak

ACADEMIC ABSTRACT

In this thesis, a continuum sensitivity analysis method is presented for calculation of shape sensitivities of an elastic bar. The governing differential equations and boundary conditions for the elastic bar are differentiated with respect to the shape design parameter to derive the continuum sensitivity equations. The continuum sensitivity equations are linear ordinary differential equations in terms of local or material shape design derivatives, otherwise known as shape sensitivities. One of the novelties of this work is the derivation of three variational formulations for obtaining shape sensitivities, one in terms of the local sensitivity and two in terms of the material sensitivity. These derivations involve evaluating (a) the variational form of the continuum sensitivity equations, or (b) the sensitivity of the variational form of the analysis equations. We demonstrate their implementation for various combinations of design velocity and global basis functions. These variational formulations are further solved using finite element analysis. The order of convergence of each variational formulation is determined by comparing the sensitivity solutions with the exact solutions for analytical test cases. This research focusses on 1-D structural equations. In future work, the three variational formulations can be derived for 2-D and 3-D structural and fluid domains.

Continuum Analytical Shape Sensitivity Analysis of 1-D Elastic Bar

Soumya Sambit Nayak

GENERAL AUDIENCE ABSTRACT

When solving an optimization problem, the extreme value of the performance metric of interest is calculated by tuning the values of the design variables. Some optimization problems involve shape change as one of the design variables. Change in shape leads to change in the boundary locations. This leads to a change in the domain definition and the boundary conditions. We consider a 1-D structural element, an elastic bar, for this study. Subsequently, we demonstrate a method for calculating the sensitivity of solution (e.g. displacement at a point) to change in the shape (length for 1-D case) of the elastic bar. These sensitivities, known as shape sensitivities, are critical for design optimization problems. We make use of continuum analytical shape sensitivity analysis to derive three variational formulations to compute these shape sensitivities. The accuracy and convergence of solutions is verified using a finite element analysis code. In future, the approach can be extended to multi-dimensional structural and fluid domain problems

Dedication

This thesis is dedicated to my parents.

Acknowledgements

This journey would not have been fruitful without the guidance and support from a number of people. To begin with, I am deeply grateful to my advisor and mentor, Dr. Mayuresh Patil. He bestowed me with counsel and direction in my coursework as well as my career path. His enlightening ideas and advices carried me through various research roadblocks. It was an honor working under Dr. Patil.

Co-chair, Dr. Pinar Acar always had valuable feedbacks and constructive comments for my research. Her course on optimization led the foundation for my research. I would like to take this opportunity to thank Dr. Danesh Tafti for agreeing to be a member of my advisor committee and for our insightful discussions on various research topics. I would also like to thank the Department of Mechanical Engineering and all my coursework professors.

My stay at Virginia Tech wouldn't have been this enjoyable without all the friends I made here in Blacksburg. They are my family away from family. A big shout out to all my friends both in US and India.

Finally, I am forever indebted to my parents and sister for their love and sacrifice that has gotten me this far. Also, I am really thankful to all my cousins and extended family for all their encouraging words and their unswerving confidence in me.

Contents

List of Figures ix

List of Tables xii

Chapter 1

Introduction 1

1.1 Motivation 1

1.2 Research Objectives 2

1.3 Thesis Outline 2

Chapter 2

Background 4

2.1 Sensitivity Analysis 4

2.2 Shape Sensitivity Analysis 5

2.3 Discrete Sensitivity Analysis 6

2.4 Continuum Sensitivity Analysis 8

2.5 Literature Review 10

Chapter 3

Theory 13

3.1 Problem Definition 13

3.2 Design velocity 14

3.3 Material and Local Derivatives 15

3.4 Material Sensitivity ODE 16

3.5 Local Sensitivity ODE 17

3.6 Variational Form of Analysis ODE 17

3.7 Material Sensitivity of the Analysis Variational Form 18

3.8 Variational Form of Material Sensitivity Equation 20

3.9 Equivalence of the Variational Forms for Material Sensitivity . 21

3.10 Local Sensitivity of the Analysis Variational Form 22

3.11 Variational Form of Local Sensitivity Equation 23

Chapter 4

Results 25

4.1 Key Sensitivity Equations 25

4.2 Design Velocities and Trial Functions 26

4.3 Exact Sensitivity Results from Sensitivity ODE 27

4.4 Approximate Sensitivity Results using the Variational Forms .. 32

4.4.1 Approximate Analysis Solution using Variational Form 32

4.4.2 Material Sensitivity Results 33

4.4.3 Local Sensitivity Results 35

4.4.4 Singular Term for Piecewise Continuous Design Velocity ... 36

4.4.5 Tabulated Results 37

4.5 Comparison Plots 43
4.5.1 Linear Design Velocity Plots 43
4.5.2 Quadratic Design Velocity Plots 46
4.5.3 Piecewise Design Velocity Plots 48
4.6 Convergence in Finite Element Results 51
4.6.1 Exact Solutions from ODEs 51
4.6.2 Analysis Solutions 54
4.6.3 Local Sensitivity Results 55
4.6.4 Material Sensitivity Results 59
Chapter 5
Conclusions and Future Work 69
5.1 Research Summary 69
5.2 Future Work 70
Bibliography 71

List of Figures

Figure 2.1 Design Sensitivity Analysis Classification..5

Figure 3.1 Elastic bar with axial load...14

Figure 3.2 Different parametric representations of the elastic bar: (a) parameterization 1 (material points move to the right), and (b) parameterization 2 (material points move to the left). (Kulkarni M.D. et al., 2020) 14

Figure 4.1 Material sensitivity solutions for linear design velocity..................44

Figure 4.2 Material sensitivity solutions for quadratic design velocity............46

Figure 4.3 Material sensitivity solutions for piecewise design velocity............49

Figure 4.4 Convergence of analysis solution for (a) constant forcing function, (b) quadratic forcing function, and (c) sinusoidal forcing function. 55

Figure 4.5 Convergence of local sensitivity solution using Eq[5] for (a) constant forcing function, (b) quadratic forcing function, and (c) sinusoidal forcing function...56

Figure 4.6 Log-log plots of error norms of local sensitivity solutions using Eq[5] for (a) quadratic forcing function, and (b) sinusoidal forcing function. 58

Figure 4.7 Convergence of material sensitivity solution with linear design velocity for (a) constant forcing function using Eq[6], (b) quadratic forcing function using Eq[6], (c) sinusoidal forcing function using Eq[6], (d) constant forcing function using Eq[7], (e) quadratic forcing function using Eq[7], and (f) sinusoidal forcing function using Eq[7]. 61

Figure 4.8 Convergence of material sensitivity solution with quadratic design velocity for (a) constant forcing function using Eq[6], (b) quadratic forcing function using Eq[6], (c) sinusoidal forcing function using Eq[6], (d) constant forcing function using Eq[7], (e) quadratic forcing function using Eq[7], and (f) sinusoidal forcing function using Eq[7]. 63

Figure 4.9 Log-log plots of error norms of material sensitivity solutions with quadratic design velocity for (a) constant forcing function using Eq[6], (b) quadratic forcing function using Eq[6], (c) sinusoidal forcing function using Eq[6], (d) constant forcing function using Eq[7], (e) quadratic forcing function using Eq[7], and (f) sinusoidal forcing function using Eq[7]. 66

Figure 4.10 Convergence of material sensitivity solution with quadratic design velocity for (a) constant forcing function using Eq[6], (b) quadratic forcing function using Eq[6], (c) sinusoidal forcing function using Eq[6], (d) constant forcing function using Eq[7], (e) quadratic forcing function using Eq[7], and (f) sinusoidal forcing function using Eq[7]. 68

List of Tables

Table 4.1 Exact analysis solution, local and material sensitivity using ODEs Eq[1], Eq[2] and Eq[3] for constant forcing..30

Table 4.2 Local and material sensitivity solutions using Eq[5], Eq[6] and Eq[7] for trial functions $\psi = \frac{x}{L}, \left(\frac{x}{L}\right)^2$ and constant forcing 38

Table 4.3 Local and material sensitivity solutions using Eq[5], Eq[6] and Eq[7] for finite element trial functions for two elements for constant forcing...........39

Table 4.4 Analysis solution, local and material sensitivity using Eq[4], Eq[5], Eq[6] and Eq[7] for finite element trial functions for two elements for constant forcing...41

Table 4.5 Analysis solution, local and material sensitivity using Eq[4], Eq[5], Eq[6] and Eq[7] for finite element trial functions for four element and constant forcing...42

Table 4.6 Material sensitivity solutions for linear design velocity...................45

Table 4.7 Material sensitivity solutions for quadratic design velocity.............47

Table 4.8 Material sensitivity solutions for piecewise design velocity 50

Table 4.9 Exact analysis solution, local and material sensitivity using ODEs

Eq[1], Eq[2] and Eq[3] for quadratic forcing 52

Table 4.10 Exact analysis solution, local and material sensitivity using ODEs

Eq[1], Eq[2] and Eq[3] for sinusoidal forcing..53

Chapter 1

Introduction

1.1 Motivation

Solving an optimization problem requires the calculation of the change in the response of a performance metric to change in design parameters in a given domain. Some design parameters involve changes in the configuration or change in domain shape. Structural optimization variables can be classified into three basic categories, namely, sizing, shape and topology. Sizing optimization is incorporated to obtain optimum size of a specific member in a fixed domain. Topology optimization determines optimal distribution of material inside a given design domain. While, in shape optimization, the domain definition changes. This leads to change in boundary locations as well as domain material points with respect to the shape design variable(s). The current work is focused on calculation of shape sensitivities for shape optimization.

Shape optimization finds its applications in various engineering domains involving structures, fluids and fluid-structure interaction. This requires evaluation of shape sensitivities that influence the improvement in design performance in each iteration. Developing an accurate and efficient method for computing these shape sensitivities is necessary for the application of shape design optimization to complex systems. This thesis demonstrates the derivation of variational formulations in order to calculate shape sensitivities for the case of a 1-D elastic bar. The main research objectives are highlighted next.

1.2 Research Objectives

The goal of this research is to derive formulations and estimate their accuracy in predicting local and material shape sensitivities. This necessitates the following research objectives.

1. Develop continuum sensitivity equations for local and material shape sensitivities.
2. Derive variational formulations for shape sensitivities using the above equations.
3. Discuss the accuracy of shape sensitivity solutions and determine its order of convergence using Finite Element Analysis (FEA).

These objectives are addressed in Chapter 3 and Chapter 4.

1.3 Thesis Outline

The organization of this thesis is as follows.

Chapter 2 (Background) talks about sensitivity analysis with shape as a parameter followed by a brief discussion about discrete and continuum sensitivity analysis. This chapter concludes with a literature review of shape sensitivity analysis in structures and fluids.

Chapter 3 (Theory) deals with the definition of a 1-D elastic bar and its governing equations. The continuum sensitivity equations are then developed. The most important part of this chapter is the derivation of variational formulations for local and material sensitivities.

Chapter 4 (Results) discusses the accuracy of these formulations by comparison with exact solutions. Initially, we maintain constant forcing and manipulate the design velocities and basis functions to generate comparison plots. FEA is then used to determine the order of convergence of local and material sensitivity solutions from these variational formulations. This is performed for multiple design velocities and forcing terms.

Chapter 5 (Conclusion) briefly summarizes the key highlights of the current research. Finally, we discuss the potential applications of the thesis research in multi-dimensional problems and future scope of this research.

Chapter 2

Background

2.1 Sensitivity Analysis

Sensitivity is the effect of change in a system parameter on the system's response. Basic examples of sensitivity include effect of change in material stiffness on the deflection in a bar, effect of change in Reynolds number on the pressure at a point on a body, and effect of change in shape on the drag force on a body.

Sensitivity Analysis can be used for:

a) Hypothesis generation (understand the problem and solution)
b) Gradient-based optimization
c) Reliability calculation
d) Uncertainty quantification

Sensitivity analysis can be classified into two primary methods – Analytical and Numerical. The system equations are differentiated with respect to the parameter for sensitivities in analytical sensitivity calculation. On the other hand, in finite difference (numerical) sensitivity calculation, the change in system's response due to a small perturbation in the parameter is evaluated. Complex step method is another type of numerical method where the parameter is assumed to be complex and the perturbation is on the imaginary axis.

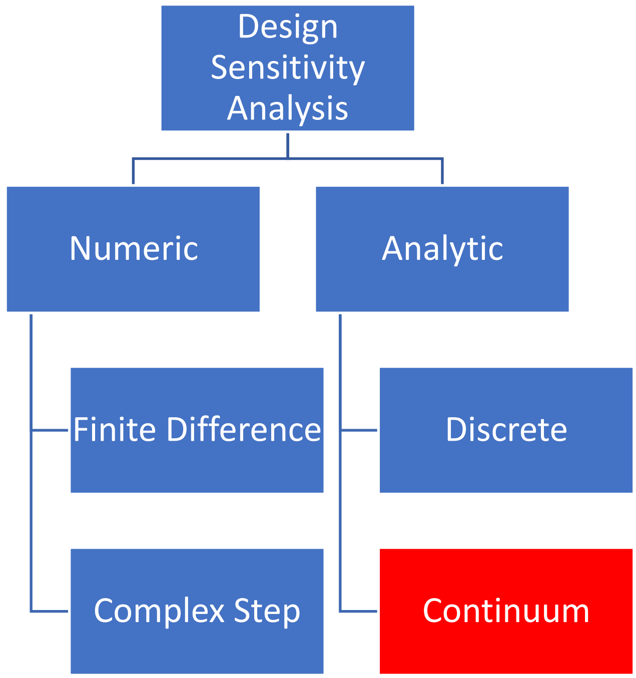


Figure 2.1 Design Sensitivity Analysis Classification

Finite difference and complex step methods cannot conduct sensitivity for multiple parameters at the same time, that is, they are unable to handle adjoint sensitivity calculations. While, analytical sensitivity analysis can be formulated to conduct multiple sensitivity analyses at the same time.

2.2 Shape Sensitivity Analysis

Shape change is a considerably different problem than a change in coefficient. Change in shape leads to change in the boundary locations. This leads to a change in the domain definition and the boundary conditions. The domain equations remain the same as typically PDEs do not change. The change in position of points in the domain with respect to the shape parameter is called the design velocity. The two classifications mentioned in Figure 2.1, numerical and analytical methods, can deal with any parameter including a shape parameter. We use Analytical Shape Sensitivity Analysis because of the computational and accuracy advantage of analytical sensitivity over finite difference. There are two types of shape sensitivities, local and material. Local or Eulerian sensitivity corresponds to the change in the response at a fixed location in space while Material or Lagrangian sensitivity denotes the change in the response at a location moving with shape change. The two derivatives are related by:

$$\dot{q} = q' + \frac{dq^{as}}{dx} D \qquad 2.1$$

where, $\dot{q}$ represents the material derivative and q' represents the local derivative. q^{as} is the analysis solution. Thus, $\frac{dq^{as}}{dx}$ becomes the spatial derivative of the analysis solution. Lastly, D is the domain derivative with respect to shape change, in other words, the design velocity.

Analytical Shape Sensitivity Analysis can be primarily classified into two methods: Discrete and Continuum as illustrated in Figure 2.1.

2.3 Discrete Sensitivity Analysis

Discrete Analytical Shape Sensitivity Analysis (DASSA) is based on the change in discretized domain equations due to change in domain. Thus, this leads to material sensitivity and needs the design velocity throughout the domain.

Consider a PDE in space given by:

$$F(q(x), c) = 0 \tag{2.2}$$

$q(x)$ here is a field variable, c is the shape parameter, and F represents a differential operator.

We have boundary conditions (BCs):

$$B(q(x), c) = 0 \tag{2.3}$$

DASSA involves discretization followed by sensitivity analysis. Consider the discretized system of equations for the above PDE along with BCs given by:

$$\{f(\{q(x)\}, c)\} = \{0\} \tag{2.4}$$

where $\{f\}$ denotes the vector of discretized equations and $\{q\}$ denotes the vector of discretized field variable.

