CHAPTER 8

Laminated bars of fiber-reinforced polymer composites

8.1 Fibrous composites

A composite material consists of two or more constituents that are chemically distinct on a macroscopic scale and have a recognizable interface between them. An important class of composites for aerospace applications are fiber-reinforced polymer composites (FRP). Fiber-reinforced polymer composites consist of continuous and aligned fibers embedded in a polymer matrix. Continuous glass fibers are 3-20 μm in diameter, with most about 12 μm. The diameter of carbon and graphite fibers is about 8 μm. Fibers are inherently much stiffer and stronger than the same material in bulk form. The polymer matrix supports, protects, and transfers stresses to the fibers. Typically the matrix is of considerably lower density, stiffness, and strength than the fibers. Polymers are subdivided into thermosets and thermoplastics. Thermoset polymers, such as epoxies, become cross-linked during fabrication and do not soften on reheating. Thermoplastic polymers, such as PEEK, soften on heating and can be reshaped with heat and pressure. Usually fibers are bundled in tows, which can consist of 3,000 to 30,000 fibers.

The unidirectional lamina is the basic form of a continuous fiber composite (i.e., one with all fibers in the same direction as shown in figure. 8.1). It can be fabricated from pre-impregnated tape (filament tows pre-impregnated with epoxy), filament winding, pultrusion, or resin transfer molding (RTM). The thickness of lamina, denoted by t_{ply}, in a laminate is typically about 0.127 mm (0.005 in.). Laminates are fabricated by stacking

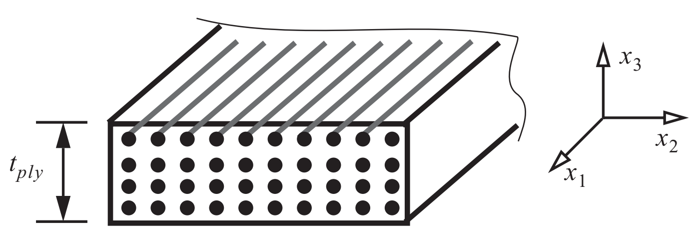


Fig. 8.1 Unidirectional lamina of a continuous fiber composite.

unidirectional lamina at different fiber orientations followed by curing. Curing is a drying process of the matrix material to form a bond between the fibers and between the lamina.

The advantage of polymer-composites aerospace structures are many: They weigh less than equivalent-strength aluminum, do not corrode or fatigue, require less maintenance, and reduce the need for drilled holes and parts. Composite parts generally cost more than equivalent metal parts, but that premium is decreasing. And the cost premium is offset by operating savings in fuel and maintenance (Canaday, 2015).

8.1.1 Material law in principal directions

Fiber-reinforced composites are usually treated as a linear elastic material with orthotropic material properties in the material principal directions (i.e directions parallel and perpendicular to the fibers). In a right-handed Cartesian system denoted by x_1-x_2-x_3, let the x_1-axis be parallel to the fibers, the x_2-axis be transverse to the fibers, and the x_3-axis be parallel to the thickness of the lamina. (Also, refer to discussion with respect to eq. (A.131) in the appendix.) In the discussion of the material law, it is convenient to use a contracted notation for strain components and the corresponding stress components. The contracted notation defines the 6-by-1 engineering strain vector in principal material directions as

$$\{\gamma_m\} = \begin{bmatrix}\varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & \gamma_{23} & \gamma_{31} & \gamma_{12}\end{bmatrix}^T, \tag{8.1}$$

where the normal strains are denoted by ε_{11}, ε_{22}, and ε_{33}, and the shear strains are denoted by γ_{23}, γ_{31}, and γ_{12}. The corresponding 6-by-1 stress vector in principal material directions is

$$\{\sigma_m\} = \begin{bmatrix}\sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{23} & \sigma_{31} & \sigma_{12}\end{bmatrix}^T, \tag{8.2}$$

where the normal stresses are denoted by σ_{11}, σ_{22}, and σ_{33}, and the shear stresses are denoted by σ_{23}, σ_{31}, and σ_{12}. See figure. 8.2.

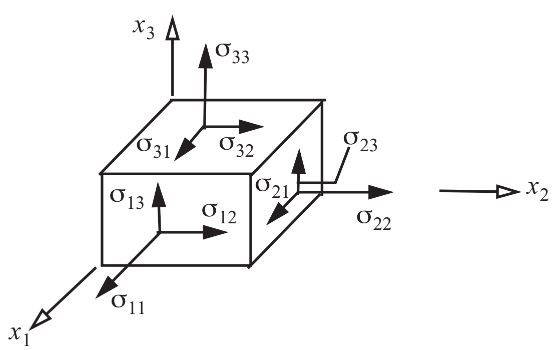


Fig. 8.2 Stresses in material principal directions.

Hooke's law for an orthotropic material in the contracted notation is

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{bmatrix} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0\\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & C_{66}\end{bmatrix}}_{[C]} \begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\end{bmatrix}, \tag{8.3}$$

in which $[C]$ is the symmetric 6X6 compliance matrix. The non-zero elements of the compliance matrix in terms of engineering constants are

$$C_{11} = \frac{1}{E_1} \qquad C_{22} = \frac{1}{E_2} \qquad C_{33} = \frac{1}{E_3} \qquad C_{44} = \frac{1}{G_{23}} \qquad C_{55} = \frac{1}{G_{13}} \qquad C_{66} = \frac{1}{G_{12}}\text{, and} \tag{8.4}$$

$$\begin{aligned} C_{21} &= -\nu_{21}/E_1 = C_{12} = -\nu_{12}/E_2 \\ C_{31} &= -\nu_{31}/E_1 = C_{13} = -\nu_{13}/E_3\,. \\ C_{32} &= -\nu_{32}/E_2 = C_{23} = -\nu_{23}/E_3 \end{aligned} \tag{8.5}$$

The nine independent engineering constants are described as follows:

- Moduli of elasticity in the fiber direction, transverse direction, and thickness direction are denoted by E_1, E_2, and E_3, respectively.
- The principal Poisson's ratios are ν_{21}, ν_{31}, and ν_{32}.
- Shear moduli in the 2-3 plane, 3-1 plane, and 1-2 plane are denoted by G_{23}, G_{31}, and G_{12}, respectively.

The three minor Poisson's ratios, ν_{12}, ν_{13}, and ν_{23}, are determined from symmetry of the compliance matrix. Material characterization tests are conducted to measure the nine independent engineering constants. However, the most accurate measurements are made for the in-plane properties E_1, E_2, ν_{21}, and G_{12}.

8.1.2 Compliance matrix in bar coordinate directions

Consider the thin-walled bar, or beam, analysis presented in article 3.2 to article 3.5. Instead of a the wall composed of a homogeneous, linear elastic material as in article 3.7, we now take the wall to be composed of a fibrous composite material. The fibers are parallel and contained in thin layers, or lamina, that are normal to the thickness coordinate direction ζ of the wall. Within a lamina the bar contour coordinate direction s, and longitudinal direction z, are not, in general, aligned with the material principal coordinate directions x_1 and x_2. Define a positive angle φ by the counterclockwise rotation from the positive z-axis to the positive x_1-axis as shown in figure. 8.3.

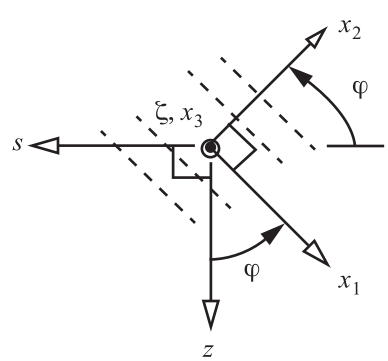


Fig. 8.3 Material principal directions x_1, x_2, and x_3 with respect to bar axes s, z, and ζ.

The direction cosines between the principal material coordinate directions x_1-x_2-x_3 and the bar coordinate directions s-z-ζ are listed in table 8.1.

Table 8.1 Direction cosines

	s	z	ζ
x_1	$\cos(90° + \varphi)$	$\cos\varphi$	0
x_2	$\cos(180° - \varphi)$	$\cos(90° + \varphi)$	0
x_3	0	0	1

Let $m = \cos\varphi$ and $n = \sin\varphi$. Then, the matrix relations between the coordinate directions are written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -n & m & 0 \\ -m & -n & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[\lambda]} \begin{bmatrix} s \\ z \\ \zeta \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} s \\ z \\ \zeta \end{bmatrix} = \underbrace{\begin{bmatrix} -n & -m & 0 \\ m & -n & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[\lambda]^T} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} . \tag{8.6}$$

Matrix [λ] is an orthogonal matrix so that $[\lambda][\lambda]^T = [\lambda]^T[\lambda] = [I]$, and $Det[\lambda] = 1$. In the material coordinate directions the symmetric, Cartesian strain tensor is denoted by the 3X3 matrix $[\varepsilon]$, and the symmetric, stress tensor is denoted by the 3X3 matrix $[\sigma]$. The elements of these matrices are

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{11} & \gamma_{12}/2 & \gamma_{13}/2 \\ \gamma_{12}/2 & \varepsilon_{22} & \gamma_{23}/2 \\ \gamma_{13}/2 & \gamma_{23}/2 & \varepsilon_{33} \end{bmatrix} \text{ and } [\sigma] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} . \tag{8.7}$$

In the bar coordinate directions, the strain matrix $[\varepsilon']$ and stress matrix $[\sigma']$ are denoted by

$$[\varepsilon'] = \begin{bmatrix} \varepsilon_{ss} & \gamma_{zs}/2 & \gamma_{\zeta s}/2 \\ \gamma_{zs}/2 & \varepsilon_{zz} & \gamma_{\zeta z}/2 \\ \gamma_{\zeta s}/2 & \gamma_{\zeta z}/2 & \varepsilon_{\zeta\zeta} \end{bmatrix} \text{ and } [\sigma'] = \begin{bmatrix} \sigma_{ss} & \sigma_{zs} & \sigma_{\zeta s} \\ \sigma_{zs} & \sigma_{zz} & \sigma_{\zeta z} \\ \sigma_{\zeta s} & \sigma_{\zeta z} & \sigma_{\zeta\zeta} \end{bmatrix} . \tag{8.8}$$

From eq. (A.63) and eq. (A.65) in the appendix the transformation relations between the Cartesian strain matrices are

$$[\varepsilon'] = [\lambda][\varepsilon][\lambda]^T \text{ and } [\varepsilon] = [\lambda]^T[\varepsilon'][\lambda] . \tag{8.9}$$

From eq. (A.96) and eq. (A.97) in the appendix the transformation relations between the stress matrices are

$$[\sigma'] = [\lambda][\sigma][\lambda]^T \text{ and } [\sigma] = [\lambda]^T[\sigma'][\lambda] . \tag{8.10}$$

After performing the matrix operations indicated for the strain matrices (8.9), we can establish the contracted notation for the transformation of the strain vectors. The results are as follows:

$$\begin{bmatrix}\varepsilon_{ss}\\ \varepsilon_{zz}\\ \varepsilon_{\zeta\zeta}\\ \gamma_{z\zeta}\\ \gamma_{\zeta s}\\ \gamma_{sz}\end{bmatrix} = \underbrace{\begin{bmatrix} n^2 & m^2 & 0 & 0 & 0 & -mn\\ m^2 & n^2 & 0 & 0 & 0 & mn\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -n & -m & 0\\ 0 & 0 & 0 & m & -n & 0\\ 2mn & -2mn & 0 & 0 & 0 & -m^2+n^2\end{bmatrix}}_{[T_{\varepsilon 1}]}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{bmatrix} \tag{8.11}$$

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\\ \gamma_{23}\\ \gamma_{31}\\ \gamma_{12}\end{bmatrix} = \underbrace{\begin{bmatrix} n^2 & m^2 & 0 & 0 & 0 & mn\\ m^2 & n^2 & 0 & 0 & 0 & -mn\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -n & m & 0\\ 0 & 0 & 0 & -m & -n & 0\\ -2mn & 2mn & 0 & 0 & 0 & -m^2+n^2\end{bmatrix}}_{[T_{\varepsilon 2}]}\begin{bmatrix}\varepsilon_{ss}\\ \varepsilon_{zz}\\ \varepsilon_{\zeta\zeta}\\ \gamma_{z\zeta}\\ \gamma_{\zeta s}\\ \gamma_{sz}\end{bmatrix} . \tag{8.12}$$

Note that $\mathrm{Det}[T_{\varepsilon 1}] = \mathrm{Det}[T_{\varepsilon 2}] = 1$, and $[T_{\varepsilon 1}][T_{\varepsilon 2}] = [I]$. That is, $[T_{\varepsilon 2}] = [T_{\varepsilon 1}]^{-1}$. After performing the matrix operations indicated for the stress matrices (8.10), we can establish the contracted notation for the transformation of the stress vectors. The results are as follows:

$$\begin{bmatrix}\sigma_{ss}\\ \sigma_{zz}\\ \sigma_{\zeta\zeta}\\ \sigma_{z\zeta}\\ \sigma_{\zeta s}\\ \sigma_{sz}\end{bmatrix} = \underbrace{\begin{bmatrix} n^2 & m^2 & 0 & 0 & 0 & -2mn\\ m^2 & n^2 & 0 & 0 & 0 & 2mn\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -n & -m & 0\\ 0 & 0 & 0 & m & -n & 0\\ mn & -mn & 0 & 0 & 0 & -m^2+n^2\end{bmatrix}}_{[T_{\sigma 1}]}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\end{bmatrix} \tag{8.13}$$

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{33}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\end{bmatrix} = \underbrace{\begin{bmatrix} n^2 & m^2 & 0 & 0 & 0 & 2mn\\ m^2 & n^2 & 0 & 0 & 0 & -2mn\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -n & m & 0\\ 0 & 0 & 0 & -m & -n & 0\\ -mn & mn & 0 & 0 & 0 & -m^2+n^2\end{bmatrix}}_{[T_{\sigma 2}]}\begin{bmatrix}\sigma_{ss}\\ \sigma_{zz}\\ \sigma_{\zeta\zeta}\\ \sigma_{z\zeta}\\ \sigma_{\zeta s}\\ \sigma_{sz}\end{bmatrix} . \tag{8.14}$$

Note that $\mathrm{Det}[T_{\sigma 1}] = \mathrm{Det}[T_{\sigma 2}] = 1$, and $[T_{\sigma 1}][T_{\sigma 2}] = [I]$. That is, $[T_{\sigma 2}] = [T_{\sigma 1}]^{-1}$. Additionally, from the foregoing eqs. (8.11) to (8.14) the following matrix relations can be shown:

$$[T_{\varepsilon 1}]^T = [T_{\sigma 2}] \text{ and } [T_{\varepsilon 2}]^T = [T_{\sigma 1}]. \tag{8.15}$$

The elements of the 6X6 matrices in eq. (8.15) are as follows:

$$\underbrace{\begin{bmatrix} n^2 & m^2 & 0 & 0 & 0 & -mn \\ m^2 & n^2 & 0 & 0 & 0 & mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -n & -m & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 2mn & -2mn & 0 & 0 & 0 & -m^2+n^2 \end{bmatrix}^T}_{[T_{\varepsilon 1}]^T} = \underbrace{\begin{bmatrix} n^2 & m^2 & 0 & 0 & 0 & 2mn \\ m^2 & n^2 & 0 & 0 & 0 & -2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -n & m & 0 \\ 0 & 0 & 0 & -m & -n & 0 \\ -mn & mn & 0 & 0 & 0 & -m^2+n^2 \end{bmatrix}}_{[T_{\sigma 2}]}$$

, and

$$\underbrace{\begin{bmatrix} n^2 & m^2 & 0 & 0 & 0 & mn \\ m^2 & n^2 & 0 & 0 & 0 & -mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -n & m & 0 \\ 0 & 0 & 0 & -m & -n & 0 \\ -2mn & 2mn & 0 & 0 & 0 & -m^2+n^2 \end{bmatrix}^T}_{[T_{\varepsilon 2}]^T} = \underbrace{\begin{bmatrix} n^2 & m^2 & 0 & 0 & 0 & -2mn \\ m^2 & n^2 & 0 & 0 & 0 & 2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -n & -m & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ mn & -mn & 0 & 0 & 0 & -m^2+n^2 \end{bmatrix}}_{[T_{\sigma 1}]}.$$

To determine the off-axis compliance material law we pre-multiply the on-axis material law (8.3) by matrix $[T_{\varepsilon 1}]$, followed by substituting of eq. (8.14) for the on-axis stresses on the right-hand side of eq. (8.3). Use the fact that $[T_{\varepsilon 1}]^T = [T_{\sigma 2}]$ from eq. (8.15). Denote the 6X6 off-axis compliance matrix by $[C']$ and we find that $[C'] = [T_{\varepsilon 1}][C][T_{\varepsilon 1}]^T$. The form of the off-axis material law is

$$\begin{bmatrix} \varepsilon_{ss} \\ \varepsilon_{zz} \\ \varepsilon_{\zeta\zeta} \\ \gamma_{z\zeta} \\ \gamma_{\zeta s} \\ \gamma_{sz} \end{bmatrix} = \begin{bmatrix} C'_{11} & C'_{12} & C'_{13} & 0 & 0 & C'_{16} \\ C'_{21} & C'_{22} & C'_{23} & 0 & 0 & C'_{26} \\ C'_{31} & C'_{32} & C'_{33} & 0 & 0 & C'_{36} \\ 0 & 0 & 0 & C'_{44} & C'_{45} & 0 \\ 0 & 0 & 0 & C'_{54} & C'_{55} & 0 \\ C'_{61} & C'_{62} & C'_{63} & 0 & 0 & C'_{66} \end{bmatrix} \begin{bmatrix} \sigma_{ss} \\ \sigma_{zz} \\ \sigma_{\zeta\zeta} \\ \sigma_{z\zeta} \\ \sigma_{\zeta s} \\ \sigma_{sz} \end{bmatrix}. \tag{8.16}$$

Matrix $[C']$ is symmetric with the compliance coefficients in terms of the engineering constants and the directions cosines given by eq. (8.17) to (8.23) below.

$$C'_{11} = \frac{m^4}{E_2} + \frac{m^2n^2}{G_{12}} + \frac{n^4 - 2m^2n^2\nu_{21}}{E_1} \qquad C'_{21} = C'_{12} = \frac{m^2n^2(G_{12} - E_2)}{E_2G_{12}} + \frac{m^2n^2 - (m^4 + n^4)\nu_{21}}{E_1} \tag{8.17}$$

$$C'_{31} = C'_{13} = \frac{-n^2\nu_{31}}{E_1} - \frac{m^2\nu_{32}}{E_2} \qquad C'_{61} = C'_{16} = mn\left(\frac{-2m^2}{E_2} + \frac{m^2 - n^2}{G_{12}} - \frac{2m^2\nu_{21}}{E_1} + \frac{2n^2(1 + \nu_{21})}{E_1}\right) \tag{8.18}$$

$$C'_{22} = \frac{m^2n^2}{G_{12}} + \frac{n^4}{E_2} + \frac{m^4 - 2m^2n^2\nu_{21}}{E_1} \qquad C'_{23} = C'_{32} = \frac{-m^2\nu_{31}}{E_1} - \frac{n^2\nu_{32}}{E_2} \tag{8.19}$$

$$C'_{26} = C'_{62} = mn\left(\frac{-2n^2}{E_2} + \frac{n^2 - m^2}{G_{12}} + \frac{2m^2(1 + \nu_{21}) - 2n^2\nu_{21}}{E_1}\right) \tag{8.20}$$

$$C'_{33} = \frac{1}{E_3} \qquad C'_{63} = C'_{36} = 2mn\left(\frac{\nu_{32}}{E_2} - \frac{\nu_{31}}{E_1}\right) \tag{8.21}$$

$$C'_{44} = \frac{m^2}{G_{13}} + \frac{n^2}{G_{23}} \qquad C'_{54} = C'_{45} = mn\left(\frac{1}{G_{13}} - \frac{1}{G_{23}}\right) \qquad C'_{55} = \frac{m^2}{G_{23}} + \frac{n^2}{G_{13}} \tag{8.22}$$

$$C'_{66} = \frac{(m^2 - n^2)^2}{G_{12}} + \frac{4m^2n^2(E_1 + (1 + 2\nu_{21})E_2)}{E_1E_2}. \tag{8.23}$$

8.1.3 Plane stress

Since composites used in many structural applications are thin plates or thin shells, the assumption of a plane stress state as used plate and shell theory is also made for a composite plate. In figure. 8.2 the in-plane stress components are σ_{11}, σ_{22}, and σ_{12}. Thus, the following stress components are assumed negligible with respect to the in-plane stress components and set equal to zero in eq. (8.3):

$$\sigma_{33} = \sigma_{31} = \sigma_{13} = \sigma_{32} = \sigma_{23} = 0. \tag{8.24}$$

Hence, the compliance form of the orthotropic material law for a unidirectional lamina subject to plane stress is

$$\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{bmatrix} = \underbrace{\begin{bmatrix}C_{11} & C_{12} & 0\\ C_{21} & C_{22} & 0\\ 0 & 0 & C_{66}\end{bmatrix}}_{[C]}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \begin{bmatrix}1/E_1 & -\nu_{12}/E_2 & 0\\ -\nu_{21}/E_1 & 1/E_2 & 0\\ 0 & 0 & 1/G_{12}\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix}, \tag{8.25}$$

and the thickness normal strain is

$$\varepsilon_{33} = C_{31}\sigma_{11} + C_{32}\sigma_{22}. \tag{8.26}$$

The stress-strain form of the material law (8.25) is written as

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{12}\end{bmatrix} = \underbrace{\begin{bmatrix}Q_{11} & Q_{12} & 0\\ Q_{21} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}}_{[Q]}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{bmatrix} = \begin{bmatrix}E_1/(1 - \nu_{21}\nu_{12}) & (\nu_{12}E_1)/(1 - \nu_{21}\nu_{12}) & 0\\ (\nu_{21}E_2)/(1 - \nu_{21}\nu_{12}) & E_2/(1 - \nu_{21}\nu_{12}) & 0\\ 0 & 0 & G_{12}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{bmatrix}, \tag{8.27}$$

where the matrix $[Q]$ is called the reduced stiffness matrix. Matrix $[Q]$ is symmetric since $\nu_{21}E_2 = \nu_{12}E_1$. (refer to eq. (8.5)). It follows from eq. (8.3) that the transverse shear strains $\gamma_{23} = \gamma_{31} = 0$, which leads to transverse shear strains $\gamma_{z\zeta} = \gamma_{\zeta s} = 0$ by eq. (8.11). Also, the normal strain $\varepsilon_{\zeta\zeta} = \varepsilon_{33}$. From eq. (8.13) the stresses $\sigma_{\zeta\zeta} = \sigma_{z\zeta} = \sigma_{\zeta s} = 0$.

