

Applications of Castigliano's Theorems

6.1 Coplanar trusses

6.1.1 Castigliano's first theorem

Consider a truss idealized as an assemblage of uniform bars connected by smooth ball-and-socket joints in three-dimensional trusses, or by smooth hinge joints in a coplanar truss. External forces are assumed to act only at the joints. The line connecting the joints at the end of each bar is assumed to coincide with the reference axis of the bar. Hence, the axial force and strain in each bar is uniform along its length, and the bar is either in tension or compression.

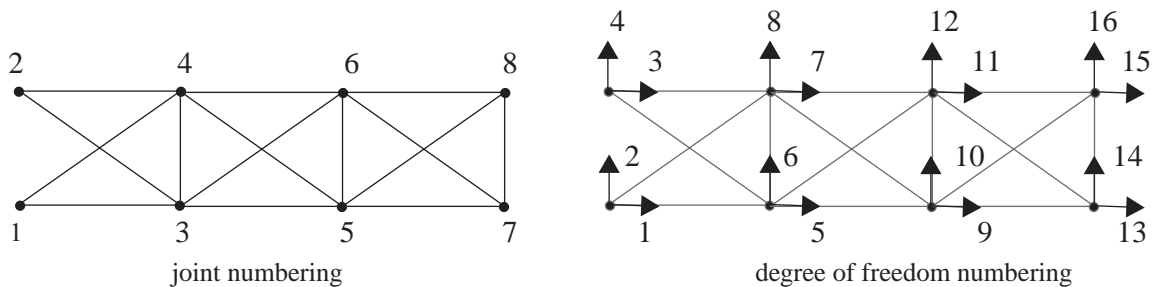


Fig. 6.1 A fifteen-bar truss.

A coplanar truss consisting of fifteen bars and eight joints is shown in figure 6.1. Each joint in a coplanar truss has two degrees of freedom, one horizontal displacement and the one vertical displacement. Hence, there are sixteen displacement degrees of freedom for this truss. At joint i , $i = 1, 2, \dots, 8$, the horizontal displacement is denoted by q_{2i-1} and the vertical displacement is denoted by q_{2i} . The positive directions for the displacements and corresponding forces in the fifteen bar truss are shown in figure 6.1. The original coordinates of the joints and the sixteen displacements completely define the configuration of the truss in the deformed state.

A typical bar in a truss connecting joints labeled i and j is shown in figure 6.2(a). The location of the bar in a X - Y coordinate system is established by the coordinates of joint i (X_i, Y_i) and those of joint j (X_j, Y_j). The angle of

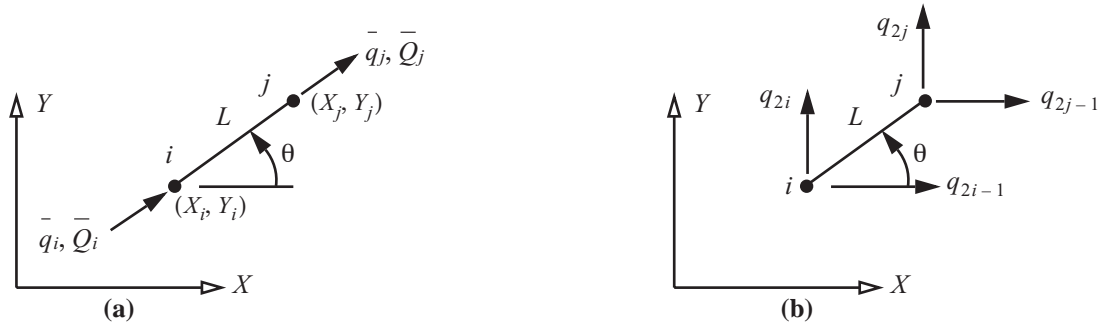


Fig. 6.2 (a) Truss bar i - j subject to axial displacements. (b) Truss bar i - j subject to horizontal and vertical displacements.

the bar with respect to the X -axis is denoted by θ . Trigonometric functions of the angle θ are related to the coordinates of the joints and length L of the bar by

$$\cos\theta = (X_j - X_i)/L \quad \sin\theta = (Y_j - Y_i)/L \quad L = \sqrt{(X_j - X_i)^2 + (Y_j - Y_i)^2}. \quad (6.1)$$

As shown in figure 6.2(a), the axial displacement of the bar at joint i is \bar{q}_i and that of joint j is \bar{q}_j . Assume the axial displacement $w(z) = \bar{q}_i(1 - z/L) + \bar{q}_j(z/L)$, $0 \leq z \leq L$. The axial strain $\epsilon_{zz} = w' = (\bar{q}_j - \bar{q}_i)/L$, and denote the elongation $\Delta_{i-j} = \bar{q}_j - \bar{q}_i$. Also, assume the temperature change is uniform in z . From eq. (3.79) on page 46 the axial force in the bar is

$$N_{i-j} = \left(\frac{EA}{L}\right)_{i-j} \Delta_{i-j} - (N_T)_{i-j}. \quad (6.2)$$

The differential equation of equilibrium $\frac{dN}{dz} = 0$ (eq. (3.53) on page 42) is satisfied under assumptions of uniform axial strain and uniform axial change in temperature. The strain energy (5.81) on page 145 of the bar reduces to

$$U = \frac{1}{2} \frac{EA}{L} \Delta_{i-j}^2 - N_T \Delta_{i-j}. \quad (6.3)$$

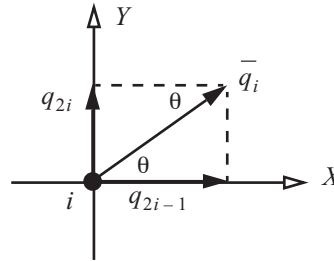
Castigliano's first theorem determines the force \bar{Q}_i corresponding to displacement \bar{q}_i , and force \bar{Q}_j corresponding to displacement \bar{q}_j . The results are

$$\bar{Q}_i = \frac{\partial U}{\partial \bar{q}_i} = \frac{EA}{L}(\bar{q}_i - \bar{q}_j) + N_T \quad \text{and} \quad \bar{Q}_j = \frac{EA}{L}(-\bar{q}_i + \bar{q}_j) - N_T. \quad (6.4)$$

Note that $\bar{Q}_i + \bar{Q}_j = 0$, which is the condition of equilibrium.

In figure 6.2(b) truss displacements of joint i are (q_{2i-1}, q_{2i}) and for joint j are (q_{2j-1}, q_{2j}) . At joint i the

Fig. 6.3 Relation between the displacements components at joint i .



truss and axial displacements are related by $q_{2i-1} = \bar{q}_i \cos \theta$ and $q_{2i} = \bar{q}_i \sin \theta$ as shown in figure 6.3. Likewise at joint j $q_{2j-1} = \bar{q}_j \cos \theta$ and $q_{2j} = \bar{q}_j \sin \theta$. These relations can be solved for the axial displacement in terms of the truss displacements to get

$$\bar{q}_i = q_{2i-1} \cos \theta + q_{2i} \sin \theta \quad \bar{q}_j = q_{2j-1} \cos \theta + q_{2j} \sin \theta. \quad (6.5)$$

The elongation of the truss bar i - j in terms of the joint displacements is

$$\Delta_{i-j} = \bar{q}_j - \bar{q}_i = (q_{2j-1} - q_{2i-1}) \cos \theta + (q_{2j} - q_{2i}) \sin \theta. \quad (6.6)$$

The elongation (6.6) is the sum of the projections of the relative displacements onto the reference axis of the undeformed bar which is depicted in figure 6.4.

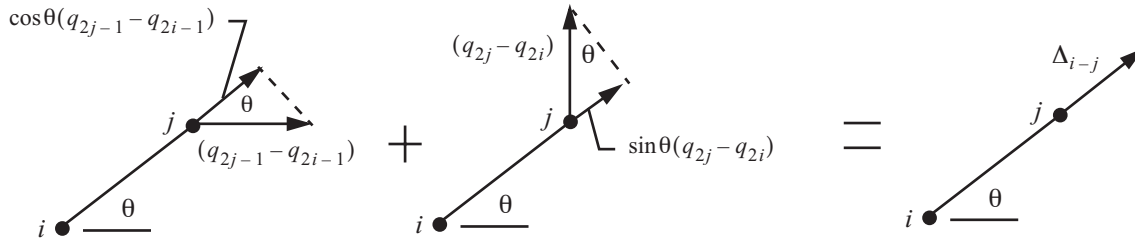


Fig. 6.4 Elongation of the bar as the sum of projections of the relative horizontal and vertical displacements along the direction of the undeformed bar.

For the m -th bar of the truss shown in figure 6.1, where, $m = 1, 2, \dots, 15$, denote its extension stiffness by $(EA/L)_m$, its elongation by Δ_m , and denote the thermal force by N_{Tm} . The temperature change is uniform in each bar, but can be different from bar to bar. The relation between bar index m and the joints i and j of the bar are defined by assignment. For example in figure 6.1, the bar identified by $m = 2$ may be selected as the bar connecting joint 1 to joint 4, so its elongation (6.6) is

$$\Delta_2 = \Delta_{1-4} = (q_7 - q_1) \cos \theta_2 + (q_8 - q_2) \sin \theta_2.$$

The sine and cosine of angle θ_2 are determined from eq. (6.1). The strain energy of the assemblage is simply the sum of the strain energies in each bar, where (6.3) is the energy for one bar. Hence, the total strain energy is

$$U = \sum_{m=1}^{15} \left\{ \frac{1}{2} \left(\frac{EA}{L} \right)_m \Delta_m^2 - N_{Tm} \Delta_m \right\} \quad (6.7)$$

The displacements q_n and the corresponding forces Q_n , $n = 1, 2, \dots, 16$, used in the formulation of Castigliano's theorem are the displacements and corresponding forces at the joints. Hence, Castigliano's first theorem for the truss shown in figure 6.1 is

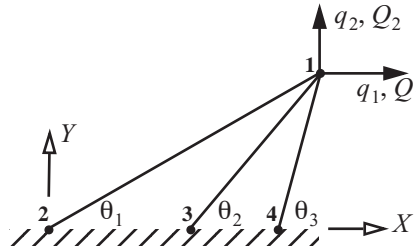
$$Q_n = \sum_{m=1}^{15} \left[\left(\frac{EA}{L} \right) \Delta_m - N_{Tm} \right] \frac{\partial \Delta_m}{\partial q_n} \quad n = 1, 2, \dots, 16. \quad (6.8)$$

Example 6.1 Three-bar coplanar truss

The coplanar truss shown in figure 6.5 consists of three bars ($m = 1, 2, 3$) and four joints 1, 2, 3, 4. Beginning joint i and end joint j for each bar are listed in the figure. Joints 2, 3, and 4 are fixed so their displacements equal zero, and joint 1 is movable. The change in the thermal force in each bar is equal to zero. The spring stiffness of the bars are denoted by $(EA/L)_m$. Determine the 2×2 stiffness matrix using Castigliano's first theorem.

Fig. 6.5 Three-bar truss.

m	joint i	joint j
1	2	1
2	3	1
3	4	1



Solution. The elongation of each bar as determined from eq. (6.6) is

$$\Delta_m = \cos(\theta_m)q_1 + \sin(\theta_m)q_2 \quad m = 1, 2, 3. \quad (a)$$

Castigliano's theorem (6.8) applied to this example yields

$$Q_1 = \sum_{m=1}^3 \left(\frac{EA}{L} \right)_m [\cos(\theta_m)q_1 + \sin(\theta_m)q_2][\cos(\theta_m)] \quad (b)$$

$$Q_2 = \sum_{m=1}^3 \left(\frac{EA}{L} \right)_m [\cos(\theta_m)q_1 + \sin(\theta_m)q_2][\sin(\theta_m)] \quad (c)$$

These results are written in the matrix form

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \quad (d)$$

where the elements of the stiffness matrix are

$$k_{11} = \sum_{m=1}^3 \left(\frac{EA}{L} \right)_m \cos^2(\theta_m) \quad k_{22} = \sum_{m=1}^3 \left(\frac{EA}{L} \right)_m \sin^2(\theta_m) \quad k_{12} = k_{21} = \sum_{m=1}^3 \left(\frac{EA}{L} \right)_m \cos(\theta_m)\sin(\theta_m). \quad (e)$$

Note that this example is statically indeterminate, since there are only two equilibrium equations at the movable

joint 1 but three unknown bar forces. For specified nodal forces Q_1 and Q_2 , matrix eq. (d) is solved for the nodal displacements q_1 and q_2 . From eq. (a) the elongation of each bar is then computed, and from these elongations the bar forces are determined from

$$N_m = \left(\frac{EA}{L}\right)_m \Delta_m \quad m = 1, 2, 3 \quad \blacksquare \quad (f)$$

Example 6.2 Three-bar truss with lack of fit

Consider the same three bar-truss of example 6.1, but now assume that bar 1 was too short and had to be stretched an amount $\bar{\Delta}_1$ in order to connect it to joint 1. This is a case of lack of fit, and lack of fit is common in the fabrication of structures. That is, before the external loads are applied ($Q_1 = Q_2 = 0$), the truss bars experience initial forces due to the lack of fit of bar 1. Determine the initial forces in the bars using Castigliano's first theorem.

Solution. Lack of fit can be included in the energy analysis by modifying the specified thermal force term in the strain energy (6.7). For uniform material properties and uniform change in temperature, the thermal force in a truss bar is $N_T = EA\alpha\Delta T$. (Refer to eq. (3.75) on page 46.) The factor $\alpha\Delta T = \varepsilon_0$ is the initial strain due to the temperature change. Note ε_0 is dimensionless. Now interpret ε_0 as the initial strain specified due to lack of fit.

