Teachers' Reflection on Inquiry-Oriented Instruction in Online Professional Development

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In light of the expansion of student-centered instructional approaches in mathematics education and a brightening spotlight on virtual teacher supports, I look to Inquiry-Oriented Instruction (IOI) and explore how instructors reflect on and plan for their implementation of IOI in online professional development. I focus specifically on two teachers' comments on their implementation of IOI materials covering Abstract Algebra topics in online work groups developed to support teachers in implementing IOI. I analyze both reflection and enactment through the components of IOI characterized through the Instructional Triangle. Analysis of the teachers' reflections, viewed through their participation in the roles of sense maker, inquirer, and builder, revealed interesting differences in the teachers' approaches to IOI. I detail these two teachers' approaches to IOI and ultimately shed light on the intricacies of IOI and online professional development. Such findings support the growing bodies of research centered around IOI and corresponding professional development.

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Introduction

Recently there have been calls for reform in the teaching of undergraduate mathematics. Policy makers have particularly called for instructors to implement student-centered instructional approaches, as they have been shown to benefit student affect and achievement (e.g., Freeman et al., 2014; Laursen, Hassi, Kogan, & Weston, 2014). Moreover, there are longstanding calls for more interactive approaches to curriculum design (Saxe & Braddy, 2015; Ball & Cohen, 1996). Researchers at the K12 level have, for quite some time now, been advocating for curriculum materials that "draw on teachers' understanding and students' thinking, and… depend on engaging ways to represent the material and develop the intellectual environment of a class" (Ball & Cohen, 1996, p. 8). One type of student-centered instruction utilizing special curriculum design and shown to support undergraduate students' learning in mathematics is Inquiry-Oriented Instruction (IOI; Kuster, Johnson, Keene, & Andrews-Larson, 2017). The demand for such student-centered instruction is not met with the simple dissemination of best practices in the classroom, but rather professional development that supports a radical change in instruction and student engagement (Henderson, Beach, & Finkelstein, 2011).

Therefore, in addition to the calls for more reform-oriented instruction, there is also a growing spotlight on support and professional development in mathematics education (Fortune & Keene, 2019). Finding ways to support instructional change is seen by some in the mathematics education community as the most critical question facing education researchers (Andrews-Larson et al., 2019, p. 2). In fact, some researchers claim that

> "learning to support instructors effectively in relation to the aforementioned challenges [of shifting toward student-centered pedagogies] is particularly important and thus implies a need for professional development programs that foster the development of undergraduate mathematics instructors' pedagogical reasoning" (Andrews-Larson et al., 2019, p. 3).

The need for resources in virtual environments has developed from an emerging population of teachers that seek to implement reform-oriented instruction, yet are distanced from similar communities that can provide support. Such resources must also "include opportunities for active interpretive processes that examine the complex contexts of classrooms" (Burbank & Kauchak, 2002, p. 500). This need for virtual engagement has become even more crucial as multiple communities move online in response to the COVID-19 pandemic.

Researchers in undergraduate mathematics education have aimed to support instructors in implementing IOI by developing course curriculum and offering professional development opportunities for mathematics instructors (e.g., Larsen, Johnson, & Scholl, 2016; Rasmussen, Keene, Dunmyre, & Fortune, 2018; Wawro, Zandieh, Rasmussen, & Andrews-Larson, 2013). One such professional development project is Teaching Inquiry-Oriented Instruction: Establishing Supports (TIMES; NFS Awards: #143195, #1431641, #1431393), which was designed to support instructors as they implemented IOI. Through this project, participating instructors, called TIMES Fellows, were given access to research-based curriculum materials, IOI training workshops, and online working groups (OWG), in which they planned and reflected on their instruction throughout the semester with the support of their fellow instructors and a facilitator from project personnel. To evaluate the effectiveness of this project, video data was collected of the teachers' OWG meetings for analysis.

In this study, I focused analysis on two TIMES Fellows who implemented Inquiry-Oriented Abstract Algebra (IOAA). These two instructors implemented the isomorphism and quotient group tasks from the Teaching Abstract Algebra for Understanding curriculum materials (Larsen, Johnson, & Scholl, 2016). They also both participated in the same OWG as they implemented IOI. While these teachers used the same curriculum materials and participated in the same OWG, they reflected on the instructional sequences and their own instruction differently. I sought to gain insight into the differences observed in their interactions in the OWG by investigating differences in the instructors' discussion with respect to the various components of IOI. I particularly focused on these instructors' discussions as they planned for or reflected on their implementation of IOI for the instructional units on isomorphism and quotient groups. To do this, I leveraged the Instructional Triangle (Cohen & Ball, 1999) and the key components of IOI (Kuster et al., 2019) to identify differences among the instructors as they planned for and reflected on their instruction. I addressed the following research question: How does two instructors' participation progress and vary over time in online professional development as they discuss and reflect on their approaches to IOI?

Literature Review

Several studies have looked at how teachers have or may implement and think about IOI. Fewer studies look at the structure and supports of online work groups as professional development for undergraduate instructors. The current study takes place within the context of a larger study: TIMES. TIMES was a professional development research grant specifically looking at how to provide support for instructors interested in implementing IOI. In this project, IOI was rather narrowly defined and the majority of the instructional support was provided through online working groups. Here I will: describe IOI and discuss some of the instructional challenges that have been documented in the IOI research literature; discuss some of the online/virtual professional development, especially those studies that are relevant for our context; and discuss a related study from the TIMES project that investigated an in person component of their professional development program. The current literature places this study at an important intersection of the principals of IOI and how teachers think about and plan to teach reformed curriculum.

Challenges of Implementing IOI

Here, I review studies that explicate the roles students and teachers play in IOI classrooms, the tendencies of inquiry-oriented instructors, and the challenges they face in implementing IOI. For this study, I use a rather narrowly defined version of IOI. TIMES was developed to support IOI differential equations, IOI abstract algebra and IOI linear algebra— all of which were informed by Realistic Mathematics Education (RME; Gravemeijer & Doorman, 1999) and have a significant research base that characterizes their intended instructional implantation and associated instructional challenges (IODE, Rasmussen, Keene, Dunmyre, & Fortune, 2018; IOAA; Larsen, Johnson, & Scholl, 2016; IOLA; Wawro, Zandieh, Rasmussen, & Andrews-Larson, 2013). The IOI curricula at the center of TIMES were all informed by the instructional design theory of RME, which laid the foundation for IOI wherein students are given opportunities to reinvent key mathematical ideas and methods and presented with challenging, experientially real tasks to promote their own inquiry. The goal is that after students reinvent mathematics through their own mathematical activity, they will take ownership of their knowledge. Moreover, Gravemeijer and Doorman (1999) claimed that "the overall goal of instructional design is to support the gradual emergence of a taken-as-shared mathematical reality" (p. 127). These ideas shared by Gravemeijer and Doorman support, if not originate, the main ideas of IOI.

Implementing these RME inspired curricular materials necessitates inquiry. This inquiry, both on the part of the students and the teacher, is the cornerstone of IOI. In a 2007 study, Rasmussen and Kwon (2007) first coined the term *inquiry-oriented* learning and investigated teaching through the implementation of IOI in a differential equations course. As explained by Rasmussen and Kwon, teacher inquiry serves three main purposes: "it enables teachers to construct models for how their students interpret and generate mathematical ideas…it provides opportunities for teachers to learn something new about particular mathematical ideas, in light of student thinking… [and] it better positions teachers to build on students' thinking by posing new questions and tasks" (p. 190). In their study, they also sought to better understand students' thinking when they engaged in IOI. Students are positioned to learn through inquiry in mathematical discussions, proposing conjectures, and explaining their thinking in the inquiry setting. Rasmussen and Kwon stated that inquiry can "empower learners to see themselves as capable of reinventing mathematics and to see mathematics itself as a human activity" (p. 190). Rasmussen and Kwon also related IOI to teacher knowledge; "as our understanding of student thinking evolves, so does our understanding of the kinds of teacher knowledge that would be important for promoting student learning" (Rasmussen and Kwon, 2007, p. 192). They identified such knowledge as awareness of students' informal ways of reasoning, strategies for connecting teaching to student thinking, and mathematical content.

Functionally, through IOI, teachers give students opportunities to reinvent key mathematical ideas and methods and present them with challenging, experientially real tasks to promote their own inquiry. Kuster, Johnson, Keene, and Andrews-Larson (2018) looked at the progress of three teachers' IOI implementation in linear algebra classrooms over a year's time. Four key ideas of IOI were later identified by Andrews-Larson and colleagues that fit well into the context of the classrooms in this study:

> "(1) generating and eliciting student ideas (by engaging students on carefully designed mathematical problem-solving activities and having students share their tentative ideas, strategies, and approaches in whole class discussion), (2) building on student ideas, (3) developing a shared understanding (e.g. just because one student or group shares an idea in whole class discussion does not mean every person in the room followed and made sense of that idea), and (4) formalizing mathematical language and notation in ways that connect to and leverage the tentative ideas and approaches put forth by students." (Andrews-Larson et al., 2019, p. 2).

These are the tasks that teachers of IOI focus on in the classroom and often find difficult implementing (Wagner, Speer, & Rossa, 2007).