Transform eq. (8.27) to an off-axis material law as follows: For plane stress the stress transformation equation (8.13) reduces to

$$\begin{bmatrix} \sigma_{ss} \\ \sigma_{zz} \\ \sigma_{sz} \end{bmatrix} = \underbrace{\begin{bmatrix} n^2 & m^2 & -2mn \\ m^2 & n^2 & 2mn \\ mn & -mn & (-m^2+n^2) \end{bmatrix}}_{[T_{\sigma 1}]} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}, \tag{8.28}$$

and the strain transformation eq. (8.11) reduces to

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{bmatrix} = \underbrace{\begin{bmatrix} n^2 & m^2 & mn \\ m^2 & n^2 & -mn \\ -2mn & 2mn & (-m^2+n^2) \end{bmatrix}}_{[T_{\varepsilon 2}]} \begin{bmatrix} \varepsilon_{ss} \\ \varepsilon_{zz} \\ \gamma_{sz} \end{bmatrix}. \tag{8.29}$$

Pre-multiply eq. (8.27) $[T_{\sigma 1}]$, and substitute the strain transformation eq. (8.29) on the right-hand side of eq. (8.27). Use the fact that $[T_{\varepsilon 2}] = [T_{\sigma 1}]^T$ from eq. (8.15). These matrix manipulations result in the off-axis material law in plane stress given by

$$\begin{bmatrix} \sigma_{ss} \\ \sigma_{zz} \\ \sigma_{sz} \end{bmatrix} = [\bar{Q}] \begin{bmatrix} \varepsilon_{ss} \\ \varepsilon_{zz} \\ \gamma_{sz} \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{21} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{61} & \bar{Q}_{62} & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{ss} \\ \varepsilon_{zz} \\ \gamma_{sz} \end{bmatrix}, \tag{8.30}$$

where the transformed reduced stiffness matrix is given by $[\bar{Q}] = [T_{\sigma 1}][Q][T_{\sigma 1}]^T$. Since the on-axis reduced stiffness matrix $[Q]$ is symmetric, it follows from these matrix relations that the transformed stiffness matrix is symmetric ($[\bar{Q}]^T = [\bar{Q}]$). Elements of the transformed reduced stiffness matrix in terms of the reduced matrix elements are given by

$$\begin{bmatrix} \bar{Q}_{11} \\ \bar{Q}_{22} \\ \bar{Q}_{21} \\ \bar{Q}_{66} \\ \bar{Q}_{16} \\ \bar{Q}_{26} \end{bmatrix} = \begin{bmatrix} n^4 & m^4 & 2m^2n^2 & 4m^2n^2 \\ m^4 & n^4 & 2m^2n^2 & 4m^2n^2 \\ m^2n^2 & m^2n^2 & (m^4+n^4) & -4m^2n^2 \\ m^2n^2 & m^2n^2 & -2m^2n^2 & (-m^2+n^2)^2 \\ mn^3 & -m^3n & mn(m^2-n^2) & 2mn(m^2-n^2) \\ m^3n & -mn^3 & -mn(m^2-n^2) & -2mn(m^2-n^2) \end{bmatrix} \begin{bmatrix} Q_{11} \\ Q_{22} \\ Q_{21} \\ Q_{66} \end{bmatrix}. \tag{8.31}$$

8.1.4 Nomenclature of composite materials

Composite materials are identified by the name of the fiber followed by the name of the matrix. For example, AS4/3501-6 denotes the carbon fiber AS4 and the epoxy matrix 3501-6. The data in table 8.2 is taken from Herakovich (1998, p.14), and it lists typical properties for AS4/3501-6 and T300/5208 carbon fiber-reinforced epoxy composites.

Table 8.2 Material properties of selected CFRP lamina

Property	Units	AS4/3501-6	T300/5208
Axial modulus E_1	GPa	148	132
	Msi	21.5	19.2
Transverse modulus E_2	GPa	10.5	10.8
	Msi	1.46	1.56
Major Poisson's ratio ν_{21}	dimensionless	0.30	0.24
Major Poisson's ratio ν_{23}	dimensionless	0.59	0.59
Shear modulus G_{12}	GPa	5.61	5.65
	Msi	0.81	0.82
Shear modulus G_{23}	GPa	3.17	3.38
	Msi	0.46	0.49
Density	g/cm^3	1.52	1.54
	lb./in.^{3}	0.055	0.056
Ply thickness t_{ply}	mm	0.127	0.127
	in.	0.005	0.005
Fiber volume fraction V_f	dimensionless	0.62	0.62

Example 8.1 Transformed reduced stiffness matrix for an off-axis ply

Determine the transformed reduced stiffness matrix of T300/5208 carbon/epoxy for a 30-degree off-axis lamina in U.S. customary units.

Solution. From table 8.2 $E_1 = 19.2\ \text{Msi}$, $E_2 = 1.56\ \text{Msi}$, $\nu_{21} = 0.24$, and $G_{12} = 0.82\ \text{Msi}$. The minor Poisson's ratio is $\nu_{12} = 0.24[(1.56\ \text{Msi})/(19.2\ \text{Msi})] = 0.0195$. The reduced stiffness matrix is computed from eq. (8.30) and eq. (8.31); i.e.,

$$[Q] = \begin{bmatrix} 19.3 & 0.376 & 0 \\ 0.376 & 1.57 & 0 \\ 0 & 0 & 0.82 \end{bmatrix} \text{Msi}. \tag{a}$$

The transformed reduced stiffness matrix is given by $[\bar{Q}] = [T_{\sigma 1}][Q][T_{\sigma 1}]^T$, the reduced stiffness by eq. (8.27), and the transform matrix $[T_{\sigma 1}]$ by eq. (8.28). The matrix product is

$$[\bar{Q}] = \begin{bmatrix} 1/4 & 3/4 & -\sqrt{3}/2 \\ 3/4 & 1/4 & \sqrt{3}/2 \\ \sqrt{3}/4 & -\sqrt{3}/4 & -1/2 \end{bmatrix} \begin{bmatrix} 19.3 & 0.376 & 0 \\ 0.376 & 1.57 & 0 \\ 0 & 0 & 0.82 \end{bmatrix} \begin{bmatrix} 1/4 & 3/4 & \sqrt{3}/4 \\ 3/4 & 1/4 & -\sqrt{3}/4 \\ -\sqrt{3}/2 & \sqrt{3}/2 & -1/2 \end{bmatrix}, \tag{b}$$

and the result is

$$[\bar{Q}] = \begin{bmatrix} 2.84537 & 3.53313 & 2.01589 \\ 3.53313 & 11.7104 & 5.66142 \\ 2.01589 & 5.66142 & 3.97713 \end{bmatrix} \text{Msi}. \blacksquare \tag{c}$$

8.1.5 Laminated wall

Laminates are made by stacking the unidirectional lamina, also called plies, at different fiber orientations. The plies are usually bound together by the same matrix material that is used within the lamina. Laminates are designated by the ply angle stacking sequence. A $[45\ {-45}\ 0\ 90]$ stacking sequence denotes a 4-ply laminate with plies at 45, – 45, 0, and 90 degrees with respect to the longitudinal *z*-axis. A $[45\ {-45}\ 0\ 90]_2$ stacking sequence denotes an 8-ply laminate with plies at 45, – 45, 0, 90, 45,– 45, 0, and 90 degrees. A $[45\ {-45}\ 0\ 90]_S$ stacking sequence denotes an 8-ply symmetric laminate with plies at 45, – 45, 0, 90, 90, 0, – 45, and 45 degrees. The assumptions of lamination theory are

- The laminate consists of perfectly bonded layers or lamina.
- Each layer is a homogeneous material with known effective properties.
- Each layer is in a state of plane stress.
- Individual layers can be isotropic or orthotropic.

Consistent with thin-walled bar theory in chapter 3, we **assume that the strains** ε_{zz}, ε_{ss}, **and** γ_{zs} **are spatially uniform through the thickness of the wall**. That is, there is no local bending of the laminated wall. The laminate can stretch and shear in-plane as membrane. For a laminate of *Np*-plies, the material law for the *k*-th ply, where $k = 1, 2, \ldots, Np$, is obtained from eq. (8.30) as

$$\begin{bmatrix} \sigma_{ss} \\ \sigma_{zz} \\ \sigma_{sz} \end{bmatrix}^{(k)} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{21} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{61} & \bar{Q}_{62} & \bar{Q}_{66} \end{bmatrix}^{(k)} \begin{bmatrix} \varepsilon_{ss} \\ \varepsilon_{zz} \\ \gamma_{sz} \end{bmatrix} \qquad k = 1, 2, \ldots, Np. \tag{8.32}$$

Even though the strains are uniform through the thickness of the wall, note that the stresses are piecewise constant through the thickness of the wall since the transformed reduced stiffness matrix changes from ply to ply. Let the origin of the thickness coordinate ζ be at the midplane of the laminate, such that $-t/2 \le \zeta \le t/2$, where t denotes the total thickness of the laminated wall. The stress resultant n_s, the axial stress resultant n_z, and the shear flow q are defined by integrals through the thickness of the wall of the corresponding stresses; i.e.,

$$\begin{bmatrix} n_s \\ n_z \\ q \end{bmatrix} = \int_{-t/2}^{t/2} \begin{bmatrix} \sigma_{ss} \\ \sigma_{zz} \\ \sigma_{zs} \end{bmatrix} d\zeta = \sum_{k=1}^{Np} \left(\int_{\zeta_k}^{\zeta_{k+1}} \begin{bmatrix} \sigma_{ss} \\ \sigma_{zz} \\ \sigma_{zs} \end{bmatrix}^{(k)} d\zeta \right), \tag{8.33}$$

where $\zeta = \zeta_k$ at the bottom of the *k-th* ply, and $\zeta = \zeta_{k+1}$ at the top of the *k-th* ply. Denote the thickness of the *k-th* ply by t_k such that $\zeta_{k+1} - \zeta_k = t_k$. Substitute for the stresses from Hooke's law (8.32) into eq. (8.33) to get

$$\begin{bmatrix} n_s \\ n_z \\ q \end{bmatrix} = \sum_{k=1}^{Np} \left\{ \int_{\zeta_k}^{\zeta_{k+1}} \left(\begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{21} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{61} & \bar{Q}_{62} & \bar{Q}_{66} \end{bmatrix}^{(k)} \right) d\zeta \right\} \begin{bmatrix} \varepsilon_{ss} \\ \varepsilon_{zz} \\ \gamma_{sz} \end{bmatrix}. \tag{8.34}$$

The last result is written as

$$\begin{bmatrix} n_s \\ n_z \\ q \end{bmatrix} = \underbrace{\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{21} & A_{22} & A_{26} \\ A_{61} & A_{62} & A_{66} \end{bmatrix}}_{[A]} \begin{bmatrix} \varepsilon_{ss} \\ \varepsilon_{zz} \\ \gamma_{sz} \end{bmatrix}, \tag{8.35}$$

where $[A]$ is the in-plane stiffness matrix. Elements of the in-plane stiffness matrix are computed by the sum

$$\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{21} & A_{22} & A_{26} \\ A_{61} & A_{62} & A_{66} \end{bmatrix} = \sum_{k=1}^{Np} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{21} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{61} & \bar{Q}_{62} & \bar{Q}_{66} \end{bmatrix}^{(k)} t_k. \tag{8.36}$$

Stiffness elements A_{11} and A_{22} correspond to in-plane extensional stiffnesses in the *s*- and *z*-directions, respectively. Element A_{66} corresponds to a shear stiffness in the *s-z* plane, stiffnesses $A_{21} = A_{12}$ are Poisson's type terms, and stiffnesses $A_{61} = A_{16}$ and $A_{62} = A_{26}$ couple in-plane shear and extension. The in-plane stiffness matrix depends on the content of the layers in the laminate, and is independent of the stacking sequence of the layers through the thickness of the laminate.

Example 8.2 In-plane stiffness matrix for a laminate with two plies

Consider a two-ply $[\varphi\ -\varphi]$ laminate with plies of equal thickness *t*/2.

(a) Determine the [A] matrix
(b) Evaluate the [A] matrix for T300/5208 with $\varphi = 30°$ and $t/2 = 0.005$ in.

Solution to part (a). The transformed reduced stiffnesses are given by eq. (8.31) in which $m = \cos\varphi$ and $n = \sin\varphi$. Note that stiffnesses $\bar{Q}_{11}$, $\bar{Q}_{22}$, $\bar{Q}_{66}$, and $\bar{Q}_{21}$ are even functions of the ply angle φ, and stiffnesses $\bar{Q}_{61}$ and $\bar{Q}_{62}$ are odd functions of φ. Thus,

$$[A] = \begin{bmatrix} \bar{Q}_{11}(\varphi) & \bar{Q}_{12}(\varphi) & \bar{Q}_{16}(\varphi) \\ \bar{Q}_{21}(\varphi) & \bar{Q}_{22}(\varphi) & \bar{Q}_{26}(\varphi) \\ \bar{Q}_{61}(\varphi) & \bar{Q}_{62}(\varphi) & \bar{Q}_{66}(\varphi) \end{bmatrix} \frac{t}{2} + \begin{bmatrix} \bar{Q}_{11}(\varphi) & \bar{Q}_{12}(\varphi) & -(\bar{Q}_{16}(\varphi)) \\ \bar{Q}_{21}(\varphi) & \bar{Q}_{22}(\varphi) & -(\bar{Q}_{26}(\varphi)) \\ -(\bar{Q}_{61}(\varphi)) & -(\bar{Q}_{62}(\varphi)) & \bar{Q}_{66}(\varphi) \end{bmatrix} \frac{t}{2}$$

$$[A] = \begin{bmatrix} \bar{Q}_{11}(\varphi) & \bar{Q}_{12}(\varphi) & 0 \\ \bar{Q}_{21}(\varphi) & \bar{Q}_{22}(\varphi) & 0 \\ 0 & 0 & \bar{Q}_{66}(\varphi) \end{bmatrix} t\,.$$

Solution to part (b). From the T300/5208 example on page 231 $\bar{Q}_{11}(30) = 2.843\ \text{Msi}$, $\bar{Q}_{22}(30) = 11.7\ \text{Msi}$, $\bar{Q}_{66}(30) = 3.975\ \text{Msi}$, and $\bar{Q}_{21}(30) = 3.531\ \text{Msi}$. Thus,

$$[A] = \begin{bmatrix} 2.843 & 3.531 & 0 \\ 3.531 & 11.7 & 0 \\ 0 & 0 & 3.975 \end{bmatrix} (10^6\ \text{lb./in.}^2)(0.01\ \text{in.}) = \begin{bmatrix} 28.43 & 35.3 & 0 \\ 35.3 & 117. & 0 \\ 0 & 0 & 39.7 \end{bmatrix} 10^3\ \text{lb./in.}\ \blacksquare.$$

8.1.6 Balanced and specially orthotropic laminates

A laminate consisting of off-axis plies with positive fiber angles φ_i and off-axis plies with negative fiber angles $-\varphi_i$, $i = 1, 2, 3, \ldots, i_{max}$, with each φ_i-ply and $-\varphi_i$-ply having the same thickness and material properties, is called a **balanced laminate**. For example, a stacking sequence $[30/\text{-}30]_{2S}$ is a balanced laminate consisting of eight plies if each $30°$-ply and -30-ply have the same thickness and material properties. For a balanced laminate the in-plane stiffness coefficients $A_{16} = A_{61} = 0$, and $A_{26} = A_{62} = 0$ as example 8.2 illustrates. The in-plane material law for a balanced laminate reduces to the form

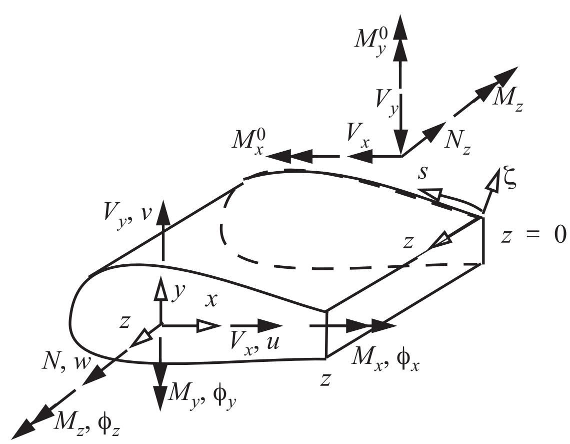
$$\begin{bmatrix} n_s \\ n_z \\ q \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{ss} \\ \varepsilon_{zz} \\ \gamma_{zs} \end{bmatrix} . \tag{8.37}$$

In eq. (8.37) resultants n_s and n_z are independent of the shear strain γ_{zs}, and the shear flow q is independent of the normal strains ε_{ss} and ε_{zz}. That is, there is no coupling between in-plane extension and shear. Laminates whose material law is given by (8.37) are also said to be **specially orthotropic**. Laminates consisting of only $0°$ and $90°$ plies are specially orthotropic laminates, since the product $mn = \cos\varphi\sin\varphi = 0$ in the last two rows of (8.31) results in $\bar{Q}_{16} = \bar{Q}_{26} = 0$ for these laminates. Hence, a $[0/90]$ laminate has coupling stiffnesses $A_{16} = A_{26} = 0$ as can be recognized from eq. (8.36). Another example of a specially orthotropic laminate is a stacking sequence $[\pm 45/0/90]_S$.

