

Analysis of the Benefits of Resource Flexibility,  
Considering Different Flexibility Structures

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(ABSTRACT)

We study the benefits of resource flexibility, considering two different flexibility structures. First, we want to understand the impact of the firm's pricing strategy on its resource investment decision, considering a partially flexible resource. Secondly, we study the benefits of a flexible resource strategic approach, considering a resource flexibility structure that has not been studied in the previous literature.

First, we study the capacity investment decision faced by a firm that offers two products/services and that is a price-setter for both products/services. The products offered by the firm are of varying levels (complexities), such that the resources that can be used to produce the higher level product can also be used to produce the lower level one. Although the firm needs to make its capacity investment decision under high demand uncertainty, it can utilize this limited (downward) resource flexibility, in addition to pricing, to more effectively match its supply with demand. Sample applications include a service company, whose technicians are of different capabilities, such that a higher level technician can perform all tasks performed by a lower level technician; a firm that owns a main plant, satisfying both end-product and intermediate-product demand, and a subsidiary, satisfying the intermediate-product demand

only. We formulate this decision problem as a two-stage stochastic programming problem with recourse, and characterize the structural properties of the firm's optimal resource investment strategy when resource flexibility and pricing flexibility are considered in the investment decision.

We show that the firm's optimal resource investment strategy follows a *threshold* policy. This structure allows us to understand the impact of coordinated decision-making, when the resource flexibility is taken into account in the investment decision, on the firm's optimal investment strategy, and establish the conditions under which the firm invests in the flexible resource. We also study the impact of demand correlation on the firm's optimal resource investment strategy, and show that it may be optimal for the firm to invest in both flexible and dedicated resources when product demand patterns are perfectly positively correlated. Our results offer managerial principles and insights on the firm's optimal resource investment strategy as well as extend the newsvendor problem with pricing, by allowing for multiple resources (suppliers), multiple products, and resource pooling.

Secondly, we study the benefits of a delayed decision making strategy under demand uncertainty, considering a system that satisfies two demand streams with two capacitated and flexible resources. Resource flexibility allows the firm to delay its resource allocation decision to a time when partial information on demands is obtained and demand uncertainty is reduced. We characterize the structure of the firm's optimal delayed resource allocation strategy. This characterization allows us to study how the revenue benefits of the delayed resource allocation strategy depend on demand and capacity parameters, and the length of the selling season. Our study shows that the revenue benefits of this strategy can be significant, especially when

demand rates of the different types are close, while resource capacities are much different. Based on our analysis, we provide guidelines on the utilization of such strategies.

Finally, we incorporate the uncertainty in demand parameters into our models and study the effectiveness of several delayed capacity allocation mechanisms that utilize the resource flexibility. In particular, we consider that demand forecasts are uncertain at the start of the selling season and are updated using a Bayesian framework as early demand figures are observed. We propose several heuristic capacity allocation policies that are easy to implement as well as a heuristic procedure that relies on a stochastic dynamic programming formulation and perform a numerical study. Our study determines the conditions under which each policy is effective.

Dedicated to my parents and family.

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## Chapter 1

# Introduction and Motivation

It is important to reduce the gap between a firm's supply and demand in any industry. The firm's resource investment and management decisions greatly impact its ability to match supply with demand. However, the resource investment decision in many industries is characterized by long investment lead-times and economies of scale in investment costs. As a result, this decision needs to be made early, using highly uncertain long-term demand forecasts, and is highly expensive and difficult to change later on. An example is the automotive industry, where the resource investment decision needs to be made 3-5 years before production starts, at a time when demand is highly uncertain - in the automotive industry, the demand forecast in the capacity investment stage deviates from the actual sales by 40% on average [Biller, Bish, and Muriel (2002); Jordan and Graves (1995)]. Similar examples can be found in several manufacturing and service industries.

Utilizing *resource flexibility*, which refers to the ability of a resource to produce multiple products (satisfy multiple service types), is one of the strategies that can yield large benefits

by hedging against demand uncertainty. However, it requires a high cost to acquire and operate a flexible system. Hence, we need to understand the benefits of resource flexibility in different environments and the key factors that affect the firm's resource investment strategy. In this dissertation, we study the benefits of flexible resource strategic approaches in managing various systems.

Resource flexibility allows the firm to delay its resource allocation decision to a time when more information on demands is obtained and demand uncertainty is reduced, thus providing a *risk pooling* effect. The value of delayed decision making strategies has been extensively studied in the academic literature in the context of delayed product differentiation, distribution, and component commonality [see, for instance, Aviv and Federgruen (2001a, 2001b); Garg and Tang (1997); Lee and Tang (1997); and the many references in Tayur, Ganeshan, and Magazine (1999)]. However, academic research that studies the investment and management issues of flexible resources is quite limited, and generally makes the assumption that the firm does not have pricing power. This assumption is being relaxed in some recent papers [Bish and Wang (2002); Chod and Rudi (2002); Fine and Freund (1990)]. In addition, most research that studies the value of resource flexibility focuses on the flexibility structures that are commonly encountered in manufacturing environments. The management of resource flexibility in a service environment will be quite different from that of a manufacturing environment due to the different characteristics/constraints that the service environment has. In most service environments, resources are perishable; that is, unused service capacity cannot be stored in terms of inventory. Thus, making use of resource flexibility in a service environment can offer potential benefits in matching supply with demand.

Based on these observations, in this dissertation we attempt to contribute to the academic literature on resource flexibility by studying its benefits, considering two different flexibility structures. First, we want to understand the impact of the firm's pricing strategy on its resource investment decision, considering a partially flexible resource. Secondly, we want to study the benefits of a flexible resource strategic approach, considering a resource flexibility structure that has not been previously studied in the literature. In the following, we present these two research directions in detail.

## 1.1 A Limited Resource Flexibility Structure

First, we study the resource investment decision faced by a firm that manufactures products of different levels (complexities), such that the resources that can be used to produce a higher level product can also be used to produce the lower level products, but not vice versa. Thus, the firm can utilize this limited resource flexibility, as needed, to satisfy demands for the multiple products that it offers. As an example, consider a firm that operates a main plant, which has the capability to produce for both the end-product and intermediate-product markets, and a subsidiary, which has only the capability to produce for the intermediate-product market (i.e., it can only produce components/parts of the end-product). This type of a relationship is illustrated in Figure 1.1. We consider the case where both the main plant and the subsidiary are owned by the same decision-maker/parent company. As an example of this situation, consider the case of GM and Allison Transmission. GM owns Allison Transmission, which supplies components, such as transit systems, to GM [<http://www.allisontransmission.com/>]. GM also has the capability to produce these components in house however. On the other hand,

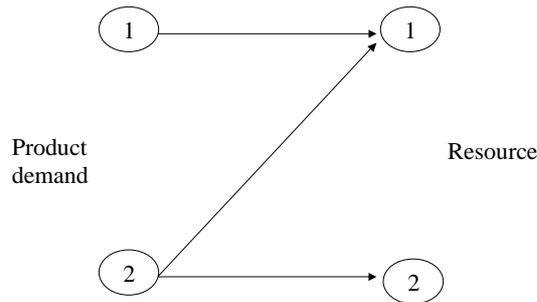


Figure 1.1: The relationship between resources and products under a limited flexibility structure

Allison Transmission lacks the capability of assembly and marketing, and thus, cannot compete in the end-product market. Hence, in this example, the end-product market represents the market for cars and the intermediate-product market that for components. Several other examples exist in supply chain networks. Our focus is on understanding the impact that coordination (centralized decision making) has on the firm's resource investment decision when the resources are owned by a single decision-maker, rather than studying the mechanism of sub-contracting when multiple players are involved. The latter situation is analyzed in Van Mieghem (1999) for a limited resource flexibility case, similar to the one described here.

Consequently, in this research we would like to answer the following research questions. What is the structure of the firm's resource investment decision in the *centralized* system, when both supply- and demand-side flexibilities (i.e., resource flexibility and the firm's pricing power) are considered and coordinated in the investment decision? How does the resource investment decision in the *centralized* system compare with that in the *de-centralized* system, in which the resource flexibility is not considered in the investment? How is the firm's resource investment decision affected by correlations between product demands? Our intent is to build new theory and intuition on the benefits of *centralized* decision making on the resource investment decision when the inherent flexibilities in the system, such as resource flexibility and pricing flexibility, are considered.

## 1.2 A Swappable Resource Flexibility Structure

Secondly, we study the benefits of resource flexibility in the context of a delayed resource allocation strategy, considering a different type of a resource flexibility structure. We consider a system that utilizes capacitated resources to satisfy demands for the different demand types such that each resource needs to be allocated entirely to one demand type. However, resources are flexible in that each resource can be allocated to each demand type. This type of flexibility can be found not only in manufacturing environments, but also in service environments. An example in a service environment includes a multiplex, consisting of theaters of different capacities, where each theater needs to be allocated to a different movie/play. In this case, one needs to determine which resource (i.e., theater) to allocate to which demand type (i.e., movie/play). As an example from a manufacturing environment, consider a plant that builds

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two products using two flexible production lines, each of which can manufacture each product. However, each product line may need to be assigned to only one product due to set-ups. As stated above, utilizing this type of a resource flexibility enables postponing the resource allocation decision to a time when more information on demands is obtained and demand uncertainty is reduced.

Implementing this type of a delayed allocation strategy may lead to some costs in certain settings. For example, if the capacity allocation decision at a movie theater complex is postponed to a time when some customers are already seated, then those customers may need to be reseated, leading to loss of goodwill. Consequently, our objective is to understand the impact of important factors, such as demand and capacity parameters, on the benefits of resource flexibility so as to derive insights and guidelines on the utilization of such strategies.

The remainder of this dissertation is organized as follows. Chapter 2 provides a brief overview of the related literature. Then, Chapters 3 and 4 discuss how we model and study these two problems and provide suggestions for further research in these areas.

## Chapter 2

# Literature Review

Our research spans several streams of literature, including research on resource flexibility and substitutable inventory systems, the price-setting newsvendor problem, delayed decision making, demand information updating, and coordination in supply chains. This chapter briefly reviews the literature that is most related to our study. Section 2.1 introduces the literature that concerns resource flexibility, while Section 2.2 focuses on the other relevant literature.

### 2.1 Resource Flexibility

*Flexibility* has received much attention in the economics and manufacturing literature [see the references in Beach et al. (2000); Jones and Ostroy (1984); Sethi and Sethi (1990)]. However, it is only recently that flexible resource investment and management issues have been incorporated into operations management models; see Van Mieghem (2003) for an excellent review of research in this area. Most recently, Bish and Wang (2002) consider a two-product

firm that operates in a monopolistic situation and employs a postponed pricing strategy. They study the structure of the firm's optimal resource investment portfolio in the presence of a *fully flexible* resource option, which refers to a resource with the ability to produce both products. They characterize the conditions under which the firm invests in the flexible resource. Their analysis shows that the firm can invest in the fully flexible resource even when demand patterns for the two products are perfectly positively correlated, but only when one product is *preferred* over the other one, where the definition of *preferred* depends on demand function parameters and dedicated resource investment costs. Their result extends the results in Van Mieghem (1998), who considers a similar problem under exogenously determined prices, and shows that flexible resource investment in the case of perfect positive correlation requires a price differential between the products. In addition, both papers show that in the case of perfectly negatively correlated demands, the firm does not always invest in the flexible resource. Fine and Freund (1990) study a similar problem using a scenario-based approach, and determine the conditions under which the firm invests in the flexible resource. Chod and Rudi (2002) quantify the value of resource flexibility considering price flexibility and clearing (i.e., the firm utilizes all the available resource capacity to satisfy demands). For this purpose, they consider that the firm can invest in one fully flexible resource, and compare it with the case where the firm invests in two dedicated resources. Goyal and Netessine (2004) consider two competing firms and study the firm's optimal technology choice (i.e., flexible versus dedicated capacity). In addition to the technology choice, each firm needs to determine its capacity investment and production levels. Goyal and Netessine characterize the optimal strategy of each firm using a game-theoretic approach and study the impact of competition on the firm's flexible resource

investment level.

Recognizing that the flexible resource can also be seen as a financial option that can be exercised after demand uncertainty is resolved, several researchers have used financial options theory to study this problem [see, for instance, Andreou (1990); Dangl (1999); Triantis and Hodder (1990)]. Specifically, Triantis and Hodder (1990) study a two-product model, considering that the firm only invests in one fully flexible resource. Their numerical results suggest that the value of the flexible resource increases as demand variability increases and the correlation between the demands decreases. In addition, they numerically show that flexibility does have some value when the demands are perfectly positively correlated, but have different variabilities. See also Caulkins and Fine (1990), Eberly and Van Mieghem (1997), and Harrison and Van Mieghem (1999) for multi-period extensions of the resource investment problem, considering a fully flexible resource; and Jordan and Graves (1995) for a multi-product multi-plant system, but with fixed capacity levels.

The above literature studies the resource investment decision considering a fully flexible resource structure, which can also apply to an arbitrary resource flexibility structure. *Limited resource flexibility*, such as the downward resource flexibility structure, in which resources that can produce higher level products can also be used to produce the lower level products, but not vice versa, also arises frequently in applications, such as in the context of the multi-plant, multi-market firm described in Section 1.1, as well as a rental car company, where customer requests can be upgraded and satisfied by a higher level resource (car) [Netessine, Dobson, and Shumsky (2002)]; and a semi-conductor company that produces different levels of circuits such that higher level circuits can be substituted for lower level circuits if there

is shortage for the latter [Bassok, Anupindi, and Akella (1999)]. However, all research that has considered the partially flexible resources has assumed that product/service prices are exogenously set, and thus are inputs to the resource investment/inventory ordering decision. Netessine, Dobson, and Shumsky (2002) study the impact of demand correlation on the firm's resource investment decision, considering *partially flexible* resources, where a higher value resource can be used to satisfy the demand for the next lower level resource. They show that for the case of two demand types, increasing correlation causes a shift from flexible to dedicated resource in the investment decision. Bassok, Anupindi, and Akella (1999) consider a similar flexibility structure, where demand for a specific product can be substituted by any product of higher value, and determine the optimal ordering levels. There is also some earlier literature on *substitutable inventory systems* [see Khouja (1999) for a review as well as Parlar (1988); Pasternak and Drezner (1991)]. Van Mieghem (1999) studies the resource investment decision in the presence of two resources and a limited flexibility structure, similar to the one described in Section 1.1, but in a different setting, since he considers that each resource is owned by a different player. Thus, his focus is on analyzing the value of sub-contracting under different contract types in a game-theoretic setting.

Finally, the *swappable resource flexibility*, presented in Section 1.2, is much different from that studied in the academic literature. To our knowledge, research that studies the important factors that affect the value of swappable resource flexibility, as is done in this dissertation, is non-existent, with the exception of Bish, Suwandechochai, and Bish (2002), who analyze swappable resources in the airline industry, which operates quite differently from the service industries considered here.

## 2.2 Other Relevant Literature

Another stream that is related to our research is the literature on the price-setting newsvendor problem [see Petruzzi and Dada (1999) for a review]. Several variations of the price-setting newsvendor problem have been studied, including Petruzzi and Dada (2001), who study a two-supplier, two-market variation of it. However, their problem is different from the one studied in this dissertation in the sense that the offering of products to the markets is sequential and thus, the firm can update its forecasts based on the information obtained in the first market.

There is extensive research that studies the value of decision postponement strategies and coordination in supply chains [see Tayur, Ganeshan, and Magazine (2000) for an extensive review of research in this area as well as Van Mieghem and Dada (1999)]. Finally, although some researchers have recently started studying operations management issues in some service industries, including movie theaters [see, for instance, Eliashberg and Sawhney (1994); Eliashberg et al. (2000, 2001); Swami, Eliashberg, and Weinberg (1999); Swami, Puterman, and Weinberg (2001)], none of these researchers have studied the benefits of a delayed resource allocation strategy, as considered in this dissertation.

## Chapter 3

# The Firm's Investment Decision under a Limited Flexibility Structure

In this chapter, we describe how we model and analyze the firm's resource investment decision problem, considering the limited resource flexibility structure presented in Section 1.1. We consider a firm that seeks a coordinated resource investment, pricing, and resource allocation decision under demand uncertainty so as to maximize its expected profit. We model this decision problem as two-stage stochastic programming problem, as commonly done in the literature [see, for instance, Bish and Wang (2002); Chod and Rudi (2002); Fine and Freund (1990); Gupta, Gerchak, and Buzacott (1992); Van Mieghem (1998)]. In the first stage, the firm determines its resource investment decision under demand uncertainty so as to maximize its expected profit. Then, in the second stage, demand uncertainty is resolved and the firm jointly determines its pricing and resource allocation decisions. Our objective is to study the impact that considering resource flexibility has on the firm's investment decision.

The remainder of this chapter is organized as follows. Section 3.1 presents our model and assumptions. Section 3.2 introduces the *de-centralized* system, which acts as the base case, while Section 3.3 considers the *centralized* system, in which the option to invest in the flexible resource is considered in the investment decision. In Section 3.4, we analyze the impact of demand correlation on the value of resource flexibility considering perfectly positively correlated and perfectly negatively correlated demand patterns. Finally, in Section 3.5, we present our conclusions and suggest directions for further research in this area.

### 3.1 Model Description

We consider a firm that offers two products, since the case of two products is analytically tractable, while capturing the essential elements of the problem. We assume that each product serves a different market so that there is no consumer-driven substitution between the products (i.e., a consumer, demanding a specific product, will not switch to the other product based on the prices). As discussed in Section 1.1, an example of this situation includes the two-market firm, where the markets respectively correspond to the end-product market and the intermediate-product market. We consider that the firm acts as a price-setter in a monopolistic situation in both markets, and model the demand for each product as a downward sloping linear function of its own price, as is common in the literature [see for instance, Van Mieghem and Dada (1999) and the references there in]; that is, for  $i = 1, 2$ ,

$$d_i = \xi_i - \alpha_i p_i,$$

where  $d_i$  and  $p_i$  denote the demand and price of Product  $i$ , respectively,  $\alpha_i$  ( $> 0$ ) is the slope, and  $\xi_i$  is the intercept of the demand curve. The firm has the option to invest in a *flexible resource* (*Resource 1*) that can produce both products and a *dedicated resource* (*Resource 2*) that can produce Product 2 only; see Figure 1.1.

We model the firm's resource investment decision as a two-stage stochastic programming problem: In the first stage, the firm determines its resource investment decision,  $\vec{K} = (K_1, K_2)$ , where  $K_i$  denotes the investment level for Resource  $i$ , under long-term uncertainty for both product demands, so as to maximize its expected profit. In this stage, locations of the demand curves are uncertain; that is,  $\xi_i, i = 1, 2$ , are continuous random variables, each with positive support. This represents the uncertainty in the market sizes. Then, in the second stage, demand uncertainty is resolved (i.e., the realization,  $\epsilon_i$ , of random variable,  $\xi_i$ , is observed for  $i = 1, 2$ ), and the firm jointly determines its pricing (or contribution margin, given by the sales price minus the variable cost of production) and resource allocation decisions so as to maximize its revenue. We define  $s_i, i = 1, 2$ , as the total amount of Product  $i$  demand satisfied by the firm in the second stage using both the flexible and the dedicated resources. Let  $f_i(\cdot)$ ,  $F_i(\cdot)$ , and  $\bar{F}_i(\cdot)$  respectively denote the probability density function (pdf), cumulative distribution function (cdf), and the tail distribution of  $\xi_i, i = 1, 2$ . Thus, the firm seeks a coordinated resource investment, pricing, and resource allocation strategy to maximize its expected profit. As in the earlier literature, we assume that resource investment costs are linear – all our results readily extend to convex investment costs – and that the variable cost of satisfying each product is the same for each resource. Hence, without loss of generality, the variable cost of production is not incorporated into our revenue function, and prices correspond to

the contribution margins. Let  $c_i$  denote the unit investment cost of Resource  $i$ ,  $i = 1, 2$ . Throughout the chapter, we assume that the flexible resource is more expensive to invest in than the dedicated resource; that is,  $c_1 > c_2$ . This is a reasonable assumption, since the flexible resource is capable of producing both products, while the dedicated resource can only produce Product 2.

Our objective is to understand the impact of centralized decision making on the firm's optimal investment strategy. In the *centralized* system, the investment decision for both resources is made jointly. Thus, the firm considers the *flexibility*, which allows Resource 1 to be allocated to Product 2 demand if needed, in the investment decision. On the other hand, in the *de-centralized* system, the investment decision for each resource is made separately. Let  $Z^C$  and  $Z^D$  denote the expected profit in Stage 1, given by the expected revenue less the investment cost, in the centralized and de-centralized systems, respectively. In the following sections, we first analyze the structure of the firm's optimal investment strategy in the de-centralized and centralized systems.