The initial strain due to the specified displacement $\bar{\Delta}$ required to connect a bar to a joint is $\varepsilon_0 = \bar{\Delta}/L$. Let $N_T \rightarrow N_{\bar{\Delta}} = EA(\bar{\Delta}/L)$. The strain energy is modified to

$$U = \sum_{m=1}^3 \left\{ \frac{1}{2} \left(\frac{EA}{L}\right)_m \Delta_m^2 - N_{\bar{\Delta}m} \Delta_m \right\} \quad (a)$$

The specified initial strain is only for bar 1, so

$$U = \sum_{m=1}^3 \frac{1}{2} \left(\frac{EA}{L}\right)_m \Delta_m^2 - \left(\frac{EA}{L}\right)_1 \bar{\Delta}_1 \Delta_1 \quad (b)$$

Castigliano's theorem (6.8) leads to

$$Q_n = \sum_{m=1}^3 \left[\left(\frac{EA}{L}\right)_m \Delta_m \right] \frac{\partial \Delta_m}{\partial q_n} - \left(\frac{EA}{L}\right)_1 \bar{\Delta}_1 \left(\frac{\partial \Delta_1}{\partial q_n}\right) \quad n = 1, 2 \quad (c)$$

The matrix form of eq. (c) is

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} - \left(\frac{EA}{L}\right)_1 \bar{\Delta}_1 \begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \end{bmatrix} \quad (d)$$

Elements of the stiffness matrix are the same as given by eq. (e) of example 6.1. Set $Q_1 = 0$ and $Q_2 = 0$, since no external forces are applied to the joint just after assembly. Then solve the matrix equation (d) for the joint displacements to get

$$\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}^{-1} \begin{bmatrix} \cos(\theta_1) \\ \sin(\theta_1) \end{bmatrix} \left(\frac{EA}{L}\right)_1 \bar{\Delta}_1. \quad (e)$$

From this solution for the displacements we can calculate the elongation of each bar after assembly from eq. (a) in example 6.1. The initial bar forces after assembly are computed from

$$N_1 = \left(\frac{EA}{L}\right)_1 (\Delta_1 - \bar{\Delta}_1) \quad N_2 = \left(\frac{EA}{L}\right)_2 \Delta_2 \quad N_3 = \left(\frac{EA}{L}\right)_3 \Delta_3. \quad (f)$$

A specific case: $\theta_1 = 30^\circ$, $\theta_2 = 45^\circ$, $\theta_3 = 60^\circ$, and EA is the same for each bar. Take $L_1 = L$, so that $L_2 = L/\sqrt{2}$, and $L_3 = L/(\sqrt{3})$. The solution for the displacements from eq. (b) are $q_1 = 1.458\bar{\Delta}_1$ and $q_2 = -\bar{\Delta}_1$. The elongations are $\Delta_1 = 0.763\bar{\Delta}_1$, $\Delta_2 = 0.324\bar{\Delta}_1$, and $\Delta_3 = -0.137\bar{\Delta}_1$, and the bar forces from eq. (f) are

$$N_1 = -0.237EA(\bar{\Delta}_1/L) \quad N_2 = 0.458EA(\bar{\Delta}_1/L) \quad N_3 = -0.237EA(\bar{\Delta}_1/L). \blacksquare \quad (g)$$

6.1.2 Castigliano's second theorem for a statically determinate truss

Example 6.3 Truss displacements

The truss shown in figure 6.6 consists of three bars labeled 1-2, 1-3, and 2-3. Joint 1 is a fixed pin, and pin joint 3 is free to move vertically but not horizontally. A downward applied force of a 84,000 N acts at joint 2. The cross-sectional areas of bars are $A_{1-2} = 900\text{mm}^2$, $A_{1-3} = 300\text{mm}^2$, and $A_{2-3} = 1,200\text{mm}^2$. Each bar has a modulus of elasticity $E = 70,000 \text{ N/mm}^2$. The degree of freedom numbering is shown the figure. Determine displacements q_3 and q_4 by Castigliano's second theorem

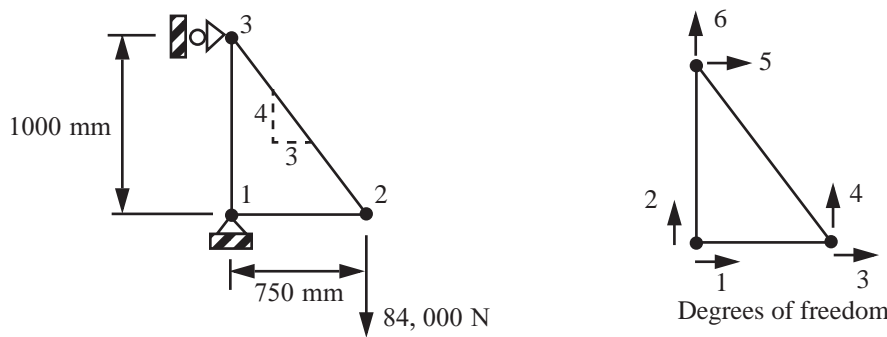


Fig. 6.6 A statically determinate three-bar truss.

A note on static determinacy: Let m = the number of unknown bar forces, r = the number of support reactions, and let j = the number of joints. There are two independent equilibrium equations per joint. For a statically determinate truss, the number of unknown forces is equal to number of independent equilibrium equations (i.e., $2j = m$

+ r). For the truss in this example $j = 3$, $m = 3$, and $r = 3$. So it is statically determinate. For the truss in example 6.1, $j = 3$, $m = 3$, and $r = 6$, and $6 < 3 + 6$. So the truss in example 6.1 is statically indeterminate.

Solution. Free body diagrams of joints 2 and 3 are shown figure 6.7. The diagrams are drawn assuming each bar is in tension, so the reaction of the bar force acting on a joint is an arrow aligned with the bar and pointing away from the joint. The objective is to determine each bar force in terms of external forces Q_3 and Q_4 . Note that $Q_3 = 0$ and $Q_4 = -84,000$ N, but we will wait to substitute these numerical values after the derivatives are evaluated in Castigliano's second theorem.

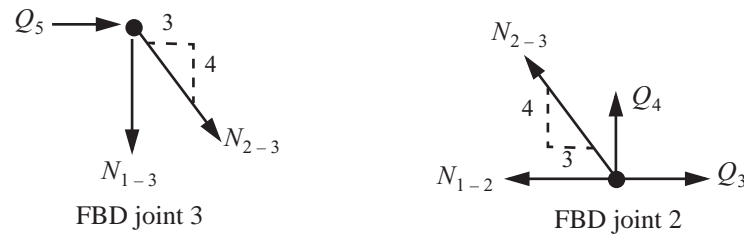


Fig. 6.7 Free body diagrams of two joints of the three-bar truss.

The only contribution to the complementary strain energy in (5.84) on page 145 is the axial normal force N , which is spatially uniform along the length of the bar. Also, there is no change in temperature from the reference state. Hence, the complementary strain energy for the truss is

$$U^* = \frac{1}{2} \left[\left(\frac{L}{EA} \right)_{1-2} N_{1-2}^2 + \left(\frac{L}{EA} \right)_{1-3} N_{1-3}^2 + \left(\frac{L}{EA} \right)_{2-3} N_{2-3}^2 \right], \tag{a}$$

Castigliano's second theorem for the displacement q_3 is

$$q_3 = \frac{\partial U^*}{\partial Q_3} = \left(\frac{L}{EA} \right)_{1-2} N_{1-2} \frac{\partial N_{1-2}}{\partial Q_3} + \left(\frac{L}{EA} \right)_{1-3} N_{1-3} \frac{\partial N_{1-3}}{\partial Q_3} + \left(\frac{L}{EA} \right)_{2-3} N_{2-3} \frac{\partial N_{2-3}}{\partial Q_3}. \tag{b}$$

The terms in eq. (b) are listed in table 6.1. Replace the derivatives of the bar forces with their values listed in the

Table 6.1 Terms in Castigliano's theorem for displacements q_3 and q_4

Bar	L, mm	A, mm ²	L/(EA), mm/N	N	$\partial N / \partial Q_3$	$\partial N / \partial Q_4$
1-2	750	900	11.90×10^{-6}	$Q_3 + 3Q_4/4$	1	3/4
1-3	1000	300	47.62×10^{-6}	Q_4	0	1
2-3	1250	1200	14.88×10^{-6}	$-5Q_4/4$	0	-5/4

table to get

$$q_3 = \left(\frac{L}{EA} \right)_{1-2} N_{1-2}(1) + \left(\frac{L}{EA} \right)_{1-3} N_{1-3}(0) + \left(\frac{L}{EA} \right)_{2-3} N_{2-3}(0). \tag{c}$$

Substitute the equation for bar force N_{1-2} from the table in the previous equation and note that $Q_3 = 0$ and

$Q_4 = -84,000$ N to get

$$q_3 = \left(\frac{L}{EA}\right)_{1-2} (Q_3 + 3Q_4/4) = (11.90 \times 10^{-6} \text{ mm/N}) \left(0 + \frac{3}{4}(-84,000 \text{ N})\right) = -0.750 \text{ mm} . \quad (\text{d})$$

Castigliano's second theorem for the displacement q_4 is

$$q_4 = \frac{\partial U^*}{\partial Q_4} = \left(\frac{L}{EA}\right)_{1-2} N_{1-2} \frac{\partial N_{1-2}}{\partial Q_4} + \left(\frac{L}{EA}\right)_{1-3} N_{1-3} \frac{\partial N_{1-3}}{\partial Q_4} + \left(\frac{L}{EA}\right)_{2-3} N_{2-3} \frac{\partial N_{2-3}}{\partial Q_4} . \quad (\text{e})$$

Replace the bar forces and their derivatives in eq. (e) with their values listed in table 6.1 to get

$$q_4 = \left(\frac{L}{EA}\right)_{1-2} (0 + 3Q_4/4) \left(\frac{3}{4}\right) + \left(\frac{L}{EA}\right)_{1-3} (Q_4)(1) + \left(\frac{L}{EA}\right)_{2-3} (-5Q_4/4) \left(-\frac{5}{4}\right) . \quad (\text{f})$$

Substitute numerical values into eq. (f) to get

$$q_4 = \left[(11.90 \times 10^{-6} \text{ mm/N}) \frac{9}{16} + (47.62 \times 10^{-6} \text{ mm/N}) + (14.88 \times 10^{-6} \text{ mm/N}) \frac{25}{16} \right] (-84,000 \text{ N}) . \quad (\text{g})$$

The final result from eq. (g) is

$$q_3 = -0.750 \text{ mm} \quad q_4 = -6.516 \text{ mm} \blacksquare \quad (\text{h})$$

6.2 Beam structures

Example 6.4 Cross-sectional properties of a thin-walled tube

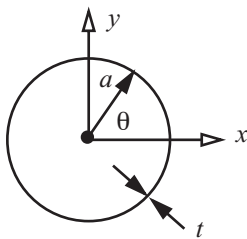


Fig. 6.8.

The cross section is a thin-walled tube with a circular contour of radius a and wall thickness t , as shown in figure 6.8.

Solution. The x - and y - axes are axes of symmetry in the cross section, so the centroid and shear center coincide with the center of the circular contour. The parametric coordinates of the circular contour are $x = a \cos \theta$ and $y = a \sin \theta$, and the arc length along the contour $s = a\theta$; $0 \leq \theta \leq 2\pi$. The cross-sectional area and first area moments are

$$A = \int_0^{2\pi} t a d\theta = 2\pi a t \quad Q_x = \int_0^{2\pi} y t a d\theta = 0 \quad Q_y = \int_0^{2\pi} x t a d\theta = 0 . \quad (\text{a})$$

The first area moments equal zero since the center of the circle is the centroid. The second area moments are

$$I_{xx} = \int_0^{2\pi} y^2 t a d\theta = \pi a^3 t \quad I_{yy} = \int_0^{2\pi} x^2 t a d\theta = \pi a^3 t \quad I_{xy} = \int_0^{2\pi} x y t a d\theta = 0 . \quad (\text{b})$$

Since the product area moment is zero, then coefficients $n_x = n_y = 0$ and $k = 1$ from eq. (4.4) on page 79. To

compute the transverse shear compliances given in eq. (4.30) on page 83, we need to compute the distribution functions from eq. (4.19) and eq. (4.26). The distribution functions for the first area moments for a segment of the contour from $s = 0$ to $s = a\theta$ are given by

$$Q_x(\theta) = \int_0^\theta y t a d\theta = a^2 t (1 - \cos\theta) \quad Q_y(\theta) = \int_0^\theta x t a d\theta = a^2 t \sin\theta. \quad (\text{c})$$

The coordinates normal and tangent to the contour with respect to the shear center are

$$r_n = x \left(\frac{1}{a} \frac{dy}{d\theta} \right) - y \left(\frac{1}{a} \frac{dx}{d\theta} \right) = a \quad r_t = x \left(\frac{1}{a} \frac{dx}{d\theta} \right) + y \frac{1}{a} \frac{dy}{d\theta} = 0. \quad (\text{d})$$

(Refer to eq. (3.10) on page 34.) The area enclosed by the contour is

$$A_c = \frac{1}{2} \int_0^{2\pi} r_n a d\theta = \pi a^2. \quad (\text{e})$$

The shear flow distribution functions given by eqs.(4.19) and (4.26) on page 82 for the closed section are

$$F_x = \frac{1}{I_{yy}} \left[Q_y(\theta) - \frac{1}{2A_c} \int_0^{2\pi} r_n Q_y(\theta) a d\theta \right] = \frac{a^2 t \sin\theta}{I_{yy}} = \frac{\sin\theta}{a\pi}, \text{ and} \quad (\text{f})$$

$$F_y = \frac{1}{I_{xx}} \left[Q_x(\theta) - \frac{1}{2A_c} \int_0^{2\pi} r_n Q_x(\theta) a d\theta \right] = -\left(\frac{a^2 t}{I_{xx}} \right) \cos\theta = \frac{-\cos\theta}{a\pi}. \quad (\text{g})$$

Finally, the transverse shear compliances are

$$c_{xx} = \frac{1}{Gt} \int_0^{2\pi} F_x^2(\theta) a d\theta = \frac{\pi a^5 t}{G I_{yy}^2} = \frac{1}{\pi a t G} \quad c_{yy} = \frac{1}{Gt} \int_0^{2\pi} F_y^2(\theta) a d\theta = \frac{1}{\pi a t G}, \text{ and} \quad (\text{h})$$

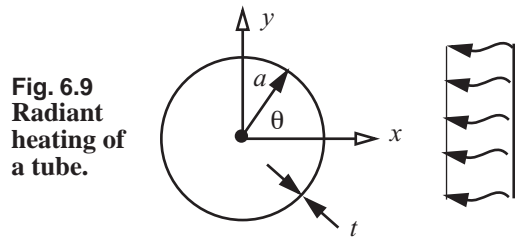
$$c_{xy} = \frac{1}{Gt} \int_0^{2\pi} [F_x(\theta) F_y(\theta)] a d\theta = 0. \quad (\text{i})$$

For a uniform shear modulus around the contour, the torsion constant is determined from eq. (3.161) on page 70 as

$$J = \frac{4A_c^2}{\oint \frac{ds}{t}} = \frac{4(\pi a^2)^2}{\frac{2\pi a}{t}} = 2\pi a^3 t \blacksquare \quad (\text{j})$$

6.2.1 Castigliano’s second theorem

Example 6.5 Thin-walled tube subject to radiant heating



A common structural member in orbiting space structures is a thin-walled tube. Tubes are used as truss members and for satellite booms. Solar heating combined with heat conduction results in the distribution of temperature around the perimeter and along the length of the tube. The data in this example is for an aluminum 6061-T6 tube taken from Thornton (1996, pp. 118-121).