8.2 Composite thin-walled bar with a closed cross-sectional contour

The analysis in this section was published by Johnson, et al., (2001), and it is also reviewed by Vasiliev and Morozov (2013). We consider free bending and torsion of a thin-walled bar with a closed cross-sectional contour as depicted in figure. 8.4. The laminated wall consists of unidirectional FRP layers. The external traction compo-

Fig. 8.4 Closed cross-sectional bar subject to free bending and torsion.



nents acting on the lateral surface of the bar $p_n(s, z)$, $p_s(s, z)$, and $p_z(s, z)$ appearing in eq. (3.42) on page 41 are prescribed to be zero for all values of s and z. Thus, distributed force intensities $f_x = f_y = f_z = 0$ in eq. (3.42), and distributed moment intensities $m_x = m_y = m_z = 0$ in eq. (3.45) all vanish. The differential equilibrium equations (3.53), (3.56), (3.54), and (3.61) on page 42 are satisfied for

$$\frac{dN}{dz} = 0 \qquad \frac{dV_x}{dz} = 0 \qquad \frac{dV_y}{dz} = 0 \qquad \frac{dM_z}{dz} = 0 \qquad 0 \le z \le L . \tag{8.38}$$

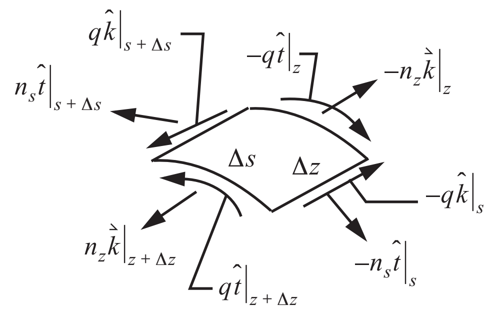
Hence, the axial force N, shear forces V_x and V_y, and the torque M_z are uniform along the length L of the bar. Bending moment equilibrium equations (3.55) and (3.57) on page 43 are satisfied by

$$M_x = M_x^0 + V_y z \qquad M_y = M_y^0 + V_x z \qquad 0 \le z \le L\,, \tag{8.39}$$

where M_x^0 and M_y^0 are the bending moments acting on the cross section at $z = 0$.

Consider a free body diagram of the stress resultants acting on a segment of the wall with dimensions Δs-by-Δz is shown in figure. 8.5.

Fig. 8.5 Stress resultants acting on an element of the wall



Force equilibrium leads to

$$[n_z\Delta s\hat{k}|_{z+\Delta z} - n_z\Delta s\hat{k}|_z] + [q\Delta z\hat{k}|_{s+\Delta s} - q\Delta z\hat{k}|_s] + [q\Delta s\hat{t}|_{z+\Delta z} - q\Delta s\hat{t}|_z] + [n_s\Delta z\hat{t}|_{s+\Delta s} - n_s\Delta z\hat{t}|_s] = 0\,. \tag{8.40}$$

Expand the functions $n(s,z)$, $q(s,z)$, and $n_s(s,z)$ in a Taylor series about s and z to get

$$\left(n_z + \frac{\partial n_z}{\partial z}\Delta z - n_z\right)\Delta s\hat{k} + \left(q + \frac{\partial q}{\partial s}\Delta s - q\right)\Delta z\hat{k} + \left(q + \frac{\partial q}{\partial z}\Delta z - q\right)\Delta s\hat{t} + \left(n_s\hat{t} + \frac{\partial n_s\hat{t}}{\partial s}\Delta s - n_s\hat{t}\right)\Delta z + O(\Delta s^2, \Delta z^2) = 0\,. \tag{8.41}$$

Division of eq. (8.41) by the product $\Delta s \cdot \Delta z$, followed by taking the limit as $\Delta s \to 0$ and $\Delta z \to 0$ leads to the differential equations

$$\left(\frac{\partial n_z}{\partial z} + \frac{\partial q}{\partial s}\right)\hat{k} + \left(\frac{\partial q}{\partial z} + \frac{\partial n_s}{\partial s}\right)\hat{t} + n_s\frac{\partial \hat{t}}{\partial s} = 0\,.$$

From eq. (3.6) on page 34 the derivative of the unit tangent vector is $\frac{\partial \hat{t}}{\partial s} = \frac{-\hat{n}}{R_s}$, where R_s is the radius of curvature of the contour. The differential equations of equilibrium at coordinates s and z in the wall are

$$\frac{\partial n_z}{\partial z} + \frac{\partial q}{\partial s} = 0 \qquad \frac{\partial q}{\partial z} + \frac{\partial n_s}{\partial s} = 0 \qquad \frac{-n_s}{R_s} = 0\,. \tag{8.42}$$

From the last two equations in (8.42) we get

$$n_s = 0 \qquad \frac{\partial q}{\partial z} = 0\,. \tag{8.43}$$

That is, the circumferential stress resultant vanishes and the shear flow is independent of the longitudinal coordinate z.

8.2.1 Anisotropic Hooke's law for the cross section

Set $n_s = 0$ in eq. (8.35), and solve for the normal strain ε_{ss} to eliminate it in the material law. We write the resulting material law in several forms to be used in subsequent developments:

$$n_z = B\varepsilon_{zz} + bq \qquad q = B_s\gamma_{sz} + B_z\varepsilon_{zz}\text{, and} \tag{8.44}$$

$$\varepsilon_{zz} = \frac{1}{B}(n_z - bq) \qquad \gamma_{sz} = \frac{1}{B}(aq - bn_z)\,. \tag{8.45}$$

The coefficients in the previous equations are

$$B = A_{22} - A_{12}A_{21}/A_{11} - bB_z\,, \tag{8.46}$$

$$B_s = A_{66} - A_{61}A_{16}/A_{11} \qquad B_z = A_{62} - A_{12}A_{61}/A_{11}\text{, and} \tag{8.47}$$

$$a = \frac{1}{B_s}(A_{22} - A_{12}A_{21}/A_{11}) \qquad b = B_z/B_s\,. \tag{8.48}$$

The stiffness parameters b and B_z represent the shear-extension coupling of the laminated wall, since they are directly related to stiffnesses A_{61} and A_{62} by eqs. (8.47) and (8.48). In a specially orthotropic laminate $A_{61} = A_{62} = 0$, so $b = B_z = 0$. There is no material coupling between shear and extension in a specially orthotropic laminate.

The second assumption is traditional for the beam theory and states that the axial strain is a linear function of coordinates *x* and *y*. From eq. (3.30) on page 38 the axial normal strain along the contour ($\zeta = 0$) is

$$\varepsilon_{zz} = \frac{dw}{dz} + y(s)\frac{d\phi_x}{dz} + x(s)\frac{d\phi_y}{dz}\,. \tag{8.49}$$

where $w(z)$ is the axial displacement of the cross section, $\phi_x(z)$ is the rotation of the cross section about the *x*-axis, and $\phi_y(z)$ is the rotation of the cross section about the negative *y*-axis. Refer to figure. 8.5. Substitute eq. (8.49) for the strain in the first equation of (8.44) to get the normal stress resultant as

$$n_z = B\left(\frac{dw}{dz} + y(s)\frac{d\phi_x}{dz} + x(s)\frac{d\phi_y}{dz}\right) + bq\,. \tag{8.50}$$

Substitute the previous expression for the normal stress resultant into the definition of the bar resultant *N* in eq. (3.39) on page 40 to get

$$N = \oint n_z ds = S\left(\frac{dw}{dz}\right) + S_x\left(\frac{d\phi_x}{dz}\right) + S_y\left(\frac{d\phi_y}{dz}\right) + \oint (bq)ds\,, \tag{8.51}$$

where

$$S = \oint B(s)ds \qquad S_x = \oint B(s)y(s)ds \qquad S_y = \oint B(s)x(s)ds\,. \tag{8.52}$$

In eq. (8.52) the modulus-weighted extensional stiffness of the cross section of the beam is denoted by S, the modulus-weighted first moment of the cross-sectional area about the *x*-axis by S_x, and the modulus-weighted first moment of the cross-sectional area about the *y*-axis by S_y. We now locate the origin of the *x*-*y* coordinates at the modulus-weighted centroid of the cross section. Let $x(s) = X(s) - X_c$ and $y(s) = Y(s) - Y_c$, where $X(s)$ and $Y(s)$ are the Cartesian coordinates of the contour with respect to an arbitrary origin at point O (see Fig. 3.1 on page 31). The coordinates (X_c, Y_c) of the modulus-weighted centroid are determined from

$$S_x = 0 = \oint B(s)Y(s)ds - Y_cS \qquad S_y = 0 = \oint B(s)X(s)ds - X_cS. \tag{8.53}$$

Since $S_x = S_y = 0$, eq. (8.51) is written as

$$\bar{N} = S\left(\frac{dw}{dz}\right) \qquad \text{where} \qquad \bar{N} = N - \oint (bq)ds. \tag{8.54}$$

The bending moments M_x and M_y acting in the cross section are determined from the normal stress resultant n_z by

$$M_x = \oint (yn_z)ds \qquad M_y = \oint (xn_z)ds. \tag{8.55}$$

Substitute eq. (8.50) for the normal stress resultant into these expressions for the bending moments to get

$$M_x = D_{xx}\left(\frac{d\phi_x}{dz}\right) + D_{xy}\left(\frac{d\phi_y}{dz}\right) + \oint (ybq)ds \qquad M_y = D_{xy}\left(\frac{d\phi_x}{dz}\right) + D_{yy}\left(\frac{d\phi_y}{dz}\right) + \oint (xbq)ds. \tag{8.56}$$

The modulus-weighted second moments of the cross section appearing in eq. (8.56) are defined by

$$\begin{bmatrix} D_{xx} & D_{yy} & D_{xy} \end{bmatrix} = \oint \begin{bmatrix} y^2 & x^2 & xy \end{bmatrix} Bds. \tag{8.57}$$

Solve for the gradients of the bending rotations eq. (8.56) and write the result as

$$\begin{bmatrix} \dfrac{d\phi_x}{dz} \\ \dfrac{d\phi_y}{dz} \end{bmatrix} = k \begin{bmatrix} \dfrac{1}{D_{xx}} & \dfrac{-n_x}{D_{yy}} \\ \dfrac{-n_y}{D_{xx}} & \dfrac{1}{D_{yy}} \end{bmatrix} \begin{bmatrix} \bar{M}_x \\ \bar{M}_y \end{bmatrix}, \tag{8.58}$$

where

$$n_x = \frac{D_{xy}}{D_{xx}} \qquad n_y = \frac{D_{xy}}{D_{yy}} \qquad k = \frac{1}{1 - n_x n_y}, \tag{8.59}$$

and

$$\bar{M}_x = M_x - \oint (ybq)ds \qquad \bar{M}_y = M_y - \oint (xbq)ds. \tag{8.60}$$

Substitute eq. (8.54) for the axial displacement gradient, and eq. (8.58) for the bending rotation gradients, into eq. (8.50) to express the normal stress resultant as

$$n_z = \frac{B}{S}\bar{N} + B\bar{y}(s)\frac{k}{D_{xx}}\bar{M}_x + B\bar{x}(s)\frac{k}{D_{yy}}\bar{M}_y + bq, \tag{8.61}$$

where

$$\bar{x}(s) = x(s) - n_x y(s) \qquad \bar{y}(s) = y(s) - n_y x(s). \tag{8.62}$$

8.2.2 Expressions for the shear flow and normal stress resultant

Substitute the normal stress resultant from eq. (8.61) into the equilibrium differential equation (8.42), to get

$$\frac{dq}{ds} = -\left(\frac{B}{S}\right)\frac{d\bar{N}}{dz} - B\bar{y}\frac{k}{D_{xx}}\frac{d\bar{M}_x}{dz} - B\bar{x}\frac{k}{D_{yy}}\frac{d\bar{M}_y}{dz}. \tag{8.63}$$

Recall that the stiffness parameter *b* and the shear flow *q* are independent of coordinate *z*. Derivatives of $\bar{N}$, $\bar{M}_x$, and $\bar{M}_y$ with respect to *z* are determined from eqs. (8.54) and (8.60) as

$$\frac{d\bar{N}}{dz} = \frac{dN}{dz} \qquad \frac{d\bar{M}_x}{dz} = \frac{dM_x}{dz} \qquad \frac{d\bar{M}_y}{dz} = \frac{dM_y}{dz}. \tag{8.64}$$

Derivatives of the bar resultants are given by equilibrium equations (8.38) and (8.39). Thus,

$$\frac{dN}{dz} = 0 \qquad \frac{dM_x}{dz} = V_y \qquad \frac{dM_y}{dz} = V_x. \tag{8.65}$$

The derivative of the shear flow with respect to the contour coordinate reduces to

$$\frac{dq}{ds} = -B\bar{y}\frac{k}{D_{xx}}V_y - B\bar{x}\frac{k}{D_{yy}}V_x. \tag{8.66}$$

Now we integrate the previous result with respect to the contour coordinate from $s = 0$ to $s = s$ and write the result as

$$q(s) = q_0 - \bar{S}_x(s)\frac{k}{D_{xx}}V_y - \bar{S}_y(s)\frac{k}{D_{yy}}V_x, \tag{8.67}$$

where

$$q_0 = q(0) \qquad \bar{S}_x(s) = \int_0^s B\bar{y}(s)ds \qquad \bar{S}_y(s) = \int_0^s B\bar{x}(s)ds. \tag{8.68}$$

Substitute eq. (8.62) for $\bar{x}(s)$ and $\bar{y}(s)$ into eq. (8.68) to get

$$\bar{S}_x(s) = S_x(s) - n_y S_y(s) \qquad \bar{S}_y(s) = S_y(s) - n_x S_x(s), \tag{8.69}$$

where the modulus-weighted first moments of a segment of the cross-sectional area from $s = 0$ to $s = s$ are defined by

$$S_x(s) = \int_0^s B(s)y(s)ds \qquad S_y(s) = \int_0^s B(s)x(s)ds. \tag{8.70}$$

Note that $S_x(s)$ and $S_y(s)$ evaluated at the end point of the closed contour vanish, which is consistent with eq. (8.53). The shear flow at the contour origin q_0 is determined by torque equivalence of the shear flow with respect to the modulus-weighted centroid. That is,

$$M_{zc} = \oint r_{nc}(s)q(s)ds, \tag{8.71}$$

where $r_{nc}(s)$ is the coordinate normal to the contour as depicted in Fig. 3.3 on page 33, and it is determined from eq. (3.11) on page 34. Substitute eq. (8.67) for the shear flow in eq. (8.71) and solve for q_0 to find

$$q_0 = \frac{1}{2A_c}\left[M_{zc} + \left(\frac{kV_y}{D_{xx}}\right)\oint(r_{nc}\bar{S}_x)ds + \left(\frac{kV_x}{D_{yy}}\right)\oint(r_{nc}\bar{S}_y)ds\right], \tag{8.72}$$

where the area enclosed by the contour is given by

$$A_c = \frac{1}{2}\oint r_{nc}ds. \tag{8.73}$$

With q_0 determined, we write the final expression for the shear flow in eq. (8.67) as

$$\boxed{q(s) = \frac{M_{zc}}{2A_c} - F_{xc}(s)V_x - F_{yc}(s)V_y}, \tag{8.74}$$

where the shear flow distribution functions are defined by

$$F_{xc}(s) = \frac{k}{D_{yy}}\left[\bar{S}_y(s) - \frac{1}{2A_c}\oint(r_{nc}\bar{S}_y)ds\right] \qquad F_{yc}(s) = \frac{k}{D_{xx}}\left[\bar{S}_x(s) - \frac{1}{2A_c}\oint(r_{nc}\bar{S}_x)ds\right]. \tag{8.75}$$

Substitute eq. (8.54) for the normal stress resultant $\bar{N}$ in eq. (8.61), and substitute for $\bar{M}_x$ and $\bar{M}_y$ from eq. (8.60) into eq. (8.61), to get

$$n_z = \frac{B}{S}\left[N - \oint(bq)ds\right] + B\bar{y}\frac{k}{D_{xx}}\left[M_x - \oint(ybq)ds\right] + B\bar{x}\frac{k}{D_{yy}}\left[M_y - \oint(xbq)ds\right] + bq. \tag{8.76}$$

Substitute eq. (8.74) for the shear flow into the previous equation for the normal stress resultant. After some algebraic manipulations we write the result as

$$\boxed{n_z = \frac{B}{S}N + B\bar{y}(s)\frac{k}{D_{xx}}M_x + B\bar{x}(s)\frac{k}{D_{yy}}M_y + \Phi_x(s)V_x + \Phi_y(s)V_y + \Phi(s)\frac{M_{zc}}{2A_c}}. \tag{8.77}$$

The functions $\Phi_x(s)$, $\Phi_y(s)$, and $\Phi(s)$ are a consequence of the coupling between extension and shear in the material law (i.e., $b \neq 0$). If the stiffness parameter $b = 0$ over the entire contour, then $\Phi_x(s) = \Phi_y(s) = \Phi(s) = 0$. Equations for these functions are

$$\Phi_x(s) = -bF_{xc}(s) + \frac{B}{S}\left(\oint bF_{xc}ds\right) + \frac{Bk\bar{y}(s)}{D_{xx}}\left(\oint byF_{xc}ds\right) + \frac{Bk\bar{x}(s)}{D_{yy}}\left(\oint bxF_{xc}ds\right), \tag{8.78}$$

$$\Phi_y(s) = -bF_{yc}(s) + \frac{B}{S}\left(\oint bF_{yc}ds\right) + \frac{Bk\bar{y}(s)}{D_{xx}}\left(\oint byF_{yc}ds\right) + \frac{Bk\bar{x}(s)}{D_{yy}}\left(\oint bxF_{yc}ds\right), \text{ and} \tag{8.79}$$

$$\Phi(s) = b - \frac{B}{S}\oint bds - \frac{Bk\bar{y}(s)}{D_{xx}}\oint byds - \frac{Bk\bar{x}(s)}{D_{yy}}\oint bxds. \tag{8.80}$$

8.2.3 Complementary work and energy

Consider a free bending and torsion state of the bar as shown in figure. 8.4 where the displacements, strains, and forces satisfy the compatibility conditions, Hooke's law, and the equilibrium conditions. In this state, the actual displacements of the modulus-weighted centroid are $u_c(z)$, $v_c(z)$, and $w(z)$, and the corresponding forces are

$V_x(z)$, $V_y(z)$, and $N(z)$, respectively. The actual rotations of a cross section with respect to the modulus-weighted centroid are $\phi_x(z)$, $\phi_y(z)$, and $\phi_z(z)$, and the corresponding moments are $M_x(z)$, $M_y(z)$, and $M_{zc}(z)$, respectively. Now consider infinitesimal increments in the forces and moments denoted by δV_x, δV_y, δN, δM_x, δM_y, and δM_{zC} from the equilibrium state. For an element of the bar of length Δz, the complementary work is given by

$$\delta \bar{U}^* \Delta z = [\delta V_x u_c + \delta V_y v_c + \delta N w + \delta M_x \phi_x + \delta M_y \phi_y + \delta M_{zC} \phi_z]\Big|_z^{z+\Delta z}, \tag{8.81}$$

where $\delta \bar{U}^*$ denotes the increment in the complementary work per unit axial length. Divide eq. (8.81) by Δz, and let $\Delta z \to 0$, to get in the limit

$$\delta \bar{U}^* = \frac{d}{dz}[\delta V_x u_c + \delta V_y v_c + \delta N w + \delta M_x \phi_x + \delta M_y \phi_y + \delta M_{zC} \phi_z]. \tag{8.82}$$