## 3.2 De-centralized System

We first study the *de-centralized* system, in which the firm makes its resource investment decision independently for each resource; that is, the investment decision decomposes by product. The purpose of studying this system is to understand the impact of resource flexibility on the investment decision by comparing it with the *centralized* system. Thus, it acts as the base case. The investment problem for Product  $i$ ,  $i = 1, 2$ , in the *de-centralized* system can be

formulated as the following two-stage stochastic programming problem:

$$\textbf{Stage 1:} \quad \max_{K_i} \quad Z_i^D = E[\Pi_i^D(K_i)] - c_i K_i$$

$$\text{subject to} \quad K_i \geq 0$$

$$\textbf{Stage 2:} \quad \Pi_i^D(K_i) = \max_{p_i, s_i} \Pi_i(K_i) = \max_{p_i, s_i} p_i s_i$$

$$\text{subject to} \quad s_i \leq \epsilon_i - \alpha_i p_i \tag{3.1}$$

$$s_i \leq K_i \tag{3.2}$$

$$s_i, p_i \geq 0. \tag{3.3}$$

Thus, in Stage 1, the firm determines its resource investment level,  $K_i, i = 1, 2$ , so as to maximize its expected profit. Then, in Stage 2, given realization  $\epsilon_i$  of random variable  $\xi_i$  and the resource investment decision,  $K_i$ , the firm determines the price and the amount of demand to satisfy so as to maximize its expected revenue. In Stage 2 Problem, constraint (3.1) ensures that the demand satisfied does not exceed the induced market demand, constraint (3.2) is the resource capacity constraint, and constraints (3.3) are the non-negativity constraints for the price and demand satisfied. We observe that constraint (3.1) holds as an equality (i.e.,  $s_i = \epsilon_i - \alpha_i p_i$ ) in an optimal solution to Stage 2 Problem, since any solution with excess demand would be sub-optimal for the problem (i.e., price could be further increased to reduce demand to the capacity level). Thus, Stage 2 Problem for Product  $i, i = 1, 2$ , reduces to the following problem:

$$\begin{aligned} \text{Stage 2: } \Pi_i^D(K_i) &= \max_{p_i} \Pi_i(K_i) = \max_{p_i} p_i(\epsilon_i - \alpha_i p_i) \\ \text{subject to } \epsilon_i - \alpha_i p_i &\leq K_i \quad \leftarrow \gamma_i \end{aligned} \quad (3.4)$$

$$p_i \leq \frac{\epsilon_i}{\alpha_i} \quad (3.5)$$

$$p_i \geq 0, \quad (3.6)$$

where  $\gamma_i$  denotes the corresponding Lagrangian multiplier, and constraint (3.5) corresponds to the non-negativity constraint for the demand satisfied. It is easy to show that constraints (3.5) and (3.6) are not needed in the formulation; the optimal solution will always satisfy them.

Let  $p_i^D$  and  $\gamma_i^D$  denote the optimal solution to Product  $i$ 's Stage 2 Problem, and let  $s_i^D = \epsilon_i - \alpha_i p_i^D$ . Since  $\Pi_i(K_i)$  is strictly concave in  $p_i$  and constraint (3.4) is linear, the first-order Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient for optimality. These conditions lead to the following results.

$$p_i^D = \max\left(\frac{\epsilon_i}{2\alpha_i}, \frac{\epsilon_i - K_i}{\alpha_i}\right) \Rightarrow s_i^D = \min\left(\frac{\epsilon_i}{2}, K_i\right). \quad (3.7)$$

$$\gamma_i^D = \frac{1}{\alpha_i}(2\alpha_i p_i^D - \epsilon_i)^+ = \begin{cases} \frac{1}{\alpha_i}(\epsilon_i - 2K_i), & \text{if } \epsilon_i > 2K_i, \\ 0, & \text{otherwise.} \end{cases} \quad (3.8)$$

We let  $\vec{K}^D = (K_1^D, K_2^D)$  denote the optimal investment vector in the *de-centralized* system.

The following result characterizes the optimal investment decision in the *de-centralized* system.

**Theorem 1** *An investment level  $K_i^D$  is optimal to Product  $i$ 's problem,  $i = 1, 2$ , in the de-*

centralized system if and only if there exists a Lagrangian multiplier  $w_i \geq 0$  that satisfies the following first-order KKT conditions for the Stage 1 Problem:

$$E[\xi_i - 2K_i^D | \xi_i > 2K_i^D] \Pr(\xi_i > 2K_i^D) = \alpha_i(c_i - w_i) \quad (3.9)$$

$$K_i^D w_i = 0.$$

*Proof:* Consider the Stage 1 Problem:

$$\begin{aligned} \text{Stage 1: } \quad & \max_{K_i} \quad Z_i^D = E[\Pi_i^D(K_i)] - c_i K_i \\ & \text{subject to } \quad K_i \geq 0 \quad \leftarrow w_i, \end{aligned}$$

where  $w_i$  is the corresponding Lagrangian multiplier.

Using Equation (3.7), the expected profit for Product  $i, i = 1, 2$ , in Stage 1 can be written as:

$$\begin{aligned} Z_i^D &= E[\Pi_i^D(K_i)] - c_i K_i \\ &= \int_0^{2K_i} \frac{\epsilon_i^2}{4\alpha_i} f_i(\epsilon_i) d\epsilon_i + \int_{2K_i}^{\infty} \frac{K_i(\epsilon_i - K_i)}{\alpha_i} f_i(\epsilon_i) d\epsilon_i - c_i K_i. \end{aligned}$$

By Leibniz's rule,

$$\begin{aligned} \frac{\partial Z_i^D}{\partial K_i} &= \frac{\partial E[\Pi_i^D(K_i)]}{\partial K_i} - c_i = \int_{2K_i}^{\infty} \frac{(\epsilon_i - 2K_i)}{\alpha_i} f_i(\epsilon_i) d\epsilon_i - c_i \\ \Rightarrow \frac{\partial^2 Z_i^D}{\partial K_i^2} &= \frac{-2}{\alpha_i} \bar{F}_i(2K_i) < 0, \text{ for any } K_i > 0, \end{aligned}$$

considering any continuous distribution of  $\xi_i$  in  $[0, \infty)$ . Thus,  $Z_i^D$  is strictly concave in  $K_i$ , for  $K_i > 0$ . Therefore, since the constraint is linear, the following first-order KKT conditions are necessary and sufficient for optimality:

$$E[\xi_i - 2K_i^D | \xi_i > 2K_i^D] \Pr(\xi_i > 2K_i^D) = \alpha_i(c_i - w_i)$$

$$K_i^D w_i = 0.$$

$$w_i \geq 0.$$

This completes the proof. ■

Theorem 1 implies that if  $c_i < \frac{E[\xi_i]}{\alpha_i}$ ,  $i = 1, 2$ , then  $K_i^D$ , the optimal investment level for Resource  $i$  in the dedicated system, is the unique solution to:

$$E[\xi_i - 2K_i^D | \xi_i > 2K_i^D] \Pr(\xi_i > 2K_i^D) = \alpha_i c_i.$$

Otherwise (if  $c_i \geq \frac{E[\xi_i]}{\alpha_i}$ ),  $K_i^D = 0$ .

### 3.3 Centralized System

Next, we study the *centralized* system, which considers the resource flexibility in the investment decision. The two-stage stochastic programming formulation for the centralized system is as follows.

$$\text{Stage 1 (P1):} \quad \max_{K_1, K_2} \quad Z^C = E[\Pi^C(\vec{K})] - \sum_{i=1}^2 c_i K_i$$

subject to  $K_i \geq 0, i = 1, 2$

$$\text{Stage 2 (P2): } \Pi^C(\vec{K}) = \max_{p_1, p_2, s_1, s_2} \Pi(\vec{K}) = \max_{p_1, p_2, s_1, s_2} \sum_{i=1}^2 p_i s_i$$

$$\text{subject to } s_1 \leq K_1 \tag{3.10}$$

$$s_1 + s_2 \leq K_1 + K_2 \tag{3.11}$$

$$s_i \leq \epsilon_i - \alpha_i p_i, \quad i = 1, 2 \tag{3.12}$$

$$s_i, p_i \geq 0, \quad i = 1, 2.$$

In the Stage 2 Problem, constraints (3.10) and (3.11) are the resource capacity constraints, and constraints (3.12) are the demand constraints. Similar to the de-centralized system, constraints (3.12) would hold as equalities in an optimal solution, since any solution with excess demand would be sub-optimal for Problem P2. Hence, we can rewrite Problem P2 as follows.

$$\text{P2*}: \Pi^C(\vec{K}) = \max_{p_1, p_2} \Pi(\vec{K}) = \max_{p_1, p_2} \sum_{i=1}^2 p_i (\epsilon_i - \alpha_i p_i)$$

$$\text{subject to } \epsilon_1 - \alpha_1 p_1 \leq K_1 \tag{3.13}$$

$$\epsilon_1 - \alpha_1 p_1 + \epsilon_2 - \alpha_2 p_2 \leq K_1 + K_2 \tag{3.14}$$

$$p_1 \leq \frac{\epsilon_1}{\alpha_1} \tag{3.15}$$

$$p_2 \leq \frac{\epsilon_2}{\alpha_2} \tag{3.16}$$

$$p_i \geq 0, \quad i = 1, 2$$

where constraints (3.15) and (3.16) are the non-negativity constraints for the demands satisfied. Thus, Problem P2 reduces to an optimization problem having prices as the only decision variables. For convenience, we are going to use this equivalent formulation to characterize the structural properties of the Stage 2 Problem.

### 3.3.1 Characterization of the Optimal Pricing and Resource Allocation Strategy in Stage 2

In this section, we determine the optimal solution to Problem P2\*, the joint Stage 2 Problem in the centralized system. For this purpose, we decompose the demand space into the following disjoint sets (See Figure 3.1):

$$\begin{aligned}
\Omega_0 &= \{\xi_1 < 2K_1, \quad \xi_1 + \xi_2 < 2K_1 + 2K_2\} \\
\Omega_1 &= \{\xi_1 + \xi_2 > 2K_1 + 2K_2, \quad \alpha_1(\xi_2 - 2K_2) > \alpha_2(\xi_1 - 2K_1), \quad \alpha_1(\xi_2 - 2K_1 - 2K_2) < \alpha_2\xi_1\} \\
\Omega_2 &= \{\xi_2 > 2K_2, \quad \alpha_1(\xi_2 - 2K_2) < \alpha_2(\xi_1 - 2K_1)\} \\
\Omega_3 &= \{\xi_1 > 2K_1, \quad \xi_2 < 2K_2\} \\
\Omega_4 &= \{\alpha_1(\xi_2 - 2K_1 - 2K_2) > \alpha_2\xi_1\}, \tag{3.17}
\end{aligned}$$

where “,” corresponds to the logical operator “and”.

It is easy to show that function  $\Pi(\vec{K})$ , given by  $p_1(\epsilon_1 - \alpha_1 p_1) + p_2(\epsilon_2 - \alpha_2 p_2)$ , is strictly, jointly concave in  $p_1$  and  $p_2$ . Therefore, the first-order KKT conditions are necessary and sufficient for optimality. This leads to the following lemma.

**Lemma 1** *Given realizations  $\epsilon_1$  and  $\epsilon_2$  of random variables  $\xi_1$  and  $\xi_2$  and a resource invest-*

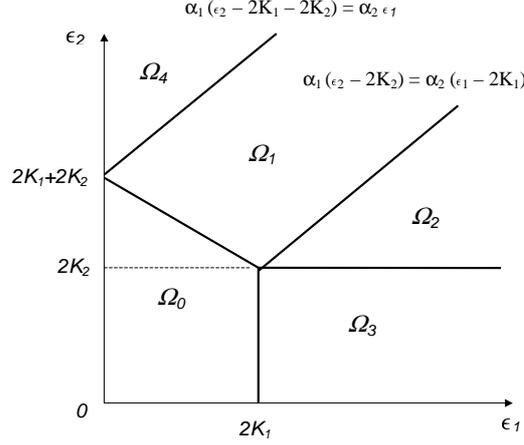


Figure 3.1: The demand space for Stage 2 Problem in the centralized system

ment vector  $\vec{K}$ ,  $\Pi^C(\vec{K})$ , the optimal solution value to Problem  $P2^*$ , can be expressed as:

$$\Pi^C(\vec{K}) = \begin{cases} \sum_{i=1}^2 \frac{\epsilon_i^2}{4\alpha_i}, & \text{if } \Omega_0 \\ \sum_{i=1}^2 \left( \frac{\epsilon_i}{2\alpha_i} + \frac{\epsilon_1 + \epsilon_2 - 2(K_1 + K_2)}{2(\alpha_1 + \alpha_2)} \right) \left( \frac{\epsilon_i}{2} - \frac{\alpha_i(\epsilon_1 + \epsilon_2 - 2(K_1 + K_2))}{2(\alpha_1 + \alpha_2)} \right), & \text{if } \Omega_1 \\ \sum_{i=1}^2 K_i \left( \frac{\epsilon_i - K_i}{\alpha_i} \right), & \text{if } \Omega_2 \\ K_1 \left( \frac{\epsilon_1 - K_1}{\alpha_1} \right) + \frac{\epsilon_2^2}{4\alpha_2}, & \text{if } \Omega_3 \\ (K_1 + K_2) \left( \frac{\epsilon_2 - K_1 - K_2}{\alpha_2} \right), & \text{if } \Omega_4 \end{cases}$$

*Proof:* See Appendix A. ■

In Lemma 1,  $\Omega_0$  corresponds to the set of demand realizations for which the optimal solution to the unconstrained problem is also optimal for the constrained problem, while all other sets correspond to optimal solutions on the boundary lines of the feasible region. Specifically,  $\Omega_1, \Omega_2$ , and  $\Omega_4$  are the sets whose optimal solutions use both resources fully, whereas in  $\Omega_3$ , only the flexible resource (Resource 1) is completely utilized. In sets  $\Omega_2$  and  $\Omega_3$ , each resource satisfies only the demand for its own product, whereas in set  $\Omega_4$ , both resources are used to satisfy the demand for Product 2.

### 3.3.2 Characterization of the Optimal Investment Strategy in Stage 1

Using the optimal solution to Problem P2\*, stated in Lemma 1, we can now characterize the optimal investment strategy in Stage 1. The firm seeks a coordinated strategy of investment, resource allocation, and pricing decisions in order to maximize its expected profit in Stage 1, given by:

$$\begin{aligned} \mathbf{P1:} \quad & \max_{K_1, K_2} \quad Z^C(\vec{K}) = E[\Pi(\vec{K})] - \sum_{i=1}^2 c_i K_i \\ & \text{subject to} \quad K_i \geq 0, \quad i = 1, 2 \quad \longleftarrow v_i \end{aligned}$$

Let  $v_i, i = 1, 2$ , represent the corresponding Lagrangian multipliers. We denote the maximizer of  $Z^C(\vec{K})$  by  $\vec{K}^C = (K_1^C, K_2^C)$ , which is the optimal investment vector in Stage 1 in the centralized system. We have the following results.

**Lemma 2**  $Z^C(\vec{K})$  is strictly, jointly concave in  $K_1, K_2$  for all continuous distributions of  $\xi_1$  and  $\xi_2$  in  $[0, \infty)$ .

*Proof:* See Appendix B. ■

Thus,  $\vec{K}^C$  is unique and the first-order KKT conditions are necessary and sufficient for optimality. These conditions lead to the following theorem.

**Theorem 2** *An investment vector  $\vec{K} = (K_1, K_2)$  is optimal for the centralized system if and only if there exists a Lagrangian multiplier vector  $\vec{v} = (v_1, v_2) \geq 0$  that satisfies the following conditions:*

$$\begin{aligned}
 \mathbf{KKT-1:} \quad & E\left[\frac{\xi_1 + \xi_2 - 2(K_1 + K_2)}{\alpha_1 + \alpha_2} \middle| \Omega_1\right] Pr(\Omega_1) + E\left[\frac{\xi_1 - 2K_1}{\alpha_1} \middle| \Omega_2\right] Pr(\Omega_2) \\
 & + E\left[\frac{\xi_1 - 2K_1}{\alpha_1} \middle| \Omega_3\right] Pr(\Omega_3) + E\left[\frac{\xi_2 - 2K_1 - 2K_2}{\alpha_2} \middle| \Omega_4\right] Pr(\Omega_4) = c_1 - v_1 \\
 \mathbf{KKT-2:} \quad & E\left[\frac{\xi_1 + \xi_2 - 2(K_1 + K_2)}{\alpha_1 + \alpha_2} \middle| \Omega_1\right] Pr(\Omega_1) \\
 & + E\left[\frac{\xi_2 - 2K_2}{\alpha_2} \middle| \Omega_2\right] Pr(\Omega_2) + E\left[\frac{\xi_2 - 2K_1 - 2K_2}{\alpha_2} \middle| \Omega_4\right] Pr(\Omega_4) = c_2 - v_2
 \end{aligned}$$

**Complementary Slackness:**  $v_i K_i = 0, \quad i = 1, 2.$

*Proof:* The proof directly follows from the first-order KKT conditions for the Stage 1 Problem.

■

Observe that the optimal investment strategy can be one of the following forms, each of which corresponds to a boundary solution for the Stage 1 Problem:

1.  $\vec{K}^{CN} = (K_1^{CN} = 0, K_2^{CN} = 0);$
2.  $\vec{K}^{C1} = (K_1^{C1} > 0, K_2^{C1} = 0);$
3.  $\vec{K}^{C2} = (K_1^{C2} = 0, K_2^{C2} > 0);$

$$4. \vec{K}^{CB} = (K_1^{CB} > 0, K_2^{CB} > 0).$$

Let  $\Omega_i^{CN}, \Omega_i^{C1}, \Omega_i^{C2}$ , and  $\Omega_i^{CB}$  respectively denote set  $\Omega_i$ , defined in Equation (3.17), at boundary solutions  $\vec{K}^{CN}, \vec{K}^{C1}, \vec{K}^{C2}$ , and  $\vec{K}^{CB}$ , for  $i = 0, 1, \dots, 4$ .

The next question is to determine the conditions under which each of these boundary solutions will be optimal. Theorem 3 shows that the optimal investment strategy follows a *threshold* policy.

**Theorem 3** Let  $c_1^{CN} \equiv E[\frac{\xi_1}{\alpha_1}] + E[\frac{\alpha_1 \xi_2 - \alpha_2 \xi_1}{\alpha_1 \alpha_2} | \Omega_4^{CN}] Pr(\Omega_4^{CN})$ ,  $c_2^{CN} \equiv E[\frac{\xi_2}{\alpha_2}]$ ,

$$c_1^d \equiv c_2 + E[\frac{\alpha_2(\xi_1 - 2K_1^{C1}) - \alpha_1 \xi_2}{\alpha_1 \alpha_2} | \Omega_2^{C1}] Pr(\Omega_2^{C1}), \text{ and}$$

$$c_1^u \equiv c_2 + E[\frac{\alpha_2 \xi_1 - \alpha_1(\xi_2 - 2K_2^{C2})}{\alpha_1 \alpha_2} | \Omega_2^{C2}] Pr(\Omega_2^{C2}) + E[\frac{\xi_1}{\alpha_1} | \Omega_3^{C2}] Pr(\Omega_3^{C2}).$$

Then, the optimal investment strategy is the following threshold policy:

If  $c_2 \geq c_2^{CN}$ , then

$$\vec{K}^C = \begin{cases} \vec{K}^{CN}, & \text{if } c_1 \geq c_1^{CN}, \\ \vec{K}^{C1}, & \text{if } c_1 < c_1^{CN}. \end{cases}$$

If  $c_2 < c_2^{CN}$ , then

$$\vec{K}^C = \begin{cases} \vec{K}^{C1}, & \text{if } c_1 \leq c_1^d, \\ \vec{K}^{C2}, & \text{if } c_1 \geq c_1^u, \\ \vec{K}^{CB}, & \text{if } c_1^d < c_1 < c_1^u. \end{cases}$$

*Proof:* See Appendix C. ■

Theorem 3 states that the firm does not invest in any resource if the unit investment costs for both resources are too expensive (i.e.,  $c_2 \geq c_2^{CN}$  and  $c_1 \geq c_1^{CN}$ ); invests only in the flexible resource if the unit investment cost for the dedicated resource is too expensive, but that of the flexible resource is not (i.e., if  $c_2 \geq c_2^{CN}$  and  $c_1 < c_1^{CN}$  or if  $c_2 < c_2^{CN}$  and  $c_1 \leq c_1^l$ ); invests only in the dedicated resource if the unit investment cost for the flexible resource is too expensive, but that of the dedicated resource is not (i.e.,  $c_2 < c_2^{CN}$  and  $c_1 \geq c_1^u$ ); and invests in both resources otherwise (i.e.,  $c_2 < c_2^{CN}$  and  $c_1^l < c_1 < c_1^u$ ).

Next we make use of Theorem 3 to gain insight on cases where the firm never invests in the flexible resource only. The following result will be used subsequently in our analysis.

