A Combinatorial Proof of the Positivity of Lusztig's q-Analogue of Weight Multiplicity for Rank 2 Lie Algebras

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Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy
in
Mathematics

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December 2, 2003
Blacksburg, Virginia

Keywords: Root Systems, Lie Algebras, Lusztig q-analogue, Combinatorics

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Abstract

We prove the positivity of Lusztig's q-analogue of weight multiplicity in a purely combinatorial way for rank 2 Lie algebras. Each summand in the polynomial can be interpreted as a linear combination of positive roots. We prove that all negative coefficients are cancelled in the polynomial. Further, the analysis of the root systems allows us to state formulae for every coefficient in Lusztig's q-analogue for rank 2 Lie algebras.

Contents

List of Figures

1 Notation

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank n, $I = \{1, 2, ..., n\}$ and $\{\alpha_i \mid i \in I\} \subset V$ the set of simple roots which form a basis for V, an n-dimensional real vector space with symmetric nondegenerate scalar product $(,)$.

The set of simple coroots is $\{\alpha_i^\vee = \frac{2}{(\alpha_i,\alpha_i)}\alpha_i \mid i \in I\}$, and its dual basis $\{\omega_i \mid i \in I\}$ is the set of fundamental weights.

These bases provide several important subsets of V, which are the weight lattice $P = \sum_{i\in I} \mathbb{Z}\omega_i$, root lattice $Q = \sum_{i\in I} \mathbb{Z}\alpha_i$, $Q^+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}\alpha_i$, the set of dominant weights $P^+ = \sum_{i\in I} \mathbb{Z}_{\geq 0}\omega_i$, the dominant chamber $C = \sum_{i\in I} \mathbb{R}^+\omega_i$ and its closure $\overline{C}$.

For $i \in I$, define $s_i \in GL(V)$ by $s_i(\lambda) = \lambda - (\lambda, \alpha_i^\vee)\alpha_i$ for $\lambda \in V$. s_i is called a simple reflection. The Weyl group W is the subgroup of $GL(V)$ generated by the s_i. The set of positive weights R^+ is the set of images of the simple roots under action by W that lie in the positive span of the simple roots. Lastly, $\rho = \sum_i \omega_i$, and $l(w)$ is the smallest number of simple reflections whose composition is $w \in W$.

2 Introduction

For λ, $\mu \in P^+$, Lusztig's q-analogue of the multiplicity of the weight μ in the irreducible $\mathfrak{g}$-module of highest weight λ is defined by the following formula[3]:

$$K_{\lambda\mu}(q) = \sum_{w\in W} (-1)^{l(w)} P(\beta^w; q) \tag{2.1}$$

where

$$\beta^w = w^{-1}(\lambda + \rho) - (\mu + \rho) \tag{2.2}$$

and $P(\beta; q) \in \mathbb{Z}_{\geq 0}[q]$ is defined by

$$P(\beta; q) = \sum_{\substack{p:R^+\to\mathbb{Z}_{\geq 0} \\ \sum_{\gamma\in R^+} p(\gamma)\gamma=\beta}} q^{\sum_{\gamma\in R^+} p(\gamma)} \tag{2.3}$$

It is known that Lusztig's q-analogues have nonnegative integer coefficients due to their interpretation involving intersection cohomology[3]. But (2.1) has negative terms. The goal of this paper is to establish a purely combinatorial proof of the nonnegativity of Lusztig's q-analogue for $\mathfrak{g}$ of rank 2 by exhibiting an explicit cancellation of all negative terms appearing in (2.1). In addition, the analysis of the root systems will yield a direct formula for the coefficients in the Lusztig q-analogue.

To accomplish the goal, we must define a set whose elements correspond to the individual powers of q in equation (2.3). To this end we fix a specific ordering of the positive roots. Let $R^+ = \{\gamma_i \mid 1 \leq i \leq N\}$ and $\Gamma = \mathbb{Z}^N_{\geq 0}$. We call an element of Γ a **root combination**. The **weight** of $p \in \Gamma$ is $wt(p) = \sum_j p_j\gamma_j \in Q^+$. The **size** of $p \in \Gamma$ is $\mid p \mid= \sum_j p_j$.

Let Γ^β be the set of root combinations of weight β and $\Gamma^w = \Gamma^{\beta(w)}$ with respect to fixed weights $\lambda, \mu \in P^+$. Further, let $\Gamma^{\beta,s}$ be the set of root combinations of weight β and size s.

Remark 1. $\Gamma^\beta = \varnothing$ if $\beta \notin Q^+$.

We may now rewrite equation (2.3) and (2.1) as:

$$P(\beta^w; q) = \sum_{p \in \Gamma^w} q^{|p|} \tag{2.4}$$

$$K_{\lambda\mu}(q) = \sum_{w \in W} \Big[(-1)^w \sum_{p \in \Gamma^w} q^{|p|}\Big] \tag{2.5}$$

For the analysis of the individual root systems, we begin with a realization of the root system. A fixed numbering for the positive roots is chosen, and we indicate the set of $w \in W$ for which $\Gamma^w \neq \varnothing$.

At this point, cancellation mappings between two sets of root combinations are presented. These well-defined mappings are designed to be injective, size-preserving, and when the images of two different mappings are in the same set, then their images are disjoint. All of these properties are then proven for structural reason inherent in each root system. Once this has been accomplished, a positive formula for the Lusztig q-analogue will be stated, thereby satisfying the goal of this paper. This will be followed by a presentation of explicit formulae for the coefficients of the Lusztig q-analogue. These latter formulae however will not provide a means to demonstrate the positivity of (2.1).

3 General Properties

Before we consider any specific root system, we examine a few results that are true for all of them.

Let $\lambda, \mu \in P^+$. Let $\lambda_i, \mu_i \in \mathbb{Z}_{\geq 0}$ be such that:

$$\lambda = \sum_i \lambda_i \omega_i \tag{3.1}$$

$$\mu = \sum_i \mu_i \omega_i \tag{3.2}$$

We observe that $\Gamma^{\beta,s}$ is contained in a cone with generators that do not depend on β and s.

Proposition 2. *There is a finite set $\{\chi_i\}_{i=1}^k \subset \mathbb{Z}^N$ such that for any $\beta \in Q^+$, $s \in \mathbb{Z}_{\geq 0}$, if $\Gamma^{\beta,s} \neq \varnothing$, then there is a $\tau \in \Gamma^{\beta,s}$ such that*

$$\Gamma^{\beta,s} \subset \tau - \sum_{i=1}^k \mathbb{Z}_{\geq 0}\chi_i$$

*τ is called the **top root combination** for $\Gamma^{\beta,s}$.*

The details of this proof are left to the individual sections. From this proposition, we have that for any two root combinations p and q of the same weight and size, $p = q + \sum_i r_i \chi_i$ for some $r_i \in \mathbb{Z}$. Further, we say that $q \preccurlyeq_i p$ if $r_i \geq 0$, and $q \preccurlyeq p$ if $r_i \geq 0$ for all i. From this partial ordering, it is clear that given any $p \in \Gamma^{\beta,s}$, $p \preccurlyeq \tau$.

Lemma 3. *For $w = s_i$, $i = 1, 2$, $\beta^{s_i} - \beta^{id} = -(\lambda_i + 1)\alpha_i$.*

Proof.

$$\begin{aligned} \beta^{s_i} - \beta^{id} &= s_i(\lambda + \rho) - (\lambda + \rho) \\ &= -\langle \lambda + \rho \, , \, \alpha_i^\vee \rangle \alpha_i \\ &= -(\lambda_i + 1)\alpha_i \end{aligned}$$

□

For all rank 2 root systems but G_2, only the generators for the Weyl group (along with the identity) will produce root combinations. So, the philosophy will be to find mappings ψ^{s_i} from the sets Γ^{s_i} that take them to disjoint subsets of Γ^{id}, as pictured. Here, the sets $\Gamma^{id}_{s_i} \subset \Gamma^{id}$ are the images of Γ^{s_i} under the mappings ψ^{s_i}. Thus, root combinations from Γ^{s_i} and $\Gamma^{id}_{s_i}$ will cancel with each other in the Lusztig q-analogue. Thus, it is the set $\Gamma^{id}_0 \subset \Gamma^{id}$ that completely describes the Lusztig q-analogue.

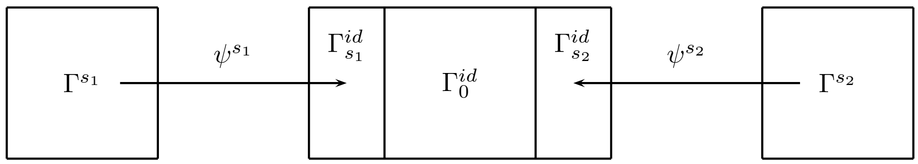


Figure 1: Basic cancellation diagram for types A_2, B_2, and C_2

And now we present a couple basic, long established results about root systems.

Lemma 4. *[2] Let $i \in I$, $w \in W$, and $\lambda \in C$. Then $l(s_i w) > l(w)$ if and only if $\langle w\lambda \, , \, \alpha_i^\vee \rangle > 0$ and $l(s_i w) < l(w)$ if and only if $\langle w\lambda \, , \, \alpha_i^\vee \rangle < 0$.*

Proposition 5. *If $\lambda - \mu \notin Q^+$ then $\Gamma^w = \varnothing$ for all $w \in W$, and so $K_{\lambda\mu}(q) = 0$.*

Proof. We induct on the length of $w \in W$. When $l(w) = 0$, $\beta^{w^{-1}} = \lambda - \mu \notin Q^+$. So Remark 1 implies that $\Gamma^{w^{-1}} = \varnothing$ for $l(w) = 0$.

Now suppose that for $v \in W$, $l(v) > 0$, that $\beta^{v^{-1}} \notin Q^+$ and thus $\Gamma^{v^{-1}} = \varnothing$. Suppose $w = s_i v$ and $l(w) > l(v)$. So by Lemma 4 it follows that $\langle v(\lambda + \rho) \, , \, \alpha_i^\vee \rangle > 0$. Also, we know that $w(\lambda + \rho) = v(\lambda + \rho) - \langle v(\lambda + \rho \, , \, \alpha_i^\vee \rangle \alpha_i$. It follows that:

$$\begin{aligned} \beta^{w^{-1}} &= w(\lambda + \rho) - (\mu + \rho) \\ &= v(\lambda + \rho) - (\mu + \rho) - \langle v(\lambda + \rho) \, , \, \alpha_i^\vee \rangle \alpha_i \\ &= \beta^{v^{-1}} - \langle v(\lambda + \rho) \, , \, \alpha_i^\vee \rangle \alpha_i \end{aligned}$$

Thus, as $\beta^{v^{-1}} \notin Q^+$ and $\langle v(\lambda + \rho) \, , \, \alpha_i^\vee \rangle > 0$, it follows that $\beta^{w^{-1}} \notin Q^+$, which, by Remark 1, implies that $\Gamma^{w^{-1}} = \varnothing$ for all $w \in W$. □

Remark 6. Let $\beta \in Q$. Then $\beta \in Q^+$ if and only if $\langle \beta \, , \, \omega_i \rangle \geq 0$ for all $i \in I$.

4 C_2

First, we consider the Lie algebra C_2. One realization of C_2 in $\mathbb{R}^2$ gives simple roots $\alpha_1 = (1, -1)$ and $\alpha_2 = (0, 2)$. Thus, the coroots are $\alpha_1^\vee = (1, -1)$ and $\alpha_2^\vee = (0, 1)$, and the fundamental weights are $\omega_1 = (1, 0)$ and $\omega_2 = (1, 1)$.

Figure 2: The simple roots, fundamental weights, and positive roots of C_2

The positive roots for C_2 are indexed as follows:

$$\begin{array}{rcl} \gamma_1 = & 2\omega_1 - \omega_2 & = \alpha_1 \\ \gamma_2 = & 2\omega_1 & = 2\alpha_1 + \alpha_2 \\ \gamma_3 = & \omega_2 & = \alpha_1 + \alpha_2 \\ \gamma_4 = & -2\omega_1 + 2\omega_2 & = \alpha_2 \end{array}$$

The generators for the Weyl group in C_2 act on P as follows:

$$\begin{array}{rcl} s_1(a\omega_1 + b\omega_2) & = & -a\omega_1 + (a+b)\omega_2 \\ s_2(a\omega_1 + b\omega_2) & = & (a+2b)\omega_1 - b\omega_2 \end{array}$$

We may restrict our inspection to three elements of the Weyl group.

Lemma 7. *For any $\tau \in P^+$ and any $w \in W$ not in $\{id, s_1, s_2\}$, $w^{-1}(\tau + \rho) - \rho \notin Q^+$. Thus, given $w \in W$, $\beta^w \notin Q^+$ if $w \notin \{id, s_1, s_2\}$*

Proof. Consider the image of the fundamental chamber (which contains $\tau + \rho$), and its image under every $w \in W$

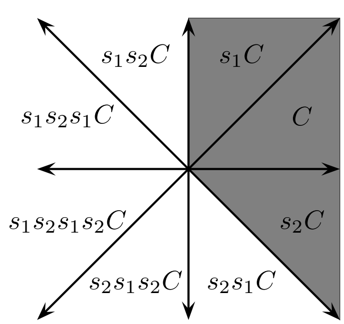


Figure 3: Weyl elements acting on the fundamental chamber of C_2

The shaded region in Figure 3 contains Q^+. $\tau+\rho \in C$ so $w^{-1}(\tau+\rho) \in w^{-1}C$. And since subtracting an element of P^+ will leave one in the nonnegative real span of Q^+ only if one was there to begin with, we see immediately that for $w \in W$, if $w \notin \{id, s_1, s_2\}$, then $w^{-1}(\tau+\rho)-\rho \notin Q^+$. The second statement follows immediately from this. $\square$

So now we may compute β for each of the three useful members of the Weyl group:

$$\begin{aligned}
\beta^{id} &= (\lambda_1-\mu_1)\omega_1+(\lambda_2-\mu_2)\omega_2 \\
\beta^{s_1} &= (-\lambda_1-\mu_1-2)\omega_1+(\lambda_1+\lambda_2-\mu_2+1)\omega_2 \\
\beta^{s_2} &= (\lambda_1+2\lambda_2-\mu_1+2)\omega_1+(-\lambda_2-\mu_2-2)\omega_2
\end{aligned}$$

Naturally, we are interested in values $\beta^w \in Q^+$. From Remark 6 we gain the following two conditions on β^w: $\beta_1^w+\beta_2^w \geq 0$ and $\beta_1^w+2\beta_2^w \geq 0$. When $w=s_1$, $\beta_1^{s_1}<0$, so the first condition implies the second. Similarly, when $w=s_2$, $\beta_2^{s_2}<0$, so the second condition implies the first. So we have the following conditions:

$$\beta^{s_1} \in Q^+ \quad \text{if and only if} \quad \lambda_2-\mu_2 \geq \mu_1+1 \tag{4.1}$$
$$\beta^{s_2} \in Q^+ \quad \text{if and only if} \quad \lambda_1-\mu_1 \geq 2\mu_2+2 \tag{4.2}$$

As $\sum_i p_i\gamma_i = \sum_i \beta_i\omega_i$ for any $p \in \Gamma^\beta$, we may write each coordinate of β in terms of the p_i:

$$\beta_1 = 2p_1+2p_2-2p_4 \tag{4.3}$$
$$\beta_2 = -p_1+p_3+2p_4 \tag{4.4}$$

Equations (4.3) and (4.4) show how to determine if a root combination p is of weight β. With this in mind, we consider the following relationship amongst the positive roots.

$$2\gamma_3 = \gamma_2+\gamma_4 \tag{4.5}$$

We define the vector χ based on this relation:

$$\chi = (0,-1,+2,-1) \tag{4.6}$$

Proposition 8. *The vector χ satisfies Proposition 2. $\tau \in \Gamma^{\beta,s}$ is the top root combination if and only if $\tau_2=0$ or $\tau_4=0$.*

Proof. Let $p \in \Gamma^{\beta,s}$. This means that p satisfies equations (4.3), (4.4) and $p_1+p_2+p_3+p_4=s$. These are three linearly independent equations with four unknowns. So the solution space is one-dimensional. To show that the vector χ is a basis for this solution space, it is sufficient to demonstrate that $p+\chi$ is also in the solution space. Testing $p+\chi$ in each relevant equation gives:

$$\begin{aligned}
2p_1+2(p_2-1)-2(p_4-1) = {} & 2p_1+2p_2-2p_4 & = \beta_1 \\
-p_1+(p_3+2)+2(p_4-1) = {} & -p_1+p_3+2p_4 & = \beta_2 \\
p_1+(p_2-1)+(p_3+2)+(p_4-1) = {} & p_1+p_2+p_3+p_4 & = s
\end{aligned}$$

Thus $p+\chi$ is of weight β with size s.

Let $\tau = p+\min(p_2,p_4)\chi$. Then $\tau = (p_1, p_2-\min(p_2,p_4), p_3+2\min(p_2,p_4), p_4-\min(p_2,p_4))$. So we know that τ is of weight β and size s. Further, $\tau \in \Gamma$, and so $\tau \in \Gamma^{\beta,s}$. We see that for τ, either $\tau_2=0$ or $\tau_4=0$.

This means that $\tau + \chi \notin \Gamma^{\beta,s}$. And as the solution space is one dimensional, all elements $p \in \Gamma^{\beta,s}$ must be of the form $\tau - r\chi$ where $r \in \mathbb{Z}_{\geq 0}$. Thus, τ is the top root combination in $\Gamma^{\beta,s}$.

Now, suppose that τ is the top root combination for $\Gamma^{\beta,s}$. If both τ_2 and τ_4 are positive, then $\tau+\chi \in \Gamma^{\beta,s}$, which implies that τ is not the top root combination. Thus, either τ_2 or τ_4 must be zero. $\square$

Since both s_1 and s_2 are elements of W with negative sign, every contribution they make must be cancelled by root combinations of weight β^{id}. We seek mappings ψ^{s_1} and ψ^{s_2} that map their domains into disjoint subsets of Γ^{id}, as pictured in Figure 1.

We define the functions $\psi^{s_1} : \Gamma^{s_1} \to \Gamma$ and $\psi^{s_2} : \Gamma^{s_2} \to \Gamma$ in the following manner:

$$\psi^{s_1}(p) = \begin{cases} (p_1, p_2 + \frac{\lambda_1+1}{2}, p_3, p_4 - \frac{\lambda_1+1}{2}) & \text{if } \lambda_1 \text{ is odd} \\ (p_1, p_2 + \frac{\lambda_1}{2}, p_3 + 1, p_4 - \frac{\lambda_1}{2} - 1) & \text{if } \lambda_1 \text{ is even.} \end{cases} \tag{4.7}$$

$$\psi^{s_2}(p) = (p_1 - \lambda_2 - 1, p_2, p_3 + \lambda_2 + 1, p_4) \tag{4.8}$$

It is clear that $| p |=| \psi^{s_i}(p) |$ for $i = 1, 2$. It is also clear that both functions are injective. It remains to be seen that the functions are well-defined, the images are contained in Γ^{id} and are disjoint, and that they are surjective on their respective images, given by:

$$\Gamma^{id}_{s_1} = \begin{cases} \{p \in \Gamma^{id} \mid p_2 \geq \frac{\lambda_1+1}{2}\} & \text{if } \lambda_1 \text{ is odd} \\ \{p \in \Gamma^{id} \mid p_2 \geq \frac{\lambda_1}{2} \text{ and } p_3 \geq 1\} & \text{if } \lambda_1 \text{ is even} \end{cases} \tag{4.9}$$

$$\Gamma^{id}_{s_2} = \{p \in \Gamma^{id} \mid p_3 \geq \lambda_2 + 1\} \tag{4.10}$$

Once these are proven, a characterization of all positive contributions to $K_{\lambda\mu}(q)$ for C_2 may be stated.

Proposition 9. *The maps ψ^{s_i} are well-defined for $i \in \{1, 2\}$.*

Proof. Let $p \in \Gamma^{s_i}$. We show that $\psi^{s_i}(p) \in \Gamma$.

i=1: For s_1, equation (4.3) becomes $-\lambda_1 - \mu_1 - 2 = 2p_1 + 2p_2 - 2p_4$. As $p \in \Gamma^{s_1}$, all its entries are nonnegative, so $2p_4 \geq \lambda_1 + \mu_1 + 2 > \lambda_1 + 1$. Thus ψ^{s_1} is well-defined on Γ^{s_1}.

i=2: For s_2, equation (4.4) becomes $-\lambda_2 - \mu_2 - 2 = -p_1 + p_3 + 2p_4$. Thus $p_1 \geq \lambda_2 + \mu_2 + 2 > \lambda_2 + 1$. And so ψ^{s_2} is well-defined on Γ^{s_2}. $\square$

Proposition 10. *If $p \in \Gamma^{s_i}$, $i = 1, 2$, then $\psi^{s_i}(p) \in \Gamma^{id}$.*

Proof. **i=1:** We first assume that $\lambda_1 + 1$ is even. So $wt(p) - wt(\psi^{s_1}(p)) = \frac{\lambda_1+1}{2}(\gamma_4 - \gamma_2) = -(\lambda_1 + 1)\alpha_1 = \beta^{s_1} - \beta^{id}$ which implies that $\psi^{s_i}(p) \in \Gamma^{id}$ by Lemma 3.

Now assume that λ_1+1 is odd. Here, we have $wt(p)-wt(\psi^{s_1}(p)) = \frac{\lambda_1}{2}(\gamma_4-\gamma_2)+(\gamma_4-\gamma_3) = -(\lambda_1+1)\alpha_1 = \beta^{s_1} - \beta^{id}$ so $\psi^{s_i}(p) \in \Gamma^{id}$ by Lemma 3.

i=2: In this case, $wt(p) - wt(\psi^{s_2}(p)) = -(\lambda_2 + 1)\alpha_2 = \beta^{s_2} - \beta^{id}$ which implies that $\psi^{s_2}(p) \in \Gamma^{id}$ by Lemma 3. $\square$

Proposition 11. *For $i = 1, 2$, $\psi^{s_i}(\Gamma^{s_i}) = \Gamma^{id}_{s_i}$.*

Proof. The mapping $\phi^{s_i} : \Gamma_{s_i}^{id} \to \Gamma$ is the inverse of ψ^{s_i}.

$$\begin{aligned}
\phi^{s_1}(p) &= \begin{cases} (p_1, p_2 - \frac{\lambda_1+1}{2}, p_3, p_4 + \frac{\lambda_1+1}{2}) & \text{if } \lambda_1 \text{ is odd} \\ (p_1, p_2 - \frac{\lambda_1}{2}, p_3 - 1, p_4 + \frac{\lambda_1}{2} + 1) & \text{if } \lambda_1 \text{ is even.} \end{cases} && (4.11)\\
\phi^{s_2}(p) &= (p_1 + \lambda_2 + 1, p_2, p_3 - \lambda_2 - 1, p_4) && (4.12)
\end{aligned}$$

Given the similar nature of the mappings it is clear that $\phi^{s_i} = (\psi^{s_i})^{-1}$. Thus, $\psi^{s_i} : \Gamma^{s_i} \to \Gamma_{s_i}^{id}$ is bijective. □

Proposition 12. $\Gamma_{s_1}^{id} \cap \Gamma_{s_2}^{id} = \varnothing$

Proof. Suppose $p \in \Gamma_{s_1}^{id} \cap \Gamma_{s_2}^{id}$. Then $p_2 \geq \frac{\lambda_1}{2}$ and $p_3 \geq \lambda_2 + 1$. From equations (4.3) and (4.4) we know that $p_1 + 2p_2 + p_3 = \beta_1^{id} + \beta_2^{id}$. And so it follows that:

$$\begin{aligned}
\beta_1^{id} + \beta_2^{id} &\geq 2p_2 + p_3 \\
&\geq \lambda_1 + \lambda_2 + 1 \\
&\geq (\lambda_1 - \mu_1) + (\lambda_2 - \mu_2) + 1 \\
&> \beta_1^{id} + \beta_2^{id}
\end{aligned}$$

Thus no such p exists and $\Gamma_{s_1}^{id} \cap \Gamma_{s_2}^{id} = \varnothing$. □

Theorem 13. *In C_2,*

$$K_{\lambda\mu}(q) = \sum_{p \in \Gamma_0^{id}} q^{|p|} \tag{4.13}$$

where

$$\Gamma_0^{id} = \begin{cases} \{p \in \Gamma^{id} \mid p_2 < \frac{\lambda_1+1}{2} \ \mathit{AND}\ p_3 < \lambda_2 + 1\} & \mathit{if}\ \lambda_1\ \mathit{is\ odd} \\ \{p \in \Gamma^{id} \mid (p_2 < \frac{\lambda_1}{2}\ \mathit{OR}\ p_3 = 0)\ \mathit{AND}\ p_3 < \lambda_2 + 1\} & \mathit{if}\ \lambda_1\ \mathit{is\ even} \end{cases} \tag{4.14}$$

Now we discern, from the structure of the root combinations, formulae that directly compute the coefficients of the polynomials in (2.1). The philosophy for this will be straightforward: taking into consideration the size-preserving relationships between the positive roots, we discern a method to count the number of root combinations in each Γ^β of a particular size. This method will rely only on being able to determine what the top root combination is per size.