Statically admissible incremental actions requires that they satisfy the equilibrium differential equations (8.38) and (8.39): i.e.,

$$\frac{d}{dz}\delta V_x = \frac{d}{dz}\delta V_y = \frac{d}{dz}\delta N = \frac{d}{dz}\delta M_{zc} = 0 \qquad \frac{d}{dz}\delta M_x = \delta V_y \qquad \frac{d}{dz}\delta M_y = \delta V_x. \tag{8.83}$$

Imposing equilibrium (8.83) reduces eq. (8.82) to

$$\delta \bar{U}^* = \psi_{xc}\delta V_x + \psi_{yc}\delta V_y + \left(\frac{dw}{dz}\right)\delta N + \left(\frac{d\phi_x}{dz}\right)\delta M_x + \left(\frac{d\phi_y}{dz}\right)\delta M_y + \left(\frac{d\phi_z}{dz}\right)\delta M_{zC}, \tag{8.84}$$

where shear strains averaged over the cross section of the bar relative to the centroid are defined by

$$\psi_{xc} = \frac{du_c}{dz} + \phi_y \qquad \psi_{yc} = \frac{dv_c}{dz} + \phi_x. \tag{8.85}$$

An elastic material is defined by the existence of a complementary strain energy function per unit axial length with the form $\bar{U}^*(M_x, M_y, N, V_x, V_y, M_{zC})$. Then, the total increment in function $\bar{U}^*$ is

$$\delta \bar{U}^* = \frac{\partial \bar{U}^*}{\partial M_x}\delta M_x + \frac{\partial \bar{U}^*}{\partial M_y}\delta M_y + \frac{\partial \bar{U}^*}{\partial N}\delta N + \frac{\partial \bar{U}^*}{\partial V_x}\delta V_x + \frac{\partial \bar{U}^*}{\partial V_y}\delta V_y + \frac{\partial \bar{U}^*}{\partial M_{zC}}\delta M_{zC}, \tag{8.86}$$

Identify the complementary work (8.84) with the complementary energy (8.86) to get the important properties of complementary strain energy function. That is,

$$\frac{d\phi_x}{dz} = \frac{\partial \bar{U}^*}{\partial M_x} \qquad \frac{d\phi_y}{dz} = \frac{\partial \bar{U}^*}{\partial M_y} \qquad \frac{dw}{dz} = \frac{\partial \bar{U}^*}{\partial N} \qquad \psi_{xc} = \frac{\partial \bar{U}^*}{\partial V_x} \qquad \psi_{yc} = \frac{\partial \bar{U}^*}{\partial V_y} \qquad \frac{d\phi_z}{dz} = \frac{\partial \bar{U}^*}{\partial M_{zC}}. \tag{8.87}$$

Now consider the complementary work for the free bending and torsion state of an element of the wall Δs-by-Δz as shown in Fig. 8.5. On the contour ($\zeta = 0$) the axial displacement $u_z(s, z)$ corresponds to the stress resultant n_z and tangential displacement $u_s(s, z)$ corresponds to the shear flow q. Let δU_o^* denote the increment in the complementary work per unit area for increments in the stress resultants δn_z and δq acting on the element Δs-by-Δz of Fig. 8.5. Then, the complementary work is

$$\delta U_o^* \Delta s \Delta z = [(\delta n_z \Delta s)u_z + (\delta q \Delta s)u_s]\Big|_z^{z+\Delta z} + [(\delta q \Delta z)u_z]\Big|_{ss}^{s+\Delta s}. \tag{8.88}$$

Divide eq. (8.88) by $\Delta s \Delta z$, and let $\Delta s \to 0$ and $\Delta z \to 0$, to get in the limit

$$\delta U_o^* = \frac{\partial}{\partial z}[(\delta n_z)u_z + (\delta q)u_s] + \frac{\partial}{\partial s}[(\delta q)u_z], \tag{8.89}$$

which expands to

$$\delta U_o^* = \left[\frac{\partial}{\partial z}(\delta n_z) + \frac{\partial}{\partial s}(\delta q)\right]u_z + \left[\frac{\partial}{\partial z}(\delta q)\right]u_s + \frac{\partial u_z}{\partial z}\delta n_z + \left(\frac{\partial u_z}{\partial s} + \frac{\partial u_s}{\partial z}\right)\delta q. \tag{8.90}$$

Statically admissible increments in the stress resultants δn_z and δq requires that they satisfy equilibrium equations (8.42) and (8.43), which are

$$\frac{\partial}{\partial z}(\delta n_z) + \frac{\partial}{\partial s}(\delta q) = 0 \qquad \frac{\partial}{\partial z}(\delta q) = 0. \tag{8.91}$$

From the strain-displacement relations (3.27) and (3.28) on page 37 we identify the axial normal strain ε_{zz} and the shear strain γ_{sz} as

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \qquad \gamma_{sz} = \frac{\partial u_z}{\partial s} + \frac{\partial u_s}{\partial z}. \tag{8.92}$$

Substitute eqs. (8.91) and (8.92) into eq. (8.90) to get the increment in the complementary work per unit area as

$$\delta U_o^* = \varepsilon_{zz}\delta n_z + \gamma_{sz}\delta q. \tag{8.93}$$

For an elastic material we identify δU_o^* with the increment in the complementary strain energy function per unit area, which is a function of the stress resultants, or $U_o^*(n_z, q)$, with the properties

$$\varepsilon_{zz} = \frac{\partial U_o^*}{\partial n_z} \qquad \gamma_{sz} = \frac{\partial U_o^*}{\partial q}. \tag{8.94}$$

Now substitute Hooke's law (8.45) for the normal strain ε_{zz} and for the shear strain γ_{sz} in the previous equation to get

$$\frac{\partial U_o^*}{\partial n_z} = \frac{1}{B}(n_z - bq) \qquad \frac{\partial U_o^*}{\partial q} = \frac{1}{B}(aq - bn_z). \tag{8.95}$$

The complementary strain energy function per unit area consistent with these properties (8.95) is

$$U_o^* = \frac{1}{2B}(n_z^2 - 2bn_zq + aq^2). \tag{8.96}$$

The increment in the complementary energy per unit axial length $\delta\bar{U}^*$ is defined $\delta\bar{U}^* = \oint \delta U_o^* ds$. Hence,

$$\bar{U}^* = \frac{1}{2}\oint \frac{1}{B}(n_z^2 - 2bn_zq + aq^2)ds. \tag{8.97}$$

Equations (8.74) and (8.77) relate the shear flow and normal stress resultant to the bar forces N, V_x, and V_y and the moments M_x, M_y, and M_z. Imposing the properties of the complementary strain energy for the bar given by (8.87) to the expression (8.97) for the complementary strain energy, we get the following relations:

$$\frac{d\phi_x}{dz} = \oint \frac{1}{B}\left[(n_z - bq)\frac{\partial n_z}{\partial M_x} + (aq - bn_z)\frac{\partial q}{\partial M_x}\right]ds \tag{8.98}$$

$$\frac{d\phi_y}{dz} = \oint \frac{1}{B}\left[(n_z - bq)\frac{\partial n_z}{\partial M_y} + (aq - bn_z)\frac{\partial q}{\partial M_y}\right]ds \tag{8.99}$$

$$\frac{dw}{dz} = \oint \frac{1}{B}\left[(n_z - bq)\frac{\partial n_z}{\partial N} + (aq - bn_z)\frac{\partial q}{\partial N}\right]ds \tag{8.100}$$

$$\psi_{xc} = \oint \frac{1}{B}\left[(n_z - bq)\frac{\partial n_z}{\partial V_x} + (aq - bn_z)\frac{\partial q}{\partial V_x}\right]ds \tag{8.101}$$

$$\psi_{yc} = \oint \frac{1}{B}\left[(n_z - bq)\frac{\partial n_z}{\partial V_y} + (aq - bn_z)\frac{\partial q}{\partial V_y}\right]ds \tag{8.102}$$

$$\frac{d\phi_z}{dz} = \oint \frac{1}{B}\left[(n_z - bq)\frac{\partial n_z}{\partial M_{zC}} + (aq - bn_z)\frac{\partial q}{\partial M_{zC}}\right]ds\ . \tag{8.103}$$

Equations (8.98) to (8.103) are statements of Castigilano's second theorem.

8.2.4 Cross-sectional compliance matrix

Substitute eq. (8.74) for the shear flow, and substitute eq. (8.77) for the normal stress resultant, into eqs. (8.98) to (8.103), followed by integration over the contour. The result from the integration leads to compliance form of the material law:

$$\begin{bmatrix} d\phi_x/dz \\ d\phi_y/dz \\ dw/dz \\ \psi_{xc} \\ \psi_{yc} \\ d\phi_z/dz \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ N \\ V_x \\ V_y \\ M_{zC} \end{bmatrix}\ . \tag{8.104}$$

Elements of the compliance matrix are given below.

$$c_{11} = \frac{k}{D_{xx}} \qquad c_{21} = \frac{-kn_y}{D_{xx}} = c_{12} = \frac{-kn_x}{D_{yy}} \qquad c_{13} = c_{31} = 0 \qquad c_{14} = c_{41} = \left(\frac{k}{D_{xx}}\right)\oint(b\bar{y}F_{xc})ds \tag{8.105}$$

$$c_{15} = c_{51} = \left(\frac{k}{D_{xx}}\right)\oint(b\bar{y}F_{yc})ds \qquad c_{16} = c_{61} = \left(\frac{-k}{2A_cD_{xx}}\right)\oint(b\bar{y})ds \tag{8.106}$$

$$c_{22} = \frac{k}{D_{yy}} \qquad c_{23} = c_{32} = 0 \qquad c_{24} = c_{42} = \left(\frac{k}{D_{yy}}\right)\oint(b\bar{x}F_{xc})ds \tag{8.107}$$

$$c_{25} = c_{52} = \left(\frac{k}{D_{yy}}\right)\oint(b\bar{x}F_{yc})ds \qquad c_{26} = c_{62} = \left(\frac{-k}{2A_cD_{yy}}\right)\oint(b\bar{x})ds \qquad c_{33} = 1/S \tag{8.108}$$

$$c_{34} = c_{43} = \left(\frac{1}{S}\right)\oint(bF_{xc})ds \qquad c_{35} = c_{53} = \left(\frac{1}{S}\right)\oint(bF_{yc})ds \qquad c_{36} = c_{63} = \left(\frac{-1}{2A_cS}\right)\oint(b)ds \tag{8.109}$$

$$c_{44} = \oint\frac{1}{B}(aF_{xc}^2 + 2bF_{xc}\Phi_x + \Phi_x^2)ds \qquad c_{45} = c_{54} = \oint\frac{1}{B}(aF_{xc}F_{yc} + bF_{yc}\Phi_x + bF_{xc}\Phi_y + \Phi_x\Phi_y)ds \tag{8.110}$$

$$c_{46} = c_{64} = \left(\frac{1}{2A_c}\right)\oint\frac{1}{B}(-aF_{xc} + bF_{xc}\Phi - b\Phi_x + \Phi\Phi_x)ds \qquad c_{55} = \oint\frac{1}{B}(aF_{yc}^2 + 2bF_{yc}\Phi_y + \Phi_y^2)ds \tag{8.111}$$

$$c_{56} = c_{65} = \left(\frac{1}{2A_c}\right)\oint\frac{1}{B}(-aF_{yc} + bF_{yc}\Phi - b\Phi_y + \Phi\Phi_y)ds \qquad c_{66} = \left(\frac{1}{4A_c^2}\right)\oint\frac{1}{B}(a - 2b\Phi + \Phi^2)ds\,. \tag{8.112}$$

Matrix $\left[c_{ij}\right]$ is symmetric so that twenty-one of the coefficients are independent. The fifteen of the off-diagonal coefficients correspond to different types of coupling effects as described in table 8.3.

Table 8.3 Description of coupling coefficients

Coefficients	**Coupling effects**	**Comment**
c_{21}	combined bending about *x*- and *y*-axes	$c_{21} = 0$ if $D_{xy} = 0$
c_{31} & c_{32}	bending-extension	$c_{31} = c_{32} = 0$, since origin is at modulus weighted centroid
c_{41}, c_{51}, c_{42}, & c_{52}	bending-shearing	are zero if parameter $b = 0$ over entire contour
c_{61} & c_{62}	bending-torsion	are zero if parameter $b = 0$ over entire contour
c_{43} & c_{53}	shearing-extension	are zero if parameter $b = 0$ over entire contour
c_{63}	torsion-extension	is zero if parameter $b = 0$ over entire contour
c_{64} & c_{65}	torsion-shearing	
c_{45}	combined transverse shear in *x*-*z* and *y*-*z* planes	

Example 8.3 Graphite-epoxy circular tube

Nixon (1987) conducted experiments on thin-walled tubes fabricated from T300/5208 graphite/epoxy. The test specimens had a mean radius R = 20.32 mm, a wall thickness t = 1.016 mm, and were composed of two unidirectional layers with angles $\varphi_1 = -20°$ and $\varphi_2 = 70°$. The thickness of both layers is the same, and the properties of the material are E_1 = 146.7 GPa (21.3 ksi), E_2 = 11.0 GPa (1.6 ksi), G_{12} = 6.41 GPa (0.93 ksi), principal Poisson's ration $\nu_{21} = 0.38$ and the minor Poisson's ratio $\nu_{12} = 0.0285$. The twist per unit length $d\phi_z/dz$ was measured in the experiment for an applied axial force N and an applied torque M_z. Determine this relationship from the composite bar theory.

Solution. The in-plane stiffness matrix is determined from $[A] = [\bar{Q}(\varphi_1)]\frac{t}{2} + [\bar{Q}(\varphi_2)]\frac{t}{2}$, where the formulas for the elements of transformed reduced stiffness matrix are listed in eq. (8.35). The result is

$$\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{21} & A_{22} & A_{26} \\ A_{61} & A_{62} & A_{66} \end{bmatrix} = \begin{bmatrix} 67.8649 & 17.4628 & 15.6839 \\ 17.4628 & 67.8649 & -15.6839 \\ 15.6839 & -15.6839 & 19.6701 \end{bmatrix} \text{MN/m} . \tag{a}$$

From eqs. (8.47) and (8.48) the stiffness parameters of the composite bar theory are

$$B_s = 16.0454 \text{ MN/m} \qquad B_z = -19.7196 \text{ MN/m} \qquad b = -1.22899 \qquad B = 39.1363 \text{ MN/m} \qquad a = 3.9495 . \tag{b}$$

Note that the stiffness parameters are spatially uniform over the entire contour. Cartesian coordinates relative to the center of the circular contour are $x = R\cos\theta$ and $y = R\sin\theta$, $0 \le \theta < 2\pi$. From eq. (8.52) the axial stiffness is $S = 4.99669 \text{ MN}$, and the modulus-weighted first moments (8.70) are

$$S_x = \oint B(s)y(s)ds = B\int_0^{2\pi} y(\theta)Rd\theta = 0 \qquad S_y = \oint B(s)x(s)ds = B\int_0^{2\pi} x(\theta)Rd\theta = 0 . \tag{c}$$

As a consequence of eq. (**c**) the modulus-weighted centroid coincides with the center of the circular contour. The modulus-weighed second moments are computed from eq. (8.57), and the results are

$$D_{xx} = D_{yy} = \pi BR^3 = 1,031.57 \text{ N-m}^2 \qquad D_{xy} = 0 . \tag{d}$$

Thus, from eq. (8.59) and eq. (8.62) we find $n_x = n_y = 0$, $k = 1$, $\bar{x} = x$, and $\bar{y} = y$. The combined first moment functions in eq. (8.69) are computed from the first moment functions in eq. (8.70). The results are

$$\bar{S}_x(\theta) = BR^2(1-\cos\theta) \qquad \bar{S}_y(\theta) = BR^2\sin\theta . \tag{e}$$

Note that $\bar{S}_x(0) = \bar{S}_y(0) = \bar{S}_x(2\pi) = \bar{S}_y(2\pi) = 0$. The shear flow distribution functions F_{xc} and F_{yc} are computed from eq. (8.75). Functions Φ_x, Φ_y, Φ, which couple shear and torsion to the normal stress resultant (8.77), are computed from eqs. (8.78) to (8.80). In this example, the results for these functions are

$$F_{xc} = \frac{\sin\theta}{\pi R} \qquad F_{yc} = \frac{-\cos\theta}{\pi R} \qquad \Phi_x = \Phi_y = \Phi = 0 \qquad 0 \le \theta < 2\pi . \tag{f}$$

Elements of the compliance matrix are determined by eqs. (8.105) to (8.112). In this example the result for the cross-sectional compliance matrix is

$$\begin{Bmatrix} d\phi_x/dz \\ d\phi_y/dz \\ dw/dz \\ \psi_x \\ \psi_y \\ d\phi_z/dz \end{Bmatrix} = \frac{1}{\pi B R^3}\begin{bmatrix} 1 & 0 & 0 & bR & 0 & 0 \\ 0 & 1 & 0 & 0 & -bR & 0 \\ 0 & 0 & \frac{R^2}{2} & 0 & 0 & \frac{-bR}{2} \\ bR & 0 & 0 & aR^2 & 0 & 0 \\ 0 & -bR & 0 & 0 & aR^2 & 0 \\ 0 & 0 & \frac{-bR}{2} & 0 & 0 & \frac{a}{2} \end{bmatrix}\begin{Bmatrix} M_x \\ M_y \\ N \\ V_x \\ V_y \\ M_z \end{Bmatrix}. \tag{g}$$

The twist per unit length, or unit twist, for the circular tube is equal to $c_{63}N + c_{66}M_z$. The unit twist evaluates as

$$\frac{d\phi_z}{dz} = \left(\frac{-b}{2\pi B R^2}\right)N + \left(\frac{a}{2\pi B R^3}\right)M_z = (1.21043\times10^{-5})N + (1.91431\times10^{-3})M_z. \tag{h}$$

The unit twist in eq. (**h**) is plotted with respect to the torque in figure. 8.6 for two values of the axial force. Discrete measurements from the experiment reported by Nixon (1987) are shown by filled circles in the plot. ■

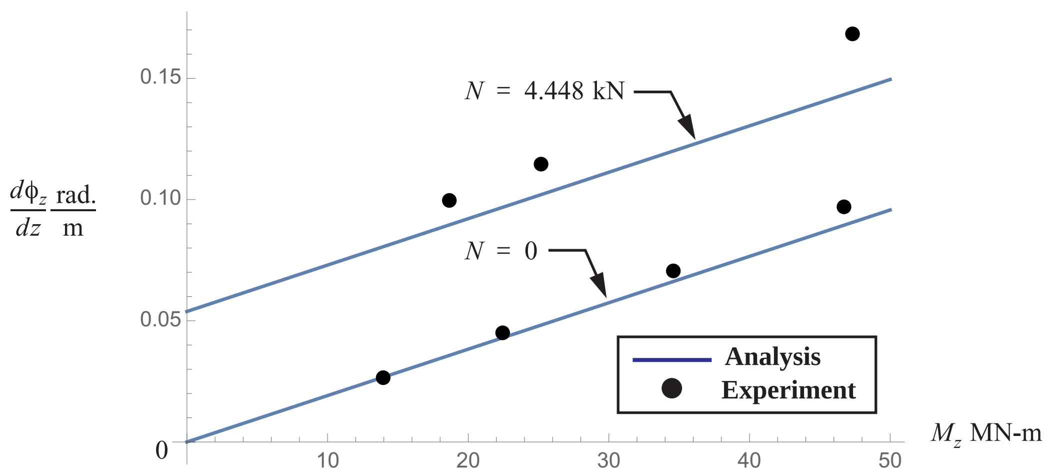


Fig. 8.6 **example 8.3: Unit twist versus torque for the two values of the axial force.**

Example 8.4 Composite box beam

Consider the composite box beam in the experiments conducted by Smith and Chopra (1991) and Chandra et al. (1990). As shown in figure. 8.7, the beam is clamped at its left end where the axial coordinate $z = 0$, $0 \le z \le L$, where the length of the beam L = 762 mm. The cross-sectional dimensions of the rectangular contour are $b_x = 24.2$ mm and $b_y = 13.6$ mm, and the wall thickness $t = 0.76$ mm over the entire contour. The material is unidirectional tape of carbon-epoxy with properties $E_1 = 142$ GPa, $E_2 = 9.8$ GPa, $G_{12} = 6$ GPa, $\nu_{21} = 0.42$, and $\nu_{12} = 0.029$.