To find the shape sensitivities we evaluate the material derivative of the discretized system of equations with respect to the shape parameter.

$$\frac{d\{f\}}{dc} = \frac{\partial\{f\}}{\partial c} + \left[\frac{\partial\{f\}}{\partial\{q\}}\right]\frac{d\{q\}}{dc} = \{0\} \tag{2.5}$$

Here $\left[\frac{\partial\{f\}}{\partial\{q\}}\right]$ is the Jacobian or coefficient matrix $[J]$.

Eq[2.5] can be solved for shape sensitivity as:

$$[J]\{\dot{q}\} = -\left[\frac{\partial\{f\}}{\partial c}\right] \tag{2.6}$$

Though the equation seems simple to solve for a normal parameter, it is not the case for a shape parameter. Change in shape parameter leads to change in domain. This, in turn, requires computation of mesh Jacobian sensitivities (in RHS) which is computationally inefficient. Hence, we need alternate approaches like continuum sensitivity analysis.

2.4 Continuum Sensitivity Analysis

Continuum Analytical Shape Sensitivity Analysis (CASSA) involves evaluating the sensitivity of the PDE with respect to the shape parameter followed by discretization. Here it is easier to write sensitivity-PDE with respect to local sensitivity because the spatial derivatives in the PDE will change with shape transformation for material sensitivity. But the BCs are given on the boundary and move with change in boundary, thus the sensitivity-BCs are easily represented with respect to material sensitivity. We can solve the equations by converting everything either into local sensitivity variables or material sensitivity variables.

Considering the same system of PDE and BCs:

$$F(q(x), c) = 0 \tag{2.7}$$

$$B(q(x), c) = 0 \tag{2.8}$$

Taking the derivative of PDE with respect to the shape parameter c in the local frame, we have:

$$F' + \frac{\partial F}{\partial q} q' = 0 \tag{2.9}$$

If there is no change in PDE with the shape parameter, $F' = 0$.

Taking the derivative of BCs with respect to the shape parameter c in the material frame, we have:

$$\dot{B} + \frac{\partial B}{\partial q}\dot{q} = 0 \qquad 2.10$$

Replacing the value of $\dot{q}$ from Eq[2.1]:

$$\dot{B} + \frac{\partial B}{\partial q}(q' + q_x^{as} D) = 0 \qquad 2.11$$

The type of BC determines, sensitivity-BC relationship. $\dot{B} = 0$ for Dirichlet BC.

Here $\left[\frac{\partial B}{\partial q}\right]$ and $\left[\frac{\partial F}{\partial q}\right]$ together represent the Jacobian or coefficient matrix $[J]$.

Thus, the CASSA Equation in terms of local sensitivity looks similar to DASSA Equation in terms of material sensitivity Eq[2.6]:

$$[J][q'] = -[b] \qquad 2.12$$

But unlike DASSA, the RHS in CASSA depends on the term $q_x^{as} D$, that is, the spatial derivative of the analysis solution and the design velocity at boundary where the shape is changing. Thus, the accuracy of spatial derivatives determines the accuracy of local derivatives and sensitivity calculation. Furthermore, if we need the material sensitivity (typically at the boundary), we need the spatial derivative of the analysis solution again.

2.5 Literature Review

Accurate and efficient sensitivity or design derivative analysis is the foundation for solving gradient-based design optimization problems. Though numerical sensitivity methods discussed in the previous sections are easy to implement, they can be computationally expensive and inaccurate. On the other hand, analytical sensitivity methods (DASSA and CASSA) are highly accurate but their implementation depends on the complexity of governing equations of the physical problem. This literature review focuses primarily on CASSA. The development of CASSA in recent times can be categorized into three primary applications: structures, fluids and fluid-structure interaction. This section summarizes major research contributions in structures. CASSA for fluids and fluid-structure interaction is mentioned briefly.

Haug and Arora (1978) introduced CASSA in structural applications. They developed formulations for design sensitivity analysis to treat three classes of symmetrical mechanical systems: (a) finite element structural systems, (b) boundary value problems, and (c) initial value (dynamic) problems. The early 1980s witnessed multiple papers on material sensitivity formulations for structural mechanics problems. A noteworthy contribution during this period was the introduction of two methods: (a) boundary integral method (local CASSA), and (b) domain integral method (total CASSA or material sensitivity CASSA). Boundary integral method involves transforming domain integrals into boundary integrals. Chun and Haug (1983) used this for shape optimal design problems governed by equations of 2-D elasticity and noted that it can be computationally cheaper than traditional total CASSA. Braibant and Fleury

(1984, 1985) provided another direct formulation of total CASSA, the domain integral method. Initially, they used it in simple examples such as beam, fillet and plate with a hole which they later extended to more bidimensional problems. A parametrical representation of the structure was used to predict the movement of material points in the domain which helped to establish an analytical formulation of the sensitivity analysis. Wickert and Canfield (2009) adopted the term “continuum sensitivity analysis” (CSA) and utilized it for a fluid-structure interaction problem. A major difference between local and total CASSA is in their design velocity definition. While it is necessary to define design velocity throughout the entire domain in total CASSA, just prescribing the design velocity on the boundaries is enough for local CASSA. Consequently, local CASSA can be computationally less expensive than total CASSA (Liu and Canfield, 2013).

CASSA was used for fluid sensitivity applications by formulating sensitivity equations in terms of local design derivative. Turner and Patil (2018) presented a local form CASSA approach to the incompressible Navier-Stokes equations. Local CASSA is convenient as it requires design velocity definition on just the boundaries, but it has one drawback. The boundary conditions of local sensitivity equations have terms with higher order derivatives. Cross and Canfield (2014, 2015) developed a technique to provide accurate approximations of these derivatives by reconstructing a more accurate solution at the boundaries. They introduced it in linear and non-linear systems and termed it as “spatial gradient reconstruction” (SGR). The SGR only depends on analysis output, which enables a general formulation of all boundary conditions (of local sensitivity equations).

Hence, local CASSA with SGR is shown to be a non-intrusive method. Kulkarni et al. (2020) employed the same for various fluid applications.

Chapter 3

Theory

3.1 Problem Definition

Consider a 1-D Elastic Bar ODE with constant EA:

$$EAu_{xx} + f = 0 \qquad 3.1$$

where E is the Young's modulus

A is the cross-sectional area

u is the displacement at a point

f is the axial force.

Consider Boundary conditions (BCs):

$$\Gamma_e :\ u(x = 0) = u_0 \text{ (Dirichlet/Essential BC)} \qquad 3.2$$

$$\Gamma_n :\ EAu_x(x = L) = N_L \text{ (Neumann/Natural BC)} \qquad 3.3$$

Figure 3.1 below illustrates this elastic bar problem.

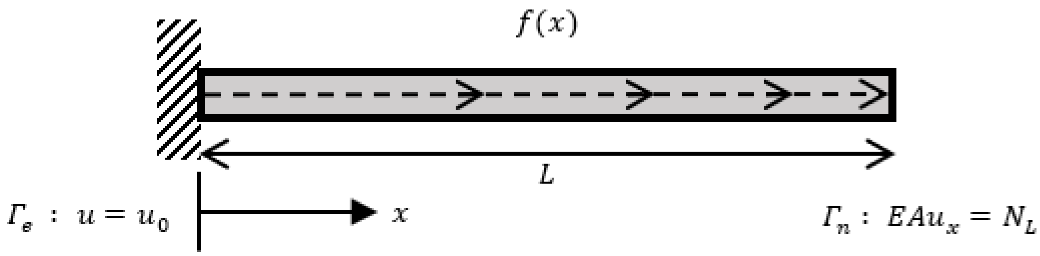


Figure 3.1 Elastic bar with axial load

3.2 Design velocity

As explained in Section 2.2, design velocity is an integral part of shape sensitivity analysis. We can define it as the movement of material points with respect to shape parameter due to the change in boundary locations. Let us further illustrate for the case of an elastic bar (Figure 3.2).

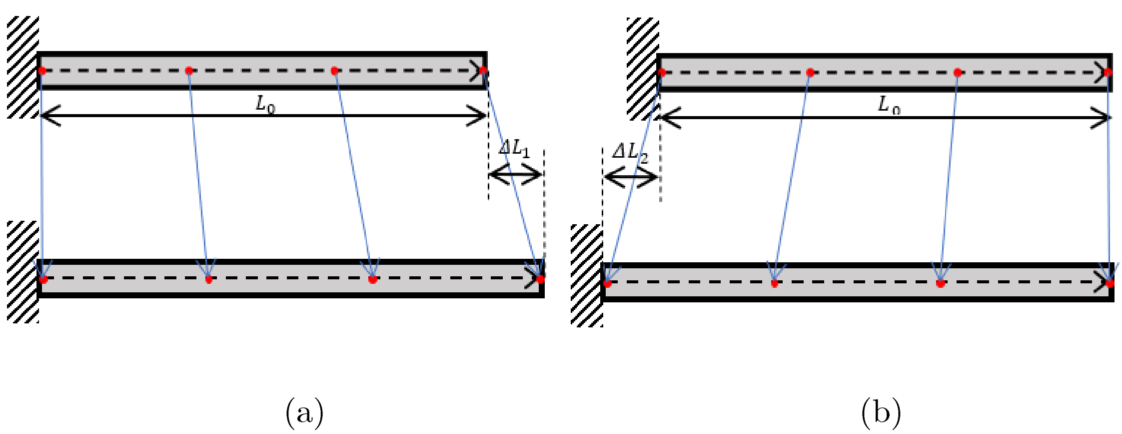


Figure 3.2 Different parametric representations of the elastic bar: (a) parameterization 1 (material points move to the right), and (b) parameterization 2 (material points move to the left). (Kulkarni M.D. et al., 2020)

The shape parameter is the length of the bar, L for the elastic bar shown in Figure 3.2. We can observe how the material points inside the domain shift due to a change in boundary location or the change in length, L. This change in position of each point with respect to the shape parameter, L represents the design velocity. Hence, with the convention of right side as the direction of positive x-axis, design velocities for parameterization 1 and parameterization 2 turn out to be $\frac{x}{L}$ and $\frac{x}{L} - 1$ respectively.

3.3 Material and Local Derivatives

Let us derive some identity relations for local and material sensitivities.

For a design velocity D, the local form – material form relationship becomes:

$$\dot{u} = u' + Du_x^{as} \tag{3.4}$$

Similarly, for $\dot{\overline{u_x}}$ the material sensitivity of u_x,

$$\dot{\overline{u_x}} = u_x' + Du_{xx}^{as} \tag{3.5}$$

Note here we have used the fact that the order of spatial derivative and local sensitivity are interchangeable and denoted by u_x':

$$u_x' = (u')_x = (u_x)' \tag{3.6}$$

Taking derivative w.r.t x in Eq[3.4],

$$\dot{u}_x = u_x' + D_x u_x^{as} + Du_{xx}^{as} \tag{3.7}$$

From Eq[3.5] and Eq[3.7] above, we have a relationship between the material sensitivity of u_x, that is $\dot{\overline{u_x}}$ and the spatial derivative of $\dot{u}$, that is $\dot{u}_x$,

$$\dot{\overline{u_x}} = \dot{u}_x - D_x u_x^{as} \tag{3.8}$$

Further, taking derivative w.r.t x in Eq[3.6],

$$\dot{u}_{xx} = u'_{xx} + D_{xx}u_x^{as} + 2D_x u_{xx}^{as} + Du_{xxx}^{as} \qquad 3.9$$

The local and material form relationship for $\dot{\overline{u_{xx}}}$,

$$\dot{\overline{u_{xx}}} = u'_{xx} + Du_{xxx}^{as} \qquad 3.10$$

From Eq[3.9] and Eq[3.10] above, we have,

$$\dot{\overline{u_{xx}}} = \dot{u}_{xx} - D_{xx}u_x^{as} - 2D_x u_{xx}^{as} \qquad 3.11$$

3.4 Material Sensitivity ODE

For simplicity, let us consider $EA = 1$. So, the 1-D Elastic Bar Analysis ODE becomes:

$$u_{xx} + f = 0 \qquad 3.12$$

The material Sensitivity-ODE becomes:

$$\dot{\overline{u_{xx}}} + \dot{f} = 0 \qquad 3.13$$

This can be further simplified using Eq[3.10]:

$$\dot{u}_{xx} - D_{xx}u_x^{as} - 2D_x u_{xx}^{as} + \dot{f} = 0 \qquad 3.14$$

The material form Sensitivity-BCs are:

$$\dot{u}(0) = \dot{u}_0 \qquad 3.15$$

$$\dot{\overline{u_x(L)}} = \dot{N}_L \qquad 3.16$$

Eq[3.16] above can be further simplified as:

$$\dot{u}_x(L) = \dot{N}_L + D_x(L)u_x^{as}(L) \tag{3.17}$$

3.5 Local Sensitivity ODE

The local form Sensitivity-ODE becomes:

$$u'_{xx} + f' = 0 \tag{3.18}$$

We can notice here that:

$$u'_x = (u')_x = (u_x)' \tag{3.19}$$

And local form Sensitivity-BCs:

$$u'(0) = \dot{u}(0) - u_x^{as}(0)D(0) \tag{3.20}$$

$$u'_x(L) = \dot{\overline{u_x(L)}} - u_{xx}^{as}(L)D(L) \tag{3.21}$$

Eq[3.20] and Eq[3.21] above can further be simplified as:

$$u'(0) = \dot{u}_0 - u_x^{as}(0)D(0) \tag{3.22}$$

$$u'_x(L) = \dot{N}_L - u_{xx}^{as}(L)D(L) \tag{3.23}$$

3.6 Variational Form of Analysis ODE

Rewriting the Analysis ODE for a 1-D Elastic Bar with $EA = 1$,

$$u_{xx} + f = 0 \quad 3.24$$

With boundary conditions,

$$u(0) = u_0 \text{ (Dirichlet/Essential BC)} \quad 3.25$$

$$u_x(L) = N_L \text{ (Neumann/Natural BC)} \quad 3.26$$

The variational form of this Analysis ODE Eq[3.24] would be,

$$\int_0^L (u_{xx} + f)\psi \, \mathrm{d}x = 0$$

$$\Rightarrow \int_0^L (-u_x\psi_x + f\psi)\, dx + u_x\psi|_0^L = 0 \quad 3.27$$

Here ψ is the weight function.