Recalling Proposition 8 and χ, it is easy to see at this point just how many root combinations there are in $\Gamma^{\beta,s}$. We simply count, starting from τ, down until $p_3 = 0$ or 1. Let p be the root combination with $p_3 = 0$ or 1. Then clearly, $p = \tau - \lfloor \frac{\tau_3}{2} \rfloor \chi$. So, the total number of root combinations of a particular size is:

$$\Omega(\tau) = \Omega(\tau_3) = \lfloor \frac{\tau_3}{2} \rfloor + 1 \tag{4.15}$$

Now we need to craft clear formulae that state what τ_3 is for any size and any β. First, we observe that $\beta_1 + \beta_2 - \mid p \mid = p_2 - p_4$. This tells us that $p_2 \geq p_4$ when $\beta_1 + \beta_2 \geq \mid p \mid$, and $p_2 \leq p_4$ when $\beta_1 + \beta_2 \leq \mid p \mid$. So, the formula for the top root combinations is piecewise linear, depending on s and β.

So for given weight β and size s, let $x = s - (\beta_1 + \beta_2)$. Then:

$$\tau = \begin{cases} (\frac{\beta_1}{2} + x, -x, \frac{\beta_1}{2} + \beta_2 + x, 0) & x \leq 0 \\ (\frac{\beta_1}{2} + x, 0, \frac{\beta_1}{2} + \beta_2 - x, x) & 0 \leq x \end{cases} \tag{4.16}$$

Proposition 14. *The top root combination in* $\Gamma^{\beta,s}$ *is given by* (4.16).

Proof. First, both cases in equation (4.16) have size $x+\beta_1+\beta_2=s$. To verify that the root combinations presented in equation (4.16) are indeed of weight β is a simple matter of checking them against equations (4.3) and (4.4). First, for $x \le 0$, we have:

$$\begin{aligned} 2(\frac{\beta_1}{2}+x)+2(-x)-2(0) &= \beta_1 \\ -(\frac{\beta_1}{2}+x)+(\frac{\beta_1}{2}+\beta_2+x)+2(0) &= \beta_2 \end{aligned}$$

And for $0 \le x$ we have:

$$\begin{aligned} 2(\frac{\beta_1}{2}+x)+2(0)-2(x) &= \beta_1 \\ -(\frac{\beta_1}{2}+x)+(\frac{\beta_1}{2}+\beta_2-x)+2(x) &= \beta_2 \end{aligned}$$

And, as in both cases, either τ_2 or $\tau_4=0$, by Proposition 8 it follows that τ is the top root combination of size s and weight β. □

Theorem 15. *The coefficient for* q^s *in the Lusztig q-analogue in* C_2 *is:*

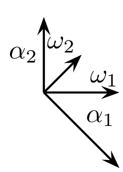
$$\sum_{w \in W} (-1)^{l(w)} \Omega(\tau^w) \tag{4.17}$$

where $\tau^w \in \Gamma^{\beta(w),s}$ *is the top root combination.*

Example 1. Let $\lambda_1=10$, $\lambda_2=8$, $\mu_1=2$, $\mu_2=1$. We want to know what the coefficient in equation (2.1) of q^{13} is. So, using equation (4.16), we find that $\tau=(2,2,9,0),(2,0,2,9)$ and $(11,2,0,0)$ in $\Gamma^{id}, \Gamma^{s_1}$ and Γ^{s_2} respectively. Thus, the coefficient is: $\Omega(9)-\Omega(2)-\Omega(0)=(\lfloor\frac{9}{2}\rfloor+1)-(\lfloor\frac{2}{2}\rfloor+1)-(\lfloor\frac{0}{2}\rfloor+1)=5-2-1=2$.

5 B_2

Next we consider B_2. The realization we will use gives simple roots $\alpha_1=(1,-1)$ and $\alpha_2=(0,1)$, and so the coroots are $\alpha_1^\vee=(1,-1)$ and $\alpha_2^\vee=(0,2)$, with fundamental weights $\omega_1=(1,0)$ and $\omega_2=(\frac{1}{2},\frac{1}{2})$.



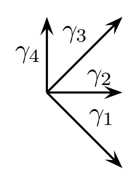


Figure 4: The simple roots, fundamental weights, and positive roots of B_2

We write out the positive roots in terms of the fundamental weights and simple roots as before:

$$\begin{aligned} \gamma_1 &= 2\omega_1-2\omega_2 &&= \alpha_1 \\ \gamma_2 &= \omega_1 &&= \alpha_1+\alpha_2 \\ \gamma_3 &= 2\omega_2 &&= \alpha_1+2\alpha_2 \\ \gamma_4 &= -\omega_1+2\omega_2 &&= \alpha_2 \end{aligned}$$

As the simple roots are on the same lines in B_2 as they are in C_2, the Weyl group will be precisely the same. However, as the fundamental weights are different in B_2, the action of the simple reflections upon them will differ slightly.

$$\begin{aligned} s_1(a\omega_1 + b\omega_2) &= -a\omega_1 + (2a+b)\omega_2 \\ s_2(a\omega_1 + b\omega_2) &= (a+b)\omega_1 - b\omega_2 \end{aligned}$$

As the Weyl group for B_2 is the same as in C_2, and P^+ and Q^+ are bounded by the same lines, Lemma 7 from the previous section also applies in this case. With λ and μ defined by equations (3.1) and (3.2), we consider the form for β^{id}, β^{s_1}, and β^{s_2}.

$$\begin{aligned} \beta^{id} &= (\lambda_1 - \mu_1)\omega_1 + (\lambda_2 - \mu_2)\omega_2 \\ \beta^{s_1} &= (-\lambda_1 - \mu_1 - 2)\omega_1 + (2\lambda_1 + \lambda_2 - \mu_2 + 2)\omega_2 \\ \beta^{s_2} &= (\lambda_1 + \lambda_2 - \mu_1 + 1)\omega_1 + (-\lambda_2 - \mu_2 - 2)\omega_2 \end{aligned}$$

From Remark 6, we gain the following conditions: $2\beta_1^w + \beta_2^w \geq 0$ and $\beta_1^w + \beta_2^w \geq 0$. As in C_2, when $w = s_1$, the first condition implies the second, and when $w = s_2$, the second condition implies the first. So we have the following conditions:

$$\beta^{s_1} \in Q^+ \quad \text{if and only if} \quad \lambda_2 - \mu_2 \geq 2\mu_1 + 2 \tag{5.1}$$
$$\beta^{s_2} \in Q^+ \quad \text{if and only if} \quad \lambda_1 - \mu_1 \geq \mu_2 + 1 \tag{5.2}$$

As $\sum_i p_i\gamma_i = \sum_i \beta_i^w \omega_i$ for any $p \in \Gamma^w$, it follows that:

$$\beta_1^w = 2p_1 + p_2 - p_4 \tag{5.3}$$
$$\beta_2^w = -2p_1 + 2p_3 + 2p_4 \tag{5.4}$$

In fact, as in C_2, there is only one size-preserving relation between the roots:

$$2\gamma_2 = \gamma_1 + \gamma_3 \tag{5.5}$$

And as in C_2, with only one relation it will not be needed to find the cancellation mappings, though it will become useful when we create a direct computation of the coefficients for the Lusztig q-analogue at the end of the section. So we write:

$$\chi = (-1, +2, -1, 0) \tag{5.6}$$

It should come as no surprise that we arrive at a characterization of a top root combination similar to the one in C_2.

Proposition 16. *The vector χ satisfies Proposition 2. τ is the top root combination in $\Gamma^{\beta,s}$ if and only if $\tau_1 = 0$ or $\tau_3 = 0$.*

Proof. The proof is analogous to that of Proposition 8. □

Once again, s_1 and s_2 are odd Weyl group elements, so every element of Γ^{s_i} must be cancelled by something in Γ^{id}, as in Figure 1.

By Lemma 3 we know that:

$$\begin{aligned}\beta^{s_1} - \beta^{id} &= -(\lambda_1 + 1)\gamma_1 \\ \beta^{s_2} - \beta^{id} &= -(\lambda_2 + 1)\gamma_4\end{aligned}$$

So now we define the functions $\psi^{s_i} : \Gamma^{s_i} \to \Gamma$.

$$\psi^{s_1}(p) = (p_1, p_2 + \lambda_1 + 1, p_3, p_4 - \lambda_1 - 1) \tag{5.7}$$

$$\psi^{s_2}(p) = \begin{cases}(p_1 - \frac{\lambda_2+1}{2}, p_2, p_3 + \frac{\lambda_2+1}{2}, p_4) \text{ if } \lambda_2 \text{ is odd} \\ (p_1 - \frac{\lambda_2}{2} - 1, p_2 + 1, p_3 + \frac{\lambda_2}{2}, p_4) \text{ if } \lambda_2 \text{ is even}\end{cases} \tag{5.8}$$

Once again it is clear from their definitions that $\mid p \mid = \mid \psi^{s_i}(p) \mid$ for $i \in I$. It is also clear that both functions are injective. It remains to be shown that they map $\Gamma^{s_i} \to \Gamma^{id}$ and are well-defined, their images are disjoint, and that they map onto the following subsets of Γ^{id}:

$$\Gamma^{id}_{s_1} = \{p \in \Gamma^{id} \mid p_2 \geq \lambda_1 + 1\} \tag{5.9}$$

$$\Gamma^{id}_{s_2} = \begin{cases}\{p \in \Gamma^{id} \mid p_3 \geq \frac{\lambda_2+1}{2}\} & \text{if } \lambda_2 \text{ is odd} \\ \{p \in \Gamma^{id} \mid p_2 \geq 1 \text{ and } p_3 \geq \frac{\lambda_2}{2}\} & \text{if } \lambda_2 \text{ is even}\end{cases} \tag{5.10}$$

Proposition 17. *The maps ψ^{s_i} are well-defined for $i \in \{1, 2\}$.*

Proof. Let $p \in \Gamma^{s_i}$. As before we simply need to show that $\psi^{s_i}(p) \in \Gamma$.

i=1: For s_1, equation (5.3) becomes $-\lambda_1 - \mu_1 - 2 = 2p_1 + p_2 - p_4$. Thus $p_4 \geq \lambda_1 + \mu_1 + 2 > \lambda_1 + 1$. This shows that ψ^{s_1} is well-defined on Γ^{s_1}.

i=2: For s_2, equation (5.4) becomes $-\lambda_2 - \mu_2 - 2 = -2p_1 + 2p_3 + 2p_4$, which implies that $2p_1 \geq \lambda_2 + \mu_2 + 2 > \lambda_2 + 1$. Thus, ψ^{s_2} is well-defined on Γ^{s_2}. □

Proposition 18. *If $p \in \Gamma^{s_i}$, $i = 1, 2$, then $\psi^{s_i}(p) \in \Gamma^{id}$.*

Proof. **i=1:** Here we have that $wt(p) - wt(\psi^{s_1}(p)) = (\lambda_1 + 1)(\gamma_4 - \gamma_2) = -(\lambda_1 + 1)\alpha_1 = \beta^{s_1} - \beta^{id}$ by Lemma 3. Thus $\psi^{s_1}(p) \in \Gamma^{id}$.

i=2: First we assume that λ_2 is odd. In this case, $wt(p) - wt(\psi^{s_2}(p)) = \frac{\lambda_2+1}{2}(\gamma_1 - \gamma_3) = -(\lambda_2 + 1)\alpha_2 = \beta^{s_2} - \beta^{id}$ by Lemma 3. Thus $\psi^{s_2}(p) \in \Gamma^{id}$.

Now we assume that λ_1 is even. So, $wt(p) - wt(\psi^{s_2}(p)) = \frac{\lambda_2}{2}(\gamma_1 - \gamma_3) + (\gamma_1 - \gamma_2) = -(\lambda_2 + 1)\alpha_2 = \beta^{s_2} - \beta^{id}$ by Lemma 3. Thus $\psi^{s_2}(p) \in \Gamma^{id}$. □

Proposition 19. *For $i = 1, 2$, $\psi^{s_i}(\Gamma^{s_i}) = \Gamma^{id}_{s_i}$.*

Proof. We construct the inverse to ψ^{s_i}, $\phi^{s_i} : \Gamma^{id}_{s_i} \to \Gamma$.

$$\phi^{s_1}(p) = (p_1, p_2 - \lambda_1 - 1, p_3, p_4 + \lambda_1 + 1) \tag{5.11}$$

$$\phi^{s_2}(p) = \begin{cases}(p_1 + \frac{\lambda_2+1}{2}, p_2, p_3 - \frac{\lambda_2+1}{2}, p_4) \text{ if } \lambda_2 \text{ is odd} \\ (p_1 + \frac{\lambda_2}{2} + 1, p_2 - 1, p_3 - \frac{\lambda_2}{2}, p_4) \text{ if } \lambda_2 \text{ is even}\end{cases} \tag{5.12}$$

Like with C_2, it is clear that $\phi^{s_i} = (\psi^{s_i})^{-1}$. Thus, $\psi^{s_i} : \Gamma^{s_i} \to \Gamma^{id}_{s_i}$ is bijective. $\square$

Proposition 20. $\Gamma^{id}_{s_1} \cap \Gamma^{id}_{s_2} = \varnothing$.

Proof. Suppose $p \in \Gamma^{id}_{s_1} \cap \Gamma^{id}_{s_2}$. Then $p_2 \geq \lambda_1 + 1$ and either $2p_3 \geq \lambda_2 + 1$ or ($p_2 \geq 1$ and $2p_3 \geq \lambda_2$). Thus in either case, $p_2 + 2p_3 \geq \lambda_1 + \lambda_2 + 1$. But we also know from equations (5.3) and (5.4) that:

$$\begin{aligned} \beta_1^{id} + \beta_2^{id} &= p_2 + 2p_3 + p_4 \\ \lambda_1 - \mu_1 + \lambda_2 - \mu_2 &\geq \lambda_1 + \lambda_2 + 1 \\ -\mu_1 - \mu_2 &\geq 1 \end{aligned}$$

This is a contradiction because μ_1 and μ_2 are nonnegative integers. Therefore no such p could exist, and $\Gamma^{id}_{s_1} \cap \Gamma^{id}_{s_2} = \varnothing$. $\square$

Theorem 21. *In B_2,*

$$K_{\lambda\mu}(q) = \sum_{p \in \Gamma_0^{id}} q^{|p|} \tag{5.13}$$

where

$$\Gamma_0^{id} = \begin{cases} \{p \in \Gamma^{id} \mid p_2 < \lambda_1 + 1 \ \mathit{AND}\ p_3 < \frac{\lambda_2+1}{2}\} & \text{if } \lambda_2 \text{ is odd} \\ \{p \in \Gamma^{id} \mid p_2 < \lambda_1 + 1 \ \mathit{AND}\ (p_2 = 0 \ \mathit{OR}\ p_3 < \frac{\lambda_2}{2})\} & \text{if } \lambda_2 \text{ is even} \end{cases} \tag{5.14}$$

Now we consider formulae for the direct computation of the Lusztig q-analogue in B_2.

The following three equations define all root combinations in a particular $\Gamma^{\beta,s}$.

$$\begin{aligned} p_1 + p_2 + p_3 + p_4 &= s \\ 2p_1 + p_2 + 0p_3 - p_4 &= \beta_1 \\ -2p_1 + 0p_2 + 2p_3 + 2p_4 &= \beta_2 \end{aligned}$$

And so at this point, the number of root combinations of size s in Γ^{β} is:

$$\Omega(\tau) = \Omega(\tau_2) = \lfloor \frac{\tau_2}{2} \rfloor + 1 \tag{5.15}$$

Since, $\beta_1 + \beta_2 - \mid p \mid = p_3 - p_1$ we know that $p_3 \geq p_1$ when $\beta_1 + \beta_2 \geq \mid p \mid$, and $p_3 \leq p_1$ when $\beta_1 + \beta_2 \leq \mid p \mid$. So, we make a piecewise linear definition again.

Given weight β and size s, set $x = s - (\beta_1 + \beta_2)$. Then

$$\tau = \begin{cases} (0, \beta_1 + \frac{\beta_2}{2} + x, -x, \frac{\beta_2}{2} + x) & x \leq 0 \\ (x, \beta_1 + \frac{\beta_2}{2} - x, 0, \frac{\beta_2}{2} + x) & 0 \leq x \end{cases} \tag{5.16}$$

The proof that this equation does satisfy the necessary equations is analogous to the proof of Proposition 14 for C_2.

Theorem 22. *The coefficient for q^s in the Lusztig q-analogue in B_2 is:*

$$\sum_{w \in W} (-1)^{l(w)} \Omega(\tau^w) \tag{5.17}$$

where $\tau^w \in \Gamma^{\beta(w),s}$ is the top root combination.

6 A_2

Let $\alpha_1 = (\sqrt{2}, 0)$ and $\alpha_2 = (-\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2})$. It is easier to see the structure of A_2 if they are written in polar notation $(r;\theta)$: $\alpha_1 = (\sqrt{2}; 0)$ and $\alpha_2 = (\sqrt{2}; \frac{2\pi}{3})$. Since both simple roots have length $\sqrt{2}$, the coroots are equal to the roots. Lastly, for this realization, the fundamental weights are $\omega_1 = (\frac{\sqrt{2}}{\sqrt{3}}; \frac{\pi}{6})$ and $\omega_2 = (\frac{\sqrt{2}}{\sqrt{3}}; \frac{\pi}{2})$.

Figure 5: The simple roots, fundamental weights, and positive roots of A_2

The positive roots are indexed as follows:

$$\begin{aligned} \gamma_1 &= \quad 2\omega_1 - \omega_2 &&= \alpha_1 \\ \gamma_2 &= \quad \omega_1 + \omega_2 &&= \alpha_1 + \alpha_2 \\ \gamma_3 &= \quad -\omega_1 + 2\omega_2 &&= \alpha_2 \end{aligned}$$

The Weyl group for A_2 is generated by reflections s_1 and s_2 across the lines $\theta = \frac{\pi}{2}$ and $\theta = \frac{\pi}{6}$ respectively. The action of the generators of the Weyl group on the fundamental weights is:

$$\begin{aligned} s_1(a\omega_1 + b\omega_2) &= -a\omega_1 + (a+b)\omega_2 \\ s_2(a\omega_1 + b\omega_2) &= (a+b)\omega_1 - b\omega_2 \end{aligned}$$

Proposition 23. *Given $w \in W$, $\beta^w \notin Q^+$ if $w \notin \{id, s_1, s_2\}$.*

Proof. Recall that $\overline{C}$ is the closure of the fundamental chamber. The boundaries for Q^+, the lines $\mathbb{R}^+\alpha_1$ and $\mathbb{R}^+\alpha_2$ are contained in the chambers $s_1\overline{C}$ and $s_2\overline{C}$ respectively as can be seen.

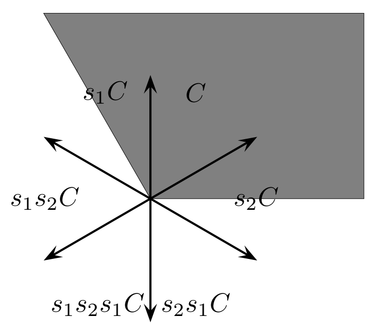


Figure 6: Weyl elements acting on the fundamental chamber of A_2

Here in Figure 6 the shaded region contains Q^+. Thus, $Q^+ \subset \cup_{w\in\{id,s_1,s_2\}} w\overline{C}$ and Q^+ is not contained in any smaller subset. The desired result follows from this. $\square$

Now we define β for each of these members of the Weyl group:

$$\begin{aligned}
\beta^{id} &= (\lambda_1 - \mu_1)\omega_1 + (\lambda_2 - \mu_2)\omega_2 \\
\beta^{s_1} &= (-\lambda_1 - \mu_1 - 2)\omega_1 + (\lambda_1 + \lambda_2 - \mu_2 + 1)\omega_2 \\
\beta^{s_2} &= (\lambda_1 + \lambda_2 - \mu_1 + 1)\omega_1 + (-\lambda_2 - \mu_2 - 2)\omega_2
\end{aligned}$$

From Remark 6 we gain the following conditions on β^w: $2\beta_1^w + \beta_2^w \geq 0$ and $\beta_1^w + 2\beta_2^w \geq 0$. When $w = s_1$, $\beta_1^{s_1}$ is always negative, so the first condition implies the second, and when $w = s_2$, $\beta_2^{s_2}$ is always negative, so the second condition implies the first. So we obtain the following conditions:

$$\beta^{s_1} \in Q^+ \quad \text{if and only if} \quad \lambda_2 - \mu_2 \geq \lambda_1 + 2\mu_1 + 3 \tag{6.1}$$
$$\beta^{s_2} \in Q^+ \quad \text{if and only if} \quad \lambda_1 - \mu_1 \geq \lambda_2 + 2\mu_2 + 3 \tag{6.2}$$

As $\sum_i p_i\gamma_i = \sum_i \beta_i^w \omega_i$ for any $p \in \Gamma^w$, it follows that:

$$\beta_1^w = 2p_1 + p_2 - p_3 \tag{6.3}$$
$$\beta_2^w = -p_1 + p_2 + 2p_3 \tag{6.4}$$

We can see immediately that there are no relations between equal numbers of the positive roots, as $\gamma_2 = \gamma_1 + \gamma_3$.