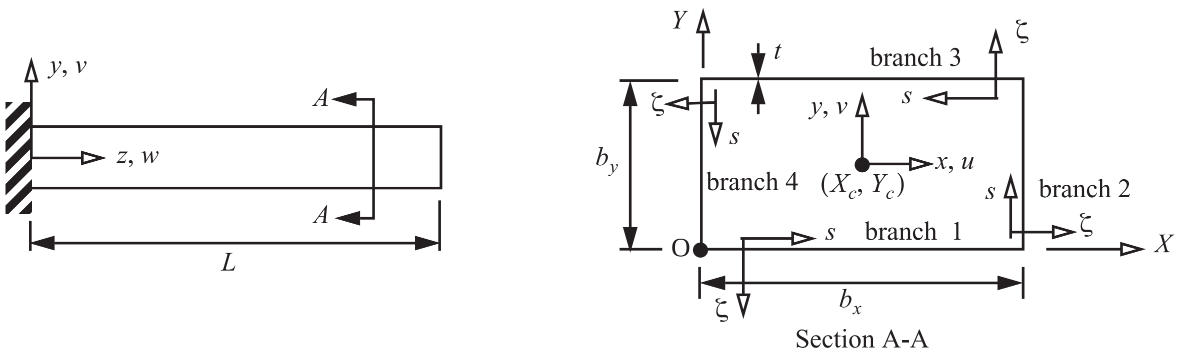
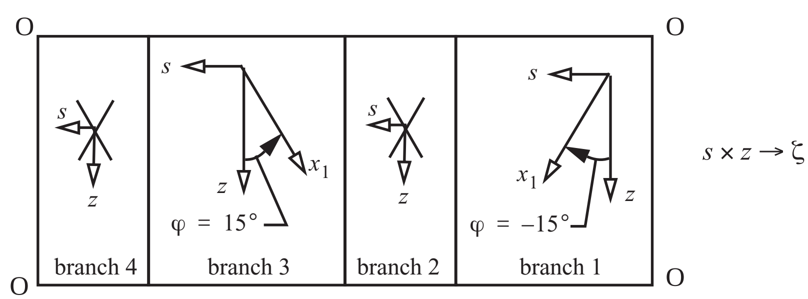


Fig. 8.7 Cantilever, thin-walled box beam.

The lower horizontal flange, or branch 1, is a unidirectional laminate with a ply angle $\varphi = -15°$, and the upper horizontal flange, or branch 3, is also a unidirectional laminate with a ply angle of $\varphi = 15°$. The vertical webs, or branches 2 and 4, are angle-ply laminates with a layup of $(15°, -15°)$. Imagine cutting the box beam parallel to the z-axis through point O. Then unfold the laminated walls and lay them flat such that the outside surface is facing up. The fiber directions with respect to s-z-ζ coordinates in each branch are shown in figure. 8.8.

Fig. 8.8 Outside surface of the unfolded box.



a) Determine the torsional rotation $\phi_z(z)$ under transverse bending $V_y = Q$.

b) Determine the slope dv/dz of the reference axis due to a torque M_z.

Solution. Stiffness coefficients of the four branches comprising the contour are listed in table 8.4.

Table 8.4 Stiffness coefficients for each branch of the box beam

Stiffness coefficient	Branch 1	Branch 3	Branches 2 & 4
A_{11}, MN/m	8.59	8.59	8.59
A_{12}, MN/m	8.93	8.93	8.93
A_{22}, MN/m	9.67	9.67	9.67
A_{66}, MN/m	10.3	10.3	10.3
A_{16}, MN/m	–2.73	2.73	0
A_{26}, MN/m	–2.27	2.27	0
B_z, MN/m	-19.9	19.8	0
B_s, MN/m	9.46	9.455	10.3
B, MN/m	45.7	45.7	87.4
b, (–)	–2.1	2.1	0
a, (–)	9.242	9.242	8.465

Note that $B_3 = B_1$, $B_4 = B_2$, $b_3 = -b_1$, $a_3 = a_1$, and $a_4 = a_2$.The origin of the contour coordinate where $s = 0$ is at point O of section A-A of figure. 8.7. The Cartesian coordinate functions $(X(s), Y(s))$ with origin also at point O are listed in table 8.5.

Table 8.5 Parametric equations of the contour for the box beam

Branch no.	Range of s, in.	$X(s)$	$Y(s)$
1	$0 \le s \le b_x$	s	0
2	$b_x \le s \le b_x + b_y$	b_x	$s - b_x$
3	$b_x + b_y \le s \le 2b_x + b_y$	$2b_x + b_y - s$	b_y
4	$2b_x + b_y \le s \le 2(b_x + b_y)$	0	$2(b_x + b_y) - s$

The axial stiffness is

$$S = \oint B ds = 2(B_1 b_x + B_2 b_y) = 4.58819 \text{ MN}. \tag{a}$$

The first moments with respect to the *X*-*Y* coordinates are

$$S_X = \oint BY ds = b_y(B_1 b_x + B_2 b_y) \qquad S_Y = \oint BX ds = b_x(B_1 b_x + B_2 b_y). \tag{b}$$

Consequently, the modulus-weighted centroid is located at

$$X_c = S_Y/S = b_x/2 \qquad Y_c = S_X/S = b_y/2. \tag{c}$$

In this example the modulus-weighted centroid coincides with the geometric centroid of the cross section. The Cartesian coordinates of the contour with respect to the modulus-weighted centroid are determined from $x(s) = X(s) - X_c$ and $y(s) = Y(s) - Y_c$. The modulus-weighted second moments computed from eq. (8.57) are

$$D_{xx} = \frac{b_y^2}{6}(3B_1 b_x + B_2 b_y) = 138.884 \text{ Nm}^2 \qquad D_{yy} = \frac{b_x^2}{6}(B_1 b_x + 3B_2 b_y) = 455.927 \text{ Nm}^2 \qquad D_{xy} = 0\,. \quad \textbf{(d)}$$

The values of the parameters listed in eq. (8.59) are $n_x = n_y = 0$ and $k = 1$. Hence, from eq. (8.62) we find $\bar{x}(s) = x(s)$ and $\bar{y}(s) = y(s)$. Also, from eq. (8.69) $\bar{S}_x(s) = S_x(s)$ and $\bar{S}_y(s) = S_y(s)$. The modulus-weighted distribution functions $S_x(s)$ and $S_y(s)$ with respect to a segment of the cross-sectional area are defined in eq. (8.70), and the results for these functions are listed in table 8.6.

Table 8.6 Modulus-weighted distribution functions for the first area moments

Branch	$\bar{S}_x(s) = S_x(s)$	$\bar{S}_y(s) = S_y(s)$
1	$(-B_1 b_y s)/2$	$B_1 s[(-b_x + s)/2]$
2	$[-B_1 b_x b_y + B_2(b_x - s)(b_x + b_y - s)]/2$	$[B_2 b_x(-b_x + s)]/2$
3	$[-B_1 b_y(2b_x + b_y - s)]/2$	$[B_2 b_x b_y - B_1(2b_x^2 + 3b_x(b_y - s) + (b_y - s)^2)]/2$
4	$-B_2([4b_x^2 + 6b_x b_y + 2b_y^2 - 4b_x s - 3b_y s + s^2]/2)$	$[B_2 b_x(2b_x + 2b_y - s)]/2$

The procedure to determine $S_x(s)$ and $S_y(s)$ is the same procedure used to determine the first area moments $Q_x(s)$ and $Q_y(s)$ for a cross section with a wall made of an isotropic material. See example 3.4 on page 71. For the composite wall $S_x(s)$ is analogous to $Q_x(s)$ of the isotropic wall, and $S_y(s)$ is analogous to $Q_y(s)$. Note that $\bar{S}_x(0) = 0$ in branch 1, and that $\bar{S}_x[2(b_x + b_y)] = 0$ in branch 4, which are necessary conditions for the first moment about the centroidal *x*-axis. Similarly, $\bar{S}_y(0) = 0$ in branch 1, and $\bar{S}_y[2(b_x + b_y)] = 0$ in branch 4, which are necessary conditions for the first moment about the centroidal *y*-axis.

The coordinates normal to the contour for each branch with respect to the centroid given by eq. (3.10) on page 34, and the area enclosed by the contour, are as follows:

$$r_{nc1} = r_{nc3} = b_y/2 = 6.8 \text{ mm}, \; r_{nc2} = r_{nc4} = b_x/2 = 12.1 \text{ mm}, \text{ and } A_c = b_x b_y = 329.12 \text{ mm}^2. \quad \textbf{(e)}$$

The numerical evaluation of the shear flow distribution functions in eq. (8.75) can now be computed with the results shown in table 8.7.

Table 8.7 Shear flow distribution functions for the box beam

Branch	F_{xc}, m^{-1} (s in meters)	F_{yc}, m^{-1} (s in meters)
1	$-15.771 - 1,212.5s + 50,102.5s^2$	$-27.066 - 2,236.88s$
2	$-71.896 + 2,319.2s$	$-260.728 - 19,505.9s + 314,612s^2$

Table 8.7 Shear flow distribution functions for the box beam

Branch	F_{xc}, m^{-1} (s in meters)	F_{yc}, m^{-1} (s in meters)
3	$-101.65 + 5,000.2s - 50,102.5s^2$	$-111.62 + 2,236.88s$
4	$-159.564 - 2,319.24s$	$-1,447.58 + 43,290.5s - 314,612s^2$

For anisotropic wall properties, the normal stress resultant (8.77) is related to shear and torsion in addition to the axial normal force and bending moments. The coefficient functions of the shear terms ($\Phi_x(s)$ and $\Phi_y(s)$) and torsion term ($\Phi(s)$) are given by eqs. (8.78) to (8.80), and the numerical results for these functions are listed below.

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Table 8.8 Coefficient functions for shear and torsion for the box beam (refer to eqs. (8.78) to (8.80)).

Branch	Φ_x, m^{-1} (s in meters)	Φ_y, m^{-1} (s in meters)	$\Phi(s)$, **dimensionless and s in meters**
1	$-12.209 - 2,546.37s + 105,222s^2$	$56.8427 - 4,697.75s$	-0.554009
2	40.0001	0	$13.482 - 434.916s$
3	$234.389 - 10,501.1s + 105,222s^2$	$234.418 - 4,697.75s$	0.554009
4	40.0001	0	$-29.9222 + 434.916s$

The numerical result for the compliance matrix (8.104) is

$$\begin{bmatrix} d\phi_x/dz \\ d\phi_y/dz \\ dw/dz \\ \psi_x \\ \psi_y \\ d\phi_z/dz \end{bmatrix} = 10^{-3}\begin{bmatrix} 7.200 & 0 & 0 & 0 & 0 & -7.561 \\ 0 & 2.193 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2.180\times10^{-4} & 4.577\times10^{-4} & 0 & 0 \\ 0 & 0 & 4.577\times10^{-4} & 3.389\times10^{-3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 4.861\times10^{-3} & 0 \\ -7.561 & 0 & 0 & 0 & 0 & 25.83 \end{bmatrix}\begin{bmatrix} M_x \\ M_y \\ N \\ V_x \\ V_y \\ M_z \end{bmatrix}.$$

The non-zero compliance coefficient c_{16} couples the torsional and bending responses of the beam. This coupling is illustrated in the following numerical examples.

a. Take the beam subject to transverse shear with $V_y = Q$, $0 \le z \le L$, and no other actions. The bending moment is $M_x = -Q(L-z)$. The twist per unit length under transverse bending is $d\phi_z/dz = c_{61}M_x$. The torsional rotation is given by

$$\phi_z = c_{61}\left[-Q\left(Lz - \frac{z^2}{2}\right)\right] = -c_{61}L^2Q\left[\frac{z}{L} - \frac{1}{2}\left(\frac{z}{L}\right)^2\right] = 0.00439Q\left[\frac{z}{L} - \frac{1}{2}\left(\frac{z}{L}\right)^2\right].$$

The distributions of the torsional rotation for Q = 4.448 N (1 lb.) from the present analysis, and from the experiment conducted by Smith and Chopra (1991), are shown in figure. 8.9.

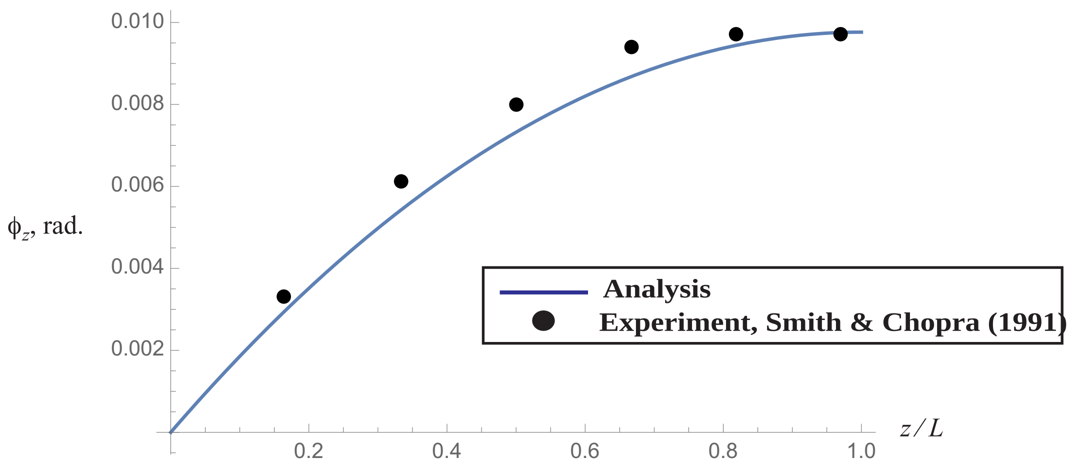


Fig. 8.9 Spanwise distribution of the torsional rotation for $V_y = Q = 4.448$ N (1 lb.).

b. Take the beam subject to a torque M_z and no other actions. From the compliance matrix we find $\psi_y = c_{55} \cdot 0 = 0$. From eq. (8.85) the slope of the reference axis $dv/dz = -\phi_x$, and from the compliance matrix $d\phi_x/dz = c_{16}M_z$, and $\phi_x = c_{16}M_z z$. Since $c_{16} = c_{61}$, the expression for the slope is

$$\frac{dv}{dz} = -c_{61}M_z z = -(-7.561\times10^{-3})LM_z(z/L) = 0.005761\left(\frac{1}{\text{Nm}}\right)M_z(z/L).$$

The distributions of the slope of the reference axis from the present analysis, and from the experiment conducted by Chandra et al. (1990), for the torque $M_z = 0.113$ Nm (1.0 lb.-in.) are shown in figure. 8.10.

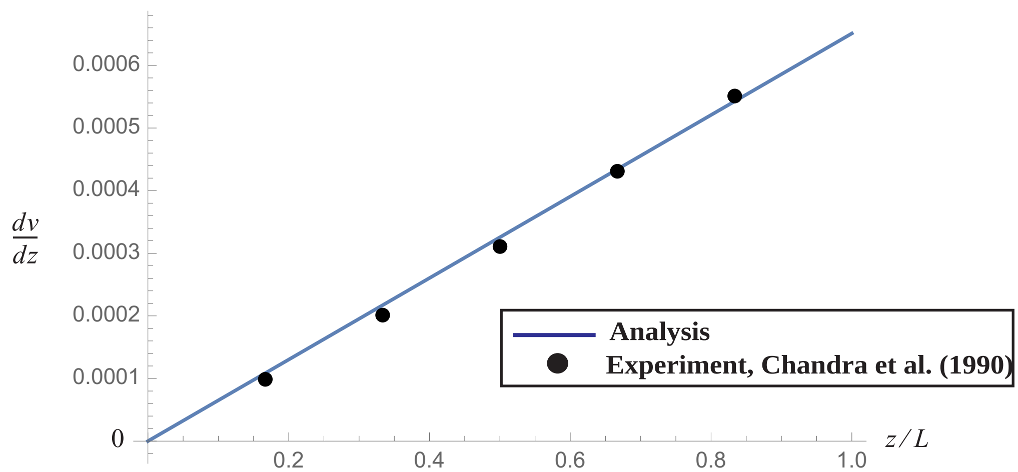


Fig. 8.10 Slope of the reference axis for an applied torque of 0.113 Nm (1.0 lb.-in.)

8.3 Open cross-sectional contour

For an open cross-sectional contour the shear flow is obtained from eq. (8.67) on page 239. The shear flow at the contour origin $q_0 = 0$ if the origin is located at intersection with a longitudinal free edge. (Refer to the discussion in article 3.8.1 on page 51.) The equation for the shear flow for the FRP composite bar is

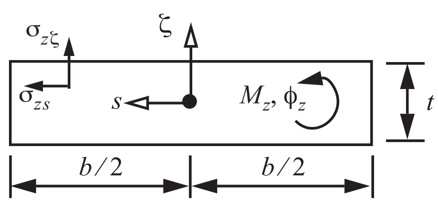
$$q(s) = -\bar{S}_x(s)\frac{k}{D_{xx}}V_y - \bar{S}_y(s)\frac{k}{D_{yy}}V_x . \tag{8.113}$$

The notes concerning the shear center in article 3.8.3 on page 57 apply as well to a bar made of an FRP composite. In particular from these notes, the resultant of the shear flow distribution over the contour is a force with components V_x and V_y acting through the shear center such that there is no torque acting at the shear center. If the cross section is subject to a torque, this torque cannot be balanced by the shear flow, which, according to eq. (8.113), is uniquely determined by the shear forces V_x and V_y. Part (b) of example 8.5 on page 263 shows how to find the shear center for an open section starting with eq. (8.113). After locating the shear center for the open cross-sectional contour, a material law for the torque acting at the shear center remains to be determined. This material law for torsion is developed in the next section.

8.4 Uniform torsion of an FRP bar with a rectangular cross section

We consider the uniform torsion of a prismatic bar with a rectangular cross section composed of a linear elastic, anisotropic material. Cartesian coordinates of the bar are denoted by $s - z - \zeta$, where the coordinate z is parallel to the longitudinal axis of the bar. The origin of the coordinates s and ζ is at the center of the cross section; $-b/2 \le s \le b/2$ where the width of the cross section is denoted by b, and $-t/2 \le \zeta \le t/2$ where the thickness by t. See figure. 8.11.

Fig. 8.11 Bar with a rectangular cross section subject to uniform torsion.



The only applied load is a torque M_z about the z-axis, and the rotation about the z-axis corresponding to the torque is denoted by ϕ_z. The torque and rotation are positive counterclockwise as shown in figure. 8.11. The shear stress components acting on the cross section are denoted by σ_{zs} and $\sigma_{z\zeta}$, and the torque is related to the shear stresses by the following integral over the cross section:

$$M_z = \int_{-b/2}^{b/2}\int_{-t/2}^{t/2}(\zeta\sigma_{zs} - s\sigma_{z\zeta})d\zeta ds . \tag{8.114}$$

The lateral surfaces of the bar are not subject any loads or tractions. Hence, the stress components must satisfy the following conditions at the boundaries of the cross section:

$$\sigma_{\zeta s} = \sigma_{\zeta z} = \sigma_{\zeta\zeta} = 0 \text{, for } -b/2 \le s \le b/2 \text{ at } \zeta = \pm t/2 . \tag{8.115}$$

$$\sigma_{ss} = \sigma_{sz} = \sigma_{s\zeta} = 0 \text{, at } s = \pm b/2 \text{ for } -t/2 \le \zeta \le t/2 \,. \tag{8.116}$$

Under uniform torsion all stress components and their corresponding strains are independent of the axial coordinate *z*. The exact elasticity formulation for the anisotropic bar is given in the monograph by Lekhnitskii (1981). We seek an approximate solution based on the following assumptions:

- Stress components σ_{ss}, $\sigma_{s\zeta}$, and $\sigma_{\zeta\zeta}$ are equal to zero in the domain of the cross section.
- The cross section is rigid in its own plane.