**Lemma 3** (i) Boundary solution  $\vec{K}^{C1}$  is not a possible solution when  $\Omega_2^{C1} = \emptyset$ ; and (ii) boundary solution  $\vec{K}^{CB}$  is not a possible solution when  $\Omega_2^{CB} = \emptyset$  and  $\Omega_3^{CB} = \emptyset$ .

*Proof:* Consider boundary solution  $\vec{K}^{C1} = (K_1^{C1} > 0, K_2^{C1} = 0)$  with  $v_1 = 0$ . Observe that at this boundary solution,  $\Omega_3^{C1} = \{\xi_1 > 2K_1^{C1}, \xi_2 < 2K_2^{C1} = 0\} = \emptyset$ . Hence, when we also have  $\Omega_2^{C1} = \emptyset$ , the KKT conditions, given in Theorem 2, reduce to the following at solution  $\vec{K}^{C1}$ :

$$\begin{aligned} \mathbf{KKT-1:} \quad & E\left[\frac{\xi_1 + \xi_2 - 2K_1^{C1}}{\alpha_1 + \alpha_2} | \Omega_1^{C1}\right] Pr(\Omega_1^{C1}) + E\left[\frac{\xi_2 - 2K_1^{C1}}{\alpha_2} | \Omega_4^{C1}\right] Pr(\Omega_4^{C1}) = c_1 \\ \mathbf{KKT-2:} \quad & E\left[\frac{\xi_1 + \xi_2 - 2K_1^{C1}}{\alpha_1 + \alpha_2} | \Omega_1^{C1}\right] Pr(\Omega_1^{C1}) + E\left[\frac{\xi_2 - 2K_1^{C1}}{\alpha_2} | \Omega_4^{C1}\right] Pr(\Omega_4^{C1}) = c_2 - v_2 \\ & v_2 \geq 0. \end{aligned}$$

Then,  $c_1 = c_2 - v_2 \Rightarrow c_1 \leq c_2$ , which contradicts with our assumption that  $c_1 > c_2$ . Thus,  $\vec{K}^{C1}$  is not a possible solution in this case. The proof for boundary solution  $\vec{K}^{CB}$  is similar. ■

Next, we analyze a special case where  $Pr(\frac{\xi_1}{\alpha_1} < \frac{\xi_2}{\alpha_2}) = 1$ . This corresponds to the case where, if resource capacities were not constraining, then the price for Product 2 would be optimally set higher than that for Product 1; that is,  $p_2^{(u)} = \frac{\epsilon_2}{2\alpha_2} > p_1^{(u)} = \frac{\epsilon_1}{2\alpha_1}$ . Thus, the dedicated resource is “more desirable” to invest in than the flexible resource. As a result, it will never be optimal for the firm to invest in the flexible resource only (boundary solution  $\vec{K}^{C1}$ ). The following theorem describes the structure of the optimal investment strategy in this case.

**Theorem 4** *Consider the case where  $Pr(\frac{\xi_1}{\alpha_1} < \frac{\xi_2}{\alpha_2}) = 1$ . Let  $c_2^{CN} = E[\frac{\xi_2}{\alpha_2}]$  and  $c_1^u = c_2 + E[\frac{\alpha_2\xi_1 - \alpha_1(\xi_2 - 2K_2^{C2})}{\alpha_1\alpha_2} | \Omega_2^{C2}]Pr(\Omega_2^{C2}) + E[\frac{\xi_1}{\alpha_1} | \Omega_3^{C2}]Pr(\Omega_3^{C2})$ . Then, the structure of the optimal investment strategy is as follows:*

*If  $c_2 \geq c_2^{CN}$ , then  $\vec{K}^C = \vec{K}^{CN}$ .*

*If  $c_2 < c_2^{CN}$ , then*

$$\vec{K}^C = \begin{cases} \vec{K}^{C2}, & \text{if } c_1 \geq c_1^u, \\ \vec{K}^{CB}, & \text{if } c_1 < c_1^u. \end{cases}$$

*Furthermore, when the optimal solution is given by  $\vec{K}^{CB}$ , then  $\frac{K_1^{CB}}{\alpha_1} < \frac{K_2^{CB}}{\alpha_2}$ .*

*Proof:* See Appendix D. ■

Hence, Theorem 4 states that the firm does not invest in any resource if the unit investment cost for the dedicated resource is too expensive (i.e.,  $c_2 \geq c_2^{CN}$ ); invests only in the dedicated resource if the unit investment cost for the flexible resource is expensive, but that of the

dedicated resource is not (i.e.,  $c_2 < c_2^{CN}$  and  $c_1 \geq c_1^u$ ); and invests in both resources otherwise (i.e.,  $c_2 < c_2^{CN}$  and  $c_1 < c_1^u$ ). It is never optimal to invest in the flexible resource only in this case.

### 3.3.3 The Effect of Centralized Decision Making on the Resource Investment Strategy

In this section, we study the impact of considering resource flexibility on the firm's optimal resource investment strategy. For this purpose, we compare the optimal solutions in the de-centralized and centralized systems. Observe that when the optimal solution to the centralized system does not invest in either resource or invests only in the dedicated resource (i.e.,  $\vec{K}^C = \vec{K}^{CN}$  or  $\vec{K}^C = \vec{K}^{C2}$ ), then the investment levels under both the de-centralized and centralized systems will be identical. Thus, in the following we analyze the case where  $\vec{K}^C = \vec{K}^{C1}$  or  $\vec{K}^C = \vec{K}^{CB}$ , and compare the optimal investment levels in both systems. This is characterized in the following lemma.

**Lemma 4** *If  $\vec{K}^C = \vec{K}^{C1}$ , then  $K_1^D < K_1^{C1}$ . If  $\vec{K}^C = \vec{K}^{CB}$ , then  $K_1^D < K_1^{CB}$  and  $K_2^D > K_2^{CB}$ .*

*Proof:* See Appendix E. ■

Thus, if any investment in the flexible resource is made in the centralized system, then its investment level will be greater than that in the de-centralized system (i.e.,  $K_1^{C1} > K_1^D$ ,  $K_1^{CB} > K_1^D$ ), whereas the investment level for the dedicated resource will be lower in the centralized system (i.e.,  $K_2^{CB} < K_2^D$ ). This result is intuitive: Considering the resource flexibility in the investment decision increases the value of the flexible resource. Our findings extend the results

of Netessine, Dobson, and Shumsky [2002], who analyze the case when pricing flexibility is not considered in the investment decision.

	$c_2$	0.4	0.3	0.2	0.1
Threshold Values	$c_1^l$	0.6	0.45	0.3	0.15
	$c_1^u$	0.7	0.65	0.6	0.515
Optimal policy $(\vec{K}^{C1})$	$c_1$	0.5	0.4	0.25	0.12
	$\vec{K}^{C1}$	(0.203, 0)	(0.314, 0)	(0.549, 0)	(0.916, 0)
Optimal policy $(\vec{K}^{C2})$	$c_1$	0.8	0.7	0.65	0.55
	$\vec{K}^{C2}$	(0, 0.056)	(0, 0.128)	(0, 0.229)	(0, 0.402)
Optimal policy $(\vec{K}^{CB})$	$c_1$	0.65	0.5	0.4	0.3
	$\vec{K}^{CB}$	(0.053, 0.029)	(0.178, 0.039)	(0.255, 0.101)	(0.347, 0.229)

Optimal policy in the de-centralized system

$c_1$	$K_1^D$	$c_2$	$K_2^D$
0.12	0.714	0.1	0.402
0.25	0.347	0.2	0.229
0.3	0.255	0.3	0.128
0.4	0.112	0.4	0.056
$\geq 0.5$	0	$\geq 0.5$	0

Table 3.1: Threshold values and the optimal investment levels for the numerical example in Section 3.3.4

### 3.3.4 A Numerical Study

In this section, we verify our analytical results on the structure of the optimal investment strategy by a numerical study. In our numerical study, we consider that  $\alpha_1 = 2$ ,  $\alpha_2 = 1$ , and  $\xi_i$ ,  $i = 1, 2$ , are independent, exponentially distributed random variables having respective rates of  $\lambda_1 = 1$  and  $\lambda_2 = 2$ .

Using Theorem 3, we obtain the threshold values for the optimal investment policy as  $c_1^{CN} = 0.75$ ,  $c_2^{CN} = 0.5$ , and  $c_1^l$  and  $c_1^u$  as shown in Table 3.1. Figure 3.2 depicts the structure of the firm's optimal investment strategy: In region  $K^{C1}$ , the values of  $(c_1, c_2)$  are such that it is optimal for the firm to invest in Resource 1 only; in region  $K^{C2}$  it is optimal to invest in Resource 2 only; in region  $K^{CB}$  it is optimal to invest in both resources; and in region  $K^{CN}$  no resource should be invested in, as stated in Theorem 3.

Specifically, refer to Table 3.1, which shows how the optimal investment strategy changes as the values of  $c_1$  and  $c_2$  are varied. We observe the following in the centralized system.

1. Both threshold values,  $c_1^l$  and  $c_1^u$ , are increasing in  $c_2$  (refer to the first three rows of Table 3.1), thus increasing the region in which the firm invests in Resource 1 only, while decreasing the region in which the firm invests in Resource 2 only.
2. For a given value of  $c_2$  ( $< c_2^{CN}$ ), in the centralized system the firm invests in Resource 1 only (solution  $\vec{K}^{C1}$ ) if  $c_1 \leq c_1^l$ ; in Resource 2 only (solution  $\vec{K}^{C2}$ ) if  $c_1 \geq c_1^u$ ; and in both resources (solution  $\vec{K}^{CB}$ ) if  $c_1^l < c_1 < c_1^u$ .
3. Keeping the value of  $c_2$  constant, the investment level of Resource 1 is decreasing in its unit investment cost,  $c_1$ , up to the level where  $c_1$  becomes so expensive that the investment level for Resource 1 drops to zero (i.e., consider the case where  $c_2 = 0.40$ :  $K_1 = 0.203$  when  $c_1 = 0.50$ ; drops to 0.053 when  $c_1 = 0.65$ ; and drops to zero when  $c_1 = 0.80$ ).

Similarly, in the de-centralized system, we observe that the investment level of each resource is decreasing in its own unit investment cost up to the level where the investment becomes so

expensive that the investment level of the resource drops to zero.

Finally, comparing the optimal solutions in the centralized and de-centralized systems, we observe that the firm invests more in Resource 1 and less in Resource 2 in the centralized system, as stated in Lemma 4. For example, when  $(c_1, c_2) = (0.12, 0.10)$ , the optimal solution in the centralized system is  $(K_1, K_2) = (0.916, 0)$ , while that in the de-centralized system is  $(K_1, K_2) = (0.714, 0.402)$ . Similarly, when  $(c_1, c_2) = (0.25, 0.20)$ , the optimal solution in the centralized system is  $(K_1, K_2) = (0.549, 0)$ , while that in the de-centralized system is  $(0.347, 0.229)$ .

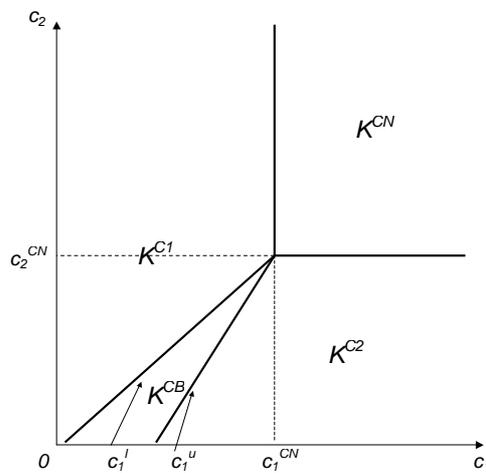


Figure 3.2: Structure of the optimal investment strategy in the numerical example in Section 3.3.4

### 3.4 Impact of Demand Correlation on the Optimal Investment Strategy

In this section, we investigate the impact of correlation in product demand patterns on the optimal investment strategy. Let  $\rho$  denote the correlation coefficient between  $\xi_1$  and  $\xi_2$ . We analyze the two extreme cases to get insight: cases where  $\xi_1$  and  $\xi_2$  are perfectly positively correlated ( $\rho = +1$ ) and perfectly negatively correlated ( $\rho = -1$ ).

#### 3.4.1 Demand Patterns with Perfect Positive Correlation

We first analyze the case where the demand patterns for the two products are perfectly positively correlated. For this purpose, we consider that  $Pr(\xi_1 = q\xi_2) = 1$  for any  $q > 0$ ; that is,  $\rho = +1$ . To simplify the notation, let  $\xi_2 = \xi$  and  $\xi_1 = q\xi$ . In this case, the demand space, given in Equation (3.17), reduces to the following:

$$\Omega_0 = \left\{ \xi < \min\left(\frac{2K_1}{q}, \frac{2K_1 + 2K_2}{q+1}\right) \right\}$$

$$\Omega_1 = \left\{ \xi > \frac{2K_1 + 2K_2}{q+1}, \quad \alpha_1(\xi - 2K_2) > \alpha_2(q\xi - 2K_1), \quad \alpha_1(\xi - 2K_1 - 2K_2) < q\alpha_2\xi \right\}$$

$$\Omega_2 = \left\{ \xi > 2K_2, \quad \alpha_1(\xi - 2K_2) < \alpha_2(q\xi - 2K_1) \right\}$$

$$\Omega_3 = \left\{ \frac{2K_1}{q} < \xi < 2K_2 \right\}$$

$$\Omega_4 = \left\{ \alpha_1(\xi - 2K_1 - 2K_2) > q\alpha_2\xi \right\}.$$

Then, the first-order KKT conditions, given in Theorem 2, reduce to the following:

$$\begin{aligned} \text{KKT-1:} \quad & E\left[\frac{(q+1)\xi - 2(K_1 + K_2)}{\alpha_1 + \alpha_2} \middle| \Omega_1\right] Pr(\Omega_1) + E\left[\frac{q\xi - 2K_1}{\alpha_1} \middle| \Omega_2\right] Pr(\Omega_2) \\ & + E\left[\frac{q\xi - 2K_1}{\alpha_1} \middle| \Omega_3\right] Pr(\Omega_3) + E\left[\frac{\xi - 2K_1 - 2K_2}{\alpha_2} \middle| \Omega_4\right] Pr(\Omega_4) = c_1 - v_1 \end{aligned}$$

$$\begin{aligned} \text{KKT-2:} \quad & E\left[\frac{(q+1)\xi - 2(K_1 + K_2)}{\alpha_1 + \alpha_2} \middle| \Omega_1\right] Pr(\Omega_1) \\ & + E\left[\frac{\xi - 2K_2}{\alpha_2} \middle| \Omega_2\right] Pr(\Omega_2) + E\left[\frac{\xi - 2K_1 - 2K_2}{\alpha_2} \middle| \Omega_4\right] Pr(\Omega_4) = c_2 - v_2 \end{aligned}$$

**Complementary Slackness:**  $v_i K_i = 0, \quad i = 1, 2$

$$v_i \geq 0, \quad i = 1, 2.$$

Our main result on the structure of the optimal investment strategy in the case of perfectly positively correlated demand patterns is given in the following theorem.

**Theorem 5** *Consider the case where  $Pr(\xi_1 = q\xi_2) = 1$  for any  $q > 0$ ; that is, the demand patterns for the two products are perfectly positively correlated. Let the threshold values,  $c_1^{CN}$ ,  $c_1^l$ ,  $c_2^{CN}$ , and  $c_1^u$ , be defined as in Theorem 3. Then, the optimal investment strategy is the following threshold policy:*

(i)  $q\alpha_2 > \alpha_1$ :

If  $c_2 \geq c_2^{CN}$ , then

$$\vec{K}^C = \begin{cases} \vec{K}^{CN}, & \text{if } c_1 \geq c_1^{CN}, \\ \vec{K}^{C1}, & \text{if } c_1 < c_1^{CN}. \end{cases}$$

If  $c_2 < c_2^{CN}$ , then

$$\vec{K}^C = \begin{cases} \vec{K}^{C1}, & \text{if } c_1 \leq c_1^l, \\ \vec{K}^{C2}, & \text{if } c_1 \geq c_1^u, \\ \vec{K}^{CB}, & \text{if } c_1^l < c_1 < c_1^u. \end{cases}$$

(ii)  $q\alpha_2 \leq \alpha_1$ :

If  $c_2 \geq c_2^{CN}$ , then  $\vec{K}^C = \vec{K}^{CN}$ .

If  $c_2 < c_2^{CN}$ , then

$$\vec{K}^C = \begin{cases} \vec{K}^{C2}, & \text{if } c_1 \geq c_1^u, \\ \vec{K}^{CB}, & \text{if } c_1 < c_1^u. \end{cases}$$

Furthermore, if  $\vec{K}^C = \vec{K}^{CB}$  is the optimal solution, then  $\frac{K_1^{CB}}{\alpha_1} < \frac{K_2^{CB}}{\alpha_2}$ .

*Proof:* Writing down the corresponding first-order KKT conditions at each of the boundary solutions  $\vec{K}^{CN}$ ,  $\vec{K}^{C1}$ ,  $\vec{K}^{C2}$ , and  $\vec{K}^{CB}$ , one can see that each of these boundary solutions is possible when  $q\alpha_2 > \alpha_1$ . Thus, the first part directly follows by Theorem 3. Consider now the case where  $q\alpha_2 \leq \alpha_1$  and the boundary solution  $\vec{K}^{C1}$ . Then, we have  $\Omega_2^{C1} = \{\xi > 0, \alpha_1\xi < \alpha_2(q\xi - 2K_1^{C1})\} = \emptyset$ , and therefore,  $\vec{K}^{C1}$  cannot be an optimal solution by Lemma 3, and the optimal solution reduces to that given in the theorem. To prove the last part, consider the case where  $q\alpha_2 \leq \alpha_1$  and where the optimal solution is given by  $\vec{K}^{CB}$ . Assume, to the contrary, that  $\frac{K_1^{CB}}{\alpha_1} \geq \frac{K_2^{CB}}{\alpha_2}$ . Then, we have  $\Omega_2^{CB} = \emptyset$  and  $\Omega_3^{CB} = \emptyset$ . Then, it follows by Lemma 3 that  $\vec{K}^{CB}$  is not a possible optimal solution in this case, which is a contradiction.

Therefore, we must have  $\frac{K_1^{CB}}{\alpha_1} < \frac{K_2^{CB}}{\alpha_2}$ . This completes the proof. ■

Thus, when the flexible resource is more expensive to invest in than the dedicated resource and  $q\alpha_2 > \alpha_1$ , all investment strategies ( $\vec{K}^{CN}$ ,  $\vec{K}^{C1}$ ,  $\vec{K}^{C2}$ , and  $\vec{K}^{CB}$ ) are still possible in the centralized system, even when the demand patterns for the two products are perfectly positively correlated. However, this is no longer the case when  $q\alpha_2 \leq \alpha_1$ , which corresponds to the situation where the unconstrained price set for Product 1 is less than or equal to that for Product 2; that is,  $p_1^{(u)} = \frac{q\epsilon}{2\alpha_1} \leq p_2^{(u)} = \frac{\epsilon}{2\alpha_2}$ . Since we also have  $c_1 > c_2$ , the dedicated resource becomes more desirable to invest in, and thus, the optimal solution never invests in the flexible resource only (i.e., boundary solution  $\vec{K}^{C1}$  is no longer optimal). Observe that this reasoning is analogous to that of the special case analyzed in Theorem 4.

### 3.4.2 Demand Patterns with Perfect Negative Correlation

We next analyze the case where the demand patterns are perfectly negatively correlated. For this purpose, we consider that  $Pr(\xi_1 + \xi_2 = q) = 1$  for any  $q > 0$ ; that is  $\rho = -1$ . To simplify the notation, let  $\xi_2 = \xi$  and  $\xi_1 = q - \xi$ . In this case, the demand space, given in Equation (3.17), reduces to the following:

$$\begin{aligned}\Omega_0 &= \{\xi > q - 2K_1, \quad q < 2K_1 + 2K_2\} \\ \Omega_1 &= \{q > 2K_1 + 2K_2, \quad \frac{2\alpha_1 K_2 - 2\alpha_2 K_1 + q\alpha_2}{\alpha_1 + \alpha_2} < \xi < \frac{2\alpha_1(K_1 + K_2) + q\alpha_2}{\alpha_1 + \alpha_2}\} \\ \Omega_2 &= \{2K_2 < \xi < \frac{2\alpha_1 K_2 - 2\alpha_2 K_1 + q\alpha_2}{\alpha_1 + \alpha_2}\} \\ \Omega_3 &= \{\xi < \min(q - 2K_1, 2K_2)\}\end{aligned}$$

$$\Omega_4 = \left\{ \xi > \frac{2\alpha_1(K_1 + K_2) + q\alpha_2}{\alpha_1 + \alpha_2} \right\}.$$

Then, the first-order KKT conditions, given in Theorem 2, reduce to the following:

$$\begin{aligned} \text{KKT-1:} \quad & E\left[\frac{q - 2(K_1 + K_2)}{\alpha_1 + \alpha_2} \mid \Omega_1\right]Pr(\Omega_1) + E\left[\frac{q - \xi - 2K_1}{\alpha_1} \mid \Omega_2\right]Pr(\Omega_2) \\ & + E\left[\frac{q - \xi - 2K_1}{\alpha_1} \mid \Omega_3\right]Pr(\Omega_3) + E\left[\frac{\xi - 2K_1 - 2K_2}{\alpha_2} \mid \Omega_4\right]Pr(\Omega_4) = c_1 - v_1 \\ \text{KKT-2:} \quad & E\left[\frac{q - 2(K_1 + K_2)}{\alpha_1 + \alpha_2} \mid \Omega_1\right]Pr(\Omega_1) \\ & + E\left[\frac{\xi - 2K_2}{\alpha_2} \mid \Omega_2\right]Pr(\Omega_2) + E\left[\frac{\xi - 2K_1 - 2K_2}{\alpha_2} \mid \Omega_4\right]Pr(\Omega_4) = c_2 - v_2 \end{aligned}$$

**Complementary Slackness:**  $v_i K_i = 0, \quad i = 1, 2$

$$v_i \geq 0, \quad i = 1, 2.$$

Our result on the structure of the optimal investment strategy in the case of perfectly negatively correlated demand patterns is given in the following theorem.