Proposition 24. *The set $\{\chi_i\}_{i=1}^k$ described in Proposition 2 is empty. So, there is only one root combination for any size in A_2.*

Proof. Let $p \in \Gamma^{\beta,s}$. Then p satisfies equations (6.3), (6.4) and $p_1 + p_2 + p_3 = s$. So we have three linearly independent equations in three unknowns. Thus, p is the only possible solution, and the basis for the solution space is empty. □

Proposition 24 implies that the mappings should be very straightforward, and will greatly simplify the construction of formula to directly compute the value of the Lusztig q-analogue.

By Lemma 3 we know that

$$\begin{aligned}
\beta^{s_1} - \beta^{id} &= -(\lambda_1 + 1)\gamma_1 \\
\beta^{s_2} - \beta^{id} &= -(\lambda_2 + 1)\gamma_3
\end{aligned}$$

Now we define the functions $\psi^{s_i} : \Gamma^{s_i} \to \Gamma$:

$$\psi^{s_1}(p) = (p_1, p_2 + \lambda_1 + 1, p_3 - \lambda_1 - 1) \tag{6.5}$$
$$\psi^{s_2}(p) = (p_1 - \lambda_2 - 1, p_2 + \lambda_2 + 1, p_3) \tag{6.6}$$

It is clear from their definitions that both functions are injective and $\mid p \mid = \mid \psi^{s_i}(p) \mid$ for $i \in \{1, 2\}$. We must show that they map $\Gamma^{s_i} \to \Gamma^{id}$, they are well-defined, their images are disjoint, and that they map onto the following subsets of Γ^{id}:

$$\Gamma_{s_i}^{id} = \{p \in \Gamma^{id} \mid p_2 \geq \lambda_i + 1\} \text{ for } i \in \{1, 2\} \tag{6.7}$$

Proposition 25. *The maps ψ^{s_i} are well-defined for $i \in \{1, 2\}$.*

Proof. Let $p \in \Gamma^{s_i}$. To prove that both maps are well defined we need simply show that $\psi^{s_i}(p) \in \Gamma$.

i=1: For s_1, equation (6.3) becomes $-\lambda_1 - \mu_1 - 2 = 2p_1 + p_2 - p_3$, thus $p_3 \geq \lambda_1 + \mu_1 + 2 > \lambda_1 + 1$. So $\psi^{s_1} \in \Gamma$.

i=2: For s_2, equation (6.4) becomes $-\lambda_2 - \mu_2 - 2 = -p_1 + p_2 + 3p_3$, thus $p_1 \geq \lambda_2 + \mu_2 + 2 > \lambda_2 + 1$. And so $\psi^{s_2} \in \Gamma$. ☐

Proposition 26. *If $p \in \Gamma^{s_i}$, $i = 1, 2$, then $\psi^{s_i}(p) \in \Gamma^{id}$.*

Proof. **i=1:** In this case we have $wt(p) - wt(\psi^{s_1}(p)) = -(\lambda_1 + 1)\alpha_1 = \beta^{s_1} - \beta^{id}$ by Lemma 3. Thus $\psi^{s_1}(p) \in \Gamma^{id}$.

i=2: Here we see that $wt(p) - wt(\psi^{s_2}(p)) = -(\lambda_2 + 1)\alpha_2 = \beta^{s_2} - \beta^{id}$ by Lemma 3. Thus $\psi^{s_2}(p) \in \Gamma^{id}$. ☐

Proposition 27. *For $i = 1, 2$, $\psi^{s_i}(\Gamma^{s_i}) = \Gamma^{id}_{s_i}$*

Proof. To prove this proposition we construct the inverse to ψ^{s_i}.

$$\begin{aligned} \phi^{s_1}(p) &= (p_1, p_2 - \lambda_1 - 1, p_3 + \lambda_1 + 1) && (6.8) \\ \phi^{s_2}(p) &= (p_1 + \lambda_2 + 1, p_2 - \lambda_2 - 1, p_3) && (6.9) \end{aligned}$$

The mapping ϕ^{s_i} merely undoes the action of ψ^{s_i}, and so clearly $\phi^{s_i} = (\psi^{s_i})^{-1}$. Thus, $\psi^{s_i} : \Gamma^{s_i} \to \Gamma^{id}_{s_i}$ is bijective. ☐

Proposition 28. $\Gamma^{id}_{s_1} \cap \Gamma^{id}_{s_2} = \varnothing$.

Proof. Suppose $p \in \Gamma^{id}_{s_1} \cap \Gamma^{id}_{s_2}$. Then $p_2 \geq \lambda_1 + 1$ and $p_2 \geq \lambda_2 + 1$. Thus, it follows that $2p_2 \geq \lambda_1 + \lambda_2 + 2$. So, from equations (6.3) and (6.4) we have that:

$$\begin{aligned} \beta^{id}_1 + \beta^{id}_2 &= p_1 + 2p_2 + p_3 \\ (\lambda_1 - \mu_1) + (\lambda_2 - \mu_2) &\geq \lambda_1 + \lambda_2 + 2 \\ -\mu_1 - \mu_2 &\geq 2 \end{aligned}$$

This is a contradiction because both μ_i are nonnegative. So no such p exists and $\Gamma^{id}_{s_1} \cap \Gamma^{id}_{s_1} = \varnothing$. ☐

Theorem 29. *In A_2*

$$K_{\lambda\mu}(q) = \sum_{p \in \Gamma^{id}_0} q^{|p|} \tag{6.10}$$

where

$$\Gamma^{id}_0 = \{p \in \Gamma^{id} \mid p_2 < \lambda_1 + 1 \textit{ AND } p_2 < \lambda_2 + 1\} \tag{6.11}$$

Now that the cancellation has been fully established, we turn to developing formulae to directly compute the Lusztig q-analogue. Since for every β and s, $\Gamma^{\beta,s}$ contains either one element or is empty, the formula is easy to state.

Theorem 30. *The coefficient for q^s in the Lusztig q-analogue in A_2 is:*

$$\sum_{w \in W} (-1)^{l(w)} \Omega(w, s) \tag{6.12}$$

where $\Omega(w, s) = 1$ if $\Gamma^{\beta(w),s}$ is nonempty, 0 otherwise.

Because this formula is so easy to state, we can find a stronger result. In fact, we can state formulae to completely describe the Lusztig q-analogue.

We first note that $\gamma_2 = \gamma_1 + \gamma_3$. This means, for any root combination in which $p_2 \geq 1$, we can replace a single p_2 with a p_1 and a p_3 and have the root combination of the same weight, but of one larger size. From here it follows that we can state precisely the root combinations of largest and smallest size, and consequently, know that every size in between also has a root combination. More precisely, the maximum size occurs when $p_2 = 0$, and the minimum size occurs when either $p_1 = 0$ or $p_3 = 0$. We merely compute what each of those values are in terms of the weight β.

First, the maximum size occurs when $p_2 = 0$. So from equations (6.3) and (6.4) we see that $\beta_1 + \beta_2 = p_1 + p_3$. So the maximum size for any weight will be $\beta_1 + \beta_2$.

The minimum size depends on whether $p_1 = 0$ or $p_3 = 0$. So first assume that $p_1 = 0$. Then, we see that $\beta_1 + 2\beta_2 = 3p_2 + 3p_3$. Thus, the minimum size when $p_1 = 0$ is $(\beta_1 + 2\beta_2)/3$. Similarly, when $p_3 = 0$, we have that $2\beta_1 + \beta_2 = 3p_1 + 3p_2$, and so the minimum size is $(2\beta_1 + \beta_2)/3$. Altogether then, then minimum size is $\min((\beta_1 + 2\beta_2)/3, (2\beta_1 + \beta_2)/3)$.

To use these formulae, for each weight, we compute the minimum and maximum sizes. For those, and every size in between, there will be a single root combination of that weight. The only thing to remember now is that either $\Gamma^{s_1} = \varnothing$ or $\Gamma^{s_2} = \varnothing$ according to Proposition 28. Thus, for whichever one is empty, the minimum size will be larger than the maximum size. When this occurs, no contribution to the Lusztig q-analogue is to be computed.

Example 2. Let $\lambda_1 = 7$, $\lambda_2 = 15$, $\mu_1 = 1$ and $\mu_2 = 0$. We want to state $K_{\lambda\mu}(q)$ under these conditions. First, we see that the minimum size for Γ^{id} is 9 while the maximum is 21. And in Γ^{s_1}, the minimum size is 12 and the maximum 13. But in Γ^{s_2}, the minimum size is 9 while the maximum is 5, so we ignore it. Putting this all together, we list the exponents 9 through 21, but omitting 12 through 13. So:

$$K_{\lambda\mu}(q) = q^9 + q^{10} + q^{11} + q^{14} + q^{15} + q^{16} + q^{17} + q^{18} + q^{19} + q^{20} + q^{21}$$

7 G_2

For the exceptional root system G_2 we use polar notation $(r; \theta)$ to state the representation. The simple roots are $\alpha_1 = (\sqrt{6}; 0)$, and $\alpha_2 = (\sqrt{2}; \frac{5\pi}{6})$ and the fundamental weights are $\omega_1 = (\sqrt{6}; \frac{\pi}{3})$, and $\omega_2 = (\sqrt{2}; \frac{\pi}{2})$.

Under the representation of G_2 that is given, the positive roots are indexed as follows:

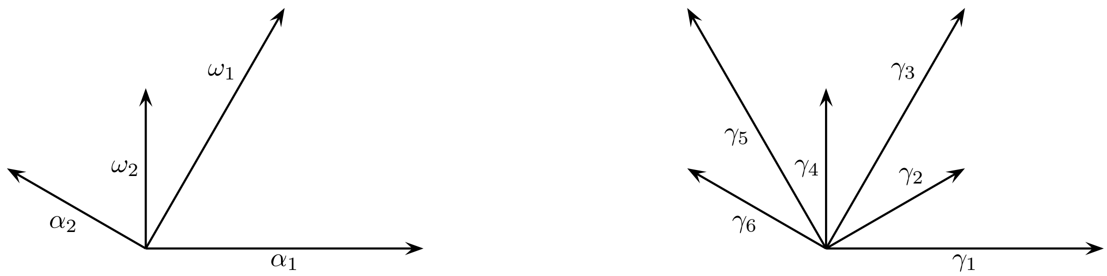


Figure 7: The simple roots, fundamental weights, and positive roots of G_2

$$\begin{aligned}
\gamma_1 &= 2\omega_1 - 3\omega_2 &&= \alpha_1 \\
\gamma_2 &= \omega_1 - \omega_2 &&= \alpha_1 + \alpha_2 \\
\gamma_3 &= \omega_1 &&= 2\alpha_1 + 3\alpha_2 \\
\gamma_4 &= \omega_2 &&= \alpha_1 + 2\alpha_2 \\
\gamma_5 &= -\omega_1 + 3\omega_2 &&= \alpha_1 + 3\alpha_2 \\
\gamma_6 &= -\omega_1 + 2\omega_2 &&= \alpha_2
\end{aligned}$$

The generators for the Weyl group associated with G_2 act as follows:

$$\begin{aligned}
s_1(a\omega_1 + b\omega_2) &= -a\omega_1 + (3a+b)\omega_2 \\
s_2(a\omega_1 + b\omega_2) &= (a+b)\omega_1 - b\omega_2
\end{aligned}$$

Now, as before, we restrict the analysis to the relevant members of the Weyl group.

Proposition 31. *Given $w \in W$, $\beta^w \notin Q^+$ if $w \notin \{id, s_1, s_2, s_1s_2, s_2s_1\}$*

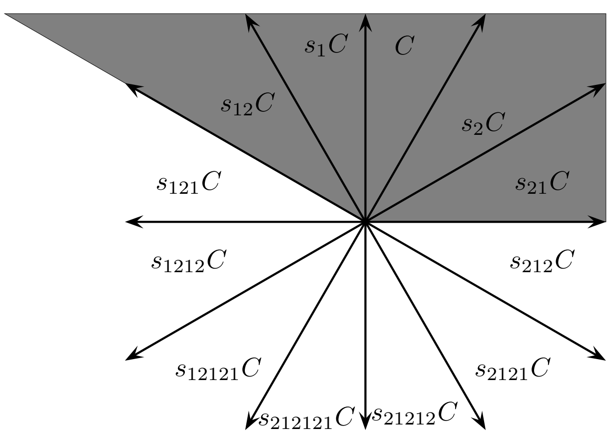


Figure 8: Weyl elements acting on the fundamental chamber of G_2

Proof. Here, s_{12} is the same as s_1s_2, and so forth. As the shaded region in Figure 8 contains Q^+, we observe that $\{id, s_1, s_2, s_1s_2, s_2s_1\}$ is the smallest subset $W' \subset W$ such that $Q^+ \subset \bigcup_{w \in W'} w\overline{C}$.

Since $\mu + \rho$ is in P^+, and Q^+ is closed under adding positive roots, subtracting it out cannot move $w(\lambda + \rho)$ into Q^+ unless it is already within Q^+. This proves the theorem. $\square$

For G_2, the β^w are as follows:

$$\begin{aligned}
\beta^{id} &= (\lambda_1 - \mu_1)\omega_1 + (\lambda_2 - \mu_2)\omega_2 && (7.1)\\
\beta^{s_1} &= (-\lambda_1 - \mu_1 - 2)\omega_1 + (3\lambda_1 + \lambda_2 - \mu_2 + 3)\omega_2 && (7.2)\\
\beta^{s_2} &= (\lambda_1 + \lambda_2 - \mu_1 + 1)\omega_1 + (-\lambda_2 - \mu_2 - 2) && (7.3)\\
\beta^{s_2 s_1} &= (-\lambda_1 - \lambda_2 - \mu_1 - 3)\omega_1 + (3\lambda_1 + 2\lambda_2 - \mu_2 + 4)\omega_2 && (7.4)\\
\beta^{s_1 s_2} &= (2\lambda_1 + \lambda_2 - \mu_1 + 2)\omega_1 + (-3\lambda_1 - \lambda_2 - \mu_2 - 5)\omega_2 && (7.5)
\end{aligned}$$

And since $\sum_i p_i\gamma_i = \sum_i \beta_i\omega_i$ for any $p \in \Gamma^\beta$, we have:

$$\begin{aligned}
\beta_1 &= 2p_1 + p_2 + p_3 - p_5 - p_6 && (7.6)\\
\beta_2 &= -3p_1 - p_2 + p_4 + 3p_5 + 2p_6 && (7.7)
\end{aligned}$$

As in the previous sections, $p \in \Gamma^\beta$ will be an element of Γ that satisfies equations (7.6) and (7.7).

An immediate result of equations (7.6) and (7.7) is that $p_3 - p_6 = 2\beta_1 + \beta_2 - \mid p \mid$. In other words, given $p, q \in \Gamma^\beta$ of the same size, then $p_3 - p_6 = q_3 - q_6$.

We elaborate on the structure of the root combinations that the positive roots create. Consider the following relations:

$$\begin{aligned}
2\gamma_2 &= \gamma_1 + \gamma_4 && (7.8)\\
2\gamma_4 &= \gamma_2 + \gamma_5 && (7.9)\\
2\gamma_4 &= \gamma_3 + \gamma_6 && (7.10)
\end{aligned}$$

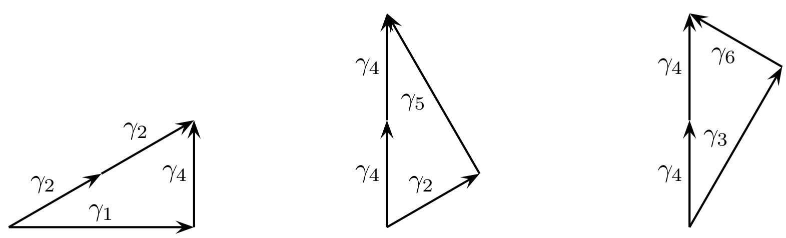


Figure 9: The three weight and size-preserving relations in G_2

Consider the following vectors:

$$\begin{aligned}
\chi_1 &= (-1, +2, 0, -1, 0, 0)\\
\chi_2 &= (0, -1, 0, +2, -1, 0)\\
\chi_3 &= (0, 0, -1, +2, 0, -1)
\end{aligned}$$

Proposition 32. *The χ_i satisfy Proposition 2. $\tau \in \Gamma^{\beta,s}$ is the top root combination if and only if one member of each of the following pairs is zero: τ_1 or τ_4, τ_2 or τ_5, τ_3 or τ_6.*

Proof. Let $p \in \Gamma^{\beta,s}$. Then p satisfies equations (7.6), (7.7) and $p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = s$. So we have three linearly independent equations in six unknowns. So the resulting solution space is three-dimensional.

Clearly, the χ_i are linearly independent vectors. To show that they span the solution space, it is sufficient to show that $p + \chi_i$ is of weight β and size s for $i \in \{1, 2, 3\}$. For $p + \chi_1$ we have:

$$\begin{array}{rcl} 2(p_1-1)+(p_2+2)+p_3-p_5-p_6 = & 2p_1+p_2+p_3-p_5-p_6 & =\beta_1 \\ -3(p_1-1)-(p_2+2)+(p_4-1)+3p_5+2p_6 = & -3p_1-p_2+p_4+3p_5+2p_6 & =\beta_2 \\ (p_1-1)+(p_2+2)+p_3+(p_4-1)+p_5+p_6 = & p_1+p_2+p_3+p_4+p_5+p_6 & =s \end{array}$$

The verification of $p + \chi_2$ and $p + \chi_3$ are similar.

Let $q = p + \min(p_1, p_5)(\chi_1 + \chi_2) + \min(p_3, p_6)\chi_3$. Then either q_3 or q_6 is zero, and either q_1 or q_5 is zero. If $q_1 = 0$, let $\tau = q + \min(q_2, q_5)\chi_2$. And thus $\tau_1 = 0$, either τ_2 or τ_5 is zero, and either τ_3 or τ_6 is zero. If instead $q_5 = 0$, let $\tau = q + \min(q_1, q_4)\chi_1$. Here, τ_1 or τ_4 is zero, $\tau_5 = 0$, and either τ_3 or τ_6 is zero. In either case, $\tau \in \Gamma^{\beta,s}$ and $\tau + \sum_i r_i\chi_i \notin \Gamma^{\beta,s}$ when $r_i > 0$ for some $i \in \{1, 2, 3\}$. Since no root combinations in the solution space to the linearly independent equations for weight and size can be reached from τ by any combination of the vectors χ_i in which any one of them is added to τ, it must be the case that all root combinations in the solution space can only be reached from τ by subtracting combinations of the vectors χ_i. So given any $p \in \Gamma^{\beta,s}$, $p = \tau - \sum_i r_i\chi_i$ for nonnegative integers r_i. Therefore τ is the top root combination for $\Gamma^{\beta,s}$.

Suppose that τ is the top root combination in $\Gamma^{\beta,s}$. If both τ_1 and τ_4 are positive, then $\tau + \chi_1 \in \Gamma^\beta$, which contradicts the assumption that τ was the top root combination. Similarly, when τ_2 and τ_5 are both positive and when τ_3 and τ_6 are both positive, then $\tau + \chi_2$ and $\tau + \chi_3 \in \Gamma^\beta$ respectively, also a contradiction. $\square$

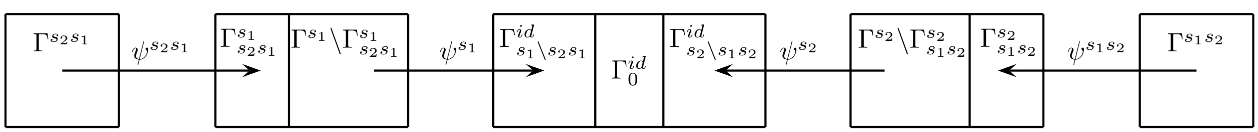


Figure 10: Cancellation diagram for G_2

The structure of the cancellation in G_2 can be visualized as in Figure 10.

First, we tackle the two outside mappings. We can see how that is to be done by comparing the difference in weight between elements of the domain and range.

$$\beta^{s_2s_1} - \beta^{s_1} = \quad (-\lambda_2 - 1)\omega_1 + (\lambda_2 + 1)\omega_2 \quad = -(\lambda_2 + 1)\gamma_2 \tag{7.11}$$

$$\beta^{s_1s_2} - \beta^{s_2} = \quad (\lambda_1 + 1)\omega_1 + (-3\lambda_1 - 3)\omega_2 \quad = -(\lambda_1 + 1)\gamma_5 \tag{7.12}$$

First, consider $\Gamma^{s_2s_1}$. Moving from $\Gamma^{s_2s_1}$ to Γ^{s_1} is the same as moving $\lambda_2 + 1$ steps in the γ_2 direction. A single step in the γ_2 direction either changes a γ_6 into a γ_4 or a γ_4 into a γ_3. It turns out, the former is all that is needed.

Moving from $\Gamma^{s_1s_2}$ to Γ^{s_2} constitutes moving $\lambda_1 + 1$ steps in the γ_5 direction. The only way to move one step in the γ_5 direction is by replacing a γ_1 with a γ_3.

We define functions $\psi^{s_2s_1} : \Gamma^{s_2s_1} \to \Gamma$ and $\psi^{s_1s_2} : \Gamma^{s_1s_2} \to \Gamma$ as:

$$\psi^{s_2s_1}(p) \quad = \quad (p_1, p_2, p_3, p_4 + \lambda_2 + 1, p_5, p_6 - \lambda_2 - 1) \tag{7.13}$$

$$\psi^{s_1s_2}(p) \quad = \quad (p_1 - \lambda_1 - 1, p_2, p_3 + \lambda_1 + 1, p_4, p_5, p_6) \tag{7.14}$$

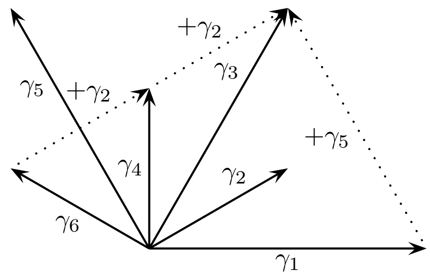


Figure 11: The possible changes of s_2s_1 and s_1s_2

Clearly, both functions are one-to-one and size-preserving. It remains to be shown that both functions map to the proper location and that they are well-defined.

Proposition 33. *The function $\psi^{s_2s_1}$ is well-defined and $\psi^{s_2s_1} : \Gamma^{s_2s_1} \to \Gamma^{s_1}$.*

Proof. From equations (7.6) and (7.7) we see that $3\beta_1^w + \beta_2^w = 3p_1 + 2p_2 + 3p_3 + p_4 - p_6$. Thus, as $3\beta_1^{s_2s_1} + \beta_2^{s_2s_1} = -\lambda_2 - 3\mu_1 - \mu_2 - 5$, it follows that $p_6 \geq \lambda_2 + 3\mu_1 + \mu_2 + 5 > \lambda_2 + 1$. Therefore $\psi^{s_2s_1}$ is well-defined on $\Gamma^{s_2s_1}$.