Assuming the weight function satisfies the homogeneous Dirichlet BC, $\psi(0) = 0$, and replacing the value of Neumann BC in the variational form, we get,

$$\int_0^L u_x\psi_x \, dx = \int_0^L f\psi \, dx + N_L\psi(L) \quad 3.28$$

3.7 Material Sensitivity of the Analysis Variational Form

Now, let us evaluate the material sensitivity of the variational equation.

$$\int_0^L \overline{\dot{u_x}}\psi_x \, dx + \int_0^L u_x^{as}\overline{\dot{\psi_x}} \, dx + \int_0^L u_x^{as}\psi_x \, \overline{\dot{dx}}$$

$$= \int_0^L \dot{f}\psi \, dx + \int_0^L f\dot{\psi} \, dx + \int_0^L f\psi\overline{\dot{dx}} + \dot{N}_L\psi(L) + N_L\dot{\psi}(L) \qquad 3.29$$

It can be shown that,

$$\overline{\dot{dx}} = D_x dx \qquad 3.30$$

Similar to previously derived equation Eq[3.8], we can show,

$$\overline{\dot{\psi_x}} = \dot{\psi}_x - D_x\psi_x \qquad 3.31$$

Rearranging and simplifying Eq[3.29] using Eq[3.30], Eq[3.31] and Eq[8],

$$\int_0^L \dot{u}_x\psi_x \, dx + \left[\int_O^L u_x^{as}\dot{\psi}_x \, dx - \int_0^L f\dot{\psi} \, dx - N_L\dot{\psi}(L)\right]$$

$$= \int_0^L u_x^{as}D_x\psi_x \, dx + \int_0^L u_x^{as}D_x\psi_x \, dx - \int_0^L u_x^{as}D_x\psi_x \, dx \qquad 3.32$$

$$+ \int_0^L \dot{f}\psi \, dx + \int_0^L f\psi D_x \, dx + \dot{N}_L\psi(L)$$

It can be noted that the part of the equation inside square brackets turns out to be zero as it satisfies the variational form of the actual ODE with weight function $\dot{\psi}$. The simplified form of the above Eq[3.32] becomes:

$$\int_0^L \dot{u}_x\psi_x \, dx = \int_0^L (u_x^{as}\, D_x\psi_x + \dot{f}\psi + f\psi D_x) \, dx + \dot{N}_L\psi(L) \qquad 3.33$$

3.8 Variational Form of Material Sensitivity Equation

Rewriting material form sensitivity equation:

$$\dot{u}_{xx} - D_{xx}u_x^{as} - 2D_x u_{xx}^{as} + \dot{f} = 0 \tag{3.34}$$

With boundary conditions:

$$\dot{u}(0) = \dot{u}_0 \tag{3.35}$$

$$\dot{u}_x(L) = \dot{N}_L + D_x(L)u_x^{as}(L) \tag{3.36}$$

The variational form of the material sensitivity ODE Eq[3.34] would be,

$$\begin{aligned} &\int_0^L \left(\dot{u}_{xx} - D_{xx}u_x^{as} - 2D_x u_{xx}^{as} + \dot{f}\right)\psi\, dx = 0 \\ \Rightarrow &\int_0^L \left(-\dot{u}_x\psi_x - D_{xx}u_x^{as}\psi + 2u_x^{as}(D_x\psi)_x + \dot{f}\psi\right) dx \\ &+\dot{u}_x\psi|_0^L - 2u_x^{as}D_x\psi|_0^L = 0 \end{aligned} \tag{3.37}$$

We need to use the integration by parts for the u_{xx}^{as} term because it is typically not available in an approximate solution.

Applying the boundary conditions,

Assuming $\psi(0) = 0$ for homogeneous Dirichlet BC and replacing the value of Neumann BC in the variational form, we get,

$$\int_0^L \dot{u}_x \psi_x \, dx = \int_0^L (\dot{f}\,\psi + D_{xx} u_x^{as} \psi + 2u_x^{as} \psi_x D_x) \, dx \qquad 3.38$$

$$+\dot{N}_L \psi(L) - D_x(L) N_L \psi(L)$$

3.9 Equivalence of the Variational Forms for Material Sensitivity

Subtracting the equation Eq[3.38] from Eq[3.33], we get:

$$\int_0^L (f\psi D_x - D_{xx} u_x^{as} \psi - u_x^{as} \psi_x D_x) \, dx + D_x(L) N_L \psi(L) = 0 \qquad 3.39$$

Eq[3.39] above can be rearranged as:

$$\int_0^L (-u_x^{as} D_{xx} \psi - u_x^{as} D_x \psi_x + f\psi D_x) \, dx + u_x^{as} D_x \psi |_0^L = 0$$

$$\Rightarrow \int_0^L (u_{xx}^{as} + f) D_x \psi \, dx = 0 \qquad 3.40$$

Eq[3.40] above is satisfied for all cases as it represents the variational form of the governing equations Eq[3.27] except with a weight function of $D_x \psi$ instead of ψ. Hence, the equation for material sensitivity of variational form Eq[3.33] can be shown to be analytically similar to Eq[3.38].

3.10 Local Sensitivity of the Analysis Variational Form

Let us recall the variational form of the analysis ODE Eq[3.28].

$$\int_0^L u_x^{as}\psi_x\,dx - \int_0^L f\psi\,dx - N_L\psi(L) = 0 \qquad 3.41$$

Now, let us evaluate the local sensitivity of the variational equation.

$$\int_0^L u_x'\psi_x\,dx + \int_0^L u_x^{as}\psi_x'\,dx + \int_0^L u_x^{as}\psi_x\,(dx)'$$
$$= \int_0^L f'\psi\,dx + \int_0^L f\psi'\,dx + \int_0^L f\psi(dx)' + N_L'\psi(L) + N_L\psi'(L) \qquad 3.42$$

As $(dx)' = 0$, two of the terms with $(dx)'$, $\int_0^L u_x\psi_x\,(dx)'$ and $\int_0^L f\psi(dx)'$, can be eliminated. Eliminating these terms and rearranging to further simplify the equation,

$$\int_0^L u_x'\psi_x\,dx + \left[\int_0^L u_x^{as}\psi_x'\,dx - \int_0^L f\psi'\,dx - N_L\psi'(L)\right]$$
$$= \int_0^L f'\psi\,dx + N_L'\psi(L) \qquad 3.43$$

It can be noted that the part of the equation inside square brackets turns out to be zero as it satisfies the variational form of the analysis ODE with weight function ψ'. The simplified form of the Eq[3.43] above becomes:

$$\int_0^L u'_x \psi_x \, dx = \int_0^L f' \psi \, dx + N'_L \psi(L) \qquad 3.44$$

N'_L or $u'_x(L)$ can be further written as:

$$u'_x(L) = \dot{u}(L) - u^{as}_{xx}(L)D(L) \Rightarrow N'_L = \dot{N}_L - u^{as}_{xx}(L)D(L) \qquad 3.45$$

So, another form of the final equation Eq[3.44] would be,

$$\int_0^L u'_x \psi_x \, dx = \int_0^L f' \psi \, dx + \dot{N}_L \psi(L) - u^{as}_{xx}(L)D(L)\psi(L) \qquad 3.46$$

3.11 Variational Form of Local Sensitivity Equation

Rewriting local form sensitivity equation:

$$u'_{xx} + f' = 0 \qquad 3.47$$

With boundary conditions:

$$u'(0) = \dot{u}_0 - u^{as}_x(0)D(0) \qquad 3.48$$

$$u'_x(L) = \dot{N}_L - u^{as}_{xx}(L)D(L) \qquad 3.49$$

The variational form of this local sensitivity ODE would be,

$$\int_0^L (u'_{xx} + f')\psi \, dx = 0 \qquad 3.50$$

$$\Rightarrow \int_0^L u_x' \psi_x \, dx = \int_0^L f' \psi \, dx + [\psi u_x']_0^L$$

Applying the boundary conditions,

Assuming $\psi(0) = 0$ for homogeneous Dirichlet BC and replacing the value of Neumann BC in the variational form.

$$\int_0^L u_x' \psi_x \, dx = \int_0^L f' \psi \, dx + \psi(L) \dot{N}_L - \psi(L) u_{xx}^{as}(L) D(L) \qquad 3.51$$

So, Eq[3.46] and Eq[3.51], are exactly identical.

Finally, the order of accuracy of sensitivity solutions from each of these variational formulations depends on the least accurate term(s) in the formulation. Chapter 4 illustrates the same in detail.

Chapter 4

Results

4.1 Key Sensitivity Equations

Let us revisit the seven key sensitivity equations derived in Chapter 3.

Key ODEs with BCs:

Eq[1] Bar Analysis ODE:

$$u_{xx} = -f;$$
$$u(0) = u_0; \quad u_x(L) = N_L \tag{4.1}$$

Eq[2] Local Sensitivity ODE:

$$u'_{xx} = -f';$$
$$u'(0) = \dot{u}_0 - u_x^{as}(0)D(0); \quad u'_x(L) = \dot{N}_L - u_{xx}^{as}(L)D(L) \tag{4.2}$$

Eq[3] Material Sensitivity ODE:

$$\dot{u}_{xx} = -\dot{f} + 2u_{xx}^{as}D_x + u_x^{as}D_{xx};$$
$$\dot{u}(0) = \dot{u}_0; \quad \dot{u}_x(L) = \dot{N}_L + D_x(L)u_x^{as}(L) \tag{4.3}$$

Key Variational formulations:

Eq[4] Analysis Variational form:

$$\int_0^L u_x \psi_x \, dx = \int_0^L f\psi \, dx + N_L \psi(L) \qquad 4.4$$

Eq[5] Local Sensitivity Variational form:

$$\int_0^L u_x' \psi_x \, dx = \int_0^L f' \psi \, dx + \psi(L)\dot{N}_L - \psi(L) u_{xx}^{as}(L) D(L) \qquad 4.5$$

Eq[6] Material Sensitivity Variational form (Material Sensitivity of Analysis variational form):

$$\int_0^L \dot{u}_x \psi_x \, dx = \int_0^L (u_x^{as} \, \psi_x D_x + \dot{f}\psi + f\psi D_x) \, dx + \dot{N}_L \psi(L) \qquad 4.6$$

Eq[7] Material Sensitivity Variational form (Variation of Material Sensitivity ODE):

$$\int_0^L \dot{u}_x \psi_x \, dx = \int_0^L (\dot{f}\,\psi + D_{xx} u_x^{as} \psi + 2u_x^{as} \psi_x D_x) \, dx + \dot{N}_L \psi(L) + D_x(L) N_L \psi(L) \qquad 4.7$$

4.2 Design Velocities and Trial Functions

Let us consider a simple case where the forcing term is a constant. To evaluate and compare local and material sensitivities, we need to define two parameters, the design velocity and trial function (for approximate solutions via variational formulations). In this study, we have included results for three types of functions for design velocity: linear, quadratic and piecewise linear.

$$D = \frac{x}{L} \tag{4.8}$$

$$D = \left(\frac{x}{L}\right)^2 \tag{4.9}$$

$$D = \begin{cases} 0 & \left(0, \frac{L}{2}\right) \\ \frac{2x}{L} - 1 & \left(\frac{L}{2}, L\right) \end{cases} \tag{4.10}$$

Furthermore, we have examined a few sub-cases for linear design velocity.

$$D = 1 - \frac{x}{L} \tag{4.11}$$

$$D = \frac{x}{L} - 1 \tag{4.12}$$

$$D = -\frac{x}{L} \tag{4.13}$$

For solving the variational form sensitivity equations, we have chosen the trial function $\psi = \frac{x}{L}, \left(\frac{x}{L}\right)^2$ and $\psi =$ finite element hat functions.

4.3 Exact Sensitivity Results from Sensitivity ODE

Let us derive local and material sensitivities for a forcing term:

$$f = b \tag{4.14}$$

where b is a constant.

So, the local and material derivatives of f would be:

$$f' = 0 \tag{4.15}$$

$$\dot{f} = 0 \tag{4.16}$$

Let us choose design velocity:

$$D = \frac{x}{L} \tag{4.17}$$

Consider Bar ODE with BCs:

$$u_{xx} + f = 0 \tag{4.18}$$

$$u(0) = 0 \text{ (Dirichlet/Essential BC)} \tag{4.19}$$

$$u_x(L) = 0 \text{ (Neumann/Natural BC)} \tag{4.20}$$

Solving this ODE, we get:

$$u = -\frac{bx^2}{2} + cx + d \tag{4.21}$$

where c and d are constants.

Applying BCs, we get the values for c and d as:

$$u(0) = 0 \Rightarrow d = 0 \tag{4.22}$$

$$u_x(L) = 0 \Rightarrow -bL + c = 0 \Rightarrow \; c = bL \tag{4.23}$$

Finally, we get the analysis solution:

$$u^{\mathrm{as}} = -\frac{bx^2}{2} + bLx = \frac{1}{2}b(2L - x)x \qquad 4.24$$

Material sensitivity equations with BCs derived in the previous section simplify:

$$\dot{u}_{xx} - D_{xx}u_x^{as} - 2D_x u_{xx}^{as} + \dot{f} = 0 \qquad 4.25$$

$$\dot{u}(0) = 0 \qquad 4.26$$

$$\dot{u}_x(L) = 0 \qquad 4.27$$

Solving this material sensitivity ODE, we get:

$$\dot{u}_{xx} = -\dot{f} + 2u_{xx}^{as}D_x + u_x^{as}D_{xx} = \frac{-2b}{L} \qquad 4.28$$

$$\Rightarrow \dot{u} = \frac{1}{L}(-bx^2 + rx + s) \qquad 4.29$$

Applying BCs, we calculate the values of r and s to get:

$$\dot{u} = \frac{-b}{L}x^2 + 2bx \qquad 4.30$$

Local sensitivity equation with BCs derived in the previous section simplify to:

$$u'_{xx} + f' = 0 \qquad 4.31$$

$$u'(0) = 0 \qquad 4.32$$

$$u'_x(L) = -u_{xx}^{as}(L)D(L) = b \qquad 4.33$$

Solving this local sensitivity ODE, we get:

$$u'_{xx} = -f' = 0 \qquad 4.34$$

$$\Rightarrow u' = rx + s \qquad 4.35$$

Applying BCs, we calculate the values of r and s to get:

$$u' = bx \qquad 4.36$$

We can check the calculated material and local sensitivities by

$$\dot{u} = u' + Du_x^{as}$$
$$\frac{-b}{L}x^2 + 2bx = bx + (-bx + bL)\frac{x}{L} \qquad 4.37$$

Similarly, we have evaluated the local and material sensitivities for other design velocities. The results are tabulated below (Table 4.1).

<table>
<tr><th>D</th><th>u^{as} from
ODE Eq1</th><th>u' from
ODE Eq2</th><th>$\dot{u}$ from
ODE Eq3</th></tr>
<tr><td>$\frac{x}{L}$</td><td rowspan="3">$-\frac{bx^2}{2} + bLx$</td><td>bx</td><td>$-\frac{b}{L}x^2 + 2bx$</td></tr>
<tr><td>$\left(\frac{x}{L}\right)^2$</td><td>$bx$</td><td>$\frac{-b}{L^2}x^3 + \frac{b}{L}x^2 + bx$</td></tr>
<tr><td>$\begin{cases} 0 & \left(0, \frac{L}{2}\right) \\ \frac{2x}{L} - 1 & \left(\frac{L}{2}, L\right) \end{cases}$</td><td>$bx$</td><td>$\begin{cases} bx \\ -\frac{2b}{L}x^2 + 4bx - bL \end{cases}$</td></tr>
</table>

Table 4.1 Exact analysis solution, local and material sensitivity using ODEs Eq[1], Eq[2] and Eq[3] for constant forcing

In this case, local sensitivity solution u' solely depends on the boundary values as the integral term in the equation is zero. This applies for the variational forms also. As the values of the above three design velocities at the boundaries is same, u' gives the same result. The material sensitivity solutions depend on the design velocity. The $\dot{u}$ obtained for the piecewise function is found to be continuous. The results obtained from these ODEs Eq[1], Eq[2] and Eq[3] are the exact solutions.

Let us further solve for other linear design velocities. We get,

$$u' = -bL \text{ for } D = 1 - \frac{x}{L} \tag{4.38}$$

and

$$u' = bL \text{ for } D = \frac{x}{L} - 1 \tag{4.39}$$

as u' here depends on the value of design velocity at $x = L$.

For $D = -\frac{x}{L}$, we get

$$u' = -bx \tag{4.40}$$

Just the negative of its value for $D = \frac{x}{L}$.

On the other hand, solution of material sensitivity $\dot{u}$ varies with the coefficient (or sign) of $\frac{x}{L}$ in the design velocity. That is,

$$\dot{u} = \frac{b}{L}x^2 - 2bx \text{ for } D = 1 - \frac{x}{L}\ ; \tag{4.41}$$

$$\dot{u} = -\frac{b}{L}x^2 + 2bx \text{ for } D = \frac{x}{L} - 1\ ; \tag{4.42}$$

and

$$\dot{u} = \frac{b}{L}x^2 - 2bx \text{ for } D = -\frac{x}{L}\,. \qquad 4.43$$

4.4 Approximate Sensitivity Results using the Variational Forms

We will first derive the results for a sample case for material and local sensitivities respectively. We will then discuss about a correction term that has to be added in the Variational form of Material Sensitivity ODE in case of a piecewise design velocity. Finally, we will tabulate the results for all these cases.

4.4.1 Approximate Analysis Solution using Variational Form

Let us derive the results for a forcing term:

$$f = b \qquad 4.44$$

where b is a constant.