**Theorem 6** *Consider the case where  $Pr(\xi_1 + \xi_2 = q) = 1$  for any  $q > 0$ ; that is, the demand patterns of the two products are perfectly negatively correlated. Then, the optimal investment strategy follows the threshold policy given in Theorem 3. Furthermore, the total investment level in the optimal solution is such that  $K_1^C + K_2^C \leq \frac{q}{2}$ .*

*Proof:* See Appendix F. ■

Similar to the perfect positive correlation case, Theorem 6 shows that all investment strategies ( $\vec{K}^{CN}$ ,  $\vec{K}^{C1}$ ,  $\vec{K}^{C2}$ , and  $\vec{K}^{CB}$ ) are still possible in the centralized system, when the demand patterns are perfectly negatively correlated. The second part of Theorem 6 is intuitive. Recall

that the solution to the unconstrained Problem P2\* is given by  $p_i^{(u)} = \frac{\epsilon_i}{2\alpha_i}$ ,  $i = 1, 2$ , which leads to a demand of  $d_i = \frac{\epsilon_i}{2}$ . Then,  $E[d_1 + d_2] = E[\frac{\xi_1 + \xi_2}{2}] = \frac{q}{2}$ . Thus,  $\frac{q}{2}$  is the total number of units that the firm would be willing to satisfy if there were no resource capacity constraints, and hence, is the maximum total resource capacity the firm would be willing to invest in; that is, we must have  $K_1^C + K_2^C \leq \frac{q}{2}$ .

Optimal policy in the centralized system  
q  $\alpha_2 \leq \alpha_1$  case

	$c_2$	0.40	0.30	0.20	0.10
Threshold value	$c_1^u$	0.59	0.55	0.52	0.51
Optimal policy	$c_1$	0.70	0.70	0.70	0.70
$(\vec{K}^{C2})$	$\vec{K}^{C2}$	(0, 0.458)	(0, 0.602)	(0, 0.805)	(0, 1.151)
Optimal policy	$c_1$	0.50	0.50	0.50	0.50
$(\vec{K}^{CB})$	$\vec{K}^{CB}$	(0.087, 0.422)	(0.046, 0.588)	(0.020, 0.801)	(0.005, 1.150)

q  $\alpha_2 > \alpha_1$  case

	$c_2$	0.40	0.30	0.20	0.10
Threshold value	$c_1^l$	0.48	0.32	0.20	0.10
	$c_1^u$	1.00	1.00	1.00	1.00
Optimal policy	$c_1$	0.45	0.32	0.20	0.10
$(\vec{K}^{C1})$	$\vec{K}^{C1}$	(0.499, 0)	(0.781, 0)	(1.198, 0)	(1.895, 0)
Optimal policy	$c_1$	1.50	1.50	1.50	1.50
$(\vec{K}^{C2})$	$\vec{K}^{C2}$	(0, 0.112)	(0, 0.255)	(0, 0.458)	(0, 0.805)
Optimal policy	$c_1$	0.70	0.50	0.30	0.15
$(\vec{K}^{CB})$	$\vec{K}^{CB}$	(0.181, 0.107)	(0.352, 0.246)	(0.618, 0.432)	(0.965, 0.778)

Table 3.2: Threshold values and the optimal investment levels for the perfectly positively correlated demand patterns in Section 3.4.3

### 3.4.3 A Numerical Study

In this section, we perform a numerical study to verify our analytical results on the structure of the optimal investment strategy when demand patterns are perfectly positively or perfectly negatively correlated.

We first study the case where demand patterns are perfectly positively correlated. We consider that  $q = 1$ , which leads to  $\xi_1 = \xi_2 = \xi$  (see Section 3.4.1), and assume that  $\xi$  follows an exponential distribution with a rate of  $\lambda = 1$ . We first consider  $\alpha_1 = 2, \alpha_2 = 1$  ( $q\alpha_2 \leq \alpha_1$  case) and then  $\alpha_1 = 1, \alpha_2 = 2$  ( $q\alpha_2 > \alpha_1$  case).

Using Theorem 5, we obtain the threshold values for the optimal investment policy as  $c_2^{CN} = 1.0$  for the  $q\alpha_2 \leq \alpha_1$  case and  $c_1^{CN} = 1.0$  and  $c_2^{CN} = 0.5$  for the  $q\alpha_2 > \alpha_1$  case; in addition,  $c_1^l$  and  $c_1^u$  are as shown in Table 3.2. Our numerical results show that the optimal investment strategy in the centralized system follows the threshold policy in Theorem 5.

We next study the case where demand patterns are perfectly negatively correlated. We consider that  $q = 2$ ; that is  $\xi_1 + \xi_2 = 2$  with probability one (see Section 3.4.2), and assume that  $\xi$  follows an exponential distribution with a rate of  $\lambda = 1$  (then  $\xi_1 = 2 - \xi_2$ ). We consider that  $\alpha_1 = 1$  and  $\alpha_2 = 1$ . Using Theorem 6, we obtain the threshold values of the optimal investment policy as  $c_1^{CN} = 1.74, c_2^{CN} = 1.0$  and  $c_1^l$  and  $c_1^u$  as shown in Table 3.3. Our observations are similar to those in Section 3.3.4, except that the total investment level is such that  $K_1^C + K_2^C \leq \frac{q}{2}$ , as proven in Theorem 6. Thus, the optimal investment strategy in the centralized system follows the threshold policy in Theorem 6.

Optimal policy in the centralized system

	$c_2$	0.80	0.60	0.40	0.20
Threshold Values	$c_1^l$	1.23	0.80	0.46	0.20
	$c_1^u$	1.66	1.57	1.47	1.33
Optimal policy $\vec{K}^{C1}$	$c_1$	1.23	0.80	0.46	0.20
	$\vec{K}^{C1}$	(0.269, 0)	(0.520, 0)	(0.745, 0)	(0.942, 0)
Optimal policy $\vec{K}^{C2}$	$c_1$	1.70	1.60	1.50	1.40
	$\vec{K}^{C2}$	(0, 0.112)	(0, 0.255)	(0, 0.458)	(0, 0.805)
Optimal policy $\vec{K}^{CB}$	$c_1$	1.50	1.00	1.00	1.00
	$\vec{K}^{CB}$	(0.095, 0.069)	(0.365, 0.079)	(0.289, 0.288)	(0.198, 0.640)

Table 3.3: Threshold values and the optimal investment levels for the perfectly negatively correlated demand patterns in Section 3.4.3

### 3.5 Conclusions and Future Research Directions

In this chapter, we study the resource investment decision faced by a firm that builds two products utilizing a flexible and a dedicated resource, and that is a price-setter for both products. We characterize the structure of the firm's optimal resource investment strategy for both the centralized system, where the investment decision for the two resources is coordinated and the resource flexibility is considered in the investment decision, and the de-centralized system, where the resource flexibility is not taken into account in the investment decision. We show that the optimal investment strategy in the centralized system follows a threshold

policy. This characterization allows us to establish conditions under which the firm invests in the flexible resource, and to compare the optimal investment levels in the centralized and de-centralized systems in order to understand the impact of considering resource flexibility in the investment decision. Our analysis shows that considering resource flexibility in the investment decision increases the investment level of the flexible resource, while decreasing that of the dedicated resource. In addition, we investigate the impact of demand correlation on the firm's resource investment decision in the centralized system by considering perfectly positively correlated and perfectly negatively correlated demand patterns for the two products, and identify the conditions under which the firm invests in the flexible resource in each case.

Several extensions to our model deserve further analysis. Our model considers that the firm sets the prices for the two products independently. This is a reasonable assumption when the products are considerably different from the consumers' point of view so that consumers, demanding a specific product, can only be satisfied by that product, while the firm can allocate the flexible resource to either product; that is, all substitution is firm-driven and not consumer-driven. However, there are other applications of the substitution structure considered here, where customers may be satisfied by either product and may switch from their first-choice product to the other product based on the prices of both products. In this case, the firm needs to set prices for the different products jointly. As an example, consider a rental car company that leases compact cars and luxury cars. While the firm can substitute a luxury car for a customer demanding a compact car, customers can also switch between cars, based on the prices of both cars. Analyzing firm-driven substitution in the presence of consumer-driven substitution would be an interesting future research direction.

In this study, we considered that the firm is a price-setter for both markets, and analyzed the value of resource flexibility under the assumption that the firm can also affect its demands with its pricing policy. The next step would be to consider the impact of pricing flexibility on the value of resource flexibility.

In this chapter, we analyzed extreme cases of demand correlation, including perfectly positively and perfectly negatively correlated demand patterns for the two products, to get insight. Studying demand patterns having other correlation values would be a worthwhile future research area. For analytical tractability, our analysis considered only two products and assumed that the demand function for each product is a linear function of its own price. An important future research area would be to extend this analysis to consider more than two products as well as perform a sensitivity analysis to different demand functions.

In our models, we assumed that the fixed investment cost is zero for each resource and variable production costs are the same for both resources. It would also be interesting to consider fixed resource investment costs and different variable production costs so that one can understand how these values affect the firm's optimal resource investment strategy.

Finally, we consider, in our models, that both resources are owned by a single decision-maker. However, there are cases where the resources can be owned by different players, each with their own interests. It would be interesting to study the optimal resource investment and pricing strategies of the two players. A related study is by Van Mieghem (1999), who studies the value of sub-contracting when the resources are owned by different players, but considering that prices are exogenously determined by the market place. His models need to be extended to consider the firm's pricing decision.

## Chapter 4

# Analysis of a Swappable Resource Flexibility Structure

In this chapter, we describe our analysis on the value of the swappable resource flexibility structure presented in Section 1.2. As discussed in Chapter 1, resource flexibility allows the firm to delay its resource allocation decision to a later time when demand uncertainty is reduced. In this chapter, we characterize the structure of the firm's optimal delayed resource allocation strategy for certain cases and study the effectiveness of heuristic capacity allocation mechanisms for other cases.

The remainder of this chapter is organized as follows. In Section 4.1, we present our model and assumptions. In Section 4.2, we determine the structure of the firm's optimal delayed resource allocation policy for a "one-time decision model" that is suitable for settings in which customers line up to purchase their products/services. Section 4.3 presents a numerical study for the one-time decision model. Then, in Section 4.4 we incorporate the uncertainty in

demand parameters into our model. Section 4.5 extends our model to a “dynamic decision model” that is suitable for settings in which sales take place over the phone or the Internet so customers are unaware of the status of sales. Finally, in Section 4.6 we conclude with a summary of our findings and suggest directions for future research in this area.

## 4.1 Model and Assumptions

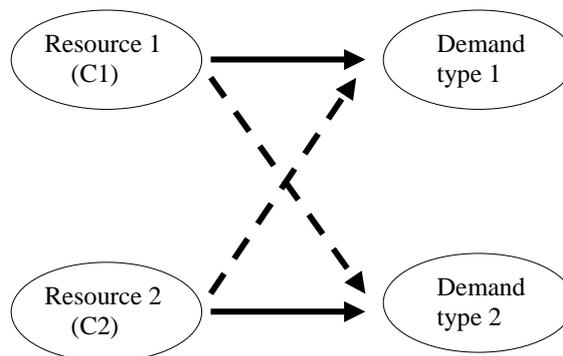


Figure 4.1: Resource allocation schemes under the swappable resource flexibility structure

We consider a system having two capacitated resources, with respective capacities of  $C_1$  and  $C_2$  units ( $C_1 < C_2$ ), and satisfying two demand types. Resources are *indivisible* in the

sense that the total capacity of each resource needs to be allocated to one demand type only. However, resources are also *flexible* such that each resource can be allocated to either demand type; see Figure 4.1, where either the solid line or the dashed line allocation is possible. Thus, capacity assignment can be made as late as possible to retain flexibility as long as no booked customers are spilled. Let  $N_i(t)$  denote the number of customers of demand type  $i$ ,  $i = 1, 2$ , arriving in a period of  $t$  time units. We assume that  $\{N_i(t), t \geq 0\}$  follows an independent, homogeneous Poisson process with a unit arrival rate of  $\Lambda_i$ ,  $i = 1, 2$ . Assume, without loss of generality, that  $\Lambda_1 < \Lambda_2$ . We consider a selling season of  $T$  periods. All unused capacity at the end of the selling season will perish.

In the following, we first consider the case where there is no forecast error in the system; that is, parameters  $\Lambda_i$ ,  $i = 1, 2$ , are known with certainty at the outset. Then, in Section 4.4 we extend our analysis to the case where only prior distributions on  $\Lambda_i$ ,  $i = 1, 2$ , are available upfront, and the posterior distributions are determined based on the early demand figures observed. Our objective is to analyze the benefits of utilizing the flexible capacity in the system, which allows the capacity allocation decision to be delayed to a time when more demand information is gathered so as to maximize the expected total revenue subject to the following constraints:

- (A1) Every customer, who has already been accepted to the system until the time of the delayed allocation decision, should be satisfied with the demand type that she desires (i.e., spilled customers are not allowed); and
- (A2) Once a customer of a demand type is rejected from the system (due to capacity limita-

tions), no future customers of that type will be accepted.

Observe that the second assumption leads to a conservative policy (clearly, higher profits can be realized by relaxing this constraint). However, it might be an appropriate assumption in systems where customers line up for purchasing their products/services, such as a movie theater, since it may not be a good company policy to reject the first customer in line while accepting the next one for the same type of product/service. Moreover, in Section 4.5 we relax Assumption **(A2)** and study the resulting model.

We assume that there is no demand substitution, i.e., if a customer cannot be satisfied because the product/service type of her choice is sold out, then the customer is lost (she will not switch to the other type). We consider the case where prices are exogenously determined and given and assume that the price for each type is the same (our analysis can easily be extended to the case with different prices). Hence, revenue will be simply given in terms of total sales. In the following, we use the terms product/service type and demand type interchangeably.

## 4.2 One-Time Decision Model

We first consider the system under Assumptions **(A1)** and **(A2)** and study the optimal delayed capacity allocation decision.

Let  $m_i(t)$ ,  $i = 1, 2$ , denote the number of bookings of Type  $i$  by time  $t$ ,  $t \geq 0$ . Observe that if  $m_i(t) < C_1$ , then it is clearly optimal to accept an arriving Type  $i$  customer,  $i = 1, 2$ . On the other hand, if a Type  $i$  customer arrives when  $m_i(t) = C_1$ , then either the booking of

the other type has exceeded  $C_1$  and therefore all future Type  $i$  customers have to be rejected, or the manager has to decide whether to accept this arriving Type  $i$  customer. If the manager accepts the current customer, then she commits capacity  $C_2$  to Type  $i$  customers; on the other hand if she rejects her, then she cannot accept any future Type  $i$  customers by Assumption **(A2)**. In either case, no further decision needs to be made. Thus, if we let  $t_i$  denote the arrival time of the  $(C_1 + 1)^{st}$  Type  $i$  customer,  $i = 1, 2$ , and  $t^* \equiv \min\{t_1, t_2\}$ , then the seller's decision problem under Assumption **(A2)** reduces to a “one-time decision” problem, which requires only one decision epoch, if any, at time  $t^*$ .

The information state at the decision epoch is given by  $\Omega = (i^*, t^*, m_j(t^*))$ , where  $t^* = t_{i^*}$  and  $j = \{1, 2\} \setminus \{i^*\}$ . Also, by definition of  $t^*$  we have  $m_j(t^*) \leq C_1$ . At this time, the decision is to either accept the current customer, who is the  $(C_1 + 1)^{st}$  customer of demand stream  $i^*$  (decision **A**), or reject the current customer (decision **R**). The expected sales given **A** is:

$$E^A[i^*, t^*, m_j(t^*)] = C_1 + 1 + m_j(t^*) + E[\min\{N_{i^*}(T - t^*)|\Omega, C_2 - C_1 - 1\}] \quad (4.1)$$

$$+ E[\min\{N_j(T - t^*)|\Omega, C_1 - m_j(t^*)\}].$$

Similarly, the expected sales given **R** is:

$$E^R[i^*, t^*, m_j(t^*)] = C_1 + m_j(t^*) + E[\min\{N_j(T - t^*)|\Omega, C_2 - m_j(t^*)\}]. \quad (4.2)$$

Thus, the decision (**A** or **R**) depends on the difference,

$$\Delta[i^*, t^*, m_j(t^*)] = E^A[i^*, t^*, m_j(t^*)] - E^R[i^*, t^*, m_j(t^*)] \quad (4.3)$$

$$\begin{aligned}
&= 1 + E[\min\{N_{i^*}(T - t^*)|\Omega, C_2 - C_1 - 1\}] \\
&\quad - E[\min\{N_j(T - t^*)|\Omega, C_2 - m_j(t^*)\} - \min\{N_j(T - t^*)|\Omega, C_1 - m_j(t^*)\}].
\end{aligned}$$

We have the following result.

**Lemma 5**  $\Delta[i^*, t^*, m_j(t^*)]$  is (1) decreasing in  $m_j(t^*)$ , (2) increasing in  $\Lambda_{i^*}$ , and (3) decreasing in  $\Lambda_j$ .

*Proof.* The result follows by definition of  $\Delta[i^*, t^*, m_j(t^*)]$  (see Equation (4.3)) and can be proved by stochastic coupling. ■

In the following, we determine the optimal capacity allocation policy for the one-time decision model. This is given in the following theorem, which states that the optimal delayed capacity allocation policy follows a *threshold* policy.

**Theorem 7** Consider the case where  $\Lambda_1 \leq \Lambda_2$  and  $C_1 < C_2$ . The optimal policy in state  $(i^*, t^*, m_j(t^*))$  is as follows:

1. If  $i^* = 2$ , then accept the customer and assign capacity  $C_1$  to Type 1 and  $C_2$  to Type 2.
2. Otherwise ( $i^* = 1$ ), the optimal allocation is the following threshold policy:

$$\begin{cases} \text{accept the customer and assign } C_2 \text{ to Type 1 and } C_1 \text{ to Type 2,} & \text{if } m_2(t^*) < \bar{m}_2(t), \\ \text{reject the customer and assign } C_1 \text{ to Type 1 and } C_2 \text{ to Type 2,} & \text{otherwise,} \end{cases}$$

where the threshold value  $\bar{m}_2(t) \in \{0, 1, \dots, C_1 + 1\}$ .

*Proof.* Observe that at state  $(i^* = 2, t^*, m_1(t^*))$ ,  $\Delta[2, t^*, m_1(t^*)] \geq 0$ , for all  $m_1(t^*) \leq C_1$ ; see Equation (4.3). That is, it is better to accept the current Type 2 customer, thus assigning

capacity  $C_2$  to Type 2 and  $C_1$  to Type 1. This follows because  $\Lambda_2 > \Lambda_1$ . Thus, the only remaining question is the decision at state  $(i^* = 1, t^*, m_2(t^*))$ . By Lemma 5,  $\Delta[1, t^*, m_2(t^*)]$  is decreasing in  $m_2(t^*)$ . Therefore, the decision rule is monotonic in  $m_2(t^*)$ . Thus, if the state space is given by  $(1, t^*, m_2(t^*))$ , then we define:

$$\bar{m}_2(t) \equiv \min_{m_2 \in Z^+} : \Delta[1, t^*, m_2(t^*)] > 0 \text{ and } \Delta[1, t^*, m_2(t^*)] \leq 0. \quad (4.4)$$

If  $\bar{m}_2(t)$  does not exist (i.e.,  $\Delta[1, t^*, m_2(t^*)] < 0$ , for all  $m_2(t^*)$ ), then we set  $\bar{m}_2(t)$  to 0. Then, the optimal allocation policy in the theorem follows.