The procedure to develop the material law is as follows: (a) determine the displacements of the bar using the strain-displacement relations and the anisotropic form of Hooke's law, (b) satisfy the differential equation of equilibrium using a separable form of the stress function, (c) use static equivalence to determine the resultants of the axial normal stress, and (d) impose the principle of complementary virtual work to find the unknown part of the stress function. The final result for the material law in torsion is given by eqs. (8.193) and (8.194) on page 263.

8.4.1 Displacements

The non-zero stress components are the axial normal stress σ_{zz}, and the shear stresses σ_{zs} and $\sigma_{z\zeta}$. To effect the rigidity assumption consider Hooke's law (8.16) for the strain components ε_{ss}, $\varepsilon_{\zeta\zeta}$, and $\gamma_{\zeta s}$. Write these material laws as

$$\begin{aligned}
\varepsilon_{ss} &= \frac{\partial u_s}{\partial s} = \frac{1}{E_{ss}}\left[\sigma_{ss} + \left(\frac{C'_{12}}{C'_{11}}\right)\sigma_{zz} + \left(\frac{C'_{13}}{C'_{11}}\right)\sigma_{\zeta\zeta} + \left(\frac{C'_{16}}{C''_{11}}\right)\sigma_{zs}\right] \\
\varepsilon_{\zeta\zeta} &= \frac{\partial u_\zeta}{\partial \zeta} = \frac{1}{E_{\zeta\zeta}}\left[\left(\left(\frac{C'_{31}}{C'_{33}}\right)\sigma_{ss} + \left(\frac{C'_{32}}{C'_{33}}\right)\sigma_{zz}\right) + \sigma_{\zeta\zeta} + \left(\frac{C'_{36}}{C'_{33}}\right)\sigma_{zs}\right], \\
\gamma_{\zeta s} &= \frac{\partial u_\zeta}{\partial s} + \frac{\partial u_s}{\partial \zeta} = \frac{1}{G_{\zeta s}}\left[\left(\frac{C'_{54}}{C'_{55}}\right)\sigma_{z\zeta} + \sigma_{\zeta s}\right]
\end{aligned} \tag{8.117}$$

where E_{ss} is the modulus of elasticity for tension/compression along the *s*-axis, $E_{\zeta\zeta}$ is the modulus along the ζ-axis, and $G_{\zeta s}$ is the shear modulus in the plane of the cross section. These moduli are related to the compliance coefficients by $E_{ss} = (C'_{11})^{-1}$, $E_{\zeta\zeta} = (C'_{33})^{-1}$, and $G_{\zeta s} = (C'_{55})^{-1}$. Invoke the rigidity of the cross section by letting $E_{ss} \to \infty$, $E_{\zeta\zeta} \to \infty$, and $G_{\zeta s} \to \infty$. The assumption of a rigid cross-sectional plane leads to the vanishing of the following strain-displacement relations:

$$\varepsilon_{ss} = \frac{\partial u_s}{\partial s} = 0 \qquad \varepsilon_{\zeta\zeta} = \frac{\partial u_\zeta}{\partial \zeta} = 0 \qquad \gamma_{\zeta s} = \frac{\partial u_\zeta}{\partial s} + \frac{\partial u_s}{\partial \zeta} = 0 \,. \tag{8.118}$$

The normal strains in eq. (8.118) mean displacement functions $u_s = u_s(z, \zeta)$ and $u_\zeta = u_\zeta(z, s)$. Hooke's law (8.16) for the remaining strains reduces to

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} = C'_{22}\sigma_{zz} + C'_{26}\sigma_{sz} \,, \tag{8.119}$$

$$\gamma_{z\zeta} = \frac{\partial u_\zeta}{\partial z} + \frac{\partial u_z}{\partial \zeta} = C'_{44}\sigma_{z\zeta} \text{, and} \tag{8.120}$$

$$\gamma_{sz} = \frac{\partial u_z}{\partial s} + \frac{\partial u_s}{\partial z} = C'_{62}\sigma_{zz} + C'_{66}\sigma_{sz}\,. \qquad \textbf{(8.121)}$$

Let the axial normal strain $\varepsilon_{zz} = D(s, \zeta)$, where the function $D(s, \zeta)$ is to be determined from the independence of the strains on axial coordinate *z*. Begin by integrating the strain-displacement equation for the axial normal strain with respect to *z* to determine the axial displacement as

$$u_z = zD(s, \zeta) + w(s, \zeta)\,. \qquad \textbf{(8.122)}$$

Solve eq. (8.119), for the axial normal stress to get

$$\sigma_{zz} = (D - C'_{26}\sigma_{zs})/(C'_{22})\,. \qquad \textbf{(8.123)}$$

Substitute the axial displacement (8.122) into eq. (8.120) to find

$$\frac{\partial u_\zeta}{\partial z} = -z\frac{\partial D}{\partial \zeta} + \left(-\frac{\partial w}{\partial \zeta} + C'_{44}\sigma_{z\zeta}\right). \qquad \textbf{(8.124)}$$

Integrate eq. (8.124) with respect to *z* to get

$$u_\zeta = \frac{-z^2}{2}\frac{\partial D}{\partial \zeta} + z\left(-\frac{\partial w}{\partial \zeta} + C'_{44}\sigma_{z\zeta}\right) + v(s)\,. \qquad \textbf{(8.125)}$$

Substitute eq. (8.125) for the displacement u_ζ for the expression for the strain $\varepsilon_{\zeta\zeta}$ in eq. (8.118) to get

$$\varepsilon_{\zeta\zeta} = \frac{\partial u_\zeta}{\partial \zeta} = 0 = \frac{-z^2}{2}\frac{\partial^2 D}{\partial \zeta^2} + z\frac{\partial}{\partial \zeta}\left(-\frac{\partial w}{\partial \zeta} + C'_{44}\sigma_{z\zeta}\right), \text{ for all values of } z. \qquad \textbf{(8.126)}$$

Substitute eq. (8.123) for the axial normal stress into eq. (8.121) to get

$$\gamma_{sz} = \frac{\partial u_z}{\partial s} + \frac{\partial u_s}{\partial z} = (C'_{62}D)/C'_{22} + \beta_{66}\sigma_{sz}\,, \qquad \textbf{(8.127)}$$

where $\beta_{66} = (C'_{66} - C'^2_{26}/C'_{22})$. Substitute eq. (8.122) for u_z into eq. (8.127) to get

$$\frac{\partial u_s}{\partial z} = -z\frac{\partial D}{\partial s} + \left(-\frac{\partial w}{\partial s} + (C'_{62}D)/C'_{22} + \beta_{66}\sigma_{sz}\right). \qquad \textbf{(8.128)}$$

Integrate eq. (8.128) the with respect to *z* to find the displacement u_s as

$$u_s = \frac{-z^2}{2}\frac{\partial D}{\partial s} + z\left(-\frac{\partial w}{\partial s} + \beta_{66}\sigma_{sz} + (C'_{62}D)/C'_{22}\right) + u(\zeta)\,. \qquad \textbf{(8.129)}$$

Substitute eq. (8.129) for displacement u_s in the expression for the strain ε_{ss} in eq. (8.118) to get

$$\varepsilon_{ss} = \frac{\partial u_s}{\partial s} = 0 = \frac{-z^2}{2}\left(\frac{\partial^2 D}{\partial s^2}\right) + z\frac{\partial}{\partial s}\left(-\frac{\partial w}{\partial s} + \beta_{66}\sigma_{sz} + (C'_{62}D)/C'_{22}\right), \text{ for all values of } z. \qquad \textbf{(8.130)}$$

Substitute displacement u_ζ from eq. (8.125), and substitute displacement u_s from eq. (8.129), into the expression for the shear strain $\gamma_{\zeta s}$ in eq. (8.118) to get

$$\gamma_{\zeta s} = 0 = -z^2\frac{\partial^2 D}{\partial s \partial \zeta} + z\left[\frac{\partial}{\partial s}\left(-\frac{\partial w}{\partial \zeta} + C'_{44}\sigma_{z\zeta}\right) + \frac{\partial}{\partial \zeta}\left(-\frac{\partial w}{\partial s} + \beta_{66}\sigma_{sz} + (C'_{62}D)/C'_{22}\right)\right] + \frac{dv}{ds} + \frac{du}{d\zeta}\,. \qquad \textbf{(8.131)}$$

Equations (8.126), (8.130), and (8.131) are to be satisfied for all values of z, from which we conclude the following results:

$$\frac{\partial^2 D}{\partial \zeta^2} = 0 \qquad \frac{\partial^2 D}{\partial s^2} = 0 \qquad \frac{\partial^2 D}{\partial s \partial \zeta} = 0 \tag{8.132}$$

$$\frac{\partial}{\partial \zeta}\left(-\frac{\partial w}{\partial \zeta} + C'_{44}\sigma_{z\zeta}\right) = 0 \qquad \frac{\partial}{\partial s}\left(-\frac{\partial w}{\partial s} + \beta_{66}\sigma_{sz} + (C'_{62}D)/C'_{22}\right) = 0 \tag{8.133}$$

$$\frac{\partial}{\partial s}\left(-\frac{\partial w}{\partial \zeta} + C'_{44}\sigma_{z\zeta}\right) + \frac{\partial}{\partial \zeta}\left(-\frac{\partial w}{\partial s} + \beta_{66}\sigma_{sz} + (C'_{62}D)/C'_{22}\right) = 0 \qquad \frac{dv}{ds} + \frac{du}{d\zeta} = 0\,. \tag{8.134}$$

To satisfy the vanishing of partial derivatives of D in eq. (8.132), we find that function D is linear in the coordinates. That is,

$$D = \bar{A}s + \bar{B}\zeta + \bar{C}\,, \tag{8.135}$$

where $\bar{A}$, $\bar{B}$, and $\bar{C}$ are constants that will be determined later. Integrate the second expression eq. (8.133) with respect to s, and then integrate the first expression in eq. (8.133) with respect to ζ. The results of these integrations are

$$-\frac{\partial w}{\partial s} + \beta_{66}\sigma_{sz} + (C'_{62}D)/C'_{22} + F_1(\zeta) = 0\text{, and} \tag{8.136}$$

$$-\frac{\partial w}{\partial \zeta} + C'_{44}\sigma_{z\zeta} + F_2(s) = 0\,. \tag{8.137}$$

Substitute eq. (8.136) and eq. (8.137) into the first expression in eq. (8.134) to find

$$\frac{dF_1}{d\zeta} + \frac{dF_2}{ds} = 0\,. \tag{8.138}$$

Equation (8.138) is satisfied by $F_1(\zeta) = -\lambda\zeta$ and $F_2(s) = \lambda s$, where λ is called a separation constant. Substitute the result for F_1 into eq. (8.136) to find

$$\frac{\partial w}{\partial s} = \beta_{66}\sigma_{sz} + (C'_{62}D)/C'_{22} - \lambda\zeta\,. \tag{8.139}$$

Substitute the result for F_2 into eq. (8.137) to find

$$\frac{\partial w}{\partial \zeta} = C'_{44}\sigma_{z\zeta} + \lambda s\,. \tag{8.140}$$

Substitute the derivative of displacement w with respect to ζ from eq. (8.140) into the displacement u_ζ given in eq. (8.125) to get

$$u_\zeta = \frac{-z^2}{2}\bar{B} - \lambda z s + v(s)\,. \tag{8.141}$$

Substitute the derivative of displacement w with respect to s from eq. (8.139) into the displacement u_s given in eq. (8.129) to get

$$u_s = \frac{-z^2}{2}\bar{A} + \lambda z\zeta + u(\zeta)\,. \tag{8.142}$$

From eq. (8.134) consider the relation $\frac{dv}{ds} + \frac{du}{d\zeta} = 0$. The latter relation is satisfied by $\frac{dv}{ds} = -\omega$ and $\frac{du}{d\zeta} = \omega$, where ω is a second separation constant. Thus, $v(s) = -\omega s + v(0)$ and $u(\zeta) = \omega\zeta + u(0)$. Substitute the equations for $v(s)$ and $u(\zeta)$ into eqs. eq. (8.141)and eq. (8.142) to get

$$u_\zeta = \frac{-z^2}{2}\bar{B} - \lambda zs - \omega s + v(0) \qquad u_s = \frac{-z^2}{2}\bar{A} + \lambda z\zeta + \omega\zeta + u(0). \tag{8.143}$$

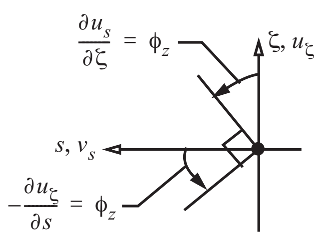


Fig. 8.12 Rotation of the cross section about the *z*-axis.

From eq. (8.118) the in-plane shear strain $\gamma_{\zeta s} = \frac{\partial u_\zeta}{\partial s} + \frac{\partial u_s}{\partial\zeta} = 0$. As shown in figure. 8.12 the partial derivative terms appearing in the shear strain can be related to rotation ϕ_z of the cross section. Let $\frac{\partial u_s}{\partial\zeta} = \phi_z$ and let $-\frac{\partial u_\zeta}{\partial s} = \phi_z$. The partial derivatives of the displacements in eq. (8.143) are equated to the rotation to get

$$\phi_z = \frac{\partial u_s}{\partial\zeta} = \lambda z + \omega\text{, and } \phi_z = -\frac{\partial u_\zeta}{\partial s} = \lambda z + \omega. \tag{8.144}$$

Thus, $\phi_z = \lambda z + \omega$, from which we identify the separation constant $\lambda = \frac{d\phi_z}{dz}$. The separation constant ω represents a rigid body rotation of the bar about the *z*-axis. To prevent rigid body rotation and displacement of the cross section set $\omega = 0$, $v(0) = 0$, and $u(0) = 0$. The final results for the displacements are

$$u_z = z(\bar{A}s + \bar{B}\zeta + \bar{C}) + w(s, \zeta), \tag{8.145}$$

$$u_\zeta = \frac{-z^2}{2}\bar{B} - zs\frac{d\phi_z}{dz}\text{, and} \tag{8.146}$$

$$u_s = \frac{-z^2}{2}\bar{A} + z\zeta\frac{d\phi_z}{dz}. \tag{8.147}$$

8.4.2 Equilibrium

The differential equation for axial equilibrium is

$$\frac{\partial\sigma_{sz}}{\partial s} + \frac{\partial\sigma_{\zeta z}}{\partial\zeta} + \underbrace{\frac{\partial\sigma_{zz}}{\partial z}}_{=\,0} = 0 \quad . \tag{8.148}$$

The axial normal stress σ_{zz} does not contribute to eq. (8.148) since it is independent of coordinate *z*. Equation (8.148) is identically satisfied by the introduction of the stress function $\psi(s, \zeta)$ where the stress components are related to the stress function by

$$\sigma_{sz} = -\frac{\partial\psi}{\partial\zeta} \qquad \sigma_{\zeta z} = \frac{\partial\psi}{\partial s}. \tag{8.149}$$

For shear stress $\sigma_{\zeta z}$ to satisfy the boundary conditions (8.115) at $\zeta = \pm t/2$ the stress function $\partial\psi/\partial s = 0$.

For shear stress σ_{sz} to satisfy the boundary conditions (8.116) at $s = \pm b/2$ the stress function $\partial\psi/\partial\zeta = 0$. That is the stress function is a constant on the boundaries, and for convenience we take $\psi = 0$ on the boundaries of the rectangular domain.

Substitute eq. (8.149) for the shear stresses in the expression (8.114) for the torque to get

$$M_z = \int_{-b/2}^{b/2}\left[\int_{-t/2}^{t/2}\left(\zeta\left(-\frac{\partial\psi}{\partial\zeta}\right) - s\left(\frac{\partial\psi}{\partial s}\right)\right)d\zeta\right]. \tag{8.150}$$

Integrate eq. (8.150) by parts with respect to s and ζ to get

$$M_z = \int_{-b/2}^{b/2}\left\{-\zeta\psi\big|_{t/2}^{t/2} + \int_{-t/2}^{t/2}\psi d\zeta\right\}ds + \int_{-t/2}^{t/2}\left\{[-s\psi]\big|_{-b/2}^{b/2} + \int_{-b/2}^{t/2}\psi ds\right\}d\zeta. \tag{8.151}$$

Since the stress function is equal to zero on the boundaries we find that the torque is given by integral of the stress function over the cross-sectional area:

$$M_z = 2\int_{-b/2}^{b/2}\int_{-t/2}^{t/2}\psi d\zeta ds. \tag{8.152}$$

We make an additional assumption for the stress function that

$$\psi(s, \zeta) = \psi_1(s)[(t/2)^2 - \zeta^2], \tag{8.153}$$

which satisfies the boundary condition that $\psi(s, \pm t/2) = 0$. Function $\psi_1(s)$ must satisfy the boundary condition $\psi_1(\pm b/2) = 0$. The shear stresses for this assumption are given by

$$\sigma_{sz} = 2\psi_1(s)\zeta \qquad \sigma_{\zeta z} = \frac{d\psi_1}{ds}[(t/2)^2 - \zeta^2]. \tag{8.154}$$

Substitute the stress function (8.153) into the torque (8.152) to get

$$M_z = \frac{t^3}{3}\int_{-b/2}^{b/2}\psi_1(s)ds. \tag{8.155}$$

8.4.3 Static equivalence

In general, the resultants of the axial normal stress σ_{zz} acting over the cross section are a normal force denoted by *N*, a bending moment about the *s*-axis by M_s, and a bending moment about the ζ-axis by M_ζ. For a laminated wall these resultants are given by

$$(N, M_\zeta, M_s) = \int_{-b/2}^{b/2}\left[\sum_{k=1}^{Np}\int_{\zeta_k}^{\zeta_{k+1}}(1, \zeta, s)\sigma_{zz}^{(k)}d\zeta\right]ds, \tag{8.156}$$

where N_p is the number of plies, and $k = 1, 2, \ldots, N_p$. At the bottom of the *k-th* ply $\zeta = \zeta_k$, and at the top of the *k-th* ply $\zeta = \zeta_{k+1}$, $\zeta_{k+1} > \zeta_k$. From eqs. (8.123) and (8.149) the axial normal stress in the *k-th* ply is

$$\sigma_{zz}^{(k)} = \frac{D}{C'^{(k)}_{22}} - \left(\frac{C'_{26}}{C'_{22}}\right)^{(k)} \sigma_{sz} = \frac{(\bar{A}s + \bar{B}\zeta + \bar{C})}{C'^{(k)}_{22}} - \left(\frac{C'_{26}}{C'_{22}}\right)^{(k)} 2\zeta\psi_1(s)\,. \tag{8.157}$$

Substitute eq. (8.157) for the axial normal stress into the equations for the axial force and bending moments given by eq. (8.156). In the process of computing the resultants, integrals that are explicit in coordinates s and ζ are performed. Integrals of the stress function also appear in this process and from eq. (8.155), and we use the fact that

$$\int_{-b/2}^{b/2} \psi_1(s)ds = \frac{3M_z}{t^3}\,.$$

The results are

$$N = (bB_{22})\bar{B} + (bA_{22})\bar{C} - \frac{6}{t^3}\theta_{26}M_z\,, \tag{8.158}$$

$$M_s = (bD_{22})\bar{B} + (bB_{22})\bar{C} - \eta_{26}M_z\,, \text{ and} \tag{8.159}$$

$$M_\zeta = \left(\frac{b^3}{12}A_{22}\right)\bar{A} - 2\theta_{26}\int_{-b/2}^{b/2} s\psi_1(s)ds\,. \tag{8.160}$$