There have been multiple studies investigating the implementation of the three IOI curricula, IOAA, IOLA, and IODE. Several of these studies identify the teacher knowledge and attention needed to implement IOI. In 2013, Johnson conducted a study to investigate teachers' mathematical activity as they implemented IOI in abstract algebra courses. Johnson looked at teachers' mathematical activity in relation to students' mathematical activity and studied classroom data from four mathematicians over four years of implementing IOI in abstract algebra classrooms. She sought to "(1) identify the mathematical activities that teachers implementing [IOI abstract algebra] curriculum engage in during classroom teaching in response to the mathematical activity of their students, and (2) investigate the ways in which teachers' mathematical activity interacts with students' mathematical activity" (p. 761). Johnson found significant evidence of teacher mathematical activity in response to student mathematical activity. Specifically, this included "interpreting students' mathematical reasoning and contributions; analyzing and evaluating students' mathematical contributions, conjectures, and arguments; and identifying mathematical connections" (p. 773). Altogether, this study also shows that teachers' mathematical activity and knowledge plays an important role in students' mathematical development when engaging with IOI.

A 2019 study by Kuster, Johnson, Rupnow, and Wilhelm looked at what professional development centered on IOI should attend to. To effectively train teachers in IOI, they proposed instructional measures that they describe to "provide a shared vernacular and specific descriptions of instructional practices that can promote instructional change" (p. 2). These instructional measures are grouped together as the inquiry-oriented instructional measure (IOIM). The development of the IOIM actually stems from the TIMES project aim to provide supports to teachers seeking to reform their instruction. Kuster and colleagues highlighted that IOI relies heavily on a classroom discourse in which the teacher prompts students to question and develop their own understandings from their pre-existing, informal knowledge. They identified four principal components of IOI: "generating student ways of reasoning, building on student contributions, developing a shared understanding, and connecting to standard mathematical language and notation" (p. 5). Behind these four principals are seven teaching practices that Kuster et al. detail in the paper. The authors argue that these practices support the implementation of IOI. Practice one states that teachers engage students in mathematical tasks significant to the mathematical point of the instruction. Kuster et al. make the important distinction that practice one is about the mathematical activity of the classroom, rather than just the mathematical activities planned by the teacher. Practice two requires that teachers prompt and build on student thinking. Practice three is similar to practice two in that it asks teachers to "actively inquire into student thinking" (p. 5). Teachers should also use student thinking to advance the mathematical agenda of the classroom (practices four and six). Practice five states that teachers should facilitate students' engagement with their peers reasoning in an effort to gain shared understanding. Lastly, in practice seven teachers are to introduce mathematical notation and language to formalize students' understanding at appropriate time during instruction. In the pilot testing of the TIMES data, the IOIM was found to be both reliable and viable for measuring inquiry-oriented instruction. I include this study with those seeking to enhance professional development for undergraduate instructors because the IOIM serves as a proxy between the important ideas of IOI and how those ideas are best carried out in the classroom, which is a focus of professional development.

In a 2006 study, Rasmussen and Marrongelle proposed two specific pedagogical content tools, transformational records, and generative alternatives, to help teachers engage students' mathematical activity while still advancing the mathematical agenda. They describe transformational records as "notations, diagrams, or other graphical representations that are initially used to record student thinking and that are later used by students to solve new problems" (Rasmussen & Marrongelle, 2006, p. 389). These types of materials can be viewed in conjunction with the emergent models of RME theory that first model students' thinking, and then progress to model more formal mathematics. Generative alternatives are "alternate symbolic expressions or graphical representations that a teacher uses to foster particular social norms for explanation and that generate student justifications for the validity of these alternatives," or promote guided reinvention (p. 389). Rasmussen and Marrongelle explain that using these pedagogical content tools (PCTs) effectively requires content knowledge, pedagogical knowledge, and expertise. It is no surprise that these PCTs are found in the instructional sequences of this study, because generating student ideas while moving forward the mathematical agenda is central to IOI. Rasmussen and Marrongelle found that teachers' savvy use of transformational records and generative alternatives in differential equations courses furthered mathematics in the classroom using student thinking and also developed meaningful mathematical discourse communities.

Similarly, Andrews-Larson, McCrackin, and Kasper (2019) used video data to analyze teachers' implementation of IOI in linear algebra classrooms for two consecutive years. These researchers claimed that one of the greatest challenges in implementing IOI for teachers was "making sense of and building on student reasoning" and sought to understand how teachers' instruction shifted through the use of IOI over time (p. 1). They found that, in general, teachers better elicited and built on student contributions, and classroom discussions became more "mathematically robust" over the course of a year (p. 1). They suggest that "shifts in one instructor's framing of mathematical discussion and coordination of social and analytic scaffolding fostered an increase in student contributions to mathematical arguments" (p. 1). Altogether, as the teachers in this study continued to practice IOI, they became more student focused and more skillfully elicited student thinking to build the mathematical agenda of their classrooms. From Andrews-Larson and colleagues work, it is clear that IOI requires a great deal of teacher attention and response to student thinking throughout instruction. This study also suggests that teachers implementing IOI can benefit from supports that help build their knowledge on students and teaching.

Wagner, Speer, and Rossa (2007) looked into the knowledge, other than content knowledge, required to implement IOI in a differential equations course. They wanted to understand what challenges mathematicians face when implementing IOI and what these challenges reveal about the knowledge needed for such instruction. To reach this understanding they followed the case of instructor Rossa as he implemented IOI in a differential equations course for the first time. They concluded that a teacher implementing IOI may face struggles in knowing how students will respond, guiding discussion, and knowing the extent of students' understanding. Moreover, to address these struggles, teachers had to attend to unfamiliar instructional goals of IOI centered around students. That is, the skills and teacher knowledge required to successfully implement IOI differs from knowledge developed in traditional mathematics instruction. Importantly, Wagner, Speer, and Rossa call for "effective support for teachers who desire to implement reform-minded practices in their classrooms," such as professional development, instructional materials, and collaboration between instructors and mathematics education researchers studying IOI (p. 265). Altogether, the challenges facing teachers implementing IOI include understanding students' informal thinking, using student ideas to advance the mathematical agenda, using effective mathematical representations, noticing student contributions, and, not to mention, content knowledge and other pedagogical content knowledge. These studies indicate the need for professional development to assist instructors in overcoming these challenges.

Reform-Oriented Instruction Teacher Supports

There is less literature on professional development for undergraduate mathematics instructors, specifically in the context of online work groups. However, the few studies that have been done on this topic pay particular attention to the type of progress that can be made in collaborative teacher professional development.

In a 2009 study, van Es looked at the "nature of teacher participation" in what she termed *video clubs* (p. 100). The *video clubs* were meetings of teachers for professional development in which the teachers shared and discussed videos of themselves teaching, much like this study's online work groups (OWGs). Van Es looked at roles such as prompter, proposer, builder, supporter, and critic, that teachers took on in the video clubs and how the teachers' participation in these roles progressed over time. The role of prompter was one that initiated discussion around certain issues; the role of proposer offered explanations or topics for discussion; the role of supporter often supported other teachers' ideas. The role of critic challenged others' claims in the video clubs, and the builder helped develop ideas. Van Es determined that the teachers learned to support the goals of the video club by changing in these roles over the course of the club meetings. The teachers grew to prompt more thought on student ideas and propose more contemplative approaches to understanding student ideas. They also used the studied videos to build on broader approaches to instruction and learned to critique in a way that focused more on student thinking and less on a certain teacher's actions in the classroom. Van Es's findings support the cause for professional development utilizing video clips of teachers' instruction. Moreover, she claims that "designing professional development with attention to both the content of teacher learning as well as teacher participation may show promise for helping teachers develop new practices for noticing student thinking and adopting a student-centered, responsive approach to mathematics instruction" (van Es, 2009, p. 130).

Similarly, in a 2019 study, Fortune and Keene looked at how a mathematician's instruction developed over the course of participating in an online faculty collaboration. The goal of this faculty collaboration was to support instructors in their understanding of and reform towards IOI. They analyzed the instructor's talk in the online faculty collaboration (OFC) by characterizing their talk as pedagogical, mathematics centered, student centered, or OFC centered. They then looked at what type of roles, active/passive and speaker/listener, the instructor played while talking in each of these categories. Fortune and Keene found that the mathematician's own view of mathematics and his determination to bring his students to this same understanding actually hindered inquiry in his classroom. The instructor's insistence of his own ideas did not align with one of the tenants of IOI, a focus on students' ideas (Fortune & Keene, 2019). Despite these findings, Fortune and Keene's results support that faculty collaborations can be helpful in supporting instructional change. Moreover, time and effort by participating teachers greatly impacts the degree of such instructional change.

In a 2002 review of the emergence of online professional development, Barnett stated, "if teachers are going to improve their practice, they need to have access to on-going, quality professional development that is situated in their everyday instructional environment that provides opportunities to communicate, collaborate, and reflect on their teaching" (p. 2). This is because, in reviewing the available literature on professional development in K-12 mathematics education, Barnett found that short-term or one-time professional development wasn't likely to result in long term change in participating teachers' instruction. Moreover, he found that consistent online resources "can reduce teacher isolation and support sharing… foster reflection on practice … [and] support the formation of communities of practice" (p. 2).

Studies have also shown the significance of the use of classroom video in professional development. In 2008, Borko and colleagues published a study on how middle school mathematics teachers' conversations around video studies grew over the course of a 2-year professional development program. They posited that "video can support collaborative learning focused on reflection, analysis, and consideration of alternative pedagogical strategies in the context of a shared common experience" (Borko, Jacobs, Eiteljorg, & Pittman, 2008, p. 419). Upon analyzing the two-year period, the researchers noted that the professional development group's conversation became more focused and in-depth. They suggest this was due to the growing professional development community and willingness to learn through the study of classroom videos. Participants of the study even commented that they grew most from recording and analyzing their own classroom videos. That is, the use of video lifted them to a new level of self-reflection not experienced in typical professional development.