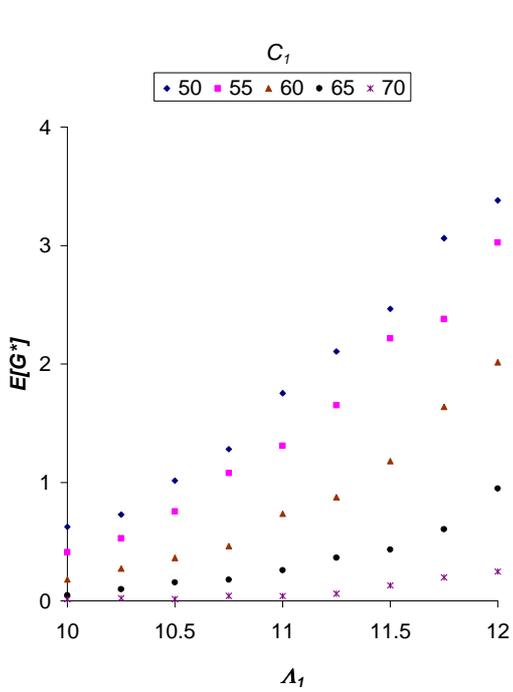
To prove the last part, substitute  $m_2(t^*) = C_1 + 1$  in Equation (4.3):

$$\begin{aligned} \Delta[1, t^*, m_2(t^*)] &= E[\min\{N_1(T - t^*), C_2 - C_1 - 1\} \\ &\quad - \min\{N_2(T - t^*), C_2 - C_1 - 1\}] \leq 0, \text{ since } N_1(T - t^*) \geq_{ST} N_2(T - t^*) \end{aligned}$$

Since  $\Delta[1, t^*, m_2(t^*)]$  is decreasing in  $m_2(t^*)$ , we have that  $\bar{m}_2(t) \in \{0, 1, \dots, C_1 + 1\}$ . This completes the proof. ■

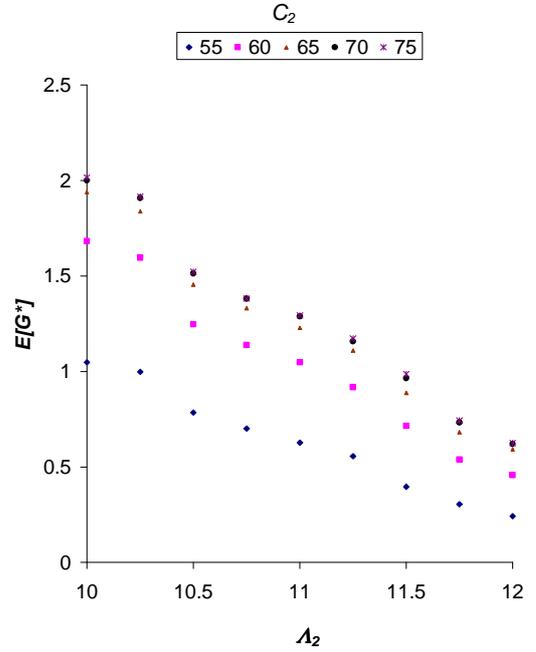
**Corollary 1** *Consider the special case where  $\Lambda_1 \leq \Lambda_2$  and  $C_2 = C_1 + 1$ . Then, the optimal delayed capacity allocation policy is always to accept the first  $(C_1 + 1)^{st}$  customer of either type, assigning capacities accordingly.*

*Proof.* Observe that  $\bar{m}_2(t) = C_1 + 1$  when  $\Lambda_1 \leq \Lambda_2$  and  $C_2 = C_1 + 1$ . The result then follows by Theorem 7. ■



$$C_2 = 75, \Lambda_2 = 12, T = 5$$

Figure 4.2:  $E[G^*]$  in  $\Lambda_1$  and  $C_1$



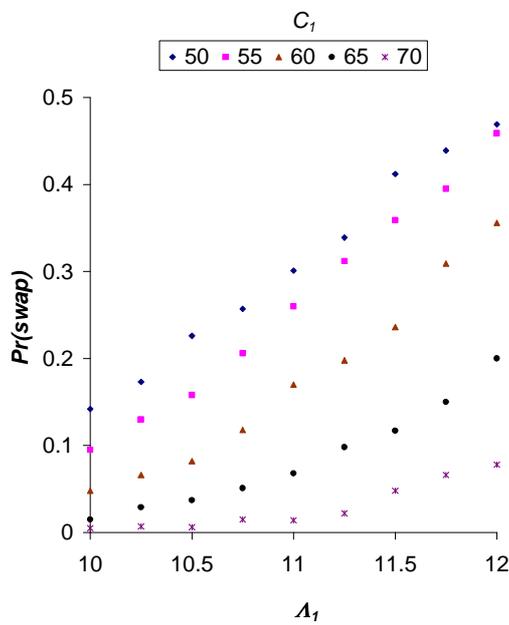
$$C_1 = 50, \Lambda_1 = 10, T = 5$$

Figure 4.3:  $E[G^*]$  in  $\Lambda_2$  and  $C_2$

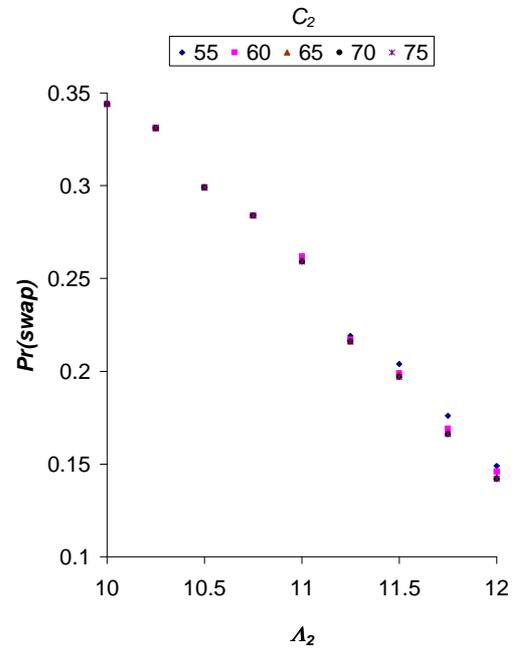
### 4.3 Numerical Study

In this section, we carry out a numerical study to analyze how the revenue increase of the optimal delayed capacity allocation policy (the threshold policy) depends on demand and capacity parameters and the length of the selling season. For this purpose, we compare the revenue obtained under the optimal delayed capacity allocation policy with the “base case”, in which capacities are assigned to demand types at the beginning of the selling season and are not changed (i.e., the non-delayed allocation policy).

Specifically, we consider two performance measures: (1) The expected revenue increase



$$C_2 = 75, \Lambda_2 = 12, T = 5$$

Figure 4.4:  $Pr(\text{swap})$  in  $\Lambda_1$  and  $C_1$ 

$$C_1 = 50, \Lambda_1 = 10, T = 5$$

Figure 4.5:  $Pr(\text{swap})$  in  $\Lambda_2$  and  $C_2$ 

under the delayed policy over the base case, given by  $E[G^*]$ ; and (2) the proportion of time the expected revenue of the delayed policy is higher than the base case, given by  $Pr(\text{swap})$ . For each scenario, characterized by  $(\Lambda_1, \Lambda_2, C_1, C_2, \text{ and } T)$ , we consider that arrivals follow a Poisson process, implement the optimal delayed allocation policy presented in Section 4.2, and compare the revenue of the delayed policy with that of the base case. Each scenario is replicated 1,000 times and the average of each performance measure is determined over all replications. In the following, we present our results for a specific set of parameters. However, we obtained similar results for a wide variety of parameters.

$C_2 \setminus \Lambda_2$	10	10.25	10.5	10.75	11	11.25	11.5	11.75	12
55	0.344	0.331	0.299	0.284	0.262	0.219	0.204	0.176	0.149
60	0.344	0.331	0.299	0.284	0.262	0.217	0.199	0.169	0.146
65	0.344	0.331	0.299	0.284	0.260	0.216	0.198	0.167	0.143
70	0.344	0.331	0.299	0.284	0.259	0.216	0.197	0.166	0.142
75	0.344	0.331	0.299	0.284	0.259	0.216	0.197	0.166	0.142

$$C_1 = 50, \Lambda_1 = 10, T = 5$$

Table 4.1:  $Pr(\text{swap})$  in  $\Lambda_2$  and  $C_2$ 

### 4.3.1 Impact of Demand and Capacity Parameters on the Revenue Gain of the Delayed Allocation

First we analyze how  $E[G^*]$  and  $Pr(\text{swap})$  depend on demand and capacity parameters; see Figures 4.2 - 4.5 and Table 4.1. We observe that  $E[G^*]$  is nondecreasing in  $\Lambda_1$ , nonincreasing in  $\Lambda_2$ , nonincreasing in  $C_1$ , and nondecreasing in  $C_2$ . Thus, the expected increase in revenue under the delayed allocation policy decreases as capacities,  $C_1$  and  $C_2$ , get closer, and increases as the arrival rates of the two demand types,  $\Lambda_1$  and  $\Lambda_2$ , get closer. These results are intuitive. When the arrival rates of the two demand types are close, the decision under the base case is likely to change as some demand observations are obtained. In addition, as  $C_1$  increases, the need to swap decreases, while as  $C_2$  increases, the potential gain of a swap increases.

Next, we study the behavior of  $Pr(\text{swap})$ . We observe that the opportunity of a swap increases as the two demand rates get closer, and is the highest when  $\Lambda_1 = \Lambda_2$ . In addition, we observe that  $Pr(\text{swap})$  is nonincreasing in both  $C_1$  and  $C_2$  (see Figures 4.4, 4.5 and Table 4.1), although the rate of change is much smaller for  $C_2$ . It is intuitive that  $Pr(\text{swap})$  is nonincreasing in  $C_1$ . All other parameters fixed, the need to swap decreases as  $C_1$  increases. However, our results also show that  $Pr(\text{swap})$  is nonincreasing in  $C_2$ . This is because all other

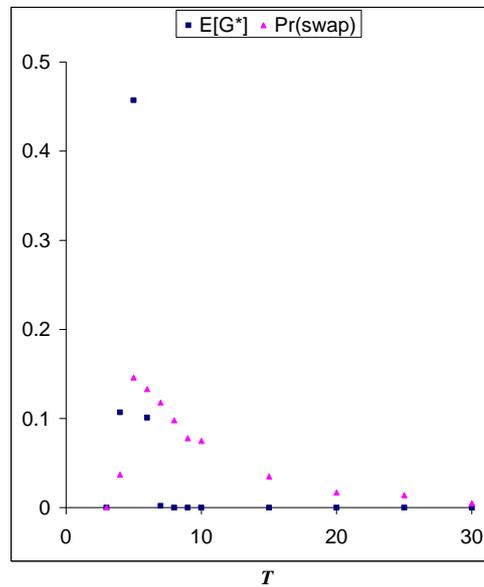


Figure 4.6:  $E[G^*]$  and  $Pr(\text{swap})$  in  $T$  ( $\Lambda_1 = 10$ ,  $\Lambda_2 = 12$ ,  $C_1 = 50$ ,  $C_2 = 60$ )

parameters fixed, the possibility that the type with the smaller arrival rate (Type 1) will fill all the additional capacity ( $C_2 - C_1$ ) decreases as  $C_2$  increases, thus reducing the probability of a swap.

#### 4.3.2 Impact of the Length of the Selling Season on the Revenue Gain of the Delayed Allocation

Next, we analyze how  $E[G^*]$  and  $Pr(\text{swap})$  change in  $T$ , the length of the selling season; see Figure 4.6. As shown in the figure, both  $E[G^*]$  and  $Pr(\text{swap})$  are unimodal in  $T$ , achieving their highest values at some value of  $T$ , after which point they start decreasing. This is because

Percent increase (%)					
$C_2 \backslash \Lambda_2$	10	10.5	11	11.5	12
55	2.15	1.49	0.91	0.50	0.30
60	2.74	1.91	1.26	0.72	0.45
65	2.98	2.08	1.37	0.87	0.52
70	3.03	2.14	1.41	0.93	0.55
75	3.04	2.15	1.41	0.94	0.56

$C_1 = 45, \Lambda_1 = 10, T = 5$

Percent increase (%)					
$C_2 \backslash \Lambda_2$	10	10.5	11	11.5	12
55	1.10	0.85	0.62	0.34	0.25
60	1.62	1.32	1.04	0.56	0.46
65	1.80	1.49	1.18	0.69	0.59
70	1.85	1.53	1.23	0.73	0.63
75	1.87	1.54	1.25	0.73	0.63

$C_1 = 50, \Lambda_1 = 10, T = 5$

Percent increase (%)					
$C_2 \backslash \Lambda_2$	10	10.5	11	11.5	12
55	0.00	0.00	0.00	0.00	0.00
60	0.54	0.51	0.32	0.30	0.21
65	0.72	0.70	0.44	0.43	0.32
70	0.76	0.75	0.48	0.46	0.35
75	0.78	0.75	0.49	0.46	0.36

$C_1 = 55, \Lambda_1 = 10, T = 5$

Table 4.2: Percent increase in the expected revenue for the optimal delayed capacity allocation policy over the base case

when all other parameters are fixed, if  $T$  is very small, then the possibility that we will observe the  $(C_1 + 1)^{st}$  customer of either type during the selling season is low, thus eliminating the need for a swap. Thus, both  $E[G^*]$  and  $Pr(\text{swap})$  are first increasing in  $T$ . However, as  $T$  becomes large, so does  $T - t^*$ , the remaining time to fill the capacities after the decision time,  $t^*$ . Since  $\Lambda_2 > \Lambda_1$ , the allocation decision favors the demand type with the larger arrival rate and hence both  $Pr(\text{swap})$  and  $E[G^*]$  drop. Observe that as  $T$  goes to  $\infty$ , both measures converge to 0.

Finally, Table 4.2 depicts the percent increase in expected revenue for the optimal delayed allocation policy over the non-delayed policy, considering different values of  $C_2$  and  $\Lambda_2$ , for cases where the total expected demand for Type 1 over the selling horizon, given by  $\Lambda_1 T = 50$ , is (i) greater than  $C_1$  ( $C_1 = 45$ ); (ii) equal to  $C_1$  ( $C_1 = 50$ ); and (iii) less than  $C_1$  ( $C_1 = 55$ ).

Recall that the non-delayed allocation policy will assign the smaller capacity to the demand type with the smaller demand rate (i.e.,  $C_1$  to Type 1 and  $C_2$  to Type 2). We observe the following.

1. The optimal delayed allocation policy is most beneficial when (i)  $C_1$  is below the expected demand level for Type 1; (ii) demand rates for the two types are close ( $\Lambda_1 = \Lambda_2$ ); and (iii)  $C_2 - C_1$ , the difference between the resource capacities, is large.
2. Its revenue benefits are decreasing in  $C_1$ , increasing in  $C_2$ , and decreasing in  $\Lambda_2$ .
3. Its revenue benefits can be significant (i.e., a 3.04% increase in expected revenue when  $C_1 = 45, \Lambda_1 = 10, C_2 = 75, \Lambda_2 = 10$ , and  $T = 5$ ).

#### 4.4 Optimal Delayed Allocation Policy When Demand Forecast is Uncertain

So far we have assumed that demand parameters are known with certainty at the start of the selling season. This assumption may not be realistic in certain environments. Therefore, in this section we relax this assumption and consider that demand parameters are uncertain at the start of the selling season. We consider the “one-time decision model” discussed in Section 4.2, and perform a numerical study to analyze the effectiveness of two heuristic capacity allocation policies as well as understand the magnitude of the revenue increase due to resource flexibility.

Suppose each  $\Lambda_i, i = 1, 2$ , has a prior distribution  $Gamma(k_i, a_i)$  with  $E[\Lambda_i] = \frac{k_i}{a_i}$ . Assume, without loss of generality, that  $E[\Lambda_1] < E[\Lambda_2]$ . At decision epoch  $t^*$ , the prior distributions are updated based on  $m_i(t^*)$ , the number of bookings obtained up to that point. Observe that

by definition of  $t^*$ ,  $m_i(t^*)$  also denotes the number of arrivals up to time  $t^*$ . Then, it is well known that the posterior distribution on  $\Lambda_i$  is  $Gamma(k_i + m_i(t^*), a_i + t^*)$ ,  $i = 1, 2$ . In order to simplify the notation, in the following we let  $\tau = T - t^*$  and denote  $m_i(t^*)$  as  $m_i$ ,  $i = 1, 2$ .

Then, for  $i = 1, 2$ , the distribution of the arrival process in the remaining time periods can be determined as follows:

$$\begin{aligned}
Pr(N_i(\tau) = n) &= \int_0^\infty \frac{e^{-\lambda_i \tau} (\lambda_i \tau)^n}{n!} \cdot \frac{(a_i + t^*) e^{-(a_i + t^*) \lambda_i} ((a_i + t^*) \lambda_i)^{k_i + m_i - 1}}{(k_i + m_i - 1)!} d\lambda_i \\
&= \frac{\tau^n (a_i + t^*)^{k_i + m_i}}{n! (k_i + m_i - 1)!} \int_0^\infty e^{-(a_i + t^* + \tau) \lambda_i} \lambda_i^{k_i + m_i + n - 1} d\lambda_i \\
&\quad \left( \text{Let } q_i = (a_i + t^* + \tau) \lambda_i \Rightarrow d\lambda_i = \frac{dq_i}{(a_i + t^* + \tau)} \right) \\
&= \frac{\tau^n (a_i + t^*)^{k_i + m_i}}{n! (k_i + m_i - 1)!} \int_0^\infty e^{-q_i} q_i^{k_i + m_i + n - 1} \frac{1}{(a_i + t^* + \tau)^{k_i + m_i + n - 1} (a_i + t^* + \tau)} dq_i \\
&= \frac{\tau^n (a_i + t^*)^{k_i + m_i}}{n! (k_i + m_i - 1)! (a_i + t^* + \tau)^{k_i + m_i + n}} \int_0^\infty e^{-q_i} q_i^{k_i + m_i + n - 1} dq_i \\
&\quad \left( \int_0^\infty e^{-\tau} \tau^n dt = \Gamma(n + 1) = n! \right) \\
&= \frac{\tau^n (a_i + t^*)^{k_i + m_i} (k_i + m_i + n - 1)!}{n! (k_i + m_i - 1)! (a_i + t^* + \tau)^{k_i + m_i + n}} \\
&= \binom{k_i + m_i + n - 1}{n} \left( \frac{a_i + t^*}{a_i + t^* + \tau} \right)^{k_i + m_i} \left( \frac{\tau}{a_i + t^* + \tau} \right)^n \\
&= \binom{r_i + n - 1}{r_i - 1} p_i^{r_i} (1 - p_i)^n, \quad n = 0, 1, 2, \dots
\end{aligned}$$

Thus, each  $N_i(\tau)$ ,  $i = 1 = 2$ , follows a Negative Binomial distribution with  $p_i = \frac{a_i + t_i}{a_i + t^* + \tau}$  and  $r_i = k_i + m_i$ , and therefore  $E[N_i(\tau)] = \frac{(k_i + m_i)(a_i + t^* + \tau)}{a_i + t^*}$ .

Bish, Lin, and Hong (2004) show that the threshold policy, given in Theorem 7, is optimal for the forecast error case only under certain conditions. Characterization of the optimal delayed allocation policy when these conditions are not satisfied remains an open research

question. Therefore, in the following we present two heuristic capacity allocation policies as well as an upper bound on the revenue possible under any capacity allocation mechanism, as proposed in Bish, Lin, and Hong (2004). Then, using these policies we study the revenue increase that can be obtained through the utilization of resource flexibility when there is demand forecast error. The following upper bound on the maximum revenue possible under any capacity allocation mechanism will be used subsequently to evaluate the performance of the heuristic policies.

$$\text{upper bound} = \max \left\{ \min(C_1, N_1(T)) + \min(C_2, N_2(T)), \min(C_1, N_2(T)) + \min(C_1, N_1(T)) \right\}.$$

The two heuristic capacity allocation policies considered are as follows:

1. The threshold policy, given in Theorem 7.
2. A myopic policy in which the seller does not consider the future arrivals and always accepts the first  $(C_1 + 1)^{st}$  customer of either type, thus committing the larger capacity of  $C_2$  to the corresponding demand type. Such a policy would be attractive in practice, since it is very easy to implement.

Observe that the myopic policy would be optimal when  $C_2 = C_1 + 1$ . In addition, the myopic policy should perform well when capacities and/or demand rates are close.