Let $p \in \Gamma^{s_2s_1}$. From equation (7.13) and (7.11) we have:

$$\begin{aligned} wt(\psi^{s_2s_1}(p)) - wt(p) &= (\lambda_2+1)\gamma_4 - (\lambda_2+1)\gamma_6 \\ &= (\lambda_2+1)\gamma_2 \\ &= \beta^{s_1} - \beta^{s_2s_1} \end{aligned}$$

Thus $\psi^{s_2s_1} : \Gamma^{s_2s_1} \to \Gamma^{s_1}$. □

Proposition 34. *The function $\psi^{s_1s_2}$ is well-defined and $\psi^{s_1s_2} : \Gamma^{s_1s_2} \to \Gamma^{s_2}$.*

Proof. The proof proceeds in the same manner as Proposition 33. Note that $\beta_1^w + \beta_2^w = -p_1 + p_3 + p_4 + 2p_5 + p_6$ from equations (7.6) and (7.7). And since $\beta_1^{s_1s_2} + \beta_2^{s_1s_2} = -\lambda_1 - \mu_1 - \mu_2 - 3$ it follows that $p_1 \geq \lambda_1 + \mu_1 + 3 > \lambda_1 + 1$. Therefore $\psi^{s_1s_2}$ is well-defined on $\Gamma^{s_1s_2}$.

Let $p \in \Gamma^{s_1s_2}$. From equation (7.14) and (7.12) we have:

$$\begin{aligned} wt(\psi^{s_1s_2}(p)) - wt(p) &= -(\lambda_1+1)\gamma_1 + (\lambda_1+1)\gamma_3 \\ &= (\lambda_1+1)\gamma_5 \\ &= \beta^{s_2} - \beta^{s_1s_2} \end{aligned}$$

Thus $\psi^{s_1s_2} : \Gamma^{s_1s_2} \to \Gamma^{s_2}$. □

As both of the functions are straight-forward, characterizations of their images are also easy to state:

$$\psi^{s_2s_1}(\Gamma^{s_2s_1}) = \Gamma^{s_1}_{s_2s_1} = \{p \in \Gamma^{s_1} \mid p_4 \geq \lambda_2 + 1\} \tag{7.15}$$
$$\psi^{s_1s_2}(\Gamma^{s_1s_2}) = \Gamma^{s_2}_{s_1s_2} = \{p \in \Gamma^{s_2} \mid p_3 \geq \lambda_1 + 1\} \tag{7.16}$$

We now turn our attention to root combinations generated by s_1 and s_2. It is not hard to find an example for each that is not in the above two sets, thus a large number of the root combinations in each set need to be cancelled in the identity. This proves to be a far more complex undertaking, so we tackle them one at a time. First, with s_1, from Lemma 3 we know that $\beta^{s_1} - \beta^{id} = -(\lambda_1+1)\alpha_1 = -(\lambda_1+1)\gamma_1$. A step in the γ_1 direction requires either a change from γ_6 to γ_2 or a change from γ_5 to γ_3. But what combination of the two is needed? That depends on the structure of the root combinations in s_1.

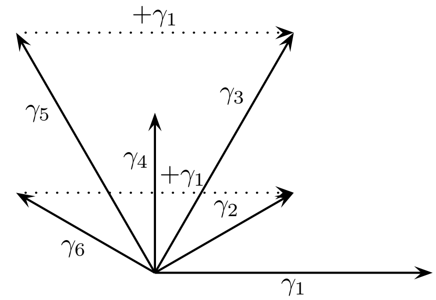


Figure 12: The possible changes of s_1

Let $z = 2\beta_1^{id} + \beta_2^{id} - \mid p \mid = 2\lambda_1 + \lambda_2 - 2\mu_1 - \mu_2 - \mid p \mid$. This tells us that $p_6 = p_3 + \lambda_1 + 1 - z$. Replace this in equation (7.6) and we have:

$$2p_1 + p_2 - p_5 = -\mu_1 - 1 - z \tag{7.17}$$

The value of p_5 will increase whenever either $\mid p \mid$ is made smaller or p_1 is made larger. Further, as when $\mid p \mid$ grows, z shrinks, it follows that p_6 will increase, as $p_6 = p_3 + \lambda_1 + 1 - z$. But what does this tell us?

First, as the function must move a sum from p_5 and p_6 totaling $\lambda_1 + 1$, this combination will change depending on the size. Secondly, as p_5 grows when p_1 does, it makes sense to also alter the combination moved for larger values of p_1. In other words, for each size, all root combinations that differ only by sums of χ_2 and χ_3 will move in the same manner. We call root combinations of the same size where the root combinations all differ only by χ_2 and χ_3 an α_1-level, as the number of $\gamma_1 (= \alpha_1)$ remains constant. And since moving from Γ^{s_1} to Γ^{id} does not alter the number of γ_1, two root combinations from different α_1-levels will always land in different places.

With this background in mind, we state the desired function. Given $p \in \Gamma^{s_1}$, let:

$$n = 2\lambda_1 + \lambda_2 + 2p_1 - 2\mu_1 - \mu_2 - \mid p \mid \tag{7.18}$$

$$\psi^{s_1}(p) = \begin{cases} (p_1, p_2, p_3 + \lambda_1 + 1, p_4, p_5 - \lambda_1 - 1, p_6) & \text{if } \lambda_1 \leq n \\ (p_1, p_2 + \lambda_1 - n, p_3 + n + 1, p_4, p_5 - n - 1, p_6 - \lambda_1 + n) & \text{if } 0 \leq n < \lambda_1 \\ (p_1, p_2 + \lambda_1 + 1, p_3, p_4, p_5, p_6 - \lambda_1 - 1) & \text{if } n < 0 \end{cases} \tag{7.19}$$

As has always been the case, this function is clearly size-preserving, and by previous discussion, is also one-to-one. It only remains to be proven that it is well-defined and that it maps $\Gamma^{s_1} \to \Gamma^{id}$. The first of these is already half-proven, and the latter is not difficult.

Proposition 35. *The function ψ^{s_1} is well-defined and $\psi^{s_1} : \Gamma^{s_1} \to \Gamma^{id}$.*

Proof. Let $p \in \Gamma^{s_1}$. To show that ψ^{s_1} is well-defined means that all root combinations in its image will have nonnegative entries. The only place a negative entry could occur is in p_5 and p_6. Recall that $n = z + 2p_1$. Replacing that in equation (7.17) gives $p_2 = p_5 - (n + \mu_1 + 1)$ which implies that $p_5 \geq n + \mu_1 + 1 \geq n + 1$. Thus, when $\lambda_1 \leq n$, it follows that $p_5 \geq \lambda_1 + 1$. Thus p_5 will never be sent to a negative number by ψ^{s_1}.

Also, recall that $p_6 = p_3 + \lambda_1 + 1 - n + 2p_1$. This implies that $p_6 \geq \lambda_1 - n$ as $p_3 + 1 + 2p_1 \geq 0$. When $n < 0$, this implies that $p_6 \geq \lambda_1 + 1$, so it follows that p_6 is never sent to a negative number by ψ^{s_1}.

Thus ψ^{s_1} is well-defined on Γ^{s_1}.

No matter which case p falls under, $wt(\psi^{s_1}(p)) = wt(p) + (\lambda_1 + 1)\gamma_1$. From equation (7.2) and Lemma 3, we have:

$$\begin{aligned} wt(\psi^{s_1}(p)) - wt(p) &= (\lambda_1 + 1)\gamma_1 \\ &= (\lambda_1 + 1)\alpha_1 \\ &= \beta^{id} - \beta^{s_1} \end{aligned}$$

Thus $\psi^{s_1} : \Gamma^{s_1} \to \Gamma^{id}$. $\square$

We may then state a characterization for the image of $\psi^{s_1}(\Gamma^{s_1})$. Let n be as in equation (7.18).

$\Gamma^{id}_{s_1} = \{p \in \Gamma^{id} \mid p_2 \geq \min(\lambda_1 + 1, \lambda_1 - n), p_3 \geq \min(\lambda_1 + 1, n + 1)\}$

However, this image eliminates too many root combinations from Γ^{id}. We must also consider the image of $\Gamma^{s_1}_{s_2 s_1}$ under ψ^{s_1}. Since $\Gamma^{s_1}_{s_2 s_1} \subset \Gamma^{s_1}$, it follows that $\psi^{s_1}(\Gamma^{s_1}_{s_2 s_1}) \subset \psi^{s_1}(\Gamma^{s_1})$. And as ψ^{s_1} does not alter p_4, the image of $\Gamma^{s_1}_{s_2 s_1}$ merely adds the requirement that $p_4 \geq \lambda_2 + 1$. So, the subset of Γ^{id} that is eliminated by root combinations from $\Gamma^{s_1} \backslash \Gamma^{s_1}_{s_2 s_1}$ can be described as follows:

Let n be as in equation (7.18). Then let

$$\Gamma^{id}_{s_1 \backslash s_2 s_1} = \{p \in \Gamma^{id} \mid p_2 \geq \min(\lambda_1 + 1, \lambda_1 - n), p_3 \geq \min(\lambda_1 + 1, n + 1), p_4 \leq \lambda_2\} \tag{7.20}$$

We now turn our attention to the last relevant member of the Weyl group, s_2. From Lemma 3 we know that $\beta^{s_2} - \beta^{id} = -(\lambda_2 + 1)\alpha_2 = -(\lambda_2 + 1)\gamma_6$. Moving by γ_6 follows the longest string in the root system: $\gamma_1 \to \gamma_2 \to \gamma_4 \to \gamma_5$. Each arrow represents one move in the γ_6 direction, so one p_1 changed to a p_5 would count as three moves in the γ_6 direction. This presents a large variety of choices in order to define the necessary mapping. It is thus imperative to examine the structure of the root combinations even further to piece together the proper approach.

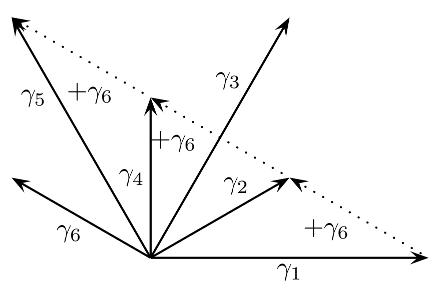


Figure 13: The possible changes of s_2

As always, we want to consider what happens for all root combinations of the same size. As with ψ^{s_1} leaving p_1 fixed, the desired mapping in this case will leave p_3 and p_6 fixed. Since $\gamma_6 = \alpha_2$, we will constrain our focus upon α_2-levels. An α_2-level is the set of root combinations of the same size that differ only by sums of χ_1 and χ_2. Thus, for all root combinations in a particular α_2-level, p_3 and p_6 will be constant.

We have only four positive roots left to consider. Since each change by χ_1 changes p_1 by one but not p_5 and each change by χ_2 changes p_5 by one, but not p_1, each α_2-level can be interpreted as a two dimensional grid. Let leftward horizontal motion be synonymous with positive changes by χ_2, and upwards vertical motion be synonymous with positive changes by χ_1. Then each entry in the grid would automatically record entries for p_1 and p_5, and only one root combination per α_2-level could fit within each grid location. Whether a grid location is actually filled or not depends on p_2 and p_4.

To make this clear, we consider an example:

p_1 \ p_5	3	2	1	0
1				10,0
2				8,1
3			7,0	6,2
4			5,1	4,3
5		4,0	3,2	2,4
6		2,1	1,3	0,5
7	1,0	0,2		

Key $\boxed{p_2, p_4}$

Figure 14: Grid representation for the α_2-level of $\lambda = 3\omega_1 + 8\omega_2$, $\mu = \omega_1 + \omega_2$, $\mid p \mid = 12$, and $p_6 = 1$

Example 3. Let $\lambda = 3\omega_1 + 8\omega_2$, and $\mu = \omega_1 + \omega_2$. Let the α_2-level be in Γ^{s_2} and have size 12, with $p_6 = 1$. Then the top root combination $\tau = (1, 10, 0, 0, 0, 1)$, and the α_2-level may be represented in grid coordinates as in Figure 14.

We say a root combination is an edge if motion by either χ_1 or χ_2 in the positive or negative directions is not a root combination. In terms of the grid location, this means that one of the four adjoining locations is empty. Clearly, a root combination is an edge if either $p_1 = 0$, $p_5 = 0$, $p_2 = 0$ or 1, or $p_4 = 0$ or 1. As the edge conditions on p_1 and p_5 are also the limits of the grid, the most useful information will be found by studying the behavior of edges associated with p_2 and p_4.

First, consider a root combination p^1 in which $p^1_4 = 0$. Then neither $p^1 + \chi_1$ nor $p^1 - \chi_2$ will be root combinations, as both have negative values. Now consider the following root combinations derived from p^1:

$$\begin{aligned} p^2 = &\quad p^1 - \chi_1 &= (p^1_1 + 1, p^1_2 - 2, p^1_3, p^1_4 + 1, p^1_5, p^1_6) \\ p^3 = &\quad p^1 - 2\chi_1 &= (p^1_1 + 2, p^1_2 - 4, p^1_3, p^1_4 + 2, p^1_5, p^1_6) \\ p^4 = &\quad p^1 - 2\chi_1 - \chi_2 &= (p^1_1 + 2, p^1_2 - 3, p^1_3, p^1_4, p^1_5 + 1, p^1_6) \end{aligned}$$

As we can see, a general pattern emerges that defines the shape of the p_4 edge. Since, for two root combinations p and p' of the same size, $p - 2\chi_1 - \chi_2 = p' \to p_4 = p'_4$, this pattern repeats itself for as long as the changes by the χ_i produce no negative terms. In the case of the root combinations p^1, p^2, p^3, and p^4, together they shall be termed a **knight**, as the grid locations they fill is the same shape as a knight's move in chess. Another characterization of this is that a root combination p is in a knight if and only if $p + \chi_1 - \chi_2 \notin \Gamma^w$.

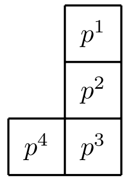


Figure 15: The basic knight

Similarly, we can define a reverse knight along the edge associated with p_2. In this case, if $p' = p - \chi_1 - 2\chi_2$, then $p' = (p_1 + 1, p_2, p_3, p_4 - 3, p_5 + 2, p_6)$.

Now for any p, $p + \chi_1 - \chi_2 = (p_1 - 1, p_2 + 3, p_3, p_4 - 3, p_5 + 1, p_6)$. Thus, if $p' = p + k(\chi_1 - \chi_2)$ for any integer k, it follows that $p_2 - 3p_5 = p'_2 - 3p'_5$. This result will be the guiding principle behind the mapping for $w = s_2$. But before the mapping itself is stated, we consider which of the possible motions should be preferred. As, a change from p_1 to p_5 represents a shift of three, it makes sense to break $\lambda_2 + 1$ into two pieces. Let x, y be integers with $0 \leq y < 3$ and $\lambda_2 + 1 = 3x + y$. Then, the idea will be to move as much from p_1 to p_5 as possible in sections of the α_2-level where p_1 is large, and to move as much as possible from p_2 to p_4 in sections where p_2 is large. In between, some combination will be necessary.

Let $1 \leq k < x$. Then all p that satisfy $3(k+1) + y > p_2 - 3p_5 \geq 3k + y$ are said to be in the k-boundary. Further, all p for which $p_2 - 3p_5 \geq 3x + y$ is true are in the top region, while all p for which $p_2 - 3p_5 < 3 + y$ are in the bottom region. The number k represents a boundary condition. Because of this, it splits up the α_2-level into x+1 regions, and in each region, ψ^{s_2} acts differently.

With that idea in mind, we now present the mapping for $w = s_2$ that accomplishes this. It is presented as eleven separate cases.

1. p in top region then $\psi^{s_2}(p) = (p_1, p_2 - 3x - y, p_3, p_4 + 3x + y, p_5, p_6)$
2. p in k-boundary then $\psi^{s_2}(p) = (p_1 - (x - k), p_2 - 3k - y, p_3, p_4 + 3k + y, p_5 + (x - k), p_6)$
3. p in bottom region, $p_2 \geq y$ then $\psi^{s_2}(p) = (p_1 - x, p_2 - y, p_3, p_4 + y, p_5 + x, p_6)$
4. p in bottom region, $y > 0$, $p_2 = 0$ and $p_4 = 0 \mod 3$ then $\psi^{s_2}(p) = (p_1 - x - (\frac{p_4}{3} + 1), p_4 + 1 - (y - 1), p_3, y - 1, p_5 + x + \frac{p_4}{3}, p_6)$
5. p in bottom region, $y > 0$, $p_2 = 0$, $p_4 = 1 \mod 3$ and ($p_4 \neq 1$ OR $y = 1$) then $\psi^{s_2}(p) = (p_1 - x - \frac{p_4 - 1}{3}, p_4 - 1 - (y - 1), p_3, y - 1, p_5 + x + (\frac{p_4 - 1}{3} + 1), p_6)$
6. p in bottom region, $y > 0$, $p_2 = 0$, $p_4 = 1 \mod 3$ and ($p_4 = 1$ AND $y = 2$) then $\psi^{s_2}(p) = (p_1 - x - 1, 1, p_3, 0, p_5 + x + 1, p_6)$
7. p in bottom region, $y > 0$, $p_2 = 0$ and $p_4 = 2 \mod 3$ then $\psi^{s_2}(p) = (p_1 - x - \frac{p_4 + 1}{3}, p_4 - (y - 1), p_3, y - 1, p_5 + x + \frac{p_4 + 1}{3}, p_6)$
8. p in bottom region, $y = 2$, $p_2 = 1$ and $p_4 = 0 \mod 3$ then $\psi^{s_2}(p) = (p_1 - x - \frac{p_4}{3}, p_4, p_3, 0, p_5 + x + \frac{p_4}{3} + 1.p_6)$
9. p in bottom region, $y = 2$, $p_2 = 1$ and $p_4 = 1 \mod 3$ then $\psi^{s_2}(p) = (p_1 - x - \frac{p_4 + 2}{3}, p_4 + 1, p_3, 0, p_5 + x + \frac{p_4 + 2}{3}.p_6)$
10. p in bottom region, $y = 2$, $p_2 = 1$, $p_4 = 2 \mod 3$ and $p_5 \neq 0$ then $\psi^{s_2}(p) = (p_1 - x - (\frac{p_4 + 1}{3} + 1), p_4 + 2, p_3, 0, p_5 + x + \frac{p_4 + 1}{3}, p_6)$

11. p in bottom region, $y = 2$, $p_2 = 1$, $p_4 = 2 \mod 3$ and $p_5 = 0$ then $\psi^{s_2}(p) = (p_1 - x - \frac{p_4+1}{3}, p_4, p_3, 1, p_5 + x + \frac{p_4+1}{3}, p_6)$

Later in this section, certain cases are referred to as exceptional or extraordinary. The exceptional cases refer to situations that will occur in roughly one third of all α_2-levels, but will only do so for a small minority of the root combinations within those α_2-levels. Thus, the exceptional cases are cases 4, 5, 7, 8, 9, and 10.

The extraordinary cases are the cases that only occur for a single root combination within an α_2-level, and these are cases 6 and 11.

Example 4. The following diagram is the grid representation of the α_2-level and it image under the mapping described for s_2, with each region colored to distinguish them. The top region is grey, the k-boundary's are red and blue, and the bottom region is white.

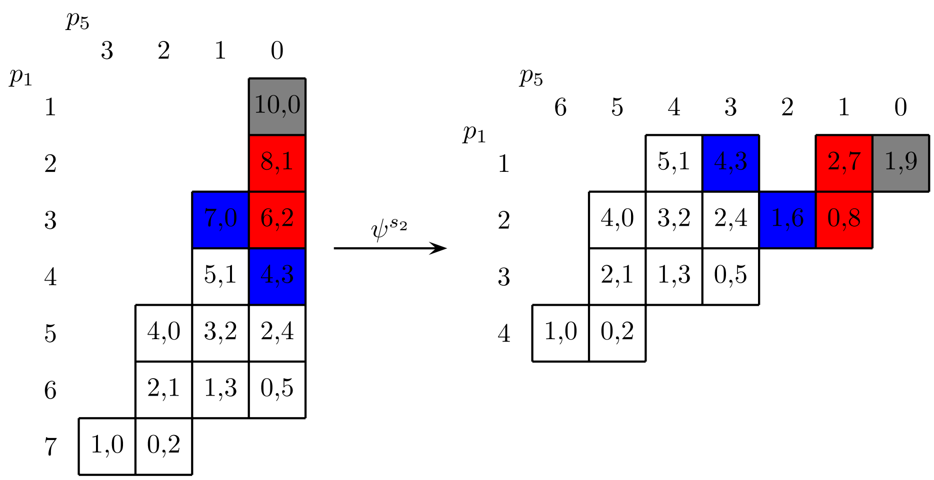


Figure 16: Grid representation for the α_2-level of $\lambda = 3\omega_1 + 8\omega_2$, $\mu_1 = \omega_1 + \omega_2$, $\mid p \mid = 12$, and $p_6 = 1$ and its image under the mapping ψ^{s_2}

It is not at all obvious that ψ^{s_2} is one-to-one or well-defined. About the only thing obvious is that ψ^{s_2} is size-preserving. In fact, to prove that ψ^{s_2} is well-defined will require an even greater examination of the structure of root combinations in Γ^{s_2}.

But first we concern ourselves with proving that ψ^{s_2} is one-to-one. We note that for two root combinations in the top region, the mapping is clearly one-to-one. Two root combinations in the same k-boundary are also sent to different locations by the mapping. Further, for two root combinations in the bottom region with $p_2 \geq y$ it is obviously one-to-one. For all else, we need to do a bit of work.

First, we prove a lemma.

Lemma 36. *If $p' = \psi^{s_2}(p)$ then $p'_2 - 3p'_5 = p_2 - 3p_5 - (\lambda_2 + 1) + F(p)$ where $F(p) : \Gamma^{s_2} \to \mathbb{Z}$ in the following manner:*

$$
\begin{array}{lllll}
F(p) = 2 & \text{If } p_2 - 3p_5 < 3 + y,\ y > 0,\ p_2 = 0, \text{ and} & p_4 = 0 & \text{mod } 3 & \\
 & \text{OR If } p_2 - 3p_5 < 3 + y,\ y = 2,\ p_2 = 1, \text{ and} & p_4 = 2 & \text{mod } 3 & \text{AND } p_5 \neq 0 \\
F(p) = -2 & \text{If } p_2 - 3p_5 < 3 + y,\ y > 0,\ p_2 = 0, \text{ and} & p_4 = 1 & \text{mod } 3 & \text{AND } (p_4 \neq 1 \text{ OR } y = 1) \\
 & \text{OR If } p_2 - 3p_5 < 3 + y,\ y = 2,\ p_2 = 1, \text{ and} & p_4 = 0 & \text{mod } 3 & \\
F(p) = 0 & \text{Otherwise} & & &
\end{array}
$$

Proof. This is merely a direct computation from each possible resulting map in the definition of ψ^{s_2}.

If p is in the top region, then $p_2' - 3p_5' = p_2 - 3x - y - 3p_5 \to F(p) = 0$.

If p is in a k-boundary, then $p_2' - 3p_5' = p_2 - 3k - y - 3(p_5 + (x - k)) = p_2 - y - 3p_5 - 3x \to F(p) = 0$.

If p is in the bottom region and $p_2 \geq y$, then $p_2' - 3p_5' = p_2 - y - 3(p_5 + x) \to F(p) = 0$.