Similarly, Sherin and Han conducted a study of four middle school mathematics teachers as they participated in a year-long video club. They open coded to find four main topics discussed by the video clubs over the course of the year: pedagogy, student conceptions, classroom discourse, and mathematics. They found that "discourse in the video clubs shifted from a primary focus on the teacher to increased attention to students' actions and ideas" (Sherin & Han, 2004, p. 163). Moreover, teachers' comments about their students' thinking became more nuanced and analytical to focus on pedagogical issues in response to student thinking.

Once again, these researchers evidence the role of the video club to enhance teacher reflection, promote pedagogical inquiry, and encourage “critical colleagueship” (p. 163). That is, the teachers established a healthy environment in which they would analyze each other’s videos and pose pedagogical solutions.

A Similar Study

Lastly, I review a study, also stemming from the TIMES project, that used the Instructional Triangle to analyze teachers’ reasoning with IOI. Andrews-Larson, Johnson, Peterson, and Keller (2019) characterized teachers’ reasoning on IOI through the vertices of the Instructional Triangle. They studied the responses of two break out groups of teachers discussing IOI in in-person, four-hour workshops on IOI at a national mathematical conference. Their aim was to study the “nature of mathematicians’ talk when asked to engage in mathematical problem-solving tasks designed to support student learning, to consider student reasoning, and to speculate about instructional choices in relation to mathematical goals of said tasks” (p. 3). These researchers found that the teachers engaged more when discussing the mathematics of certain tasks rather than the instruction of these tasks. They also proposed that “deep mathematical engagement is better achieved by asking participants to work through task sequences from a disciplinary perspective (their own mathematical lens) rather than from an instructional perspective” and “a productive way of engaging participants in students’ mathematical reasoning is to present empirical evidence that emphasizes partially formed student understandings rather than students’ presentations of their final solutions” (p. 27). Their analysis resulted in an emergent model “for supporting rich pedagogical thinking about IOI” (p. 24). This model demonstrates that teachers first engage with curriculum materials through their own mathematical activity and then engage their students’ mathematical activity which informs their pedagogical reasoning. This model of teacher mathematical activity, then student mathematical activity, then pedagogical reasoning helped guide my investigation of the OWG in this study.

Theoretical Framework

Given that the goal of inquiry-oriented instruction is to "provide students with opportunities to participate in the reinvention of important mathematical ideas through problem solving" and shift mathematical authority from instructors to students (Andrews-Larson et al., 2019, p. 1), discussion in the OWG about students' thinking was an important point of analysis. Andrews-Larson and colleagues describe inquiry-oriented instruction as complex because "as students inquire into the mathematics, instructors inquire into students' mathematical thinking so it can be leveraged as a resource for moving forward the development of the class's mathematics" (p. 1). Other studies also share this view that IOI is focused around instructors helping students formalize their own mathematical thoughts (Johnson, Caughman, Fredericks, & Gibson, 2013). Therefore, the teachers' pedagogical responses to student thinking was another important point of analysis.

The OWG of this study met for hour-long weekly meetings where a small group of teachers, TAAFU Fellows, would first address issues and concerns with implementing the IOI curriculum and then complete lesson studies. In these studies, the OWG "attended to the critical components of IOI – generating student ways of reasoning, building on student contributions, developing a shared understanding, and connecting to standard mathematical language and notation" (Johnson et al., 2020). The goal of this OWG was to support the TAAFU fellows in their understanding and implementation of IOI (Fortune & Keene, 2019).

The reviewed literature highlights understanding students' informal thinking, using student ideas to advance the mathematical agenda, using effective mathematical representations, noticing student contributions, content knowledge and pedagogical content knowledge as important in implementing IOI. Each of these components can be characterized as either math, student, or pedagogy focused. For this reason, in analyzing the teachers' OWG interactions, I first looked to the Instructional Triangle introduced by Cohen and Ball (1999) to study components of instruction. The vertices of the Instructional Triangle capture all important aspects of IOI and thus provide a useful way of analyzing and comparing the teachers' utterances in the OWGs. Moreover, the edges of the Instructional Triangle capture the intricacies of IOI that compound attention to multiple components at the same time in the classroom. The triangle consists of three vertices—mathematics, students, and instruction—and has edges connecting these vertices that represent the connections between all three. Cohen and colleagues (2003) described the Instructional Triangle as "a system of interactive mutual adjustment" (p. 138) and suggested that underlying factors of student achievement are not found solely in resource allocation but largely inside instruction that they define as "interactions among teachers and students around content, in environments" (p. 122). The edges of the Instructional Triangle pinpoint the different interactions between students and teachers of mathematics classrooms.

The mathematics vertex of the triangle includes the materials students engage in and how teachers engage with these materials themselves. The materials teachers use influence instruction by determining what type of role the instructor plays (i.e., active learning materials) and often require less lecture time from instructors and more small group work. In the context of our study, these materials are the activities of Larsen's (2013) Group and Isomorphism Lessons and Larsen and Lockwood's (2013) Quotient Group lessons, which will be detailed in the methods section. The mathematics of the instructional sequences was a crucial discussion point in all the meetings of the OWG. This is because the curriculum materials intentionally leverage intuitive thinking and higher reasoning that may not have been required in the prior ways these teachers taught

Abstract Algebra. That is, the materials demanded more mathematical activity of both students and teachers; therefore, it was necessary to engage the teachers in discussions that would bring light to the additional skills needed to implement IOI (Johnson, 2013; Johnson & Larson, 2012; JMTE, 2019) in the OWG meetings. Moreover, teachers' mathematical activity when engaging with these materials is suggested to support students' mathematical activity "indirectly in the sense that teachers' mathematical activity would inform their pedagogical activity" (Johnson, 2013, p. 762).

IOI holds the overwhelmingly popular belief that "mathematics education should take its point of departure primarily in mathematics as an activity" (Gravemeijer & Doorman, 1999, p. 116). Relatedly, in a study of teachers' mathematical activity in IOI, Johnson (2013) states, "as students engage in such mathematical activity, one would expect that teachers would need to engage in mathematical activity in response" (p. 762). Turning to the Instructional Triangle, Cohen and Ball (1999) described the significance of the student vertex of the triangle, by saying,

> Students' experiences, understandings, interests, commitments, and engagement are also crucial to instructional capacity. One way to consider the matter is that the resources that students bring influence what teachers can accomplish. Students bring experience, prior knowledge, and habits of mind, all of which influence how they apprehend, interpret, and respond to materials and teachers. (p. 3)

Unsurprisingly, "successful implementation of inquiry-oriented curricula requires a focus on students" (Johnson, 2013, p. 761). Inquiry-oriented instructors guide the progression of the course based on students' thinking (Kuster et al., 2017), so the resources students bring in terms of their own mathematical understandings and experiences have a major impact on the implementation of IOI. The student vertex of the Instructional Triangle captures an integral component of IOI in that it represents the interactive nature of teachers' and students' mathematical activity.

On the instruction vertex, Cohen and Ball (1999) stated, "Teachers' intellectual and personal resources influence instructional interactions by shaping how teachers apprehend, interpret, and respond to materials and students" (p. 3). Teachers' resources include their knowledge of students' understanding, ability to make connections to student thinking, and ability to establish desired classroom environments. Teachers implementing IOI particularly draw on these resources as they inquire into student thinking and leverage students' contributions in advancing the mathematical agenda of the lesson (e.g., Kuster et al., 2017).

In summary, I used the Instructional Triangle to guide the analysis of the teachers' OWG reflections and discussions. Using the Instructional Triangle framework allowed for several things to happen in data analysis. First, I was able to categorize the teachers' comments and reflections from the OWGs in terms of the mathematics, instruction, and students. This, in turn, allowed for comparisons to be made between teachers in terms of how much they were attending to each vertex and in what ways they are talking about them.

To describe how the teachers' interactions changed over time, I disaggregated the data by each vertex of the Instructional Triangle and teacher and studied how each teacher's attention to the different aspects of IOI changed over the course of the OWG. To study how each teacher's attention to the different vertices changed over time, I adopted van Es's (2009) framework of participation patterns through teacher roles. Van Es captures different levels of participation in video clubs (in this study, OWG) by looking at teacher roles, or "positions group members assume in order to participate," and how teachers participate in these roles to varying degrees over time (van Es, 2009, p. 103). These roles follow Andrews-Larson and colleagues' emergent model on IOI that links teachers' own mathematical engagement, their engagement with student ideas, and the resulting pedagogical reasoning. They present each of these topics as entry points for "rich pedagogical reasoning about IOI," the type of conversation that was a goal for the OWG of this study (Andrews-Larson et al., 2019, p. 24). In this way, I characterize teachers' attention to the vertices of the triangle, and resultantly the tenants of IOI, by looking at how they participated in different roles throughout the course of the OWG.

If a teacher shares an interaction in the OWG that falls on the student vertex, I am conceptualizing this as the teacher participating in the role of *inquirer* of student ideas, as they "routinely inquire into their students' mathematical thinking and reasoning" (Rasmussen & Kwon, 2007, p. 190). As the OWG progresses and the teacher shares more/less interactions on the student vertex, they can be seen as participating more/less in the role of inquirer, and hence attending more/less to the tenants of IOI that include engaging with student ideas. I align the role of *sense maker* (of mathematical content) to the mathematics vertex. When teachers play the role of sense maker of mathematics, this guides their own mathematical activity which is shown to support students' mathematical activity (Johnson, 2013). This is supported by the findings of Andrews-Larson and colleagues that teachers "explicitly drew on their own mathematical engagement with the tasks as they worked to make sense of student reasoning" (Andrews-Larson et al., 2019, p. 25). Lastly, I align the role of *builder* (of the IOI classroom objectives) to the instruction vertex. As builders, "teachers facilitate student engagement in meaningful tasks and mathematical activity related to an important mathematical point" (Kuster et al., 2019, p. 4). Moreover, an inquiry-oriented instructor "has the obligation of enculturating students into the discourse and conventional representational forms of the broader community while honoring and building on students' contributions." (Rasmussen & Marrongelle, 2006, p. 395). A primary focus of the inquiry-oriented instructor is moving the mathematical agenda forward by building on student ideas, thus pedagogy and instruction in IOI center on building on ideas presented in the classroom. Through this framework of teacher participation in roles determined by the Instructional Triangle and aligned to IOI, I can analyze how the teachers' talk on IOI changed throughout the semester. For the purposes of my analysis of teacher participation, these roles served as a way to align the teachers' participation to their attention to each vertex on the Instructional Triangle.