A thin-walled tube with a circular contour of radius a , and wall thickness t is subjected to radiant heating as shown in figure 6.9. The tube is cantilevered, that is, fixed at $z = 0$ and free at $z = L$, where L is the length of tube. The change in temperature from the reference state is uniform along the length but it varies around the perimeter, and is specified by

$$\Delta T(\theta) = \bar{T} + T_m \cos \theta \quad \theta \in [0, 2\pi]. \tag{a}$$

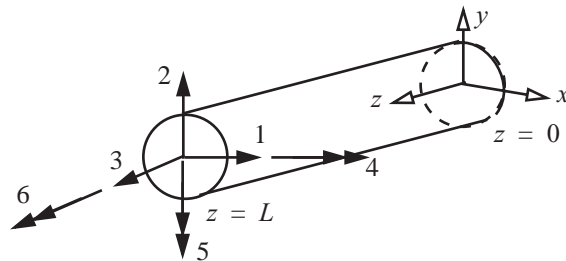
where the average temperature is denoted by \bar{T} and the perturbation in temperature is denoted by T_m . Data for this example are listed in table 6.2.

Table 6.2 Numerical data for example 6.5

radius	$a = 0.03812 \text{ m}$	Poisson’s ratio	$\nu = 0.33$
wall thickness	$t = 7.14 \times 10^{-4} \text{ m}$	coefficient of thermal expansion	$\alpha = 23 \times 10^{-6} \text{ }^\circ\text{K}$
tube length	$L = 0.8 \text{ m}$	average temperature	$\bar{T} = 462 \text{ }^\circ\text{K}$
modulus of elasticity	$E = 68.3 \text{ GPa}$	perturbation temperature	$T_m = 34 \text{ }^\circ\text{K}$

Determine the displacements q_1 and q_3 , and rotation q_5 of the cross section at the free end using Castigliano’s second theorem. The degree of freedom numbering is shown in figure 6.10.

Fig. 6.10 Degree of freedom numbering at the free end of the cantilevered tube.



Solution. From eqs. (5.83) to (5.85) on page 145 the complementary strain energy in this example is

$$U^* = \frac{1}{2} \int_0^L \left[\frac{(N + N_T)^2}{EA} + \frac{(M_x + M_{xT})^2}{EI_{xx}} + \frac{(M_y + M_{yT})^2}{EI_{yy}} + c_{xx} V_x^2 + c_{yy} V_y^2 + \frac{M_z^2}{GJ} \right] dz. \quad (\text{b})$$

The cross-sectional properties were determined in example 6.4, and they are

$$A = 2\pi at = 2\pi(0.03812\text{m})(7.14 \times 10^{-4}\text{m}) = 171.014 \times 10^{-6}\text{m}^2, \quad (\text{c})$$

$$I_{xx} = I_{yy} = \pi a^3 t = \pi(0.03812\text{m})^3(7.14 \times 10^{-4}\text{m}) = 124.25 \times 10^{-9}\text{m}^4, \text{ and } J = 2\pi a^3 t = 248.50 \times 10^{-9}\text{m}^4. \quad (\text{d})$$

The shear modulus is given by the isotropic formula $G = E/(2(1 + \nu)) = 25.6 \text{ GPa}$, so the transverse shear compliances are

$$c_{xx} = c_{yy} = \frac{1}{\pi at G} = \frac{1}{\pi(0.03812\text{m})(7.14 \times 10^{-4}\text{m})(25.6 \times 10^9\text{N/m}^2)} = \frac{456.61 \times 10^{-9}}{\text{N}}. \quad (\text{e})$$

The thermal axial force is given by eq. (3.75) on page 46, and thermal moments are given in eq. (3.78). Material properties are uniform along the contour and $x = a \cos \theta$, and $y = a \sin \theta$ in the thermal action formulas. The results for these thermal actions are

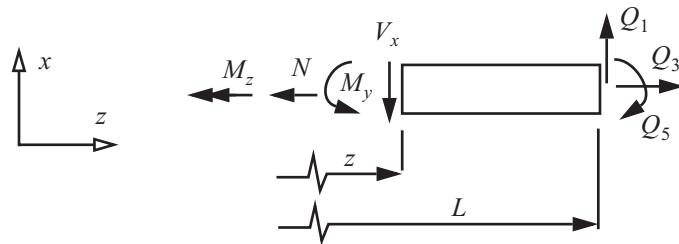
$$N_T = E\alpha \int_0^{2\pi} \Delta T t a d\theta = 2\pi at E\alpha \bar{\Delta T} \quad M_{xT} = E\alpha \int_0^{2\pi} \Delta T y t a d\theta = 0 \quad M_{yT} = E\alpha \int_0^{2\pi} \Delta T x t a d\theta = \pi a^2 t E\alpha \bar{\Delta T}_m, \quad (\text{f})$$

$$N_T = 2\pi(0.03812\text{m})(7.14 \times 10^{-4}\text{m})(68.3 \times 10^9\text{N/m}^2)(23 \times 10^{-6}\text{K})(462^\circ\text{K}) = 124.11 \times 10^3\text{N}, \text{ and} \quad (\text{g})$$

$$M_{yT} = \pi(0.03812\text{m})^2(7.14 \times 10^{-4}\text{m})(68.3 \times 10^9\text{N/m}^2)(23 \times 10^{-6}\text{K})(34^\circ\text{K}) = 174.10\text{N}\cdot\text{m}. \quad (\text{h})$$

The free body diagram of the tube in the x - z plane is shown in figure 6.11. Generalized forces Q_1 , Q_3 , and Q_5 are introduced at the free end to facilitate computing the corresponding displacements via Castigliano's theorem, and they are set equal to zero at the end of the procedure (i.e., they are fictitious actions).

Fig. 6.11 Fictitious actions acting at the free end of the cantilevered tube.



Equilibrium of the free body diagram in the x - z plane yields

$$N = Q_3 \quad V_x = Q_1 \quad M_y = Q_5 - Q_1(L - z) \quad M_z = 0 \quad z \in [0, L]. \quad (\text{i})$$

Since generalized forces $Q_2 = Q_4 = 0$, equilibrium in the y - z plane yields $V_y = M_x = 0$ for $z \in [0, L]$. The complementary strain energy reduces to

$$U^* = \frac{1}{2} \int_0^L \left[\frac{(Q_3 + N_T)^2}{EA} + \frac{[Q_5 - Q_1(L-z) + M_{yT}]^2}{EI_{yy}} + \frac{Q_1^2}{\pi atG} \right] dz. \quad (j)$$

Displacement q_1 is determined from

$$q_1 = \frac{\partial U^*}{\partial Q_1} = \int_0^L \left[\frac{(Q_3 + N_T) \partial(Q_3 + N_T)}{EA} + \frac{[Q_5 - Q_1(L-z) + M_{yT}] \partial[Q_5 - Q_1(L-z) + M_{yT}]}{EI_{yy}} + \frac{Q_1}{\pi atG} \left(\frac{\partial Q_1}{\partial Q_1} \right) \right] dz, \quad (k)$$

where we interchanged the derivative and integral since our functions are continuous. Performing the derivative inside the integral we get

$$q_1 = \frac{1}{EI_{yy}} \int_0^L [Q_5 - Q_1(L-z) + M_{yT}] [- (L-z)] dz + \frac{1}{\pi atG} \int_0^L Q_1(1) dz. \quad (l)$$

Now set Q_1 and Q_5 equal to zero and find

$$q_1 = \left(\frac{1}{EI_{yy}} \right) \int_0^L M_{yT} [- (L-z)] dz = \frac{-L^2 M_{yT}}{2EI_{yy}} = \frac{-(0.8 \text{ m})^2 (174.10 \text{ N-m})}{2(68.3 \times 10^9 \text{ N/m}^2)(124.25 \times 10^{-9} \text{ m}^4)} = -6.565 \times 10^{-3} \text{ m}. \quad (m)$$

Axial displacement q_3 is given by

$$q_3 = \frac{\partial U^*}{\partial Q_3} = \int_0^L \left[\frac{(Q_3 + N_T) \partial(Q_3 + N_T)}{EA} + \frac{[Q_5 - Q_1(L-z) + M_{yT}] \partial[Q_5 - Q_1(L-z) + M_{yT}]}{EI_{yy}} + \frac{Q_1}{\pi atG} \left(\frac{\partial Q_1}{\partial Q_3} \right) \right] dz. \quad (n)$$

Since $Q_3 = 0$, the latter equation reduces to

$$q_3 = \left(\frac{1}{EA} \right) \int_0^L (Q_3 + N_T)(1) dz \Bigg|_{Q_3=0} = \frac{L}{EA} N_T = \frac{(0.8 \text{ m})(124.11 \times 10^3 \text{ N})}{(68.3 \times 10^9 \text{ N/m}^2)(171.014 \times 10^{-6} \text{ m}^2)} = 8.5 \times 10^{-3} \text{ m}. \quad (o)$$

Finally, the rotation in radians about the y -axis is given by

$$q_5 = \left(\frac{1}{EI_{yy}} \right) \int_0^L [Q_5 - Q_1(L-z) + M_{yT}] [1] dz \Bigg|_{Q_1=Q_5=0},$$

$$q_5 = \frac{LM_{yT}}{EI_{yy}} = \frac{(0.8 \text{ m})(174.10 \text{ N-m})}{(68.3 \times 10^9 \text{ N/m}^2)(124.25 \times 10^{-9} \text{ m}^4)} = 16.41 \times 10^{-3} \text{ rad}. \quad (p)$$

Example 6.6 Wing spar subject to a distributed spanwise air load

A light airplane experiences a total lift $L = 12,000$ lb. in a certain symmetric maneuver. Thus, the lift acting on each wing is $L/2$. Assume the airload is distributed elliptically over the wing, so that the airload intensity f_L per unit span is given as

$$f_L = \frac{2L}{\pi z_{max}} \sqrt{1 - \left(\frac{z}{z_{max}}\right)^2} \quad 0 \leq z \leq z_{max}, \quad (\text{a})$$

where z is the spanwise coordinate, $z = 0$ at the root, and $z = z_{max}$ at the tip of the wing. See figure 6.12 (a). The spar of the wing is a uniform, longitudinal, thin-walled beam with a closed section stiffened by four longitudinal stringers as shown in figure 6.12 (b). This cross-section is the same one shown in figure 3.24 and analyzed in example 3.4 on page 71. Assume the spar is clamped at the root and free at the tip (i.e., a cantilever spar). At the tip of the spar we will use Castigliano's second theorem to find the vertical displacement of the shear center denoted by q_2 , and to find the torsional rotation of the cross section denoted by q_6 . To use the theorem, we introduce a fictitious force Q_2 corresponding to displacement q_2 , and a fictitious torque Q_6 corresponding to rotation q_6 . A typical cross section of the spar with the locations of the centroid (X_C), the shear center (X_{SC}), and the line of action of the airload (X_L) with respect to the vertical web are shown in the left-hand sketch of figure 6.13. The right-hand sketch in figure 6.13 illustrates that the airload is statically equivalent to the external line load intensity f_y and line moment intensity m_z resolved at the shear center.

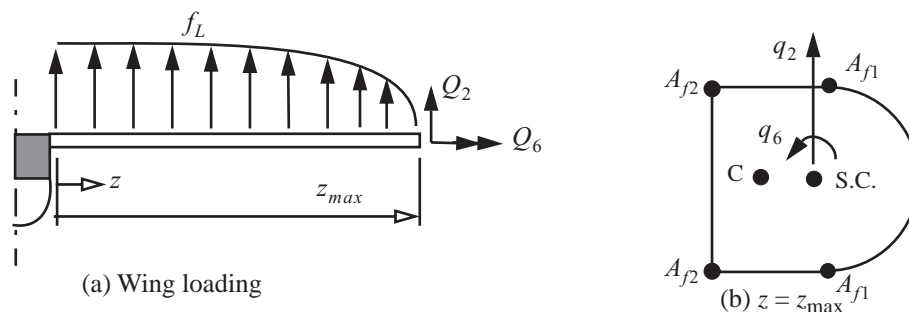


Fig. 6.12 : (a) Wing spanwise airload intensity and fictitious actions Q_2 and Q_6 of example 6.6. (b) Wing tip cross section and the corresponding generalized displacements q_2 and q_6 .

Numerical data for the cross-sectional dimensions are listed in table 6.3. The material is an aluminium alloy with

Table 6.3 Cross-sectional data for the wing spar

Dimensional data of the cross section			
A_{f1} , stringer 1 flange area	0.30 in. ²	b , length horizontal web	7.0 in.
A_{f2} , stringer 2 flange area	0.70 in. ²	t , wall thickness	0.030 in.
a , nose web radius	6.0 in	X_L , location of the airload	10.0 in.

a Young's modulus $E = 10.5 \times 10^3$ ksi, a shear modulus $G = 4.0 \times 10^3$ ksi, and with a yield strength $\sigma_{yield} = 44$ ksi. Additional cross-sectional properties computed from example 3.4 on page 71 are listed in

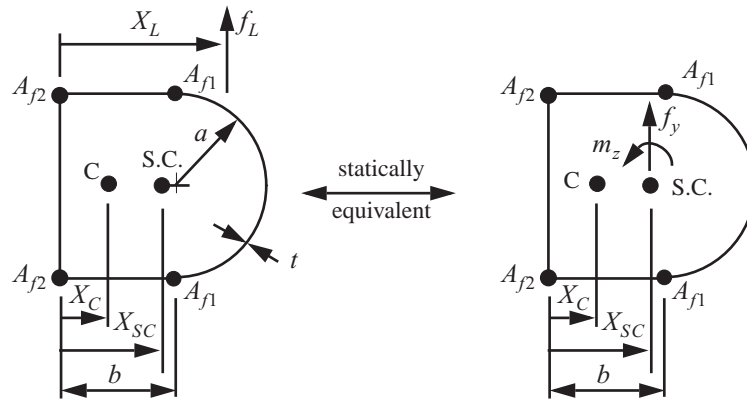


Fig. 6.13 example 6.6: Typical cross section of the uniform spar.

table 6.4.