Stiffness coefficients in the previous equations are defined by

$$A_{22} = \sum_{k=1}^{Np} \frac{1}{C'^{(k)}_{22}}(\zeta_{k+1} - \zeta_k) \qquad B_{22} = \sum_{k=1}^{Np} \frac{1}{2C'^{(k)}_{22}}(\zeta_{k+1}^2 - \zeta_k^2) \qquad D_{22} = \sum_{k=1}^{Np} \frac{1}{3C'^{(k)}_{22}}(\zeta_{k+1}^3 - \zeta_k^3)\,. \tag{8.161}$$

Shear-extension coupling coefficients are defined by

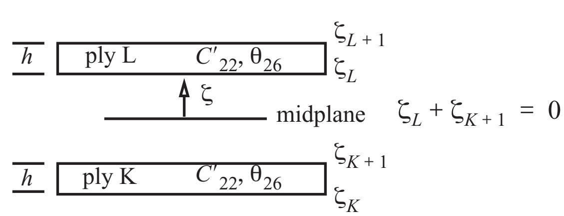
$$\theta_{26} = \sum_{k=1}^{Np}\left(\frac{C'_{26}}{C'_{22}}\right)^{(k)} \frac{1}{2}(\zeta_{k+1}^2 - \zeta_k^{2'}) \qquad \eta_{26} = \left(\frac{6}{t^3}\right)\sum_{k=1}^{Np}\left(\frac{C'_{26}}{C'_{22}}\right)^{(k)} \frac{1}{3}(\zeta_{k+1}^3 - \zeta_k^3)\,. \tag{8.162}$$

We limit consideration to a **symmetric laminate** in which the stacking sequence of the plies is symmetric about the midplane. Symmetry leads to coefficients

$$B_{22} = \theta_{26} = 0\,. \tag{8.163}$$

To illustrate that symmetry results in the previous property consider two identical plies labeled K and L in figure. 8.13. The two plies have the same material properties, same thickness denoted by h, and are symmetrically

Fig. 8.13 Identical plies symmetric about the midplane.



located with respect to the midplane. Symmetry requires the coordinates $\zeta_L + \zeta_{K+1} = 0$. The remaining coordinates are $\zeta_K = \zeta_{K+1} - h$ and $\zeta_{L+1} = \zeta_L + h$. The sum the of plies K and L that contribute to coefficient B_{22} are

$$\frac{1}{2C'_{22}}[\zeta_{L+1}^2 - \zeta_L^2 + \zeta_{K+1}^2 - \zeta_K^2] = \frac{1}{2C'_{22}}[(\zeta_L + h)^2 - \zeta_L^2 + (-\zeta_L)^2 - (-\zeta_L - h)^2] = 0. \tag{8.164}$$

Hence, for a symmetric laminate the normal force $N = 0$ leads to coefficient $\bar{C} = 0$, bending moment $M_\zeta = 0$ leads to coefficient $\bar{A} = 0$, and bending moment $M_s = 0$ leads to coefficient B given by

$$\bar{B} = \frac{\eta_{26}}{bD_{22}} M_z. \tag{8.165}$$

The transverse shear resultants acting on the cross section are denoted by V_s and V_ζ. They are given by

$$V_s = \int_{-b/2}^{b/2}\left[\int_{-t/2}^{t/2} \sigma_{zs} d\zeta\right] ds \text{ and } V_\zeta = \int_{-t/2}^{t/2}\left[\int_{-b/2}^{b/2} \sigma_{z\zeta} ds\right] d\zeta. \tag{8.166}$$

Substitute eq. (8.154) for the shear stresses in the integrals of the previous equations to get

$$V_s = \int_{-b/2}^{b/2} 2\psi_1\left[\int_{-t/2}^{t/2} \zeta d\zeta\right] ds = \int_{-b/2}^{b/2} 2\psi_1[0] ds = 0, \text{ and} \tag{8.167}$$

$$V_\zeta = \int_{-t/2}^{t/2}\left\{[t/2^2 - \zeta^2]\left[\int_{-b/2}^{b/2} \frac{d\psi_1}{ds} ds\right]\right\} d\zeta = \int_{-t/2}^{t/2} \{[t/2^2 - \zeta^2][\psi_1(b/2) - \psi_1(-b/2)]\} d\zeta = 0. \tag{8.168}$$

Therefore, the resultants acting on the cross section of the bar are $N = V_s = V_\zeta = M_s = M_\zeta = 0$ and an applied torque $M_z \neq 0$.

8.4.4 Principle of complementary virtual work

Consider uniform torsion state of the bar as shown in figure. 8.11 where the displacements, strains, and forces satisfy the compatibility conditions, Hooke's law, and the equilibrium conditions. In this state, the actual displacements are $u_z, u_\zeta,$ and u_s given by eqs. (8.145), (8.146), and (8.147), respectively. The actual non-zero strains are $\varepsilon_{zz}, \gamma_{z\zeta},$ and γ_{sz} and the corresponding stresses are $\sigma_{zz}, \sigma_{z\zeta},$ and σ_{sz}, respectively. The only cross-sectional resultant is the torque M_z and its corresponding rotation is ϕ_z. Now consider infinitesimal increments in the stresses denoted by $\delta\sigma_{zz}, \delta\sigma_{z\zeta},$ and $\delta\sigma_{sz}$ that satisfy equilibrium. For a bar of length L, $0 \le z \le L$, the increment in the internal complementary work is given by

$$\delta W_{\text{int}} = \int_0^L\left\{\int_{-b/2}^{b/2}\int_{-t/2}^{t/2} (\varepsilon_{zz}\delta\sigma_{zz} + \gamma_{z\zeta}\delta\sigma_{z\zeta} + \gamma_{sz}\delta\sigma_{sz}) d\zeta ds\right\} dz = L\int_{-b/2}^{b/2}\int_{-t/2}^{t/2} (\varepsilon_{zz}\delta\sigma_{zz} + \gamma_{z\zeta}\delta\sigma_{z\zeta} + \gamma_{sz}\delta\sigma_{sz}) d\zeta ds, \tag{8.169}$$

Note that the strains and stresses are independent of the axial coordinate z. The increment in the external complementary work is

$$\delta W_{\text{ext}} = \int_{-b/2}^{b/2}\int_{-t/2}^{t/2} (u_z\delta\sigma_{zz} + u_\zeta\delta\sigma_{z\zeta} + u_s\delta\sigma_{zs})\big|_{z=L} d\zeta ds - \int_{-b/2}^{b/2}\int_{-t/2}^{t/2} (u_z\delta\sigma_{zz} + u_\zeta\delta\sigma_{z\zeta} + u_s\delta\sigma_{zs})\big|_{z=0} d\zeta ds. \tag{8.170}$$

From eqs. (8.145) to (8.147), and with displacement coefficients $\bar{A} = \bar{C} = 0$, the displacements at the end cross sections are as follows:

$$\text{At } z = 0,\ u_z = w(s, \zeta),\ u_\zeta = 0, \text{ and } u_s = 0. \tag{8.171}$$

$$\text{At } z = L,\ u_z = L\bar{B}\zeta + w(s, \zeta),\ u_\zeta = -\frac{L^2\bar{B}}{2} - Ls\frac{d\phi_z}{dz}, \text{ and } u_s = L\zeta\frac{d\phi_z}{dz}. \tag{8.172}$$

Substitute the displacements at $z = 0$ and $z = L$ into the increment in the external work (8.170) to get

$$\delta W_{\text{ext}} = \int_{-b/2}^{b/2}\int_{-t/2}^{t/2}\left[(L\bar{B}\zeta + w(s, \zeta))\delta\sigma_{zz} + \left(-\frac{L^2\bar{B}}{2} - Ls\frac{d\phi_z}{dz}\right)\delta\sigma_{z\zeta} + \left(L\zeta\frac{d\phi_z}{dz}\right)\delta\sigma_{zs}\right]d\zeta ds - \int_{-b/2}^{b/2}\int_{-t/2}^{t/2}[w(s, \zeta)\delta\sigma_{zz}]d\zeta ds,$$

in which we used the fact that the increments in the stresses are independent of the coordinate z. The integrals involving $w\delta\sigma_{zz}$ add to zero. Hence,

$$\delta W_{\text{ext}} = \int_{-b/2}^{b/2}\int_{-t/2}^{t/2}\left[(L\bar{B}\zeta)\delta\sigma_{zz} + \left(-\frac{L^2\bar{B}}{2} - Ls\frac{d\phi_z}{dz}\right)\delta\sigma_{z\zeta} + \left(L\zeta\frac{d\phi_z}{dz}\right)\delta\sigma_{zs}\right]d\zeta ds.$$

Rearrange the terms in the last equation, and note displacement coefficient $\bar{B}$ is a constant, to get

$$\delta W_{\text{ext}} = L\bar{B}\int_{-b/2}^{b/2}\int_{-t/2}^{t/2}\zeta\delta\sigma_{xx}d\zeta ds - \frac{L^2\bar{B}}{2}\int_{-b/2}^{b/2}\int_{-t/2}^{t/2}\delta\sigma_{z\zeta}d\zeta ds + L\frac{d\phi_z}{dz}\int_{-b/2}^{b/2}\int_{-t/2}^{t/2}(\zeta\delta\sigma_{zs} - s\delta\sigma_{z\zeta})d\zeta ds. \tag{8.173}$$

Integrals of the increments in the stresses are identified as increments in the resultants δM_s, δV_ζ, and δM_z. Then, we find

$$\delta W_{\text{ext}} = L\bar{B}\delta M_s - \frac{L^2\bar{B}}{2}\delta V_\zeta + L\frac{d\phi_z}{dz}\delta M_z. \tag{8.174}$$

Since the bending moment M_s is prescribed then $\delta M_s = 0$. Similarly, shear force V_ζ is prescribed, so $\delta V_\zeta = 0$. The final expression for the increment in the external work is

$$\delta W_{\text{ext}} = L\frac{d\phi_z}{dz}\delta M_z. \tag{8.175}$$

Equate the increment in the external work (8.175) to the increment in internal work (8.169), followed by division by L, to get the principle of complementary work as

$$\frac{d\phi_z}{dz}\delta M_z = \int_{-b/2}^{b/2}\int_{-t/2}^{t/2}(\varepsilon_{zz}\delta\sigma_{zz} + \gamma_{z\zeta}\delta\sigma_{z\zeta} + \gamma_{sz}\delta\sigma_{sz})d\zeta ds. \tag{8.176}$$

The strain-stress relations are given by Hooke's law in eqs. (8.119) to (8.121). In Hooke's law for the strains ε_{zz} and γ_{sz} we substitute eq. (8.123) for the axial normal stress. After the process of eliminating the axial normal stress, we get the strain relations as

$$\varepsilon_{zz} = \bar{B}\zeta \qquad \gamma_{z\zeta} = C'_{44}\sigma_{z\zeta} \qquad \gamma_{sz} = \left(\frac{C'_{26}}{C'_{22}}\right)\bar{B}\zeta + \beta_{66}\sigma_{sz}, \tag{8.177}$$

where the compliance coefficient $\beta_{66} = C'_{66} - C'^2_{26}/C'_{22}$. Let $\bar{B} \to \bar{B} + \delta\bar{B}$ and $\sigma_{sz} \to \sigma_{sz} + \delta\sigma_{sz}$ in eq. (8.123) to find that the increment in the normal stress is

$$\delta\sigma_{zz} = \frac{\zeta}{C'_{22}}\delta\bar{B} - \frac{C'_{26}}{C'_{22}}\delta\sigma_{sz}. \tag{8.178}$$

Substitute eq. (8.177) for the strains in eq. (8.176), followed by substitution of eq. (8.178) for the increment in the normal stress. The result of these substitutions is the following form for the principle of complementary work:

$$\frac{d\phi_z}{dz}\delta M_z = \int_{-b/2}^{b/2}\left\{\int_{t/2}^{t/2}\left[\left(\frac{\zeta^2}{C'_{22}}\bar{B}\right)\delta\bar{B} + (C'_{44}\sigma_{z\zeta})\delta\sigma_{z\zeta} + (\beta_{66}\sigma_{sz})\delta\sigma_{sz}\right]d\zeta\right\}ds. \tag{8.179}$$

Statically admissible increments $\delta\sigma_{zs}$ and $\delta\sigma_{z\zeta}$ in eq. (8.179) are defined in terms of the increment in the stress function $\delta\psi_1$ by

$$\delta\sigma_{sz} = 2\zeta\delta\psi_1(s) \qquad \delta\sigma_{z\zeta} = \delta\left(\frac{d\psi_1}{ds}\right)\left[\left(\frac{t}{2}\right)^2 - \zeta^2\right]. \tag{8.180}$$

Substitute eq. (8.180) for the increments in the stresses in eq. (8.179), followed by the substituting of eq. (8.154) for the stresses σ_{sz} and $\sigma_{z\zeta}$ in eq. (8.179). The result of these substitutions is

$$\frac{d\phi_z}{dz}\delta M_z = \int_{-b/2}^{b/2}\left\{\int_{t/2}^{t/2}\left[\left(\frac{\zeta^2}{C'_{22}}\bar{B}\right)\delta\bar{B} + \left(C'_{44}\left[\frac{t^2}{2} - \zeta^2\right]^2\frac{d\psi_1}{ds}\right)\delta\left(\frac{d\psi_1}{ds}\right) + (4\beta_{66}\zeta^2\psi_1)\delta\psi_1\right]d\zeta\right\}ds. \tag{8.181}$$

In the case of laminated cross section the last equation is written as

$$\frac{d\phi_z}{dz}\delta M_z = \int_{-b/2}^{b/2}\left\{\sum_{k=1}^{Np}\int_{\zeta_k}^{\zeta_{k+1}}\left[\left(\frac{\zeta^2}{C'^{(k)}_{22}}\bar{B}\right)\delta\bar{B} + \left(C'^{(k)}_{44}\left[\frac{t^2}{2} - \zeta^2\right]^2\frac{d\psi_1}{ds}\right)\delta\left(\frac{d\psi_1}{ds}\right) + (4\beta^{(k)}_{66}\zeta^2\psi_1)\delta\psi_1\right]\right\}ds.$$

The integrations with respect to ζ are carried out in the previous equation, and the result is written as

$$\frac{d\phi_z}{dz}\delta M_z = \int_{-b/2}^{b/2}\left[(D_{22}\bar{B})\delta\bar{B} + \left[\left(\frac{t^5}{30}a_{44}\right)\frac{d\psi_1}{ds}\right]\delta\left(\frac{d\psi_1}{ds}\right) + \left[\left(4\frac{t^3}{12}a_{66}\right)\psi_1\right]\delta\psi_1\right]ds, \tag{8.182}$$

where the stiffness coefficient D_{22} is given in eq. (8.161). The laminate compliance coefficients in eq. (8.182)are defined by

$$a_{44} = \left(\frac{30}{t^5}\right)\sum_{k=1}^{Np}C'^{(k)}_{44}\int_{\zeta_k}^{\zeta_{k+1}}\left[\frac{t^2}{2} - \zeta^2\right]^2 d\zeta = \left(\frac{30}{t^5}\right)\sum_{k=1}^{Np}C'^{(k)}_{44}\left[\frac{t^4}{16}(\zeta_{k+1} - \zeta_k) - \frac{t^2}{6}(\zeta^3_{k+1} - \zeta^3_k) + \frac{1}{5}(\zeta^5_{k+1} - \zeta^5_k)\right], \text{ and} \tag{8.183}$$

$$a_{66} = \left(\frac{12}{t^3}\right)\sum_{k=1}^{Np}\beta_{66}^{(k)}\int_{\zeta_k}^{\zeta_{k+1}}\zeta^2 d\zeta = \left(\frac{12}{t^3}\right)\sum_{k=1}^{Np}\beta_{66}^{(k)}\frac{1}{3}(\zeta_{k+1}^3 - \zeta_k^3)\,. \tag{8.184}$$

The terms involving the derivatives of ψ_1 and $\delta\psi_1$ in eq. (8.182) are integrated by parts with respect to *s*. Note that the boundary term vanishes since $\delta\psi_1(\pm b/2) = 0$. ($\psi_1$ is specified at $s = \pm b/2$.) After integration by parts, eq. (8.182) reduces to

$$\frac{d\phi_z}{dz}\delta M_z = (bD_{22}\bar{B})\delta\bar{B} + \int_{-b/2}^{b/2}\left[-\left(\frac{t^5}{30}a_{44}\right)\frac{d^2\psi_1}{ds^2} + \frac{t^3}{3}a_{66}\psi_1\right]\delta\psi_1 ds\,. \tag{8.185}$$

From eq. (8.165) we substitute $\delta\bar{B} = \frac{\eta_{26}}{bD_{22}}\delta M_z$ in eq. (8.185) and collect the terms multiplying δM_z to get

$$\delta M_z\left[\frac{d\phi_z}{dz} - \eta_{26}\bar{B}\right] = \int_{-b/2}^{b/2}\left[-\left(\frac{t^5}{30}a_{44}\right)\frac{d^2\psi_1}{ds^2} + \frac{t^3}{3}a_{66}\psi_1\right]\delta\psi_1 ds\,. \tag{8.186}$$

Finally, substitute $\delta M_z = \frac{t^3}{3}\int_{-b/2}^{b/2}\delta\psi_1 ds$ in eq. (8.186) to get the complementary work statement as

$$\int_{-b/2}^{b/2}\left[\frac{t^3}{3}\left(\frac{d\phi_z}{dz} - \eta_{26}\bar{B}\right)\right]\delta\psi_1 ds = \int_{-b/2}^{b/2}\left[-\left(\frac{t^5}{30}a_{44}\right)\frac{d^2\psi_1}{ds^2} + \frac{t^3}{3}a_{66}\psi_1\right]\delta\psi_1 ds\,. \tag{8.187}$$

Since the increment in complementary work (8.187) holds for every continuous function $\delta\psi_1(s)$ such that $\delta\psi_1(\pm b/2) = 0$, we find the following differential equation governing function $\psi_1(s)$:

$$-\left(\frac{t^5}{30}a_{44}\right)\frac{d^2\psi_1}{ds^2} + \frac{t^3}{3}a_{66}\psi_1 = \frac{t^3}{3}\left(\frac{d\phi_z}{dz} - \eta_{26}\bar{B}\right) \qquad -b/2 < s < b/2\,. \tag{8.188}$$

Simplify eq. (8.188) by multiplying by $-3/t^3$ to write the differential equation as

$$\frac{t^2}{10}a_{44}\frac{d^2\psi_1}{ds^2} - a_{66}\psi_1 = -\left(\frac{d\phi_z}{dz} - \eta_{26}\bar{B}\right) \qquad -b/2 < s < b/2\,. \tag{8.189}$$

The solution of eq. (8.189) subject to $\psi_1(\pm b/2) = 0$ is

$$\psi_1(s) = \frac{1}{a_{66}}\left(\frac{d\phi_z}{dz} - \eta_{26}\bar{B}\right)\left[1 - \frac{\cosh\mu s}{\cosh\mu b/2}\right], \text{ where } \mu = \frac{1}{t}\sqrt{10a_{66}/a_{44}}\,. \tag{8.190}$$

The torque is computed from eq. (8.155) to find

$$M_z = \frac{bt^3}{3a_{66}}\left(\frac{d\phi_z}{dz} - \eta_{26}\bar{B}\right)g\left(\mu\frac{b}{2}\right), \tag{8.191}$$

where the function $g(\mu b/2)$ is defined as

$$g\left(\mu\frac{b}{2}\right) = 1 - \left(\mu\frac{b}{2}\right)^{-1}\tanh\left(\mu\frac{b}{2}\right), \text{ and } \mu\frac{b}{2} = \frac{b}{2t}\sqrt{10(a_{66}/a_{44})}. \quad (8.192)$$

Substitute eq. (8.165) for coefficient $\bar{B}$ in eq. (8.191) to find

$$M_z = \frac{bt^3}{3a_{66}}\left(\frac{d\phi_z}{dz} - \frac{\eta_{26}^2}{bD_{22}}M_z\right)g\left(\mu\frac{b}{2}\right)$$

Solve the latter equation for the torque and write the result as

$$M_z = D_T\frac{d\phi_z}{dz}. \quad (8.193)$$

where D_T is the torsional stiffness of the bar given by

$$D_T = \frac{bt^3}{3}\frac{g\left(\mu\frac{b}{2}\right)}{a_{66} + \left[t^3\eta_{26}^2 g\left(\mu\frac{b}{2}\right)\right]/(3D_{22})}. \quad (8.194)$$

From eq. (8.146) and eq. (8.165), the lateral displacement of the bar is

$$u_\zeta = -\frac{z^2}{2}\left(\frac{\eta_{26}}{bD_{22}}\right)M_z - zs\left(\frac{M_z}{D_T}\right). \quad (8.195)$$

Under the action of torsion the axis of the bar does not remain straight, but it is curved as shown in figure. 8.14(a).