Context, Data, and Methodology

In order to give context for this study I will provide an overview of the mathematical content the teachers worked with and describe the atmosphere of their OWG. Then I will describe the data collected and methodology used to analyze this data. This analysis took place within the larger TIMES project, which was designed to support the implementation of three IOI curricula: IODE, IOAA, IOLA (TIMES; NFS Awards: #143195, #1431641, #1431393). The TIMES supports were: curricula material, summer workshops, and OWG. The instructional materials were task sequences with rationale examples of students' work and suggestions for implementation. The summer workshop was three days long with the goal of helping instructors "develop an understanding of IOI and their role as the teacher; develop a shared vision of instruction and student learning goals; and develop a familiarity with the curriculum materials, task sequences, and online resources" (Kuster, Johnson, Rupnow, & Wilhelm, 2019, p. 15). The OWG met for one hour a week throughout the following semester and utilized a modified Japanese Lesson Study format. Teachers would first discuss their progress through the curricula and ask any pressing questions about mathematics or guiding small group work in the classroom. Then they would anticipate how students may think through the material. Lastly, they would review clips of each other teaching in the classroom and discuss what aspects of IOI were carried out well or could be improved. The OWG focused on two units in the IOAA curriculum: Isomorphism (Larsen, 2013) and Quotient groups (Larsen & Lockwood, 2013). First, I detail the isomorphism and quotient group lessons and the supporting theory of RME. Both local instructional theories followed the theory of RME that seeks to "allow learners to come to regard the knowledge that they acquire as their own private knowledge, knowledge for which they themselves are responsible" (Larsen, 2013, p. 2). They specifically used the frameworks of emergent models (Gravemeijer & Doorman, 1999) and proofs and refutations (Larsen & Zandieh, 2007). Emergent models hinge on developing *models-of* students' informal thinking that later transform into *models-for* more formal mathematical reasoning. Proofs and refutations (Larsen & Zandieh, 2007) frames the classroom as a place where students prove and disprove conjectures (either proposed by students or the teacher) to develop a shared understanding (Kuster, et al., 2017) of concepts. The isomorphism instructional sequence was developed through a series of design experiments (Larsen, 2013). After the isomorphism tasks were initially designed, in a cyclic way, data was collected on students engaging in the tasks, the data was analyzed to study students' mathematical activity, and the tasks were accordingly refined. The instructional sequence on quotient groups was also revised through a series of similar design experiments (Larsen & Lockwood, 2013) that included an initial design experiment with two students engaging in preliminary instruction, an experimental teaching in which the materials were first used in a full classroom, and a whole class teaching experiment where data was collected as a class engaged with the instructional materials.

The local instructional theory on isomorphism begins by having students "identify, describe, and symbolize the set of symmetries of (usually) an equilateral triangle" (p. 715). The resulting group structure is seen as a model-of students' mathematical activity. After this, students explore how to compute combinations of the symmetries of the triangle and begin to reinvent group axioms. The students then refine these axioms by considering other symmetric groups. The use of operation tables in this step helps transition from a model-of the students' thinking to a model-for more formal group notation. The goal of this process is for students to develop an understanding of the definition of a group and group properties. Next students are challenged to form a more abstract understanding of groups through the concept of isomorphism. They are given a "mystery table" and asked if this could represent the operation table for the symmetries of an equilateral triangle. Students' matchings of the elements between two tables can be seen as a model-of their understanding that progresses to a model-for (Gravemeijer & Doorman, 1999) the formal mathematics when they identify the properties necessary for isomorphism.

Students begin the quotient group instructional sequence by reinventing quotient groups based on their intuitive understanding of parity. Ultimately, they come to understand how subsets of a group can form another group whose elements are subsets. They discover how these subsets are formed using a subgroup of the original group, how they operate with each other, and the conditions of this subgroup that cause the subsets to form a quotient group. This local instructional theory developed by Larsen and Lockwood (2013) aims to help students reinvent these ideas using their preexisting knowledge of groups and sets, similar to how they used their knowledge of the symmetries of a triangle to reinvent the group and isomorphism concepts.

To this aim, teachers use these instructional materials to prompt students to reinvent the idea of cosets of a group by considering subsets defined by parity in the group D_8. Larsen and Lockwood (2013) focused this task on D_8, the group of symmetries of the square, because previous research has identified that students can leverage the idea of parity by considering symmetries of objects, and the previous isomorphism instructional sequence in the IOAA curriculum also leverage students' understanding of symmetry. In Lesson 1 of the quotient group materials, students are asked to find similar subsets to the even/odd group—the groups whose elements are "even" and "odd" and operate on each other as one might think of even and odd numbers operating on each other through addition—in the group of symmetries of the square, D_8. Here, it is expected that students use the parity structure in D_8 as a model-of partitioning groups that develops into a more general model-for partitioning groups by cosets. The goal is for students to "generalize and abstract the notion of parity" (p. 5). Larsen and Lockwood (2013) described this task as the first step of reinventing the quotient group concept through the generalization of the even/odd group, that is, moving from a model-for to a model-of viewpoint (Gravemeijer & Doorman, 1999). In a later task, students are challenged to start conjecturing necessary conditions for the partitions of the group to successfully form groups of subsets. This is an instance of Larsen and Zandieh's (2007) framework of proofs and refutations being used to move students toward a shared understanding by proving or disproving their own conjectures about quotient groups.

As part of the enhancement of the TAAFU curriculum, Johnson, Larsen, and Lockwood (2013) developed web-based instructor support materials to help instructors enact the curriculum in an IOI setting. The materials are found on an "interactive website that provides instructors with a number of resources to help them implement the curriculum effectively and faithfully." (Johnson, Larsen, & Lockwood, 2013, p. 777). The goal was to provide any resources that teachers may find helpful in enacting the tasks without overwhelming them with information. The instructor support materials provide explanations on rationale, student thinking, and implementation for each task in the TAAFU curriculum.

The OWG was designed to provide support for the instructors through discussion and reflection on the curriculum materials and the instructors' implementation of IOI. Participants were routinely asked to comment on aspects of their instruction that went well or needed improvement. These OWG sessions were recorded and transcribed for retrospective analysis. I focused analysis on the OWG meetings in which the instructors discussed their plans for and their implementations of the curriculum materials of the isomorphism and quotient group units of the TAAFU curriculum. The instructors in this study were introduced to these materials in the previously described summer workshop, and they had access to them as they taught throughout the semester. The two instructors that I analyzed here are Dr. Bradley and Dr. Jackson. Both instructors taught Abstract Algebra courses at bachelor's-granting, small, four-year institutions. This was their first experience in implementing IOI, though they had experience teaching Abstract Algebra. Dr. Bradley and Dr. Jackson participated in the same OWG, which consisted of five instructors and a facilitator who met weekly over the course of the semester. I chose to focus on Dr. Bradley and Dr. Jackson because they both consistently participated in the OWG conversations and had differing IOI implementation scores in a prior analysis. This made these two instructors a nice representation of the character of the whole OWG and an interesting point of analysis on the reflection of these teachers' IOI.

I first used the vertices of the Instructional Triangle (Cohen & Ball, 1999) to code instances in the transcript in which the instructors' comments focused on mathematics, students, or instruction. I used turns of talk as the primary unit of analysis; thus, if a participant talked about students' reasoning in one turn of talk, this would be coded on the student vertex. If a participant talked about their students' reasoning and then proposed a change in their instruction to better support students, this would be coded along the edge between students and instruction. Then, I disaggregated the data by grouping together all of the coded transcripts according to each code and instructor. Within each of these groups of coded transcripts, I drew comparisons between Dr. Bradley's and Dr. Jackson's comments for each vertex of the Instructional Triangle. In an additional layer of analysis, I looked at how Dr. Bradley and Dr. Jackson's participation in the roles of sense maker, inquirer, and builder changed over the course of the semester to study the development of the OWG culture. I wrote analytic memos (Maxwell, 2013), through which I analyzed the similarities and differences in the instructors' comments on their plan for or reflection on their instruction throughout the semester. In writing these memos, I focused on the nature of the teachers' discussion and how it changed along each vertex and role with respect to time in the OWG. I describe these similarities and differences between the instructors in the following section.

The Math Vertex

An example of interaction characterized by the math vertex is found in the beginning of the isomorphism unit when the teachers are discussing different ways to find isomorphisms between the group of triangle symmetries and D_6. One teacher posed a question to Dr. Bradley, asking how he would find the different isomorphic mappings between the two groups. Dr. Bradley responded by describing how he would view the physical triangle from different perspectives.

