Appendix C: Properties of the Squared, EWLS and MMRR Means Model Residuals

Appendix C.1: Distribution of $\mathbf{e}_i^{2\text{(mmrr)}}$

Recall that we write the robust means model fit as:

$$\hat{\mathbf{y}}^{\text{(mmrr)}} = \mathbf{H}^{\text{(mmrr)}}\mathbf{y}$$

where

$$\mathbf{H}^{\text{(mmrr)}} = \mathbf{H}^{\text{(ewls)}} + \lambda_\mu \mathbf{H}_{b_\mu}^{\text{(llr)}}(\mathbf{I} - \mathbf{H}^{\text{(ewls)}}) .$$

Notice from above that if there is no means model misspecification, $\lambda_\mu = 0$, and thus, $\mathbf{H}^{\text{(mmrr)}} = \mathbf{H}^{\text{(ewls)}}$. The properties of the squared, MMRR means model residuals which are discussed in this Appendix can therefore be applied to the EWLS squared residuals simply by replacing $\mathbf{H}^{\text{(mmrr)}}$ with $\mathbf{H}^{\text{(ewls)}}$ (since $\mathbf{H}^{\text{(mmrr)}} = \mathbf{H}^{\text{(ewls)}}$ when λ_μ is zero).

We have that $\mathbf{e}^{\text{(mmrr)}} = \left(\mathbf{I} - \mathbf{H}^{\text{(mmrr)}}\right)\mathbf{y}$ where $\mathbf{y} \sim \text{N}\left[\mathbf{X\beta} + \mathbf{f}, \mathbf{V}\right]$. Thus, $\mathbf{e}^{\text{(mmrr)}}$ has a Normal distribution with mean $\left(\mathbf{I} - \mathbf{H}^{\text{(mmrr)}}\right)\mathbf{f}$ and variance $\left(\mathbf{I} - \mathbf{H}^{\text{(mmrr)}}\right)\mathbf{V}\left(\mathbf{I} - \mathbf{H}^{\text{(mmrr)}}\right)'$. It follows that the distribution of the i^{th} means model residual is given by:

$$\text{e}_i^{\text{(mmrr)}} \sim \text{N}\left[\mu_{\text{e}_i}, \sigma^2_{\text{e}_i}\right]$$

where

$$\mu_{\text{e}_i} = \left(1 - \text{h}_{ii}^{\text{(mmrr)}}\right)\text{f}_i - \sum_{j \neq i}^{n} \text{h}_{ij}^{\text{(mmrr)}}\text{f}_j$$

and

$$\sigma^2_{\text{e}_i} = \left(1 - \text{h}_{ii}^{\text{(mmrr)}}\right)^2 \sigma_i^2 + \sum_{i \neq j}^{n} \text{h}_{ij}^{2\,\text{(mmrr)}}\sigma_j^2 .$$

Rescaling $\text{e}_i^{\text{(mmrr)}}$ we have:

$$\frac{e_i^{(mmrr)}}{\sigma_{e_i}} \sim N\left[\frac{\mu_{e_i}}{\sigma_{e_i}}, 1\right].$$

Squaring both sides results in:

$$\frac{e_i^{2(mmrr)}}{\sigma_{e_i}^2} \sim \chi'^{2}_{(1),\ \delta_i}$$

where

$$\delta_i = \frac{\mu_{e_i}^2}{\sigma_{e_i}^2}.$$

Thus, it follows that $e_i^{2(mmrr)}$ has a distribution which is a multiple of a non-central Chi-Square distribution:

$$e_i^{2(mmrr)} \sim \sigma_{e_i}^2 \chi'^{2}_{(1),\ \delta_i}.$$

Appendix C.2: Expected Value of $e_i^{2(mmrr)}$

Since $e_i^{2(mmrr)}$ has a distribution which is a multiple of a non-central Chi-Square distribution, we can write

$$E\left(e_i^{2(mmrr)}\right) = \sigma_{e_i}^2 E\left(\chi'^{2}_{(1),\ \delta_i}\right)$$

$$= \sigma_{e_i}^2 \left(1 + \delta_i\right).$$

Substituting for δ_i we have:

$$E\left(e_i^{2(mmrr)}\right) = \sigma_{e_i}^2 \left(1 + \frac{\mu_{e_i}^2}{\sigma_{e_i}^2}\right)$$

$$= \sigma^2_{e_i} + \mu^2_{e_i}$$

Appendix C.3: Variance of $e_i^{2(\mathrm{mmrr})}$

In a similar fashion to the derivation of the $E\left(e_i^{2(\mathrm{mmrr})}\right)$, the variance of $e_i^{2(\mathrm{mmrr})}$ proceeds as follows:

$$\mathrm{Var}\left[e_i^{2(\mathrm{mmrr})}\right] = \mathrm{Var}\left(\sigma^2_{e_i}\,\chi'^{\,2}_{(1),\ \delta_i}\right)$$

$$= \left[\sigma^2_{e_i}\right]^2 \left[2\left(1 + 2\delta_i\right)\right]$$

$$= 2\left[\sigma^2_{e_i}\right]^2 \left\{1 + 2\left\{\frac{\mu^2_{e_i}}{\sigma^2_{e_i}}\right\}\right\}$$

$$\mathrm{Var}\left(e_i^{2(\mathrm{mmrr})}\right) = 2\left[\sigma^2_{e_i}\right]^2 + 4\left[\left(\sigma^2_{e_i}\right)\left(\mu^2_{e_i}\right)\right].$$