In the following, we present a numerical study to understand the effectiveness of the myopic policy as well as the effectiveness of the threshold policy under conditions when it is not optimal. For this purpose, we simulate the system and implement the two swapping policies, the myopic policy and the threshold policy, as well as the “base case” in which the capacity

allocation decision is made at the start of the selling season, considering the initial demand forecasts, and is not revised later; thus resource flexibility is not considered in the capacity allocation. We report our results in terms of the percent increase in revenue under policy  $\mathbf{X}$  over the base case, for  $\mathbf{X} = \text{threshold, myopic, upper bound}$ . Each scenario, characterized by  $(C_i, k_i, a_i, i = 1, 2, T)$ , is replicated 1,000 times and the percent deviation of each policy is averaged over all replications.

In particular, we consider that  $C_1 = 50, a_1 = 1, k_1 = 11, k_2 = 10, T = 5$ , and vary the upper bound from the “base the values of  $a_2$  and  $C_2$ ”. In Table 4.3, the first number in each cell is the percent deviation of policy. Not surprisingly, our results indicate that the myopic policy works well when capacities and demand rates of the two demand types are close. Its performance degrades as the difference between the capacities and/or demand rates increase. The threshold policy also works very well. The percent deviation of the threshold policy from the upper bound is less than 0.2% in all scenarios tested.

Our results also suggest that implementing a delayed capacity allocation policy that takes advantage of the resource flexibility can yield to large increases in revenue. Of course, higher benefits can be realized in settings where it is appropriate to relax Assumption **(A2)**; this will be further discussed in the following section.

## 4.5 Dynamic Decision Model

In settings where customers purchase their products/services over the phone or the Internet, relaxing Assumption **(A2)** may be appropriate since customers will not be aware of the status of sales. Consequently, in this section we relax Assumption **(A2)** and develop a heuristic

$E[\Lambda_2]$	12			14			16			18			20		
$C_2$   $a_2$	0.833			0.714			0.625			0.556			0.5		
60	2.00	1.99	1.99	1.06	1.04	1.04	0.78	0.77	0.77	0.37	0.37	0.37	0.30	0.29	0.29
70	3.61	3.56	3.56	1.97	1.89	1.89	1.49	1.44	1.44	0.81	0.78	0.78	0.52	0.50	0.49
80	4.61	4.53	4.53	2.48	2.36	2.36	2.01	1.91	1.91	1.25	1.20	1.20	0.66	0.62	0.62
90	5.06	4.96	4.96	2.76	2.62	2.62	2.28	2.15	2.15	1.49	1.43	1.43	0.77	0.72	0.72
100	5.23	5.21	5.12	2.88	2.73	2.73	2.48	2.33	2.33	1.61	1.55	1.55	0.82	0.77	0.77
110	5.28	5.16	5.16	2.91	2.76	2.76	2.56	2.40	2.40	1.64	1.58	1.58	0.82	0.77	0.77
120	5.28	5.17	5.17	2.92	2.77	2.77	2.58	2.41	2.41	1.67	1.60	1.60	0.76	0.71	0.71
130	5.28	5.16	5.16	2.92	2.77	2.77	2.58	2.41	2.41	1.69	1.62	1.61	0.75	0.70	0.70
140	5.28	5.16	5.16	2.91	2.77	2.77	2.57	2.41	2.41	1.69	1.63	1.62	0.75	0.69	0.69
150	5.28	5.16	5.16	2.91	2.77	2.77	2.57	2.41	2.41	1.70	1.63	1.63	0.74	0.69	0.69

Table 4.3: Percent increase in the expected revenue of the upper bound, the threshold policy and the myopic policy over the base case

delayed capacity allocation strategy.

As in the previous case, if  $m_i < C_1$ , then it is optimal to accept an arriving Type  $i$  customer. On the other hand, if a Type  $i$  customer arrives at a time when  $m_i = C_1$ , then either the booking of the other type has exceeded  $C_1$ , and therefore all future Type  $i$  customers have to be rejected, or the manager has to decide whether to accept this arriving Type  $i$  customer. If the manager accepts her, she commits capacity  $C_2$  to Type  $i$  customers and no further decision needs to be made; on the other hand if she rejects it, she then postpones that commitment decision to a later time.

We present an **approximate** method to determine the firm's delayed capacity allocation decision considering the dynamic time decision model. For this purpose, we formulate the firm's dynamic capacity allocation problem using a stochastic dynamic programming formulation, where the stages correspond to periods. We divide the selling season into time intervals small enough so that the number of arrivals in each period is either zero or one. Note that

this formulation is an **approximation** of the continuous time model, in which decisions are made at time of arrivals. Note that the accuracy of the discrete-time model can be increased to the desired level by reducing the length of each period.

The state of the system at an arrival at time  $t$  is given by  $\vec{m} = (m_1(t), m_2(t))$ , the number of customers booked for each demand type  $i, i = 1, 2$ , just before the arrival at time  $t$ ; and the type of arrival at time  $t$  (a 0 type arrival corresponds to no arrival). Let  $q_i, i = 0, 1, 2$ , denote the probability of a Type  $i$  arrival in a period.

We define function  $F_t(\vec{m}, i)$  as the optimal expected revenue to go from time  $t$  until the end of the horizon, if the number of customers accepted in the system by time  $t$  is given by vector  $\vec{m}$ , and an arrival of Type  $i$  has occurred at time  $t$ . In the following,  $\vec{e}_i$  represents a row vector with 1 in position  $i$  and 0's elsewhere,  $i = 1, 2$ . We have the following recursive relationships:

$$\underline{m_i \leq C_1 - 1, \forall i; k = 0, 1, 2,}$$

$$F_t(\vec{m}, i) = 1 + E[F_{t+1}(\vec{m} + \vec{e}_i, k)],$$

that is, the current arrival is always accepted.

$$\underline{m_i = C_1, m_j \leq C_1, i = 1, 2; i \neq j; k = 0, 1, 2,}$$

$$F_t(\vec{m}, i) = \max \begin{cases} 1 + E[F_{t+1}(\vec{m} + \vec{e}_i, k)] \\ E[F_{t+1}(\vec{m}, k)] \end{cases}$$

thus the current arrival is accepted only if the corresponding expected revenue is larger than the expected revenue when she is rejected.

$$\underline{C_1 + 1 \leq m_i \leq C_2 - 1, m_j \leq C_1; i = 1, 2; i \neq j; k = 0, 1, 2,}$$

$$F_t(\vec{m}, i) = 1 + E[F_{t+1}(\vec{m} + \vec{e}_i, k)] = 1 + E[\min\{C_2 - m_i - 1, N_i(\tau)\} + \min\{C_1 - m_j, N_j(\tau)\}],$$

where  $\tau$  denotes the number of periods from  $t$  until the end of the horizon, and  $N_j(\tau), j = 1, 2$ , denotes the number of arrivals until the end of the horizon. It follows that  $N_j(\tau), j = 1, 2$ , is binomial with  $(\tau, q_j)$ . Thus the current arrival is always accepted.

Therefore, the company faces a non-trivial decision problem only when  $m_i = C_1, m_j \leq C_1, i = 1, 2; i \neq j$ . Thus, if upon the arrival of a Type  $i$  customer at time  $t$ , the system is in state  $m_i = C_1, m_j \leq C_1, i = 1, 2; i \neq j$ , then we call this a “decision epoch”. (The decisions for customers arriving to all other states are known.) In this state, defining  $\alpha_t(i) \equiv E_j[F_{t+1}(\vec{m}, j)] - E_j[F_{t+1}(\vec{m} + \vec{e}_i, j)]$ , the decision is such that:

$$\begin{cases} \text{accept the current customer and assign } C_2 \text{ to Type } i^* \text{ and } C_1 \text{ to Type } j, & \text{if } \alpha_t(i^*) \leq 1, \\ \text{reject the current customer,} & \text{if } \alpha_t(i^*) > 1. \end{cases}$$

In the following, a numerical study is presented to understand the effectiveness of the dynamic decision model. For this purpose, we compare the benefits of the dynamic policy with that of the threshold policy given in Section 4.4. We report our results in terms of the percent increase in revenue under policy  $\mathbf{X}$  over the base case, for  $\mathbf{X} = \text{threshold, dynamic policy}$ . As in Section 4.4, we consider that  $C_1 = 50, a_1 = 1, k_1 = 11, k_2 = 10, T = 5$ , and vary

$E[\Lambda_2]$	12		14		16		18		20	
$C_2$   $a_2$	0.833		0.714		0.625		0.556		0.5	
60	1.99	1.99	1.04	1.04	0.77	0.77	0.37	0.37	0.29	0.29
70	3.56	3.56	1.89	1.89	1.44	1.44	0.78	0.78	0.50	0.50
80	4.53	4.53	2.37	2.36	1.92	1.91	1.20	1.20	0.62	0.62
90	4.96	4.96	2.62	2.62	2.15	2.15	1.43	1.43	0.72	0.72
100	5.12	5.12	2.73	2.73	2.33	2.33	1.55	1.55	0.79	0.77
110	5.16	5.16	2.76	2.76	2.40	2.40	1.59	1.58	0.79	0.77
120	5.17	5.17	2.77	2.77	2.41	2.41	1.62	1.60	0.73	0.71
130	5.16	5.16	2.77	2.77	2.41	2.41	1.64	1.62	0.72	0.70
140	5.16	5.16	2.77	2.77	2.41	2.41	1.65	1.63	0.71	0.69
150	5.16	5.16	2.77	2.77	2.41	2.41	1.65	1.63	0.71	0.69

Table 4.4: Percent increase in the expected revenue of the dynamic policy and the threshold policy over the base case

the values of  $a_2$  and  $C_2$ . In Table 4.4, the first number in each cell is the percent deviation of the dynamic policy from the “base case”, while the second number is the deviation for the threshold policy. Our results show that the dynamic policy performs better than the threshold policy, as we have conjectured. Specially, the dynamic policy outperforms the threshold policy when the differences in capacities and the demand rates are large.

## 4.6 Conclusions and Future Research Directions

In this chapter, we study the benefits of a delayed decision making strategy, which makes use of the resource flexibility in the system and delays the resource allocation decision to a time when partial information on demands is gathered and demand uncertainty is reduced.

Considering a simple two demand type model that is amenable to analytical analysis, we characterize the structure of the firm’s optimal delayed resource allocation strategy for the “one-time decision model” when there is no forecast error in the system and customers line

up to purchase their products/services. This characterization enables us to study the impact of parameters such as demand, capacity, and the length of the selling season, on the revenue benefits of the delayed allocation strategy. Our study shows that the revenue benefits of this strategy can be significant, especially when demand rates are close, while resource capacities are much different. Based on our analysis, we provide guidelines and insights on the utilization of such strategies. Although in our analysis we considered that prices are the same for both demand types, our analysis can be easily extended to the case with different prices.

We find that under certain conditions the structure of the firm's optimal delayed allocation strategy for the "one-time decision model" remains the same when demand forecasts are uncertain. Moreover, we suggest two heuristic policies and compare their revenue benefits with an upper bound in order to understand their performance. We observe that both heuristics perform very well and yield large revenue benefits.

Finally, we extend the one-time decision model to a dynamic time model using a stochastic dynamic programming formulation. Our numerical results suggest that higher benefits can be realized in a dynamic time model, especially when the differences in capacities and the demand rates are large.

Numerous extensions to our models deserve further analysis. In our analysis, we consider that the firm satisfies two demand types with two capacitated and flexible resources. Extending this analysis to any number of demand types and resources would be a worthwhile extension of this research.

In our analysis, we assume that prices of the demand types are exogenously determined and are given inputs. However, in practice, the firm may be able to utilize pricing control and

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adjust prices based on market demand estimates. Studying the benefits of resource flexibility, when it is integrated with pricing flexibility, would be an interesting future research direction. Finally, we consider only the revenue side of the delayed decision making. Incorporating the cost of delaying the decision, such as incorporating the cost associated with the loss of goodwill due to a possible reseating of the customers in movie theaters, would provide a cost/benefit analysis. However, these costs are very difficult to quantify in practice. Therefore, additional research is needed to understand these costs.

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# Appendix

In the following,  $E^c$  denotes the complement of event  $E$ ,  $\phi$  denotes the empty set, and  $U$  denotes the universal set.

## Appendix A

### Proof of Lemma 1

Let  $p_i^C$ ,  $i = 1, 2$ , denote the optimal solution to Problem P2\* (see Section 3.3),  $s_i^C = \epsilon_i - \alpha_i p_i^C$  denote the corresponding demand for product  $i$ ,  $i = 1, 2$ , in the optimal solution. Observe that the following is a lower bound on  $s_2^C$ :

$$s_2^C \geq \min\left(\frac{\epsilon_2}{2}, K_2\right) \geq 0,$$

or equivalently,

$$p_2^C \leq \max\left(\frac{\epsilon_2}{2\alpha_2}, \frac{\epsilon_2 - K_2}{\alpha_2}\right).$$

Thus, there exists a  $(s_2^C, p_2^C)$  pair that provides a non-negative contribution,  $s_2^C p_2^C \geq 0$ , to the profit function, without affecting the capacity reserved for the first product. Hence, constraint (3.16) and  $p_2 \geq 0$  constraint are redundant, since the optimal solution will always satisfy them. In addition, observe that  $p_1 \geq 0$  constraint is not needed. Consider a feasible pair  $(p_1 \geq 0, s_1 \geq 0)$ . Observe that  $p_1$  would become negative only when  $s_1$  increases further. However, increasing  $s_1$  would not change the resource allocation to Product 2 and hence, it cannot be optimal. Thus, Problem P2\* reduces to the following:

$$\begin{aligned} \mathbf{P2*}: \quad & \max_{p_1, p_2} \quad \Pi(\vec{K}) = p_1(\epsilon_1 - \alpha_1 p_1) + p_2(\epsilon_2 - \alpha_2 p_2) \\ \text{subject to} \quad & p_1 \geq \frac{\epsilon_1 - K_1}{\alpha_1} \quad \longleftarrow \lambda \end{aligned} \quad (\text{A.1})$$

$$\alpha_1 p_1 + \alpha_2 p_2 \geq \epsilon_1 + \epsilon_2 - K_2 - K_1 \quad \longleftarrow \mu \quad (\text{A.2})$$

$$p_1 \leq \frac{\epsilon_1}{\alpha_1} \quad \longleftarrow \gamma \quad (\text{A.3})$$

where  $\lambda$ ,  $\mu$ , and  $\gamma$  are the corresponding Lagrangian multipliers.

We derive the first-order KKT conditions for Problem P2\* as follows. To simplify the notation, in what follows we denote  $\Pi(\vec{K})$  simply as  $\Pi$ .

**First-order KKT conditions:**

$$\epsilon_1 - 2\alpha_1 p_1 + \lambda + \alpha_1 \mu - \gamma = 0 \Rightarrow p_1 = \frac{\epsilon_1 + \lambda + \alpha_1 \mu - \gamma}{2\alpha_1} \quad (\text{A.4})$$

$$\epsilon_2 - 2\alpha_2 p_2 + \alpha_2 \mu = 0 \Rightarrow p_2 = \frac{\epsilon_2 + \alpha_2 \mu}{2\alpha_2} \quad (\text{A.5})$$

$$(\epsilon_1 - K_1 - \alpha_1 p_1) \lambda = 0 \quad (\text{A.6})$$

$$(\epsilon_1 + \epsilon_2 - K_1 - K_2 - \alpha_1 p_1 - \alpha_2 p_2) \mu = 0 \quad (\text{A.7})$$

$$(\epsilon_1 - p_1 \alpha_1) \gamma = 0. \quad (\text{A.8})$$

Observe that the unconstrained solution,  $p_i^{(u)} = \frac{\epsilon_i}{2\alpha_i}$ ,  $i = 1, 2$ , is optimal if it satisfies all constraints of Problem P2\*, i.e., if  $\frac{\epsilon_1}{2\alpha_1} \geq \frac{\epsilon_1 - K_1}{\alpha_1} \Rightarrow \epsilon_1 \leq 2K_1$  and  $\epsilon_1 + \epsilon_2 \leq 2K_1 + 2K_2$ . Observe

that constraint (A.3), given by  $\frac{\epsilon_1}{2\alpha_1} \leq \frac{\epsilon_1}{\alpha_1}$ , is already satisfied.

Define event  $E = \{\epsilon_1 \leq 2K_1, \epsilon_1 + \epsilon_2 \leq 2K_1 + 2K_2\}$ . Thus, for a given  $(\epsilon_1, \epsilon_2)$ , if event  $E$  is satisfied, then  $\vec{p}^{\vec{C}} = (\frac{\epsilon_1}{2\alpha_1}, \frac{\epsilon_2}{2\alpha_2})$  is the optimal solution to Problem P2\*. Otherwise (i.e., if  $E^c$ ), the optimal solution must be on a boundary line of the feasible region, i.e., one of the constraints must be tight. First, consider the boundary line  $\alpha_1 p_1 + \alpha_2 p_2 = \epsilon_1 + \epsilon_2 - K_1 - K_2$ , which corresponds to Lagrangian multipliers of  $\mu \geq 0, \lambda = 0, \gamma = 0$ . Then, from (A.4) and (A.5), we obtain:

$$\begin{aligned} \frac{\epsilon_1}{2} + \frac{\alpha_1 \mu}{2} + \frac{\epsilon_2}{2} + \frac{\alpha_2 \mu}{2} &= \epsilon_1 + \epsilon_2 - K_1 - K_2 \\ \Rightarrow \frac{\mu}{2}(\alpha_1 + \alpha_2) &= \frac{\epsilon_1}{2} + \frac{\epsilon_2}{2} - K_1 - K_2 \\ \Rightarrow \mu &= \frac{\epsilon_1 + \epsilon_2 - 2K_1 - 2K_2}{\alpha_1 + \alpha_2}. \end{aligned} \tag{A.9}$$

Substituting (A.9) into (A.4) and (A.5), we obtain:

$$\begin{aligned} p_1 &= \frac{\epsilon_1}{2\alpha_1} + \frac{\mu}{2} = \frac{\epsilon_1}{2\alpha_1} + \frac{\epsilon_1 + \epsilon_2 - 2K_1 - 2K_2}{2(\alpha_1 + \alpha_2)} \\ p_2 &= \frac{\epsilon_2}{2\alpha_2} + \frac{\mu}{2} = \frac{\epsilon_2}{2\alpha_2} + \frac{\epsilon_1 + \epsilon_2 - 2K_1 - 2K_2}{2(\alpha_1 + \alpha_2)}. \end{aligned}$$

If this solution also satisfies constraints (A.1) and (A.3), then it must be optimal. For constraint (A.1),

$$p_1 \geq \frac{\epsilon_1 - K_1}{\alpha_1} \Rightarrow \frac{\epsilon_1}{2\alpha_1} + \frac{\epsilon_1 + \epsilon_2 - 2K_1 - 2K_2}{2(\alpha_1 + \alpha_2)} \geq \frac{\epsilon_1 - K_1}{\alpha_1}$$

$$\begin{aligned} &\Rightarrow \frac{\epsilon_1 + \epsilon_2 - 2K_1 - 2K_2}{2(\alpha_1 + \alpha_2)} \geq \frac{\epsilon_1 - 2K_1}{2\alpha_1} \\ &\Rightarrow \alpha_1(\epsilon_2 - 2K_2) \geq \alpha_2(\epsilon_1 - 2K_1). \end{aligned}$$

Similarly, for constraint (A.3),

$$\begin{aligned} &\frac{\epsilon_1}{2\alpha_1} + \frac{\epsilon_1 + \epsilon_2 - 2K_1 - 2K_2}{2(\alpha_1 + \alpha_2)} \leq \frac{\epsilon_1}{\alpha_1} \\ &\Rightarrow \alpha_1(\epsilon_2 - 2K_1 - 2K_2) \leq \alpha_2\epsilon_1. \end{aligned}$$

Define event  $F = \{\alpha_1(\epsilon_2 - 2K_2) \geq \alpha_2(\epsilon_1 - 2K_1), \alpha_1(\epsilon_2 - 2K_1 - 2K_2) \leq \alpha_2\epsilon_1\}$ . Thus, if  $E^c F$ , then  $\vec{p}^C = (\frac{\epsilon_1}{2\alpha_1} + \frac{\epsilon_1 + \epsilon_2 - 2K_1 - 2K_2}{2(\alpha_1 + \alpha_2)}, \frac{\epsilon_2}{2\alpha_2} + \frac{\epsilon_1 + \epsilon_2 - 2K_1 - 2K_2}{2(\alpha_1 + \alpha_2)})$  is optimal. Otherwise (i.e., if  $E^c F^c$ ), the optimal solution must be one of the following boundary line solutions (i.e., the one with the higher objective function value):

1.  $p_1 = \frac{\epsilon_1 - K_1}{\alpha_1}$  (see constraint (A.1)) with  $\lambda \geq 0$ ,  $\mu = 0$ , and  $\gamma = 0$ ;
2.  $p_1 = \frac{\epsilon_1}{\alpha_1}$  (see constraint (A.3)) with  $\lambda = 0$ ,  $\mu = 0$ , and  $\gamma \geq 0$ .