> "I was thinking you could think about it with rotations for instance… alright if I'm looking at something that's being turned counterclockwise, and I want to call that R, if someone else is facing on the opposite side of the triangle then the other one is going to be the counterclockwise rotation there… So that explains the arbitrariness there and you can just turn your head to get the same thing with flips"

He also pointed out that the orders of elements in the two groups (three of order three and one of order two) cause the isomorphic mappings to not be entirely arbitrary. I place this interaction on the math vertex because Dr. Bradley is focusing on the details of the isomorphism unit from his own mathematical perspective. Dr. Bradley's way of viewing isomorphisms of the triangle symmetries group as simply different perspectives of the group itself shows his own abstract notion of isomorphism. This is also an example of Dr. Bradley participating in the role of sense maker for the OWG. He was providing an alternative way to consider the mathematics that may help others' mathematical activity.

The Student Vertex

On the student vertex, I place interactions where Dr. Bradley and Dr. Jackson are discussing *how* students reason or engage with the instructional materials and each other. Here is a time when Dr. Jackson described how his students interpreted a problem asking them to prove an aspect of the sudoku property for the group of D_6.

> "The problem was that when the students got the assignment, part two, they saw the words 'prove that each symmetry must appear at least once in each row,' and then they saw 'suppose that symmetry b appears in row for symmetry a' and they basically said, 'Why do we need to prove this? b is already supposed to be there, so we're done…' and they would not let go of the idea that they had finished the problem before they had started."

Dr. Jackson went on to explain to the OWG that his students tended to assume the conclusion when attempting to prove statements. Dr. Jackson's reflection on how the students interpreted a problem's wording and how this gave them difficulty throughout the lesson showed his attention to how his students reasoned through problems. Dr. Jackson's attention to his students' thinking showed his participation in the role of inquirer, especially when he conjectured that his students often assumed the conclusion in proving problems.

The Instruction Vertex

An example of talk characterized by the instruction vertex is found when Dr. Jackson discussed how he would like to present different groups to his students throughout the semester.

> "Since it's not clear in the notes as I go through them which groups are actually covered [in the instructional materials], I wanted to know which ones I should pay special attention to… so like when I'm creating the homework, I want to know which ones should… if I don't see them in the lessons I should try to put them in the homework so they actually see these problems."

Here, Dr. Jackson was looking at how he could present material so that by the end of the semester his students have been exposed to the major groups typically covered in an introductory Abstract Algebra course. Dr. Jackson's care to plan out the semester in this way showed his attention to the planning aspect of instruction, so I categorized this interaction in the OWG to be on the instruction vertex. Dr. Jackson's participation in the role of builder is also clear here, because he is attending to the mathematical trajectory of his classroom through his planning.

The Edges

A turn of talk characterized by more than one vertex is found on the edge of the triangle in between two vertices. Dr. Bradley showed an interaction falling on the edge connecting the instruction and student vertices when he considered altering his instruction to better accommodate his students' understanding of well-definedness.

> "Mirroring Friday's email conversation, I am wondering about how to better introduce the coset notation and how it relates to the original idea of subsets. They [the students] are good with writing $\{r, 3r\} = \{I, 2r\} + r = h + r$ and $\{r, 3r\} + \{f, f + 2r\}$is clearly

well-defined. However, I am not sure if I made it clear why we need to check to see [if] $(h + r) + (h + f)$is well-defined."

Here, Dr. Bradley is reflecting on how his students understand the idea of well-defined operations differently when using different notation, and considering how he might change his instructional decisions to best address this. By coordinating thought around both students and instruction, Dr. Bradley was simultaneously participating in the roles of builder and inquirer.

Results

Now, I give an account of Dr. Bradley and Dr. Jackson's interactions in the OWG sessions, categorized by the vertices of the Instructional Triangle. I describe the roles each teacher played side-by-side to best illustrate their participation in these roles and how they contrasted with each other over time. I describe overarching progressions and themes of each teacher's participation in the roles found on the math, students, and instruction vertices in the discussion following the results. Figure 1, below, shows the distribution of each teacher's talk turns first in the OWGs focused on the isomorphism instructional unit, and then in the OWGs focused on the quotient group instructional unit. The movement of these talk turns on the triangle over time will be expanded on in the results and discussion sections.

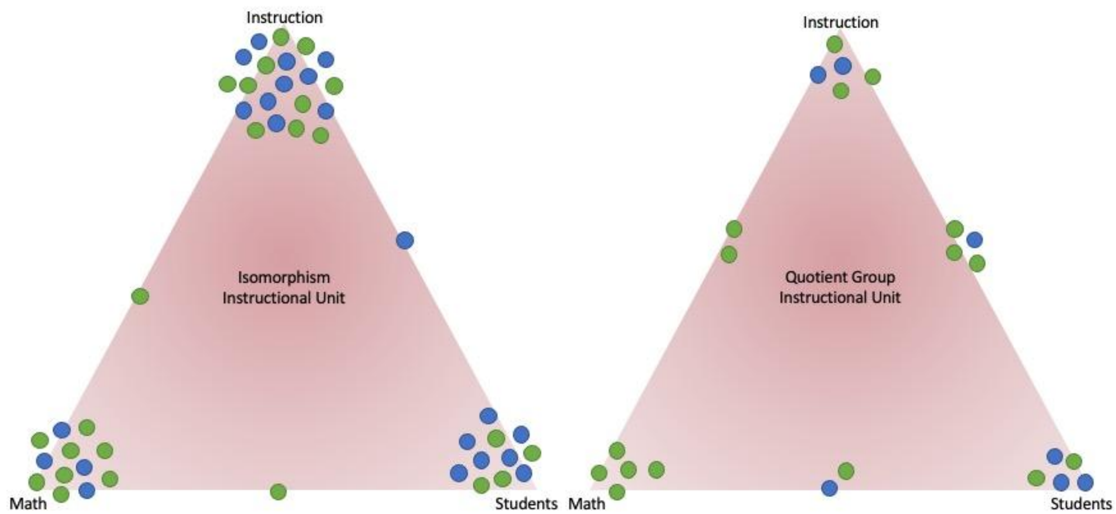

Figure 1. Distribution of Talk Turns
Dr. Bradley: green, Dr. Jackson: blue

Analysis of the Isomorphism Instructional Unit

The Math Vertex

Recall that the OWG sessions were structured around the modified Japanese Lesson Studies that started with participants working through the mathematics of the instructional units, then anticipating student thinking on the tasks, and then reviewing clips of each other's teaching. When first talking about the isomorphism unit with other teachers in the OWG, Dr. Bradley explained that he would do a "quick check" of the identity and inverses to see if a set of particular elements formed a group. In this situation, Dr. Bradley was confidently responding to another teacher's question on how he would begin the problem of the isomorphism unit that asked students to find an isomorphic mapping between the mystery group and the group of triangle symmetries. Here, Dr. Bradley was playing the role of sense maker for the OWG. His mathematical reasoning to first determine the identity and inverse relationships in a set of elements also mirrored the desired student reasoning for the task.

On the same task, Dr. Jackson made a comment about the possibility of finding multiple isomorphic mappings. He said, "I think this example is too small for there to be any problems… as in, I don't think you could choose wrong." When the OWG facilitator prompted Dr. Jackson to explain his reasoning, he described how the choice of one element arbitrarily mapped to another would then "force" the mapping of the other elements in the set. However, he then questioned himself saying, "… maybe I'm off base." When the facilitator continued to inquire into Dr. Jackson's reasoning, he confidently responded using examples of different mappings that worked as isomorphisms, based on his arbitrary choice of a single element. In fact, it is possible to "choose wrong" for an isomorphic mapping between the two groups, and as Dr. Jackson explained his reasoning, he was relying on the relationships between elements of one group to map the elements of the other group. That is, he was inadvertently using the homomorphism or operation preservation property. Here, Dr. Jackson played the role of sense maker when he first commented on the (albeit, incorrect) arbitrariness of mappings and also when he explained his thinking to the group. Similar to Dr. Bradley, Dr. Jackson is mirroring a possible approach that students may take to this problem. Dr. Bradley also responded when the facilitator asked the rest of the OWG how they would approach the problem saying, "well once you make a choice you need to make sure that the multiplications completely correspond… and you can either do that by checking (Dr. Bradley holds up to the camera a piece of paper with the homomorphism property written on it) … or I would probably prefer to just see if the (operation) tables can be overlaid onto each other in a way that makes sense…" As the facilitator continued to inquire into the teachers' reasoning, both Dr. Jackson and Dr. Bradley questioned whether their methods were productive ways to choose isomorphic mappings between the two groups. In this probing, each teacher challenged their own mathematical activity. This type of participation as sense makers of mathematics in the OWG can be seen as useful in preparing the teachers for the type of ideas students possibly had when first engaging with these tasks in the classroom.

In a following OWG session, one teacher asked the group how they would approach the idea of closure when introducing groups. They were unsure of choosing to approach closure as a condition or axiom of algebraic groups. Dr. Bradley responded stating, "I would consider closure an axiom" and justified this by saying, "my reasons are largely pedagogical I suppose… it pays off when you get to subgroups." Here, Dr. Bradley was alluding to the fact that subgroups are subsets of elements of a group that are also closed under the group's operation. Again, Dr. Bradley was offering a solution to the group and made sense of his choice by connecting it to decisions he planned to make in later instruction. This particular interaction can be characterized by the edge of the Instructional Triangle between mathematics and instruction because Dr. Bradley played the roles of both sense maker and builder of future mathematical ideas his students would need to understand subgroups.