Table 6.4 Data from example 3.4

A , area of the cross section	3.3455 in. ²		
X_C , horizontal location of the centroid	3.52367 in.	c_{yy} , compliance coefficient in transverse shear ^a	$\left(\frac{1.64758}{\text{in.}^2}\right) \frac{1}{G}$
X_{SC} , horizontal location of the shear center	6.39638 in.	c_{zz} , compliance coefficient in torsion ^b	$\left(\frac{0.0189201}{\text{in.}^4}\right) \frac{1}{G}$
I_{xx} , second area moment about the x -axis	101.619 in. ⁴	A_c , area enclosed by the contour	140.549 in. ²

a. Note: from eq. (5.65) on page 143 $c_{yy} = \sum_{i=1}^4 \int_0^{(s_i)_{max}} \frac{F_{yi}^2(s_i)}{Gt} ds_i$, where the shear flow distribution functions $F_{yi}(s_i)$ are given by eqs. (d) to (g) in part c of example 3.4.

b. Note: $c_{zz} = (GJ)^{-1}$, and from eq. (3.160) on page 70 the torsion constant $J = (4A_c^2) / \left(\oint \frac{ds}{t}\right)$.

- Determine the statically admissible bar resultants in the spar for $0 \leq z \leq z_{max}$.
- Determine the generalized displacements q_2 and q_6 of the shear center at $z = z_{max}$
- Tabulate the displacement q_2 , percentage of the displacement q_2 due to transverse shear, and the rotation q_6 of part (b) for the following spar lengths: $z_{max} = 12, 24, 60, 120, 1800, 240,$ and 300 in. Also, tabulate the ratio of the maximum von Mises effective stress (eq. (4.31) on p. 84) to the yield strength in the semi-circular web, or branch 1, at $z = 0$ for the same set of spar lengths.

Solution to part (a). The external distributed line load intensities resolved at the shear center are shown figure 6.13. In terms of the specified airload $f_y = f_L$ and $m_z = ef_L$, where $e = X_L - X_{SC}$. The differential equation for

the transverse shear force V_y is given by eq. (3.54) on page 43. Substitute the expression for the airload to get the shear force as

$$V_y = \int \left[-\left(\frac{2L}{\pi z_{max}}\right) \sqrt{1 - \left(\frac{z}{z_{max}}\right)^2} \right] dz = \left(\frac{L}{\pi}\right) \left[\operatorname{acos}\left(\frac{z}{z_{max}}\right) - \left(\frac{z}{z_{max}}\right) \sqrt{1 - \left(\frac{z}{z_{max}}\right)^2} \right] + c_1. \quad (\text{a})$$

Note that the integration is facilitated by the substitution $z/z_{max} = \cos\theta$, and using trigonometric identities.

The constant of integration c_1 is determined by the boundary condition $V_y(z_{max}) = Q_2$. Hence $c_1 = Q_2$, and the final result for the shear force is

$$V_y = \left(\frac{L}{\pi}\right) \left[\operatorname{acos}\left(\frac{z}{z_{max}}\right) - \left(\frac{z}{z_{max}}\right) \sqrt{1 - \left(\frac{z}{z_{max}}\right)^2} \right] + Q_2. \quad (\text{b})$$

The shear force at the root for $Q_2 = 0$ is $V_y(0)|_{Q_2=0} = L/2$.

The bending moment M_x is determined by eq. (3.55) on page 43. Substitute the result for the shear force V_y into eq. (3.55) to find

$$M_x = \int \left\{ \left(\frac{L}{\pi}\right) \left[\operatorname{acos}\left(\frac{z}{z_{max}}\right) - \left(\frac{z}{z_{max}}\right) \sqrt{1 - \left(\frac{z}{z_{max}}\right)^2} \right] + Q_2 \right\} dz.$$

Again, the integration for M_x is facilitated by the substitution $z/z_{max} = \cos\theta$, and using trigonometric identities to get

$$M_x = \left(\frac{Lz_{max}}{\pi}\right) \left[-\sqrt{1 - \left(\frac{z}{z_{max}}\right)^2} + \frac{1}{3} \left[1 - \left(\frac{z}{z_{max}}\right)^2 \right]^{3/2} + \left(\frac{z}{z_{max}}\right) \operatorname{acos}\left(\frac{z}{z_{max}}\right) \right] + Q_2 z + c_2. \quad (\text{c})$$

The constant of integration c_2 is determined by the boundary condition $M_x(z_{max}) = 0$. Hence $c_2 = -z_{max}Q_2$, and the final result for the bending moment is

$$M_x = \left(\frac{Lz_{max}}{\pi}\right) \left[-\sqrt{1 - \left(\frac{z}{z_{max}}\right)^2} + \frac{1}{3} \left[1 - \left(\frac{z}{z_{max}}\right)^2 \right]^{3/2} + \left(\frac{z}{z_{max}}\right) \operatorname{acos}\left(\frac{z}{z_{max}}\right) \right] - Q_2(z_{max} - z). \quad (\text{d})$$

The bending moment at the root of the spar for $Q_2 = 0$ is $M_x(0)|_{Q_2=0} = \frac{-2Lz_{max}}{3\pi}$.

From eq. (3.61) on page 43, and that $m_z = ef_L$, we can express the torque M_z as

$$\frac{dM_z}{dz} = -m_z = -ef_L. \quad (\text{e})$$

Hence,

$$M_z = -e \int \frac{2L}{\pi z_{max}} \sqrt{1 - \left(\frac{z}{z_{max}}\right)^2} dz + c_3 = -e \frac{L}{\pi} \left\{ \left(\frac{z}{z_{max}}\right) \sqrt{1 - \left(\frac{z}{z_{max}}\right)^2} + \operatorname{asin}\left(\frac{z}{z_{max}}\right) \right\} + c_3. \quad (\text{f})$$

The constant of integration c_3 is determined from the boundary condition $M_z(z_{max}) = Q_6$, which yields

$c_3 = \frac{eL}{2} + Q_6$. The final result for the torque is

$$M_z = \frac{eL}{2} - \frac{eL}{\pi} \left[\left(\frac{z}{z_{max}} \right) \sqrt{1 - \left(\frac{z}{z_{max}} \right)^2} + \text{asin} \left(\frac{z}{z_{max}} \right) \right] + Q_6. \quad (g)$$

The torque at the root of the spar for $Q_6 = 0$ is $M_z(0)|_{Q_6=0} = eL/2$. We have determined the statically admissible shear force V_y , bending moment M_x , and torque M_z in the wing spar in terms of the distributed airload, external force Q_2 , and external torque Q_6 .

Solution to part (b). From eqs. (5.84) and (5.85) on page 145, the total complementary strain energy for the bar in this example is

$$U^* = \frac{1}{2} \int_0^{z_{max}} \left(\frac{M_x^2}{EI_{xx}} + c_{yy} V_y^2 + c_{zz} M_z^2 \right) dz. \quad (h)$$

Castigliano's second theorem for the vertical displacement of the shear center is

$$q_2 = \frac{\partial U^*}{\partial Q_2} \Bigg|_{Q_2=Q_6=0} = \int_0^{z_{max}} \left[\frac{M_x}{EI_{xx}} \frac{\partial M_x}{\partial Q_2} + c_{yy} V_y \frac{\partial V_y}{\partial Q_2} + c_{zz} M_z \frac{\partial M_z}{\partial Q_2} \right] dz \Bigg|_{Q_2=Q_6=0}. \quad (i)$$

Note that the torque is independent of force Q_2 , so that $\partial M_z / \partial Q_2 = 0$. Let $q_2 = q_{2m} + q_{2v}$, where q_{2m} is the portion of the displacement due to bending moment M_x and q_{2v} is the portion due to transverse shear force V_y .

$$q_{2m} = \int_0^{z_{max}} \left[\frac{M_x}{EI_{xx}} \frac{\partial M_x}{\partial Q_2} \right] dz \Bigg|_{Q_2=Q_6=0} = \frac{1}{EI_{xx}} \int_0^{z_{max}} M_x [-(z_{max} - z)] dz \Bigg|_{Q_2=Q_6=0}. \quad (j)$$

Substitute eq. (d) for M_x with $Q_2 = 0$ into eq. (j), and perform the integration to get

$$q_{2m} = \frac{L(-32 + 45\pi)z_{max}^3}{720EI_{xx}\pi}. \quad (k)$$

The portion of the displacement due to transverse shear force V_y is

$$q_{2v} = \int_0^{z_{max}} \left[c_{yy} V_y \frac{\partial V_y}{\partial Q_2} \right] dz \Bigg|_{Q_2=Q_6=0} = c_{yy} \int_0^{z_{max}} \left\{ \left(\frac{L}{\pi} \right) \left[\text{acos} \left(\frac{z}{z_{max}} \right) - \left(\frac{z}{z_{max}} \right) \sqrt{1 - \left(\frac{z}{z_{max}} \right)^2} \right] \right\} \{1\} dz = \frac{2c_{yy}Lz_{max}}{3\pi}. \quad (l)$$

Add eqs. (k) and (l) to get the total vertical displacement at the shear center as

$$q_2 = \frac{L(-32 + 45\pi)z_{max}^3}{720EI_{xx}\pi} + \frac{2c_{yy}Lz_{max}}{3\pi}. \quad (m)$$

Castigliano's second theorem for the rotation of the cross-section at $z = z_{\max}$ about the shear center is

$$q_6 = \left. \frac{\partial U^*}{\partial Q_6} \right|_{Q_2 = Q_6 = 0} = \int_0^{z_{\max}} \left[\frac{k}{EI_{xx}} M_x \frac{\partial M_x}{\partial Q_6} + c_{yy} V_y \frac{\partial V_y}{\partial Q_6} + c_{zz} M_z \frac{\partial M_z}{\partial Q_6} \right] dz \bigg|_{Q_2 = Q_6 = 0}. \quad (\text{n})$$

The bending moment M_x and transverse shear force V_y are independent of the external torque Q_6 . Hence,

$$q_6 = \left. \frac{\partial U^*}{\partial Q_6} \right|_{Q_2 = Q_6 = 0} = c_{zz} \int_0^{z_{\max}} \left\{ \frac{eL}{2} - \frac{eL}{\pi} \left[\left(\frac{z}{z_{\max}} \right) \sqrt{1 - \left(\frac{z}{z_{\max}} \right)^2} + \text{asin} \left(\frac{z}{z_{\max}} \right) \right] \right\} [1] dz. \quad (\text{o})$$

The integration of eq. (o) yields

$$q_6 = \frac{2c_{zz}eLz_{\max}}{3\pi}. \quad (\text{p})$$

Solution to part (c). Numerical evaluation of the displacements yields

$$q_{2m} = \left(\frac{5.438 \times 10^{-7}}{\text{in.}^2} \right) z_{\max}^3 \quad q_{2v} = (1.0489 \times 10^{-3}) z_{\max} \quad q_6 = (4.3905 \times 10^{-5} \text{ in.}^{-1}) z_{\max}. \quad (\text{q})$$

The expression for the shear flow is given by eq. (3.163) on page 70. At the root cross section the equation for the shear flow reduces to

$$q(s, 0) = M_z(0)/(2A_c) - F_{y_i}(s_i)V_y(0). \quad (\text{r})$$

The torque results in a spatially uniform component to the shear flow around the contour equal to

$$q_t = M_z(0)/(2A_c) = 76.9188 \text{ lb/in.} \quad (\text{s})$$

(Refer to eq. (3.165) on p. 70). The total shear flow in each branch is

$$q_i(s_i, 0) = 76.9188 \text{ lb/in.} - F_{y_i}(s_i)(6000.0 \text{ lb.}) \quad i = 1, 2, 3, 4, \quad (\text{t})$$

where the contour coordinates s_i are shown in figure 3.24(b) on page 71, and the shear flow distribution functions $F_{y_i}(s_i)$ are given by eqs. (ab) to (ae) in part c of example 3.4. The shear stress distribution along the contour in each branch is given by

$$\sigma_{zsi}(s_i, 0) = [q_i(s_i, 0)]/t \quad i = 1, 2, 3, 4. \quad (\text{u})$$

In this example $I_{xy} = n_x = n_y = 0$ and $k = 1$ in the axial normal stress given by eq. (4.6) on page 79. For no change in temperature and $N = M_y = 0$, the axial normal stress eq. (4.6) at the root cross section in each branch reduces to

$$\sigma_{zsi}(s_i, 0) = \frac{M_x(0)y_i(s_i)}{I_{xx}} = \left[\left(-25.0591 \frac{\text{lb.}}{\text{in.}^4} \right) z_{\max} \right] y_i(s_i) \quad i = 1, 2, 3, 4. \quad (\text{v})$$

The parametric equations for the y-coordinates of the contour in each branch are

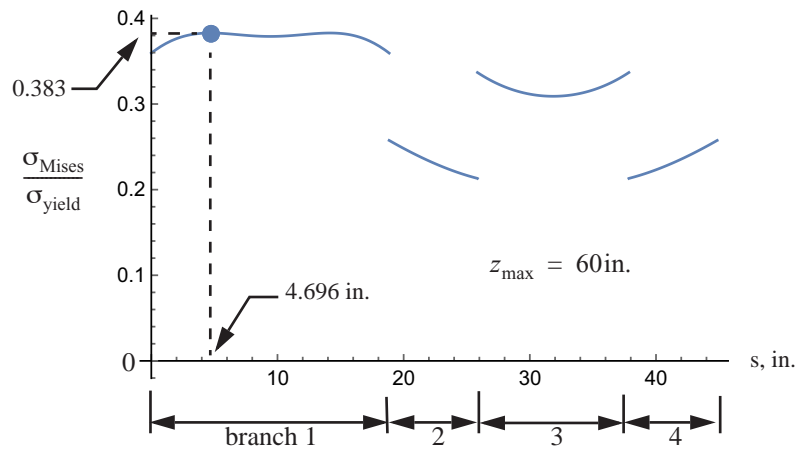
$$y_1(s_1) = -a \cos(s_1/a) \quad y_2 = a \quad y_3(s_3) = a - s_3 \quad y_4 = -a. \quad (\mathbf{w})$$

(Refer to figure 3.24 on page 71.) From eq. (4.31) the von Mises effective stress is

$$(\sigma_{\text{Mises}})_i = \sqrt{[\sigma_{zi}(s_i, 0)]^2 + 3[\sigma_{zsi}(s_i, 0)]^2} \quad i = 1, 2, 3, 4. \quad (\mathbf{x})$$

The von Mises effective stress normalized by the yield strength is plotted with respect to the contour coordinate for $z_{\text{max}} = 60$ inches in figure 6.14. As shown in the figure the maximum normalized effective stress is 0.383 at $s = 4.696$ inches in branch 1. In terms of an angular measurement in the semi-circular branch 1, we note the location as $s_1/a = (4.696/6)(180^\circ/\pi) = 44.8^\circ$. For the other values of z_{max} , the maximum value of the von Mises stress also occurs in branch 1 but at different angular locations. Discontinuities in the von Mises stress with respect to the contour coordinate are a result of the jumps in the shear flow across the stringers. (Refer to eq. (3.135) on p. 65).