If p is in the bottom region with $y > 0$, $p_2 = 0$ and $p_4 = 0 \mod 3$, then:

$$\begin{aligned} p_2' - 3p_5' &= p_4 + 1 - (y-1) - 3(p_5 + x + \frac{p_4}{3}) \\ &= 2 - 3p_5 - 3x - y \text{ and as } p_2 = 0 \to F(p) = 2 \end{aligned}$$

If instead $p_4 = 1 \mod 3$, ($p_4 \neq 1$ OR $y = 1$) then:

$$\begin{aligned} p_2' - 3p_5' &= p_4 - 1 - (y-1) - 3(p_5 + x + 1 + \frac{p_4 - 1}{3}) \\ &= -2 - 3p_5 - 3x - y \text{ and as } p_2 = 0 \to F(p) = -2 \end{aligned}$$

But if $p_4 = 1$ AND $y = 2$ then:

$$\begin{aligned} p_2' - 3p_5' &= 1 - 3(p_5 + x + 1) \\ &= -2 - 3p_5 - 3x \text{ and since } y = 2 \\ &= -3p_5 - 3x - y \text{ and as } p_2 = 0 \to F(p) = 0 \end{aligned}$$

And if $p_4 = 2 \mod 3$ then:

$$\begin{aligned} p_2' - 3p_5' &= p_4 - (y-1) - 3(p_5 + x + \frac{p_4 + 1}{3}) \\ &= -3p_5 - 3x - y \text{ and as } p_2 = 0 \to F(p) = 0 \end{aligned}$$

If p is in the bottom region with $y = 2$, $p_2 = 1$, and $p_4 = 0 \mod 3$ then:

$$\begin{aligned} p_2' - 3p_5' &= p_4 - 3(p_5 + x + 1 + \frac{p_4}{3}) \\ &= -3 - 3p_5 - 3x \text{ and since } y = 2 \\ &= -1 - 3p_5 - 3x - y \text{ and as } p_2 = 1 \to F(p) = -2 \end{aligned}$$

If instead $p_4 = 1 \mod 3$ then:

$$\begin{aligned} p_2' - 3p_5' &= p_4 + 1 - 3(p_5 + x + \frac{p_4 + 2}{3}) \\ &= -1 - 3p_5 - 3x \text{ and since } y = 2 \\ &= 1 - 3p_5 - 3x - y \text{ and as } p_2 = 1 \to F(p) = 0 \end{aligned}$$

But if $p_4 = 2 \mod 3$ with $p_5 \neq 0$ then:

$$\begin{aligned} p_2' - 3p_5' &= p_4 + 2 - 3(p_5 + x + \frac{p_4 + 1}{3}) \\ &= 1 - 3p_5 - 3x \text{ and since } y = 2 \\ &= 3 - 3p_5 - 3x - y \text{ and as } p_2 = 1 \to F(p) = 2 \end{aligned}$$

And if $p_4 = 2 \mod 3$ with $p_5 = 0$ then:

$$\begin{array}{rcl} p_2' - 3p_5' & = & p_4 - 3(p_5 + x + \frac{p_4 + 1}{3}) \\ & = & -1 - 3p_3 - 3x \text{ and since } y = 2 \\ & = & 1 - 3p_5 - 3x - y \text{ and as } p_2 = 1 \rightarrow F(p) = 0 \end{array}$$

Thus, all cases exhausted and $F(p)$ proven for each. $\square$

Lemma 37. *Let p be a root combination with $p_2 = 0$ and $p_4 = k \mod 3$ for $0 \leq k \leq 2$. Let q be a root combination in the same α_2-level as p with $q_2 = 0$. Then $q_4 = k \mod 3$.*

Proof. All root combination in the same α_2-level that have the same value for p_2 differ by integer multiples of $\chi_1 + 2\chi_2$. If $q = p + l(\chi_1 + 2\chi_2)$ for some integer l, then $q_4 = p_4 + 3l \rightarrow q_4 = p_4 \mod 3 = k \mod 3$. $\square$

Proposition 38. *The mapping ψ^{s_2} is one-to-one.*

Proof. Let p, $q \in \Gamma^{s_2}$ and in the same α_2-level. Lemma 36 implies that when $F(p) = F(q) = 0$ and p and q are not in the same region, then $\psi^{s_2}(p) \neq \psi^{s_2}(q)$. And Lemma 37 implies that in any given α_2-level, of the three main cases under the condition $y > 0$ and $p_2 = 0$, only one case can occur, with the case dependent on ($p_4 = 1$ AND $y = 2$) occurring for only one root combination if it occurs at all. It also implies that under the condition $y = 2$ and $p_2 = 1$ that only one of the cases occur (except when $p_4 = 2 \mod 3$, in which case the last two cases occur). Thus, all that remains to be proven is that the exceptional cases do not share an image with anyone in the bottom region or the k-boundaries, and the last two cases do not share images either.

First, for the image of p that satisfies an exceptional case condition to be equal to the image of a root combination from a k-boundary, $p_2' - 3p_5' \geq p_2 - 3p_5 + 4$. This is the case because $3 + y - p_2 \geq 4$. But the largest value of $F(p)$ is 2, so this can never happen.

Secondly, when $y = 1$, nonexceptional p in the bottom region will have $p_4' \geq 1$ while all exceptional p in the bottom region will have $p_4' = 0$. Thus, the images are all distinct. When $y = 2$, nonexceptional p in the bottom region will have $p_4' \geq 2$ while all exceptional p in the bottom region will have $p_4' \leq 1$. Again, the images are all distinct.

When $y = 2$, $p_2 = 0$ and $p_4 = 1$, $p' = (p_1 - x - 1, 1, p_3, 0, p_5 + x + 1, p_6)$. Suppose there exists q in the same α_2-level such that $\psi^{s_2}(q) = p'$. Then $q_2 \neq 0$ because the only case in which $q_4' = 0$ is the same case that p falls under, which would immediately show that $p = q$. So suppose that $q_2 = 1$. Then by Lemma 37 we know that $q = p - \chi_2 + l(\chi_1 + 2\chi_2)$ for some integer l. Thus $q_4 = 2 \mod 3$. First suppose that $q_5 \neq 0$. Then $q_4 + 2 = 1 \rightarrow q_4 = -1$. Then suppose that $q_5 = 0$. Then $q_4 = 1$, but $q_4 = 2 \mod 3$. Thus, no such q exists.

This leaves only the case where $y = 2$, $p_2 = 1$, $p_4 = 2 \mod 3$, and $p_5 = 0$ as a possible source of intersection. In this case $p' = (p_1 - x - \frac{p_4+1}{3}, p_4, p_3, 1, p_5 + x + \frac{p_4+1}{3}, p_6)$. This immediately shows that any source of intersection could not come from q with $q_2 = 1$, as in all other cases, $q_4' = 0$. Thus, if there exists q in the same α_2-level with $\psi^{s_2}(q) = \psi^{s_2}(p)$, it follows that $q_2 = 0$ as the only possibility. And as $q_2 = 0$, $q = p + \chi_2 + l(\chi_1 + 2\chi_2)$ for some integer l. Thus, $q_4 = 1 \mod 3$. This implies that $q_4 - 2 = p_4 \rightarrow q_4 = p_4 + 2$.

And this tells us that:

$$\begin{aligned} q_5 &= p_5 + x + \frac{p_4+1}{3} - x - \frac{q_4-1}{3} - 1 \\ &= p_5 + \frac{p_4+1}{3} - \frac{p_4+1}{3} - 1 \\ &= p_5 - 1 \end{aligned}$$

And as $p_5 = 0$ this implies that $q_5 = -1$. Thus no such q exists.

All cases are now accounted for and this proves the proposition. □

To show that ψ^{s_2} actually maps $\Gamma^{s_2} \to \Gamma^{id}$, means showing that $wt(\psi^{s_2}(p)) = \beta^{id}$ and that ψ^{s_2} is well-defined on Γ^{s_2}. The latter is difficult to prove, so we tackle the former condition first. For all but the eight exceptional and extraordinary cases the former condition is immediately obvious from Lemma 3. This Lemma also demonstrates that the exceptional and extraordinary cases give $wt(\psi^{s_2}(p)) = \beta^{id}$, but some computation is required.

Proposition 39. *For $p \in \Gamma^{s_2}$, $wt(\psi^{s_2}(p)) = \beta^{id}$*

Proof. First, consider the case when $y > 0$, $p_2 = 0$, and $p_4 = 0 \mod 3$. Then $\psi^{s_2}(p) = (p_1 - x - 1 - \frac{p_4}{3}, p_4 + 1 - (y-1), p_3, y-1, p_5 + x + \frac{p_4}{3}, p_6)$. What we need to do is count how much is moved along the string $\gamma_1 \to \gamma_2 \to \gamma_4 \to \gamma_5$, and we do that by starting at the beginning and seeing how far down each element moves. In this case, $1 + x + \frac{p_4}{3}$ is moved from p_1, it does not all go to the same place. Of that sum, 1 goes to p_2 while $x + \frac{p_4}{3}$ goes to p_5. Similarly, $p_4 - (y-1)$ moves from p_4 to p_2, a backwards step along the string. Summing these various elements up, we have:

$$\begin{aligned} wt(\psi^{s_2}(p)) - \beta^{s_2} &= [3(x + \frac{p_4}{3}) + 1 - (p_4 - (y-1))]\alpha_2 \\ &= [3x + p_4 + 1 - p_4 + y - 1]\alpha_2 \\ &= [3x + y]\alpha_2 \\ &= (\lambda_2 + 1)\alpha_2 \end{aligned}$$

This implies that $wt(\psi^{s_2}(p)) = \beta^{id}$. Similar sums exist for the other seven exceptional cases. □

All that remains to show is that ψ^{s_2} is well-defined on its domain. In each of the eleven cases, the only possibility for a negative entry exist in the γ_1 and γ_2 directions. We tackle each direction one at a time.

Lemma 40. *For $p \in \Gamma^{s_2}$, let $p' = \psi^{s_2}(p)$. Then p'_2 is always nonnegative.*

Proof. We show this case by case. If $p_2 - 3p_5 \geq 3x + y$, then $p_2 \geq 3x + y \to p'_2 = p_2 - 3x - y \geq 0$.

If p is in a k-boundary, then $p_2 - 3p_5 \geq 3k + y \to p'_2 = p_2 - 3k - y \geq 0$.

If p is in the bottom region and $p_2 \geq y$, then $p'_2 = p_2 - y \geq 0$.

If p in the bottom region with $y > 0$, $p_2 = 0$ and $p_4 = 0 \mod 3$, then $p'_2 = p_4 + 1 - (y-1) = p_4 + 2 - y \geq p_4 \geq 0$ as $y \leq 2$. Similarly, if $p_4 = 1 \mod 3$, ($p_4 \neq 1$ OR $y = 1$), then $p'_2 = p_4 - 1 - (y-1) = p_4 - y \geq 0$. And if ($p_4 = 1$ AND $y = 2$), $p'_2 = 1$. And when $p_4 = 2 \mod 3$, then $p'_2 = p_4 - (y-1) = 1 + p_4 - y \geq 1$ as $p_4 \geq 2$.

Lastly, if p in the bottom region with $y = 2$ and $p_2 = 1$ then either $p'_2 = p_4$, $p_4 + 1$, or $p_4 + 2$, which in each case leave $p'_2 \geq 0$.

Thus, p'_2 is always nonnegative. $\square$

Demonstrating that p'_1 is always nonnegative requires a far more detailed analysis of the structure of Γ^{s_2}. The philosophy will be this: starting from a particular α_2-level that we can prove is well-defined, moving to other "adjacent" α_2-levels will either on average increase the value of p_1 in the image, or it will remain the same. Thus, it will remain well-defined.

But what α_2-level will be the starting point? Recall that $p_3 - p_6 = 2\beta_1 + \beta_2 - \mid p \mid$. For $w = s_2$, equations (7.6) and (7.7) give $2\beta_1^w + \beta_2^w = 2\lambda_1 + \lambda_2 - 2\mu_1 - \mu_2$. Thus, in Γ^{s_2}, when $\mid p \mid = 2\lambda_1 + \lambda_2 - 2\mu_1 - \mu_2$, it follows that $p_3 = p_6$.

Notation 41. Let ξ be the α_2-level described by arbitrary λ, with $\mu = 0$, of weight β^{s_2} and size $\mid p \mid = 2\lambda_1 + \lambda_2 - 2\mu_1 - \mu_2$, and with $p_6 = 0$.

How many different ways then can one find an "adjacent" α_2-level from this? There are five: 1) by increasing μ_1, 2) increasing μ_2, 3) decreasing $\mid p \mid$, 4) increasing $\mid p \mid$, 5) increasing p_6. But what do each of these motions do, and how do we state that? We return to the notion of the grid location. Take a particular p and its corresponding grid location. The root combination that corresponds to p after any of these moves will be in the same grid location. Thus, we want to know how p_2 and p_4 change with each of these moves, while p_1 and p_5 are held constant, and p_3 and p_6 are determined by the resulting α_2-level.

We consider each one at a time as they move. Let $\mid p \mid = 2\lambda_1 + \lambda_2 - 2\mu_1 - \mu_2 + l$ for some integer l.

First, given p, let p' be the corresponding root combination to p removed once by μ_1. Then $\mid p' \mid = 2\lambda_1 + \lambda_2 - 2\mu_1 - \mu_2 + l = \mid p \mid -2$. So $p'_3 = p_3$ and $p'_6 = p_6$. And as $p'_1 = p_1$ and $p'_5 = p_5$, it follows from equations (7.6) and (7.7) that $p'_2 = p_2 - 1$ and $-p'_2 + p'_4 = -p_2 + p_4$. Thus, $p'_2 = p_2 - 1$ and $p'_4 = p_4 - 1$. So in total, and in general, $p' = (p_1, p_2 - 1, p_3, p_4 - 1, p_5, p_6)$.

Now, let p' be the corresponding root combination to p removed once by μ_2. Again, $\mid p' \mid = 2\lambda_1 + \lambda_2 - 2\mu_1 - \mu_2 + l = \mid p \mid -1$. So $p'_3 = p_3$, $p'_6 = p_6$, as well as $p'_1 = p_1$ and $p'_5 = p_5$. Thus from equations (7.6) and (7.7) we have that $p'_2 = p_2$ and $-p'_2 + p'_4 = -p_2 + p_4 - 1$. Thus, $p'_2 = p_2$ ad $p'_4 = p_4 - 1$. So $p' = (p_1, p_2, p_3, p_4 - 1, p_5, p_6)$.

Whenever we change only the size, it turns out to matter a great deal on whether the size is greater or less than the size of ξ. So, first suppose that $l \leq 0$. Then we have that $p_3 \geq p_6$. Let p' be the corresponding root combination to p removed once by decreasing the size. Then, $p'_3 - p'_6 = p_3 - p_6 + 1$, and since p_6 may be zero, we set $p'_3 = p_3 + 1$ and $p'_6 = p_6$. This and equations (7.6) and (7.7) imply that $p'_2 = p_2 - 1$ and $-p'_2 + p'_4 = -p_2 + p_4$. Thus, $p'_2 = p_2 - 1$ and $p'_4 = p_4 - 1$. So in general, $p' = (p_1, p_2 - 1, p_3 + 1, p_4 - 1, p_5, p_6)$.

Now suppose that $l \geq 0$, and thus $p_6 \geq p_3$. Increasing the size by one means that $p'_6 - p'_3 = p_6 - p_3 + 1$. As p_3 may be zero, we set $p'_3 = p_3$ and $p'_6 = p_6 + 1$. With this and equations (7.6) and (7.7) we know that $p'_2 = p_2 + 1$ and $-p'_2 + p'_4 = -p_2 + p_4 - 2$. Thus, $p'_2 = p_2 + 1$ and $p'_4 = p_4 - 1$. So $p' = (p_1, p_2 + 1, p_3, p_4 - 1, p_5, p_6 + 1)$.

This leaves moving the α_2-level by increasing p_6. In fact, this is the same as subtracting one χ_3. So $p' = p - \chi_3$. Thus $p' = (p_1, p_2, p_3 + 1, p_4 - 2, p_5, p_6 + 1)$.

Now we have a complete description on how we can move from α_2-levels. Of course, this motion is only in one direction, originating at ξ as defined in Notation 41. But that is all that will be needed. To use any of this, we must understand how the knights are transformed by ψ^{s_2}. And to do that, we need to know one more fact about the k-boundaries. Let p be in a k-boundary. Consider then $p' = p - 3\chi_1 = (p_1+3, p_2-6, p_3, p_4+3, p_5, p_6)$, and assume that it is a root combination. Then $p_2' - 3p_5' = p_2 - 6 - 3p_5$ which implies that p' is in the (k-2)-boundary (assuming said boundary exists). If it does not, then p' is in the bottom region. Thus, in moving three steps, we change two boundaries. As each is discrete, there must be a boundary that has only one entry per χ_1 column, and a boundary with two entries per χ_1 column. Similarly, if $p' = p - 3\chi_2 = (p_1, p_2+3, p_4-6, p_5+3, p_6)$, then $p_2' - 3p_5' = p_2 - 3p_5 - 6$. Again in moving three steps, we change two boundaries or are in the bottom region. And so, discreteness tells us that along the χ_2 row, there is a boundary with only one entry, and a boundary with two.

But, we know that $q = p + l(\chi_1 - \chi_2)$ for any integer l implies that p and q are in the same k-boundary. Thus, if a boundary is only one-thick (having only one entry) along a χ_1 column, then it will be one-thick along a χ_2 row as well. Further, the boundaries alternate one-thick and two-thick. This information will be recorded in the knight. If we number the knight in the usual manner, then we note that $p_2^1 - 3p_5^1 = p_2^4 - 3p_5^4 + 6$. So knowing how the one-thick and two-thick boundaries land upon a single knight is all that is needed to understand how the knights are transformed by the mapping ψ^{s_2}.

First, as p^1 and p^4 are in boundaries two apart, they are in boundaries of the same thickness. Thus, there are only three distinct possibilities: 1) p^1 is in the one-thick boundary, 2) p^2 is in the one-thick boundary, 3) and p^3 is in the one-thick boundary. We call these three cases a high-knight, mid-knight, and a low-knight respectfully.

Consider the high-knight first. If p^1 is in the k-boundary, then p^2 and p^3 are in the (k-1)-boundary, and p^4 is in the (k-2)-boundary. So, when we consider their images under ψ^{s_2}, we have:

$$\begin{aligned}
\psi^{s_2}(p^1) &= (p_1^1 - (x-k), p_2^1 - 3k - y, p_3^1, p_4^1 + 3k + y, p_5^1 + (x-k), p_6^1) \\
\psi^{s_2}(p^2) &= (p_1^1 + 1 - (x-(k-1)), p_2^1 - 2 - 3(k-1) - y, p_3^1, p_4^1 + 1 + 3(k-1) + y, p_5^1 + (x-(k-1)), p_6^1) \\
\psi^{s_2}(p^3) &= (p_1^1 + 2 - (x-(k-1)), p_2^1 - 4 - 3(k-1) - y, p_3^1, p_4^1 + 2 + 3(k-1) + y, p_5^1 + (x-(k-1)), p_6^1) \\
\psi^{s_2}(p^4) &= (p_1^1 + 2 - (x-(k-2)), p_2^1 - 3 - 3(k-2) - y, p_3^1, p_4^1 + 3(k-2) + y, p_5^1 + 1 + (x-(k-2)), p_6^1)
\end{aligned}$$

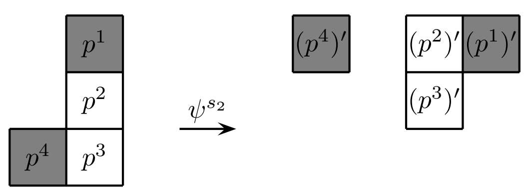


Figure 17: The high-knight and how it is changed by ψ^{s_2}

We note immediately that $(p^1)_1' = (p^2)_1' = (p^4)_1'$, and that $(p^3)_1' = (p^2)_1' + 1$. We say that the value of $(p^1)_1'$ is the **minimum** α_1. The minimum α_1 is essential to understanding why ψ^{s_2} is well-defined on Γ^{s_2}. This number will in fact be directly computable, but there is still much to show before that can be done. We will more formally define this after describing the change of all the other two knights.

Now consider the mid-knight. In this case, p^1 is in the k-boundary, p^2 is in the (k-1)-boundary, while p^3 and p^4 are in the (k-2)-boundary. So their images are:

$$\begin{aligned}
\psi^{s_2}(p^1) &= (p_1^1 - (x-k), p_2^1 - 3k - y, p_3^1, p_4^1 + 3k + y, p_5^1 + (x-k), p_6^1) \\
\psi^{s_2}(p^2) &= (p_1^1 + 1 - (x-(k-1)), p_2^1 - 2 - 3(k-1) - y, p_3^1, p_4^1 + 1 + 3(k-1) + y, p_5^1 + (x-(k-1)), p_6^1) \\
\psi^{s_2}(p^3) &= (p_1^1 + 2 - (x-(k-2)), p_2^1 - 4 - 3(k-2) - y, p_3^1, p_4^1 + 2 + 3(k-2) + y, p_5^1 + (x-(k-2)), p_6^1) \\
\psi^{s_2}(p^4) &= (p_1^1 + 2 - (x-(k-2)), p_2^1 - 3 - 3(k-2) - y, p_3^1, p_4^1 + 3(k-2) + y, p_5^1 + 1 + (x-(k-2)), p_6^1)
\end{aligned}$$

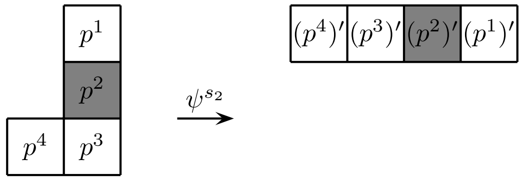


Figure 18: The mid-knight and how it is changed by ψ^{s_2}

In this case, $(p^1)'_1 = (p^2)'_1 = (p^3)'_1 = (p^4)'_1$, so the p_4 edge that formed the knights in the first place has become the row in the boundaries with the smallest value for γ_1.

Much the same works for the low-knight. In this case p^1 and p^2 are in the k-boundary, p^3 is in the (k-1)-boundary, and p^4 is in the (k-2) boundary. The images are:

$$\begin{aligned}
\psi^{s_2}(p^1) &= (p_1^1 - (x-k), p_2^1 - 3k - y, p_3^1, p_4^1 + 3k + y, p_5^1 + (x-k), p_6^1) \\
\psi^{s_2}(p^2) &= (p_1^1 + 1 - (x-k), p_2^1 - 2 - 3k - y, p_3^1, p_4^1 + 1 + 3k + y, p_5^1 + (x-k), p_6^1) \\
\psi^{s_2}(p^3) &= (p_1^1 + 2 - (x-(k-1)), p_2^1 - 4 - 3(k-1) - y, p_3^1, p_4^1 + 2 + 3(k-1) + y, p_5^1 + (x-(k-1)), p_6^1) \\
\psi^{s_2}(p^4) &= (p_1^1 + 2 - (x-(k-2)), p_2^1 - 3 - 3(k-2) - y, p_3^1, p_4^1 + 3(k-2) + y, p_5^1 + 1 + (x-(k-2)), p_6^1)
\end{aligned}$$

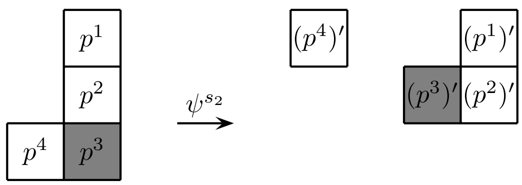


Figure 19: The low-knight and how it is changed by ψ^{s_2}

Here, the smallest values for γ_1 occur in only $(p^1)'$ and $(p^4)'$. But a smallest value is still attained, and that is what is most important.