The Student Vertex

Now I turn to the teachers' participation in the role of inquirer. When Dr. Jackson reflected on how his students approached a task in the beginning of the isomorphism unit he said,

> "The problem was that when the students got the assignment, part two, they saw the words 'prove that each symmetry must appear at least once in each row' and then they saw 'suppose that symmetry B appears in row for symmetry A' and they basically said, 'why do we need to prove this? B is already supposed to be there. So, we're done.' And they would not let go of the idea that they had finished the problem before they had started…"

Here Dr. Jackson noticed that his students were interpreting the task in a way that stymied their progress. He went on to talk about another instance when his students seemed to assume the conclusion in a proving task in the same unit, "recently they actually had another issue that was very similar, so maybe it is my students." Dr. Jackson described that his student had trouble "showing that there aren't two inverses," and how "they were very tripped up at the fact that … they wanted to use inverses to prove inverses, and I was like, 'you can't do that.'" Dr. Jackson was able to notice that his students were having difficulty in proof problems because of the way they started problems. This is important because he played the role of inquirer in this consideration of his students' reasoning. Recall that the IOI is driven by students' engagements and ideas, so Dr. Jackson's attention to his students' difficulties sought to inform his pedagogical decisions.

Dr. Jackson again played the role of inquirer into student thinking when the facilitator asked how the teachers thought students would approach task one in the isomorphism unit. He suggested that his students may "latch onto" the actions of flip and rotate, and use these to reason throughout the rest of the lesson. Dr. Bradley also responded to this question by saying, "I thought that they might look at the identity first." Both Dr. Jackson and Dr. Bradley participate in the role of inquirer preemptively in the OWG and thus prepare themselves to better engage with student reasoning in the classroom, a key component to successful IOI. When describing his students' understanding of proofs, Dr. Jackson seemed to take a negative viewpoint, stating that they "always" assumed the conclusion when starting proofs. Contrastingly, Dr. Bradley seemed more hopeful about his students learning to read and write proofs better saying, "I definitely think it's possible," about his students' potential success as proof writers.

Contrastingly, Dr. Jackson *did* seem to take a positive viewpoint when he described why he chose to present a certain clip of his teaching for the OWG to study. Recall, part of the OWG sessions focused on the teachers watching clips of each other in the classroom and discussing what happened. Describing how his students engaged in a task, Dr. Jackson said,

> "Students really generated all of the information with each other… if you looked very carefully at the group sitting down, they were discussing amongst themselves… and you could see where they come up with the question through generating their understanding… and they have shared understanding by coming up with a conclusion together."

Here, Dr. Jackson was relating what his students did to the tenant of IOI to develop a shared understanding.

The Instruction Vertex

Lastly, I look to the teachers' participation as builders on the instruction vertex. At the beginning of the discussion on the isomorphism unit, Dr. Jackson says,

> "Since it's not clear in the notes as I go through them which groups are actually covered, I wanted to know which ones I should pay special attention to… so like when I'm creating the homework I want to know which ones should… if I don't see them in the lessons I should try to put them in the homework so they actually see these problems."

This is a clear example of Dr. Jackson participating in the role of builder of the mathematical agenda in the classroom. He was making sure that his students had the opportunity to work with the main groups introduced in a typical introductory abstract algebra course. This was an example of Dr. Jackson attending to the IOI practice of connecting to standard mathematical language and notation (Kuster et al., 2017). In a similar act of proactive building, Dr. Bradley shared with the group how he must pay particular attention to the sequencing of his lessons and prepare for each class session.

Dr. Jackson also provided feedback to another teacher on how to pull in work her students had already completed in a worksheet to help them build understanding in a new classroom task. This was an example of helping students come to regard acquired knowledge as their own, by building from their previous work (Gravemeijer & Doorman, 1999). Similarly, when the facilitator asked how the teachers planned to introduce the isomorphism properties of one to one and onto to their students, Dr. Jackson suggested, "you could ask about the sudoku property… 'what does that mean in terms of a function?'" This is another example of using students' previous work to inform their reasoning for new ideas. Another teacher in the group reflected on how her students were connecting the ideas of one-to-one and onto functions to isomorphism and deciding if these properties were sufficient to make an isomorphic mapping. Dr. Jackson asked, "did any groups actually say like, 'I think that this is enough.' Like the one-to-one and onto?" When the other teacher replied by describing how her students thought about the properties of isomorphisms, Dr. Jackson said,

> "I think that's interesting because you did spend more time on it (the requirements of isomorphism), and they came to the conclusion that it wasn't enough. Whereas, with my students, we didn't go through this discussion beforehand. We had to go through it when they came to a point where they were like, 'yeah, one-to-one and onto, that's enough'… so they went through that conversation then rather than before."

Dr. Jackson was reflecting on how the other teacher and he chose to build the same mathematical ideas at different points of their lessons. He said to the other teacher, "I think it's interesting that you took a different path but eventually got to the same spot." In the same conversation, Dr. Bradley questions, "I was wondering how often they had heard the word isomorphism or isomorphism… were they comfortable with that idea at this point? It seemed like you were trying to tie their ideas to formal mathematical notation." Here, Dr. Bradley was attending to the idea of IOI of formalizing students' mathematical ideas with standard notations. This was an example where both Dr. Jackson and Dr. Bradley participated in the role of builders in the same conversation, while attending to different tenants of IOI, advancing the mathematical agenda of the classroom and formalizing students' mathematical ideas.

Dr. Bradley reflected on the types of problems he assigned his students saying, "I don't think it's a huge problem to not have them doing much difficult work right now…" and went on to explain that he wanted his students to have the expectation that problems would get more difficult later in the course. Here, Dr. Bradley was attending to the progression of the work he assigned and how this would eventually advance the mathematical agenda of the classroom, so I characterized this interaction by the role of builder. He went on to describe that when he was deciding how to introduce certain groups and material he was thinking about the "level of the class, level of the students, and department expectations," which was in line with the idea of planning his instruction with his students' understanding in mind.

While Dr. Jackson seemed confident in his instructional decisions and their impact of developing a shared understanding in the classroom, Dr. Bradley seemed unsure about how to know whether his entire class had a similar understanding. He asked Dr. Jackson after viewing a clip of Dr. Jackson's class, "At what point did you help them try to develop a shared understanding… because I need help. I'm trying to work that out still." Both teachers reflected on how their tests built the mathematical agenda of the classrooms. Dr. Bradley noticed in comparing his tests to tests from other semesters that they were covering much less material than usual. He proposed a solution to supplement the materials that were covered by assigning videos for his students to watch. Dr. Jackson actually commented that his exams may have included too much material. He said his latest test scores "could have been better," and decided to try a redemption system to encourage his students to review the missed material. He also reflected, "The problem wasn't a matter of their knowledge… it was actually time. For the ones they did do and do completely, they were actually flawless." Note that the teachers' discussions on tests have been characterized by each of the vertices. This is because their comments on tests either concerned what materials to include (math vertex), how students responded to tests (student vertex), or how to best structure and time tests within the instructional units (instruction vertex).

Analysis of the Quotient Group Instructional Sequence

The Math Vertex

Next I turn to Dr. Bradley and Dr. Jackson's interactions in the OWG focused on the quotient group instructional sequence. In response to the facilitator's question about how the teachers were feeling about beginning the quotient group material with their students, Dr. Bradley said, "I feel fine, it's the sort of thing that you just need to get, to get in the mud I think and see what happens." Dr. Bradley's response suggests that he made sense of the quotient group materials and felt confident about his own mathematical activity on the topic. Looking at both Dr. Bradley and Dr. Jackson's interactions around quotient groups that fall on the math vertex, each teacher showed concern about implementing specific parts or elements of the curriculum as they related to mathematical ideas. Dr. Bradley said he would "like to think more about" the quotient group material in the OWG, and similarly, Dr. Jackson stated, "I have many questions about the implementation of the materials regarding the quotient group unit." Recall from the methods section that the instructional sequence introducing quotient groups varies from the typical introduction to quotient groups found in most abstract algebra courses. Here, the teachers were showing a desire to make sense of the instructional materials for themselves, before presenting them to students.

Dr. Bradley took his thoughts about the quotient group lessons further by looking at the sequencing of the lessons. About this he said, "I need to figure how the tasks in Lessons 2.5, 3, and 4 interact. It seems like there is a lot of overlap between lessons 2.5 and 3 in particular." Dr. Bradley's following discussion in the OWG on this sequencing suggested that he was concerned about facilitating students' engagement with the tasks by wondering if the sequencing of the lesson worked best for students because it almost "starts over" at a base level to develop the new concept of quotient groups. His comments focused more on how the students would experience the curriculum and suggested that he was acting as sense maker of the material sequencing to best build on student ideas. Here, Dr. Bradley illustrated his attention to facilitating student engagement in meaningful mathematical tasks and advancing the mathematical agenda, which places this interaction on the edge of the Instructional Triangle between mathematics and instruction. Dr. Bradley also demonstrated a focus on guiding and advancing the mathematical agenda of the lesson in discussing the implementation of the materials and in describing his rationale for recording a student's ideas so the students could use, for instance in a specific case, the "modified Claire method" of solving a certain problem, named for a student Claire in the class who had originally proposed it. In contrast, Dr. Jackson did not exhibit a focus on these IOI practices, but that may likely be because he only made one comment about the implementation of the materials in the OWG. Altogether, Dr. Bradley's participation in the role of sense maker showed more attention to the actual quotient group material.

Both teachers were also concerned with making appropriate exams covering the quotient group material. Dr. Bradley said, "I also realize that it is difficult for me to make a quiz/exam too easy." Likewise, Dr. Jackson asked the group for "any suggestions for how to make a more appropriate exam." This focus on making suitable exams for their students centered on the IOI tenant of engaging students in meaningful tasks. I categorized these concerns as being "math" instances on the Instructional Triangle because they are situated in reference to specific mathematical topics in the context of instruction and students. It's interesting to note that at this later point in the OWG sessions, both instructors were playing the role of sense maker, however they focused more on presenting their students with tasks that made sense in the broader sense of the instructional sequence. As sense makers, both instructors were concerned with how best to organize and present the materials to their students in reference to the mathematical progression of concepts.