Fig. 6.14 Normalized von Mises effective stress plotted with respect to the contour coordinate s at the root cross section of the spar.



Numerical results are listed in table 6.5.

Table 6.5 Wing tip displacements and wing root stresses as a function of the span

$z_{\text{max}}, \text{ in.}$	Wing tip			Wing root	
	$q_2, \text{ in.}$	$q_{2v}/q_2, \%$	$q_6, \text{ deg.}$	$\sigma_{\text{Mises}}/\sigma_{\text{yield}}$	$s_1/a, \text{ deg.}$
12.	0.0135	93.1	0.0298	0.379	90.0
24.	0.0327	77.0	0.0597	0.379	90.0
60.	0.180	34.9	0.149	0.383	44.8
120.	1.066	11.8	0.298	0.509	8.82
180.	3.360	5.62	0.448	0.683	3.81
240.	7.77	3.29	0.597	0.872	2.13
300.	15.0	2.10	0.746	1.07	1.36

Note that as the length of the spar increases the percentage of the vertical displacement at the tip due to transverse shear decreases and the von Mises effective stress increases. At $z_{\text{max}} = 300$ in. the von Mises stress exceeds the yield strength of the material indicating failure by material yielding.

6.3 Coplanar Frames

Frames are also skeletal structures composed of slender bars that can transmit axial, bending, and transverse shear loads. The bars act as beams with a superimposed axial load. Joints in a frame are usually assumed rigid, which means that the rotation of all bars connected to the joint are the same. Moments can be transferred through a rigid joint, but not a hinge joint, nor ball-and-socket joint. A frame structure may also contain some hinge joints.

Example 6.7 A frame of two tubular bars

The tubular post shown in figure 6.15 supports a load of 250 N at the free end. The diameter of the cross-sectional contour is 100 mm and the wall thickness is 3 mm. The material is steel with modulus of elasticity of 206,000 N/mm² and a Poisson's ratio of 0.3. Each bar of the frame has the same uniform geometric cross section along its length. Find the vertical and horizontal displacement of the free end.

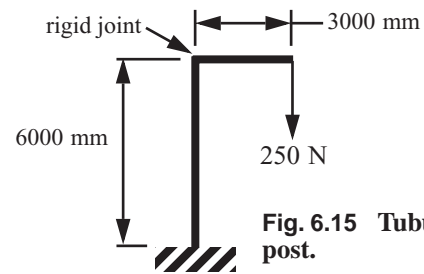


Fig. 6.15 Tubular post.

Solution. We use Castigliano's second theorem to determine the displacements of the free end for this statically determinate structure. A horizontal force Q is introduced at the free end so that the horizontal displacement can be computed from the theorem. Also, let $P = 250$ N. The complementary strain energy is determined from eqs. (5.84) and (5.85). Since there is no change in temperature nor torsion, the complementary strain energy is

$$U^* = \sum_{bars} \int_0^L \left[\frac{N^2}{2EA} + \frac{M_x^2}{2EI_{xx}} + \frac{1}{2} c_{yy} V_y^2 \right] dz. \quad (a)$$

Let u_P denote the displacement corresponding to force P , and let u_Q denote the displacement corresponding to force Q . These displacements are given by

$$u_P = \left. \frac{\partial U^*}{\partial P} \right|_{P=250, Q=0} \quad u_Q = \left. \frac{\partial U^*}{\partial Q} \right|_{P=250, Q=0}. \quad (b)$$

The coordinate system in each bar is shown in figure 6.16 (a), the free body diagram for the vertical bar in figure 6.16 (b), and the free body diagram of the horizontal bar is shown in figure 6.16 (c) The partial derivatives of the complementary strain energy for the frame with respect to the external loads are

$$\frac{\partial U^*}{\partial P} = \int_0^{L_1} \left(\frac{N_1}{EA} \frac{\partial N_1}{\partial P} + \frac{M_1}{EI_{xx}} \frac{\partial M_1}{\partial P} + c_{yy} V_1 \frac{\partial V_1}{\partial P} \right) dz_1 + \int_0^{L_2} \left(\frac{N_2}{EA} \frac{\partial N_2}{\partial P} + \frac{M_2}{EI_{xx}} \frac{\partial M_2}{\partial P} + c_{yy} V_2 \frac{\partial V_2}{\partial P} \right) dz_2, \quad (c)$$

$$\frac{\partial U^*}{\partial Q} = \int_0^{L_1} \left(\frac{N_1}{EA} \frac{\partial N_1}{\partial Q} + \frac{M_1}{EI_{xx}} \frac{\partial M_1}{\partial Q} + c_{yy} V_1 \frac{\partial V_1}{\partial Q} \right) dz_1 + \int_0^{L_2} \left(\frac{N_2}{EA} \frac{\partial N_2}{\partial Q} + \frac{M_2}{EI_{xx}} \frac{\partial M_2}{\partial Q} + c_{yy} V_2 \frac{\partial V_2}{\partial Q} \right) dz_2, \quad (d)$$

where $L_1 = 6000$ mm and $L_2 = 3000$ mm. Equilibrium determines the internal actions N, V, M in each bar.

The results are

$$N_1 = -P \quad V_1 = -Q \quad M_1 = L_2 P + (L_1 - z_1) Q \quad z_1 \in [0, L_1], \quad \text{and} \quad (e)$$

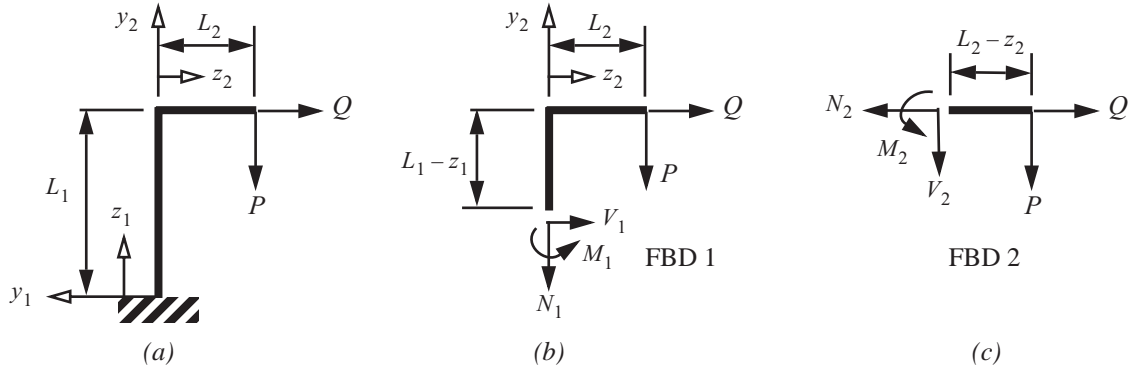


Fig. 6.16 (a) Coordinates in each bar. (b) FBD of vertical bar. (c) FBD of horizontal bar.

$$N_2 = Q \quad V_2 = -P \quad M_2 = (L_2 - z_2)P \quad z_2 \in [0, L_2]. \quad (f)$$

Evaluating the partial derivatives based on equilibrium conditions we get

$$\begin{aligned} \frac{\partial U^*}{\partial P} = & \int_0^{L_1} \left[\frac{(-P)}{EA}(-1) + \frac{(L_2 P + (L_1 - z_1)Q)}{EI_{xx}}(L_2) + c_{yy}(-Q)(0) \right] dz_1 + \\ & \int_0^{L_2} \left(\frac{Q}{EA}(0) + \frac{(L_2 - z_2)P}{EI_{xx}}(L_2 - z_2) + c_{yy}(-P)(-1) \right) dz_2 \end{aligned} \quad , \text{ and} \quad (g)$$

$$\begin{aligned} \frac{\partial U^*}{\partial Q} = & \int_0^{L_1} \left[\frac{(-P)}{EA}(0) + \frac{(L_2 P + (L_1 - z_1)Q)}{EI_{xx}}(L_1 - z_1) + c_{yy}(-Q)(-1) \right] dz_1 + \\ & \int_0^{L_2} \left[\frac{Q}{EA}(1) + \frac{(L_2 - z_2)P}{EI_{xx}}(0) + c_{yy}(-P)(0) \right] dz_2 \end{aligned} \quad (h)$$

The displacements can now be computed from the expressions for the partial derivatives as

$$u_p = \int_0^{L_1} \left[\frac{P}{EA} + \frac{L_2^2 P}{EI_{xx}} \right] dz_1 + \int_0^{L_2} \left[\frac{(L_2 - z_2)^2 P}{EI_{xx}} + c_{yy} P \right] dz_2 \Bigg|_{P=250} = \frac{L_1 P}{EA} + \underbrace{\frac{L_1 L_2^2 P}{EI_{xx}} + \frac{L_2^3 P}{3EI_{xx}}}_{\text{bending}} + c_{yy} L_2 P \Bigg|_{P=250} \quad (i)$$

$$u_Q = \frac{\partial U^*}{\partial Q} \Bigg|_{P=250, Q=0} = \int_0^{L_1} \left[\frac{(L_2 P)}{EI_{xx}}(L_1 - z_1) \right] dz_1 \Bigg|_{P=250} = \frac{L_1^2 L_2 P}{2EI_{xx}} \Bigg|_{P=250} \quad (j)$$

The formulas for the section properties are given in example 6.4 on page 160. For $a = 50$ mm and $t = 3$ mm we get

$$A = 300\pi \text{ mm}^2 \quad c_{yy} = \frac{1}{G(150\pi \text{ mm}^2)} \quad I_{xx} = 375000\pi \text{ mm}^4. \quad (\text{k})$$

For an isotropic material the shear modulus is computed from $G = E/[2(1 + \nu)]$, which evaluates to $G = 79231$ N/mm². Numerical evaluation of the displacements gives

$$u_p = \frac{(6000)250}{(206000)(300\pi)} + \frac{(6000)(3000)^2 250}{(206000)(375000\pi)} + \frac{(3000)^3 250}{3(206000)(375000\pi)} + \frac{(3000)250}{(79231)(150\pi)}, \quad (\text{l})$$

$$u_p = 0.0077259681 + 55.62697 + 9.2711617 + 0.020087459 = 64.925946 \text{ mm}, \text{ and} \quad (\text{m})$$

$$u_Q = \frac{(6000)^2(3000)250}{2(206000)(375000\pi)} = 55.62697 \text{ mm}. \quad (\text{n})$$

Note that the contribution to the displacement u_p due to bending is $55.62697 + 9.2711617 = 64.898132$ mm,

which is $\left(\frac{64.898132}{64.925946}\right) 100 = 99.957161$ % of the total displacement. **As a general rule the deflections of**

frames composed of slender bars is dominated by bending, and the contributions due to axial stretching and transverse shear deformations to the deflections can be neglected. ■

6.4 Castigliano's second theorem and statically indeterminate structures

A statically indeterminate structure is one in which the number of unknown forces exceeds the number of independent equations of static equilibrium. The excess forces are called redundants. By removing supports and/or members in a statically indeterminate structure equal to the number of redundants, a stable statically base structure can be obtained. To determine the redundants, we can imposed displacement compatibility using Castigliano's second theorem. A stable statically determinate base structure is capable of resisting the external loads. Removing a support reaction or a member in statically determinate structure renders it unstable – it is not capable of resisting external loads and it is classified as moving mechanical system (i.e., either a mechanism or linkage).

Consider a coplanar truss which consists of straight bars connected by smooth hinge joints with the external loads applied only to the joints. As discussed in example 6.3 on page 158, a truss is statically determinate if $m = 2j - r$ and statically indeterminate if $m > 2j - r$, where m denotes the number of bars or members, j the number of joints, and r denotes the number of reaction forces at the supports. Even statically determinate trusses can be unstable if the members are not arranged properly. Statical determinacy is a necessary condition for stability, not a sufficient condition. Each truss must be examined individually to determine stability. For the truss shown in part (a) of figure 6.17, $m = 9$, $r = 4$, and $j = 6$, so it is statically indeterminate. If the upper left support is removed and replaced with a horizontal force Q , then a statically determinate base structure results as shown in part (b) of figure 6.17. The force Q is the redundant and it is treated as an external load on the base structure. Equilibrium of the base structure determines the internal bar forces in terms of external forces P and Q . The solution to the truss in part (a) is effected by imposing the displacement corresponding to force Q to vanish via Castigliano's second theorem (i.e., $q = \partial U^* / \partial Q = 0$). This displacement compatibility condition determines the redundant Q .

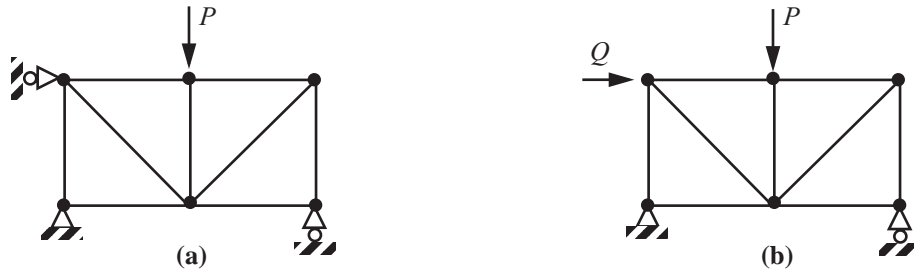


Fig. 6.17 A singly redundant truss (a), and its stable statically determinate base structure (b).