Definition 42. Let p be in a k-boundary with $p_4 = 0$. Then the minimum α_1 attained by the α_2-level containing p is $(\psi^{s_2}(p))_1$.

It is now necessary to prove that root combinations in the bottom region can never attain smaller values for γ_1 via the mapping than the minimum α_1.

Proposition 43. *Let $p' = \psi^{s_2}(p)$ for $p \in \Gamma^{s_2}$. If p is in the bottom region and p^1 in a k-boundary is in the high position on a knight, then $p'_1 \geq (p^1)'_1$.*

Proof. Consider a knight in which every member of the knight is in the bottom region. As we handle the exceptional cases separately, we may assume that no member of the knight moves exceptionally. Then each member moves according to the same rule, and so the shape of the knight is preserved in the mapping. Thus, the member of the knight whose image contains the smallest number of γ_1 is also the member who contained the smallest number of γ_1 in the bottom region. Apart for the exceptional cases, this remains true for every p in the bottom region. Thus, for any p in the bottom region that does not move exceptionally, $p'_1 \geq (p^1)'_1$ for any p^1 as in the hypothesis.

Now we consider those p in the bottom region who must move exceptionally. Note that all exceptional cases occur along the p_2 edge, and thus, upon the reverse knights. Consider any two exceptional cases p and

$q = p + \chi_1 + 2\chi_2$ such that neither p nor q fall under the extraordinary cases. Then $p_2 = q_2$, $p_1 - 1 = q_1$ and $p_4 + 3 = q_4$. Then, no matter which of the six exceptional cases is employed, $p_1' = q_1' + 2$. Thus, we want to consider for each exceptional case, the root combination with the largest value for p_4. This naturally occurs where $p_5 = 0$ or 1.

Suppose $y = 1$, $p_2 = p_5 = 0$ and $p_4 = 0 \mod 3$. Then p will move exceptionally and $p' = (p_1 - x - 1 - \frac{p_4}{3}, p_4 + 1, p_3, 0, x + \frac{p_4}{3}, p_6)$. Consider $q = p + 3\chi_1 = (p_1 - 3, 6, p_3, p_4 - 3, 0, p_6)$. Then $3(2) + y = 7 > q_2 - 3q_5 = 6 \geq 3(1) + y = 4$. So q is in the 1-boundary. Now consider $r = q + \frac{p_4-3}{3}(\chi_1 - \chi_2) = (p_1 - 2 - \frac{p_4}{3}, 3 + p_4, p_3, 0, \frac{p_4}{3} - 1, p_6)$. Since $r_4 = 0$, r is in the high position for a knight. And so we see that $\psi^{s_2}(r) = r' = (p_1 - 2 - \frac{p_4}{3} - (x-1), 3 + p_4 - (3+y), p_3, 3 + y, \frac{p_4}{3} - 1 + (x-1), p_6)$, and thus $r_1' = p_1 - 1 - x - \frac{p_4}{3} \leq p_1'$.

Now suppose that $y = 1$, $p_2 = p_5 = 0$ and $p_4 = 1 \mod 3$. Then $p' = (p_1 - x - \frac{p_4-1}{3}, p_4 - 1, p_3, 0, x + 1 + \frac{p_4-1}{3}, p_6)$. Consider $r = p + 3\chi_1 + \frac{p_4-4}{3}(\chi_1 - \chi_2) = (p_1 - 2 - \frac{p_4-1}{3}.2 + p_4, p_3, 1, \frac{p_4-1}{3} - 1, p_6)$. Then $r_2 - 3r_5 = 2 + p_4 - (p_4 - 1) + 3 = 6$. So r is in the 1-boundary, and since $r_2 - 3p_5 - 3 + y = 2$, it follows that the 1-boundary is 2-thick. And as $r_4 = 1$, it follows that the knight containing r is a high-knight, and thus r_1' is the minimum value for γ_1 obtained from the k-boundaries. And that is $r_1' = p_1 - 1 - x - \frac{p_4-1}{3} \leq p_1'$.

Now suppose that $y = 1$, $p_2 = p_5 = 0$ and $p_4 = 2 \mod 3$. Then $p' = (p_1 - x - \frac{p_4+1}{3}, p_4, p_3, 0, x + \frac{p_4+1}{3}, p_6)$. Consider $r = p + 3\chi_1 + \frac{p_4-5}{3}(\chi_1 - \chi_2) = (p_1 - 1 - \frac{p_4+1}{3}.1 + p_4, p_3, 2, \frac{p_4+1}{3} - 2, p_6)$. Again, r is in a 1-boundary, and it is 2-thick. But this time, with $r_4 = 2$, it follows that the knight containing r is a mid-knight, and thus r_1' is the minimum value obtained by the image of the k-boundaries for γ_1. And that is $r_1' = p_1 - x - \frac{p_4+1}{3} \leq p_1'$.

When $y = 1$, $p_2 = 0$, $p_5 = 1$, then let $q = p + 4\chi_1$ and $r = q + \lfloor \frac{p_4-4}{3} \rfloor (\chi_1 - \chi_2)$. Then q and r will be in the 1-boundary, and it will be 1-thick. Since r differs from q by an integer multiple of $(\chi_1 - \chi_2)$, it is automatically in the same region as q. We know q is in the 1-boundary, because $q_2 - 3q_5 = 8 - 3 = 5 \geq 4$. And we know it is 1-thick because $q + \chi_1$ would be in the 2-boundary $(10 - 3 \geq 7)$, while $q - \chi_1$ would be in the bottom region $(6 - 3 < 4)$. So r is in the 1-boundary and is on a knight.

In this case, when $p_4 = 0 \mod 3$, we have that $r_4 = 2$, and so the knight is a low-knight, and the minimum value obtained in the k-boundaries for γ_1 is $r_1' - 1 = p_1 - 2 - x - \frac{p_4}{3} \leq p_1' = p_1 - 1 - x - \frac{p_4}{3}$. When $p_4 = 1 \mod 3$, we have that $r_4 = 0$, and so the knight is a high-knight, and the minimum value obtained in the k-boundaries for γ_1 is $r_1' = p_1 - 2 - x - \frac{p_4-1}{3} \leq p_1' = p_1 - x - \frac{p_4-1}{3}$. And when $p_4 = 2 \mod 3$, we have that $r_4 = 1$, and so the knight is a mid-knight, and the minimum value obtained in the k-boundaries for γ_1 is $r_1' = p_1 - 1 - x - \frac{p_4+1}{3} \leq p_1' = p_1 - x - \frac{p_4+1}{3}$.

Now suppose that $y = 2$, and $p_2 = p_5 = 0$. Let $q = p + 3\chi_1$ and $r = q + \lfloor \frac{p_4-3}{3} \rfloor (\chi_1 - \chi_2)$. Then r and q will be in the 1-boundary, and it will be 1-thick. We know that q is in the 1-boundary because $q_2 - 3q_5 = 6 - 0 \geq 5$, and as $q_2 - 3q_5 - (3 + y) = 1$, it follows that the boundary is 1-thick. And as $r_4 \leq 2$, r is on the 1-boundary and is on a knight.

So in this case, when $p_4 = 0 \mod 3$, we have that $r_4 = 0$, and so the knight is a high-knight, and the minimum value obtained in the k-boundaries for γ_1 is $r_1' = p_1 - 1 - x - \frac{p_4}{3} \leq p_1' = p_1 - 1 - x - \frac{p_4}{3}$. When $p_4 = 1 \mod 3$, we have that $r_4 = 1$, and so the knight is a mid-knight, and the minimum value obtained in the k-boundaries for γ_1 is $r_1' = p_1 - 1 - x - \frac{p_4-1}{3} \leq p_1' = p_1 - x - \frac{p_4-1}{3}$. And when $p_4 = 2 \mod 3$, we have that $r_4 = 2$, and so the knight is a low-knight, and the minimum value obtained in the k-boundaries for γ_1 is $r_1' - 1 = p_1 - 1 - x - \frac{p_4+1}{3} \leq p_1' = p_1 - x - \frac{p_4+1}{3}$.

Now suppose that $y = 2$, and $p_2 = 0$, $p_5 = 1$. Let $q = p + 5\chi_1$ and $r = q + \lfloor \frac{p_4-5}{3} \rfloor (\chi_1 - \chi_2)$. Again, r and q will be in the 1-boundary, and this time it will be 2-thick. As $q_2 - 3q_5 = 10 - 3 = 7 \geq 5 = 3 + y$, and

$q_2 - 3q_5 - (3+y) = 2$, we know this. And as $r_4 \le 2$, r is in the 1-boundary and is on a knight.

So in this case, when $p_4 = 0 \mod 3$, we have that $r_4 = 1$, and so the knight is a mid-knight, and the minimum value obtained in the k-boundaries for γ_1 is $r'_1 = p_1 - 2 - x - \frac{p_4}{3} \le p'_1 = p_1 - 1 - x - \frac{p_4}{3}$. When $p_4 = 1 \mod 3$, we have that $r_4 = 2$, and so the knight is a low-knight, and the minimum value obtained in the k-boundaries for γ_1 is $r'_1 - 1 = p_1 - 3 - x - \frac{p_4-1}{3} \le p'_1 = p_1 - x - \frac{p_4-1}{3}$. And when $p_4 = 2 \mod 3$, we have that $r_4 = 0$, and so the knight is a high-knight, and the minimum value obtained in the k-boundaries for γ_1 is $r'_1 = p_1 - 2 - x - \frac{p_4+1}{3} \le p'_1 = p_1 - x - \frac{p_4+1}{3}$.

Now suppose that $y = 2$, $p_2 = 1$, and $p_5 = 0$. Let $q = p + 3\chi_1$ and $r = q + \lfloor \frac{p_4-3}{3} \rfloor (\chi_1 - \chi_2)$. Here, q is in the 1-boundary and it is 2-thick. We see this by $q_2 - 3q_5 = 7 \ge 5 = 3 + y$, and $q_2 - 3q_5 - (3+y) = 2$. And as $r_4 \le 2$, r is in the 1-boundary and is on a knight.

And in this case, when $p_4 = 0 \mod 3$, we have that $r_4 = 0$, and so the knight is a low-knight, and the minimum value obtained in the k-boundaries for γ_1 is $r'_1 - 1 = p_1 - 2 - x - \frac{p_4}{3} \le p'_1 = p_1 - x - \frac{p_4}{3}$. When $p_4 = 1 \mod 3$, we have that $r_4 = 1$, and so the knight is a high-knight, and the minimum value obtained in the k-boundaries for γ_1 is $r'_1 = p_1 - x - \frac{p_4+2}{3} \le p'_1 = p_1 - x - \frac{p_4+2}{3}$. And when $p_4 = 2 \mod 3$, we have that $r_4 = 2$, and so the knight is a mid-knight, and the minimum value obtained in the k-boundaries for γ_1 is $r'_1 = p_1 - x - \frac{p_4+1}{3} \le p'_1 = p_1 - x - \frac{p_4+1}{3}$.

And finally, suppose that $y = 2$, $p_2 = p_5 = 1$. Let $q = p + 4\chi_1$ and $r = q + \lfloor \frac{p_4-4}{3} \rfloor (\chi_1 - \chi_2)$. And here q is in the 1-boundary and it is 1-thick. We see this by $q_2 - 3q_5 = 9 - 3 = 6 \ge 5$. And as $r_4 \le 2$, r is in the 1-boundary and is on a knight.

In this case, when $p_4 = 0 \mod 3$, we have that $r_4 = 2$, and so the knight is a low-knight, and the minimum value obtained in the k-boundaries for γ_1 is $r'_1 - 1 = p_1 - 2 - x - \frac{p_4}{3} \le p'_1 = p_1 - x - \frac{p_4}{3}$. When $p_4 = 1 \mod 3$, we have that $r_4 = 0$, and so the knight is a high-knight, and the minimum value obtained in the k-boundaries for γ_1 is $r'_1 = p_1 - 1 - x - \frac{p_4+2}{3} \le p'_1 = p_1 - x - \frac{p_4+2}{3}$. And when $p_4 = 2 \mod 3$, we have that $r_4 = 1$, and so the knight is a mid-knight, and the minimum value obtained in the k-boundaries for γ_1 is $r'_1 = p_1 - 1 - x - \frac{p_4+1}{3} \le p'_1 = p_1 - 1 - x - \frac{p_4+1}{3}$.

Thus, in every exceptional case, the hypothesis holds. $\square$

Now that we have established that the value of p'_1 for any p in either the k-boundaries or the bottom region is at least as large as the minimum α_1, it is time to make explicit calculations for the minimum α_1. We now return to the ξ defined in Notation 41 and show that on this, ψ^{s_2} is well-defined.

Proposition 44. *For $p \in \xi$, $\psi^{s_2}(p)$ is well-defined.*

Proof. Recall that ξ is the α_2-level for arbitrary λ, $\mu = 0$, $\mid p \mid = 2\lambda_1 + \lambda_2 - 2\mu_1 - \mu_2$, and $p_6 = 0$. Our goal will be to find a knight in the k-boundaries and analyze what happens. It turns out that there are three cases: 1) $\lambda_1 = 0$, 2) $\lambda_1 > 0$ and odd, and 3) $\lambda_1 > 0$ and even.

Let $\lambda_1 = 0$. Consider $\tau = (1, \lambda_2 - 1, 0, 0, 0, 0)$. This is the top root combination in ξ with $\lambda_1 = 0$. A quick verification in equations (7.6) and (7.7) show this to be the case. However, for small values of λ_2, τ is in the bottom region. Let's consider them first. Since $\lambda_1 = 0$, $\lambda_2 = 0$ would produce no root combinations at all, we may ignore it. When $\lambda_2 = 1$, the only root combination in ξ is $\tau = (1, 0, 0, 0, 0, 0)$. And $\psi^{s_2}(\tau) = (0, 0, 0, 1, 0, 0)$. Consider $2 \le \lambda_2 \le 4$. In each case, x=1, but $\tau_2 - 3\tau_5 < 3 + y$, so τ is in the bottom region. But also in each case, $\tau_2 \ge y$, so $\psi^{s_2}(\tau) = (0, 1, 0, y, 1, 0)$. And as this is the top root combination, by Proposition 43 we know that for any other root combinations p, $p'_1 \ge 0$.

Now, for $\lambda_2 \geq 5$, τ is in a k-boundary (as $\lambda_2 + 1 > \tau_2 - 3\tau_5 \geq 5$), so the knights become useful to consider. We note that in this case, τ is in the (x-1)-boundary, and it is always 1-thick. Why? Because $\tau_2 - 3\tau_5 = \lambda_2 - 1 = (\lambda_2 - 2) + 1$. And as $\tau_4 = 0$, being in the high position on the knight, the knight is always a high knight. Thus, τ_1' is the minimum α_1 attained. And $\psi^{s_2}(\tau) = (0, 1, 0, \lambda_2 - 2, 1, 0)$, so the minimum α_1 attained is zero.

So in the case $\lambda_1 = 0$, ψ^{s_2} is well-defined on ξ.

Now let $\lambda_1 > 0$ and odd. Then $\tau = (0, \lambda_1 + \lambda_2 + 1, 0, \lambda_1 - 1, 0, 0)$. Again, a check of this against equations (7.6) and (7.7) demonstrates that τ is indeed in ξ. And from proposition 32 we know that τ is the top root combination. But here, it is quite likely that τ is not on a knight as $\tau_4 = \lambda_1 - 1$ is more likely than not to be larger than 2. Regardless what it is, let $p = \tau - \frac{1}{2}(\lambda_1 - 1)(\chi_2)$. Then $p_4 = (\lambda_1 - 1) - (\lambda_1 - 1) = 0$. So p is in the high position on a knight. Is p in a k-boundary? As $p_2 = \lambda_1 + \lambda_2 + 1 + \frac{1}{2}(\lambda_1 - 1)$ and $p_5 = \frac{1}{2}(\lambda_1 - 1)$, it follows that $p_2 - 3p_5 = \lambda_1 + \lambda_2 + 1 + \frac{1}{2}(\lambda_1 - 1) - \frac{3}{2}(\lambda_1 - 1) = \lambda_1 + \lambda_2 + 1 - (\lambda_1 - 1) = \lambda_2 + 2$. Even when $\lambda_1 = 1$ and $\tau = p$, the numbers still add up properly. So we see that in fact p is not in a k-boundary, but in the top region. As ψ^{s_2} does not change the value of γ_1 for root combinations in the top region, lemma 40 is all that is needed to see that ψ^{s_2} is well-defined on root combinations in the top region. So we need another root combination that will be in a k-boundary.

Let $q = p - \chi_1$. Then $q = (1, \frac{3}{2}(\lambda_1 - 1) + \lambda_2, 0, 1, \frac{1}{2}(\lambda_1 - 1), 0)$. In this case $q_2 - 3q_5 = \lambda_2 = (\lambda_2 - 2) + 2$. Thus, q is in the (x-1)-boundary, and it is 2-thick. Since $q_4 = 1$, it is in the mid position on the knight. And as $q - \chi_1$ is also in the (x-1)-boundary, it follows that the knight is a high-knight. So, the minimum α_1 attained is $q_1' = 1 - (x - (x - 1)) = 0$.

Now let $\lambda_1 > 0$ and even. Again, $\tau = (0, \lambda_1 + \lambda_2 + 1, 0, \lambda_1 - 1, 0, 0)$. Let $p = \tau - \frac{1}{2}(\lambda_1 - 2) = (0, \frac{3}{2}(\lambda_1 - 2) + \lambda_2 + 3, 0, 1, \frac{1}{2}(\lambda_1 - 2), 0)$. As before, we see quickly that p is in the top region. So this time, let $q = p - \chi_1 - \chi_2$. Then $q = (1, \frac{3}{2}(\lambda_1 - 2) + \lambda_2 + 2, 0, 0, \frac{1}{2}(\lambda_1 - 2) + 1, 0)$. Here, $q_2 - 3q_5 = \lambda_2 + 2 - 3 = \lambda_2 - 1 = (\lambda_2 - 2) + 1$. So q is in the (x-1)-boundary and it is 1-thick. Further, since $q_4 = 0$, it is the high position on the knight. So, the knight is a high-knight, and the minimum α_1 attained is $q_1' = 1 - (x - (x - 1)) = 0$.

Thus, ψ^{s_2} is well-defined on ξ. □

Now we consider how shifting ξ in any of the five directions alters the minimum α_1 that can be attained.

Lemma 45. *Given any α_2-level, increasing μ_1 by one leaves the minimum α_1 attained fixed.*

Proof. Let t be the value of the minimum α_1 attained for the α_2-level containing p in a 1-thick k-boundary on a knight.

When shifting μ_1 up by one, each grid location subtracts one from both γ_2 and γ_4. If p is in the high position, then its corresponding grid location will not be filled. Consider then $q = p - \chi_1$. Here, $q_4 = p_4 + 1$, and so its grid location is always filled in the shifted α_2-level. Let that root combination be $r = (q_1, q_2 - 1, q_3, q_4 - 1, q_5, q_6) = (p_1 + 1, p_2 - 3, p_3, p_4, p_5, p_6)$. Then r is in the same position on a knight as p. Note that $r_2 - 3r_5 = p_2 - 3p_5 - 3$. Thus, if r is in a k-boundary, it is also 1-thick. Further, if r is in a k-boundary, then its boundary is one smaller than that of p. In other words, if p is in the l-boundary, then r is in the (l-1)-boundary. As both p and r are in 1-thick k-boundaries and are in the same position on a knight, they are both in the same type of knights and the minimum α_1 attained in each is p_1' and r_1' for high and mid-knights, $p_1' - 1$ and $r_1' - 1$ in low-knights. But we have that $p_1' = p_1 - (x - l) = p_1 + 1 - (x - (l - 1)) = r_1 - (x - (l - 1)) = r_1'$. Thus, regardless whether $t = p_1'$ or $t = p_1' - 1$, the shifted α_2-level will have the same minimum α_1.

But what if r is in the bottom region? Then it follows that p was in the 1-boundary. Consider $h =$

$r+2\chi_1+\chi_2 = (r_1-2, r_2+3, r_3, r_4, r_5-1, r_6) = (p_1-1, p_2, p_3, p_4, p_5-1, p_6)$. We have that $h_2-3h_5 = p_2-3p_5+3$, so h is in the 2-boundary, and it is 1-thick. Further, as $h_4 = p_4$, h is in the same position as p. And we see that $h'_1 = p_1 - 1 - (x-2) = p_1 - (x-1) = p'_1$. So again, regardless of the type of knight p is on, increasing μ_1 does not alter the minimum α_1 attained. □

Lemma 46. *Given any α_2-level, increasing μ_2 by one increases the minimum α_1 attained by one if the knights are low-knights. Otherwise, the minimum α_1 remains fixed.*

Proof. Let t be the value of the minimum α_1 attained for the α_2-level containing p in a 1-thick k-boundary on a knight. When shifting μ_2 by one, each grid location subtracts one from γ_4.

Now suppose that p is in the high position. Then its corresponding grid location will not be filled in the shifted α_2-level. Let $q = p-(\alpha_1-\alpha_2) = (p_1+1, p_2-3, p_3, p_4+3, p_5-1, p_6)$. Naturally, q will be in the same k-boundary as p, and since $q_4 = 3$, it follows that its corresponding grid location will be filled. Let that root combination be $r = (p_1+1, p_2-3, p_4, p_4+2, p_5-1, p_6)$. Then, since $r_2-3r_5 = p_2-3-3(p_5-1) = p_2-3p_5$, r is in the same k-boundary as p and it is still 1-thick. But, $r_4 = 2$, so the knight is now a low-knight. Thus, the minimum α_1 attained is $r'_1 - 1 = p_1 + 1 - (x-k) - 1 = p_1 - (x-k) = p'_1 = t$. So, if the α_2-level has high-knights, shifting up by μ_2 does not alter the minimum α_1 attained.

Now suppose that p is in the mid position. Then its corresponding grid location is filled in the shifted α_2-level. Let that root combination be $r = (p_1, p_2, p_3, p_4-1, p_5, p_6)$. Since $p_4 = 1$, it follows that $r_4 = 0$. Further, as $r_2 = p_2$ and $r_5 = p_5$, r is in the same k-boundary in its α_2-level that p is in its α_2-level, and it will be 1-thick. Thus, r is in a high-knight, so the minimum α_1 attained is $r'_1 = p_1 - (x-k) = p'_1 = t$. Thus, if the α_2-level has mid-knights, then shifting up by μ_2 does not alter the minimum α_1 attained.

Lastly, suppose that p is in the low position. Then its corresponding grid location is filled by $r = (p_1, p_2, p_3, p_4-1, p_5, p_6)$. Again, r is in the same k-boundary in its α_2-level that p is in its α_2-level, and it is also 1-thick. But now we see that $r_4 = 1$, so it is a mid-knight. So the minimum α_1 attained is $r'_1 = p_1 - (x-k) = p'_1 = (p'_1-1)+1 = t+1$. Thus, if the α_2-level has low-knights, then shifting up by μ_2 increases by one the minimum α_1 attained. □

Lemma 47. *Given any α_2-level whose size is $\leq 2\lambda_1+\lambda_2-2\mu_1-\mu_2$, decreasing the size by one does not alter the minimum α_1 attained.*

Proof. Let t be the value of the minimum α_1 attained for the α_2-level containing p in a 1-thick k-boundary on a knight. When decreasing $\mid p \mid$ by one, each grid location subtracts one from γ_2 and γ_4, and adds one to γ_3.