Let us choose design velocity and trial functions as:

$$D = \frac{x}{L} \qquad 4.45$$

$$\psi = \frac{x}{L}, \left(\frac{x}{L}\right)^2 \qquad 4.46$$

Let us assume:

$$u^{as} = a_1 \left(\frac{x}{L}\right) + a_2 \left(\frac{x}{L}\right)^2 \qquad 4.47$$

where a_1 and a_2 are constants.

Note that the assumed trial functions $\frac{x}{L}$ and $\left(\frac{x}{L}\right)^2$ satisfy the Dirichlet BCs. Substituting in the analysis variational form Eq[4] and using the same weight functions as the trial functions, we get a system of linear equations in a_1 and a_2. Solving for a_1 and a_2 we get,

$$a_1 = bL^2, \qquad a_2 = -\frac{bL^2}{2} \qquad 4.48$$

Finally, we get the analysis solution which ends up being exact since the exact solution can be represented using the trial function space.

$$u^{as} = bL^2 \left(\frac{x}{L}\right) - \frac{bL^2}{2}\left(\frac{x}{L}\right)^2 = -\frac{bx^2}{2} + bLx \qquad 4.49$$

4.4.2 Material Sensitivity Results

The material sensitivity of f would be:

$$\dot{f} = 0 \qquad 4.50$$

The material sensitivity equation in a variational form is given in Eq[6]:

$$\int_0^L \dot{u}_x \psi_x \, dx = \int_0^L (u_x^{as} \, \psi_x D_x + \dot{f}\psi + f\psi D_x) \, dx + \dot{N}_L \psi(L) \qquad 4.51$$

Let us assume:

$$\dot{u} = a_3\left(\frac{x}{L}\right) + a_4\left(\frac{x}{L}\right)^2 \tag{4.52}$$

where a_3 and a_4 are constants.

Note that the assumed solution satisfies the Dirichlet BC for material sensitivity $\dot{u}(0) = 0$. Substituting this assumed solution in Eq[6] and using the same weight functions as the trial functions, we get a system of linear equations in a_3 and a_4. Solving for a_3 and a_4 we get,

$$a_3 = 2bL, \qquad a_4 = -bL \tag{4.53}$$

Finally, we get the material sensitivity $\dot{u}$,

$$\dot{u} = 2bL\left(\frac{x}{L}\right) - bL\left(\frac{x}{L}\right)^2 = \frac{-b}{L}x^2 + 2bx \tag{4.54}$$

The sensitivity equation derived using variational form of material sensitivity Eq[7]:

$$\int_0^L \dot{u}_x \psi_x \, dx = \int_0^L (\dot{f}\,\psi + D_{xx} u_x^{as} \psi + 2u_x^{as} \psi_x D_x)\, dx \tag{4.55}$$

$$+\dot{N}_L \psi(L) - D_x(L) N_L \psi(L)$$

This equation too gives us the same $\dot{u}$,

$$\dot{u} = \frac{-b}{L}x^2 + 2bx \tag{4.56}$$

This solution matches the analytical sensitivity calculated using ODE Eq[3] because the analytical sensitivity can be represented in terms of the assumed trial function space.

4.4.3 Local Sensitivity Results

The local sensitivity of f would be:

$$f' = 0 \tag{4.57}$$

Let us choose design velocity and weight functions as:

$$D = \frac{x}{L} \tag{4.58}$$

$$\psi = \frac{x}{L}, \left(\frac{x}{L}\right)^2 \tag{4.59}$$

The sensitivity equation derived using local sensitivity of variational form (or variational form of local sensitivity) Eq[5]:

$$\int_0^L u'_x \psi_x \, dx = \int_0^L f' \psi \, dx + \psi(L)\dot{N}_L - \psi(L) u_{xx}^{as}(L) D(L) \tag{4.60}$$

Solving this equation for u'_x, we get,

$$\int_0^L u'_x \psi_x \, dx = b \tag{4.61}$$

Let us assume:

$$u' = a_5 \left(\frac{x}{L}\right) + a_6 \left(\frac{x}{L}\right)^2 \tag{4.62}$$

where a_5 and a_6 are constants.

Substituting this assumed solution in Eq[5] and using the same weight functions as the trial functions, we get a system of linear equations in a_5 and a_6. Solving for a_5 and a_6 we get,

$$a_5 = bL, \qquad a_6 = 0 \tag{4.63}$$

Finally, we get the material sensitivity $\dot{u}$,

$$u' = bL\left(\frac{x}{L}\right) + 0\left(\frac{x}{L}\right)^2 = bx \tag{4.64}$$

Again, this solution matches the analytical sensitivity calculated using ODE Eq[2] because the analytical sensitivity can be represented in terms of the assumed trial function space.

4.4.4 Singular Term for Piecewise Continuous Design Velocity

Let us review the Equation we derived for Variational form of Material Sensitivity ODE.

$$\int_0^L \dot{u}_x \psi_x \, dx = \int_0^L (\dot{f}\,\psi + D_{xx} u_x^{as} \psi + 2u_x^{as} \psi_x D_x)\, dx + \dot{N}_L \psi(L) + D_x(L) N_L \psi(L) \tag{4.65}$$

Here, we observe that the second derivative term in RHS, $\int_0^L D_{xx} u_x \psi \, dx$ would be undefined at $x = \frac{L}{2}$ for the piecewise design velocity. So, we will find inaccurate results for this equation unless we integrate the singularity at $x = \frac{L}{2}$. We can assume u_x^{as} and ψ to be continuous at the point $x = \frac{L}{2}$. So, the term $\int_0^L D_{xx} u_x \psi \, dx$ at $x = \frac{L}{2}$, can be written as:

$$\int_{\frac{L}{2}-}^{\frac{L}{2}+} D_{xx} u_x^{as} \psi \, dx = u_{x_{\frac{L}{2}}} \psi_{\frac{L}{2}} \left\{ D_{x_{\frac{L}{2}+}} - D_{x_{\frac{L}{2}-}} \right\} \qquad 4.66$$

This term simplifies to be a constant b (for the case of constant forcing) which must be added to achieve accurate solutions for the piecewise design velocity case.

In case the analysis solution is evaluated using finite elements, we may have discontinuities in u_x^{as} at $x = \frac{L}{2}$. For these cases, we have considered $u_{x_{\frac{L}{2}}}$ as the average of its value at $x = \frac{L}{2} -$ and $x = \frac{L}{2} +$, that is,

$$u_{x_{\frac{L}{2}}}^{as} = \frac{u_{x_{\frac{L}{2}+}}^{as} + u_{x_{\frac{L}{2}-}}^{as}}{2} \qquad 4.67$$

4.4.5 Tabulated Results

Let us solve the three variational form equations, one for local sensitivity and two for material sensitivity, using different trial functions.

D	ψ	u^{as} from Var Eq4	u' from Var Eq5	$\dot{u}$ from Var Eq6	$\dot{u}$ from Var Eq7
$\frac{x}{L}$	$\frac{x}{L}, \left(\frac{x}{L}\right)^2$	$-\frac{bx^2}{2} + bLx$	bx	$-\frac{b}{L}x^2 + 2bx$	$same\ as\ Eq6$
$\left(\frac{x}{L}\right)^2$			bx	$-\frac{b}{2L}x^2 + \frac{3b}{2}x$	$same\ as\ Eq6$
$\begin{cases} 0 & \left(0, \frac{L}{2}\right) \\ \frac{2x}{L} - 1 & \left(\frac{L}{2}, L\right) \end{cases}$			bx	$-\frac{b}{4L}x^2 + \frac{5b}{4}x$	$\frac{b}{2L}x^2$ *neglecting the singularity $same\ as\ Eq6$ *integrating the singularity

Table 4.2 Local and material sensitivity solutions using Eq[5], Eq[6] and Eq[7] for trial functions $\psi = \frac{x}{L}, \left(\frac{x}{L}\right)^2$ and constant forcing

We have evaluated the local and material sensitivities for different design velocities. The results for $\psi = \frac{x}{L}, \left(\frac{x}{L}\right)^2$ are tabulated in Table 4.2. In Table 4.2, we have used the approximate analysis solution in the calculation of local and material sensitivities using the three variational forms Eq[5], Eq[6] and Eq[7]. For this case the approximate analysis solution matches the exact solution. The results for local sensitivity match the exact solution from Table 4.1. This is because the local sensitivity solution in this case solely depends on the boundary values of the weight function, design velocity and the second spatial derivative of the analysis solution. Comparing the material sensitivity results from both the variational forms, we find that it matches with the exact solution in the case where the design velocity is a linear function. For a design velocity of $\left(\frac{x}{L}\right)^2$, we could only reach second degree accuracy because of our choice of quadratic trial functions. Finally, for the piecewise design velocity case, we get a quadratic material sensitivity throughout the domain. The solution is not piecewise here because our trial functions are continuous. Also, we notice how we had to use a correction term for recalculating the correct result for material sensitivity when solving using Eq[7].

When the trial functions are finite element hat functions for two elements, we have the following results (Table 4.3).

$\boldsymbol{D}$	$\boldsymbol{\psi}$	$\boldsymbol{u^{as}}$ **from ODE Eq1**	$\boldsymbol{u'}$ **from Var Eq5**	$\boldsymbol{\dot{u}}$ **from Var Eq6**	$\boldsymbol{\dot{u}}$ **from Var Eq7**
$\frac{x}{L}$	2 elements Hat functions $\psi_1 = 1 - \frac{2x}{L}$ $\psi_2 = \frac{2x}{L}$ $\psi_3 = 2 - \frac{2x}{L}$ $\psi_4 = \frac{2x}{L} - 1$	$-\frac{bx^2}{2} + bLx$	bx	$\begin{cases} \frac{3bx}{2} \\ \frac{bx}{2} + \frac{bL}{2} \end{cases}$	$same\ as\ Eq6$
$\left(\frac{x}{L}\right)^2$			bx	$\begin{cases} \frac{5bx}{4} \\ \frac{3bx}{4} + \frac{bL}{4} \end{cases}$	$same\ as\ Eq6$
$\begin{cases} 0 \ \left(0, \frac{L}{2}\right) \\ \frac{2x}{L} - 1 \ \left(\frac{L}{2}, L\right) \end{cases}$			bx	bx	$\begin{cases} 0 \\ bx - \frac{bL}{2} \end{cases}$ *neglecting the singularity $same\ as\ Eq6$ *integrating the singularity

Table 4.3 Local and material sensitivity solutions using Eq[5], Eq[6] and Eq[7] for finite element trial functions for two elements for constant forcing

In Table 4.3, we have used the exact analysis solution in the calculation of local and material sensitivities using the three variational forms Eq[5], Eq[6] and Eq[7]. The results for local sensitivity match the exact solution from Table 4.1. Similar to Table 4.2, the local sensitivity solution in this case solely depends on the boundary values of the trial function, design velocity and the second spatial derivative of the analysis solution. Comparing the material sensitivity results from both the variational forms, we find that they are linear approximations of the exact solution. Finally, for the piecewise design velocity case, we get a linear material sensitivity with the same slope throughout the domain (same solution for both elements). Again, we can notice how we had to use a correction term for recalculating the corrected material sensitivity when solving using Eq[7].

When the weight functions are finite element hat functions for two elements, we have the results in Table 4.4. In Table 4.4, we have used a finite element approximation of the analysis solution in the calculation of local and material sensitivities using the three variational forms Eq[5], Eq[6] and Eq[7]. The domain is discretized into two elements. The results for local sensitivity in this case are trivial and do not match the exact solution from Table 4.1. This is because the second spatial derivative of the analysis solution, u_{xx}^{as} is zero. And similar to previous tables the local sensitivity here solely depends on the product of boundary values of the weight function, design velocity and the second spatial derivative of the analysis solution. Further, if we use the exact value of the second spatial derivate of analysis solution, $u_{xx}^{as} = -b$, we get accurate local sensitivities for all three design velocities. Comparing the material sensitivity results,

<table>
<tr><th>$\boldsymbol{D}$</th><th>$\boldsymbol{\psi}$</th><th>$\boldsymbol{u^{as}}$ from
Var Eq4</th><th>$\boldsymbol{u'}$ from
Var Eq5</th><th>$\dot{\boldsymbol{u}}$ from
Var Eq6</th><th>$\dot{\boldsymbol{u}}$ from
Var Eq7</th></tr>
<tr><td>$\frac{x}{L}$</td><td rowspan="3">2 elements
Hat functions
$\psi_1 = 1 - \frac{2x}{L}$
$\psi_2 = \frac{2x}{L}$
$\psi_3 = 2 - \frac{2x}{L}$
$\psi_4 = \frac{2x}{L} - 1$</td><td rowspan="3">$\begin{cases} \frac{3bLx}{4} \\ \frac{bL}{4}(L+x) \end{cases}$
*using hat functions</td><td>0</td><td>$\begin{cases} \frac{3bx}{2} \\ \frac{bx}{2} + \frac{bL}{2} \end{cases}$</td><td>$same\ as\ Eq6$</td></tr>
<tr><td>$\left(\frac{x}{L}\right)^2$</td><td>0</td><td>$\begin{cases} \frac{31bx}{24} \\ \frac{19bx}{24} + \frac{bL}{4} \end{cases}$</td><td>$\begin{cases} \frac{11bx}{8} \\ \frac{7bx}{8} + \frac{bL}{4} \end{cases}$</td></tr>
<tr><td>$\begin{cases} 0 & \left(0, \frac{L}{2}\right) \\ \frac{2x}{L} - 1 & \left(\frac{L}{2}, L\right) \end{cases}$</td><td>0</td><td>$bx$</td><td>$\begin{cases} 0 \\ bx - \frac{bL}{2} \end{cases}$
*neglecting the singularity
$same\ as\ Eq6$
*integrating the singularity</td></tr>
</table>

Table 4.4 Analysis solution, local and material sensitivity using Eq[4], Eq[5], Eq[6] and Eq[7] for finite element trial functions for two elements for constant forcing

we find that they are linear approximations of the exact solution. The material sensitivities found using two variational forms, Eq[6] and Eq[7], are different from each other in the case of quadratic design velocity. Finally, again we can notice how we had to use a correction term for recalculating the corrected material sensitivity when solving using Eq[7] for piecewise design velocity.

Below are the results when the weight functions are finite element hat functions for 4 elements (hat functions used to find the analysis solution) (Table 4.5).

D	ψ	u^{as} from Var Eq4	u' Var Eq5	$\dot{u}$ from Var Eq6	$\dot{u}$ from Var Eq7
$\frac{x}{L}$	4 elements Hat functions $\psi_1 = 1 - \frac{4}{L}x$ $\psi_2 = \frac{4}{L}x$ $\psi_3 = 2 - \frac{4}{L}x$ $\psi_4 = \frac{4}{L}x - 1$ $\psi_5 = 3 - \frac{4}{L}x$ $\psi_6 = \frac{4}{L}x - 2$ $\psi_7 = 4 - \frac{4}{L}x$ $\psi_8 = \frac{4}{L}x - 3$	$\begin{cases} \frac{7bLx}{8} \\ \frac{bL}{16}(L+10x) \\ \frac{3bL}{16}(L+2x) \\ \frac{bL}{8}(3L+x) \end{cases}$ *using hat functions	0	$\begin{cases} \frac{7bx}{4} \\ \frac{b}{8}(L+10x) \\ \frac{3b}{8}(L+2x) \\ \frac{b}{4}(3L+x) \end{cases}$	$same\ as\ Eq6$
$\left(\frac{x}{L}\right)^2$			0	$\begin{cases} \frac{115bx}{96} \\ \frac{b}{96}(127x-3L) \\ \frac{b}{96}(103x+9L) \\ \frac{b}{96}(43x+54L) \end{cases}$	$\begin{cases} \frac{39bx}{32} \\ \frac{b}{32}(43x-L) \\ \frac{b}{32}(35x+3L) \\ \frac{b}{32}(15x+18L) \end{cases}$
$\begin{cases} 0 & \left(0,\frac{L}{2}\right) \\ \frac{2x}{L}-1 & \left(\frac{L}{2},L\right) \end{cases}$			0	$\begin{cases} bx \\ bx \\ \frac{b}{4}(6x-L) \\ \frac{b}{2}(x+L) \end{cases}$	$\begin{cases} 0 \\ 0 \\ \frac{3b}{4}(2x-L) \\ \frac{bx}{2} \end{cases}$ *neglecting the singularity $same\ as\ Eq6$ *integrating the singularity

Table 4.5 Analysis solution, local and material sensitivity using Eq[4], Eq[5], Eq[6] and Eq[7] for finite element trial functions for four element and constant forcing

In Table 4.5, we have again used the finite element approximation of the analysis solution in the calculation of local and material sensitivities using the three variational forms Eq[5], Eq[6] and Eq[7]. But, here the domain is discretized into four elements. As expected, all our observations are similar to that of Table 4.4.