The Student Vertex

Looking at Dr. Bradley and Dr. Jackson's comments around the quotient group unit on the student vertex of the Instructional Triangle, there is again evidence of similarities and differences. Both teachers seemed excited about their students' progress on understanding quotient groups. Dr. Bradley said, "My students are also generating great conjectures about quotient groups. They volunteered the motivation for the definition of coset basically on the first day." Dr. Jackson similarly stated, "Students are understanding quotient groups. Though we just barely finished the definition of quotient groups, we are starting on the right foot." This attention to student progress is similar to the IOI practice of teachers formalizing ideas after students reach an understanding of the concepts. Both teachers also spoke of a disconnect between the mathematical and observed progress of students in the class and evidence that they are ready for formative assessments. For example, Dr. Jackson spoke of the negative effect a prior test had created in his students' class experience, saying, "My last test punched my students in the face." Similarly, Dr. Bradley spoke of the challenge he faced in decreasing the difficulty of a test, although he recognized the need to do so in order to be responsive to the academic diversity in his students, i.e., to allow the "C"-students to be successful. Both teachers spoke of the connections between mathematical activity and student progress, which seemed to suggest the notion of productive struggle, as seen in other literature (e.g., Warshauer, 2015). Here, Dr. Bradley's and Dr. Jackson's comments showed strong participation in the role of inquirer. They also both discussed students' progress in the class, but they did so in very different ways, which I explicate next.

The data showed a difference in the instructors' perspectives about students' mathematical ability. Dr. Bradley's comments demonstrated a positive outlook on students' ability, whereas Dr. Jackson exhibited a negative outlook. Dr. Bradley discussed how students were generating great conjectures and how they were good at using the correct formal notation. He took this discussion further by supposing that students could have benefitted more without struggling through the idea of well-definedness, and he presented the idea that students who worked through the IOAA quotient group materials may perform better on tests covering these topics than students who used a different set of learning materials, by saying, for instance, "I do think there are gains, but the gains don't manifest themselves on exactly the same problems." Here, Dr. Bradley was critically thinking about the benefits of engaging in the quotient group materials for his students. He also mentioned gains in his students' performance on the test. Here, Dr. Bradley seemed to see students' contributions and understandings as assets to leverage toward the mathematical agenda. He saw students' conjectures as useful for motivating the concept of quotient groups. This illustrates his reflection on the IOI tenant of being responsive to student thinking and using student contributions to inform the lesson.

Meanwhile, Dr. Jackson's language had a negative connotation in saying the students "slogged through the well-definedness stuff for a bit with little to no progress." He further exhibited a negative outlook on the students' abilities by saying, "my last exam punched my students in the face." Having a negative view of students' abilities seems counterproductive to the IOI practice of using student contributions to inform the lesson. This shows that, though Dr. Jackson was participating in the role of inquirer by considering student ideas, he often did so in a way that did not seem productive in generating student ideas.

The Instruction Vertex

Looking at Dr. Bradley's and Dr. Jackson's interactions around the quotient group material coded on the instruction vertex, I saw similarities in how both teachers were mindful of maintaining the mathematical agenda of the classroom while also adapting to the progress they saw students making. Dr. Bradley showed the intent to adapt his course schedule, stating, "I have been facing issues similar to Dr. Thompson and Dr. Jackson. I need to take a day on Friday just catching up with the grading system and some general ideas on how to prove things." This attention to the mathematical agenda was evidence of Dr. Jackson participating in the role of builder.

Also, both instructors demonstrated care for their students' experiences and success in the classroom. They both discussed ways in which they could engage students in each other's reasoning. For example, Dr. Bradley detailed an instance in class where he had a student rephrase her mathematical thought to the class, "My heart kinda broke when I said, 'Oh, Britt can you re-explain this?' I just thought that that was a missed opportunity. I could have done something else… other than [ask] Britt, I could have asked someone on her team or something." Here, Dr. Bradley was using Britt's contribution in his lesson and also reflecting on how he could have used her ideas to engage other students in the lesson by having them rephrase her thoughts. This use of students' contributions is also vital to IOI (Kuster et al., 2018). Dr. Bradley seemed reflective of how he was leveraging student thinking and engaging other students in each other's thinking, which aligns with the IOI practice of engaging students in each other's reasoning. Also, his reflection on his engagement with students' thinking seemed to align with the IOI tenants of being responsive to student contributions and using them to inform the lesson and guide the mathematical agenda of the lesson. Dr. Jackson also focused some of his explanations on engaging students in other students' thinking. For instance, he suggested an instructional approach of having students "break off into pairs and one of the two has to explain what just transpired to the other person." Thus, they both demonstrated attention to building on student thinking, though Dr. Bradley did this more often than Dr. Jackson.

While Dr. Jackson and Dr. Bradley share the similarities of self-reflection and intrinsic investment in the growth of their personal instructional practices, there are differences that separate their participation in the role of builder. Besides making comments more often than Dr. Jackson, Dr. Bradley offered more in terms of the criticality of himself. In particular, Dr. Bradley mentioned several instances of emotional responses he had to something in class or to his personal intentions for the students' experiences. His comments such as, "this is due to subtleties that I am only now recognizing," indicate his foregrounding of his own responsibility in creating and facilitating the environment which best supports them. Dr. Jackson also shared a similar reflection, but overall he was more oriented towards what the students *did* or *would do*.

Dr. Bradley seemed concerned about formalizing students' mathematical thinking in a way that stays true to the discipline of mathematics and true to students' thinking. He shared, "I am wondering about how to better introduce the coset notation and how it relates to the original idea of subsets," which may be evidence of him basing his instructional decision making on his own understanding of mathematics. This is also related to the instruction vertex in that it addressed the instructor's intellectual resources that influence instruction. Contrastingly, Dr. Jackson's statements about his instruction seem less reflective. Dr. Jackson talked about showing the students proofs after the students "slogged through" the material with little to no progress. Rather than using his mathematical knowledge to notate students' thinking as Dr. Bradley did, he instead used his mathematical knowledge to simply explain the material to students and show them how to write proofs. Just showing students how to write proofs seems counterproductive to him adhering to the principles of IOI. By contrast with Dr. Bradley, Dr. Jackson's remarks more clearly indicated that he was thinking about the character of the class's temporal occupation, rather than his specific involvement in it, but referenced specific phenomena using more clinical language.

The Edges

It is important to note that the OWG discussion was more often characterized by the edges of the instructional triangle as the OWG progressed. That is, the teachers showed more coordination of participation in multiple roles at once over time. This can be seen in the convergence of the dots to the edges of the triangle over time in Figure 1. In the beginning of the semester, when focusing on the isomorphism instructional unit, Dr. Bradley showed participation along two edges of the instructional triangle. The facilitator questioned how the teachers planned to introduce the more formal notation of operation preservation to students, and Dr. Bradley responded, "Well my best thought is that you should tell them how you overlay the two (operation) tables, you know naturally and they match up. And force them to write things out in function notation… and then they notice that (the operation must be preserved between both groups)." Here, Dr. Bradley played the role of both sense maker and inquirer, on the edge of the Instructional Triangle between math and students. This is because he was considering the best representation of operation preservation, and then conjecturing how his students would notice this relationship between two groups' operation tables. Dr. Bradley had another interaction on the edge of the Instructional Triangle when he talked about which groups were advantageous to introduce to students. "My opinion is that the quaternions (groups) are really nice because they're a great source of counterexamples… they don't work like the way the other common examples are… I put quaternions in my first homework." Here Dr. Bradley was making sense of the attributes of quaternion groups through his own mathematical activity and also considering how he could build on the mathematical agenda of the classroom by including them in homework so that his students were familiar with these groups when they became useful in later lessons.

Dr. Jackson also showed the coordination of roles while discussing the isomorphism instructional unit. In an instance of Dr. Jackson explaining how his students "assumed the conclusion was correct," he seemed frustrated with his class's proving skills and said, "they have no intuition of what is correct anymore." The OWG responded by offering ideas to help develop

Dr. Jackson's students' reasoning with proofs. He later questioned, "would it have helped to lower the level of abstraction a little bit and say instead of trying to prove that Z has an inverse or 5Z start with 35 and start with the question 'why is 35 in 5Z?'" In this instance, Dr. Jackson both reflected on his students' reasoning and inquired into ways to build their reasoning in future lessons through changes in instruction. This positions Dr. Jackson as both inquirer and builder of the mathematical agenda, represented by the edge between students and instruction on the Instructional Triangle.

In another example of reflecting on his sequencing, Dr. Bradley shared with the group that he planned to spend a few class sessions reviewing proving skills with his students because they were struggling to write proofs in their homework. Here, Dr. Bradley was participating in the role of builder, by thinking about future instructional changes, and also the role of inquirer by interpreting how his students performed on their homework and using this to guide his decision making. In a later OWG, when Dr. Bradley was asked how his classroom time dedicated to proving skills went, he said, "It was fine but I don't know if it was worth the class time given that they ended up having reasonable proofs." He then brainstormed ways to do this review more time-efficiently in the next semester.

Recall, when discussing how their students performed on tests covering the quotient group material, both Dr. Bradley and Dr. Jackson considered how they might adapt tests to better support their students' understanding. Here, they illustrated a connection between the instruction vertex and the student vertex of the Instructional Triangle in that the teachers based their pedagogical decision making about tests on their interpretations of students' experiences. In this way again, they acted as both inquirers and builders.