Since the shifting causes almost exactly the same changes as does shifting up by μ_1, (the only difference is that decreasing the size increases γ_3, which is unaffected by ψ^{s_2} anyway), we proceed in the same manner as we did with lemma 45.

We observe that if p is in the high position, then its corresponding grid location will not be filled. Consider then $q = p - \chi_1$. Here, $q_4 = p_4+1$, and so its grid location is always filled in the shifted α_2-level. Let that root combination be $r = (q_1, q_2-1, q_3+1, q_4-1, q_5, q_6) = (p_1+1, p_2-3, p_3+1, p_4, p_5, p_6)$. Then r is in the same position on a knight as p. Note that $r_2-3r_5 = p_2-3p_5-3$. Thus, if r is in a k-boundary, it is also 1-thick. Further, if r is in a k-boundary, then its boundary is one smaller than that of p. In other words, if p is in the l-boundary, then r is in the (l-1)-boundary. As both p and r are in 1-thick k-boundaries and are in the same position on a knight, they are both in the same type of knights and the minimum α_1 attained in each is p'_1 and r'_1 for high and mid-knights, p'_1-1 and r'_1-1 in low-knights. But we have that

$p_1' = p_1 - (x - l) = p_1 + 1 - (x - (l-1)) = r_1 - (x - (l-1)) = r_1'$. Thus, regardless whether $t = p_1'$ or $t = p_1' - 1$, the shifted α_2-level will have the same minimum α_1.

But what if r is in the bottom region? Then it follows that p was in the 1-boundary. Consider $h = r + 2\chi_1 + \chi_2 = (r_1 - 2, r_2 + 3, r_3, r_4, r_5 - 1, r_6) = (p_1 - 1, p_2, p_3 + 1, p_4, p_5 - 1, p_6)$. We have that $h_2 - 3h_5 = p_2 - 3p_5 + 3$, so h is in the 2-boundary, and it is 1-thick. Further, as $h_4 = p_4$, h is in the same position as p. And we see that $h_1' = p_1 - 1 - (x - 2) = p_1 - (x - 1) = p_1'$. So again, regardless of the type of knight p is on, decreasing the size does not alter the minimum α_1 attained. □

Lemma 48. *Given any α_2-level whose size is $\geq 2\lambda_1 + \lambda_2 - 2\mu_1 - \mu_2$, increasing the size by one does not alter the minimum α_1 attained if the knights are mid-knights. Otherwise, the minimum α_1 attained increases by one.*

Proof. Let t be the value of the minimum α_1 attained for the α_2-level containing p in a 1-thick k-boundary on a knight. When increasing $\mid p \mid$ by one, each grid location adds one to γ_2, and γ_6 and subtracts one from γ_4.

Suppose that p is in the high position. Let $q = p - \chi_1$. Then $q_4 = 1$, so it corresponding grid location will be filled. Let that root combination be $r = (q_1, q_2 + 1, q_3, q_4 - 1, q_5, q_6 + 1) = (p_1 + 1, p_2 - 1, p_3, p_4, p_5, p_6 + 1)$. We see immediately that r is in the high position on its knight. Further, $r_2 - 3r_5 = p_2 - 3p_5 - 1$. Since p was in a 1-thick k-boundary, then r will be in the 2-thick k-boundary. And since any member of a 2-thick k-boundary in the high position always attains the minimum α_1, we have that the minimum α_1 is $r_1' = p_1 + 1 - (x - k) = p_1' + 1 = t + 1$. So when shifting the size up by one from α_2-level with high-knights, the minimum α_1 increases by one.

Suppose that p is in the mid position. Then $p_4 = 1$, and its corresponding grid location will be filled. Let that root combination be $r = (p_1, p_2 + 1, p_3, p_4 - 1, p_5, p_6 + 1)$. We see immediately that r is in the high position on its knight. Also, $r_2 - 3r_5 = p_2 - 3p_5 + 1$. Since p was in a 1-thick k-boundary, then r will be in the 2-thick k-boundary for its α_2-level. And again, since any member of a 2-thick k-boundary in the high position always attains the minimum α_1, we have that the minimum α_1 is $r_1' = p_1 - (x - k) = p_1' = t$. So when shifting the size up by one on an α_2-level with mid-knights, the minimum α_1 attained remains fixed.

Now suppose that p is in the low position. Let $q = p - \chi_1$. Then both p and q have corresponding root combinations in the shifted α_2-level. Let them be r^1 and r^2 respectively. Then $r^1 = (p_1, p_2 + 1, p_3, p_4 - 1, p_5, p_6 + 1)$ and $r^2 = (p_1 + 1, p_2 - 1, p_3, p_4, p_5, p_6 + 1)$. Note that $r_2^1 - 3r_5^1 = p_2 - 3p_5 + 1$ and $r_2^2 - 3r_5^2 = p_2 - 3p_5 - 1$. Since p is in a 1-thick k-boundary, it follows that r^1 and r^2 are in the 2-thick k-boundary in the shifted α_2-level. Further, as $r_4^1 = 1$ and $r_4^2 = 2$, they occupy the mid and low positions on a knight respectively. Thus, the knight containing r^1 and r^2 is a high knight, so the minimum α_1 attained in the shifted α_2-level is $(r^1)_1' = p_1 - (x - k) = p_1' = t + 1$ as p was in a low-knight. Thus, when shifting the size up by one on an α_2-level with low-knights, the minimum α_1 attained increases by one. □

Lemma 49. *Given any α_2-level, increasing the value of γ_6 by one does not alter the minimum α_1 attained if the knights are mid-knights. Otherwise, the minimum α_1 attained increases by one.*

Proof. Let t be the value of the minimum α_1 attained for the α_2-level containing p in a 1-thick k-boundary on a knight. When increasing the value of γ_6 by one, each grid location subtracts two from γ_4, and adds one to γ_3 and γ_6.

Suppose that p is in the high position. Then its corresponding grid location in the shifted α_2-level is empty. Let $q = p - (\chi_1 - \chi_2) = (p_1 + 1, p_2 - 3, p_3, p_4 + 3, p_5 - 1, p_6)$. Then as $q_4 = 3$, its corresponding grid location contains a corresponding root combination. Let that root combination be $r = (p_1 + 1, p_2 -$

$3, p_3 + 1, p_4 + 1, p_5 - 1, p_6 + 1)$. We immediately see that r is in the mid position on a knight. Also $r_2 - 3r_5 = p_2 - 3 - 3(p_5 - 1) = p_2 - 3p_5$. So r and p share the same value for a k-boundary, and both are 1-thick. Thus, the knight is a mid-knight, and so the minimum α_1 attained is $r'_1 = p_1 + 1 - (x-k) = p'_1 + 1 = t+1$. So increasing the value of γ_6 by one on an α_2-level with high-knights increases the minimum α_1 attained by one.

Suppose that p is in the mid position. Then its corresponding grid location in the shifted $alpha_2$-level is empty. Let $q = p - (\chi - \chi_2) = (p_1 + 1, p_2 - 3, p_3, p_4 + 3, p_5 - 1, p_6)$. Then as $q_4 = 4$, its corresponding grid location contains a corresponding root combination. Let that root combination be $r = (p_1 + 1, p_2 - 3, p_3 + 1, p_4 + 1, p_5 - 1, p_6 + 1)$. We immediately see that r is in the low position on a knight. Also $r_2 - 3r_5 = p_2 - 3 - 3(p_5 - 1) = p_2 - 3p_5$. So r and p share the same value for a k-boundary, and both are 1-thick. Thus, the knight is a low-knight, and so the minimum α_1 attained is $r'_1 - 1 = p_1 + 1 - (x-k) - 1 = p'_1 = t$. So increasing the value of γ_6 by one on an α_2-level with mid-knights fixes the minimum α_1.

Suppose that p is in the low position. Then its corresponding grid location in the shifted α_2-level has a corresponding root combination. Let that root combination be $r = (p_1, p_2, p_3 + 1, p_4 - 2, p_5, p_6 + 1)$. We immediately see that r is in the high position on a knight. Also $r_2 - 3r_5 = p_2 - 3p_5$. So r and p share the same value for a k-boundary, and both are 1-thick. Thus, the knight is a high-knight, and so the minimum α_1 attained is $r'_1 = p_1 - (x-k) = p'_1 = t+1$ as p was in a low-knight. So increasing the value of γ_6 by one on an α_2-level with low-knights increases the minimum α_1 attained by one. □

Now that all of the possible ways of shifting have been explored, it must be shown that the order in which the shifting is accomplished does not change the minimum α_1 attained.

Proposition 50. *Of the five ways to shift a α_2-level, the ordering in which these shifts are applied does not matter.*

Proof. First off, increasing μ_1 and decreasing the size when the size is $\leq 2\lambda_1 + \lambda_2 - 2\mu_1 - \mu_2$ change neither the knight type or the minimum α_1 by lemmas 45 and 47. And as the three other types of shifting depend on the knight, these two shifts can be done in any order. So, it remains to show that increasing μ_2, increasing the size if the size is $\geq 2\lambda_1 + \lambda_2 - 2\mu_1 - \mu_2$, and increasing the value of γ_6 can be done in any order. We compare them two at a time. Since each change depending on mod 3, this leaves 3 cases with 9 subcases each to consider. But, as $+ \mid z \mid$ and $+p_6\gamma_6$ change the knights and increase the minimum α_1 in exactly the same manner by lemmas 48 and 49, they both commute with the same things. So these two are interchangeable in order. This reduces us to nine cases.

As all knights in ξ are high-knights, without loss of generality we begin on high-knights for all cases. Let t be the minimum α_1. In the cases to follow, when $z \leq 0$, the proofs work the same simply by removing all references to z.

Let $\mu_2 = 0 \mod 3$ and $p_6 + z = 0 \mod 3$. Shifting μ_2 first increases t by $\frac{\mu_2}{3}$ and makes all knights high. Then shifting by $p_6\gamma_6 + \mid z \mid$ increases $t + \frac{\mu_2}{3}$ by $2\frac{p_6+z}{3}$ and makes all knights high. Thus, after shifting μ_2 then $p_6\gamma_6 + \mid z \mid$, the minimum α_2 is $t + \frac{\mu_2}{3} + \frac{2(p_6+z)}{3}$. Now shifting $p_6\gamma_6 + \mid z \mid$ first, increases t by $\frac{2(p_6+z)}{3}$ and makes all knights high. Then shifting by μ_2 increases $t + \frac{2(p_6+z)}{3}$ by $\frac{\mu_2}{3}$ and makes all knights high. Thus, shifting by $p_6\gamma_6 + \mid z \mid$ then by μ_2 gives a minimum α_1 of $t + \frac{\mu_2}{3} + \frac{2(p_6+z)}{3}$. Thus both orderings produce the same results.

Let $\mu_2 = 0 \mod 3$ and $p_6 + z = 1 \mod 3$. Shifting μ_2 first increases t by $\frac{\mu_2}{3}$ and makes all knights high. Then shifting by $p_6\gamma_6 + \mid z \mid$ increases $t + \frac{\mu_2}{3}$ by $\frac{2(p_6+z-1)}{3} + 1$ and makes all knights mid. Thus, after shifting

μ_2 then $p_6\gamma_6+ \mid z \mid$, the minimum α_2 is $t + \frac{\mu_2}{3} + \frac{2(p_6+z-1)}{3} + 1$. Now shifting $p_6\gamma_6+ \mid z \mid$ first, increases t by $\frac{2(p_6+z-1)}{3} + 1$ and makes all knights mid. Then shifting by μ_2 increases $t + \frac{2(p_6+z-1)}{3} + 1$ by $\frac{\mu_2}{3}$ and makes all knights mid. Thus, shifting by $p_6\gamma_6+ \mid z \mid$ then by μ_2 gives a minimum α_1 of $t + \frac{\mu_2}{3} + \frac{2(p_6+z-1)}{3} + 1$. Thus both orderings produce the same results.

Let $\mu_2 = 0 \mod 3$ and $p_6 + z = 2 \mod 3$. Shifting μ_2 first increases t by $\frac{\mu_2}{3}$ and makes all knights high. Then shifting by $p_6\gamma_6+ \mid z \mid$ increases $t + \frac{\mu_2}{3}$ by $\frac{2(p_6+z-2)}{3} + 1$ and makes all knights low. Thus, after shifting μ_2 then $p_6\gamma_6+ \mid z \mid$, the minimum α_2 is $t + \frac{\mu_2}{3} + \frac{2(p_6+z-2)}{3} + 1$. Now shifting $p_6\gamma_6+ \mid z \mid$ first, increases t by $\frac{2(p_6+z-2)}{3} + 1$ and makes all knights low. Then shifting by μ_2 increases $t + \frac{2(p_6+z-2)}{3} + 1$ by $\frac{\mu_2}{3}$ and makes all knights low. Thus, shifting by $p_6\gamma_6+ \mid z \mid$ then by μ_2 gives a minimum α_1 of $t + \frac{\mu_2}{3} + \frac{2(p_6+z-2)}{3} + 1$. Thus both orderings produce the same results.

Let $\mu_2 = 1 \mod 3$ and $p_6 + z = 0 \mod 3$. Shifting μ_2 first increases t by $\frac{\mu_2-1}{3}$ and makes all knights low. Then shifting by $p_6\gamma_6+ \mid z \mid$ increases $t + \frac{\mu_2-1}{3}$ by $\frac{2(p_6+z)}{3}$ and makes all knights low. Thus, after shifting μ_2 then $p_6\gamma_6+ \mid z \mid$, the minimum α_2 is $t + \frac{\mu_2-1}{3} + \frac{2(p_6+z)}{3}$. Now shifting $p_6\gamma_6+ \mid z \mid$ first, increases t by $\frac{2(p_6+z)}{3}$ and makes all knights high. Then shifting by μ_2 increases $t + \frac{2(p_6+z)}{3}$ by $\frac{\mu_2-1}{3}$ and makes all knights low. Thus, shifting by $p_6\gamma_6+ \mid z \mid$ then by μ_2 gives a minimum α_1 of $t + \frac{\mu_2-1}{3} + \frac{2(p_6+z)}{3}$. Thus both orderings produce the same results.

Let $\mu_2 = 1 \mod 3$ and $p_6 + z = 1 \mod 3$. Shifting μ_2 first increases t by $\frac{\mu_2-1}{3}$ and makes all knights low. Then shifting by $p_6\gamma_6+ \mid z \mid$ increases $t + \frac{\mu_2-1}{3}$ by $\frac{2(p_6+z-1)}{3} + 1$ and makes all knights high. Thus, after shifting μ_2 then $p_6\gamma_6+ \mid z \mid$, the minimum α_2 is $t + \frac{\mu_2-1}{3} + \frac{2(p_6+z-1)}{3} + 1$. Now shifting $p_6\gamma_6+ \mid z \mid$ first, increases t by $\frac{2(p_6+z-1)}{3} + 1$ and makes all knights mid. Then shifting by μ_2 increases $t + \frac{2(p_6+z-1)}{3} + 1$ by $\frac{\mu_2-1}{3}$ and makes all knights high. Thus, shifting by $p_6\gamma_6+ \mid z \mid$ then by μ_2 gives a minimum α_1 of $t + \frac{\mu_2-1}{3} + \frac{2(p_6+z-1)}{3} + 1$. Thus both orderings produce the same results.

Let $\mu_2 = 1 \mod 3$ and $p_6 + z = 2 \mod 3$. Shifting μ_2 first increases t by $\frac{\mu_2-1}{3}$ and makes all knights low. Then shifting by $p_6\gamma_6+ \mid z \mid$ increases $t + \frac{\mu_2-1}{3}$ by $\frac{2(p_6+z-2)}{3} + 2$ and makes all knights mid. Thus, after shifting μ_2 then $p_6\gamma_6+ \mid z \mid$, the minimum α_2 is $t + \frac{\mu_2-1}{3} + \frac{2(p_6+z-2)}{3} + 2$. Now shifting $p_6\gamma_6+ \mid z \mid$ first, increases t by $\frac{2(p_6+z-2)}{3} + 1$ and makes all knights low. Then shifting by μ_2 increases $t + \frac{2(p_6+z-2)}{3} + 1$ by $\frac{\mu_2-1}{3} + 1$ and makes all knights mid. Thus, shifting by $p_6\gamma_6+ \mid z \mid$ then by μ_2 gives a minimum α_1 of $t + \frac{\mu_2-1}{3} + \frac{2(p_6+z-2)}{3} + 2$. Thus both orderings produce the same results.

Let $\mu_2 = 2 \mod 3$ and $p_6 + z = 0 \mod 3$. Shifting μ_2 first increases t by $\frac{\mu_2+1}{3}$ and makes all knights mid. Then shifting by $p_6\gamma_6+ \mid z \mid$ increases $t + \frac{\mu_2+1}{3}$ by $\frac{2(p_6+z)}{3}$ and makes all knights mid. Thus, after shifting μ_2 then $p_6\gamma_6+ \mid z \mid$, the minimum α_2 is $t + \frac{\mu_2+1}{3} + \frac{2(p_6+z)}{3}$. Now shifting $p_6\gamma_6+ \mid z \mid$ first, increases t by $\frac{2(p_6+z)}{3}$ and makes all knights high. Then shifting by μ_2 increases $t + \frac{2(p_6+z)}{3}$ by $\frac{\mu_2+1}{3}$ and makes all knights mid. Thus, shifting by $p_6\gamma_6+ \mid z \mid$ then by μ_2 gives a minimum α_1 of $t + \frac{\mu_2+1}{3} + \frac{2(p_6+z)}{3}$. Thus both orderings produce the same results.

Let $\mu_2 = 2 \mod 3$ and $p_6 + z = 1 \mod 3$. Shifting μ_2 first increases t by $\frac{\mu_2+1}{3}$ and makes all knights mid. Then shifting by $p_6\gamma_6+ \mid z \mid$ increases $t + \frac{\mu_2+1}{3}$ by $\frac{2(p_6+z-1)}{3}$ and makes all knights low. Thus, after shifting μ_2 then $p_6\gamma_6+ \mid z \mid$, the minimum α_2 is $t + \frac{\mu_2+1}{3} + \frac{2(p_6+z-1)}{3}$. Now shifting $p_6\gamma_6+ \mid z \mid$ first, increases t by $\frac{2(p_6+z-1)}{3} + 1$ and makes all knights mid. Then shifting by μ_2 increases $t + \frac{2(p_6+z-1)}{3} + 1$ by $\frac{\mu_2+1}{3} - 1$ and makes all knights mid. Thus, shifting by $p_6\gamma_6+ \mid z \mid$ then by μ_2 gives a minimum α_1 of $t + \frac{\mu_2+1}{3} + \frac{2(p_6+z-1)}{3}$.

Thus both orderings produce the same results.

Let $\mu_2 = 2 \mod 3$ and $p_6 + z = 2 \mod 3$. Shifting μ_2 first increases t by $\frac{\mu_2+1}{3}$ and makes all knights mid. Then shifting by $p_6\gamma_6+ \mid z \mid$ increases $t + \frac{\mu_2+1}{3}$ by $\frac{2(p_6+z-2)}{3} + 1$ and makes all knights high. Thus, after shifting μ_2 then $p_6\gamma_6+ \mid z \mid$, the minimum α_2 is $t + \frac{\mu_2+1}{3} + \frac{2(p_6+z-2)}{3} + 1$. Now shifting $p_6\gamma_6+ \mid z \mid$ first, increases t by $\frac{2(p_6+z-2)}{3} + 1$ and makes all knights low. Then shifting by μ_2 increases $t + \frac{2(p_6+z-2)}{3} + 1$ by $\frac{\mu_2+1}{3}$ and makes all knights mid. Thus, shifting by $p_6\gamma_6+ \mid z \mid$ then by μ_2 gives a minimum α_1 of $t + \frac{\mu_2+1}{3} + \frac{2(p_6+z-2)}{3} + 1$. Thus both orderings produce the same results.

Thus, the ordering for any shifting does not matter. □

It is time to put all of these pieces together.

Notation 51. The notation $\xi[+\mu_1][+\mu_2][+ \mid z \mid][+k\gamma_6]$ refers to the α_2-level of size $2\lambda_1 + \lambda_2 - 2\mu_1 - \mu_2 + z$ with $p_6 = k$ if $z \leq 0$ and $p_6 = k + z$ if $z \geq 0$.

Theorem 52. *The mapping $\psi^{s_2} : \Gamma^{s_2} \to \Gamma^{id}$ is well-defined.*

Proof. Consider any arbitrary α_2-level. Throughout this proof, μ_1, μ_2 refer to their value for the arbitrary α_2-level. By lemma 40 we know that the top region is well-defined. By proposition 43 we know that for any p in the bottom region, $p_1' \geq$ minimum α_1 attained by the k-boundaries. So if every member of the k-boundaries is well-defined, then so too will every member of the bottom region.

We know that the size of the α_2-level is equal to $2\lambda_1 + \lambda_2 - 2\mu_1 - \mu_2 + z$ for some integer z. First suppose that $z \leq 0$. By proposition 44 we know that ξ is well-defined. By lemma 45 we know that the minimum α_1 attained by $\xi[+\mu_1]$ is still zero. And by lemma 47 we know that the minimum α_1 attained by $\xi[+\mu_1][+ \mid z \mid]$ is also still zero. And by proposition 50, the following table properly lists the minimum α_1 for $\xi[+\mu_1][+\mu_2][+ \mid z \mid][+k\gamma_6]$

$(\mu_2 \mod 3)$	$(p_6 \mod 3)$	minimum α_1
0	0	$\frac{\mu_2}{3} + \frac{2(p_6)}{3}$
0	1	$\frac{\mu_2}{3} + \frac{2(p_6-1)}{3} + 1$
0	2	$\frac{\mu_2}{3} + \frac{2(p_6-2)}{3} + 1$
1	0	$\frac{\mu_2-1}{3} + \frac{2(p_6)}{3}$
1	1	$\frac{\mu_2-1}{3} + \frac{2(p_6-1)}{3} + 1$
1	2	$\frac{\mu_2-1}{3} + \frac{2(p_6-2)}{3} + 2$
2	0	$\frac{\mu_2+1}{3} + \frac{2(p_6)}{3}$
2	1	$\frac{\mu_2+1}{3} + \frac{2(p_6-1)}{3}$
2	2	$\frac{\mu_2+1}{3} + \frac{2(p_6-2)}{3} + 1$

In each case of course, $t \geq 0$.

Thus, for all α_2-levels with $z \leq 0$, ψ^{s_2} is well-defined.