4.5 Comparison Plots

Let us compare local sensitivities presented in the tables in section 4.3 and 4.4. The local sensitivity is found to be bx for Tables 4.1, 4.2 and 4.3. Also, we discussed why we had trivial solution for local sensitivity in Tables 4.4, 4.5 and how we can rectify it by replacing the exact value of u_{xx}^{as}(second spatial derivate of analysis solution) in the equation. After replacing this value for $u_{xx}^{as}(L)$, we get the same solution for local sensitivity, bx.

Unlike local sensitivity, material sensitivity varies depending on the trial functions for the same design velocity. Let us plot these material sensitivity solutions to compare their accuracy with respect to the exact solution.

4.5.1 Linear Design Velocity Plots

Table 4.6 summarizes the results for linear design velocity from Tables 4.1 to 4.5. Figure 4.1 compares the material sensitivity solutions for the linear design velocity, $D = \frac{x}{L}$. In all these plots, we have assumed the forcing term, $b = 1$ and the length of the bar, $L = 1$. We notice that for quadratic trial functions $\psi = \frac{x}{L}, \left(\frac{x}{L}\right)^2$, we are able to match exact value of material sensitivity with both variational forms. When the trial functions are finite element hat functions (while using the exact analysis solution), we get a linear approximation of the material sensitivity. We get the same approximate linear material sensitivity from

Variational Eq[6] and Eq[7] even when we use the finite element (two elements) approximation of the analysis solution instead of the exact analysis solution. When solving for four finite elements, we again get matching material sensitivity solutions from the two variational equations, Eq[6] and Eq[7].

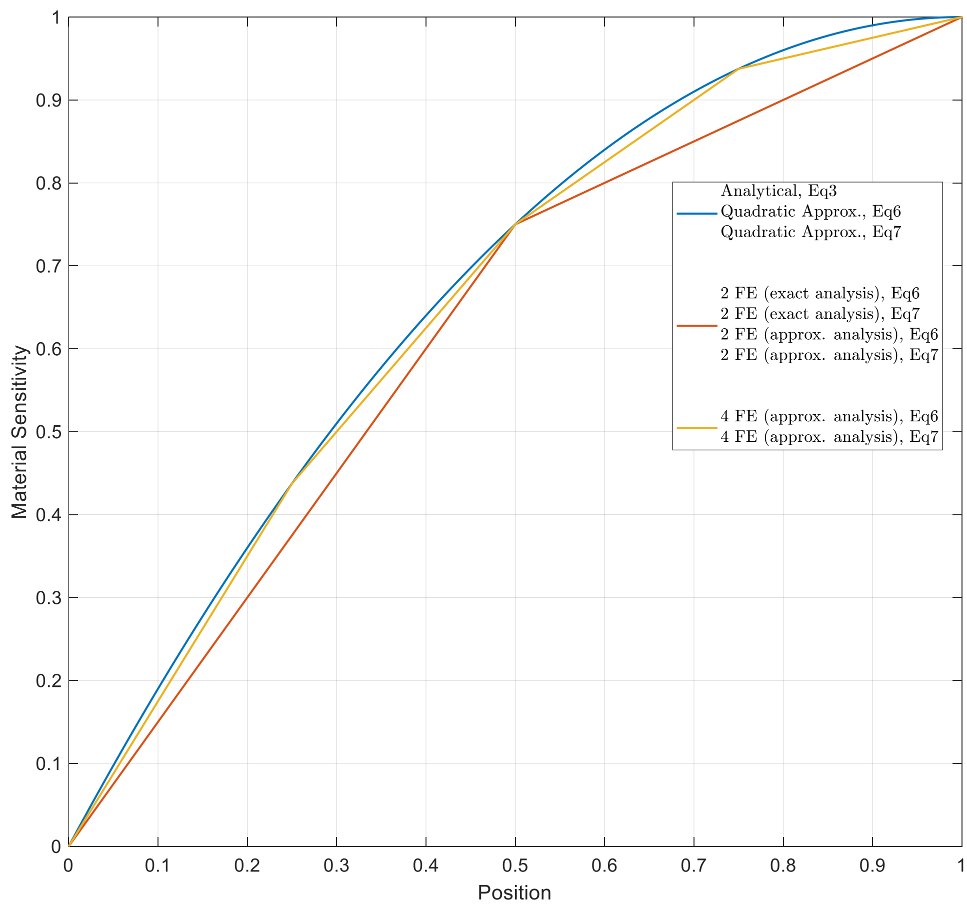


Figure 4.1 Material sensitivity solutions for linear design velocity

(different trial functions)

For $\boldsymbol{D} = \frac{x}{L}$	$\dot{\boldsymbol{u}}$ $(\boldsymbol{b} = \mathbf{1}, \boldsymbol{L} = \mathbf{1})$
Analytical, Eq3	$-\frac{b}{L}x^2 + 2bx$
Quadratic Approx., Eq6	$same\ as\ analytical$
Quadratic Approx., Eq7	$same\ as\ Eq6$
2 FE (exact analysis solution), Eq6	$\begin{cases} \frac{3bx}{2} \\ \frac{bx}{2} + \frac{bL}{2} \end{cases}$
2 FE (exact analysis solution), Eq7	$same\ as\ Eq6$
2 FE (approx. analysis solution), Eq6	$same\ as\ using\ exact\ analysis$
2 FE (approx. analysis solution), Eq7	$same\ as\ Eq6$
4 FE (approx. analysis solution), Eq6	$\begin{cases} \frac{7bx}{4} \\ \frac{b}{8}(L + 10x) \\ \frac{3b}{8}(L + 2x) \\ \frac{b}{4}(3L + x) \end{cases}$
4 FE (approx. analysis solution), Eq7	$same\ as\ Eq6$

Table 4.6 Material sensitivity solutions for linear design velocity

Upon graphical comparison, we notice that the nodal values of the FEM solution from Eq[6] and Eq[7] match the exact solution at nodes. We can understand the convergence of the material sensitivity solution by solving these two equations for more elements and comparing the errors. Further analysis on this can be found in Section 4.6.

4.5.2 Quadratic Design Velocity Plots

Table 4.7 summarizes the results for quadratic design velocity from Tables 4.1 to 4.5. Figure 4.2 compares the material sensitivity solutions for the quadratic design velocity, $D = \left(\frac{x}{L}\right)^2$.

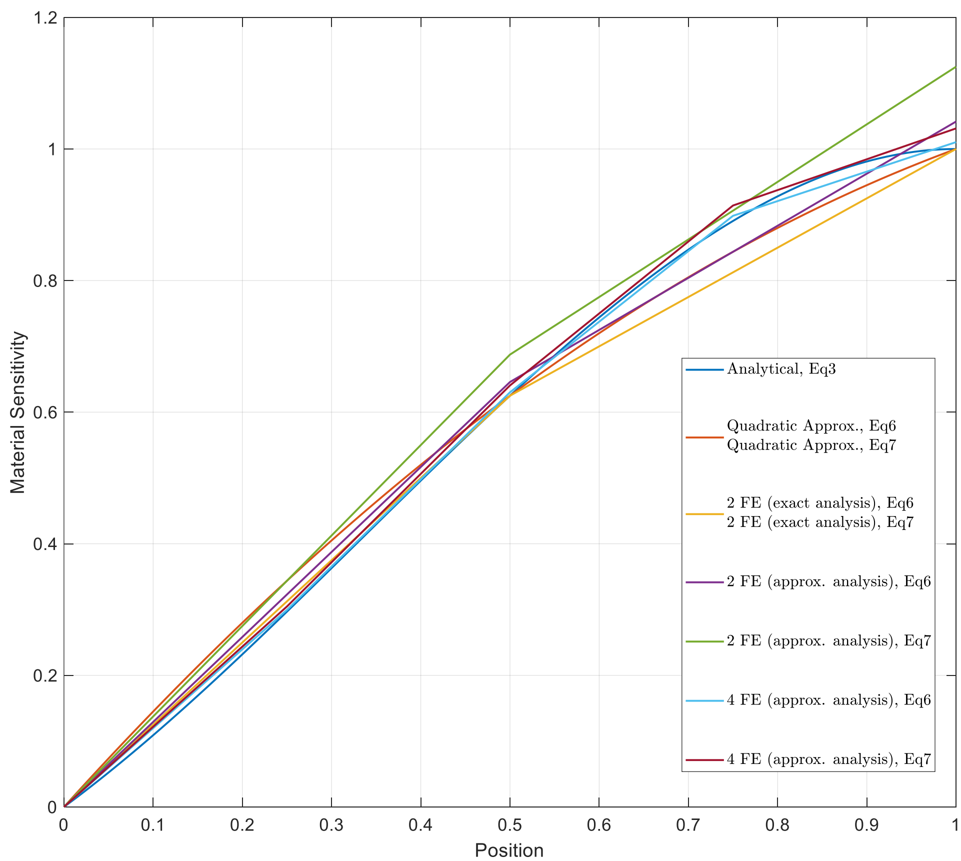


Figure 4.2 Material sensitivity solutions for quadratic design velocity (different trial functions)

For $D = \left(\frac{x}{L}\right)^2$	$\dot{u}$ $(b = 1, L = 1)$
Analytical, Eq3	$\frac{-b}{L^2}x^3 + \frac{b}{L}x^2 + bx$
Quadratic Approx., Eq6	$-\frac{b}{2L}x^2 + \frac{3b}{2}x$
Quadratic Approx., Eq7	$same\ as\ Eq6$
2 FE (exact analysis solution), Eq6	$\begin{cases} \frac{5bx}{4} \\ \frac{3bx}{4} + \frac{bL}{4} \end{cases}$
2 FE (exact analysis solution), Eq7	$same\ as\ Eq6$
2 FE (approx. analysis solution), Eq6	$\begin{cases} \frac{31bx}{24} \\ \frac{19bx}{24} + \frac{bL}{4} \end{cases}$
2 FE (approx. analysis solution), Eq7	$\begin{cases} \frac{11bx}{8} \\ \frac{7bx}{8} + \frac{bL}{4} \end{cases}$
4 FE (approx. analysis solution), Eq6	$\begin{cases} \frac{115bx}{96} \\ \frac{b}{96}(127x - 3L) \\ \frac{b}{96}(103x + 9L) \\ \frac{b}{96}(43x + 54L) \end{cases}$
4 FE (approx. analysis solution), Eq7	$\begin{cases} \frac{39bx}{32} \\ \frac{b}{32}(43x - L) \\ \frac{b}{32}(35x + 3L) \\ \frac{b}{32}(15x + 18L) \end{cases}$

Table 4.7 Material sensitivity solutions for quadratic design velocity

In all these plots, we have assumed the forcing term, $b = 1$ and the length of the bar, $L = 1$. For this case, the exact solution for material sensitivity derived from the ODE Eq[3] is a cubic polynomial. We notice that for weight functions $\psi = \frac{x}{L}$,$\left(\frac{x}{L}\right)^2$, we are able to arrive at a quadratic approximation for material sensitivity with both variational forms. When the weight functions are finite element hat functions (while using the exact analysis solution), we get a linear approximation of the material sensitivity. When we use the finite element (two elements) approximation of the analysis solution instead of the exact analysis solution, we get different solutions from Variational Eq[6] and Eq[7]. When solving for four finite elements, we again get different material sensitivity solutions from the two variational equations, Eq[6] and Eq[7]. Upon graphical comparison, we observe that the finite element material sensitivity solutions with four elements is more accurate than those with two elements. Again, further analysis on convergence and error with more elements can be found in Section 4.6.

4.5.3 Piecewise Design Velocity Plots

Table 4.8 summarizes the results for the piecewise design velocity function from Tables 4.1 to 4.5. Figure 4.3 compares the material sensitivity solutions for the piecewise design velocity. In all these plots, we have assumed the forcing term, $b = 1$ and the length of the bar, $L = 1$. For this case, the exact solution for material sensitivity derived from the ODE Eq[3] is a continuous piecewise function as well (with first half linear and second half quadratic). We notice that for weight functions $\psi = \frac{x}{L}$, $\left(\frac{x}{L}\right)^2$, we are able to arrive at a quadratic approximation for material sensitivity with both variational forms. When the weight functions are finite element hat functions (while using the exact analysis solution), we get a linear approximation which completely matches the linear part of the exact material sensitivity solution. We get the same approximate linear material sensitivity from both variational forms even when we use the finite element (two elements) approximation of the analysis solution instead of

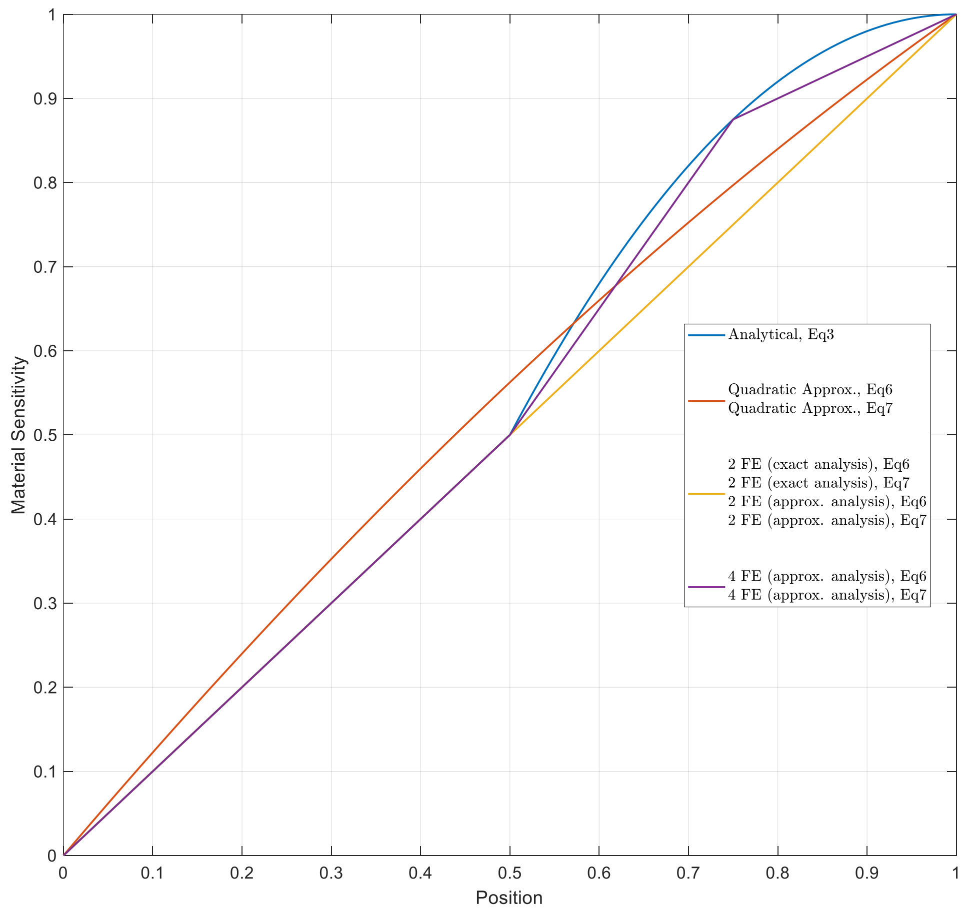


Figure 4.3 Material sensitivity solutions for piecewise design velocity (different trial functions)

For $D = \begin{cases} 0 & \left(0, \frac{L}{2}\right) \\ \frac{2x}{L} - 1 & \left(\frac{L}{2}, L\right) \end{cases}$	$\dot{u}$ $(b = 1, L = 1)$
Analytical, Eq3	$\begin{cases} bx \\ -\frac{2b}{L}x^2 + 4bx - bL \end{cases}$
Quadratic Approx., Eq6	$-\frac{b}{4L}x^2 + \frac{5b}{4}x$
Quadratic Approx., Eq7 (corrected)	$same\ as\ Eq6$
2 FE (exact analysis solution), Eq6	bx
2 FE (exact analysis solution), Eq7 (corrected)	$same\ as\ Eq6$
2 FE (approx. analysis solution), Eq6	$same\ as\ using\ exact\ analysis$
2 FE (approx. analysis solution), Eq7 (corrected)	$same\ as\ Eq6$
4 FE (approx. analysis solution), Eq6	$\begin{cases} bx \\ bx \\ \frac{b}{4}(6x - L) \\ \frac{b}{2}(x + L) \end{cases}$
4 FE (approx. analysis solution), Eq7 (corrected)	$same\ as\ Eq6$

Table 4.8 Material sensitivity solutions for piecewise design velocity

the exact analysis solution. When solving for four finite elements, we get matching material sensitivity solutions from the two variational equations, Eq[6] and Eq[7]. Upon graphical comparison, we can observe that the finite element material sensitivity solutions with four elements are more accurate than those with two elements. Again, further analysis on convergence and error with more elements can be found in Section 4.6.