Discussion

Now, I will describe some similarities and differences between Dr. Bradley and Dr. Jackson's interactions and each vertex and their change in each role as the OWG progressed. I found several interesting distinctions between Dr. Bradley and Dr. Jackson's participation in the OWG.

When analyzing the results on the math vertex, I found differences between the teachers and also changes in each teacher's participation in the role of sense maker over time. Altogether, Dr. Bradley had more interactions on the math vertex and participated in the role of sense maker more than Dr. Jackson. Dr. Bradley seemed confident about his own understanding of both the isomorphism and quotient group material. However, he did ask several questions about the best way to introduce the quotient group materials in the classroom. He seemed most concerned with best sequencing the materials to support his students' mathematical progression. Unlike Dr. Bradley, Dr. Jackson did not have confident statements about his understanding of the isomorphism and quotient group material. He *did* ask questions about implementing the quotient group materials, and both teachers showed concern about the tests they gave to their students and how these helped or hindered their progress.

Notably, Dr. Bradley had several interactions that fell on the edges of the Instructional Triangle. He demonstrated participation in both the roles of sense maker and builder in his interactions on the math vertex. This is characterized by the edge of the Instructional Triangle between the math and instruction vertices. He also had interactions characterized by the edge between the math and student vertices while participating in both the sense maker and inquirer roles. This type of participation shows that Dr. Bradley was coordinating multiple aspects of IOI simultaneously. The clearest difference between the teachers' interactions is found in the beginning of the discussions around the isomorphism unit. Dr. Bradley seems most confident in his understanding of the mathematics, while Dr. Jackson becomes unsure when his reasoning is questioned by the facilitator. This is an important distinction because inquiry-oriented instructors must use their own mathematical activity to build on their students' mathematical activity (Johnson, 2013).

Altogether, the teachers had more interactions on the math vertex at the beginning of the OWG. When discussing the mathematics in the first OWG meetings, the teachers seemed to be attending more to their own mathematical understanding. As the OWG progressed, they more often talked about how to make sense of the instructional materials *for* their students. This mirrors the IOI practice that has teachers "facilitate student engagement in meaningful tasks and mathematical activity related to an important mathematical point" (Kuster, Johnson, Rupnow, &Wilhelm, 2019).

Turning to the teachers' interactions on the student vertex, I found that Dr. Jackson participated more in the role of inquirer. In the beginning of discussions around the isomorphism unit, Dr. Jackson often commented on his students' progress or difficulties with certain ideas. It was clear in his comments, that Dr. Jackson was frustrated with his students' slow progress through the mathematics of each unit. Dr. Bradley showed less participation in the role of inquirer, rarely commenting on students at the beginning of the OWG. As the semester progressed, Dr. Bradley showed more participation in this role in later OWG sessions. He seemed most concerned with how his students engaged with the quotient group materials and how he could best write assessments for them, a concern Dr. Jackson shared. Dr. Bradley and Dr. Jackson both had some interaction on an edge of the Instructional Triangle by considering how to build the mathematical agenda based on how their students understood the material. This shows their simultaneous participation in the roles of builder and inquirer. An important distinction here is that, while Dr. Jackson participated more often in the role of inquirer, his comments about his students were usually negative, whereas Dr. Bradley's comments were *only* positive. This demonstrates that while actively inquiring into student thinking is a key component of IOI, this may only be productive when approached from a positive viewpoint of students with the intent to revise instructional decisions.

Both teachers showed strong participation in the role of builder throughout the OWG. The data shows that the entire OWG interacted most on the instructional vertex and often provided helpful feedback to each other's questions on instruction. Dr. Bradley and Dr. Jackson often commented on their own instruction and that of other teachers in productive ways. I also saw a flip in the teachers' interactions on this vertex as compared to the math vertex. While Dr. Jackson seemed confident in his instructional decisions and rarely asked for opinions in the OWG, Dr. Bradley more often asked questions about his instructional decisions. Altogether, Dr. Bradley seemed more open to instructional changes to improve his IOI and sought out suggestions for such change from the teachers in the OWG. Dr. Jackson showed less interactions suggesting such change. He more often confidently described what he did in the classroom or how he planned to introduce materials, with less self-reflection.

Perhaps most interesting, while both teachers showed change in their participation over time, they did this differently. Dr. Bradley showed the most coordination of the different aspects of IOI, because over time, his participation converged to the edges of the instructional triangle. While Dr. Jackson also showed coordination of the multiple aspects of IOI with some interaction of the edges of the triangle, he didn't do this as much as Dr. Bradley. Both teachers' coordination of multiple roles supports the idea that the OWG was effective in helping the teachers develop their IOI skills. Altogether, the OWG environment started with supportive discursive engagement and continued in this manner throughout the entire semester. Dr. Bradley and Dr. Jackson's engagement was fairly indicative of the entire OWG. That is, over time, the conversation of the whole OWG began to focus most on coordinating the key components of IOI and making instructional decisions based on students' understanding.

Conclusions

Altogether, I wanted to look at how teachers implementing IOI talked about their instruction in the context of online professional development. The goal of this study was to better understand how teachers engage with IOI and gather conclusions about how professional development may support teachers seeking to use this ambitious teaching style. To do this I first used the Instructional Triangle to characterize what the TIMES teachers focused on in the OWG meetings. Then I analyzed the teachers' participation in the roles of sense-maker, inquirer, and builder along the vertices and edges of the Instructional Triangle. In this method, the triangle served as a proxy to better understand what the teachers paid attention to in the OWG and how they intended to implement IOI in their classrooms. Necessarily, I showed how the Instructional Triangle was particularly well suited for analysis around IOI. That is, the vertices, math, students, and instruction, capture the important aspects of planning for and implementing IOI. Moreover, the lens of participation allowed me to characterize how the teacher's attention to each of these vertices changed over the course of the OWG.

I found that the teachers in the OWG thought about their instruction in different ways, and also changed in their participation in each role over the course of the OWG. In many ways, the teachers who participated as TIMES fellows were a self-selecting demographic, in that they were all teachers who cared about instruction and saw their own practice as fallible. Drs. Bradley and Jackson were no exception. In addition, they were inclined to open-mindedness towards instructional techniques. The data revealed instances of their reflection on this process of pedagogical growth. For example, Dr. Bradley's reflections showed that he was intrinsically invested in his instructional practice as a teacher because he showed strong participation in the role of builder. The data also showed that Dr. Jackson reflected on his decisions made during instruction in this same role, but with less clear intent to alter his instruction. While such discourse seems to be invaluable in both teachers' and students' academic growth, it is nonetheless ironic that such behavior is also a key and indicative component of IOI.

Perhaps most interestingly, the data showed that there were significant differences in the ways that Drs. Bradley and Jackson spoke of math in the role of sense maker. Dr. Jackson asked questions, seeking input from others, and did not appear as confident in the mathematics of the instructional sequences. Contrastingly, when Dr. Bradley participated in the role of sense maker, he was often explaining the mathematics of the lessons in response to questions from the other OWG teachers. Meanwhile, Dr. Bradley's discourse in the role of builder suggested more self-reflection, similar to his comments coded in the instruction vertex. The comments shared by Dr. Bradley suggest more mathematical, discursive engagement in his expectations for students while Dr. Jackson's feedback in this area was sparse and did not seem to be self-reflective. Even more strikingly, the analysis revealed that while both teachers participated much in the role of inquirer, the way they talked about students was very different. Dr. Jackson often took a negative approach to his students' thinking, while Dr. Bradley made more positive comments about his students' contributions to the mathematical discourse in his classroom. This shows that teachers can certainly participate in the role of inquirer and focus on students' thinking in ways that may be counterproductive to the tenants of IOI.

Research on how teachers' attitudes toward student contributions affect the inquiry-oriented classroom progress can shed light on the contrast we found here. In conclusion, the differences between Dr. Bradley and Dr. Jackson's participation in the OWG over time with respect to the different components of IOI reveals how teachers may approach IOI in very different ways. Moreover, the convergence of the teachers' remarks to the edges of the

Instructional Triangle over the course of the OWG suggest that the professional development supported their coordination of the components of IOI and developed the necessary skills for implementing IOI. To explore the discrepancies between Dr. Bradley and Dr. Jackson, more research looking at teachers' engagement with and reflection on IOI materials is needed.

A significant contribution of this paper is the characterization of IOI through the Instructional Triangle. Further, given the constant interactions between math, instruction, and students in IOI, we see the edges of the Instructional Triangle as an avenue for finer-grained analysis of IOI through these reflexive interactions. Corresponding to this, the analysis of the teachers' participation in the roles of sense-builder, inquirer, and builder over time gives a deeper look into how teachers develop the special skills required for IOI (Johnson, 2013; Johnson & Larson, 2012; JMTE, 2019). These merged frameworks can be used in further research on teachers' implementation of IOI and participation in online professional development. Moreover, this study contributed to the growing body of research on effective professional development (Fortune & Keene, 2019). My conclusions are particularly descriptive of online professional development, an increasingly important topic as universities move to more virtual settings and teachers in isolated communities seek to use more reform-oriented mathematics instruction. More broadly, this study contributed to the robust group of research on the ambitious teaching style of IOI and how teachers reflect on their instruction. As IOI becomes a more popular teaching solution for undergraduate mathematics, more data on how to support teachers in this type of instruction is useful in the mathematics education community. I suggest future research on how the teachers of this study implemented their IOI in the classroom. A cross analysis of how the teachers talked about their IOI with how they used IOI in the classroom may yield more insight on how to best support IOI teachers in and out of the classroom.

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