Now suppose that $z \geq 0$. By lemma 45 we know that the minimum α_1 attained by $\xi[+\mu_1]$ is still zero. And by proposition 50, the following table properly lists the minimum α_1 for $\xi[+\mu_1][+\mu_2][+\mid z \mid][+k\gamma_6]$

$(\mu_2 \mod 3)$	$(p_6 + z \mod 3)$	minimum α_1
0	0	$\frac{\mu_2}{3} + \frac{2(p_6+z)}{3}$
0	1	$\frac{\mu_2}{3} + \frac{2(p_6+z-1)}{3} + 1$
0	2	$\frac{\mu_2}{3} + \frac{2(p_6+z-2)}{3} + 1$
1	0	$\frac{\mu_2-1}{3} + \frac{2(p_6+z)}{3}$
1	1	$\frac{\mu_2-1}{3} + \frac{2(p_6+z-1)}{3} + 1$
1	2	$\frac{\mu_2-1}{3} + \frac{2(p_6+z-2)}{3} + 2$
2	0	$\frac{\mu_2+1}{3} + \frac{2(p_6+z)}{3}$
2	1	$\frac{\mu_2+1}{3} + \frac{2(p_6+z-1)}{3}$
2	2	$\frac{\mu_2+1}{3} + \frac{2(p_6+z-2)}{3} + 1$

In each case of course, $t \geq 0$.

Thus, for all α_2-levels, ψ^{s_2} is well-defined. But since that describes all of Γ^{s_2} for every possible combination λ, μ, it proves the theorem. $\square$

Theorem 53. *$\psi^{s_2}(\Gamma^{s_2}) \subset \Gamma^{id}$.*

Proof. This is a direct consequence of Proposition 39 and Theorem 52. $\square$

Now that all the necessary properties regarding ψ^{s_2} have been proved, all that remains to be shown is that the sets $\Gamma^{id}_{s_1 \backslash s_2 s_1}$ and $\Gamma^{id}_{s_2 \backslash s_1 s_2}$ do not intersect, where $\Gamma^{id}_{s_2 \backslash s_1 s_2} = \psi^{s_2}(\Gamma^{s_2} \backslash \Gamma^{s_2}_{s_1 s_2})$. The foundation for this has already been established in theorem 52.

Theorem 54. *$\Gamma^{id}_{s_1 \backslash s_2 s_1} \cap \Gamma^{id}_{s_2 \backslash s_1 s_2} = \varnothing$.*

Proof. We proceed by taking an α_2-level in Γ^{s_2} and showing that under the mapping ψ^{s_2}, it does not intersect $\Gamma^{id}_{s_1 \backslash s_2 s_1}$. First, if p is in the top region, then $p'_4 \geq \lambda_2 + 1$. But, equation (7.20) states that for every root combination q in $\Gamma^{id}_{s_1 \backslash s_2 s_1}$, $q_4 \leq \lambda_2$. Thus, for any p in the top region, $\psi^{s_2}(p) \notin \Gamma^{id}_{s_1 \backslash s_2 s_1}$.

Now for p anywhere else in the α_2-level, we know that $p'_1 \geq$ the minimum α_1 for the α_2-level. Thus, for there to be any possibility of intersection from the α_2-level, the condition that $p'_3 \geq n + 1$ would have to be satisfied, where n is defined by equation (7.18). This is the case, because if $p'_3 \geq \lambda_1 + 1$ for any p in Γ^{s_2}, then $p \in \Gamma^{s_2}_{s_1 s_2}$, and thus does not need to be considered. Further, since the minimum α_1 is set by the α_2-level, we simply replace p'_1 with t, where t equals the minimum α_1 attained, without loss of generality.

In the case of ξ, $t = 0$, and so $n = 0$. Thus, for intersection to occur, $p_3' \geq 1$. But $p_3' = 0$ for all p in ξ. Thus, the images do not intersect.

Now, relying on the same shifting used to prove that ψ^{s_2} was well-defined, we show that in any α_2-level, it will always be the case that $p_3' \leq n$.

First, shifting by $+\mu_1$ fixes both t and p_3. Thus, n remains constant since the size automatically shifts to accommodate the shift in μ_1 and t does not change. Thus, given any α_2-level in which $p_3' \leq n$, it follows that for any α_2-level $[+\mu_1]$, $p_3' \leq n$ still.

Next, shifting by $+\mu_2$ fixes p_3, but it increases t by $\lfloor \frac{\mu_2+1}{3} \rfloor$. And as the shift accommodates the change in weight class as well, n shifts to $n + 2(\lfloor \frac{\mu_2+1}{3} \rfloor)$. And as p_3 remains constant, we still have $p_3' \leq n$.

When decreasing the weight class by z for an α_2-level whose weight class is $\leq 2\lambda_1 + \lambda_2 - 2\mu_1 - \mu_2$, we leave t fixed, but p_3 is increased by z. Also, since $\mid p \mid$ has been reduced by z, it follows that n is increased by z. Totaled together and we have that $p_3' \leq n \rightarrow p_3' + z \leq n + z$. So, intersection still cannot occur.

When increasing the weight class by z for an α_2-level whose weight class is $\geq 2\lambda_1 + \lambda_2 - 2\mu_1 - \mu_2$, we leave p_3 fixed, but t is increased by:

$$\begin{aligned} t &= \frac{2z}{3} && \text{when } z = 0 \mod 3 \\ t &= \frac{2(z-1)}{3} + 1 && \text{when } z = 1 \mod 3 \\ t &= \frac{2(z-2)}{3} + 1 && \text{when } z = 2 \mod 3 \end{aligned}$$

And since the weight class is increasing, it follows that $n = 2t - z$. So, for each possible expression for t we have:

$$\begin{aligned} z = 0 \mod 3 &\quad \rightarrow \quad n = \frac{z}{3} \geq 0 \\ z = 1 \mod 3 &\quad \rightarrow \quad n = \frac{z+2}{3} \geq 0 \\ z = 2 \mod 3 &\quad \rightarrow \quad n = \frac{z-2}{3} \geq 0 \text{ as } z \geq 2 \end{aligned}$$

But since $p_3' = 0$, it follows that $p_3' \leq n$.

This leaves only the shift $+p_6\gamma_6$ to consider. In this case, the weight class remains fixed, p_3 increases the same amount that p_6 does, and t increases like so:

$$\begin{aligned} t &= \frac{2p_6}{3} && \text{when } p_6 = 0 \mod 3 \\ t &= \frac{2(p_6-1)}{3} + 1 && \text{when } p_6 = 1 \mod 3 \\ t &= \frac{2(p_6-2)}{3} + 1 && \text{when } p_6 = 2 \mod 3 \end{aligned}$$

And since the weight class remains the same, $n = 2t$. But as the following are true:

$$\begin{aligned} p_6 &\leq \frac{4z}{3} && \text{when } z = 0 \mod 3 \\ p_6 &\leq \frac{4(z-1)}{3} + 2 && \text{when } z = 1 \mod 3 \\ p_6 &\leq \frac{4(z-2)}{3} + 2 && \text{when } z = 2 \mod 3 \end{aligned}$$

It follows that $p' + 3 \leq n \to p'_3 + p_6 \leq n + 2t$.

Thus, if there is no intersection for a particular α_2-level, any α_2-level that can be reached via shifting will not produce any intersection either. But as all α_2-levels can be reached via shifting from ξ, given any $p \in \Gamma^{s_2} \backslash \Gamma^{s_2}_{s_1 s_2}$, $\psi^{s_2}(p) \notin \Gamma^{id}_{s_1 \backslash s_2 s_1}$. Therefore, $\Gamma^{id}_{s_1 \backslash s_2 s_1} \cap \Gamma^{id}_{s_2 \backslash s_1 s_2} = \varnothing$. □

With all the mappings properly defined, and all the requisite properties proven, we may now define the set of root combinations in Γ^{id} that are used for the Lusztig q-analogue.

Theorem 55. *In G_2, we may simplify the Lusztig q-analogue polynomial to:*

$$K_{\lambda\mu}(q) = \sum_{p \in \Gamma_0^{id}} q^{|p|} \tag{7.21}$$

where

$$\Gamma_0^{id} = \{p \in \Gamma^{id} \mid p \notin \Gamma^{id}_{s_1 \backslash s_2 s_1} \text{ AND } p \notin \Gamma^{id}_{s_2 \backslash s_1 s_2}\} \tag{7.22}$$

Now we develop formulae to compute the Lusztig q-analogue polynomials directly. This requires a detailed analysis of the structure of root combinations in G_2. But most of the work detailing that structure has already been accomplished. However, given that there are three size-preserving relations between the positive roots, the formula to count the number of root combinations per size will be far more complicated than in the case of C_2 or B_2.

First, we discern that the shape of every α_1-level in any $\Gamma^{\beta,s}$ is that of a triangle. Recall that an α_1-level is every root combination of the same size that differ only by χ_2 and χ_3. Thus, for p and q in the same α_1-level, $p_1 = q_1$.

Let τ^1 be the root combination in the α_1-level such that for any other root combination p in the α_1-level, $p \preccurlyeq \tau^1$. We say τ^1 is the top root combination in the α_1-level.

Lemma 56. *The number of root combinations in the α_1-level with top root combination in the α_1-level of τ^1 is:*

$$\Delta(\lfloor \frac{\tau_4^1}{2} \rfloor + 1) \text{ where } \Delta(x) = \frac{(x)(x+1)}{2} \tag{7.23}$$

Proof. We proceed constructively. The α_1-level can be split into rows and columns, where the rows contain root combinations that differ only by χ_2 and the columns contain root combinations that differ only by χ_3. Since moving in the negative χ_2 and χ_3 directions subtracts the number of γ_4, and nothing else, and they both always subtracts 2 γ_4, the number τ_4^1 will be all that is needed to count the root combinations.

First, consider all root combinations in the same row as τ^1. There are a total of $\lfloor \frac{\tau_4^1}{2} \rfloor + 1$ root combinations in that row. Similarly, in the row with $\tau^1 - \chi_3$, there are a total of $\lfloor \frac{\tau_4^1 - 2}{2} \rfloor + 1 = \lfloor \frac{\tau_4^1}{2} \rfloor$ root combinations. So, we see that in the row with $\tau^1 - r\chi_3$, there are a total of $\lfloor \frac{\tau_4^1}{2} \rfloor + 1 - r$ root combinations.

So, each row has one less root combination than the one before it. Thus, the sum of every root combination will be the $(\lfloor \frac{\tau_4^1}{2} \rfloor + 1)$th triangle number, as it is the largest row in the α_1-level. □

With this lemma established, the only thing left to do is to discern a method for finding the top root combination in the α_1-level for every α_1-level. Clearly, since p_3 and p_6 are only changed by χ_3, one or the other will always be zero for the top root combination in an α_1-level. Also, either p_2 or p_5 will be zero. Since whether p_3 or p_6 is zero is determined by the size of the root combinations, we focus instead on when p_2 or p_5 is zero.

Consider the top root combination for the size, τ. Suppose that $\tau_2 = 0$. Then τ lies on the p_2 edge, and so the top root combination in the α_1-level one greater will be $\tau - \chi_1 - 2\chi_2$. Further, we have that $(\tau - \chi_1 - 2\chi_2)_4 = \tau_4 - 3$. So, at each higher α_1-level, when $\tau_2^1 = 0$, the value of τ_4^1 has decreased by three.

So, when $\tau_2 = 0$, the total number of root combinations the same size is:

$$\sum_{j=0}^{\lfloor \frac{\tau_4}{3} \rfloor} \Delta\Big(\lfloor \frac{\tau_4 - 3j}{2} \rfloor + 1\Big) \tag{7.24}$$

Now, consider when $\tau_5 = 0$. Then, for $0 \leq r \leq \lfloor \frac{\tau_2}{2} \rfloor$, $\tau - r\chi_1$ is the top root combination in its α_1-level. And here, $(\tau - r\chi_1)_4 = \tau_4 + r$. Thus, the total number of root combinations in the α_1-levels spanned by τ to $\tau - r\chi_1$ is:

$$\sum_{i=\tau_4}^{\tau_4 + \lfloor \frac{\tau_2}{2} \rfloor} \Delta\Big(\lfloor \frac{i}{2} \rfloor + 1\Big) \tag{7.25}$$

But, that is not all the root combinations of that size. That is only part of them. Note that $[\tau - (\lfloor \frac{\tau_2}{2} \rfloor)\chi_1]_2 = 0$ or 1. It is 0 when τ_2 is even, and 1 when τ_2 is odd. First, we consider what happens when τ_2 is even. In this case, we have $[\tau - (\frac{\tau_2}{2})\chi_1]_2 = 0$, and so, a modified version of equation (7.24) is necessary to account for the remaining root combinations. However, as the $j = 0$ case is already accounted for in equation (7.25), we start the new sum out at $j = 1$. So, equation (7.24) becomes:

$$\sum_{j=1}^{\lfloor \frac{\tau_4 + \tau_2/2}{3} \rfloor} \Delta\Big(\lfloor \frac{\tau_4 + \tau_2/2 - 3j}{2} \rfloor + 1\Big) \tag{7.26}$$

Now, when p_2 is odd, we have $[\tau - (\frac{\tau_2 - 1}{2})\chi_1]_2 = 1$. So to reach the top root combination for the next higher α_1-level, we need to add $-\chi_1 - \chi_2$. And as $[\tau - (\frac{\tau_2}{2})\chi_1 - \chi_1 - \chi_2]_2 = 0$, we may now modify equation (7.24) to account for the remaining root combinations. Recalling that $-\chi_1 - \chi_2$ decreases the value of p_4 by one, we total the remaining root combinations by:

$$\sum_{j=0}^{\lfloor \frac{\tau_4 + (\tau_2 - 1)/2 - 1}{3} \rfloor} \Delta\Big(\lfloor \frac{\tau_4 + (\tau_2 - 1)/2 - 1 - 3j}{2} \rfloor + 1\Big) \tag{7.27}$$

Now, putting all of this together, we can state a general formula that counts every root combination of a particular size from just knowing the top root combination τ.

$$\Omega(\tau) = \Omega(\tau_2, \tau_4) = \begin{cases} \sum_{i=\tau_4}^{\tau_4 + \frac{\tau_2}{2}} \Delta\Big(\lfloor \frac{i}{2} \rfloor + 1\Big) + \sum_{j=1}^{\lfloor \frac{\tau_4 + \tau_2/2}{3} \rfloor} \Delta\Big(\lfloor \frac{\tau_4 + \tau_2/2 - 3j}{2} \rfloor + 1\Big) & \tau_2 \text{ even} \\ \sum_{i=\tau_4}^{\tau_4 + \frac{\tau_2 - 1}{2}} \Delta\Big(\lfloor \frac{i}{2} \rfloor + 1\Big) + \sum_{j=0}^{\lfloor \frac{\tau_4 + (\tau_2 - 1)/2 - 1}{3} \rfloor} \Delta\Big(\lfloor \frac{\tau_4 + (\tau_2 - 1)/2 - 1 - 3j}{2} \rfloor + 1\Big) & \tau_2 \text{ odd} \end{cases} \tag{7.28}$$

In order to develop exact statements about the top root combinations for every size in each Γ^β, we need to recall a few more facts from the earlier discussion on G_2. First, for all root combinations of size $2\beta_1 + \beta_2$,

$p_3 = p_6$. Decreasing the size below that forced $p_3 \geq p_6$, while increasing beyond it made $p_3 \leq p_6$. Now recall Proposition 32, which stated the top root combination was characterized by having one member of the following pairs always be zero: either p_1 or p_4, p_2 or p_5, and p_3 or p_6. So, we will know that we have found the top root combination when that criterion and equations (7.6) and (7.7) are satisfied.

However, Proposition 32 also implies that the formulae will differ between the root combinations. After all, the top root combination depends on which member of each pair is zero, and that will depend on the weight of the root combinations. So, while there will be similarities in the formulae, they will differ.

So, we have the following formulae for τ based on the weight β.

For $w = id$:

$$\tau = \begin{cases} (0, 0, -x, 3\beta_1^{id} + \beta_1^{id} + 3x, -\beta^{id} - x, 0) & x \leq -\beta_1^{id} \\ (0, \beta^{id} + x, -x, \beta_1^{id} + \beta_1^{id} + x, 0, 0) & -\beta_1^{id} \leq x \leq 0 \\ (0, \beta^{id} + x, 0, \beta_1^{id} + \beta_1^{id} - x, 0, x) & 0 \leq x \leq \beta_1^{id} + \beta_2^{id} \\ (-\beta_1^{id} - \beta_2^{id} + x, 3\beta_1^{id} + 2\beta_2^{id} - x, 0, 0, 0, x) & \beta_1^{id} + \beta_2^{id} \leq x \end{cases} \tag{7.29}$$

For $w = s_1$:

$$\tau = \begin{cases} (0, 0, -x, 3\beta_1^{s_1} + \beta_2^{s_1} + 3x, -\beta_1^{s_1} - x, 0) & x \leq 0 \\ (0, 0, 0, 3\beta_1^{s_1} + \beta_2^{s_1} + x, -\beta_1^{s_1} - x, x) & 0 \leq x \leq -\beta_1^{s_1} \\ (0, \beta_1^{s_1} + x, \beta_1^{s_1} + \beta_2^{s_1} - x, 0, x) & -\beta_1^{s_1} \leq x \leq \beta_1^{s_1} + \beta_2^{s_1} \\ (-\beta_1^{s_1} - \beta_2^{s_1} + x, 3\beta_1^{s_1} + 2\beta_2^{s_1} - x, 0, 0, 0, x) & \beta_1^{s_1} + \beta_2^{s_1} \leq x \end{cases} \tag{7.30}$$

For $w = s_2$ when $\beta_1^{s_2} + \beta_2^{s_2} \leq 0$:

$$\tau = \begin{cases} (-\beta_1^{s_2} - \beta_2^{s_2} - x, 3\beta_1^{s_2} + 2\beta_2^{s_2} + 3x, -x, 0, 0, 0) & x \leq 0 \\ (-\beta_1^{s_2} - \beta_2^{s_2} + x, 3\beta_1^{s_2} + 2\beta_2^{s_2} - x, 0, 0, 0, x) & 0 \leq x \end{cases} \tag{7.31}$$

And for $w = s_2$ when $\beta_1^{s_2} + \beta_2^{s_2} > 0$:

$$\tau = \begin{cases} (-\beta_1^{s_2} - \beta_2^{s_2} - x, 3\beta_1^{s_2} + 2\beta_2^{s_2} + 3x, -x, 0, 0, 0) & x \leq -\beta_1^{s_2} - \beta_2^{s_2} \\ (0, \beta_1^{s_2} + x, -x, \beta_1^{s_2} + \beta_2^{s_2} + x, 0, 0) & -\beta_1^{s_2} - \beta_2^{s_2} \leq x \leq 0 \\ (0, \beta_1^{s_2} + x, 0, \beta_1^{s_2} + \beta_2^{s_2} - x, 0, x) & 0 \leq x \leq \beta_1^{s_2} + \beta_2^{s_2} \\ (-\beta_1^{s_2} - \beta_2^{s_2} + x, 3\beta_1^{s_2} + 2\beta_2^{s_2} - x, 0, 0, 0, x) & \beta_1^{s_2} + \beta_2^{s_2} \leq x \end{cases} \tag{7.32}$$

For $w = s_2 s_1$:

$$\tau = \begin{cases} (0, 0, 0, 3\beta_1^{s_2 s_1} + \beta_2^{s_2 s_1} + x, -\beta_1^{s_2 s_1} - x, x) & 0 \leq x \leq -\beta_1^{s_2 s_1} \\ (0, \beta_1^{s_2 s_1} + x, 0, \beta_1^{s_2 s_1} + \beta_2^{s_2 s_1} - x, 0, x) & -\beta_1^{s_2 s_1} \leq x \leq \beta_1^{s_2 s_1} + \beta_2^{s_2 s_1} \\ (-\beta_1^{s_2 s_1} - \beta_2^{s_2 s_1} - x, 3\beta_1^{s_2 s_1} + 2\beta_2^{s_2 s_1} - x, 0, 0, 0, x) & \beta_1^{s_2 s_1} + \beta_2^{s_2 s_1} \leq x \end{cases} \tag{7.33}$$

Lastly, for $w = s_1 s_2$:

$$\tau = \begin{cases} (-\beta_1^{s_2 s_1} - \beta_2^{s_2 s_1} - x, 3\beta_1^{s_2 s_1} + 2\beta_2^{s_2 s_1} + 3x, -x, 0, 0, 0) & x \leq 0 \\ (-\beta_1^{s_2 s_1} - \beta_2^{s_2 s_1} + x, 3\beta_1^{s_2 s_1} + 2\beta_2^{s_2 s_1} - x, 0, 0, 0, x) & 0 \leq x \end{cases} \tag{7.34}$$

It is clear that every equation used to calculate a top root combination for G_2 satisfies the conditions of proposition 32. Further, they all satisfy equations (7.6) and (7.7) for their respective $w \in W$. Thus, this describes the necessary information to directly compute the coefficients of the Lusztig q-analogue. Of course, not every example will contain every possible size. So whenever the equations for the top root combinations produce negative entries, they can be ignored.

Theorem 57. *The coefficient for q^s in the Lusztig q-analogue in G_2 is:*

$$\sum_{w \in W} (-1)^{l(w)} \Omega(\tau^w) \tag{7.35}$$

where $\tau^w \in \Gamma^{\beta(w),s}$ is the top root combination.

Example 5. Let $\lambda = 23\omega_1 + 17\omega_2$ and $\mu = 3\omega_1 + \omega_2$. We would like to find the coefficient for q^{33}.

In Γ^{id}, $2\beta_1^{id} + \beta_2^{id} = 56$, and so $x = 33 - 56 = -23$. We see that $-23 \leq -\beta_1^{id} = -20$, so the top root combination is $(0, 0, 23, 60 + 16 - 69, -20 + 23, 0) = (0, 0, 23, 7, 3, 0)$. So, the contribution from Γ^{id} is $\Omega(0, 7)$.

In Γ^{s_1}, $2\beta_1^{s_1} + \beta_2^{s_1} = 32$, and so $x = 1 \leq -\beta_1^{s_1} = 28$. So, the top root combination is $(0, 0, 0, 3(-28) + 88 + 1, 28 - 1, 1) = (0, 0, 0, 5, 27, 1)$. So the contribution from Γ^{s_1} is $\Omega(0, 5)$.

In Γ^{s_2}, $2\beta_1^{s_2} + \beta_2^{s_2} = 56$, and so $x = -23$. Further, $\beta_1^{s_2} + \beta_2^{s_2} = 18$. So the top root combination is $(-38 + 20 + 23, 3(38) + 2(-20) + 3(-23), 23, 0, 0, 0) = (5, 5, 23, 0, 0, 0)$. Thus, the contribution from Γ^{s_2} is $\Omega(5, 0)$.

In $\Gamma^{s_2 s_1}$, $2\beta_1^{s_2 s_1} + \beta_2^{s_2 s_1} = 14$, and so $x = 19 \leq -\beta_1^{s_2 s_1} = 46$. Thus the top root combination is $(0, 0, 0, 3(-46) + 106 + 19, 46 - 19, 19) = (0, 0, 0, -13, 27, 19)$. And so there is no contribution from $\Gamma^{s_2 s_1}$.

In $\Gamma^{s_1 s_2}$, $2\beta_1^{s_1 s_2} + \beta_2^{s_1 s_2} = 32$, and so $x = 1$. Thus, the top root combination is $(-62 + 92 + 1, 3(62) + 2(-92) - 1, 0, 0, 0, 1) = (31, 1, 0, 0, 0, 1)$. So, the contribution is $\Omega(1, 0)$.

So, all together we have $\Omega(0, 7) - \Omega(0, 5) - \Omega(5, 0) + \Omega(1, 0) = 17 - 9 - 6 + 1 = 3$. So the coefficient for q^{33} in the Lusztig q-analogue is 3.

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