Example 6.8 **Statically indeterminate truss**

Consider the truss shown in part (a) of figure 6.18. The horizontal bars and the vertical bars have a length denoted by L , and each bar has the same elastic modulus E and same cross-sectional area A . For this truss $m = 6$, $r = 3$, and $j = 4$. So the truss is statically indeterminate. Note that this truss is statically determinate externally, but is statically indeterminate internally. Determine the bar forces in terms of the external applied load P .

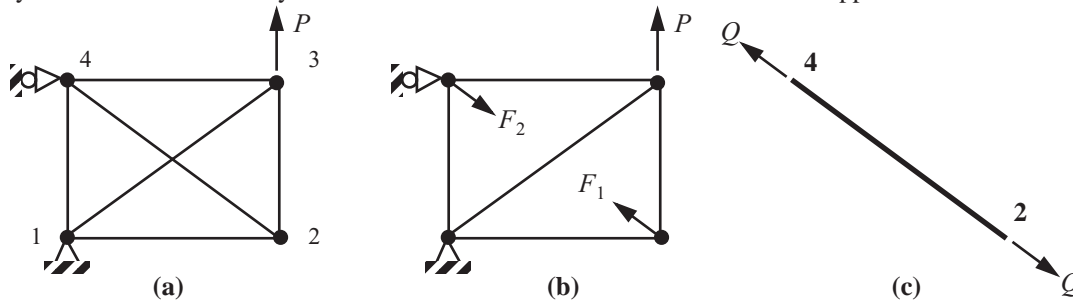


Fig. 6.18 (a) Statically indeterminate truss. (b) Statically determinant base structure with bar 2-4 replaced by forces F_1 and F_2 . (c) Bar 2-4 subject to equal and opposite forces.

Solution. Consider a statically determinate truss with bar 2-4 removed, and a force F_1 acting at joint 2 and a force F_2 acting at joint 4 as shown in figure 6.18 (b). These forces are oppositely directed along a line action coinciding with the removed bar 2-4. Let the complementary strain energy for this statically determinate, five-bar truss be denoted by \hat{U}^* . We employ Castigliano's second theorem to determine the displacement u_1 corresponding to force F_1 and displacement u_2 corresponding to F_2 . That is,

$$u_i = \frac{\partial \hat{U}^*}{\partial F_i} = \frac{1}{EA} \left[LN_{12} \frac{\partial N_{12}}{\partial F_i} + \sqrt{2} LN_{13} \frac{\partial N_{13}}{\partial F_i} + LN_{14} \frac{\partial N_{14}}{\partial F_i} + LN_{23} \frac{\partial N_{23}}{\partial F_i} + LN_{34} \frac{\partial N_{34}}{\partial F_i} \right] \quad i = 1, 2. \quad (a)$$

The bar forces are determined by joint equilibrium, and the results are shown in table 6.6. Bar forces are assumed

positive in tension.

Table 6.6 Terms in eq. (a) for Castigliano's second theorem

						$F_1 = F_2 = Q$
Bar	Length L	Axial force N	$\partial N / \partial F_1$	$LN \frac{\partial N}{\partial F_1}$	$LN \frac{\partial N}{\partial F_2}$	$LN \frac{\partial N}{\partial Q}$
1-2	L	$-F_1 / \sqrt{2}$	$-1 / \sqrt{2}$	$LF_1 / 2$	0	$LQ / 2$
1-3	$\sqrt{2}L$	$F_1 + \sqrt{2}P$	1	$\sqrt{2}LF_1 + 2LP$	0	$\sqrt{2}LQ + 2LP$
1-4	L	$-F_2 / \sqrt{2}$	0	0	$(LF_2) / 2$	$(LQ) / 2$
2-3	L	$-F_1 / \sqrt{2}$	$-1 / \sqrt{2}$	$LF_1 / 2$	0	$LQ / 2$
3-4	L	$-F_1 / \sqrt{2} - P$	$-1 / \sqrt{2}$	$LF_1 / 2 + LP / \sqrt{2}$	0	$LQ / 2 + LP / \sqrt{2}$

The sum of elements in column five divided by the product EA determines displacement u_1 , and the sum of column six divided by EA determines u_2 . Simplifying the results leads to

$$u_1 = \frac{L}{EA} \left[\frac{3+2\sqrt{2}}{2} \right] F_1 + \frac{L}{EA} \left[\frac{4+\sqrt{2}}{2} \right] P \quad \text{and} \quad u_2 = \frac{L}{2EA} F_2. \quad (\text{b})$$

The relative inward displacement between joints 2 and joint 4 is given by the sum $u_1 + u_2$. For equal and opposite forces we set $F_1 = F_2 = Q$, and then the relative inward displacement reduces to

$$\Delta_{1/2} = u_1 + u_2 \Big|_{F_1=F_2=Q} = \frac{L}{EA} [2 + \sqrt{2}]Q + \frac{L}{EA} \left[\frac{4+\sqrt{2}}{2} \right] P. \quad (\text{c})$$

The seventh column in the table is obtained by setting $F_1 = F_2 = Q$. The sum of elements in the seventh column divided by EA is derivative of \hat{U}^* with respect to Q ; i.e.,

$$\frac{\partial \hat{U}^*}{\partial Q} = \frac{L}{EA} [2 + \sqrt{2}]Q + \frac{L}{EA} \left[\frac{4+\sqrt{2}}{2} \right] P. \quad (\text{d})$$

We conclude that the relative inward displacement between joints 2 and joint 4 is given by

$$\Delta_{1/2} = \frac{\partial \hat{U}^*}{\partial Q}. \quad (\text{e})$$

The elongation of bar 2-4 is denoted by Δ_{24} and its complementary strain energy is denoted by U_{24}^* .

Hooke's law for bar 2-4 is given by eq. (6.2) on page 154, which for $N \rightarrow Q$ and $L \rightarrow \sqrt{2}L$ is solved for its elongation. The complementary strain energy is given by eq. (5.84) on page 145. These relations are

$$\Delta_{24} = \frac{\sqrt{2}L}{EA}Q \text{ and } U^*_{24} = \frac{1}{2EA} \int_0^{\sqrt{2}L} N^2 dz \Big|_{N=Q} = \frac{\sqrt{2}L}{2EA}Q^2. \quad (\text{f})$$

Castigliano's second theorem is $\frac{\partial U^*_{24}}{\partial Q} = \frac{\sqrt{2}L}{EA}Q$, which is equal to the elongation. Thus, $\Delta_{24} = \frac{\partial U^*_{24}}{\partial Q}$.

Geometric compatibility of the statically indeterminate, six-bar truss requires the relative inward displacement between joints 2 and 4 equals the negative of the elongation of bar 2-4. In other words, the sum $\Delta_{1/2} + \Delta_{24} = 0$. Hence,

$$\Delta_{1/2} + \Delta_{24} = \frac{\partial \hat{U}^*}{\partial Q} + \frac{\partial U^*_{24}}{\partial Q} = \frac{\partial U^*}{\partial Q} = 0, \quad (\text{g})$$

where the total complementary strain energy of the statically indeterminate six-bar truss is $U^* = \hat{U}^* + U^*_{24}$. Hence, Castigliano's second theorem applied to the six-bar truss is

$$\frac{\partial U^*}{\partial Q} = \frac{L}{EA}[2 + \sqrt{2}]Q + \frac{L}{EA}\left[\frac{4 + \sqrt{2}}{2}\right]P + \frac{\sqrt{2}L}{EA}Q = \frac{L}{EA}\{2(1 + \sqrt{2})Q + [(4 + \sqrt{2})P]/2\} = 0 \quad (\text{h})$$

From eq. (h) we determine the redundant as

$$Q = -\left(\frac{4 + \sqrt{2}}{4(1 + \sqrt{2})}\right)P = -0.56066P. \quad (\text{i})$$

Finally, the bar forces are

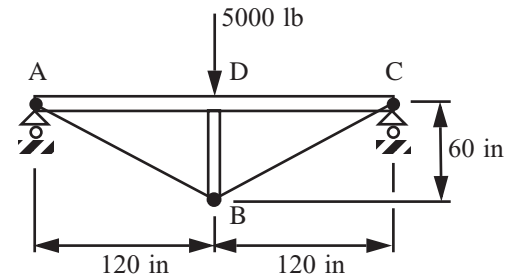
$$\begin{aligned} N_{1-2} &= 0.396447P & N_{1-3} &= 0.853553P & N_{1-4} &= 0.396447P \\ N_{2-3} &= 0.396447P & N_{2-4} &= -0.56066P & N_{3-4} &= -0.603553P \end{aligned} \quad (\text{j})$$

The condition that $\partial U^*/\partial Q = 0$ is interpreted as the relative displacement between the faces of an imaginary cut in bar 2-4 is equal to zero. ■

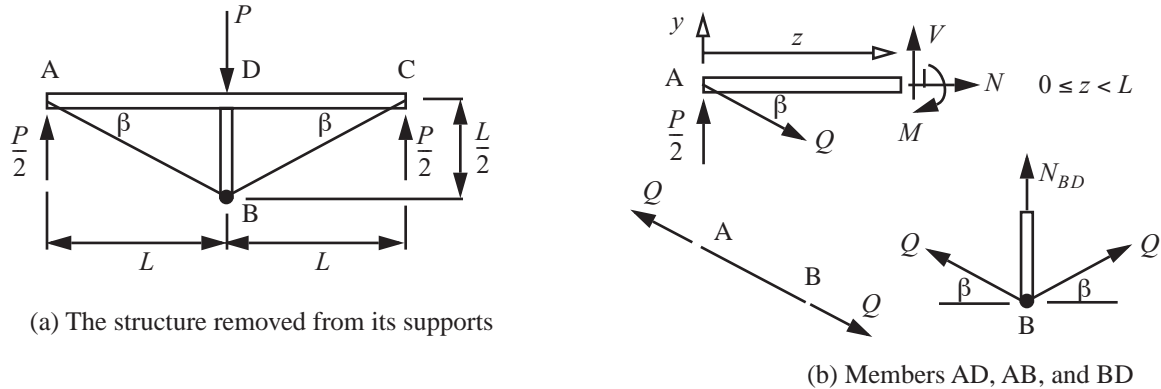
If the solution of the truss in example 6.8 was undertaken using Castigliano's first theorem, it would lead to five simultaneous linear equations for the unknown joint displacements q_3, q_4, q_5, q_6 , and q_8 in terms of the applied load P . (Refer to "Coplanar trusses" on page 153 for the displacement numbering convention.) After solving for these simultaneous equations for the joints displacements, the elongations of each bar, Δ_{i-j} , would be computed from eq. (6.6) on page 155. Lastly, the bar forces are determined from $N_{i-j} = (EA/L)_{i-j}\Delta_{i-j}$. Using Castigliano's second theorem for this singly redundant truss, we only had to solve one equation for the unknown redundant Q . The number of simultaneous equations to be solved in a statically indeterminate structure by Castigliano's second theorem is equal to the number of redundants.

Example 6.9 King Post truss

In this example we paraphrase the problem statement given in the text by Bruhn (1973, p. A8.42). The structure shown in figure 6.19 consists of members ADC, AB, BC, and BD. Continuous member ADC is simply supported at ends A and C, has an area of 9.25 in^2 , and a second area moment of 216 in^4 . Members AB, BC and BD have areas of 2 in^2 . The modulus of elasticity is the same for all members. Determine the internal actions in each member using Castigliano's second theorem.

Fig. 6.19 King Post Truss.


Solution. This structure is statically determinant externally. Also, the structure, its support conditions, and the


Fig. 6.20 Free body diagrams of the King Post truss.

external loading are symmetric about the vertical line of action of the 5,000 lb. force. The support reactions of the truss removed from its supports at A and C are shown in figure 6.20(a). Consideration of the free body diagrams of members AD, AB, and BD in figure 6.20(b) leads to the conclusion that this structure is statically indeterminate internally. The redundant Q is taken as the axial force in member AB. If Q is known, then the forces and moments in the other members are determined by equilibrium. Neglecting the energy due to transverse shear in member ADC, the complementary strain energy is

$$U^* = 2 \int_0^L \left[\frac{N^2}{2EA_{AC}} + \frac{M^2}{2EI} \right] dz + 2 \left[\frac{L_{AB} Q^2}{2EA} \right] + \frac{(L/2) N_{BD}^2}{2EA}. \quad (\text{a})$$

Note that the complementary strain energy in members AD and AB are multiplied by two to account for the energy in members DC and BC, respectively. The compatibility condition that the relative displacement of an imaginary cut in member AB vanishes is that the derivative of the complementary energy with respect to Q equals zero. Thus,

$$0 = 2 \int_0^L \left\{ \frac{N}{EA_{AC}} \left(\frac{\partial N}{\partial Q} \right) + \frac{M}{EI} \left(\frac{\partial M}{\partial Q} \right) \right\} dz + \frac{2L_{AB} Q}{EA} + \frac{(L/2) N_{BD}}{EA} \left(\frac{\partial N_{BD}}{\partial Q} \right). \quad (\text{b})$$

Equilibrium equations of member AD are

$$N + Q \cos \beta = 0 \quad V + P/2 - Q \sin \beta = 0 \quad M + (P/2 - Q \sin \beta)z = 0 \quad 0 \leq z < L. \quad (\text{c})$$

The axial equilibrium equation for member BD is

$$N_{BD} + 2Q \sin \beta = 0. \quad (\text{d})$$

Substitute member axial forces and moment from the equilibrium eq. (c) into the compatibility condition (b) to get

$$0 = 2 \int_0^L \left\{ \frac{(-Q \cos \beta)}{EA_{AC}} (-\cos \beta) + \frac{[-(P/2 - Q \sin \beta)z]}{EI} [(\sin \beta)z] \right\} dz + \frac{2L_{AB}Q}{EA} + \frac{(L/2)(-2Q \sin \beta)}{EA} (-2 \sin \beta). \quad (\text{e})$$