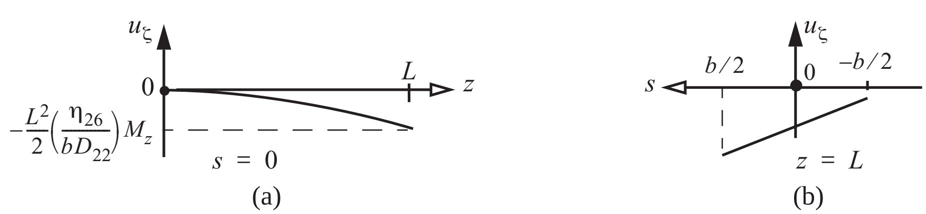


Fig. 8.14 Lateral displacement of the bar under torsion: (a) in the plane $s = 0$, and (b) in the plane $z = L$

Example 8.5 Transverse bending and torsion of a composite channel section

The cross section of the bar shown in figure. 8.15(a) is composed of a lower horizontal flange with length $b_1 = 16\text{ mm}$, an upper horizontal flange with length $b_3 = 16\text{ mm}$. The flanges are joined by a vertical web with length $b_2 = 32\text{ mm}$. The lower flange is denoted by branch 1, the web by branch 2, and the upper flange by branch 3. Each branch is fabricated from T300/5208 graphite/epoxy with material properties listed in Table 8.2 on page 231, and the dimensional units used in this example are Newtons and millimeters. The laminate in each branch consists of eight plies with a specially orthotropic, symmetric stacking sequence of $[45°/–45°/0/90]_S$. The thickness of each branch $t = 1.016\text{ mm}$. As shown in figure. 8.15(b), the cross section is symmetric about the *X*-axis both in geometry and material properties. The axial stiffness per unit length *B* is given in eq. (8.46), the torsional stiffness per unit length B_s is given in eq. (8.47), and they are the same in each branch. For a specially orthotropic laminate the coupling coefficient $b = 0$ in eq. (8.46). Numerical evaluation of these stiffness coefficients are

$$B = A_{22} - A_{12}A_{21}/A_{11} = 15,822.7 \text{ N/mm}, \text{ and } B_s = A_{66} = 5,565.23 \text{ N/mm}.$$

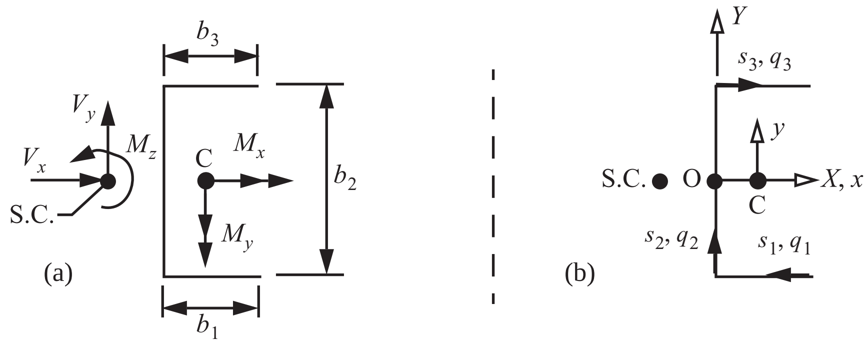


Fig. 8.15 (a) Channel section subject to transverse bending and torsion. (b) Cross-sectional coordinate systems and shear flows.

The section shown in figure. 8.15(a) is subject to an axial force N (not shown in figure. 8.15(a)), transverse shear forces V_x and V_y, bending moments M_x and M_y, and a torque M_z.

a) Determine the material law for extension and bending of the bar.

b) Determine the material law for shear and torsion of the bar.

Solution to part (a). The parametric equations of the contour in the X-Y coordinates are

$$X_1(s_1) = b_1 - s_1 \qquad Y_1 = -b_2/2 \qquad 0 \le s_1 \le b_1,$$

$$X_2(s_2) = 0 \qquad Y_2(s_2) = -b_2/2 + s_2 \qquad 0 \le s_2 \le b_2, \text{ and}$$

$$X_3(s_3) = s_3 \qquad Y_3 = b_2/2 \qquad 0 \le s_3 \le b_3.$$

To locate the modulus-weighted centroid on the X-axis, we first have to determine the modulus-weighted area S and the modulus-weighted first area moment about the Y-axis S_Y from eq. (8.52). These are given by

$$S = \int_0^{b_1} B ds_1 + \int_0^{b_2} B ds_2 + \int_0^{b_3} B ds_3 = B(b_1 + b_2 + b_3) = 1.01265 \times 10^6 \text{N}, \text{ and}$$

$$S_Y = \int_0^{b_1} BX_1(s_1) ds_1 + \int_0^{b_2} BX_2(s_2) ds_2 + \int_0^{b_3} BX_3(s_3) ds_3 = B(b_1^2 + b_3^2)/2 = 4.05061 \times 10^6 \text{N-mm}.$$

The location of the modulus-weighted centroid (8.53) is

$$X_c = S_Y/S = \frac{b_1^2 + b_3^2}{2(b_1 + b_2 + b_3)} = 4 \text{ mm} \qquad Y_c = 0.$$

The parametric equations of the contour with respect to the centroidal axes x and y are determined as follows:

$$x_1(s_1) = X_1(s_1) - X_c = 12 \text{ mm} - s_1 \qquad y_1 = -16 \text{ mm} \qquad 0 \le s_1 \le 16 \text{ mm} \tag{a}$$

$$x_2(s_2) = -4 \text{ mm} \qquad y_2(s_2) = -16 \text{ mm} + s_2 \qquad 0 \le s_2 \le 32 \text{ mm} \tag{b}$$

$$x_3(s_3) = -4 \text{ mm} + s_3 \qquad y_3 = 16 \text{ mm} \qquad 0 \le s_3 \le 16 \text{ mm} \tag{c}$$

Equations (**a**), (**b**), and (**c**) are substituted into the formulas for the modulus-weighted second moments D_{xx} and D_{yy} given by eq. (8.57) to get

$$D_{xx} = \int_0^{b_1} By_1^2 ds_1 + \int_0^{b_2} By_2^2 ds_2 + \int_0^{b_3} By_3^2 ds_3 = 1.72826 \times 10^8 \text{ N-mm}^2 \text{, and} \tag{d}$$

$$D_{yy} = \int_0^{b_1} Bx_1^2 ds_1 + \int_0^{b_2} Bx_2^2 ds_2 + \int_0^{b_3} Bx_3^2 ds_3 = 0.27004 \times 10^8 \text{ N-mm}^2 . \tag{e}$$

The modulus-weighted product moment $D_{xy} = 0$, because the *x*-axis is an axis of symmetry. The cross-sectional material law in extension and bending is

$$\begin{bmatrix} N \\ M_x \\ M_y \end{bmatrix} = \begin{bmatrix} S & 0 & 0 \\ 0 & D_{xx} & 0 \\ 0 & 0 & D_{yy} \end{bmatrix} \begin{bmatrix} \frac{dw}{dz} \\ \frac{d\phi_x}{dz} \\ \frac{d\phi_y}{dz} \end{bmatrix} = 10^6 \begin{bmatrix} 1.01265 \text{ N} & 0 & 0 \\ 0 & 172.826 \text{ N-mm}^2 & 0 \\ 0 & 0 & 27.004 \text{ N-mm}^2 \end{bmatrix} \begin{bmatrix} \frac{dw}{dz} \\ \frac{d\phi_x}{dz} \\ \frac{d\phi_y}{dz} \end{bmatrix} . \tag{f}$$

Solution to part (b). To establish the material law for shear and torsion we start with the shear flow given by eq. (8.113). For the channel section the product moment $D_{xy} = 0$, which means coefficients $n_x = n_y = 0$ and $k = 1$ in eqs. (8.59) and (8.69). At the contour origin where $s_1 = 0$ the shear flow must equal zero since the longitudinal edge is free of traction. Equation (8.113) for each branch reduces to

$$q_j(s_j) = -S_{xj}(s_j)\frac{V_y}{D_{xx}} - S_{yj}(s_j)\frac{V_x}{D_{yy}} \qquad 0 \le s_j \le b_j \qquad j = 1, 2, 3 . \tag{g}$$

The modulus-weighted, first area moments S_x and S_y are functions of the contour coordinate given by eq. (8.70), and have dimensional units of *N-mm*. The first area moment functions with respect to the *x*-axis are

$$S_{x1}(s_1) = \int_0^{s_1} By_1 ds_1 = -253,163 s_1 \qquad 0 \le s_1 \le 16 \text{ mm} , \tag{h}$$

$$S_{x2}(s_2) = S_{x1}(16) + \int_0^{s_2} By_2 ds_2 = -4.05061 \times 10^6 - 253,163 s_2 + 7,911.34 s_2^2 \qquad 0 \le s_2 \le 32 \text{ mm} \text{, and} \tag{i}$$

$$S_{x3}(s_3) = S_{x2}(32) + \int_0^{s_3} B_3 y_3 ds_3 = -4.05061\times10^6 + 253,163.s_3 \qquad 0 \le s_3 \le 16 \text{ mm}. \tag{j}$$

Note that at the free longitudinal edges $S_{x1}(0) = 0$ and $S_{x3}(16) = -4.05061\times10^6 + 4.05061\times10^6 = 0$. The first area moment functions with respect to the *y*-axis are

$$S_{y1}(s_1) = \int_0^{s_1} B x_1 ds_1 = 189,872.s_1 - 7,911.34 s_1^2 \qquad 0 \le s_1 \le 16 \text{ mm}, \tag{k}$$

$$S_{y2}(s_2) = S_{y1}(16) + \int_0^{s_2} B x_2 ds_2 = 1.01265\times10^6 - 63,290.7 s_2 \qquad 0 \le s_2 \le 32 \text{ mm, and} \tag{l}$$

$$S_{y3}(s_3) = S_{y2}(32) + \int_0^{s_3} B_3 x_3 ds_3 = -1.01265\times10^6 - 63,290.7 s_3 + 7,911.34 s_3^2 \qquad 0 \le s_3 \le 16 \text{ mm}. \tag{m}$$

Also note that $S_{y1}(0) = 0$ and $S_{y3}(16) = 0$. The resultant of the shear flow distribution is a horizontal force denoted by F_X, a vertical force F_Y, and a torque at the shear center M_z. The resultant forces are

$$F_X = \int_0^{b_3} q_3 ds_3 - \int_0^{b_1} q_1 ds_1 = \frac{V_x}{2} + \frac{3}{16}V_y - \left(-\frac{V_2}{2} + \frac{3}{16}V_y\right) = V_x \text{, and } F_Y = \int_0^{b_2} q_2 ds_2 = V_y. \tag{n}$$

Equation (**n**) yields the expected result that the horizontal force equals the shear force V_x, and the vertical force equals the shear force V_y. We cannot compute the torque until the location of the shear center is known. The coordinates of the shear center (x_{sc}, y_{sc}) are determined by letting $I_{xx} \to D_{xx}$, $I_{yy} \to D_{yy}$, $\bar{Q}_x(s) \to \bar{S}_x(s)$, and $\bar{Q}_y(s) \to \bar{S}_y(s)$ in eq. (3.106) on page 54. The transformation of eq. (3.106) to the composite laminate is

$$x_{sc} = -\left(\frac{k}{D_{xx}}\right)\int_c r_{nc}(s)\bar{S}_x(s)ds \qquad y_{sc} = \left(\frac{k}{D_{yy}}\right)\int_c r_{nc}(s)\bar{S}_y(s)ds. \tag{o}$$

The coordinate normal to the contour with respect to the centroid is denoted by $r_{nc}(s)$. It is depicted in figure. 3.3(b) on page 33, and the expression to compute it is given in eq. (3.11) on page 34. For the channel section the normal coordinates for each branch are

$$r_{nci} = x_i\frac{dy_i}{ds_i} - y_i\frac{dx_i}{ds_i} \qquad i = 1, 2, 3. \tag{p}$$

Evaluation of eq. (**p**) results in $r_{nc1} = -16$ mm, $r_{nc2} = -4$ mm, and $r_{nc3} = -16$ mm. For the channel section in this example the evaluation of eq. (**o**) is

$$x_{sc} = \left(\frac{-1}{D_{xx}}\right)\sum_{i=1}^{3}\int_0^{b_i} r_{nci}S_{xi}ds_i = -10 \text{ mm} \qquad y_{sc} = \left(\frac{1}{D_{yy}}\right)\sum_{i=1}^{3}\int_0^{b_i} r_{nci}S_{yi}ds_i = 0\,. \tag{q}$$

The torque from the shear flows with respect to the shear center is

$$M_z = \sum_{i=1}^{3}\int_0^{b_i} r_{ni}q_i ds_i\,, \tag{r}$$

where the coordinate normal to the contour with respect to the shear center is denoted by $r_n(s)$. This normal coordinate is depicted in figure. 3.3(b) on page 33, and the expression to compute it is given in eq. (3.10) on page 34. For this example the normal coordinate for each branch is given by

$$r_{ni} = r_{nci} - x_{sc}\frac{dy_i}{ds_i} \qquad i = 1, 2, 3\,. \tag{s}$$

Evaluation of eq. (**s**) yields $r_{n1} = -16$ mm, $r_{n2} = 6$ mm, and $r_{n3} = -16$ mm. Evaluating the torque given by eq. (**r**) gives

$$M_z = \underbrace{(8V_x - 3V_y)}_{\text{branch 1}} + \underbrace{(6V_y)}_{\text{branch 2}} + \underbrace{(-8V_x - 3V_y)}_{\text{branch 3}} = 0\,. \tag{t}$$

Equation (**t**) shows that the torque due to the shear flows equals zero at the shear center. Hence, the resultant of the shear flow distribution is a force with its line of action passing through the shear center having components V_x and V_y.

The material law for transverse shear relates the shear strains ψ_x and ψ_y to the shear forces V_x and V_y. For the bar made of an homogeneous, isotropic material this material law is discussed in article 5.5.3 on page 143. Referring to eq. (5.76) the form of the material law is the same for the composite material. That is,

$$\begin{bmatrix}\psi_x\\ \psi_y\end{bmatrix} = \begin{bmatrix}c_{xx} & c_{xy}\\ c_{yx} & c_{yy}\end{bmatrix}\begin{bmatrix}V_x\\ V_y\end{bmatrix}, \tag{u}$$

where the flexibility influence coefficients c_{xx}, c_{xy}, c_{yx}, and c_{yy} are determined from the complementary strain energy per unit axial length $\bar{U}^*$. For the open section $\bar{U}^*$ is obtained from eq. (8.97) on page 242, and it is

$$\bar{U}^* = \frac{1}{2}\int_c \frac{1}{A_{66}}q^2 ds\,. \tag{v}$$

The shear strains ψ_x and ψ_y are determined from derivatives of the complementary strain energy per unit axial length with respect to the shear forces. For the channel section in this example we get the following results:

$$\psi_x = \frac{\partial \bar{U}^*}{\partial V_x} = \sum_{i=1}^{3}\int_0^{b_i}\frac{1}{(A_{66})_i}q_i\frac{\partial q_i}{\partial V_x}ds_i = c_{xx}V_x + c_{xy}V_y\,,$$

$$\psi_y = \sum_{i=1}^{3} \int_0^{b_i} \frac{1}{(A_{66})_i} q_i \frac{\partial q_i}{\partial V_y} ds_i = c_{yx} V_x + c_{yy} V_y,$$

$$c_{xx} = 9.16404 \times 10^{-6}\ \text{N}^{-1} \qquad c_{xy} = c_{yx} = 0 \qquad c_{yy} = 6.73827 \times 10^{-6}\ \text{N}^{-1}. \tag{w}$$

In general, external loads cause the bar to resist a torque. For an open cross-sectional contour the shear flows cannot provide this resistance to torsion. A separate analysis for the linear elastic response to uniform torsion of a symmetrically laminated bar was developed in article 8.4. The result of this development is the material law (8.193) that equates the torque to torsional stiffness D_T times the twist per unit length. The torsional stiffness is given by eq. (8.194). To compute D_T we evaluate the following laminate properties:

- the transverse shear compliance (8.183) $a_{44} = 242.736 \times 10^{-6}\ \text{mm}^2/\text{N}$,
- the torsional compliance (8.184) $a_{66} = 200.446 \times 10^{-6}\ \text{mm}^2/\text{N}$,
- the bending stiffness (8.161) $D_{22} = 1,130.94\ \text{N-mm}$,
- the dimensionless shear-extension coefficient in bending (8.162) $\eta_{26} = -0.0459727$, and
- the solution parameter (8.190) $\mu = 2.82838\ \text{mm}^{-1}$.

The function $g(\mu b/2)$ appearing in the equation for D_T depends on the length of the branch. For the channel section the values of this function are

$$g\left(\frac{\mu b_1}{2}\right) = g\left(\frac{\mu b_3}{2}\right) = 0.955805 \qquad g\left(\frac{\mu b_2}{2}\right) = 0.977903. \tag{x}$$

The torsional stiffnesses for each branch are

$$D_{T1} = D_{T3} = \frac{16t^3}{3} \frac{g\left(\mu\frac{16}{2}\right)}{a_{66} + \left[t^3 \eta_{26}^2 g\left(\mu\frac{16}{2}\right)\right] / (3D_{22})} = 26,589\ \text{N-mm}^2, \text{ and} \tag{y}$$

$$D_{T2} = \frac{32t^3}{3} \frac{g\left(\mu\frac{32}{2}\right)}{a_{66} + \left[t^3 \eta_{26}^2 g\left(\mu\frac{32}{2}\right)\right] / (3D_{22})} = 54,403.4\ \text{N-mm}^2. \tag{z}$$

The torsional stiffness of the channel is equal to the sum of the torsional stiffnesses of each of its branches. That is,

$$D_T = D_{T1} + D_{T2} + D_{T3} = 107,581.\ \text{N-mm}^2. \tag{aa}$$

Finally, the material law for transverse shear and torsion is

$$\begin{bmatrix} V_x \\ V_y \\ M_z \end{bmatrix} = \begin{bmatrix} 109,122.\ \text{N} & 0 & 0 \\ 0 & 148,406.\ \text{N} & 0 \\ 0 & 0 & 107,581.\ \text{N-mm}^2 \end{bmatrix} \begin{bmatrix} \psi_x \\ \psi_y \\ \dfrac{d\phi_z}{dz} \end{bmatrix}. \blacksquare$$

8.5 References

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