4.6 Convergence in Finite Element Results

We are going to discuss the convergence of solutions, namely analysis solution, local sensitivity and material sensitivity, obtained from finite element method. Let us use six different grid sizes to compare the errors with respect to the exact solution. Here we have used 2, 4, 8, 16, 32 and 64 element grids. For this analysis, we have chosen three types of forcing functions: constant, quadratic and sinusoidal.

$$f = b, \qquad f = x(L - x), \qquad f = sin\left(\frac{\pi x}{L}\right) \qquad 4.68$$

where b is a constant and L represents the length of the bar. Let us analyze the convergence of solutions in each of these cases.

4.6.1 Exact Solutions from ODEs

Let us first tabulate the exact solutions for each type of forcing using ODE Eq[1], Eq[2] and Eq[3]. We have already shown the exact solutions for constant forcing in Table 4.1. Table 4.9 and Table 4.10 below show the exact solutions for quadratic and sinusoidal forcing functions respectively.

<table>
<tr><th>$\boldsymbol{D}$</th><th>$\boldsymbol{u^{as}}$ from
ODE Eq1</th><th>$\boldsymbol{u'}$ from
ODE Eq2</th><th>$\dot{\boldsymbol{u}}$ from
ODE Eq3</th></tr>
<tr><td>$\frac{x}{L}$</td><td rowspan="3">$\frac{2L^3x - 2Lx^3 + x^4}{12}$</td><td rowspan="3">$\frac{3L^2x - x^3}{6}$</td><td>$\frac{2L^3x - 2Lx^3 + x^4}{3L}$</td></tr>
<tr><td>$\left(\frac{x}{L}\right)^2$</td><td>$\frac{3L^4x + L^3x^2 - L^2x^3 - 3Lx^4 + 2x^5}{6L^2}$</td></tr>
<tr><td>$\begin{cases} 0 & \left(0, \frac{L}{2}\right) \\ \frac{2x}{L} - 1 & \left(\frac{L}{2}, L\right) \end{cases}$</td><td>$\begin{cases} \frac{3L^2x - x^3}{6} \\ \frac{-L^4 + 5L^3x + 3L^2x^2 - 9Lx^3 + 4x^4}{6L} \end{cases}$</td></tr>
</table>

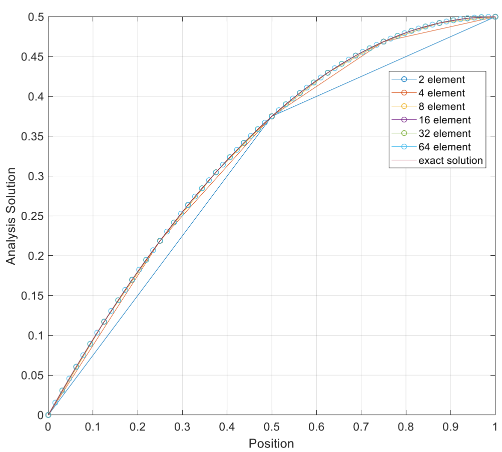
Table 4.9 Exact analysis solution, local and material sensitivity using ODEs Eq[1], Eq[2] and Eq[3] for quadratic forcing

D	u^{as} from ODE Eq1	u' from ODE Eq2	$\dot{u}$ from ODE Eq3
$\frac{x}{L}$	$\frac{L\pi x + L^2 \sin\left(\frac{\pi x}{L}\right)}{\pi^2}$	$\frac{\pi x - \pi x \cos\left(\frac{\pi x}{L}\right) + 2L \sin\left(\frac{\pi x}{L}\right)}{\pi^2}$	$\frac{2\left(\pi x + L \sin\left(\frac{\pi x}{L}\right)\right)}{\pi^2}$
$\left(\frac{x}{L}\right)^2$			$\frac{L\pi x + \pi x^2 - L\pi x \cos\left(\frac{\pi x}{L}\right) + \pi x^2 \cos\left(\frac{\pi x}{L}\right) + 2L^2 \sin\left(\frac{\pi x}{L}\right)}{L\pi^2}$
$\begin{cases} 0 & \left(0, \frac{L}{2}\right) \\ \frac{2x}{L} - 1 & \left(\frac{L}{2}, L\right) \end{cases}$			$\begin{cases} \frac{\pi x - \pi x \cos\left(\frac{\pi x}{L}\right) + 2L \sin\left(\frac{\pi x}{L}\right)}{\pi^2} \\ \frac{-L\pi + 3\pi x - L\pi \cos\left(\frac{\pi x}{L}\right) + \pi x \cos\left(\frac{\pi x}{L}\right) + 2L \sin\left(\frac{\pi x}{L}\right)}{\pi^2} \end{cases}$

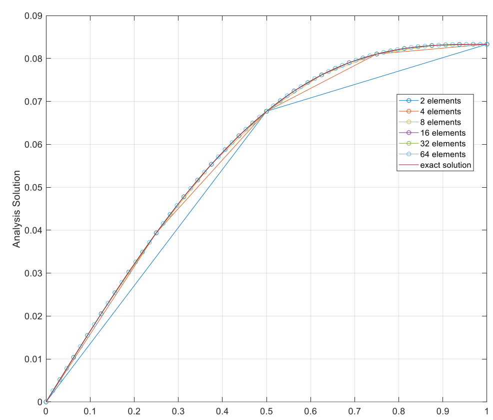
Table 4.10 Exact analysis solution, local and material sensitivity using ODEs Eq[1], Eq[2] and Eq[3] for sinusoidal forcing

4.6.2 Analysis Solutions

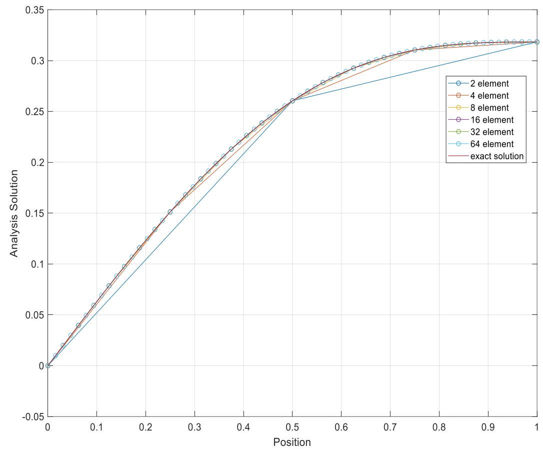
Let us analyze the convergence in the analysis solution. Figure 4.4 compares the solution from each grid size with the exact solution. Here we have assumed the forcing term, $b = 1$ and the length of the bar, $L = 1$. Upon evaluating errors in each solution, we realize that the approximate solutions match with the exact solution at every node (neglecting precision errors in the order of 10^{-15}). Hence it will be pointless to plot the log-log plots of L_1, L_2 or L_{inf} norms in these cases as error would be zero at nodes. So, for all three cases of forcing we are able to get exact value of the analysis solution at nodes for all grid sizes.



(a)



(b)



(c)

Figure 4.4 Convergence of analysis solution for (a) constant forcing function, (b) quadratic forcing function, and (c) sinusoidal forcing function.

4.6.3 Local Sensitivity Results

Figure 4.5 shows the local sensitivity solutions obtained from Eq[5] for each grid level. The local sensitivity solution is found to be the independent of design velocity. This is because in the local sensitivity variational equation, Eq[5], (a) the integral term does not have design velocity associated with it and (b) the boundary term uses the design velocity at the right boundary, $D(L)$, which is the same $(= 1)$ for all three design velocity choices.

$$\int_0^L u'_x \psi_x \, dx = \int_0^L f' \psi \, dx + \left[\psi(L)\dot{N}_L - \psi(L) u_{xx}^{as}(L) D(L)\right] \qquad 4.69$$

As we have chosen

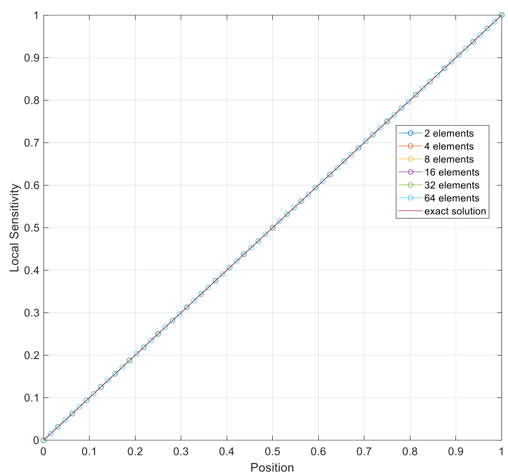
$$\dot{N}_L = 0 \qquad 4.70$$

and the second derivative of exact analysis solution

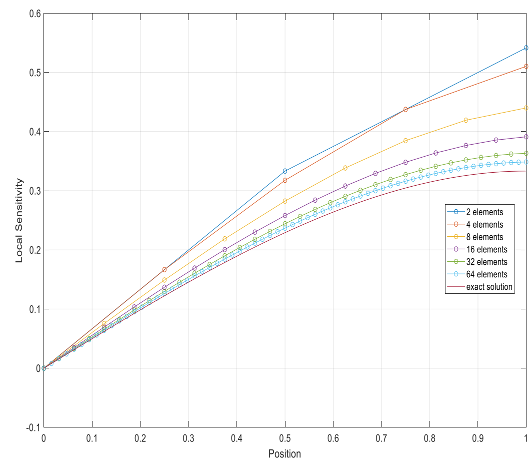
$$u_{xx}^{as}(L) = b|_{x=L} = b \quad for \quad f = b \qquad 4.71$$

$$u_{xx}^{as}(L) = (x^2 - Lx)|_{x=L} = 0 \quad for \quad f = x(L - x) \qquad 4.72$$

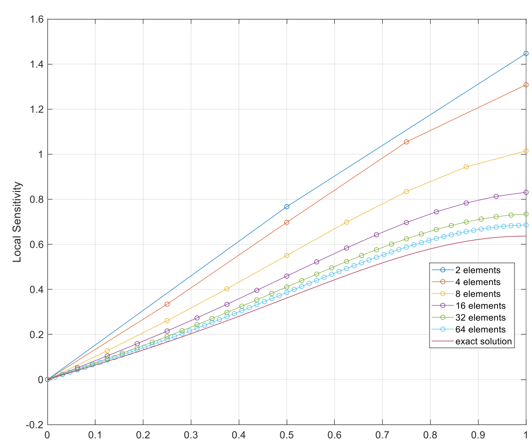
$$u_{xx}^{as}(L) = -sin\left(\frac{\pi x}{L}\right)\Big|_{x=L} = 0 \quad for \quad f = sin\left(\frac{\pi x}{L}\right) \qquad 4.73$$



(a)



(b)



(c)

Figure 4.5 Convergence of local sensitivity solution using Eq[5] for (a) constant forcing function, (b) quadratic forcing function, and (c) sinusoidal forcing function.

Since we are using a 2-node linear element FEM to evaluate the analysis solution, we obtain linear approximations of the solution for each element. The second derivative of this analysis solution would always be zero. Therefore, a three-point approximation method (closest to the right boundary) is used to obtain a quadratic polynomial for the analysis solution. Eventually, we use this approximation to evaluate the boundary value of the second derivative of the analysis solution. For constant forcing, this quadratic approximation matches the exact analysis solution as the analysis solution is quadratic. For quadratic and sinusoidal forcing, this quadratic approximation does not match the exact analysis solution which is of higher degree. In these two forcing cases, we find errors when we compare the finite element result with the exact local sensitivity solution. Let us further analyze this error using the L_1, L_2 or L_{inf} norms.

We have normalized the L_1 norm by division with the number of elements for each grid. Similarly, the L_2 norm is normalized by dividing by square root of the number of elements. L_{inf} norm requires no normalization as it just represents the maximum error value. The error norms for an error vector e are given as,

Normalized L_1 norm:

$$\|e\|_1 = \frac{1}{N}\sum_{i=1}^{N}|e_i| \qquad 4.74$$

Normalized L_2 norm:

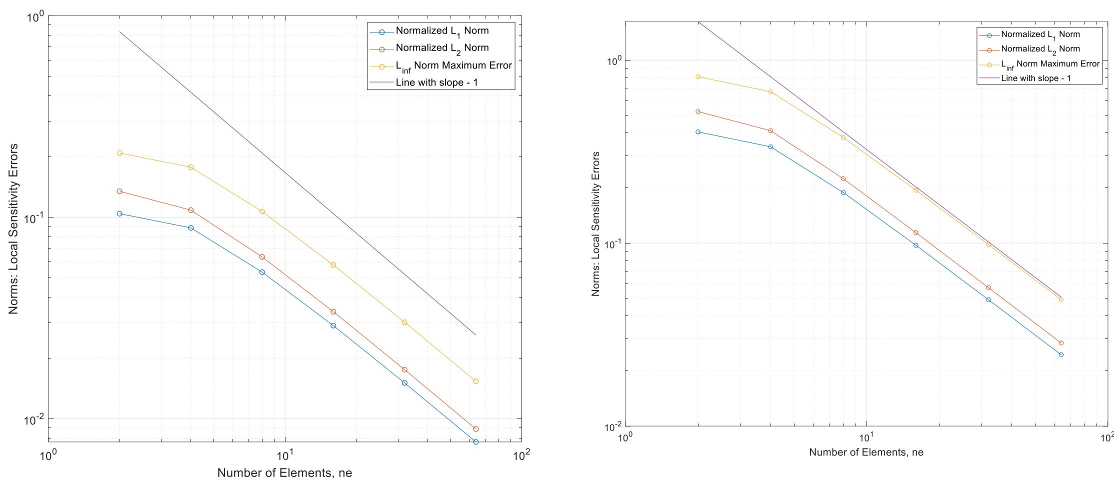
$$\|e\|_2 = \sqrt{\frac{1}{N}\sum_{i=1}|e_i|^2} \qquad 4.75$$

L_{inf} norm:

$$\|e\|_\infty = max(|e_i|) \tag{4.76}$$

We have plotted these three norms in a single log-log plot (Figure 4.6) to understand the convergence of our solution. It can be observed that all of the error norm lines are parallel to the reference line of slope '-1'. Hence, there is first order of convergence in our solutions when we solve using Eq[5]. Let us evaluate the terms of Eq[5] and our method of solving.

$$\int_0^L u'_x \psi_x \, dx = \int_0^L f' \psi \, dx + \psi(L)\dot{N}_L - \psi(L) u_{xx}^{as}(L) D(L) \tag{4.77}$$



(a) (b)

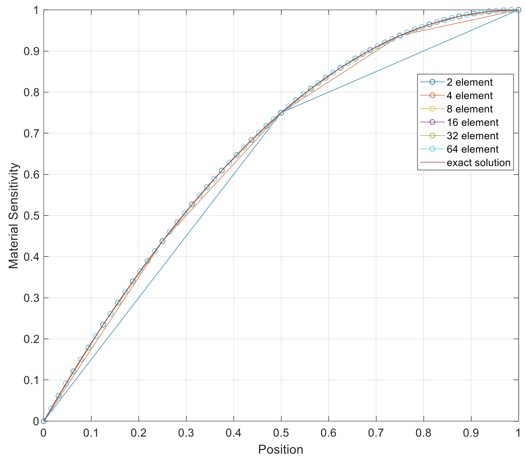
Figure 4.6 Log-log plots of error norms of local sensitivity solutions using Eq[5] for (a) quadratic forcing function, and (b) sinusoidal forcing function.