Perform the integration in eq. (e) followed by the substitutions $L_{AB} = (\sqrt{5}L)/2$, $\cos \beta = 2/(\sqrt{5})$, and $\sin \beta = 1/(\sqrt{5})$ to find

$$0 = 2 \left(\frac{-L^3 P}{6\sqrt{5}EI} + \frac{4LQ}{5EA_{AC}} + \frac{L^3 Q}{15EI} \right) + \frac{2LQ}{5EA} + \frac{\sqrt{5}LQ}{EA}. \quad (\text{f})$$

Solve (f) for the redundant Q :

$$Q = \frac{\sqrt{5}AA_{AC}L^2P}{24AI + 6A_{AC}I + 15\sqrt{5}A_{AC}I + 2AA_{AC}L^2}. \quad (\text{g})$$

Substitute the numerical values for the quantities on the right-hand side of eq. (g) to find the redundant:

$$Q = \frac{\sqrt{5}(2)(9.25)(120^2)(5000)}{24(2)(216) + 6(9.25)(216) + 15\sqrt{5}(9.25)(216) + 2(2)(9.25)(120^2)} = 4,787.18 \text{ lb.} \quad (\text{h})$$

The axial force in member BD from eq. (d) is

$$N_{BD} = -2(4787.18) \left(\frac{1}{\sqrt{5}} \right) = -4281.78 \text{ lb.} \quad (\text{i})$$

The negative value of N_{BD} means member BD is in compression. The axial force and bending moment in member AD is

$$N = -(4787.18)2/\sqrt{5} = -4281.78 \text{ lb.} \quad M = \left(-2500 + \frac{(4747.18)}{\sqrt{5}} \right) z = (-359.11 \text{ lb.})z \quad 0 \leq z < 120 \text{ in} \quad (\text{j})$$

6.4.1 Function of a Turnbuckle

A turnbuckle is a metal coupling device consisting of an oblong piece, or barrel, internally threaded at both ends into which the corresponding sections of two threaded rods are screwed in order to form a unit that can be adjusted for tension or length. A right-hand thread is used at one end and a left-hand thread at the other end. The device either lengthens or shortens when the barrel is rotated. Each full turn of the barrel causes it to travel a distance p along each screw, where p is the pitch of the threads. Tightening the turnbuckle by one turn causes the rods to be drawn closer together by a distance $2p$. That is, one turn to tighten causes the device to shorten by $2p$.

For n turns the shortening distance is $2np$, where n need not be an integer. Turnbuckles are widely used in aircraft. Biplanes may use turnbuckles to adjust the tension on structural wires bracing their wings as discussed in example 6.10 below. Turnbuckles are also widely used with flexible cables in flight control systems.

Example 6.10 Rigging biplane landing and flying wires

An acrobatic biplane has a maximum gross weight of 1,700 lbs. and a wing span of 25 feet. The cross sections of the lower and upper wings are thin, so the wing structure is strengthened by external bracing. As shown in figure 6.21 the bracing consists of landing and flying wires connecting the fuselage to the wings at the interplane strut. Turnbuckles inserted in the landing and flying wires are used to pre-tension the wires by changing their length.

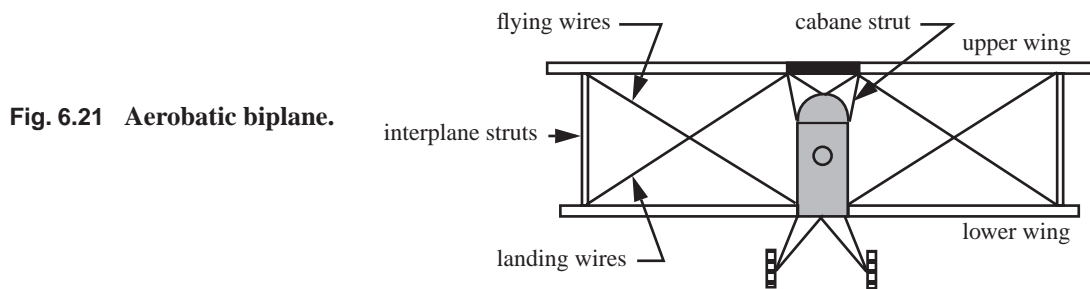


Fig. 6.21 Aerobatic biplane.

We will model the structural unit consisting of the lower wing, upper wing, interplane strut, landing wires, and flying wires as shown in figure 6.22 (a). The left-hand wings are modeled as a pin-jointed truss. Bars 1-2 and 3-4 represent the spars in the lower wing and upper wing, respectively, and are of length $L = 10$ ft. The spars are made of Sitka spruce with a Young's modulus parallel to the grain of 1.5×10^6 lb./in.², and a cross-sectional area of 1.25 in.². Bar 1-3 represents the landing wire, bar 2-4 the flying wire, and the wires are made of stainless steel with a modulus of 30×10^6 lb./in.². Each wire has a diameter of 0.125 in. Bar 1-4 is the interplane strut of length h equal to 4.3 ft., and it is assumed to be very stiff. The wings are specified to have a dihedral angle $\Gamma = 4^\circ$.

Determine the number of turns in the flying wire turnbuckle n_F , and the number of turns in the landing wire turnbuckle n_L , such that the flying wire tension is 400 lb., and the dihedral is maintained at four degrees. The pitch of the turnbuckle threads is $p = 1$ in./ $(10$ turns).

Solution. The structural model of the left-hand wing and bracing shown in figure 6.22 (a) consists of five truss bars. The turnbuckle displacements are determined from the horizontal position of the wing. Free body diagrams of joints 1 and 4 are shown in figure 6.22 (b). A vertical external force Q_2 is introduced at joint 1 so that its corresponding displacement q_2 can be determined in the application of Castigliano's theorem. Displacement q_2 is specified from the wing's required dihedral. That is $q_2 = L \sin \Gamma$, and after its determination external force Q_2 is set to zero.

From eq. (5.84) on page 145 the complementary energy for a homogenous truss bar subject to a uniform change in temperature is

$$U^* = \frac{L}{2EA}(N + N_T)^2. \quad (\text{a})$$

To account for the displacements of the turnbuckles in Castigliano's second theorem we modify the axial temper-

ature term in the complementary strain energy. The thermal axial force in the truss bar is obtained from eq. (3.75) on page 46:

$$N_T = \int_c \beta \Delta T(s, z) t(s) ds = EA \alpha \Delta T. \quad (b)$$

Let the thermal strain $\alpha \Delta T$ be replaced by the initial strain induced by the turnbuckle displacement $\Delta_T = 2np$ divided by the length of the bar containing the turnbuckle. That is, $\alpha \Delta T \rightarrow \Delta_T/L$. Then the complementary strain energy in eq. (a) that includes the displacement caused by the turnbuckle is

$$U^* = \frac{c}{2} (N + \Delta_T/c)^2, \quad (c)$$

where the flexibility influence coefficient $c = L/(EA)$.

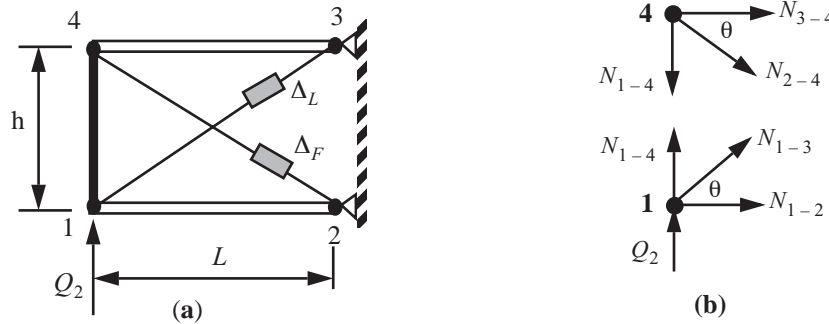


Fig. 6.22 (a) Structural model of the left-hand wing and bracing. (b) Free body diagrams.

The interplane strut subject to force N_{1-4} is assumed to be rigid. Its flexibility influence coefficients vanishes and it does not contribute the elastic complementary strain energy. The complementary strain energy is

$$U^* = \frac{1}{2} c_W (N_{1-2})^2 + \frac{1}{2} c_{ST} (N_{1-3} + \Delta_L/c_{ST})^2 + \frac{1}{2} c_{ST} (N_{2-4} + \Delta_F/c_{ST})^2 + \frac{1}{2} c_W (N_{3-4})^2. \quad (d)$$

The flexibility influence coefficients for the two wing spars is

$$c_W = \frac{L}{E_W A_W} = \frac{120}{(1.5 \times 10^6) 1.25} = 64 \times 10^{-6} \text{ in./lb}, \quad (e)$$

and the flexibility influence coefficients for the two wires is

$$c_{ST} = \frac{\sqrt{120^2 + 51.6^2}}{(30 \times 10^6) \pi (0.125/2)^2} = \frac{130.624}{(30 \times 10^6) (0.012272)} = 354.81 \times 10^{-6} \text{ in./lb}. \quad (f)$$

Equilibrium equations at joint 1 in figure 6.22(b) are

$$N_{1-3} \cos \theta + N_{1-2} = 0 \quad Q_2 + N_{1-4} + N_{1-3} \sin \theta = 0, \quad (g)$$

and equilibrium equations at joint 4 are

$$N_{3-4} + N_{2-4} \cos \theta = 0 \quad N_{1-4} + N_{2-4} \sin \theta = 0. \quad (\text{h})$$

The trigonometric functions of the angle θ are

$$\cos \theta = \lambda = \frac{120}{\sqrt{120^2 + 51.6^2}} = 0.91866919 \quad \sin \theta = \mu = \frac{51.6}{\sqrt{120^2 + 51.6^2}} = 0.39502775. \quad (\text{i})$$

Now eliminate the bar force N_{1-4} between the four equilibrium equations to get the three equations

$$N_{1-3} \lambda + N_{1-2} = 0 \quad N_{3-4} + N_{2-4} \lambda = 0 \quad Q_2 - N_{2-4} \mu + N_{1-3} \mu = 0. \quad (\text{j})$$

The force N_{2-4} in the flying wire is taken as the redundant. Solve the remaining bar forces from eq. (j) in terms of the redundant and force Q_2 to get

$$N_{1-2} = -N_{2-4} \lambda + Q_2 \lambda / \mu \quad N_{1-3} = N_{2-4} - Q_2 / \mu \quad N_{3-4} = -N_{2-4} \lambda. \quad (\text{k})$$

Substitute the results for N_{1-2} , N_{1-3} , and N_{3-4} from eq. (k) into the complementary strain energy (d) to find the energy in the form $U^*[N_{2-4}, Q_2]$ with turnbuckle displacements Δ_L and Δ_F appearing in U^* as parameters.

$$q_2 = \left. \frac{\partial U^*}{\partial Q_2} \right|_{Q_2=0} = -(c_{ST} + c_w \lambda^2)(N_{2-4} / \mu) - (\Delta L) / \mu. \quad (\text{l})$$

$$\frac{\partial U^*}{\partial N_{2-4}} = 2(c_{ST} + c_w \lambda^2)N_{2-4} - (c_{ST} + c_w \lambda^2)(Q_2 / \mu) + \Delta_L + \Delta_F = 0. \quad (\text{m})$$

Set $q_2 = L \sin \Gamma$ in eq. (l) and solve for the landing wire turnbuckle displacement, followed by solving eq. (m) for the to find flying wire turnbuckle displacement. The results are

$$\Delta L = -(c_{ST} + c_w \lambda^2)N_{2-4} - \mu L \sin \Gamma \quad \text{and} \quad \Delta F = -(c_{ST} + c_w \lambda^2)N_{2-4} + \mu L \sin \Gamma. \quad (\text{n})$$

Set $N_{2-4} = 400$ lb. to obtain the numerical results for the turnbuckle displacements and their number of turns as

$$\Delta L = -3.41912 \text{ in.} \quad n_L = -17.0956 \text{ turns} \quad \Delta F = 3.19425 \text{ in.} \quad n_F = 15.9713 \text{ turns.} \quad (\text{o})$$

The landing wire turnbuckle decreases the length between joints 1 and 3, and the flying wire turnbuckle increases the length between joints 2 and 4. The bar forces are

$$N_{1-2} = N_{3-4} = -367.468 \text{ lb.} \quad N_{1-3} = N_{2-4} = 400 \text{ lb.} \quad N_{1-4} = -158.011 \text{ lb.} \quad \blacksquare \quad (\text{p})$$

6.5 References

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Thornton, E. A., 1996, **Thermal Structures for Aerospace Applications**, AIAA Education Series, American Institute of Aeronautics and Astronautics, Inc., Reston, Virginia, pp. 118-121.

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6.6 Practice exercises

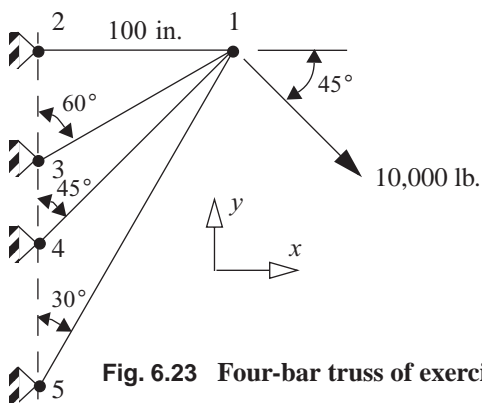


Fig. 6.23 Four-bar truss of exercise 1.

1. Each bar in the truss shown in figure 6.23 has a cross-sectional area of 1.0 in.^2 , and a modulus of elasticity of 10^7 psi . There is no change in temperature. Use Castigliano's first theorem to find
 - a) the horizontal and vertical displacements of joint 1,
 - b) the stress in psi in each bar, and
 - c) the horizontal and vertical support reactions at joint 5.

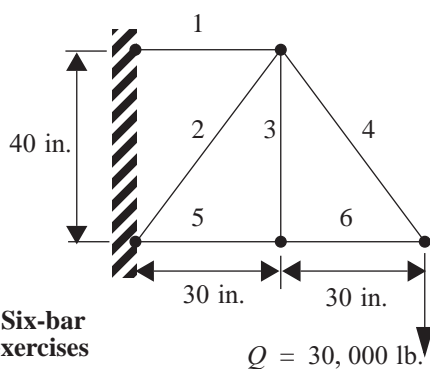


Fig. 6.24 Six-bar truss for exercises 2 and 3.