First consider the boundary line  $p_1^C = \frac{\epsilon_1 - K_1}{\alpha_1} \Rightarrow s_1^C = K_1$ . Then, P2\* reduces to:

$$\begin{aligned} &\max_{p_2} \quad \Pi(\vec{K}) = p_2(\epsilon_2 - \alpha_2 p_2) \\ &\text{subject to} \quad p_2 \geq \frac{\epsilon_2 - K_2}{\alpha_2}, \end{aligned}$$

where the optimal solution is  $p_2^C = \max(\frac{\epsilon_2}{2\alpha_2}, \frac{\epsilon_2 - K_2}{\alpha_2})$ . Therefore, if  $\epsilon_2 < 2K_2$ , then  $\vec{p} = (\frac{\epsilon_1 - K_1}{\alpha_1}, \frac{\epsilon_2}{2\alpha_2}) \Rightarrow \Pi^1 = K_1(\frac{\epsilon_1 - K_1}{\alpha_1}) + \frac{\epsilon_2^2}{4\alpha_2}$ . Otherwise (i.e., if  $\epsilon_2 > 2K_2$ ),  $\vec{p} = (\frac{\epsilon_1 - K_1}{\alpha_1}, \frac{\epsilon_2 - K_2}{\alpha_2}) \Rightarrow \Pi^2 = K_1(\frac{\epsilon_1 - K_1}{\alpha_1}) + K_2(\frac{\epsilon_2 - K_2}{\alpha_2})$ .

Next, consider the boundary line  $p_1 = \frac{\epsilon_1}{\alpha_1} \Rightarrow s_1 = 0$ . Then, P2\* reduces to:

$$\begin{aligned} \max_{p_2} \quad & \Pi(\vec{K}) = p_2(\epsilon_2 - \alpha_2 p_2) \\ \text{subject to} \quad & p_2 \geq \frac{\epsilon_2 - K_1 - K_2}{\alpha_2} \end{aligned}$$

We obtain the optimal solution as  $p_2^C = \max(\frac{\epsilon_2}{2\alpha_2}, \frac{\epsilon_2 - K_1 - K_2}{\alpha_2})$ . Therefore, if  $\epsilon_2 < 2K_1 + 2K_2$ , then  $\vec{p} = (\frac{\epsilon_1}{\alpha_1}, \frac{\epsilon_2}{2\alpha_2}) \Rightarrow \Pi^3 = \frac{\epsilon_2^2}{4\alpha_2}$ . Otherwise (i.e., if  $\epsilon_2 > 2K_1 + 2K_2$ ),  $\vec{p} = (\frac{\epsilon_1}{\alpha_1}, \frac{\epsilon_2 - K_1 - K_2}{\alpha_2}) \Rightarrow \Pi^4 = (K_1 + K_2) \frac{(\epsilon_2 - K_1 - K_2)}{\alpha_2}$ .

Let  $F = F_1, F_2$ , where  $F_1 = \{\alpha_1(\epsilon_2 - 2K_2) \geq \alpha_2(\epsilon_1 - 2K_1)\}$ , and  $F_2 = \{\alpha_1(\epsilon_2 - 2K_1 - 2K_2) \leq \alpha_2\epsilon_1\}$ . Then, if  $E^c F_1^c$  and  $\{\epsilon_2 < 2K_2\}$ , then  $\vec{p}^C = (\frac{\epsilon_1 - K_1}{\alpha_1}, \frac{\epsilon_2}{2\alpha_2})$  is optimal, since  $\epsilon_2 < 2K_1 + 2K_2$  and  $\Pi^1 > \Pi^3$ . If  $E^c F_1^c$  and  $\{\epsilon_2 > 2K_2\}$ , then  $\vec{p}^C = (\frac{\epsilon_1 - K_1}{\alpha_1}, \frac{\epsilon_2 - K_2}{\alpha_2})$  is optimal, since  $\Pi^2 > \Pi^3$  when  $\epsilon_2 < 2K_1 + 2K_2$  and  $\Pi^2 > \Pi^4$  when  $\epsilon_2 > 2K_1 + 2K_2$ . Finally, if  $E^c F_2^c$ , then  $\vec{p}^C = (\frac{\epsilon_1}{\alpha_1}, \frac{\epsilon_2 - K_1 - K_2}{\alpha_2})$  is optimal, since  $\epsilon_2 > 2K_1 + 2K_2$  and  $\Pi^4 > \Pi^2$ . We summarize the optimal solution to Problem P2\* as follows:

- If  $E$ , then  $p_i^C = \frac{\epsilon_i}{2\alpha_i}$ ,  $s_i^C = \frac{\epsilon_i}{2}$ ,  $i = 1, 2 \Rightarrow \Pi = \sum_{i=1}^2 \frac{\xi_i^2}{4\alpha_i}$ .
- If  $E^c F$ , then

$$\begin{aligned} p_i^C &= \frac{\epsilon_i}{2\alpha_i} + \frac{(\epsilon_1 + \epsilon_2 - 2K_1 - 2K_2)}{2(\alpha_1 + \alpha_2)}, \quad s_i^C = \frac{\epsilon_i}{2} - \frac{\alpha_i(\epsilon_1 + \epsilon_2 - 2K_1 - 2K_2)}{2(\alpha_1 + \alpha_2)}, \quad i = 1, 2, \\ \Rightarrow \Pi &= \sum_{i=1}^2 \left( \frac{\epsilon_i}{2\alpha_i} + \frac{\epsilon_1 + \epsilon_2 - 2(K_1 + K_2)}{2(\alpha_1 + \alpha_2)} \right) \left( \frac{\epsilon_i}{2} - \frac{\alpha_i(\epsilon_1 + \epsilon_2 - 2(K_1 + K_2))}{2(\alpha_1 + \alpha_2)} \right). \end{aligned}$$

• If  $E^c F^c = E^c F_1^c$  or  $E^c F_2^c$ , then we have the following:

- If  $E^c F_1^c$  and  $\{\epsilon_2 > 2K_2\}$ , then  $p_i^C = \frac{\epsilon_i - K_i}{\alpha_i}$ ,  $s_i^C = K_i$ ,  $i = 1, 2 \Rightarrow \Pi^C = K_1 \left( \frac{\epsilon_1 - K_1}{\alpha_1} \right) + K_2 \left( \frac{\epsilon_2 - K_2}{\alpha_2} \right)$
- If  $E^c F_1^c$  and  $\{\epsilon_2 < 2K_2\}$ , then  $p_1^C = \frac{\epsilon_1 - K_1}{\alpha_1}$ ,  $p_2^C = \frac{\epsilon_2}{2\alpha_2}$ ,  $s_1^C = K_1$ ,  $s_2^C = \frac{\epsilon_2}{2} \Rightarrow \Pi = K_1 \left( \frac{\epsilon_1 - K_1}{\alpha_1} \right) + \frac{\epsilon_2^2}{4\alpha_2}$ .
- If  $E^c F_2^c$ , then we must have  $\epsilon_2 \geq 2K_1 + 2K_2$ . Therefore,  $p_1^C = \frac{\epsilon_1}{\alpha_1}$ ,  $p_2^C = \frac{\epsilon_2 - 2K_1 - 2K_2}{\alpha_2}$ ,  $s_1^C = 0$ ,  $s_2^C = K_1 + K_2 \Rightarrow \Pi = (K_1 + K_2) \left( \frac{\epsilon_2 - K_1 - K_2}{\alpha_2} \right)$ .

This completes the proof. ■

## Appendix B

### Proof of Lemma 2

We define:

$$a \equiv \frac{2}{\alpha_1 + \alpha_2} \int \int_{\Omega_1} f_1(\epsilon_1) f_2(\epsilon_2) d\epsilon_1 d\epsilon_2, \quad b \equiv \frac{2}{\alpha_1} \int \int_{\Omega_2} f_1(\epsilon_1) f_2(\epsilon_2) d\epsilon_1 d\epsilon_2,$$

$$c \equiv \frac{2}{\alpha_1} \int \int_{\Omega_3} f_1(\epsilon_1) f_2(\epsilon_2) d\epsilon_1 d\epsilon_2, \quad d \equiv \frac{2}{\alpha_2} \int \int_{\Omega_4} f_1(\epsilon_1) f_2(\epsilon_2) d\epsilon_1 d\epsilon_2.$$

Then, we can write  $\mathbf{H}$ , the Hessian of  $E[\Pi(\vec{K})]$ , as follows:

$$\mathbf{H} = \begin{bmatrix} -a - b - c - d & -a - d \\ -a - d & -a - b - d \end{bmatrix}$$

$\mathbf{H}$  is negative definite if and only if  $-(a + b + c + d) < 0$ ,  $-(a + b + d) < 0$ , and  $(a + b + c + d)(a + b + d) - (a + d)^2 = b^2 + 2ab + ac + bc + cd + 2bd > 0$  [see, for instance, Bazarra,

Sherali, and Shetty (1993)]. All these conditions hold when the density functions of  $\xi_1$  and  $\xi_2$  are defined over  $[0, \infty)$ . Thus, we conclude that  $\mathbf{H}$  is negative definite, and therefore, function  $E[\Pi(\vec{K})]$  is strictly, jointly concave in  $K_1$  and  $K_2$  for all such distributions of  $\xi_1$  and  $\xi_2$ . Since the investment cost term,  $-\sum_{i=1}^2 c_i K_i$ , is linear in  $K_1, K_2$ , it follows that function  $Z^C(\vec{K}) = E[\Pi(\vec{K})] - \sum_{i=1}^2 c_i K_i$  is strictly, jointly concave in  $K_1$  and  $K_2$ . This completes the proof. ■

## Appendix C

### Proof of Theorem 3

We first list the demand space and the first-order KKT conditions at each boundary solution.

1. Boundary solution  $\vec{K}^{CB} = (K_1^{CB} > 0, K_2^{CB} > 0)$ :

The demand space is given in Equation (3.17) and the KKT conditions are given in Theorem 2 with  $v_1 = 0$  and  $v_2 = 0$ .

2. Boundary solution  $\vec{K}^{C1} = (K_1^{C1} > 0, K_2^{C1} = 0)$ :

$$\Omega_0^{C1} = \{\xi_1 + \xi_2 < 2K_1^{C1}\}$$

$$\Omega_1^{C1} = \{\xi_1 + \xi_2 > 2K_1^{C1}, \quad \alpha_1 \xi_2 > \alpha_2 (\xi_1 - 2K_1^{C1}), \quad \alpha_1 (\xi_2 - 2K_1^{C1}) < \alpha_2 \xi_1\}$$

$$\Omega_2^{C1} = \{\alpha_1 \xi_2 < \alpha_2 (\xi_1 - 2K_1^{C1})\}$$

$$\Omega_3^{C1} = \phi; \quad \Omega_4^{C1} = \{\alpha_1 (\xi_2 - 2K_1^{C1}) > \alpha_2 \xi_1\}$$

$$\begin{aligned} \mathbf{KKT-1:} \quad & E\left[\frac{\xi_1 + \xi_2 - 2K_1^{C1}}{\alpha_1 + \alpha_2} | \Omega_1^{C1}\right] Pr(\Omega_1^{C1}) + E\left[\frac{\xi_1 - 2K_1^{C1}}{\alpha_1} | \Omega_2^{C1}\right] Pr(\Omega_2^{C1}) \\ & + E\left[\frac{\xi_2 - 2K_1^{C1}}{\alpha_2} | \Omega_4^{C1}\right] Pr(\Omega_4^{C1}) = c_1 \end{aligned}$$

$$\begin{aligned} \mathbf{KKT-2:} \quad & E\left[\frac{\xi_1 + \xi_2 - 2K_1^{C1}}{\alpha_1 + \alpha_2} | \Omega_1^{C1}\right] Pr(\Omega_1^{C1}) \\ & + E\left[\frac{\xi_2}{\alpha_2} | \Omega_2^{C1}\right] Pr(\Omega_2^{C1}) + E\left[\frac{\xi_2 - 2K_1^{C1}}{\alpha_2} | \Omega_4^{C1}\right] Pr(\Omega_4^{C1}) = c_2 - v_2 \end{aligned}$$

$$v_2 \geq 0.$$

3. Boundary solution  $\vec{K}^{C2} = (K_1^{C2} = 0, K_2^{C2} > 0)$ :

$$\Omega_0^{C2} = \phi; \quad \Omega_1^{C2} = \phi$$

$$\Omega_2^{C2} = \{\xi_2 > 2K_2^{C2}, \quad \alpha_1(\xi_2 - 2K_2^{C2}) < \alpha_2\xi_1\}$$

$$\Omega_3^{C2} = \{\xi_2 < 2K_2^{C2}\}; \quad \Omega_4^{C2} = \{\alpha_1(\xi_2 - 2K_2^{C2}) > \alpha_2\xi_1\}$$

$$\begin{aligned} \mathbf{KKT-1:} \quad & E\left[\frac{\xi_1}{\alpha_1} | \Omega_2^{C2}\right] Pr(\Omega_2^{C2}) + E\left[\frac{\xi_1}{\alpha_1} | \Omega_3^{C2}\right] Pr(\Omega_3^{C2}) \\ & + E\left[\frac{\xi_2 - 2K_2^{C2}}{\alpha_2} | \Omega_4^{C2}\right] Pr(\Omega_4^{C2}) = c_1 - v_1 \end{aligned}$$

$$\mathbf{KKT-2:} \quad E\left[\frac{\xi_2 - 2K_2^{C2}}{\alpha_2} | \Omega_2^{C2}\right] Pr(\Omega_2^{C2}) + E\left[\frac{\xi_2 - 2K_2^{C2}}{\alpha_2} | \Omega_4^{C2}\right] Pr(\Omega_4^{C2}) = c_2$$

$$v_1 \geq 0.$$

4. Boundary solution  $\vec{K}^{CN} = (K_1^{CN} = 0, K_2^{CN} = 0)$ :

$$\Omega_0^{CN} = \phi; \quad \Omega_1^{CN} = \phi$$

$$\Omega_2^{CN} = \{\alpha_1 \xi_2 < \alpha_2 \xi_1\}; \quad \Omega_3^{CN} = \phi$$

$$\Omega_4^{CN} = \{\alpha_1 \xi_2 > \alpha_2 \xi_1\}$$

$$\mathbf{KKT-1:} \quad E\left[\frac{\xi_1}{\alpha_1} | \Omega_2^{CN}\right] Pr(\Omega_2^{CN}) + E\left[\frac{\xi_2}{\alpha_2} | \Omega_4^{CN}\right] Pr(\Omega_4^{CN}) = c_1 - v_1$$

$$\mathbf{KKT-2:} \quad E\left[\frac{\xi_2}{\alpha_2} | \Omega_2^{CN}\right] Pr(\Omega_2^{CN}) + E\left[\frac{\xi_2}{\alpha_2} | \Omega_4^{CN}\right] Pr(\Omega_4^{CN}) = c_2 - v_2$$

$$v_i \geq 0, \quad i = 1, 2.$$

Consider first the case where  $c_2 \geq E\left[\frac{\xi_2}{\alpha_2}\right] \equiv c_2^{CN}$ . At boundary solution  $\vec{K}^{C2}$ , **KKT-2** can be written as:

$$c_2 = E\left[\frac{\xi_2 - 2K_2^{C2}}{\alpha_2} | \xi_2 > 2K_2^{C2}\right] Pr(\xi_2 > 2K_2^{C2}) < E\left[\frac{\xi_2}{\alpha_2}\right],$$

which is a contradiction with our assumption that  $c_2 \geq E\left[\frac{\xi_2}{\alpha_2}\right]$ . Thus,  $\vec{K}^{C2}$  is not a possible solution in this case.

Similarly, consider the boundary solution  $\vec{K}^{CB}$ , where **KKT-2** can be written as:

$$\begin{aligned} & E\left[\frac{\xi_1 + \xi_2 - 2(K_1^{CB} + K_2^{CB})}{\alpha_1 + \alpha_2} | \Omega_1^{CB}\right] Pr(\Omega_1^{CB}) \\ & + E\left[\frac{\xi_2 - 2K_2^{CB}}{\alpha_2} | \Omega_2^{CB}\right] Pr(\Omega_2^{CB}) + E\left[\frac{\xi_2 - 2K_1^{CB} - 2K_2^{CB}}{\alpha_2} | \Omega_4^{CB}\right] Pr(\Omega_4^{CB}) = c_2 \\ \Rightarrow & E\left[\frac{\xi_2}{\alpha_2}\right] - E\left[\frac{\xi_2}{\alpha_2} | \Omega_0^{CB}\right] Pr(\Omega_0^{CB}) + \underbrace{E\left[\frac{\alpha_2(\xi_1 - 2K_1^{CB} - 2K_2^{CB}) - \alpha_1 \xi_2}{\alpha_2(\alpha_1 + \alpha_2)} | \Omega_1^{CB}\right] Pr(\Omega_1^{CB})}_{< 0 \text{ (by definition of } \Omega_1^{CB})} \end{aligned}$$

$$\begin{aligned}
& -\frac{2K_2^{CB}}{\alpha_2}Pr(\Omega_2^{CB}) - E\left[\frac{\xi_2}{\alpha_2}|\Omega_3^{CB}\right]Pr(\Omega_3^{CB}) - \frac{(2K_1^{CB} + 2K_2^{CB})}{\alpha_2}Pr(\Omega_4^{CB}) = c_2 \\
\Rightarrow c_2 & < E\left[\frac{\xi_2}{\alpha_2}\right],
\end{aligned}$$

which is a contradiction with our assumption that  $c_2 \geq E\left[\frac{\xi_2}{\alpha_2}\right]$ . Thus,  $\vec{K}^{CB}$  is not a possible solution in this case. Similarly, we can show that boundary solutions  $\vec{K}^{CN}$  and  $\vec{K}^{C1}$  are possible in this case.

Consider boundary solution  $\vec{K}^{CN}$ . By **KKT-1**, we have:

$$c_1 - v_1 = E\left[\frac{\xi_1}{\alpha_1}\right] + E\left[\frac{\alpha_1\xi_2 - \alpha_2\xi_1}{\alpha_1\alpha_2}|\Omega_4^{CN}\right]Pr(\Omega_4^{CN}) \equiv c_1^{CN}.$$

Hence,  $\vec{K}^{CN}$  is possible only if  $c_1 \geq c_1^{CN}$ . Thus, when  $c_2 \geq E\left[\frac{\xi_2}{\alpha_2}\right]$ , the structure of the optimal investment strategy is as follows:

$$\vec{K}^C = \begin{cases} \vec{K}^{CN}, & \text{if } c_1 \geq c_1^{CN}, \\ \vec{K}^{C1}, & \text{if } c_1 < c_1^{CN}. \end{cases}$$

Now consider the case where  $c_2 < c_2^{CN} = E\left[\frac{\xi_2}{\alpha_2}\right]$ . In the following, we analyze the boundary solution in this case.

Boundary solution  $\vec{K}^{C1} = (K_1^{C1} > 0, K_2^{C1} = 0)$ :

Using **KKT-1** and **KKT-2**, we can write:

$$c_1 + v_2 = c_2 + E\left[\frac{\alpha_2(\xi_1 - 2K_1^{C1}) - \alpha_1\xi_2}{\alpha_1\alpha_2}|\Omega_2^{C1}\right]Pr(\Omega_2^{C1}) \equiv c_1^l.$$

Thus,  $\vec{K}^{C1}$  is a possible solution only if  $c_1 \leq c_1^l$ .

Boundary solution  $\vec{K}^{C2} = (K_1^{C2} = 0, K_2^{C2} > 0)$ :

Using **KKT-1** and **KKT-2**, we can write:

$$c_1 - v_1 = c_2 + E\left[\frac{\alpha_2 \xi_1 - \alpha_1 (\xi_2 - 2K_2^{C2})}{\alpha_1 \alpha_2} | \Omega_2^{C2}\right] Pr(\Omega_2^{C2}) + E\left[\frac{\xi_1}{\alpha_1} | \Omega_3^{C2}\right] Pr(\Omega_3^{C2}) \equiv c_1^u.$$

Thus,  $\vec{K}^{C2}$  is a possible solution only if  $c_1 \geq c_1^u$ .

In what follows, we show that  $c_1^l < c_1^u$ . For this purpose, we need to show that:

$$E\left[\frac{\alpha_2 (\xi_1 - 2K_1^{C1}) - \alpha_1 \xi_2}{\alpha_1 \alpha_2} | \Omega_2^{C1}\right] Pr(\Omega_2^{C1}) < E\left[\frac{\alpha_2 \xi_1 - \alpha_1 (\xi_2 - 2K_2^{C2})}{\alpha_1 \alpha_2} | \Omega_2^{C2}\right] Pr(\Omega_2^{C2}) + E\left[\frac{\xi_1}{\alpha_1} | \Omega_3^{C2}\right] Pr(\Omega_3^{C2}).$$

Observing that  $\Omega_2^{C1} \subset (\Omega_2^{C2} \cup \Omega_3^{C2})$  completes the proof (see Figure A.1).