From our governing equations we can predict that the analysis solution would have second order of convergence. Also, our order of convergence will reduce for a higher derivative of our solution while it will increase where integration is involved. Hence in this case, the second and third derivatives would allow first and zero order of convergence respectively. We have used known functions or values for f', ψ, ψ_x, $\psi(L)$, $D(L)$ and $\dot{N}_L$ (from BC). The integral term in LHS, $\int_0^L u'_x \psi_x \, dx$, will have second order of accuracy as integration would improve the accuracy. The integral term in the RHS, $\int_0^L f' \psi \, dx$ is exactly known. In the boundary terms, we have $u_{xx}^{as}(L)$, which has been calculated using a three-point quadratic approximation of analysis solution at L. This, in turn, gives first order of convergence for the whole formulation. Using a four-point cubic approximation for this term might allow us to achieve second order convergence.

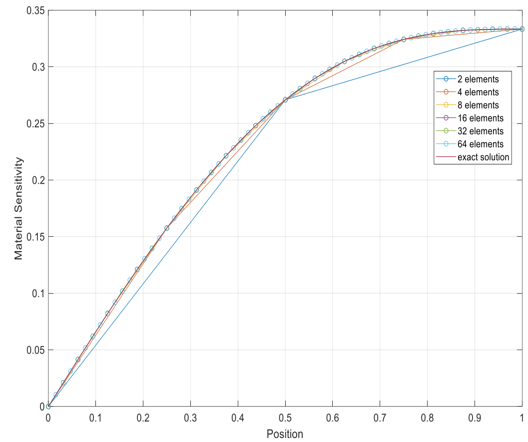
4.6.4 Material Sensitivity Results

Figure 4.7 plots the material sensitivity solutions from Eq[6] and Eq[7] for a case of linear design velocity (again assuming $b = 1$ and $L = 1$) for each grid size. Similar to analysis solution, the finite element material sensitivity solutions match the exact solution at nodes (neglecting precision errors in the order of 10^{-15}). Hence, L_1, L_2 or L_{inf} norms based on nodal errors in these cases would be trivial as error is zero everywhere.

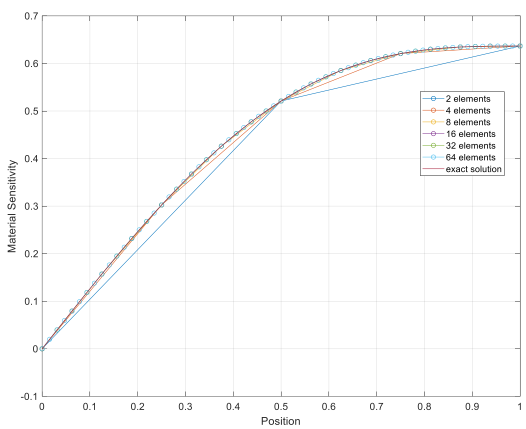
Linear design velocity



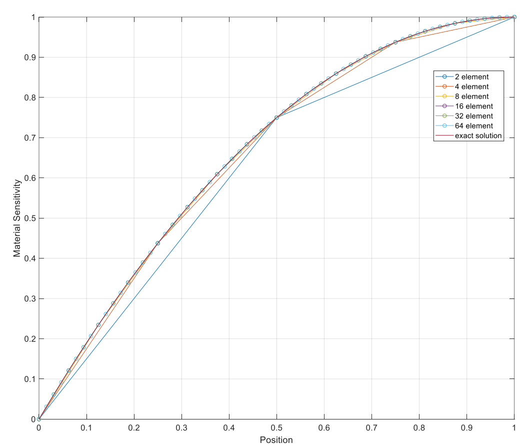
(a)



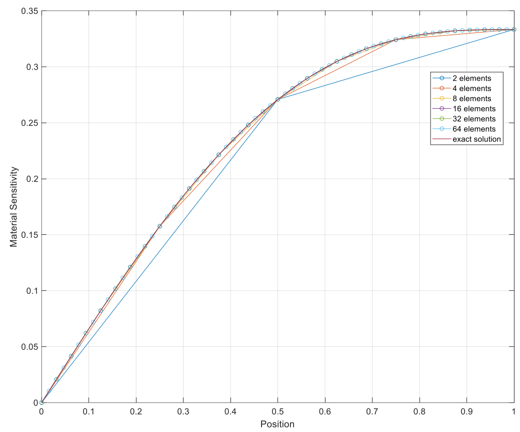
(b)



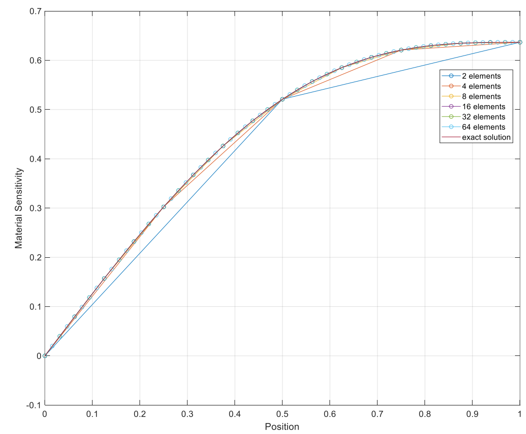
(c)



(d)



(e)

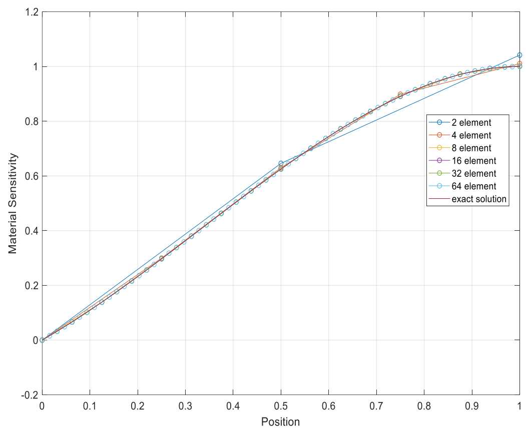


(f)

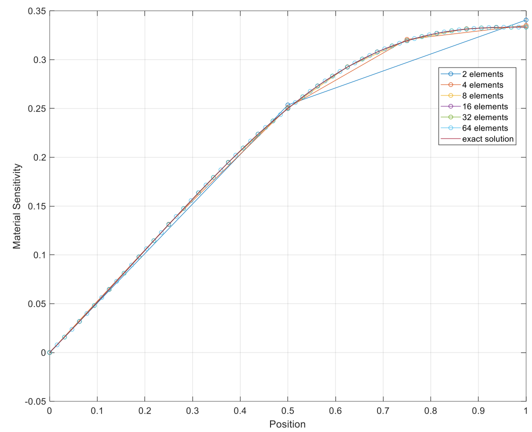
Figure 4.7 Convergence of material sensitivity solution with linear design velocity for (a) constant forcing function using Eq[6], (b) quadratic forcing function using Eq[6], (c) sinusoidal forcing function using Eq[6], (d) constant forcing function using Eq[7], (e) quadratic forcing function using Eq[7], and (f) sinusoidal forcing function using Eq[7].

Now let us plot the results for a quadratic design velocity. Figure 4.8 shows the material sensitivity solutions obtained from Eq[6] and Eq[7] for the quadratic design velocity case for each grid level. Here we find errors when compared with the exact material sensitivity solution. Let us further analyze this error using the L_1, L_2 or L_{inf} norms.

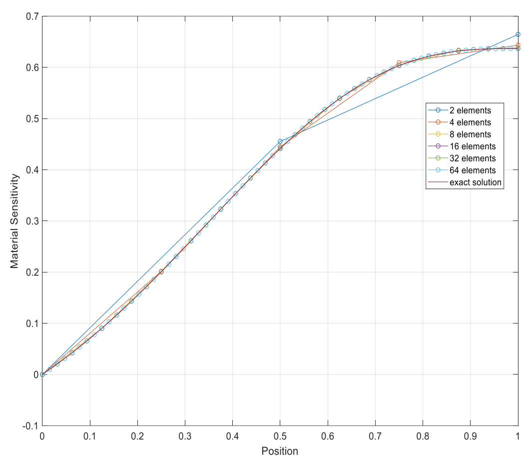
Quadratic design velocity



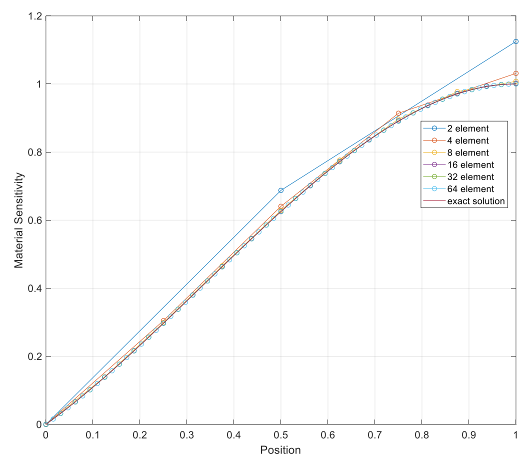
(a)



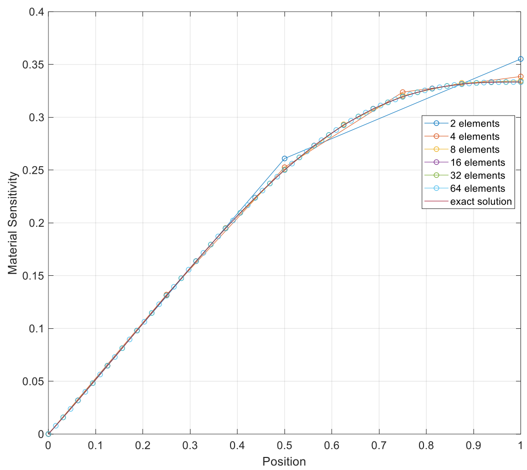
(b)



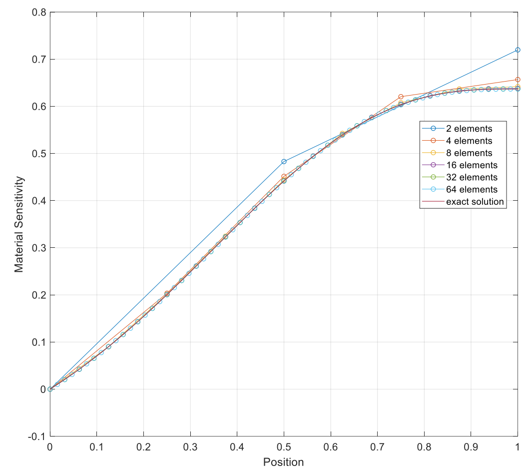
(c)



(d)



(e)



(f)

Figure 4.8 Convergence of material sensitivity solution with quadratic design velocity for (a) constant forcing function using Eq[6], (b) quadratic forcing function using Eq[6], (c) sinusoidal forcing function using Eq[6], (d) constant forcing function using Eq[7], (e) quadratic forcing function using Eq[7], and (f) sinusoidal forcing function using Eq[7].

Here, we have used the normalized L_1 and L_2 norms from Eq[4.74] and Eq[4.75] respectively. We have plotted these three norms in a single log-log plot (Figure 4.9) to understand the convergence of our solutions from Eq[6] and Eq[7]. It can be observed that all of the error norm lines are parallel to the reference line of slope '-2'. Hence, there is second order of convergence in our solutions when we solve using Eq[6] and Eq[7].

Let us evaluate the terms of Eq[6] and our method of solving.

$$\int_0^L \dot{u}_x \psi_x \, dx = \int_0^L (u_x^{as} \, \psi_x D_x + \dot{f}\psi + f\psi D_x) \, dx + \dot{N}_L \psi(L) \qquad 4.78$$

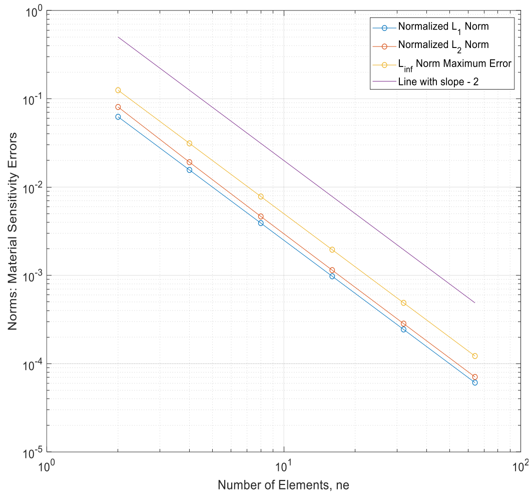
We have used known functions or values for f, $\dot{f}$, ψ, ψ_x, D_x, $\psi(L)$ and $\dot{N}_L$ (from BC). The integral term in LHS, $\int_0^L \dot{u}_x \psi_x \, dx$, will have second order of accuracy as integration would improve the accuracy. The second and third integral terms in the RHS, $\int_0^L (\dot{f}\psi + f\psi D_x) \, dx$ is exactly known. The first integral term, $\int_0^L u_x^{as} \, \psi_x D_x \, dx$, would have second order of convergence after integration. In the boundary term, we have all known values at L. Thus, we get second order of convergence for the whole formulation.

Let us now evaluate the terms of Eq[7] and our method of solving.

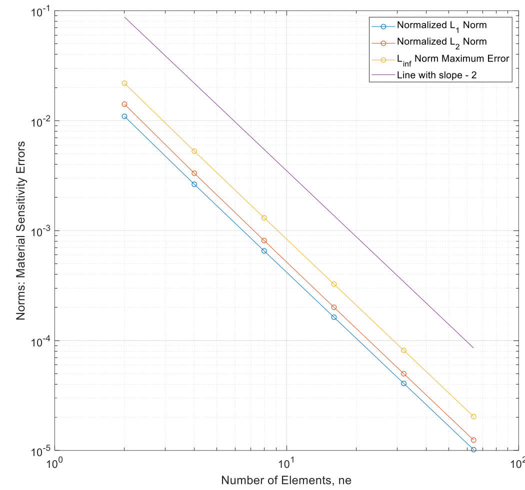
$$\int_0^L \dot{u}_x \psi_x \, dx = \int_0^L (\dot{f}\,\psi + D_{xx} u_x^{as} \psi + 2u_x^{as} \psi_x D_x) \, dx \qquad 4.79$$

$$+\dot{N}_L \psi(L) + D_x(L) N_L \psi(L)$$

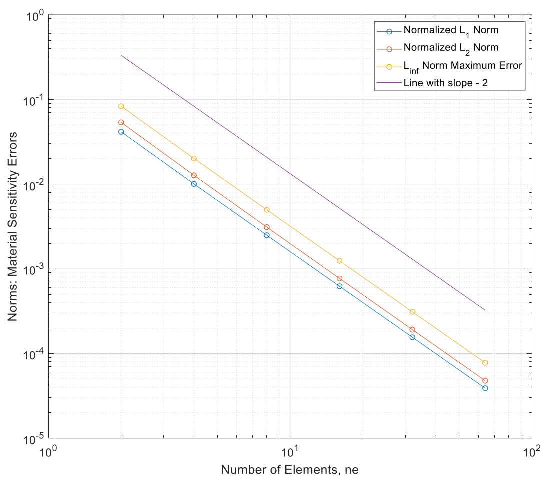
We have used known functions or values for $\dot{f}$, ψ, D_x, D_{xx}, $\psi(L)$, $D(L)$, $D_x(L)$; N_L and $\dot{N}_L$ (from BC). The integral term in LHS, $\int_0^L \dot{u}_x \psi_x \, dx$, will have second order of accuracy as integration would improve the accuracy. The first integral term in the RHS, $\int_0^L \dot{f}\psi \, dx$ is exactly known. The second and third integral terms, $\int_0^L (D_{xx} u_x^{as} \psi + 2u_x^{as} \psi_x D_x) \, dx$, would also have second order of convergence after integration. In case of piecewise design velocity, at the point of singularity we have used an average value of u_x^{as} around $x = \frac{L}{2}$ (singularity) allowing a second order convergence. In the boundary terms, we have all known values at L. Thus, we get second order of convergence from this variational formulation.