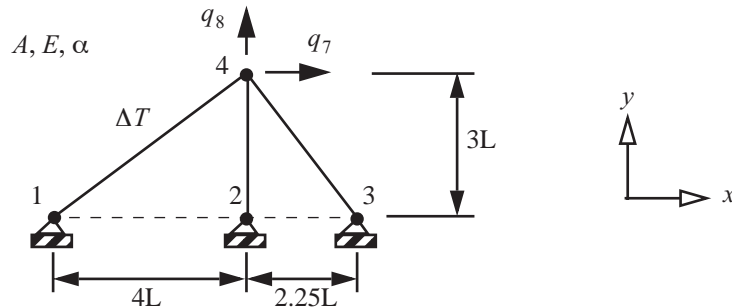
2. The bars in the truss shown in figure 6.24 have the following cross-sectional areas: $A_1 = 1.0 \text{ in.}^2$, $A_2 = A_4 = 2.0 \text{ in.}^2$, $A_3 = 1/2 \text{ in.}^2$, $A_5 = A_6 = 3/2 \text{ in.}^2$. The modulus of elasticity of each bar is 10^7 psi . Compute the vertical displacement of the right-hand joint using Castigliano's second theorem. Note this truss is statically determinate and all bar forces can be determined in terms of external load Q .

3. Use Castigliano's second theorem to compute the horizontal displacement of the right-hand joint of exercise 2.

4. The truss shown figure 6.25 consists of three bars: 1-4, 2-4, and 3-4. Each bar has the same cross-sectional area A , modulus of elasticity E , and the same coefficient of thermal expansion α . Bar 1-4 is subjected to a change in temperature ΔT from ambient temperature (the unstressed state), while bars 2-4 and 3-4 remain at ambient temperature. Use Castigliano's first theorem to deter-

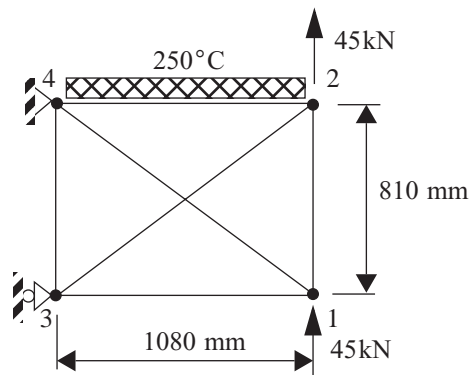
mine the horizontal displacement q_7 and the vertical displacement q_8 of joint 4.

Fig. 6.25 Three-bar truss of exercise 4.



5. The plane truss shown in figure 6.26 represents a single bay of a wing spar truss. For all bars: $E = 75 \text{ GPa}$ and $\alpha = 23.0 \times 10^{-6} / ^\circ\text{C}$. The cross-sectional areas of the bars are: 2580 mm^2 for the horizontal bars, 387 mm^2 for the vertical bars, and 2690 mm^2 for the diagonal bars. The upper horizontal bar is heated to 250°C above the zero stress temperature, and all other bars remain at the zero stress temperature. Two 45 kN lift forces act at joints 1 and 2.

Fig. 6.26 Six-bar truss in a single bay of a wing spar.



Use Castigliano's first theorem to find

- stiffness matrix in kN/mm ,
 - displacement of all joints in mm ,
 - all boundary reactions in kN , and
 - the stresses in MPa in each bar.
6. The truss shown in figure 6.27 consists of five bars: 1-2, 1-3, 1-4, 2-4, and 3-4. Each bar has the same cross-sectional area A and same modulus of elasticity E . The lengths of bars 1-2, 1-4, and 3-4 are the same, and are denoted by L . A horizontal force of magnitude P is applied to joint 1. Use Castigliano's second theorem to determine the horizontal displacement q_5 of joint 3.
7. A simply supported, uniform beam of length L is subjected to a moment Q_1 at its left end as shown in figure 6.28. The material is homogeneous and linear elastic, the cross section is symmetric ($I_{xy} = 0$), and there are no thermal strains. The bending stiffness is EI . Use Castigliano's second theorem to determine the rotation at (a) the left end, and (b) the right end. Neglect energy due to transverse shear deformation.

Fig. 6.27 Five-bar truss of exercise 6.

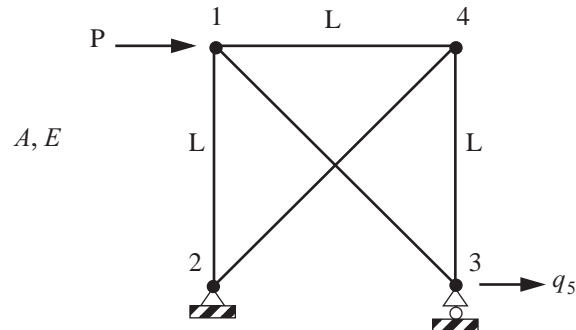


Fig. 6.28 Simply supported beam of exercise 7.

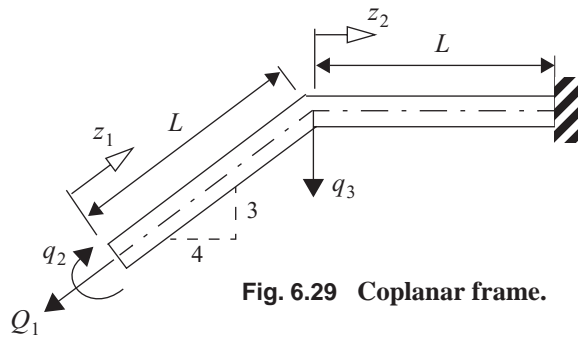
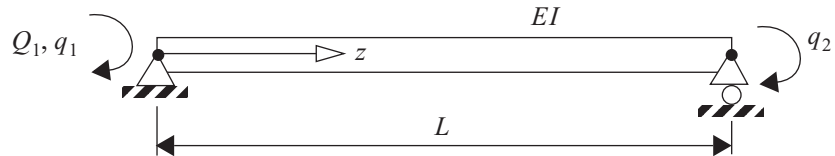
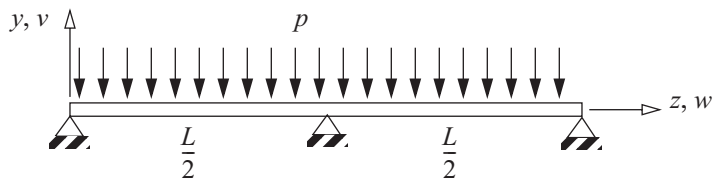


Fig. 6.29 Coplanar frame.

8. A coplanar frame is subjected to an end force Q_1 as shown in figure 6.29. The bars of the frame are uniform with axial stiffness EA and bending stiffness EI . Use Castigliano's second theorem to find
- the end rotation q_2 , and
 - the vertical displacement q_3 at the joint.

9. Consider the statically indeterminate, uniform beam shown in figure 6.30 that is subjected to a uniform, downward distributed load of intensity p . For small displacements assume that only the complementary strain energy in bending is significant. If the center support moves downward by the amount $0.01pL^4/EI_{xx}$ and remains attached to the beam, use Castigliano's second theorem to find the reactions at the left and right supports.

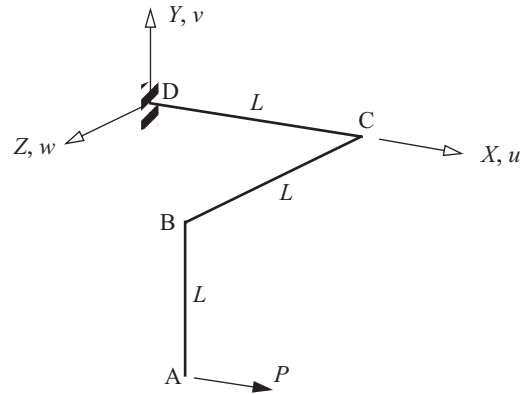
Fig. 6.30 Uniform beam of exercise 9



10. The frame consists of three slender, uniform bars of length L , and two right angle bends. Assume the bends are rigid joints. Each member has a solid circular cross section of diameter d . A force P acts in the global X -direction at point A. Find the three displacement components u_A, v_A, w_A of point A in terms of P, L, d , and E using

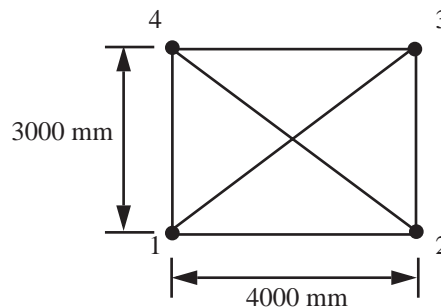
Castigliano's second theorem. Assume $G = 0.4E$. Neglect deformations due to transverse shear.

Fig. 6.31 Space frame of exercise 10.



- 11.** The rectangular space truss shown in the sketch consists of six bars: 1-2, 1-3, 1-4, 2-3, 2-4, and 3-4. The cross-sectional area of each bar is 200 mm^2 . The temperature of bar 2-3 is increased by 30°C above the stress free temperature, while the other five bars remain at the stress free temperature. Calculate the forces in all six bars. The coefficient of thermal expansion $\alpha = 7 \times 10^{-6} / ^\circ\text{C}$, and the modulus of elasticity $E = 200 \times 10^3 \text{ N/mm}^2$.

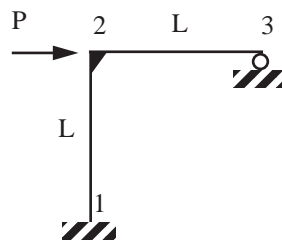
Fig. 6.32 Space truss of exercise 11



Note that $m = 6$, $j = 4$, and $r = 0$. Hence, $m < 2j - r$, and this truss cannot support an external load without accelerating. However, under the self-straining caused by the temperature change, it is statically indeterminate internally.

- 12.** Sketch the bending moment diagrams of bars 1-2 and 2-3 in the singly redundant frame shown in figure 6.33. Each bar has the same length L and flexural stiffness EI . Since the bars are slender, neglect deformations due to extension and transverse shear. Take the reaction moment at support point 1 as the redundant.

Fig. 6.33 Two-bar frame of exercise 12.



13. The aerodynamic advantages of high aspect-ratio (AR) wings are well-known—long span reduces lift-induced drag and narrow chord promotes laminar flow to reduce skin-friction drag. However, a long wing span significantly increases the structural loads at the wing root requiring heavier components to safely transmit the loading to the fuselage. The truss-braced wing (TWB) is a method to reduce the load at the wing root. (It is the subject of research in AOE at Virginia Tech under a NASA program to achieve significant fuel savings for 737 type airplanes (Warwick, 2011)). A simplified model of TWB in this exercise is a single truss bar supporting a wing spar.

A wing spar is clamped at its root and supported by a truss bar that is pinned to the support at one end and pinned to the spar at the other end. Refer to figure 6.34. The spar is subjected to a span-wise distributed air load $f_y(z)$ approximated by

$$f_y(z) = \frac{3L}{2b} \left[1 - \left(\frac{z}{b} \right)^2 \right] \quad 0 \leq \left(\frac{z}{b} \right) \leq 1. \quad (a)$$

where the lift on the wing is denoted by L and the wing span is denoted by b . The pin connection of the truss bar to the spar is at the span-wise distance $s \cdot b$ from the root, where the range of nondimensional parameter s is $0 < s \leq 1$.

The assemblage is statically indeterminate, and the statically determinate base structure is obtained by removing the lower pin support of the truss bar and replacing it by the redundant force Q which is also the tensile force in the truss bar. Refer to the right-hand sketch in figure 6.34. The condition of compatibility is the displacement corresponding to the redundant is equal to zero. Enforce compatibility by Castigliano's second theorem given by

$$q = \frac{\partial U^*}{\partial Q} = 0 \quad U^* = \int_0^{sb} \left(\frac{M_x^2}{2EI_{xx}} + \frac{N^2}{2EA} \right) dz + \int_{sb}^b \left(\frac{M_x^2}{2EI_{xx}} \right) dz + \frac{Q^2 l_s}{2EA_s}. \quad (b)$$

where l_s is the length of the strut. Numerical data are listed in table 6.7.

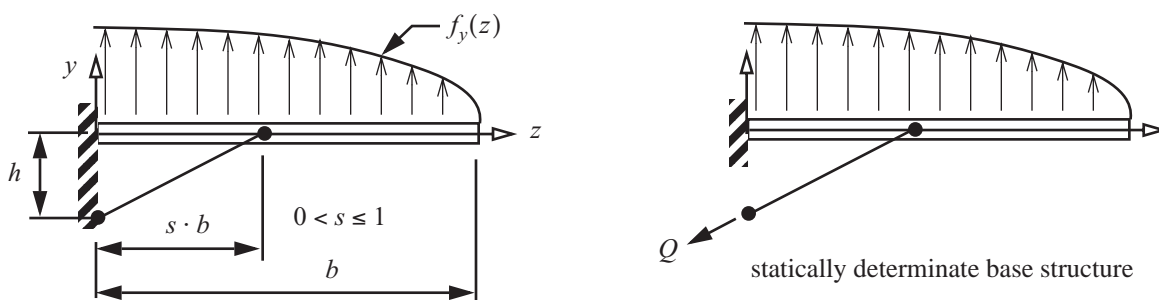


Fig. 6.34 Truss braced wing.

- Plot the normalized bending moment at the wing root $(M_x(0))/M_0$ versus s for $0.01 \leq s \leq 1$, where M_0 is the root bending moment of the cantilever wing; i.e., $M_0 = -(3/8)bL$
- Plot the tensile normal stress $\sigma = Q/A_s$ in the strut versus s for $0.01 \leq s \leq 1$.
- If the allowable tensile stress in the strut is 30 ksi, what is the value of s to yield the smallest value of the

ratio $(M_x(0))/M_0$? What is the value of $(M_x(0))/M_0$ for this particular s ?

Table 6.7 Numerical data for the strut-braced wing

b , wing span	390 in.
h , vertical distance from the spar centroid to lower strut support	72 in.
A , cross-sectional area of the spar	23.88 in. ²
I_{xx} , second area moment of the cross section of the spar	872.716 in. ⁴
A_s , cross-sectional area of the strut (1.75 in. diameter)	2.40528 in. ²
L , wing lift	50,000. lb.
E , modulus of elasticity for the spar and strut material	10×10^6 lb./in. ²