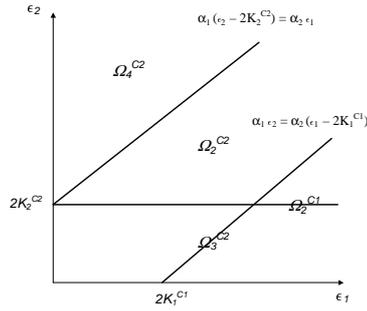


Figure A.1: The relationship between  $\Omega_2^{C1}$  and  $\Omega_2^{C2}$

Thus, when  $c_2 < E[\frac{\xi_2}{\alpha_2}]$ , the structure of the optimal investment strategy is as follows:

$$\vec{K}^C = \begin{cases} \vec{K}^{C1}, & \text{if } c_1 \leq c_1^l, \\ \vec{K}^{C2}, & \text{if } c_1 \geq c_1^u, \\ \vec{K}^{CB}, & \text{if } c_1^l < c_1 < c_1^u. \end{cases}$$

## Appendix D

### Proof of Theorem 4

Consider the special case where  $Pr(\alpha_2\xi_1 < \alpha_1\xi_2) = 1$ . Observe that  $\Omega_2^{C1} = \Omega_3^{C1} = \phi$ . Then, by Lemma 3,  $\vec{K}_1^{C1}$  is not a possible solution in this case.

Similarly, observe that  $\Omega_0^{CN} = \Omega_1^{CN} = \Omega_2^{CN} = \Omega_3^{CN} = \phi$  and  $\Omega_4^{CN} = U$ . Thus, the first-order KKT conditions reduce to the following:

$$\mathbf{KKT-1:} \quad E[\frac{\xi_2}{\alpha_2}] = c_1 - v_1 \Rightarrow c_1 = E[\frac{\xi_2}{\alpha_2}] + v_1,$$

$$\mathbf{KKT-2:} \quad E[\frac{\xi_2}{\alpha_2}] = c_2 - v_2 \Rightarrow c_2 = E[\frac{\xi_2}{\alpha_2}] + v_2.$$

Thus, this investment strategy is possible only if  $c_1 > c_2 \geq E[\frac{\xi_2}{\alpha_2}]$ . Therefore, it follows by Theorem 3 that when  $c_2 \geq E[\frac{\xi_2}{\alpha_2}]$ ,  $\vec{K}^{CN}$  is the only possible solution.

Next, we consider the case where  $c_2 < E[\frac{\xi_2}{\alpha_2}]$ .

Boundary solution  $\vec{K}^{C2} = (K_1^{C2} = 0, K_2^{C2} > 0)$ :

It follows from the first-order KKT conditions that  $\vec{K}^{C2}$  is a possible solution only if  $c_1 - v_1 =$

$$c_2 + E\left[\frac{\alpha_2 \xi_1 - \alpha_1 (\xi_2 - 2K_2^{C2})}{\alpha_1 \alpha_2} | \Omega_2^{C2}\right] Pr(\Omega_2^{C2}) + E\left[\frac{\xi_1}{\alpha_1} | \Omega_3^{C2}\right] Pr(\Omega_3^{C2}) \equiv c_1^u \Rightarrow c_1 \geq c_1^u.$$

Thus, the structure of the optimal investment strategy, when  $\alpha_2 \xi_1 < \alpha_1 \xi_2$ ,  $c_1 > c_2$ , and  $c_2 < E\left[\frac{\xi_2}{\alpha_2}\right]$ , is as follows:

$$\vec{K}^C = \begin{cases} \vec{K}^{C2}, & \text{if } c_1 \geq c_1^u, \\ \vec{K}^{CB}, & \text{if } c_1 < c_1^u. \end{cases}$$

Next, consider the case where  $\vec{K}^C = \vec{K}^{CB}$ . If  $\frac{K_1^{CB}}{\alpha_1} \geq \frac{K_2^{CB}}{\alpha_2}$ , then  $\Omega_2^{CB} = \Omega_3^{CB} = \phi$ , and  $\vec{K}^{CB}$  is not a possible solution by Lemma 3. Thus, we must have  $\frac{K_1^{CB}}{\alpha_1} < \frac{K_2^{CB}}{\alpha_2}$ .

## Appendix E

### Proof of Lemma 4

Recall, by Theorem 1, that if  $c_i < E\left[\frac{\xi_i}{\alpha_i}\right]$ , then  $K_i^D$ ,  $i = 1, 2$ , is the unique solution to

$$E\left[\frac{\xi_i - 2K_i^D}{\alpha_i} | \xi_i > 2K_i^D\right] Pr(\xi_i > 2K_i^D) = c_i. \quad (\text{A.10})$$

Otherwise,  $K_i^D = 0$ .

Boundary solution  $\vec{K}^{C1} = (K_1^{C1} > 0, K_2^{C1} = 0)$ :

From **KKT-1**,

$$\begin{aligned} & E\left[\frac{\xi_1 - 2K_1^{C1}}{\alpha_1} | \Omega_1^{C1}, \xi_1 > 2K_1^{C1}\right] Pr(\Omega_1^{C1}, \xi_1 > 2K_1^{C1}) \\ & + E\left[\frac{\xi_1 - 2K_1^{C1}}{\alpha_1} | \Omega_2^{C1}\right] Pr(\Omega_2^{C1}) + E\left[\frac{\xi_1 - 2K_1^{C1}}{\alpha_1} | \Omega_4^{C1}, \xi_1 > 2K_1^{C1}\right] Pr(\Omega_4^{C1}, \xi_1 > 2K_1^{C1}) \end{aligned}$$

$$\begin{aligned}
& + E\left[\frac{\alpha_1\xi_2 - \alpha_2\xi_1 + 2\alpha_2K_1^{C1}}{\alpha_1(\alpha_1 + \alpha_2)} \mid \Omega_1^{C1}, \xi_1 > 2K_1^{C1}\right] Pr(\Omega_1^{C1}, \xi_1 > 2K_1^{C1}) \\
& + E\left[\frac{\alpha_1(\xi_2 - 2K_1^{C1}) - \alpha_2(\xi_1 - 2K_1^{C1})}{\alpha_1\alpha_2} \mid \Omega_4^{C1}, \xi_1 > 2K_1^{C1}\right] Pr(\Omega_4^{C1}, \xi_1 > 2K_1^{C1}) \\
& + E\left[\frac{\xi_1 + \xi_2 - 2K_1^{C1}}{(\alpha_1 + \alpha_2)} \mid \Omega_1^{C1}, \xi_1 < 2K_1^{C1}\right] Pr(\Omega_1^{C1}, \xi_1 < 2K_1^{C1}) \\
& + E\left[\frac{\xi_2 - 2K_1^{C1}}{\alpha_2} \mid \Omega_4^{C1}, \xi_1 < 2K_1^{C1}\right] Pr(\Omega_4^{C1}, \xi_1 < 2K_1^{C1}) = c_1 \\
\Rightarrow & E\left[\frac{\xi_1 - 2K_1^{C1}}{\alpha_1} \mid \xi_1 > 2K_1^{C1}\right] Pr(\xi_1 > 2K_1^{C1}) \\
& + \underbrace{E\left[\frac{\alpha_1\xi_2 - \alpha_2\xi_1 + 2\alpha_2K_1^{C1}}{\alpha_1(\alpha_1 + \alpha_2)} \mid \Omega_1^{C1}, \xi_1 > 2K_1^{C1}\right] Pr(\Omega_1^{C1}, \xi_1 > 2K_1^{C1})}_{>0 \text{ (by definition of } \Omega_1^{C1})} \\
& + \underbrace{E\left[\frac{\xi_1 + \xi_2 - 2K_1^{C1}}{(\alpha_1 + \alpha_2)} \mid \Omega_1^{C1}, \xi_1 < 2K_1^{C1}\right] Pr(\Omega_1^{C1}, \xi_1 < 2K_1^{C1})}_{>0 \text{ (by definition of } \Omega_1^{C1})} \\
& + \underbrace{E\left[\frac{\alpha_1(\xi_2 - 2K_1^{C1}) - \alpha_2(\xi_1 - 2K_1^{C1})}{\alpha_1\alpha_2} \mid \Omega_4^{C1}, \xi_1 > 2K_1^{C1}\right] Pr(\Omega_4^{C1}, \xi_1 > 2K_1^{C1})}_{>0 \text{ (by definition of } \Omega_4^{C1})} \\
& + \underbrace{E\left[\frac{\xi_2 - 2K_1^{C1}}{\alpha_2} \mid \Omega_4^{C1}, \xi_1 < 2K_1^{C1}\right] Pr(\Omega_4^{C1}, \xi_1 < 2K_1^{C1})}_{>0 \text{ (by definition of } \Omega_4^{C1})} = c_1
\end{aligned}$$

Therefore,  $E\left[\frac{\xi_1 - 2K_1^{C1}}{\alpha_1} \mid \xi_1 > 2K_1^{C1}\right] Pr(\xi_1 > 2K_1^{C1}) < c_1 \Rightarrow K_1^{C1} > K_1^D$ , by Equation (A.10).

Boundary solution  $\vec{K}^{CB} = (K_1^{CB} > 0, K_2^{CB} > 0)$ :

From **KKT-1**:

$$\begin{aligned}
& E\left[\frac{\xi_1 - 2K_1^{CB}}{\alpha_1} \mid \Omega_1^{CB}, \xi_1 > 2K_1^{CB}\right] Pr(\Omega_1^{CB}, \xi_1 > 2K_1^{CB}) \\
& + E\left[\frac{\xi_1 - 2K_1^{CB}}{\alpha_1} \mid \Omega_2^{CB}\right] Pr(\Omega_2^{CB}) + E\left[\frac{\xi_1 - 2K_1^{CB}}{\alpha_1} \mid \Omega_3^{CB}\right] Pr(\Omega_3^{CB}) \\
& + E\left[\frac{\xi_1 - 2K_1^{CB}}{\alpha_1} \mid \Omega_4^{CB}, \xi_1 > 2K_1^{CB}\right] Pr(\Omega_4^{CB}, \xi_1 > 2K_1^{CB})
\end{aligned}$$

$$\begin{aligned}
& + E\left[\frac{\alpha_1(\xi_2 - 2K_2^{CB}) - \alpha_2(\xi_1 - 2K_1^{CB})}{\alpha_1(\alpha_1 + \alpha_2)} \mid \Omega_1^{CB}, \xi_1 > K_1^{CB}\right] Pr(\Omega_1^{CB}, \xi_1 > 2K_1^{CB}) \\
& + E\left[\frac{\alpha_1(\xi_2 - 2K_1^{CB} - 2K_2^{CB}) - \alpha_2(\xi_1 - 2K_1^{CB})}{\alpha_1\alpha_2} \mid \Omega_4^{CB}, \xi_1 > 2K_1^{CB}\right] Pr(\Omega_4^{CB}, \xi_1 > 2K_1^{CB}) \\
& + E\left[\frac{\xi_1 + \xi_2 - 2K_1^{CB} - 2K_2^{CB}}{(\alpha_1 + \alpha_2)} \mid \Omega_1^{CB}, \xi_1 < 2K_1^{CB}\right] Pr(\Omega_1^{CB}, \xi_1 < 2K_1^{CB}) \\
& + E\left[\frac{\xi_2 - 2K_1^{CB} - 2K_2^{CB}}{\alpha_2} \mid \Omega_4^{CB}, \xi_1 < 2K_1^{CB}\right] Pr(\Omega_4^{CB}, \xi_1 < 2K_1^{CB}) = c_1 \\
\Rightarrow & E\left[\frac{\xi_1 - 2K_1^{CB}}{\alpha_1} \mid \xi_1 > 2K_1^{CB}\right] Pr(\xi_1 > 2K_1^{CB}) \\
& + \underbrace{E\left[\frac{\alpha_1(\xi_2 - 2K_2^{CB}) - \alpha_2(\xi_1 - 2K_1^{CB})}{\alpha_1(\alpha_1 + \alpha_2)} \mid \Omega_1^{CB}, \xi_1 > 2K_1^{CB}\right] Pr(\Omega_1^{CB}, \xi_1 > 2K_1^{CB})}_{>0 \text{ (by definition of } \Omega_1^{CB})} \\
& + \underbrace{E\left[\frac{\xi_1 + \xi_2 - 2K_1^{CB} - 2K_2^{CB}}{(\alpha_1 + \alpha_2)} \mid \Omega_1^{CB}, \xi_1 < 2K_1^{CB}\right] Pr(\Omega_1^{CB}, \xi_1 < 2K_1^{CB})}_{>0 \text{ (by definition of } \Omega_1^{CB})} \\
& + \underbrace{E\left[\frac{\alpha_1(\xi_2 - 2K_1^{CB} - 2K_2^{CB}) - \alpha_2(\xi_1 - 2K_1^{CB})}{\alpha_1\alpha_2} \mid \Omega_4^{CB}, \xi_1 > 2K_1^{CB}\right] Pr(\Omega_4^{CB}, \xi_1 > 2K_1^{CB})}_{>0 \text{ (by definition of } \Omega_4^{CB})} \\
& + \underbrace{E\left[\frac{\xi_2 - 2K_1^{CB} - 2K_2^{CB}}{\alpha_2} \mid \Omega_4^{CB}, \xi_1 < 2K_1^{CB}\right] Pr(\Omega_4^{CB}, \xi_1 < 2K_1^{CB})}_{>0 \text{ (by definition of } \Omega_4^{CB})} = c_1
\end{aligned}$$

Therefore,  $E\left[\frac{\xi_1 - 2K_1^{CB}}{\alpha_1} \mid \xi_1 > 2K_1^{CB}\right] Pr(\xi_1 > 2K_1^{CB}) < c_1 \Rightarrow K_1^{CB} > K_1^D$ , by Equation (A.10).

For the second KKT condition, we obtain:

$$\begin{aligned}
& E\left[\frac{\xi_2 - 2K_2^{CB}}{\alpha_2} \mid \Omega_1^{CB}\right] Pr(\Omega_1^{CB}) + E\left[\frac{\xi_2 - 2K_2^{CB}}{\alpha_2} \mid \Omega_2^{CB}\right] Pr(\Omega_2^{CB}) \\
& + E\left[\frac{\xi_2 - 2K_2^{CB}}{\alpha_2} \mid \Omega_4^{CB}\right] Pr(\Omega_4^{CB}) - \frac{2K_1^{CB}}{\alpha_2} Pr(\Omega_4^{CB}) \\
& + E\left[\frac{\alpha_2(\xi_1 - 2K_1^{CB}) - \alpha_1(\xi_2 - 2K_2^{CB})}{\alpha_2(\alpha_1 + \alpha_2)} \mid \Omega_1^{CB}\right] Pr(\Omega_1^{CB}) = c_2
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow E\left[\frac{\xi_2 - 2K_2^{CB}}{\alpha_2} \mid \xi_2 > 2K_2^{CB}\right] Pr(\xi_2 > 2K_2^{CB}) - \underbrace{\frac{2K_1^{CB}}{\alpha_2}}_{>0} Pr(\Omega_4^{CB}) \\
&\quad + \underbrace{E\left[\frac{\epsilon_2 - 2K_2^{CB}}{\alpha_2} \mid \Omega_0^{CB}, \epsilon_2 < 2K_2^{CB}\right]}_{<0} Pr(\Omega_0^{CB}, \epsilon_2 < 2K_2^{CB}) \\
&\quad + \underbrace{E\left[\frac{\alpha_2(\xi_1 - 2K_1^{CB}) - \alpha_1(\xi_2 - 2K_2^{CB})}{\alpha_2(\alpha_1 + \alpha_2)} \mid \Omega_1^{CB}\right]}_{<0 \text{ (by definition of } \Omega_1^{CB})} Pr(\Omega_1^{CB}) = c_2
\end{aligned}$$

Therefore,  $E\left[\frac{\xi_2 - 2K_2^{CB}}{\alpha_2} \mid \xi_2 > 2K_2^{CB}\right] Pr(\xi_2 > 2K_2^{CB}) > c_2 \Rightarrow K_2^{CB} < K_2^D$ , by Equation (A.10).

This completes the proof. ■

## Appendix F

### Proof of Theorem 6

Writing down the corresponding first-order KKT conditions at each of the boundary solutions  $\vec{K}^{CN}$ ,  $\vec{K}^{C1}$ ,  $\vec{K}^{C2}$ , and  $\vec{K}^{CB}$ , one can see that each of these boundary solutions is possible.

Thus, the first part follows directly by Theorem 3. To prove the second part, assume, to the contrary, that  $K_1^C + K_2^C > \frac{q}{2}$ . Then, the optimal investment strategy must be one of the forms  $\vec{K}^{CB}$ ,  $\vec{K}^{C1}$ , or  $\vec{K}^{C2}$ .

Consider first the boundary solution  $\vec{K}^{CB} = (K_1^{CB} > 0, K_2^{CB} > 0)$ , with  $v_1 = 0, v_2 = 0$ . At this boundary solution, we have  $\Omega_0^{CB} = \{\xi > q - 2K_1^{CB}\}$ ,  $\Omega_1^{CB} = \emptyset$ ,  $\Omega_2^{CB} = \emptyset$ , since  $2K_2^{CB} > \frac{2\alpha_1 K_2^{CB} - 2\alpha_2 K_1^{CB} + q\alpha_2}{\alpha_1 + \alpha_2}$ ,  $\Omega_3^{CB} = \{\xi < q - 2K_1^{CB}\}$ , and  $\Omega_4^{CB} = \emptyset$ . To show that  $\Omega_4^{CB} = \emptyset$ , suppose, to the contrary, that  $\Omega_4^{CB} \neq \emptyset$ ; that is,

$$(\alpha_1 + \alpha_2)\xi > 2\alpha_1(K_1^{CB} + K_2^{CB}) + q\alpha_2 > 2\alpha_1q + \alpha_2q, \quad \text{since } K_1^C + K_2^C > \frac{q}{2}$$

$$\Rightarrow \xi_2 > (\xi_1 + \xi_2) \underbrace{\frac{(2\alpha_1 + \alpha_2)}{\alpha_1 + \alpha_2}}_{>1},$$

where the last inequality follows because  $\xi = \xi_2$  and  $q = \xi_1 + \xi_2$ , by definition. However, this inequality is not possible, and therefore,  $\Omega_4^{CB} = \emptyset$ . Then, **KKT-2** reduces to  $0 = c_2 - v_2 \Rightarrow v_2 = c_2$ , which contradicts with  $v_2 = 0$ . Thus,  $\vec{K}^{CB}$  is not a possible solution in this case.

Next, consider the boundary solution  $\vec{K}^{C1} = (K_1^{C1} > 0, K_2^{C1} = 0)$ , with  $v_1 = 0, v_2 \geq 0$ . In a similar way, we can show that  $\Omega_0^{C1} = U$ ,  $\Omega_1^{C1} = \emptyset$ ,  $\Omega_2^{C1} = \emptyset$ ,  $\Omega_3^{C1} = \emptyset$ , and  $\Omega_4^{C1} = \emptyset$ . Then, it follows by Lemma 3, that  $\vec{K}^{C1}$  is not a possible solution.

Finally, consider the boundary solution  $\vec{K}^{C2} = (K_1^{C2} = 0, K_2^{C2} > 0)$ , with  $v_1 \geq 0, v_2 = 0$ . Then, we obtain that  $\Omega_0^{C2} = \{\xi > q\}$ ,  $\Omega_1^{C2} = \emptyset$ ,  $\Omega_2^{C2} = \emptyset$ ,  $\Omega_3^{C2} = \{\xi < q\}$ , and  $\Omega_4^{C2} = \emptyset$ . Then, **KKT-2** reduces to  $0 = c_2 - v_2 \Rightarrow v_2 = c_2$ , which contradicts with  $v_2 = 0$ . Thus,  $\vec{K}^{C2}$  is not a possible solution in this case. Hence, we must have  $K_1^C + K_2^C \leq \frac{q}{2}$ . This completes the proof. ■

# Vita

Seong-Jong Hong was born in Jeju island, the most beautiful place in Korea, in 1973. He received his B.S. degree in Industrial Engineering at Ajou University in Korea in 1995, and M.S. degree in Industrial and Systems Engineering at Colorado State University at Pueblo in 2000. He was the recipient of the full year scholarship for the undergraduate education from Ajou university in Korea in 1992, and the R.O.T.C. scholarship from March 1993 to February 1995. He received the best Executive Officer Award in the Teaching Skill Competition from Artillery Corps Division in Korea in 1995.

His research focus is on operations management, with emphasis on the analysis of flexible systems in manufacturing and service environments. His research was supported in part by the National Science Foundation. He was selected as one of finalists in the SCALE Doctoral Dissertation Proposal Competition at the University of Florida in 2004. He worked as a Graduate Teaching Assistant and a Research Assistant in the department throughout his Ph.D. studies.