Normalized L1 Norm
Normalized L2 Norm
Linf Norm Maximum Error
Line with slope - 2
Norms: Material Sensitivity Errors
Number of Elements, ne

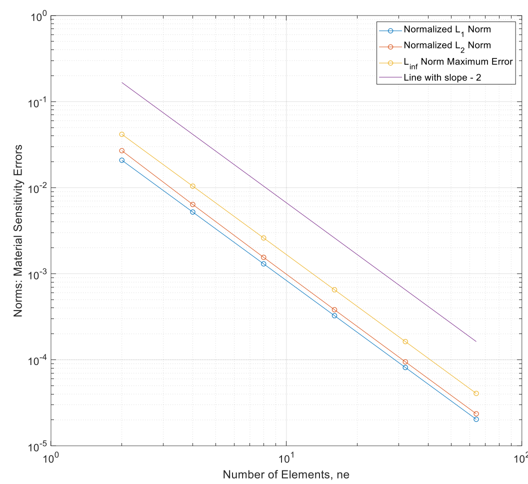
(a)


Normalized L1 Norm
Normalized L2 Norm
Linf Norm Maximum Error
Line with slope - 2
Norms: Material Sensitivity Errors
Number of Elements, ne

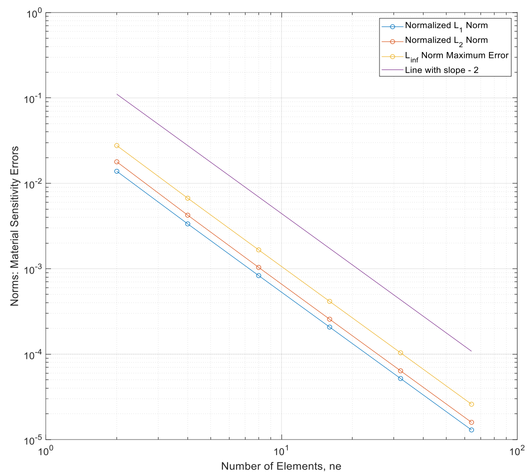
(b)


Normalized L1 Norm
Normalized L2 Norm
Linf Norm Maximum Error
Line with slope - 2
Norms: Material Sensitivity Errors
Number of Elements, ne

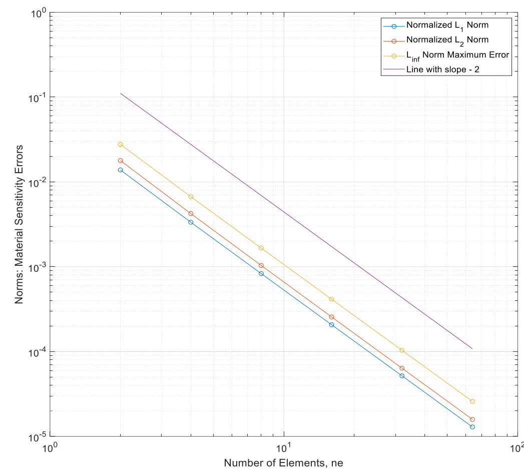
(c)


Normalized L1 Norm
Normalized L2 Norm
Linf Norm Maximum Error
Line with slope - 2
Norms: Material Sensitivity Errors
Number of Elements, ne

(d)



(e)



(f)

Figure 4.9 Log-log plots of error norms of material sensitivity solutions with quadratic design velocity for (a) constant forcing function using Eq[6], (b) quadratic forcing function using Eq[6], (c) sinusoidal forcing function using Eq[6], (d) constant forcing function using Eq[7], (e) quadratic forcing function using Eq[7], and (f) sinusoidal forcing function using Eq[7].

Now let us plot the results for a piecewise linear design velocity. Figure 4.10 plots the material sensitivity solutions for a case of piecewise design velocity obtained from Eq[6] and Eq[7] for each grid size. In these cases, we have to incorporate the correction term to account for the integral value of $\int_0^L (D_{xx} u_x^{as} \psi)\, dx$ which is zero everywhere except at singularity point, $x = \frac{L}{2}$. Combining equations Eq[4.66] and Eq[4.67] for three types of forcing, we get,

$$\int_{\frac{L}{2}-}^{\frac{L}{2}+} D_{xx} u_x^{as} \psi \, dx = u_{x_{\frac{L}{2}}}^{as} \psi_{\frac{L}{2}} \left\{ D_{x_{\frac{L}{2}+}} - D_{x_{\frac{L}{2}-}} \right\}$$
$$= \frac{u_{x_{\frac{L}{2}+}}^{as} + u_{x_{\frac{L}{2}-}}^{as}}{2} \psi_{\frac{L}{2}} \left\{ D_{x_{\frac{L}{2}+}} - D_{x_{\frac{L}{2}-}} \right\} \qquad 4.80$$

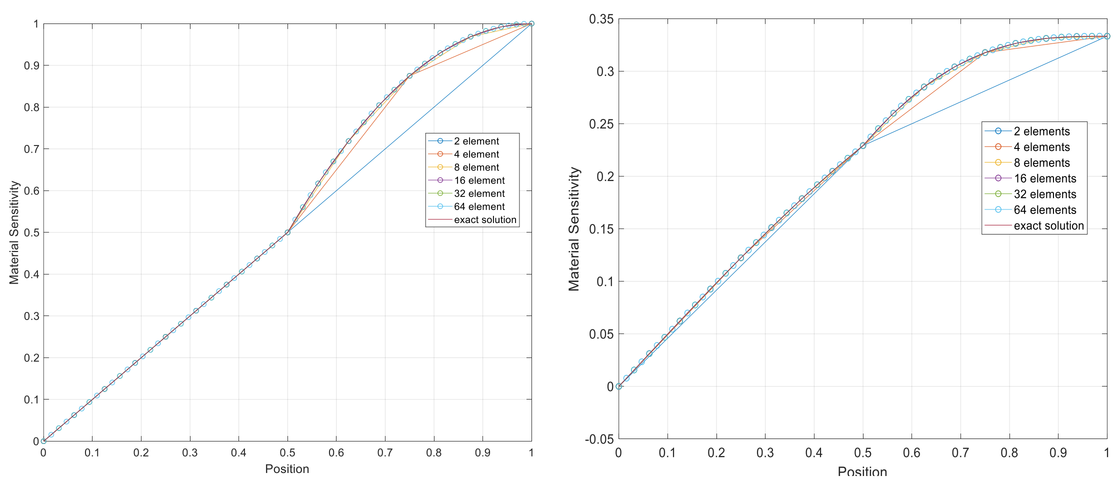
$$= b \quad for \quad f = b$$

$$= \frac{L^2}{6} \quad for \quad f = x(L - x)$$

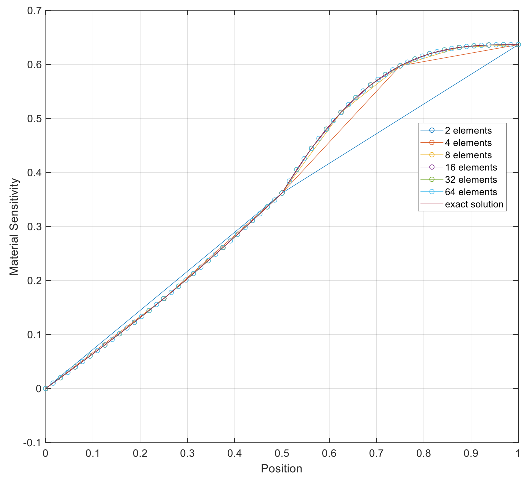
$$= \frac{2}{\pi} \quad for \quad f = sin\left(\frac{\pi x}{L}\right)$$

This value turns out to be a constant throughout all grid sizes for a given forcing term. Here, the finite element solutions completely match the exact solution at nodes (neglecting precision errors in the order of 10^{-15}). Hence, L_1, L_2 or L_{inf} norms based on nodal errors in these cases would be trivial as error is zero everywhere.

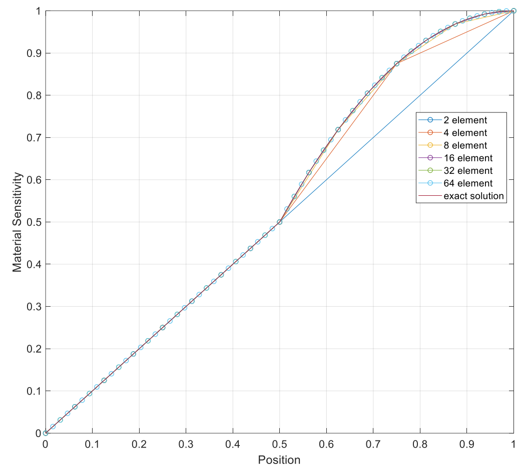
Piecewise linear design velocity



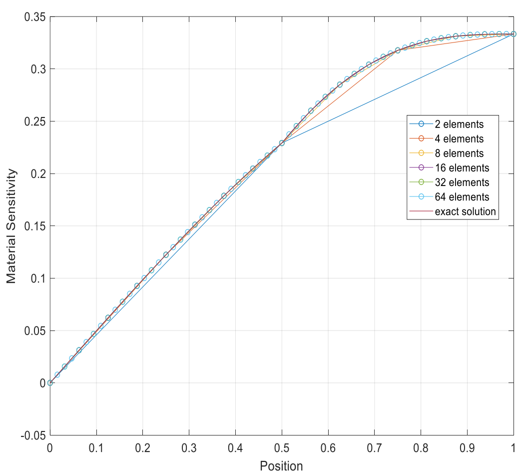
(a) (b)



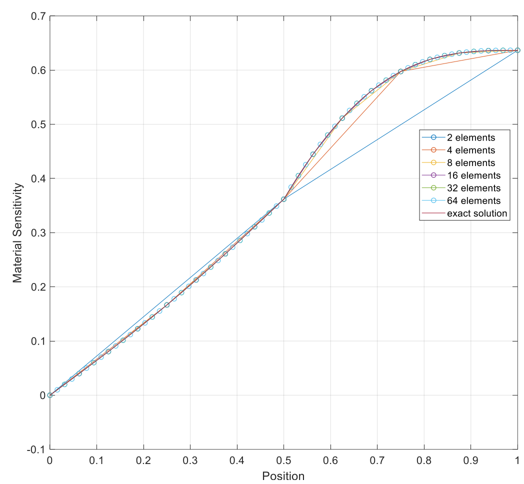
(c)



(d)



(e)



(f)

Figure 4.10 Convergence of material sensitivity solution with quadratic design velocity for (a) constant forcing function using Eq[6], (b) quadratic forcing function using Eq[6], (c) sinusoidal forcing function using Eq[6], (d) constant forcing function using Eq[7], (e) quadratic forcing function using Eq[7], and (f) sinusoidal forcing function using Eq[7].

Chapter 5

Conclusions and Future Work

5.1 Research Summary

Analytical design sensitivity analysis methods offer the advantage of accuracy and computational efficiency over numerical finite difference sensitivity methods. These methods can be classified as DASSA and CASSA. DASSA employs a “discretize then differentiate” approach, that is, the governing differential equations are first discretized and then used to calculate sensitivities. CASSA, however, involves differentiation followed by discretization. Hence, we first differentiate the governing differential equations with respect to the shape parameter to derive linear continuum sensitivity differential equations, and then solve these equations to evaluate the sensitivities of the performance metric with respect to design variables.

In this research, we have employed CASSA to calculate the shape sensitivities of an elastic bar. This has led to the derivation of three variational formulations to solve these shape design derivatives. These are:

(a) Local sensitivity variational formulation Eq[5]

(b) Material sensitivity variational formulation Eq[6]: Material sensitivity of the analysis variational form

(c) Material sensitivity variational formulation Eq[7]: Variation of material sensitivity differential equation

We have used the following methods to address various issues associated with each of these formulations.

(a) A three-point quadratic approximation of finite element analysis solution to estimate its second spatial derivative at the right boundary location. This is used in the local sensitivity variational formulation Eq[5].

(b) An approximate integration of a second order derivative term at singularity. This is applied for the material sensitivity formulation Eq[7] in the case of a piecewise continuous (linear) design velocity.

A first order of convergence in finite element results is observed for local sensitivity solutions. This can be improved to second order by using a four-point cubic approximation to estimate the spatial derivatives at the boundary. For material sensitivity results, we notice a second order of convergence.

5.2 Future Work

In this work, we have used CASSA to derive three shape sensitivity variational formulations for 1-D structural equation of an elastic bar. In future, this approach can be extended to multi-dimensional structural and fluid domain problems.

Bibliography

Braibant, V. and Fleury, C. Shape optimal design using b-splines. *Computer Methods in Applied Mechanics and Engineering*, 44(3):247-267, 1984.

Braibant, V. and Fleury, C. An approximation concepts approach to shape optimal design. *Computer Methods in Applied Mechanics and Engineering*, 53(3):119-148, 1985.

Chun, Y.W. and Haug, E.J. Shape optimization of a solid revolution. *Journal of Engineering Mechanics*, 109 (1):30-46, 1983.

Cross, D. M. and Canfield, R. A. Local continuum shape sensitivity with spatial gradient reconstruction. *Structural and Multidisciplinary Optimization*, 50(6):975–1000, 2014.

Cross, D. M. and Canfield, R. A. Local continuum shape sensitivity with spatial gradient reconstruction for nonlinear analysis. *Structural and Multidisciplinary Optimization*, 51(4):849–865, 2015.

Haug, E.J. and Arora, J.S. Design sensitivity analysis of elastic mechanical systems. *Computer Methods in Applied Mechanics and Engineering*, Vol. 15, pp. 35-62, 1978.

Kulkarni M.D., Canfield R.A. and Patil M.J. Nonintrusive continuum sensitivity analysis for fluid applications. *Journal of Computational Physics*, Vol. 403, 2020.

Kulkarni M.D., Cross D.M., Canfield R.A. Discrete Adjoint Formulation for Continuum Sensitivity Analysis. *AIAA Journal*, 54(2):758-766, 2016.

Liu, S. and Canfield, R.A., Equivalence of continuum and discrete analytic sensitivity methods for nonlinear differential equations. *Structural and Multidisciplinary Optimization*, 48(6):1173-1188, 2013.

Turner A.M. and Patil M.J., Aerodynamic Shape Optimization using Continuum Shape Sensitivity Analysis. *57th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, Kissimmee, Florida.*, AIAA 2018-1909, 2018.

Wickert, D.P and Canfield, R.A. Least-squares continuous sensitivity analysis of an example fluid-structure interaction problem. *49th AIAA/ASME/ASCE/ AHS/ASC Structures, Structural Dynamics, and Materials Conference, Schaumberg, Illinois.*, AIAA 2008-1896, 2008.